# Simple Cosmological Models

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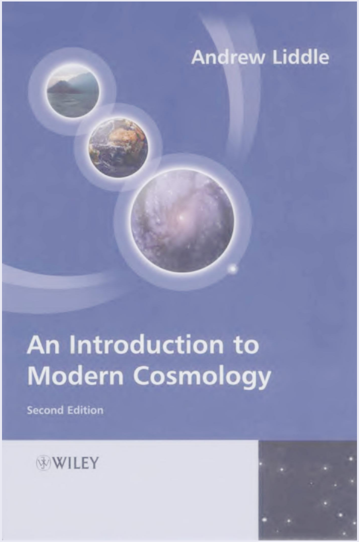
# Literature

## Our discussion is based on the book



# Simple Cosmological Models

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# Hubble's law

- The Friedmann equation allows to explain Hubble's discovery of the relation $\vec{v} = H \cdot \vec{r}$ between recession velocity $\vec{v} = d\vec{r}/dt$ and the distance $\vec{r}$.
- Since $\vec{v} \| \vec{r}$ and $\vec{r} = a\vec{x}$ (the comoving position $\vec{x}$ is a constant by definition):
$$\vec{v} = \frac{|\dot{\vec{r}}|}{|\vec{r}|}\vec{r} = \frac{\dot{a}}{a}\vec{r} = H\vec{r}\,, \quad H = \frac{\dot{a}}{a} \tag{1}$$
- The term *Hubble's constant* is a bit misleading. It is constant in space due to the cosmological principle, but *there is no reason for it to be constant in time*. Friedmann equation can be written as an evolution equation for $H(t)$:
$$H^2(t) = \tfrac{8\pi G}{3}\rho(t) - \tfrac{k}{a^2(t)} \tag{2}$$

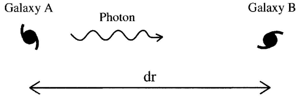
$\Rightarrow$ Best to use for $H(t)$ the term **Hubble parameter**, reserving **Hubble constant** for its present value $H_0 \equiv H(t_0)$.

- Measured value $H_0 > 0 \quad \Rightarrow$ the Universe is expanding.

# Expansion and redshift

- The redshift of spectral lines used to justify expansion of Universe can be related to the scale factor $a$.
- Light is passed between two objects which are 'very close' ($dr$). Hubble's law $\Rightarrow$ their relative velocity is:

$$dv = H \cdot dr = \frac{\dot{a}}{a} dr \qquad (3)$$




- (3) & Doppler law $\Rightarrow$ the change in wavelength between emission and observation, $d\lambda \equiv \lambda_{obs} - \lambda_{em}$ is: $\frac{d\lambda}{\lambda_{em}} = \frac{dv}{c} = \frac{\dot{a}}{a}\frac{dr}{c} = \frac{\dot{a}}{a} dt = \frac{da}{a}$ (4)

  where we have used the NR approximation for the redshift:

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \sqrt{\frac{1+v/c}{1-v/c}} - 1 \approx \frac{v}{c} \qquad (5)$$

- N.B. The wavelength is increased $\Rightarrow d\lambda > 0$
- Integrating (4) $\Rightarrow ln\lambda = \ln a + const.$, i.e. $\lambda(t) \propto a(t)$ (6)
  where $\lambda(t)$ is the instantaneous wavelength measured at time $t$.
- (6) $\Rightarrow$

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} \qquad (7)$$

# Expansion and redshift - a more rigorous derivation

- For light $ds = 0$. Plugging it into FLRW equation

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \tag{8}$$

and assuming that the light ray propagates only radially from $r = 0$ to $r = r_0$ (i.e. $d\theta = d\phi = 0$) :

$$\frac{cdt}{a(t)} = \frac{dr}{\sqrt{1-kr^2}} \tag{9}$$

(N.B. the spatial coordinates in the metric are comoving, so galaxies remain at fixed coordinates; the expansion is entirely taken care of by the scale factor $a(t)$).

- Total time the light ray takes to get from $r = 0$ to $r = r_0$:

$$\int_{t_{em}}^{t_{obs}} \frac{cdt}{a(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_{em}+dt_{em}}^{t_{obs}+dt_{obs}} \frac{cdt}{a(t)} \tag{10}$$

Third term corresponds to the ray emmitted $dt_{em}$ later and observed at time $t_{obs} + dt_{obs}$ (the galaxies are still at the same coordinates).

# Expansion and redshift - a more rigorous derivation

- In a differential form (10) says: $$\frac{c\,dt_{obs}}{a(t_{obs})} = \frac{c\,dt_{em}}{a(t_{em})} \tag{11}$$

- In an expanding Universe $a(t_{obs}) > a(t_{em}) \Rightarrow dt_{obs} > dt_{em}$.
  The time interval between the two rays increases as the Universe expands.

- Now imagine that, instead of being two separate rays, they correspond to successive maxima of a single wave.
  As the wavelength is proportional to the time between the peaks $\lambda \propto dt \propto a(t)$ and so:
  $$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} \tag{12}$$

# Solving the equations (Friedmann, fluid)

- To study the evolution specify the relationship between the mass density $\rho$ and the pressure $p$, i.e. the **equation of state** $p(\rho)$ (EoS).
- Let's first consider two possibilities:

**Matter** Shorthand for **non-relativistic matter**. Any type of material exerting negligible pressure, $p = 0$. It is the simplest assumption.
Good approximation to use for the atoms in the Universe once it has cooled down, as they are quite well separated and seldom interact.
Also a good description of a collection of galaxies in the Universe, as they have no interactions other than gravitational ones.
Occasionally the term **dust** is also used instead of **matter**.

**Radiation** Kinetic energy of massless particles leads to a pressure force.
The radiation pressure is $p = \frac{1}{3}\rho c^2$.
More generally, any particles moving with ultra-relativistic speeds $v \cong c$ have this EoS, neutrinos being an obvious example.

- Here we will concentrate on the case when in the Friedmann equation $k = 0$. This corresponding to a **flat geometry**.

# Matter: Solving the fluid equation

- Start by solving the fluid equation $$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \qquad (13)$$

  for $p = 0$: $$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = \frac{d}{dt}(\rho a^3) = 0 \qquad (14)$$

  $\Rightarrow$ density falls with volume of the Universe: $$\rho \sim a^{-3} \qquad (15)$$

- The equations we are solving ($k = 0$) have one very useful symmetry; their form is unchanged if we multiply the scale factor a by a constant, since only the combination $\frac{\dot{a}}{a}$ appears.
  - We are free to rescale $a(t)$ as we choose.
  - The normal convention is $a(t_0) = 1$ at the present time.
  - $\vec{r} = a(t_0)\vec{x} = \vec{x}$ , at present physical and comoving coordinate systems coincide.

- N.B.from now on the subscript '0' indicates the present value of quantities. Denoting the present density by $\rho_0$ fixes the proportionality constant: $$\rho = \frac{\rho_0}{a^3} \qquad (16)$$

# Matter: Solving the Friedmann equation

- The Friedmann equation: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{17}$$
- For $\rho = \rho_0 a^{-3}$ and $k = 0$: $$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a} \tag{18}$$
- (18) is separable $\Rightarrow$ directly solvable. Try instead educated guess $a = t^q$ which in the cosmology is preferred one. The l.h.s. of (18) has dependence $t^{2q-2}$, the r.h.s. has $t^{-q}$ $\Rightarrow$ $q = \frac{2}{3}$. Since $a(t_0) = 1$ we have: $$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad ; \quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} \tag{19}$$
- In this solution the expansion rate $H(t)$ decreases with time: $$H(t) \equiv \frac{\dot{a}}{a} = \frac{\rho_0}{a^3} = \frac{2}{3t} \tag{20}$$

  becoming infinitely slow for $t \to \infty$.
- Despite the pull of gravity, the material in the Universe does not recollapse but rather expands forever. This is one of the classic cosmological solutions, and will be much used later on.

# Radiation

- The fluid equation (13) with $p = \frac{1}{3}\rho c^2$ gives: $$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \qquad (21)$$

  Using the same trick as before, with the $a^3$ replaced by $a^4$ in equation (14), gives $$\rho \propto a^{-4} \qquad (22)$$

  bringing in the second classic cosmological solution:

  $$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad ; \quad \rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} \qquad (23)$$

- The Universe expands more slowly if it is radiation dominated than if it is matter dominated. $\Leftarrow$ $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$ (24)
- However, in each case the density of material falls off as $\rho \sim t^{-2}$.
- The extra $a^{-1}$ in the fall off of the radiation density with volume (compared to the matter density) is due to its non-zero pressure and hence to its work $pdV$ done during expansion.

# Mixtures

- A more general situation: mixture of both matter and radiation. There are two separate fluid equations, one for each fluid:

$$\rho_{mat} \propto a^{-3} \qquad ; \qquad \rho_{rad} \propto a^{-4} \tag{25}$$

  but only a single Friedmann equation for $\rho = \rho_{mat} + \rho_{rad}$.
- General situation quite messy. Consider instead the case where one of the densities is by far the larger. $\Rightarrow$ Friedmann equation is solved by inclusion of the dominant component only.

⋆ For $\rho_{rad} \gg \rho_{mat}$:

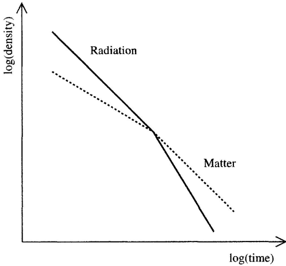
$$a(t) \propto t^{1/2}; \ \rho_{rad} \propto t^{-2}; \ \rho_{mat} \propto a^{-3} \propto t^{-3/2} \tag{26}$$

**unstable situation** $\Leftarrow$ $\rho_{mat}$ falls slower than $\rho_{rad}$. Even originally very small matter component will eventually come to dominate!!

⋆ For $\rho_{mat} \gg \rho_{rad}$:

$$a(t) \propto t^{2/3}; \ \rho_{mat} \propto t^{-2}; \ \rho_{rad} \propto a^{-4} \propto t^{-8/3} \tag{27}$$

Matter domination is a **stable situation**.




- In both cases eventually the matter comes to dominate, and as it does so the expansion rate speeds up from $a(t) \propto t^{1/2}$ to the $a(t) \propto t^{2/3}$ law.

# Particle number densities

- The number density $n$ is useful $\Leftarrow$ in most circumstances particle number is conserved.
- Even at high interaction rates the Universe is in a **state of thermal equilibrium** $\Leftrightarrow$ (for a particular type of particle) any $n$–changing interaction, must proceed at the same rate in both forward and backward directions.
- The only thing that changes $n$ is the volume: $$n \propto a^{-3} \tag{28}$$
- The energy $E_{mat}$ of NR particles is dominated by their rest mass-energy which is constant, so
$$\rho_{mat} \propto \epsilon_{mat} = n_{mat} \times E_{mat} \propto a^{-3} \times const. \propto a^{-3} \tag{29}$$
- Photons lose energy as the Universe expands and their wavelength is stretched, so their energy is $E_{rad} \propto a^{-1}$ as we have already seen.
$$\rho_{rad} \propto \epsilon_{rad} = n_{rad} \times E_{rad} \propto a^{-3} \times a^{-1} \propto a^{-4} \tag{30}$$
- Although $\epsilon_{mat}$ and $\epsilon_{rad}$ evolve differently, $n_{mat}$ and $n_{rad}$ evolve in the same way. So, apart from epochs during which the assumption of thermal equilibrium fails, **the relative number densities of the different particles** (e.g. electrons and photons) **do not change as the Universe expands**.

# Evolution including curvature: $k < 0$

- Return back to the general case of $k \neq 0$ corresponding to spherical $k > 0$ or hyperbolic $k < 0$ geometry.
- Assume: Universe is always dominated by NR matter. For $k = 0 \Rightarrow a \propto t^{2/3}$.

**Q:** Is it possible for the expansion to stop (i.e. to have $H \equiv \frac{\dot{a}}{a} = 0$)?

- Friedmann equation: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \Rightarrow$ For $k < 0$ Universe expands forever!
- $\rho_{mat} \propto a^{-3} \Rightarrow kc^2 a^{-2}$ will finally dominate: $$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{kc^2}{a^2} \qquad (31)$$ $\Rightarrow a \sim t$. So when $kc^2 a^{-2}$ term comes to dominate, the expansion of the Universe becomes yet faster.
- For $k < 0$ velocity becomes non-zero constant. This is sometimes known as **free expansion**.

# Evolution including curvature: $k > 0$

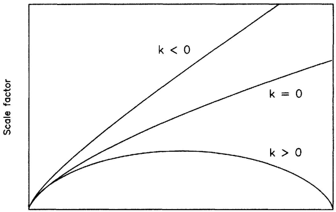
- For $k > 0$ $\exists H = 0 \Leftrightarrow$ the two terms on the r.h.s. of Friedmann eq. cancel each other. $\Rightarrow$ Expansion ends at finite $t$. As gravitational attraction persists, the **recollapse** of the Universe becomes inevitable.

- Consider Universe containing only matter ($p = 0$) so that $\rho = \rho_0 a^{-3}$.

  Friedmann equation reads: $$a(\dot{a}^2 + kc^2) = \frac{8\pi G\rho_0}{3} \qquad (32)$$

  $$\Rightarrow \quad a(\theta) = \frac{8\pi G\rho_0}{3k}(1 - cos\theta); \quad t(\theta) = \frac{8\pi G\rho_0}{3k^{3/2}}(\theta - cos\theta) \qquad (33)$$

  The solution is time reversible $\Rightarrow$
  at $t < \infty$ the Universe will come to an end in a **Big Crunch**.




- Precise analogy with escape velocity from the Earth.
- $U > 0$ – particle can escape to infinity, with a final kinetic energy $U$.
- $U = 0$ – particle can just escape but with $v = 0$.
- $U < 0$ – particle cannot escape the gravitation attraction and recollapses inwards.

# Observational Parameters: $H_0$

- Hubble constant $H_0 \equiv$ present value of Hubble parameter $H(t_0)$:
- Hubble's initial value was $H_0 = 500 kms^{-1} Mpc^{-1}$
- The expansion velocity can only be accurately distinguished from the peculiar velocity (motions of galaxies relative to one another) at large distances. While peculiar velocity is independent of distance the Hubble velocity is proportional to distance.
- Most recent value (2018) from the Hubble Space Telecope $H_0 = v/r = 100h\, kms^{-1} Mpc^{-1}$ ; $h = 73.45 \pm 1.66$.
- The precision of $h$ introduces uncertainties throughout cosmology. The actual distances to faraway objects are only known up to an uncertainty of the factor $h$, because recession velocities are the only way to estimate their distance. For this reason it is common to see distances specified in the form, for example. $100h^{-1} Mpc$ where the number 100 is accurately known but $h^{-1}$is not.

# Observational Parameters: $\Omega_0$

- For which value of $H^2 = \frac{8}{3}\pi G\rho - ka^{-2}$ the Universe is flat?

  Flat $\Leftrightarrow$ $k = 0 \Rightarrow$
  $$\rho_c(t) = (3H^2)/(8\pi G) \tag{34}$$

- $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2} \Rightarrow$
  $\rho_c(t_0) = 1.88h^2 \times 10^{-26} kgm^{-3} = 2.78h^{-1} \times 10^{11} M_\odot/(h^{-1}Mpc)^3$
- Tiny density of matter is sufficient to halt and reverse the expansion of the Universe.
- $(10^{11} - 10^{12})M_\odot \simeq M_{galaxy},\ R_{galaxy} \cong Mpc$
  $\Rightarrow$ the Universe cannot be far away from the critical density (within an order of magnitude or so).
- The Universe needs not be flat $\Rightarrow$ $\rho_c$ is not necessarily the true density of the Universe, but it sets a natural scale for the density:
  $$\Omega(t) \equiv \rho/\rho_c \tag{35}$$
  where $\Omega(t)$ is (time-dependent) density parameter and $\Omega_0 \equiv \Omega(t_0)$ is its present value.

# Observational Parameters: $\Omega_0$

- In the new variables the Friedman equation reads:

$$H^2 = \frac{8\pi G \rho_c \Omega}{3} - \frac{k}{a}^2 = H^2\Omega - \frac{k}{a}^2 \tag{36}$$

and so:

$$\Omega - 1 = \frac{k}{a^2 H^2} \tag{37}$$

- the case $\Omega = 1$ is very special, because in that case $k = 0$ and since $k$ is a fixed constant the equation $\Omega = 1$ is true for all time.

- That's true independently of the type of matter in the Universe, and is often called a **critical-density Universe**.

# Observational Parameters: $q_0$

- Consider a Taylor expansion of the scale factor around $t_0$:

$$a(t) = a(t_0) + \dot{a}(t_0)\,[t - t_0] + \tfrac{1}{2}\ddot{a}(t_0)\,[t - t_0]^2 + \dots \quad (38)$$

$$\frac{a(t)}{a(t_0)} = 1 + H_0\,[t - t_0] + \frac{q_0}{2} H_0^2\,[t - t_0]^2 + \dots \quad (39)$$

**Deceleration parameter** $$q_0 \equiv -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}(t_0)^2} \quad (40)$$

- Consider matter dominated Universe ($p = 0$). Then from the acceleration equation (24) and the definition $\rho_c$ we have:

$$q_0 = \frac{4\pi G}{3}\rho\frac{3}{8\pi G\rho_c} = \frac{\Omega_0}{2} \quad (41)$$

- N.B. $q_0$ is not independent of $\Omega_0$ and $H_0$. However, we don't know everything about the material in the Universe, so $q_0$ can provide a new way of looking at the Universe. It can be measured directly observing very large distance objects, such as the numbers of distant galaxies.
- Recent measurements obtained $q_0 < 0 \Rightarrow$ big surprise!!!!

# The Cosmological Constant: Introducing $\Lambda$

- Multiplying Einstein equations by metrical tensor

$$g^{\mu\nu}\left[R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\right] = R - 2R = -g^{\mu\nu}8\pi G T_{\mu\nu} = -8\pi G T^{\mu}_{\mu} \quad (42)$$

we obtain equivalent form: $$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}) \quad (43)$$

- In vacuum energy-momentum tensor $T_{\mu\nu}$ vanishes $\Rightarrow$ Einstein field equations in empty space reduce to $R_{\mu\nu} = 0$.

where $$R_{\mu\nu} \equiv \partial^{\nu}\Gamma^{\lambda}_{\lambda\mu} - \partial^{\lambda}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\nu\sigma} \quad (44)$$

and $$\Gamma^{\mu}_{\nu\kappa} = \tfrac{1}{2}g^{\mu\lambda}(\partial_{\kappa}g_{\lambda\nu} + \partial_{\nu}g_{\lambda\kappa} + \partial_{\lambda}g_{\nu\kappa}) \quad (45)$$

- In a space-time of $N = 2$ or 3 dimensions this would imply the vanishing of the full curvature tensor $R_{\lambda\mu\nu\kappa}$ and the consequent absence of a gravitational field.
- **For $N \geq 4$ true gravitational fields can exist in empty space**.

# The Cosmological Constant: Introducing $\Lambda$

- What about adding new terms to $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ ?
  - From $g_{\mu\nu}$ and its first derivatives no new tensor can be constructed $\Leftarrow$ at any point we can find a coordinate system in which the first derivatives of $g_{\mu\nu}$ vanish, so in this coordinate system the desired tensor must be equal to one of those that can be constructed out of the metric tensor alone, (e.g., $g_{\mu\nu}$ or $g^{\mu\nu}$ or $\varepsilon^{\mu\nu\lambda\eta}$ ..., and so on), and since this is an equality between tensors it must be true in all coordinate systems.
  - The only possibility is: $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \quad (46)$$
  - In the Friedmann equation $\Lambda$ appears as an extra term, giving: $$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \quad (47)$$
  - **Einstein's original idea**: **use $\Lambda$ and $\rho$ to balance curvature $k$ $\Rightarrow$ $H(t) = 0$ and hence a static Universe**.
    N.B. Such a balance proves to be unstable to small perturbations ....
  - Nowadays $\Lambda$ is mostly discussed in the context of Universes with the flat Euclidean geometry $k = 0$.

# The Cosmological Constant: Introducing Λ

- $\Lambda$ in acceleration equation: :
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \tag{48}$$
  $\Lambda > 0$ gives a positive contribution to $\ddot{a}$
  $\Rightarrow$ **acts effectively as a repulsive force**.
- For $\Lambda$ sufficiently large, it can overcome the gravitational attraction represented by the first term in (48) $\Rightarrow$ **accelerating Universe**.
- Define new density parameter: $$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2} \tag{49}$$
- Repeating the steps used to write the Friedmann equation in the form of (37), we find: $$\Omega + \Omega_\Lambda - 1 = ka^{-2}H^{-2} \tag{50}$$
- The flat Universe condition $k = 0$, generalizes to $\Omega + \Omega_\Lambda = 1$
  - Open Universe: $0 < \Omega + \Omega_\Lambda < 1$
  - Flat Universe: $\Omega + \Omega_\Lambda = 1$
  - Closed Universe: $\Omega + \Omega_\Lambda > 1$

# Fluid description of $\Lambda$

- Consider $\Lambda$ as if it were a fluid with energy density $\rho_\Lambda$ and pressure $p_\Lambda$. Defining $\rho_\Lambda \equiv \Lambda/(8\pi G)$ brings the Friedmann equation into the form

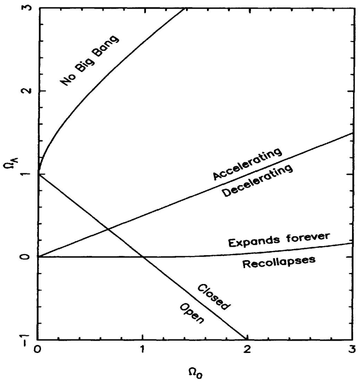
$$H^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{k}{a^2} \tag{51}$$

which automatically ensures that $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c$

- Consider the fluid equation for $\Lambda$: $\dot{\rho_\Lambda} + 3\frac{\dot{a}}{a}\left(\rho_\Lambda + \frac{p_\Lambda}{c^2}\right) = 0$ (52)
  - $\rho_\Lambda = const.$ by definition $\Rightarrow$ $p_\Lambda = -\rho_\Lambda c^2$ (53)
  - $\Lambda$ **has a negative effective pressure**.
  - As the Universe expands, work is done on the cosmological constant fluid.
  - Its energy density remains constant even though the volume of the Universe is increasing.
- Physical interpretation of $\Lambda$
  - In quantum physics: 'zero-point energy', which remains even if no particles are present. Unfortunately particle physics theories predict $\Lambda$ $10^{124}\times$ larger than observations allow ...

# Cosmological models with $\Lambda$

- $\Lambda \neq 0$ greatly increases range of possible behaviors of the Universe.
    - Closed Universe ($k > 0$) does not need to recollapse, nor an open Universe needs to expand forever.
    - For $\Lambda$ big enough there's even no need for a Big Bang: Universe begins in a collapsing phase followed by a bounce at finite size under the influence of the $\Lambda$. (Such models are now ruled out by observations).
- Focus on $\Omega(t_0)$ and $\Lambda \Leftarrow H(t)$ provides only an overall scaling factor.
- Assume $p = 0$ matter.
- Pressureless Universe with $\Lambda \neq 0$ has:
$$q_0 = \frac{1}{2}\Omega_0 - \Omega_\Lambda$$
- Assuming additionally $\Omega_0 + \Omega_\Lambda = 1$ (i.e. $k = 0$) this simplifies to $q_0 = \frac{3}{2}\Omega_0 - 1$ and acceleration occurs for $\Omega_\Lambda > 1/3$.
- For $\Omega_0 \leq 1$ the recollapse depends on $\Lambda \gtrless 0$, but for $\Omega_0 > 1$ the gravitational attraction of matter can overcome a small positive $\Lambda$ and cause recollapse.



# Cosmological models with Λ: Recent measurements

- M. Kowalski *et al.*, *Improved Cosmological Constraints from New, Old and Combined Supernova Datasets*, arXiv:0804.4142v1 [astro-ph] **Astrophysics Journal** 686:749-778,2008.
- N.B. $\Omega_m \equiv \Omega_0$ , $p_\Lambda = -\, w\rho_\Lambda c^2$