Stellar evolution: from molecular clouds to compact stars and black holes

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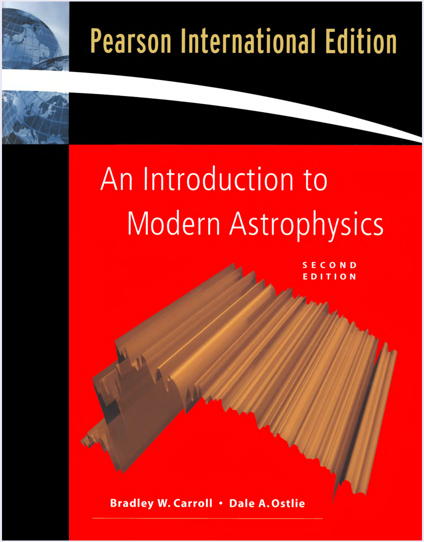
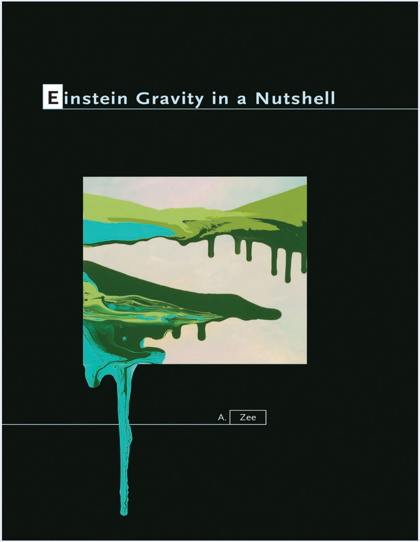
November 2, 2020

Literature

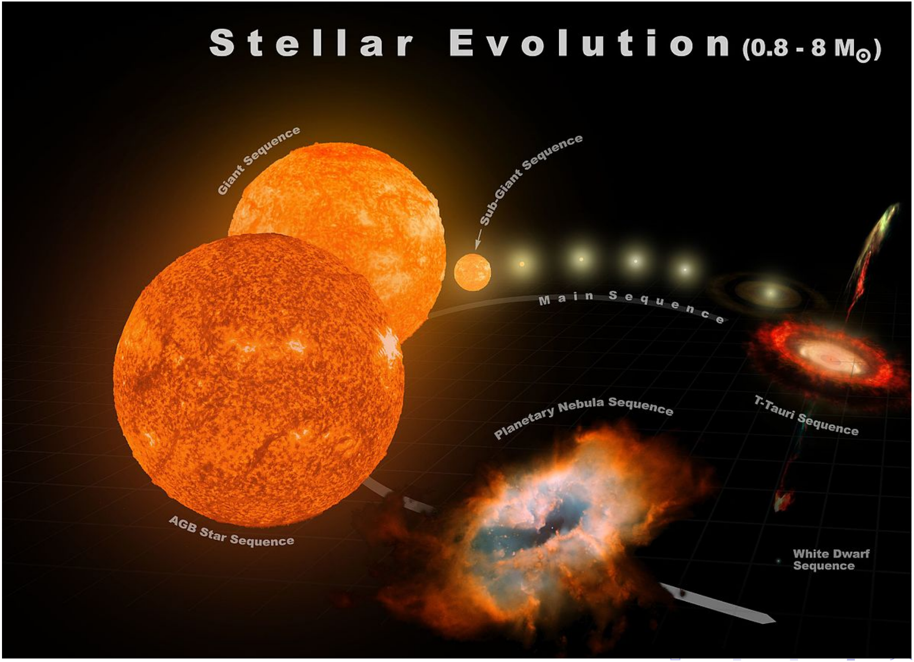
Our discussion is based on the books . . . + Wikipedia





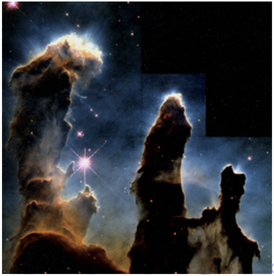
Stellar evolution

- Birth of a star
- Nuclear burning
- Star in hydrostatic equilibrium
- Schwarzschild solution
- Latest stages of stellar evolution
- Tolman-Oppenheimer-Volkov equation
- Compact stars
    - White Dwarfs
    - Supernovae
    - Neutron stars

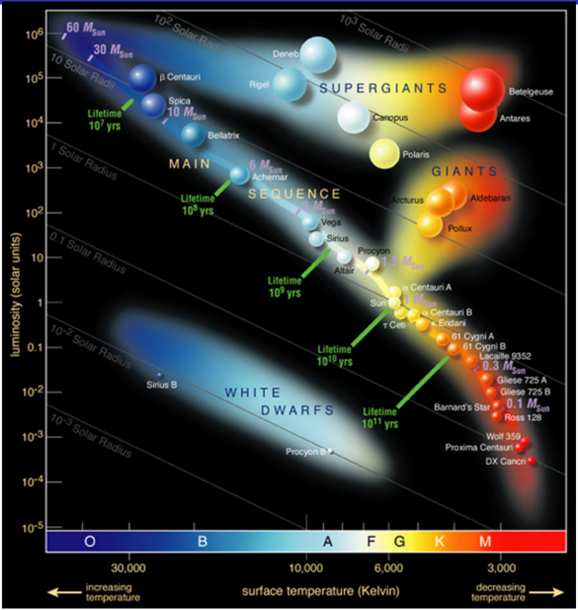

Stellar Evolution (0.8 - 8 $M_\odot$)
Giant Sequence
Sub-Giant Sequence
Main Sequence
T-Tauri Sequence
Planetary Nebula Sequence
AGB Star Sequence
White Dwarf Sequence

Birth of a star

- Begins with *gravitational collapse* and subsequent *fragmentation* of a giant molecular cloud ($2R \approx 100\ell y, M \approx 6 \times 10^6 M_\odot$).
- In each of these fragments, the collapsing gas releases gravitational potential energy as heat.
- As its temperature and pressure increases, a fragment condenses into a rotating sphere of superhot gas known as a **protostar**.
- Protostar continues to grow by *accretion* (accumulation) of gas and dust from the molecular cloud. Its further development is determined by its mass M.
- Protostars with $M \leq 0.08 M_\odot$, known as **brown dwarfs**, never reach temperatures high enough for nuclear fusion of hydrogen to begin.



Hertzsprung–Russell diagram: luminosty vs. temperature



https : //en.wikipedia.org/wiki/Luminosity

Nuclear burning stages in stars

- For a more-massive protostar, the core temperature will eventually reach $T \approx 10^6 K^*$, allowing hydrogen to fuse, first to deuterium and then to helium (PPI chain):
$p + p \rightarrow d + e^+ + \nu_e,\ d + p \rightarrow ^3_2 He + \gamma,\ ^3_2He + ^3_2 He \rightarrow ^4_2 He + 2p$
- In stars with $M_\star \gtrsim M_\odot$, the CNO cycle (Bethe 1938) contributes a large portion of the energy generation:
$^{12}_6C + p \rightarrow ^{13}_7 N + \gamma,\ ^{13}_7N \rightarrow ^{13}_6 C + e^+ + \nu_e,\ ^{13}_6C + p \rightarrow ^{14}_7 N + \gamma$
$^{14}_7N + p \rightarrow ^{15}_8 O + \gamma,\ ^{15}_8O \rightarrow ^{15}_7 N + e^+ + \nu_e,\ ^{15}_7N + p \rightarrow ^{12}_6 C + ^4_2 He$
- The onset of nuclear fusion leads relatively quickly to a hydrostatic equilibrium: energy released by the core maintains a high gas pressure, balancing the weight of the star's matter and preventing further gravitational collapse.
- The star thus evolves rapidly to a stable state, beginning the main-sequence phase of its evolution.

*) $1K = 8.621738 \times 10^{-5} eV, \quad 1eV = 11600K$

Nuclear burning stages in massive stars

- After the core of a main sequence star is transformed to ${}^4_2He$, it contracts until ($T > 6\times T_\odot$) enough for the triple alpha process.
  ${}^4_2He + {}^4_2He \rightarrow {}^8_4Be, (-92keV)$; ${}^8_4Be + {}^4_2He \rightarrow {}^{12}_6C + 2\gamma, (+7.36MeV)$.

- Carbon fusion starts in the cores of stars with $M_\star > M_\odot$ at birth.
  ${}^{12}_6C + {}^{12}_6C \rightarrow {}^{20}_{10}Ne + {}^4_2He, \quad \ldots \rightarrow {}^{23}_{11}Na + {}^1_1H, \quad \ldots \rightarrow {}^{23}_{12}Mg + n$.

- At even higher temperatures and densities neon fusion starts.
  ${}^{20}_{10}Ne + \gamma \rightarrow {}^{16}_8O + {}^4_2He, \qquad {}^{20}_{10}Ne + {}^4_2He \rightarrow {}^{24}_{12}Mg + \gamma$.

- As the neon-burning process ends, the core of the star contracts and heats until it reaches the ignition temperature for oxygen burning.
  ${}^{16}_8O + {}^{16}_8O \rightarrow {}^{20}_{10}Ne + {}^4_2He, \quad \ldots \rightarrow {}^{31}_{15}P + {}^1_1H, \quad \ldots \rightarrow {}^{31}_{16}S + n, \quad \ldots$

- Silicon burning @$M_\star > 8-10 \times M_\odot$ – adding more alphas:
  ${}^{28}_{14}Si + {}^4_2He \rightarrow {}^{32}_{16}S, \quad {}^{32}_{16}S + {}^4_2He \rightarrow {}^{36}_{18}Ar, \quad \ldots\ldots, {}^{48}_{24}Cr + {}^4_2He \rightarrow {}^{52}_{26}Fe,$
  ${}^{52}_{26}Fe + {}^4_2He \rightarrow {}^{56}_{28}Ni,$.

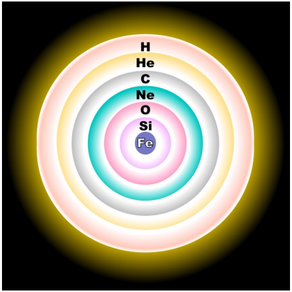
Latest stages of stellar evolution

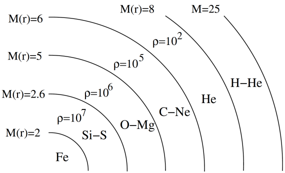
- $A = 56$ has the lowest M_A/m_N
  $\Rightarrow$ the silicon-burning sequence (lasting 1 day) stops the nuclear fusion process.

- Then ${}^{56}_{28}Ni \rightarrow {}^{56}_{27}Co \rightarrow {}^{56}_{26}Fe$ decays occur.
  The star has run out of nuclear fuel and within minutes its core begins to contract.

- The inert core will then accumulate mass from the outside layers until it reaches the limit mass $M_{Ch} \approx 1.4 M_\odot$.

- At this point, it will start to implode since no thermal pressure can balance the gravitation.



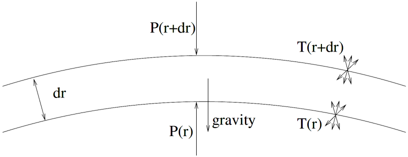


Star in hydrostatic equilibrium

- Gravitational force working for collapse is balanced by a pressure gradient:

$$\frac{dP}{dr} + \frac{G\mathcal{M}(r)\rho(r)}{r^2} = 0, \quad \mathcal{M}(r) = \int_0^r \rho(r')4\pi r'^2 dr' \qquad (1)$$

- Downward forces come from gravity and from $P(r+dr)$.
  Upward force comes only from $P(r)$.



- The pressure gradient dP/dr, through the EoS $P(T,\rho)$, generates the gradients $d\rho/dr$ and dT/dr.
- Later can be determined by considering radiation transport through the star i.e. from measurement of its luminosity L and its gradient dL/dr, respectively.
- In particular, $dL/dr = 4\pi r^2 \epsilon(r)$, is determined by the total energy release of binding energy in fusion reactions per unit volume $\epsilon(r)$.

Star's energy budget

- $$E_{tot} = \sum_{particles} (mc^2 + \frac{p^2}{2m}) + E_{grav} + E_{\gamma}$$
- For a spherical star with density $\rho_0 = const.$ and radius R_{grav} Eq.(1) gives $\mathcal{M}(r) = (4\pi/3)r^3\rho_0 = (r/R_{grav})M$:

$$E_{grav} = -\int_0^{R_{grav}} \frac{G\mathcal{M}(r)\rho_0}{r^2} 4\pi r^2 dr = -\frac{3}{5}\frac{GM^2}{R_{grav}} \quad (2)$$

- N.B. $\rho(r){=}\rho_0 \Rightarrow R \geq R_{grav}$. In particular, $R_{grav\odot} = 0.37R_{\odot}$.
- Multiplying (1) by $4\pi r^3$ and integrating by parts:

$$3\bar{P}V = -E_{grav} = (3/5)(GM^2)/R_{grav} \quad (3)$$

- Assuming $P_{\gamma} \approx 0$ and ideal gas EoS $PV = NkT$:

$$k\bar{T} = (1/5)\frac{N_b}{N_{part}}\frac{GN_b m_p^2}{R_{grav}} \approx (1/10)\frac{GN_b m_p^2}{R_{grav}} \quad (4)$$

where N_{part} is the total number of free massive particles in the star and $N_b \approx M/m_p$ is the number of baryons (nucleons) in the star.
Example: $N_{b\odot} \propto 10^{57} \Rightarrow k\bar{T} = 500eV \approx 6 \times 10^6 K$.

Star's thermodynamics

- Eq. (4) tells us that star has a negative specific heat $\bar{T} \sim 1/R_{grav}$.
  As star loses its energy and contracts, its temperature increases.
  Crucial for its ability to maintain a stable nuclear-burning regime.
- Using virial theorem $2\bar{E}_{kin} = -\bar{U}$ we obtain

$$\bar{E}_{kin} = \sum_{particles} \frac{3}{2} k\bar{T} = \sum_{particles} \frac{p^2}{2m} = -\frac{1}{2}\bar{U} = -\frac{1}{2}E_{grav} \qquad (5)$$

- For photons we have:

$$E_\gamma = \int_0^R \rho_\gamma 4\pi r^2 dr = \int_0^R \frac{2\pi^2}{30}\frac{kT^3}{\hbar c^3} 4\pi r^2 dr, \;\; n_\gamma = \frac{E_\gamma}{V} = \frac{2.5}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \approx \frac{\rho_\gamma}{3kT} \qquad (6)$$

- Using (4) and $\alpha_G \equiv (Gm_p^2)/(\hbar c) = 6.7\times 10^{-39}$ total number of photons in the star is:

$$N_\gamma = n_\gamma \bar{T} 4\pi R^3_{grav}/3 = \alpha_G^3 N_b^3 \frac{1}{1000}\frac{4\times 2.4}{3\pi} \qquad (7)$$

$\Rightarrow$ $N_\gamma/N_b \approx 10^{-3}\alpha_G^3 N_b^2$, for the sun $N_{\gamma\odot}/N_{b\odot} \approx 10^{-3}$
but for star with $M = 30M_\odot$ we have $N_\gamma/N_b \approx 30^2 \times 10^{-3} \rightarrow 1$.

Reminder: On metrics sign convention in QFT and GR

- In QFT time-dominant sign convention $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$ is used. The momentum squared $p^2 = p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu = m^2$.
- **From now on we will use the GR space-dominant sign convention** $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$ giving $p^2 = -m^2$.
- To go from one convention to the other, simply flip the sign of $g_{\mu\nu}$:

$$g_{\mu\nu}^{GR} = -g_{\mu\nu}^{QFT} \tag{8}$$

- N.B. Flipping the sign of $g_{\mu\nu}$ does not change the sign of $\Gamma^\mu_{\nu\kappa} = \frac{1}{2} g^{\mu\lambda}(\partial_\kappa g_{\lambda\nu} + \partial_\nu g_{\lambda\kappa} + \partial_\lambda g_{\nu\kappa})$

$\Rightarrow$ $R^\lambda_{\rho\mu\nu}$ and also $R_{\mu\nu}$ do not change a sign but the scalar curvature $R \equiv R^\mu_\mu = g^{\mu\nu} R_{\mu\nu}$ does.

$\Rightarrow$ $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ does not change too.

$\Rightarrow$ The GR convention: the sphere has positive scalar curvature.

Schwarzschild solution of the Einstein equations

- **Static and isotropic solution in the vacuum** (outside the mass M):

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \Leftrightarrow 8\pi G T_{\mu\nu} = 0 \qquad (9)$$

$$g_{tt} = f(r),\ g_{rr} = 1/f(r),\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2 sin^2\theta \qquad (10)$$

where $f(r) \equiv (1 - \frac{r_s}{r})$ and $r_s \equiv 2GM$, (Schwarzschild 1915)

- Represent first and and untill now the most important exact solution of Einstein equations of GR.

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \qquad (11)$$

- At $r = r_s$ (Schwarzschild radius horizon) the coefficient on radial coordinate $\rightarrow \infty$ while that for time coordidate $\rightarrow 0$.
- $f(r < r_s) < 0 \Rightarrow t$ and r exchange their roles!!!!
$\Rightarrow$ For ds^2 not to change its sign g_{rr} must change fast enough to offset the change of g_{tt}.
- No stationary observer exists for $r < r_s$. Moving forward in time requires also moving to smaller radius.

The black hole solution (1/2)

- No stationary observer exists for $r < r_s$. Moving forward in time requires also moving to smaller radius.

- **Q:** Is it real or artefact due to the wrong choice of the coordinates?
- **A:** The second possibility was believed to be true until 1960's.

- Let's look for a better set of coordinates $(t, r) \to (\tilde{t}, r)$:

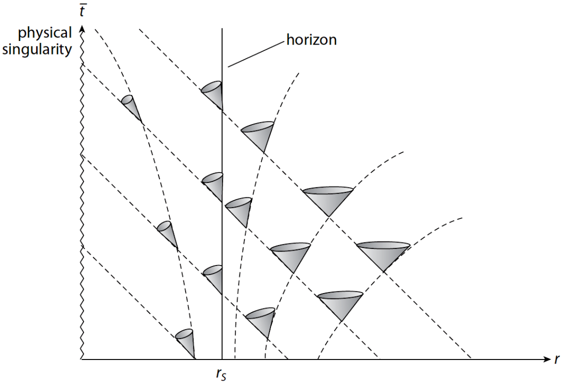
$$d\tilde{t} \equiv dt + \frac{r_s}{r-r_s}dr, \;\; ds^2 = -f(r)(d\tilde{t} + dr)(d\tilde{t} - \frac{r+r_s}{r-r_s}dr) + r^2 d\Omega^2 \qquad (12)$$

- The radial $d\Omega = 0$ light rays $ds^2 = 0$ follow:
  - $d\tilde{t} + dr = 0$, incoming, since $dr < 0$ for $d\tilde{t} > 0$

  or
  - $d\tilde{t} = \frac{r+r_s}{r-r_s}dr$, outgoing for $r > r_s$, since $dr > 0$ for $d\tilde{t} > 0$.

The black hole solution (2/2)

- The incoming rays always move at 45°.
- The outgoing ray angle with the r-axis varies. Starting at 45° for $r \gg r_s$, slowly increasing with decreasing r until it reaches 90° at the horizon $r = r_s$.



- **For $r < r_s$, we no longer have any outgoing light rays!!!**

$\Rightarrow$ Material particles cannot escape, since their worldlines have to lie inside the light cone.

Galaxy inside the horizon example

- The Schwarzschild radius r_s for a spherical mass M of uniform density $\rho(r) = \bar{\rho}$ and radius R is given by $r_s \equiv \tilde{r} = 2GM = \frac{8\pi}{3} G\bar{\rho}R^3$. For the Sun we would have
$$\tilde{r}_\odot / R_\odot \approx \frac{3}{7} \times 10^{-5}. \qquad (13)$$
- **Example (Rindler):** Consider (a quite unrealistic!) spherical and non-rotating galaxy containing $\sim 10^{11}$ suns equally spaced throughout and initially at rest:
$$\tilde{r}_{Galaxy} \approx 10^{11}\tilde{r}_\odot \approx \frac{3}{7} \times 10^6 R_\odot \qquad (14)$$
The ratio of the volume inside galaxy Schwarzschild radius $\tilde{r}_{Galaxy}$ to the volume of the sun is given by $(\tilde{r}_{Galaxy}/R_\odot)^3 \approx 10^{17}$.
- The galaxy where individual suns would be 100 stellar diameters apart would still fit into the galactic horizon $100^3 \times 10^{11} = 10^{17}$.
- $\Rightarrow$ If that galaxy were to collapse to a volume where the individual stars were 100 stellar diameters apart, it would be inside its horizon.

The BH solution in Eddington-Finkelstein coordinates

- Another set of coordinates: Eddington-Finkelstein coordinates $v = t + r + r_s \log(\frac{|r-r_s|}{r_s})$ with $f(r) \equiv (1 - r_s/r)$:

$$ds^2 = -(1 - r_s/r)dv^2 + 2dvdr + r^2 d\Omega^2 \quad (15)$$

- Radial light rays $ds^2 = 0$ follow the path $(r - r_s)dv^2 = 2rdvdr$.
  - Light rays along $dv = 0$ are always ingoing.
  - Light rays along $(r - r_s)dv = 2rdr$ are outgoing for $r > r_s$ and ingoing for $r < r_s$.
- If we put $v \equiv R + T$ and $r \equiv R - T$ then at the horizon $r = r_s$ the metric reads $ds^2 = 2dvdr = 2(dT^2 - dR^2)$.
  $\Rightarrow$ The Schwarzschild singularity is thus a mere coordinate singularity which can be transformed away.
- $g_{\mu\nu}$ of (15), its first and second derivatives are continuous at $r = r_s$. By continuity, the field equations must be satisfied there also.
  $\Rightarrow$ The entire spacetime described by the Schwarzschild metric is regular down to $r = 0$, where the curvature $R \to \infty$.

Generality of Schwarzschild solution

- **Theorem** (Birkhoff 1923): any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat.
  $\Rightarrow$ The Schwarzschild spacetime geometry is the unique spherically symmetric solution of $G_{\mu\nu} = 0$.

- **Example 1: Pulsating star**. A spherically symmetric pulsating star of fixed mass. BT $\Rightarrow$ the exterior geometry must be static. The only effect of the star's pulsation is to change the location of the stellar surface but without emission of gravitational waves.

- **Corollary:** Imploding, spherical star cannot produce any gravitational waves; such waves would break the spherical symmetry.
- By contrast, a star that implodes non-spherically or merger of two neutron stars (first detected by LIGO+Virgo 17.8.2017) or of two black holes (first detected by LIGO 14.9.2015) can produce a strong burst of gravitational waves.

The BH solution in Gullstrand–Painlevé coordinates

- $(t, r) \to (t_r, r)$, $\quad \psi(r) \equiv t - t_r$ for which the spacial part of $g_{\mu\nu}$ is simply the flat metric $dr^2 + r^2 d\Omega^2$. Metrics with $f(r) \equiv \frac{r - r_s}{r}$ reads:

$$ds^2 = -f(r)dt_r^2 + 2f(r)\psi' dt_r dr + \left[\frac{1}{f(r)} - f(r)\psi'^2\right] dr^2 + r^2 d\Omega^2 \quad (16)$$

where $\psi' \equiv \frac{d\psi}{dr}$. The condition $[\ldots] = 1 \Rightarrow \psi'(r) = \frac{r}{r - r_s}\sqrt{r_s/r}$ and

$$ds^2 = -\frac{r - r_s}{r} dt_r^2 + 2\sqrt{\frac{r_s}{r}} dt_r dr + dr^2 + r^2 d\Omega^2 \quad (17)$$

- Particle falling radially toward a black hole from rest at infinity:

$$\frac{dr}{dt} = \frac{r - r_s}{r}\sqrt{\frac{r_s}{r}} \quad (18)$$

appears (to a distant observer) slowing down as it gets nearer the event horizon and halts at r_s.

- In GP coordinates:

$$\frac{dr}{dt_r} = \sqrt{\frac{r_s}{r}} = \sqrt{\frac{2GM}{r}} \quad (19)$$

equals the Newtonian escape velocity.

The BH solution in Gullstrand–Painlevé coordinates

- Inside the event horizon $r < r_s$, dr/dt_r increases becoming infinite at the singularity. As shown below it is always less than that of light.
- Assume radial motion. The light ray geodetics is:

$$ds^2 = 0 = \left[dr + \left(1 + \sqrt{\frac{r_s}{r}}\right)dt_r\right]\left[dr + \left(1 - \sqrt{\frac{r_s}{r}}\right)dt_r\right] \quad (20)$$

and so for the light we have: $$dr/dt_r = 1 \pm \sqrt{r_s/r} \quad (21)$$

- At $r \to \infty$ $dr/dt_r = \pm 1$ it coincides with that in SR.
- At $r = r_s$ outward shining light gets stuck: $dr/dt_r = 0$.
- Inside the event horizon $r < r_s$ the external observer measures that the light moves toward the center with speed greater than 2.
- $(dr/dt_r)_{particle}/(dr/dt_r)_{light} = \sqrt{r_s/r}/(1 + \sqrt{r_s/r}) < 1$. QED.

Falling into BH: Gullstrand–Painlevé coordinates

- Using (19) compute the travel time for the particle from horizon to the BH center.

$$T_r = \int_0^{T_r} dt_r = \int_0^{r_s} \sqrt{\frac{r}{r_s}} dr = \frac{2}{3} r_s = \frac{4}{3} GM \tag{22}$$

- Using (21) the time of travel for light shining inward from event horizon to the BH center is

$$T_r = \int_0^{T_r} dt_r = \int_0^{r_s} \left(1 + \sqrt{\frac{r}{r_s}}\right)^{-1} dr = r_s(2 \ln 2 - 1) = 1.54 GM \tag{23}$$

For $M = 3M_\odot, \quad T_r = 11\mu s$.
For supermassive BH with mass $M = 3.7 \times 10^6 M_\odot$ residing at the centre of the Milky Way, $T_r = 14s$.

Interior of star as a static perfect fluid

- Energy-momentum tensor of a perfect fluid flowing with local 4-velocity u^μ in a flat spacetime: $T^{\mu\nu} = (\rho + P)u^\mu u^\nu + p\eta^{\mu\nu}$.
- Using the equivalence principle $\eta^{\mu\nu} \to g^{\mu\nu}$ it can be cast into form valid in curved space-time:
$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + pg^{\mu\nu} \tag{24}$$
- N.B. Independenty of the metric we have $T \equiv T^\lambda_\lambda = \rho - 3P$.
- Plugging (24) into Einstein equations (using GR sign convention!!):
$$R_{\mu\nu} = +\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T), \qquad \kappa \equiv 8\pi G \tag{25}$$
we obtain:
$$R_{\mu\nu} = \kappa[(\rho + p)u^\mu u^\nu + \tfrac{1}{2}(\rho - P)g_{\mu\nu}] \tag{26}$$
- Assume a static spherically symmetric interior described by the metric:
$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{27}$$
- Static $\Rightarrow$ $u^i = 0$. Normalization $g_{\mu\nu}u^\mu u^\nu = -1 \Rightarrow u^0 = A^{-1/2}$.

Solving for the spacetime inside the star

- The field eqation (25) implies:

$$R_{tt} = \tfrac{1}{2}\kappa(\rho + 3P)A, \quad R_{rr} = \tfrac{1}{2}\kappa(\rho - P)B, \quad R_{\theta\theta} = \tfrac{1}{2}\kappa(\rho - P)r^2 \tag{28}$$

- Inspired by the Schwarzschild solution (10) let's assume:

$$B^{-1} = (1 - \frac{2G\mathcal{M}(r)}{r}) \tag{29}$$

which after insertion to (27) yields:

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r), \quad \mathcal{M}(r) = 4\pi \int_0^r dr^{'} r^{'2} \rho(r^{'}) \tag{30}$$

- Condition for hydrostatic exquilibrium reads:

$$\frac{dP(r)/dr}{\rho + P} = -\frac{dA(r)/dr}{2A} \tag{31}$$

Tolman-Oppenheimer-Volkoff equation

- After some algebra:

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2}\left[1+\frac{P(r)}{\rho(r)}\right]\left[1+\frac{4\pi r^3 P(r)}{\mathcal{M}(r)}\right]\left[1-\frac{2G\mathcal{M}(r)}{r}\right]^{-1} \quad (32)$$

(Tolman, Oppenheimer-Volkoff, 1939).

- $P(r)$ steadily decreases until P(R)=0, defining the star radius.
- Star mass is $M = \mathcal{M}(R)$.
- For some simple EoS $P = P(\rho)$, analytic solutions may be found.
- In general, numerical integration of (32) is needed starting outward from $r = 0$ with the boundary condition $M(r = 0) = 0$ and some chosen value $P(r = 0) = P_0$.

Incompressible fluid

- $r_1 \le r_2 \Rightarrow \rho(r_2) \le \rho(r_1) \le \rho(r=0) \equiv \rho_0$. $\Rightarrow$ Incompressible fluid with $\rho(r) = \rho_0$ provides an upper limit on P and M.

- $\mathcal{M}(r) = (4\pi/3) r^3 \rho_0 = (r/R) M$ and analytical integration of TOV Eq. (32) yields:

$$P(r) = \rho_0 \frac{\left[1 - \frac{r_s}{R}\left(\frac{r}{R}\right)^2\right]^{1/2} - \left[1 - \frac{r_s}{R}\right]^{1/2}}{3\left[1 - \frac{r_s}{R}\right]^{1/2} - \left[1 - \frac{r_s}{R}\left(\frac{r}{R}\right)^2\right]^{1/2}} \qquad (33)$$

- Demanding $P(r=0) > 0 \Rightarrow 3\left[1 - \frac{r_s}{R}\right]^{1/2} > 1$:

$$R > \frac{9}{8} r_s = \frac{9}{4} GM \qquad (34)$$

- **Theorem** (Buchdahl 1959): The inequality (34) is true for any reasonable EoS. $\Rightarrow$ For $R \le \frac{9}{8} r_s = \frac{9}{4} GM$ star becomes the black hole.

- $r_s/R_{\oplus} = 10^{-9}$, $r_s/R_{\odot} = 10^{-6}$, $r_s/R_{wd} = 10^{-4}$, $r_s/R_{ns} = 10^{-1}$.

Incompressible fluid in NR limit

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2}\left[1+\frac{P(r)}{\rho(r)c^2}\right]\left[1+\frac{4\pi r^3 P(r)}{\mathcal{M}(r)c^2}\right]\left[1-\frac{2G\mathcal{M}(r)}{rc^2}\right]^{-1} \quad (35)$$

- In NR limit $c \to \infty$ in (35) $[\ldots] \to 1$:

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2} \quad (36)$$

  the outward force due to pressure on an infinitesimal volume of stellar material balances the inward force due to gravity.

- Integrating (36) for the case of constant density ρ_0:

$$P(r) = -\int_r^R \frac{G\mathcal{M}(r)\rho_0}{r^2}dr = -\int_r^R \frac{4}{3}\pi G\rho_0^2 r dr = \frac{2}{3}\pi G\rho_0^2(R^2-r^2) \quad (37)$$

  we obtain an expression for the pressure at the centre of star:

$$P_c \equiv P(r=0) = \tfrac{2}{3}\pi G\rho_0^2 R^2 \quad (38)$$

Compact stars

- Once the **nucleosynthesis** arrives at ^{56}Fe, its continuation **becomes endoergic**.
- When star does not burn nuclear fuel it can not support itself against gravitational collapse by generating thermal pressure.
- Survival strategy: The only way how to prevent the collapse is to use the pressure of degenerate Fermi gas.
- White dwarfs use degenerate electron gas pressure.
- Neutron stars use degenerate neutron gas pressure.

Object	M	R	$\rho/\rho_\odot$	GM/R
Sun	$M_\odot$	$R_\odot$	1	10^{-6}
White dwarf	$\lesssim M_\odot$	$\sim 10^{-2} R_\odot$	$\lesssim 10^7$	$\sim 10^{-4}$
Neutron star	$\sim 1 - 3M_\odot$	$\sim 10^{-5} R_\odot$	$\lesssim 10^{15}$	$\sim 10^1$

$M_\odot \approx 2 \times 10^{30} kg$ $\quad R_\odot \approx 7 \times 10^8 m$

Degenerate fermion gas at T=0K

- NR fermions in a box $L_x = L_y = L_z = L$ at $T = 0K$.

$$\lambda_i = \frac{2L}{N_i}; \quad p_i = \frac{hN_i}{2L}; \quad i = x, y, z \tag{39}$$

$$\frac{p^2}{2m} = \frac{h^2}{8mL^2}(N_x^2 + N_y^2 + N_z^2) = \frac{h^2 N^2}{8mL^2}; \quad N^2 \equiv N_x^2 + N_y^2 + N_z^2 \tag{40}$$

- Total number of the fermions inside the spehere of radius N occupying octant $N_x > 0, N_y > 0, N_z > 0$ is:

$$N_f = (2s+1)\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi N^3\right); \quad N = \left(\frac{3N_f}{\pi}\right)^{1/3} \tag{41}$$

- The maximum fermion energy ε_F per unit volume is

$$\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}, \quad n = N_f/L^3 \tag{42}$$

and average momentum squared and energy per fermion is:

$$\bar{p}_F^2 = \frac{\int_0^{p_F} p^2 \; p^2 dp}{\int_0^{p_F} p^2 dp} = \frac{3}{5}p_F; \qquad \bar{\varepsilon}_F = \frac{\bar{p}_F^2}{2m} = \frac{3}{5}\varepsilon_F \tag{43}$$

Degenerate fermion gas at $T > 0K$

- For white dwarf (DW) the assumption of complete degeneracy is OK even at $T > 0K$. For fully ionized atoms:

$$n_e = \left(\frac{\#\text{electrons}}{\text{nucleon}}\right)\left(\frac{\#\text{nucleons}}{\text{volume}}\right) = \left(\frac{Z}{A}\right)\frac{\rho}{m_H} \qquad (44)$$

- Demanding that an average electron lives inside the Fermi sphere $\Rightarrow$

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3} > \frac{3}{2}kT \qquad (45)$$

or

$$\frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3m_e k}\left[\frac{3\pi^2}{m_H}\left(\frac{Z}{A}\right)\right]^{2/3} = 1261 km^2 kg^{-2/3} \equiv \mathcal{D} \qquad (46)$$

- For $\odot$ at the center $T_c/\rho_c^{2/3} \approx 4\mathcal{D}$, for Sirius B: $T_c/\rho_c^{2/3} \approx 0.03\mathcal{D} \ll \mathcal{D}$.

Electron degeneracy pressure

- Separation between neighboring electrons in completely degenerate electron gas of uniform density n_e is $n_e^{-1/3}$.
- To satisfy Pauli principle also $\Delta x \approx n_e^{-1/3}$ and so electron momentum is $p = p_x \approx \hbar/\Delta x \approx \hbar n_e^{1/3}$.
- Total momentum $p = \sqrt{p_x^2 + p_y^2 + p_z^2} = \sqrt{3} p_x$. Using (44) we have:

$$p \approx \sqrt{3}\hbar n_e^{1/3} \tag{47}$$

- For NR electrons with (unrealistic) $p = const.$ $\quad P = \frac{1}{3} n_e p v \Rightarrow$

$$P \approx \frac{\hbar^2}{m_e} n_e^{5/3} \approx \frac{5}{(3\pi^2)^{2/3}} P_{exact} \approx \frac{1}{2} P_{exact} \tag{48}$$

$$P_{exact} = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_e} n_e^{5/3} \tag{49}$$

White dwarfs: mass - volume relation

- WDs lack brightness because its H-supply, main energy source of stars, has been already used up. They consist mainly of He, CO or Fe and radiate their released gravitational energy through slow contraction.
- $\approx$ Equilibrium $\Leftrightarrow$ the pressure of degenerate electron gas at the WD centre (49) more less cancels the pressure due to gravity (38):

$$\frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}n_e^{5/3} = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho_0}{m_H}\right]^{5/3} \approx \tfrac{2}{3}\pi G\rho_0^2 R_{wd}^2 \qquad (50)$$

- For compact star $M_{wd} \lesssim \frac{4\pi}{3}R_{wd}^3\rho_0 \;\Rightarrow$

$$R_{wd} \approx \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{G m_e M_{wd}^{1/3}}\left[\left(\frac{Z}{A}\right)\frac{1}{m_H}\right]^{5/3} \qquad (51)$$

- Surprising implication: $M_{wd}R_{wd}^3 \sim M_{wd}V_{wd} = const.!$
- Does it mean that there is no limit on M_{wd}? No, there's a limit!

White dwarfs: Chandrasekhar limit

- We have assumed NR electrons. However, for small enough R_{wd} the Fermi energy $\varepsilon \sim m_e$.
- Let us look at the equilibrium obtained assuming relativistic electrons. Typical electron energy is $E_e \sim p_e \sim \hbar n_e^{1/3}$.
- Total electron degeneracy energy E_{deg} of WD consisting of N_e electrons has to be comparable to its gravitational energy E_g:

$$E_{deg} \sim N_e E_e \sim N_e \hbar n_e^{1/3} \sim \frac{\hbar}{R_{wd}} \left(\frac{Z M_{wd}}{A m_H} \right)^{4/3} \sim \frac{G M_{wd}^2}{R_{wd}} \qquad (52)$$

$$\Rightarrow \qquad M_{ch} \sim \left(\frac{Z}{A m_H} \right)^2 \left(\frac{\hbar}{G} \right)^{3/2} = 1.8 \left(\frac{Z}{A} \right)^2 M_\odot \qquad (53)$$

More precisse derivation gives for $Z/A = 0.5$ famous
Chandrasekhar limit $M_{Ch} = 1.44 M_\odot$.

- Once WD becomes compact enough for the electrons to be relativistic, there is a solution with only one mass, irrespective of the radius.

Fate of the White dwarfs: alone or in environment

- WD left alone is stable once formed and will continue to cool almost indefinitely, eventually to become a black dwarf. The time required to reach this state is calculated to be longer than the current age of the universe . . . They are are thought to be the final evolutionary state of stars whose mass is not high enough to become a neutron star.

Q: How may become $M_{wd} > M_{Ch}$?

A: Either via stable accretion of material from a companion or the collision of two WDs. In this way it raises its core temperature enough to ignite carbon fusion, at which point it undergoes runaway nuclear fusion, completely disrupting it.

- There can be also iron WD sitting at the centre of giant $M_{wd} > 20M_{\odot}$. Mixture of Fe nuclei and degenerate electrons.

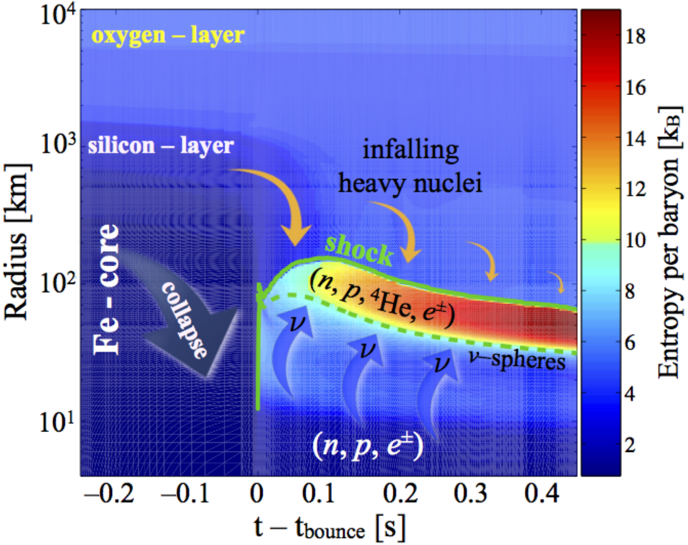
The core collapse in very massive stars

- **Implosion:** Once $M_{wd} > M_{Ch}$ the star starts to implode.
- **Nucleation:** As its temperature rises the nuclei will start to evaporate to ^4He and then to nucleons.
- **Neutronization:** At the same time, the Fermi level of the electrons becomes sufficiently high to reach $m_n - m_p - m_e = 0.78 MeV$ which is needed for electron capture $p + e^- \to n + \nu_e$.
  This will lead to a smaller electron degeneracy pressure $\Rightarrow$
  WD will be unable to support its weight against the force of gravity.
- **Lucky coincidence:** Since $m_e \approx E_{binding} \approx m_p - m_n \approx \mathcal{O}(MeV) \Rightarrow$ collapse, nuclear evaporation and neutronization all occur at roughly the same moment!!!
- Once the protons are converted to neutrons electron degeneracy pressure (49) is unable to support its weight against the force of gravity. $\Rightarrow$ The core will undergo sudden, catastrophic collapse to form a neutron star or, if it exceeds the TOV limit (34), i.e. $15 M_\odot$ it becomes a black hole.

Supernovae

- The core collapse produces a massive flux of ν which might fragment some nuclei.
- e^- capture in very dense parts of the infalling matter produces additional neutrons.
- Shock wave by rebound of some of the infalling material is started. The explosion sends a shock wave of the star's former surface zooming out at a speed of 10,000 km s^{-1}, and heating it so it shines brilliantly for about a week. Outer layers rich in intermediate mass (A=4–56) nuclei are blown off into the interstellar medium.
- The rebounding matter is bombarded by the neutrons and so some of its nuclei might transform into a spectrum of heavier-than-iron nuclei including the radioactive elements up to (and likely beyond) uranium.
- N.B. Rebound is driven by the bulk modulus $K = -V\frac{\partial P}{\partial V}$ and hence by the EoS of highly compressed and heated nuclear matter.

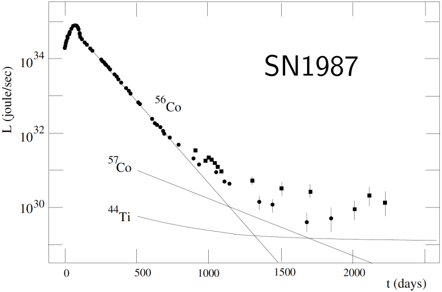
Supernova core-collapse simulation

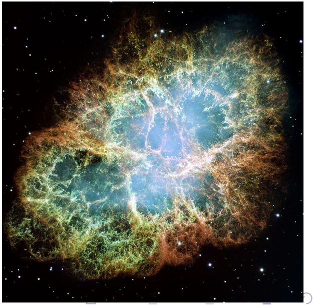


T. Fischer *et al.*, The state of matter in simulations of core-collapse supernovae – Reflections and recent developments, Publ. Astron. Soc. Austral. **34**, 67 (2017)

Energy and light from Supernova

- Energy change ΔE of the star with $M = 1.5M_\odot$ and core of $1.4M_\odot$ collapsing from $R{=}1000km \rightarrow R{=}10km$ is $\Delta E \approx \frac{3}{5}\frac{GM^2}{R} \approx 3 \times 10^{46} J$
- ΔE is almost entirely evacuated in the form of thermally produced neutrinos and antineutrinos $\gamma\gamma \leftrightarrow e^+e^- \leftrightarrow \bar{\nu}\nu$ and not in the form of those from an electron capture $e^-(A,Z) \rightarrow \nu_e(A,Z-1)$.
- Luminosity is mostly dominated by subsequent radioactive heating of the rapidly expanding ejecta. Decay ${}^{56}Ni \rightarrow {}^{56}Co \rightarrow {}^{56}Fe$ produces photons, primarily of $E_\gamma = 847\text{keV and } 1238\text{keV}$.





Neutron stars

- **Neutron star**: small densely packed objects held by gravity and supported by **neutron degeneracy pressure**. $1.4 M_\odot \leq M_{ns} < 3 M_\odot$
- Assuming $M_{ns} \approx M_{ch} = 1.4 M_\odot$ neutron star is a huge nucleus with $A = M_{ns}/m_n \approx 10^{57}$ neutrons moving with NR velocity, see (47):

$$v = \frac{p}{m_n} = \frac{\hbar}{m_n} n^{1/3} = \frac{\hbar}{m_n} \left(\frac{\rho}{m_n}\right)^{1/3} = 0.2c \qquad (54)$$

- Replacing $m_e \to m_n$ and $Z/A \to 1$ in (51) we have for neutron star:

$$R_{ns} \approx \frac{(18\pi)^{2/3}}{10} \frac{\hbar^2}{G m_n M_{ns}^{1/3}} \left(\frac{1}{m_H}\right)^{5/3} \qquad (55)$$

  which for $M_{ns} = 1.4 M_\odot$ gives $R_{ns} = 4400 m$.
- Average density $\rho = M_{ns}/(\frac{4}{3}\pi R_{ns}^3) = 6.7 \times 10^{17} kgm^{-3} \approx 3 \times \rho_N$.
  $\Rightarrow$ Neutrons in NS are more packed then in the normal nucleus.