Expansion law of the Universe

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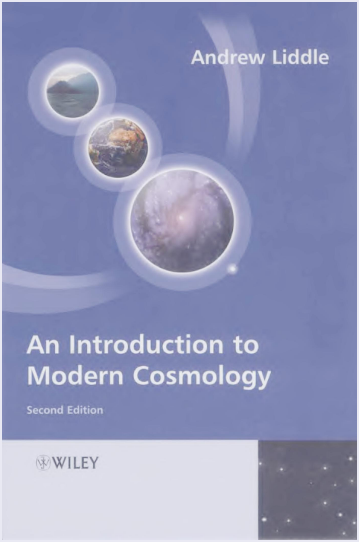
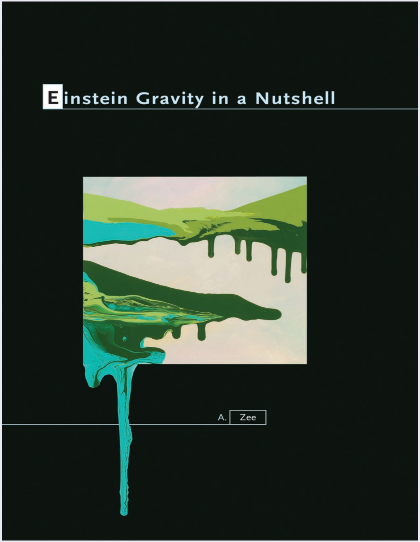
March 13, 2018

Literature

Our discussion is based on the books





- Newtonian approach to cosmology
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- Elements of relativistic cosmology
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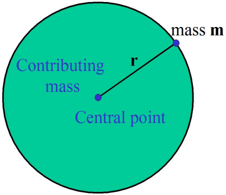
Newtonian approach to cosmology

- It is perfectly possible to discuss cosmology without having already learned general relativity.
- In fact, the most crucial equation, the Friedmann equation which describes the expansion of the Universe, turns out to be the same when derived from Newton's theory of gravity as from the equations of general relativity.
- The Newtonian derivation is not completely rigorous, and general relativity is required to fully patch it up, a detail that need not concern us at this stage.

The Friedmann equation

- Observer in a uniform expanding medium, with mass density ρ. Because the Universe looks the same from anywhere, we can consider any point to be its centre.



- **Theorem (Newton)**: in a spherically-symmetric distribution of matter, a particle with mass m at distance r from the observer feels no force at all from the material at greater radii, and the material at smaller radii (shown as the colored region) gives exactly the force which one would get if all the material was concentrated at the central point. **Prove it!**

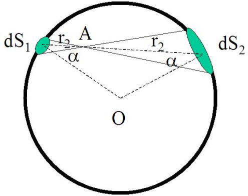
- The material has total mass $M = 4\pi\rho r^3/3$, and our particle has a gravitational potential energy

$$V = -\frac{GMm}{r} = -\frac{4\pi G\rho r^2 m}{3} \tag{1}$$

- The equation describing how the separation r changes can be derived from energy conservation

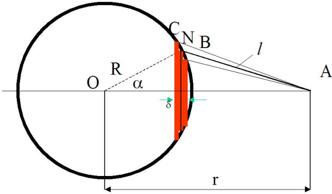
$$U = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m \tag{2}$$

Newton theorem



- Lemma 1: Gravitation field **inside** infinitely thin spherical shell of the constant surface density is zero.
- Proof: $dS_1 \cdot cos\alpha = r_1^2 d\Omega$, $dS_2 \cdot cos\alpha = r_2^2 d\Omega$

$$F_1 = \frac{Gm\rho dS_1}{r_1^2} = \frac{Gm\rho dS_2}{r_2^2} = F_2$$



- Lemma 2: Gravitation field **outside** infinitely thin spherical shell of the constant surface density is the same as one would get if all the material was concentrated at the central point.
- Proof: $\delta \approx \overline{CB} sin\alpha \Rightarrow S_{CNB} = 2\pi R sin\alpha \overline{CB} = 2\pi R\delta$

  potential due to CNB ring at point A is $-\frac{G\rho 2\pi R\delta}{\ell}$

  - Potential due to whole spherical shell of mass $M = \rho 4\pi R^2$ is thus :

$$V = -\sum \frac{G\rho 2\pi R\delta}{\ell} = -G\rho 2\pi R \sum \frac{\delta}{\ell} = -\frac{G\rho 2\pi R}{r} 2R = -\frac{GM}{r}$$

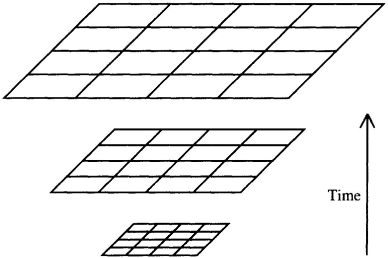
Q.E.D.

The Friedmann equation

- Universe is homogeneous ⇒ the above argument applies to any two particles.
- **Comoving coordinates**: coordinates which are carried along with the expansion. Let $\vec{r}$ and x be real and comoving distances, respectively. Expansion is uniform ⇒

$$\vec{r} = a(t)\vec{x} \tag{3}$$

- Homogeneity ⇒ **The scale factor of the Universe** $a(t)$ depends only on time. It measures the universal expansion rate. Since the coordinate distances $\vec{x}$ are by definition fixed it tells us how physical separations grow with time.



- The galaxies remain at fixed locations in the coordinate system $\vec{x}$. The original coordinate system $\vec{r}$, which does not expand, is usually known as **physical coordinates**.

The Friedmann equation

- Substituting (3) into (2): $$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi}{3}G\rho a^2x^2m \qquad (4)$$

  after some rearrangement:

  The Friedmann equation

  $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \qquad (5)$$

  $k = -2U/mc^2x^2$ is the time-independent constant (the total energy U is conserved, c is speed of light, the comoving separation x is fixed). $[k] = length^{-2}$.

- An expanding Universe has a unique value of k, which it retains throughout its evolution. Later on we will see that it is related to the geometry of the Universe (it is often called the curvature).

On the meaning of expansion

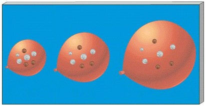
- **The expansion does not mean that**
  - the Earth's orbit is going to get further from the Sun
  - the stars within our galaxy are going to become more widely separated with time.
- **It does mean that distant galaxies are getting further apart!!**
- The distinction is whether or not the motion of objects is governed by the cumulative gravitational effect of a homogeneous distribution of matter between them.
  - The Earth's motion in its orbit is dominated the Sun (with a minor contribution from the other planets). Stars are orbiting in the common gravitational potential well which they themselves create.
  - The above environments are ones of considerable excess density, very different from the **smooth distribution of matter** used to derive the Friedmann equation.
- At large enough scales $\sim 10Mpc$, the Universe is homogeneous and isotropic. It is on these large scales that the expansion of the Universe is felt, and on which the cosmological principle applies.

Things that go faster than light

Q: If velocity is proportional to distance, then if we consider galaxies far enough away can we not make the velocity $v > c$, in violation of special relativity?

A: Distant objects can appear to move away faster than the speed of light. However, it is space itself which is expanding. There is no violation of causality, because no signal can be sent between such galaxies.

- Special relativity is not violated, because it refers to the relative speeds of objects passing each other, and cannot be used to compare the relative speeds of distant objects.



Example: Colony of ants on a balloon

- $v_{max}^{ant} = 1cms^{-1}$. Two ants in opposite directions $\Delta v_{max}^{ant} = 2cms^{-1}$.
- If the balloon is blown up fast enough $\Rightarrow$ ants which are far apart can move apart faster than Δv_{max}^{ant}. However, they will never get to tell each other about it (the balloon is pulling them apart faster than they can move)

The fluid equation

- Friedmann eq.: $\dot{a}^2 = (8\pi G\rho a^2)/3 - kc^2 \quad \Rightarrow \quad \rho \equiv \rho(t)$?
- First law of thermodynamics applied to an expanding volume V of unit comoving radius: $$dE + pdV = TdS \tag{6}$$

$$E = mc^2 = V\rho c^2 = \frac{4\pi}{3}a^3\rho c^2 \tag{7}$$

$$\frac{dE}{dt} = 4\pi a^2\rho c^2\frac{da}{dt} + \frac{4\pi}{3}a^3\frac{d\rho}{dt}c^2 \quad , \quad \frac{dV}{dt} = 4\pi a^2\frac{da}{dt} \tag{8}$$

- For **reversible expansion** $dS = 0 \Rightarrow dE + pdV = 0 \Rightarrow$

The fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{9}$$

- $3\frac{\dot{a}}{a}\rho \leftrightarrow$ dilution in the density due to volume increase.
- $3\frac{\dot{a}}{a}\frac{p}{c^2} \leftrightarrow$ loss of energy due to the pressure work. Energy lost from the fluid has gone into gravitational potential energy.

- In a homogeneous Universe ρ and p are everywhere the same $\Rightarrow$ there are no pressure forces. (Force is supplied through $\vec{\nabla}p \neq 0$). So pressure does not contribute a force helping the expansion along; its effect is solely through the work done as the Universe expands.

The equation of state

- Now we know what ρ is doing if we know what the pressure p is. Specifying the pressure we specify what kind of material our model Universe is filled with.
- To describe the evolution of the Universe we need to solve the fluid and Friedmann equations. To solve them we need to know also **the equation of state** $p \equiv p(\rho)$.
- The simplest possibility is that there is no pressure at all, and that particular case is known as (non-relativistic) matter.
- Other possibilities are matter or radiation dominated Universe.
- We will solve the equations later on.

The acceleration equation

- The third equation, not independent of the first two, of course, describes the acceleration of the scale factor a. By differentiating equation (5) with respect to time we obtain:

$$2\frac{\dot{a}}{a}\frac{a\ddot{a}-\dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{kc^2\dot{a}}{a^3} \tag{10}$$

- Substituting in for $\dot{\rho}$ from equation (9) and cancelling the factor $2\dot{a}/a$ in each term gives

$$2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2} \tag{11}$$

- Using (5) again, we arrive at

The acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{12}$$

- If the material has any pressure, this increases the gravitational force, and so further decelerates the expansion.

- N.B. The acceleration equation does not feature the constant k which appears in the Friedmann equation; it cancelled out in the derivation.

Elements of relativistic cosmology

- Based on famous Einstein equations of **General relativity**.
- Finite velocity of signal propagation ($c = 1$) brings into existence an autonomous entity – the field.
- Instead of Newton potential we have **Gravitational field** $g_{\mu\nu}(x)$ describing simultaneously the metric at spacetime point x.
- GR resuscitates the notion of **Geometrodynamic**: eqs. of particle motion under the influence of the field are equivalent to free motion eqs. in appropriately curved space time (effect of gravitation is equivalent to that of a coordinate transformation).
- Curved geometry of the space-time leads to new geometrical concepts: Affine connection, Riemanian curvature *etc.*
- In static non-relativistic limit $c \to \infty$ GR reproduces old Newton theory and its previously derived cosmological consequences.
- Most useful approximation to GR metrics is the cosmological principle leading to Friedman-Lemaître-Robertson- Walker metric.

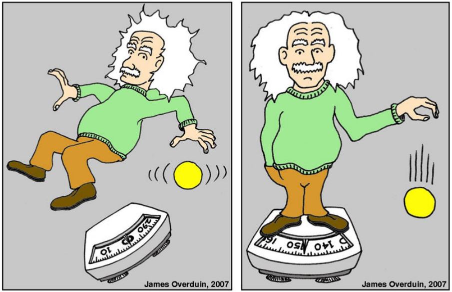
Special relativity reminder

- Covariant vector: $$A_\mu \equiv (A_0, A_i) = (A^0, -\mathbf{A}) \tag{13}$$
- Contravariant vector: $$A^\mu \equiv (A^0, A^i) = (A_0, \mathbf{A}) \tag{14}$$
- Scalar product: $$A \cdot B \equiv A_\mu A^\mu = (A^0)^2 - (\mathbf{A})^2 \equiv \eta_{\mu\nu} A^\mu A^\nu \tag{15}$$
- Metric tensor: $$\eta^{\mu\nu} = \eta_{\mu\nu} = diag(1, -1, -1, -1) \tag{16}$$
  defines distance between two points $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 d\tau^2$
- Properties: $$A_\mu = \eta_{\mu\nu} A^\nu \quad , \quad A^\mu = \eta^{\mu\nu} A_\nu \tag{17}$$
  $$\eta_{\mu\nu} = \eta_{\nu\mu} \quad , \quad \eta_{\mu\alpha}\eta^{\alpha\nu} = \delta^\nu_\mu = \eta^\nu_\mu \tag{18}$$
  Tensors $\delta^\nu_\mu, \eta_{\mu\nu}, \eta^{\mu\nu}$ are the same in all coordinate systems.
- 4-velocity $u^\mu = dx^\mu/ds$ and 4-acceleration $a^\mu = du^\mu/ds = d^2x^\mu/ds$
- Action for relativistic free particle
  $$S = -m \int \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} = -m \int ds = -m \int c d\tau \Rightarrow d^2x^\mu/ds = 0$$
- N.B. The rest mass m has a minus sign as it is part of the potential energy in non-relativistic physics.

Equivalence Principle

- General Relativity (GR) is based on the **Equivalence Principle**: at any space-time point in arbitrary gravitational field $\exists$ **locally inertial** coordinate system where effects of this field are absent.

A. Einstein 1907



For an observer falling freely from the roof of a house, the gravitation field does not exist (left). Conversely (right), an observer in a closed box – such as elevator or spaceship – cannot tell whether his weight is due to gravity or acceleration.

General relativity

- Equivalence principle: **To cancel out effect of gravity at a given point one may use locally the flat metric** $\eta_{\mu\nu}$.
- Consider freely falling observer who erects a SR coordinate frame ξ^μ in his neighbourhood. The equation of motion for nearby freely falling particle is:
$$\frac{d^2\xi^\mu}{d\tau^2} = 0 \quad ; \quad ds^2 = c^2 d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{19}$$
- Now suppose that the observer makes a transformation to some other set of coordinates x^μ:
$$d\xi^\mu = \frac{\partial \xi^\nu}{\partial x^\nu} dx^\mu \tag{20}$$
- (20)$\mapsto$ (19) $\Rightarrow$ two principal equations of GR:

$$\frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\mu}{\partial \xi^\nu} \frac{\partial^2 \xi^\nu}{\partial x^\alpha x^\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \equiv \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{21}$$

$$c^2 d\tau^2 = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} dx^\alpha dx^\beta \equiv g_{\alpha\beta}(x) dx^\alpha dx^\beta \tag{22}$$

Gravitational field in GR

- There is a metric tensor $g_{\mu\nu}(x)$ and the gravitation force is to be interpreted as arising from non-zero derivatives of this tensor.

- Taking the flat metric globally is not possible in the curved space-time under the gravitational field.
  - Space-time is curved $\eta_{\mu\nu} \to g_{\mu\nu} \equiv g_{\mu\nu}(x)$; $\quad g_{\mu\alpha}g^{\alpha\nu} = g_{\mu}^{\nu} = \delta_{\mu}^{\nu}$
  - Only distance to nearby points makes sense $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \quad (23)$

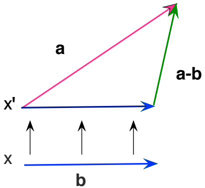
- $g_{\mu\nu}(x)$ – real symmetric matrix $\Rightarrow$ can be, with a suitable choice of coordinates, (at least locally) made into diagonal matrix $g_{\mu\nu} = diag(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$.
- If one of λ's is positive while other three are negative $\Rightarrow$ the space-time is called **Riemann space**. Then $det\ g_{\mu\nu}(x) \equiv g_{\mu\nu}(x) < 0$.
- The invariant volume element is $\sqrt{-g(x)}d^4x$

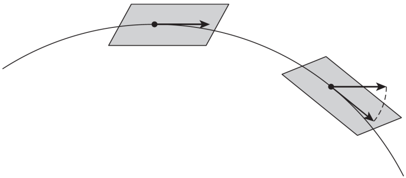
Equations of motion

- Eq. of motion of freely falling particle: $$\frac{d^2x^\mu}{d\xi^2} + \Gamma^\mu_{\nu\kappa}\frac{dx^\nu}{d\xi}\frac{dx^\kappa}{d\xi} = 0 \quad (24)$$
  where $\Gamma^\mu_{\nu\kappa}$ is **Riemann-Christoffel symbol** and ξ parametrizes position along the space-time curve.
  For massive particles $\xi \sim \tau \quad (\ d\tau \equiv \sqrt{g_{\mu\nu}dx^\mu dx^\nu}\ )$.
- A space-time path $x^\mu = x^\mu(\xi)$ satisfying (24) is called **geodesics** (i.e. $\delta\int_{x_0}^{x_1} d\tau = 0$ with end points x_0 and x_1 fixed).
- Particle at rest in these coordinates will stay at rest forever ⇒ **comoving coordinates**. t – time in the rest frame of comoving clock.

Parallel transport of tangent vectors: 2-d case

- Let $\vec{X}(x^1, x^2) \equiv \vec{X}(x)$ be a surface embeded in E^3 with 3-component vectors $\vec{e}_\mu \equiv \partial_\mu \vec{X}$ forming the basis vectors for the surface $\vec{X}(x)$.
- Vectors defined at two different points, x and $x^{'}$, can be compared only by moving the vector at x to the point $x^{'}$ by **parallel transport**.





$\Rightarrow$ Parallel transport of a vector $\vec{e}_\mu(x)$ living in the **tangent plane** at point x on a curved surface to a nearby point $x^{'} = x + \delta x$ leads to change of basis vectors: $\vec{e}_{\mu,\nu} \equiv \partial_\nu \vec{e}_\mu = \partial_\nu \partial_\mu \vec{X} = \Gamma^\lambda_{\mu\nu} \vec{e}_\lambda + K_{\mu\nu} \vec{n}$ (25)

- $\vec{e}_{\mu,\nu}$ sticks out of the surface because it has a component along (unit) normal vector to the surface $\vec{n} \equiv (\vec{e_1} \times \vec{e_2})/|\vec{e_1} \times \vec{e_2}|$. This component (dashed line) has to be projected away.

Parallel transport of a vector field

- Let $\vec{A}(x) \equiv A^{\nu}(x)\vec{e}_{\nu}(x)$ be a vector field on some general (curved) 2d-manifold (cylinder, sphere, torus, Möbius plane, Klein bottle, ...).
- $\vec{A}$ lives on the plane tangent to the manifold: $\vec{A} \cdot \vec{n} = 0$
  $\Rightarrow$ is visible only from ambient E^3.
- Infinitezimal parallel transport of $\vec{A}(x)$:
  $\partial_\nu \vec{A}(x) = \partial_\nu (A^\mu(x)\vec{e}_\mu(x)) = \partial_\nu A^\mu(x)\vec{e}_\mu(x) + A^\mu(x)\partial_\nu \vec{e}_\mu(x)$
- (25) $\Rightarrow$ $$\partial_\nu \vec{A}(x) = \partial_\nu (A^\mu)\vec{e}_\mu + A^\lambda \Gamma^\mu_{\lambda\nu}\vec{e}_\mu + A^\mu K_{\mu\nu}\vec{n} \qquad (26)$$
  where $\Gamma^\mu_{\lambda\nu}$ is so called Riemann-Christoffel symbol and $K_{\mu\nu} = \vec{e}_{\mu,\nu} \cdot \vec{n}$.
- Dropping terms proportional to $\vec{n}$ in (26) defines a **covariant derivative**: $$\nabla_\nu \vec{A} \equiv (\partial_\nu A^\mu + A^\lambda \Gamma^\mu_{\lambda\nu})\vec{e}_\mu \equiv (\nabla_\nu A^\mu)\vec{e}_\mu \qquad (27)$$
- $\nabla_\nu \vec{A}$ is nothing but the ordinary derivative $\partial_\nu \vec{A}$ adjusted for the change of the reference frame.
  It does not describe how $\vec{A}(x + \delta x)$ differs from $\vec{A}(x)$ but how $\vec{A}(x + \delta x)$ differs from $\vec{A}(x)$ parallel transported to $x + \delta x$.

Tangent frame, $\Gamma^{\mu}_{\lambda\nu}$, $K_{\mu\nu}$ and metrics on the surface

$\partial_\nu \vec{A}(x) = \partial_\nu A^\mu \vec{e}_\mu + A^\lambda \Gamma^{\mu}_{\lambda\nu} \vec{e}_\mu + A^\mu K_{\mu\nu} \vec{n}$

- $\Gamma^{\mu}_{\lambda\nu}$ tells us how the tangent plane is rotating around $\vec{n}$.
- $K_{\mu\nu}$ tell us about how the surface is curving in the ambient space (E^3 in our case).
- Since $d\vec{X} = \partial_\mu \vec{X} dx^\mu$, the distance squared between two nearby points x and $x + dx$ of the manifold is given by:

$$ds^2 = d\vec{X} \cdot d\vec{X} = (\partial_\mu \vec{X} \cdot \partial_\nu \vec{X}) dx^\mu dx^\nu = (\vec{e}_\mu \cdot \vec{e}_\nu) dx^\mu dx^\nu \qquad (28)$$

- Symmetric tensor $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$ is called **metric** on the surface $\vec{X}$.
- For 2-d surfaces $d\vec{X} = \partial_\mu \vec{X} dx^\mu = \partial_1 \vec{X} dx^1 = \partial_2 \vec{X} dx^2$ one obtains
  - $ds^2 = d\theta^2 + \sin^2\theta d\phi$ for the sphere
  - $ds^2 = r^2 d\phi^2 + dz^2$ for the cylinder
  - $ds^2 = (R + r\cos\phi)^2 d\theta^2 + r^2 d\phi^2$ for torus with tube radius r and distance from the center of the tube to the center of the torus R

Covariant derivative

- Vectors defined at two different points, x and $x^{'}$, can be compared only by moving the vector at x to the point $x^{'}$ by **parallel transport** i.e. the naive derivative of a vector is not a tensor but **the covariant derivative** $\Rightarrow \partial_\mu \mapsto \nabla_\mu$:

$$\nabla_\mu A_\nu \equiv (\delta^\lambda_\nu \partial_\mu - \Gamma^\lambda_{\mu\nu}) A_\lambda \quad , \quad \nabla_\mu A^\nu \equiv (\delta^\nu_\lambda \partial_\mu + \Gamma^\nu_{\lambda\nu}) A^\lambda \qquad (29)$$

but for the scalar they must coincide: $\partial_\mu (A^\nu A_\nu) = \nabla_\mu (A^\nu A_\nu)$

- (29) $\Rightarrow$ Bianchi identity: $\nabla_\lambda (A_\mu B_\nu) = (\nabla_\lambda) B_\nu + A_\mu (\nabla_\lambda B_\nu) \qquad (30)$
- (29) $\Rightarrow$ parallel transport mappings preserve the metric tensor:

$$\nabla_\lambda g_{\mu\nu} = 0 \qquad (31)$$

- (31) $\Rightarrow$ Coefficients $\Gamma^\mu_{\nu\kappa}$ – the **affine connection** (aka Riemann -Christoffel symbols): $\Gamma^\mu_{\nu\kappa} = \frac{1}{2} g^{\mu\lambda} (\partial_\kappa g_{\lambda\nu} + \partial_\nu g_{\lambda\kappa} + \partial_\lambda g_{\nu\kappa}) \qquad (32)$

tell us how to define parallelism between neighboring points.

Einstein equations

- **Ricci tensor**: $$R_{\mu\nu} \equiv \frac{\partial \Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\nu\sigma} \tag{33}$$
- **Scalar curvature**: $$R \equiv R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{34}$$

The Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{35}$$

- Total energy-momentum tensor of the matter, radiaton and the vacuum is conserved: $\nabla_{\mu} T^{\mu}_{\nu} = \nabla_{\mu}(g^{\mu\lambda} T_{\lambda\nu}) = 0$ (36)

  (35) & (36) $\Rightarrow \nabla_{\mu} G^{\mu}_{\nu} = \nabla_{\mu}(g^{\mu\lambda} G_{\lambda\nu}) = 0$
  N.B. this equation also follows from the Bianchi identity (30).

- The weak and static gravitational field & NR limit:
  $$g_{00} \simeq 1 + \phi_N \ \& \ T_{00} \simeq \rho c^2$$
  (35)$\mapsto \triangle \phi = 4\pi G \rho c^2$ i.e. Poisson equation for the Newton potential.

The Friedman–Lemaître–Robertson–Walker metric

- General solution of (35) i.e. $g_{\mu\nu}(x)$ does not exist (the field equations are non-linear). For example, there is no known complete solution for a space-time with two massive bodies in it (binary star).
- The cosmological principle $\Rightarrow$ dramatic simplification: at a given time the Universe should not have any preferred locations.
- $\Rightarrow$ The spatial part of the metric must have a **constant curvature** (satisfied e.g. by a flat metric which has zero curvature everywhere). The most general metric with constant curvature is:

$$ds^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \qquad (37)$$

$$g_{tt} = 1,\ g_{rr} = -\tfrac{a^2}{1-kr^2},\ g_{\theta\theta} = -a^2r^2,\ g_{\phi\phi} = -a^2r^2sin^2\theta \qquad (38)$$

  where $a(t)$ is the **scale factor of the Universe**.
- N.B. Inserting function of time $b^2(t)$ before the dt^2 in (37) just redefines time variable: $dt \mapsto dt^{'} = b(t)dt$. GR tells us that we can use any coordinate system and so this extension is not more general than (37).

Relativistic perfect fluid

- Einstein's equations (35) tell us how the presence of matter curves space-time, and one needs to describe this matter. The possible constituents of the Universe considered in these lectures are all examples of so-called perfect fluids, meaning that they have no viscosity or heat flow. $\Rightarrow \quad T^\mu_\nu = diag(\rho c^2, -p, -p, -p)$ (39)

- From (36) which expresses fact that the GR automatically encodes energy conservation: $\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha = 0$ (40)
we obtain for the $\nu = 0$ component relevant Christoffel symbols (N.B. T^μ_ν is diagonal!): $\Gamma^0_{00} = 0 \ ; \ \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}$ (41)
Substituting them in (33), (34) and (35), keeping careful track of the summation over repeated indices, gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \qquad (42)$$

which is nothing but the fluid equation (9) derived previously in the Newtonian gravitation limit.

Elements of general relativistic cosmology

- For FLRW metric (38), there are two independent Einstein equations (35), the time-time and the space-space. The derivation is too lengthy to reproduce here, but can be found in any good general relativity textbook. The time-time Einstein equation gives precisely the Friedmann equation (5):
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{43}$$
- N.B. Compared to (5) there is a difference when interpreting k.
  - In the Newton limit it's related to the total energy U of the particle with mass m having comoving coordinate x: $k = -U/(mc^2x^2)$
  - In the GR case it is via (37) and (38) related to 3d the curvature of the FLRW space-time $k = (r^2 g_{rr} - g_{\theta\theta})/(r^4 g_{rr})$
- The space-space equation: $$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -8\pi G\frac{p}{c^2} \tag{44}$$

  and if we subtract the Friedmann equation (43) from it we get precisely the acceleration equation (12).

On metrics sign convention in QFT and GR

- In particle physics and quantum field theory (QFT) time-dominant sign convention $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$ is extensively being used. The momentum squared $p^2 = p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu = (p^0)^2 - (\vec{p})^2 = m^2$.
- This should be compared with space-dominant sign convention $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$ giving $p^2 = -m^2$ which is used in the literature on gravity and string theory (GR).
- To go from one convention to the other, simply flip the sign of $g_{\mu\nu}$:

$$g_{\mu\nu}^{GR} = -g_{\mu\nu}^{QFT} \tag{45}$$

- N.B. Flipping the sign of $g_{\mu\nu}$ does not change the sign of $\Gamma^\mu_{\nu\kappa} = \frac{1}{2} g^{\mu\lambda}(\partial_\kappa g_{\lambda\nu} + \partial_\nu g_{\lambda\kappa} + \partial_\lambda g_{\nu\kappa}) \Rightarrow R^\lambda_{\rho\mu\nu}$ and also $R_{\mu\nu}$ do not change a sign but the scalar curvature $R \equiv R^\mu_\mu = g^{\mu\nu} R_{\mu\nu}$ does.
- $\Rightarrow$ $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ does not change too.
- $\Rightarrow$ The GR convention: the sphere has positive scalar curvature.

On mass, energy and vanishing factors of c^2

- From now on we will drop c from all equations moving to so called **natural units** $c = 1$.
- The constant k appearing in the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{46}$$

  will then have dimensionality $[k] = time^{-2}$[time].
- N.B. Since mass density ρ and energy density ϵ are related via $\epsilon = \rho c^2 \Rightarrow$ in natural units they became the same. For clarity we must be careful to maintain the distinction. Note that the phrase **mass density** is used in Einstein's sense – it includes the contributions to the mass from the energy of the various particles, as well as any rest mass they might have.