12) 1) důkaz sporem, předpoklad $\dim \mathcal{H} = n < \infty$

$\dim \mathcal{H} < \infty \Rightarrow P, Q$ omezené, $\mathcal{D}(P) = \mathcal{D}(Q) = \mathcal{H}$; P, Q matice

$\operatorname{tr} PQ - \operatorname{tr} QP = \frac{1}{i} \operatorname{tr} I = \frac{n}{i}$ (1)

$\operatorname{tr} PQ = \sum_{j=1}^{n} \langle \varphi_j, PQ\varphi_j \rangle = \sum_{j,k} \langle \varphi_j, P\varphi_k \rangle \langle \varphi_k, Q\varphi_j \rangle = \operatorname{tr} QP$

(1) $\Rightarrow 0 = \frac{n}{i}$ spor

2) důkaz sporem, předpoklad $\exists f \in \mathcal{H}, Pf = \lambda f \quad \lambda \in \mathbb{R}, \|f\| \neq 0$

$\langle Pf, Qf \rangle - \langle Qf, Pf \rangle = \frac{1}{i} \|f\|^2$

$\lambda \langle f, Qf \rangle - \lambda \langle Qf, f \rangle = \frac{1}{i} \|f\|^2$

$0 = \frac{1}{i} \|f\|^2$ spor

3) důkaz sporem, předpoklad P omezený

$\lambda \in \sigma(Q) \subset \mathbb{R}$ (spektrum každého s.a. operátoru je neprázdné)

$\exists (\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(Q), \|\varphi_n\| = 1, \|(Q-\lambda)\varphi_n\| \to 0$ (Weylovo kritérium)

$\frac{1}{i} \|\varphi_n\|^2 = \langle P\varphi_n, Q\varphi_n \rangle - \langle Q\varphi_n, P\varphi_n \rangle$

$1 = |\langle P\varphi_n, Q\varphi_n \rangle - \langle Q\varphi_n, P\varphi_n \rangle|$

$1 = |\langle P\varphi_n, (Q-\lambda)\varphi_n \rangle - \langle (Q-\lambda)\varphi_n, P\varphi_n \rangle|$

$1 \leq 2 \|P\varphi_n\| \|(Q-\lambda)\varphi_n\|$

$1 \leq 2 \|P\| \|(Q-\lambda)\varphi_n\| \xrightarrow[n\to\infty]{} 0$

13) $-\Delta$ s.a.

ukážeme a) $\sigma(-\Delta) \subseteq \mathbb{R}^+$ a b) $\sigma(-\Delta) \supseteq \mathbb{R}^+$

a) $\langle \varphi, -\Delta\varphi \rangle = \|\nabla\varphi\|^2 \geq 0 \Rightarrow \sigma(-\Delta) \subseteq \mathbb{R}^+$

b) $\chi(s) \in [0,1]$

$$\chi(s) = \begin{cases} 1 & |s| < \frac{1}{2} \\ 0 & |s| > 1 \end{cases}$$

$\chi(s) \in C_0^\infty(\mathbb{R})$

$\varphi_n = e^{ikx}\chi\left(\frac{|x|}{n}\right) \qquad k \in \mathbb{R}^n$

$\nabla\varphi_n = ik\, e^{ikx}\chi\left(\frac{|x|}{n}\right) + e^{ikx}\chi'\left(\frac{|x|}{n}\right)\frac{1}{n}\frac{x}{|x|}$

$\Delta\varphi_n = -k^2\varphi_n + 2ik\frac{x}{|x|}\frac{1}{n}e^{ikx}\chi'\left(\frac{|x|}{n}\right) + e^{ikx}\frac{1}{n^2}\chi''\left(\frac{|x|}{n}\right) + e^{ikx}\chi'\left(\frac{|x|}{n}\right)\frac{1}{n}\frac{2}{|x|}$

$\Rightarrow (-\Delta - k^2)\varphi_n = \mathcal{O}\left(\frac{1}{n}\right)$

$\|(-\Delta - k^2)\varphi_n\| \to 0 \quad n \to \infty \quad \Rightarrow k^2 \in \sigma(-\Delta) \quad \forall k \in \mathbb{R}^n$

tzn.: $\sigma(-\Delta) = \mathbb{R}_0^+$

14) $x \in \mathcal{H}, \|x\| = 1$ pak

$\langle x, Sx\rangle = \langle Sx, x\rangle = \overline{\langle x, Sx\rangle} \Rightarrow \langle x, Sx\rangle \in \mathbb{R}$

$|\langle x, Sx\rangle| \leq \|x\| \|Sx\| \leq \|S\|$

$\Rightarrow [m, M] \subseteq [\|-S\|, \|S\|]$

ověříme $\sigma(S) \subseteq [m, M]$

nechť $\lambda \notin \overline{W(S)}$ pak $d : d(\lambda, \overline{W(S)}) > 0$

numerický obor

$\|x\| = 1 \Rightarrow \quad d \leq |\lambda - \langle x, Sx\rangle| = |\langle x, (\lambda - S)x\rangle| \leq \|(S-\lambda)x\|$

ze cvičení 11 $\lambda - S$ injektivní, $(\lambda - S)^{-1}$ omezená, $Ran(\lambda - S)$ hustá $\Rightarrow \lambda \in \rho(S)$

$\Rightarrow \sigma(S) \subseteq [m, M] \subseteq [-\|S\|, \|S\|]$

$W(S-m) = W(S) - m$ a také $\sigma(S-m) = \sigma(S) - m$

$r(S-m) = \|S-m\| \Rightarrow \|S-m\| = M - m \in \sigma(S-m)$ tzn. $M \in \sigma(S)$

stejně

$W(S-M) = W(S) - M$ a $-\|S-M\| = m - M$ (neb $r(S-M) = \|S-M\|$) tzn. $m - M \in \sigma(S-M)$

$r(S) = \|S\| \Rightarrow$ buď $M = \|S\|$ nebo $-m = \|S\|$

15) zavedeme funkci

$$\chi_R(x) = \begin{cases} 1 & |x|<R \\ \frac{1}{2}\left(1+\cos\left(\frac{\pi}{R}(R-|x|)\right)\right) & \\ 0 & |x|>2R \end{cases}$$

$$\psi(x-h) = \chi_R(x)\,\psi(x-h) + (1-\chi_R(x))\psi(x-h)$$

$$\frac{\psi(x-h)}{|x|} = \chi_R \frac{\psi(x-h)}{|x|} + \underbrace{\frac{1}{|x|}(1-\chi_R(x))\,\psi(x-h)}_{\leq \frac{1}{R}(1-\chi_R(x))\psi(x-h)}$$

$$\left\|\frac{f}{|x|}\right\|_2 \leq 2\|\nabla f\|_2 \quad \forall f \in C_0^\infty$$

$$\int|\nabla(\chi_R\,\psi(x-h))|^2 \leq 2\left(\int|\nabla\chi_R|^2|\psi(x-h)|^2dx + \int|\chi_R|^2|\nabla\psi(x-h)|^2dx\right)$$

$$\int_{\mathbb{R}^3}|\nabla\chi_R(x)|^2|\psi(x-h)|^2dx = \int_{\mathbb{R}^3}|\nabla\chi_R(x-h)|^2|\psi(x)|^2dx \xrightarrow[h\to\infty]{} 0$$

$$\int_{\mathbb{R}^3}|\chi_R|^2|\nabla\psi(x-h)|^2dx = \int_{\mathbb{R}^3}|\chi_R(x-h)|^2|\nabla\psi(x)|^2dx \xrightarrow[h\to\infty]{} 0$$

16) 1) $\mathrm{Tr}\, W^2 \leq \|W\|\, \mathrm{Tr}\, W \leq (\mathrm{Tr}\, W)^2 \leq 1$

2) W p.o. $\Rightarrow \exists\, n$ vlastních hodnot $\lambda_1 \dots \lambda_n$

$\sum_{j=1}^{n} \lambda_j = 1$

$\mathrm{Tr}\, W^2 = \sum_{j=1}^{n} \lambda_j^2$

najdeme minimum: $\sum \lambda_j = 1$

$L = \sum \lambda_j^2 - a \sum \lambda_j$

$\Rightarrow \partial_j L = 0 \qquad 2\lambda_j - a = 0 \Rightarrow \lambda_j = \lambda_i \quad \forall i,j$

$\Rightarrow \lambda_j = \frac{1}{n}$ a $\mathrm{Tr}\, W^2 = \frac{1}{n}$

3) musíme ukázat $W' \geq 0$, $\mathrm{Tr}\, W' = 1$

$W' \geq 0$ ověříme přímo $\langle \varphi, EWE\varphi \rangle = \langle E^+\varphi, WE\varphi \rangle \geq 0$ n $W \geq 0$, $E \geq 0$

$\mathrm{Tr}\, W' = (\mathrm{Tr}(EW))^{-1}\, \mathrm{Tr}\, EWE = (\mathrm{Tr}(EW))^{-1}\, \mathrm{Tr}\, WE^2 = (\mathrm{Tr}(EW))^{-1}\, \mathrm{Tr}\, WE = 1$

4) $W = |\varphi\rangle\langle\varphi|$

$W' = (\mathrm{Tr}\, EW)^{-1}\, EWE = (\mathrm{Tr}(EW))^{-1} |E\varphi\rangle\langle E^+\varphi| = (\mathrm{Tr}(EW))^{-1} (E|\varphi\rangle\langle\varphi|E)$

stačí ověřit $\|E\varphi\|^2 \stackrel{?}{=} \mathrm{Tr}\, EW$

$\mathrm{Tr}\, EW = \sum_j \langle \varphi_j | E\varphi \rangle \langle \varphi | E\varphi_j \rangle = \langle \varphi, E\varphi \rangle = \langle \varphi, E^2 \varphi \rangle \underset{E=E^+}{=} \|E\varphi\|^2$

17) z přednášky: stav po měření $W' = \dfrac{E_A(\Delta)\, W\, E_A(\Delta)}{Tr(E_A(\Delta) W)}$ s pravděpodobností $w = Tr(E_A(\Delta) W)$

iterativně: $W,\ w = 1$

$$W_1 = \frac{E_{A_1}(\Delta_1)\, W\, E_{A_1}(\Delta_1)}{Tr(E_{A_1}(\Delta_1) W)} \qquad w_1 = Tr\big(E_{A_1}(\Delta_1) W\big)$$

$$W_2 = \frac{E_{A_2}(\Delta_2)\, W_1\, E_{A_2}(\Delta_2)}{Tr(E_{A_2}(\Delta_2) W_1)} \qquad w_2 = Tr\big(E_{A_2}(\Delta_2) W_1\big)\cdot w_1$$

$$= \frac{E_{A_2}(\Delta_2) E_{A_1}(\Delta_1)\, W\, E_{A_1}(\Delta_1) E_{A_2}(\Delta_2)}{Tr(E_{A_1}(\Delta_1) W)\; Tr\dfrac{(E_{A_2}(\Delta_2) E_{A_1}(\Delta_1) W E_{A_1}(\Delta_1))}{Tr(E_{A_1}(\Delta_1) W)}} = \frac{E_{A_2}(\Delta_2) E_{A_1}(\Delta_1)\, W\, E_{A_1}(\Delta_1) E_{A_2}(\Delta_2)}{Tr(E_{A_2}(\Delta_2) E_{A_1}(\Delta_1) W)}$$

$$Tr(EFWF) = Tr(EFWFE) = Tr(EFWF^{+}E^{+}) = Tr(EFW(EF)^{+}) = Tr((EF)^2 W) = Tr(EF$$

$$w_2 = Tr\big(E_{A_2}(\Delta_2) E_{A_1}(\Delta_1) W E_{A_1}(\Delta_1) E_{A_2}(\Delta_2)\big)$$

$\vdots$

$$W_n = \frac{E_{A_n}(\Delta_n) \dots E_{A_1}(\Delta_1) W E_{A_1}(\Delta_1) \dots E_{A_n}(\Delta_n)}{Tr(E_{A_n}(\Delta_n) \dots E_{A_1}(\Delta_1) W)} \qquad w_n = w_{n-1} \cdot \ldots \cdot w_1 = Tr(E_{A_n}(\Delta_n) \dots E_{A_1}(\Delta_1) W)$$

3) plyne z $E_A(\Delta_1) \cdot E_A(\Delta_2) = E_A(\Delta_1 \cap \Delta_2) = E_A(\Delta_2) E_A(\Delta_1)$