Elektrostatika

Fascinující děje – přenos síly na dálku

Historie – Řecko (tření jantaru, magnetovec)
Čína (kompas)
W. Gilbert (20 látek jako jantar 1600)
N. Cabeo (+ a − 1629)
H. Cavendish (náboj na povrchu vodiče)
Ch. Coulomb, A. Volta, H. Oersted, Jean Baptiste Biot,
Felix Savart, André Marie Ampère (~1830)
Michael Faraday (1831)

James Clark Maxwell (1873)

El. náboj

1) Silový účinek

2). Kvantován – ∃ elementární kvantum náboje $1{,}602 \cdot 10^{-19}\,C$

3) Zachovává se lokálně

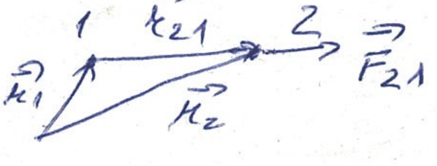
4) nemění se při Lorentzových transformacích

5) Kvazineutralita (+ a − se vyruší v dostatečně „velkém" objemu)

Coulombův zákon 1785

(1) $$\vec{F}_{21} = k\,\frac{Q_1 Q_2}{r_{21}^3}\,\vec{r}_{21} = k\,\frac{Q_1 Q_2}{r_{21}^2}\,\overbrace{\frac{\vec{r}_{21}}{r_{21}}}^{\vec{e}_{21}} = -\vec{F}_{12}$$

$$\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$$



jak působí (1) na (2)

② Princip superpozice

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32} + \vec{F}_{34} + \dots$$

$k \approx 9 \cdot 10^9 \ N m^2/C^2$ $\qquad k = c^2 \cdot 10^{-7}$

velikost c

bodový náboj – trochu idealizace (proton)

Pro počítání – hustoty

lineární $\quad \tau = \lim\limits_{\Delta l \to 0} \frac{\Delta q}{\Delta l} \ ; \ q = \int\limits_l \tau \, dl$

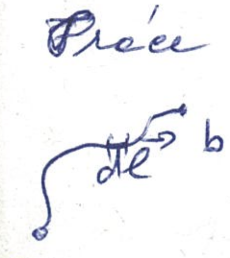
plošná $\quad \sigma = \lim\limits_{\Delta S \to 0} \frac{\Delta q}{\Delta S} \ ; \ q = \int\limits_S \sigma \, dS$

objemová $\quad \rho = \lim\limits_{\Delta V \to 0} \frac{\Delta q}{\Delta S} \ ; \ q = \int\limits_V \rho \, dV$

<u>Elektrostatická energie soustav</u>

= práce potřebná k vytvoření soustav

Práce síly $\Delta A = \vec{F} \cdot \Delta \vec{l}$



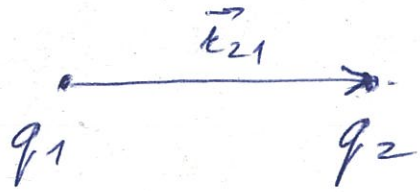
$$A = \int_a^b \vec{F} \cdot d\vec{l} = \int_a^b F \cdot e_\tau \cdot dl$$

tečné

Práce potřebná k dodat energii – proti síle

tj. $-\vec{F}_c$

pro 2 náboje q_1 q_2



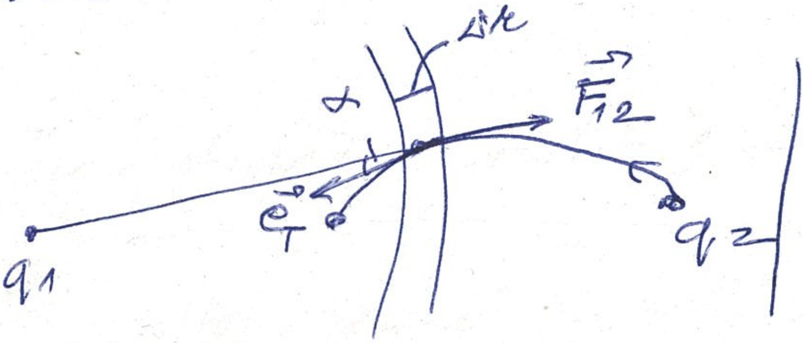
$\leftarrow \infty$

v ∞ $F = 0$

$$W = \int_{\infty}^{r_{21}} -\vec F_{21} \cdot d\vec l = -\int_{\infty}^{r_{21}} k \cdot \frac{q_1 q_2}{r^2}\, dr =$$

$$= k \left[\frac{q_1 q_2}{r} \right]_{\infty}^{r_{21}} = k \, \frac{q_1 q_2}{r_{21}} \quad [J]$$

nezávislost na dráze



$-\vec F_{21} \cdot \vec e_T \cdot \Delta l =$

$= -F_{21} \cdot 1 \cdot \cos\alpha \cdot \Delta l$

$\Delta l \cdot \cos\alpha = \Delta r$

proč nezávisí na dráze

$\Delta A = \vec F \cdot \Delta \vec l$ pokud $\vec F \parallel \Delta \vec l$,

pak $F = \dfrac{\Delta A}{\Delta l}$

pro dané $\Delta \vec l$ v místě bude ΔA maximální

$\Leftrightarrow \Delta \vec l \parallel \vec F$ $\qquad \vec F = -\operatorname{grad} W$

3 náboje

1 q_1 2 q_2 3 q_3

$$\boxed{W = k \frac{q_1 q_2}{r_{21}} + \underbrace{k \frac{q_1 q_3}{r_{31}} + k \frac{q_2 q_3}{r_{32}}}_{\text{z principu superpozice}}}$$

můžeme přepsat symetricky

$$W = \frac{1}{2}\left(k\frac{q_1 q_2}{r_{21}} + k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{31}} + k\frac{q_3 q_1}{r_{13}} + k\frac{q_2 q_3}{r_{32}} + k\frac{q_2 q_3}{r_{23}} \right)$$

pro více nábojů

$$W = k \sum_{\alpha=1}^{N} \sum_{\beta=1}^{\alpha-1} \frac{q_\alpha q_\beta}{r_{\alpha\beta}} = \frac{1}{2} k \sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha} \frac{q_\alpha q_\beta}{r_{\alpha\beta}}$$

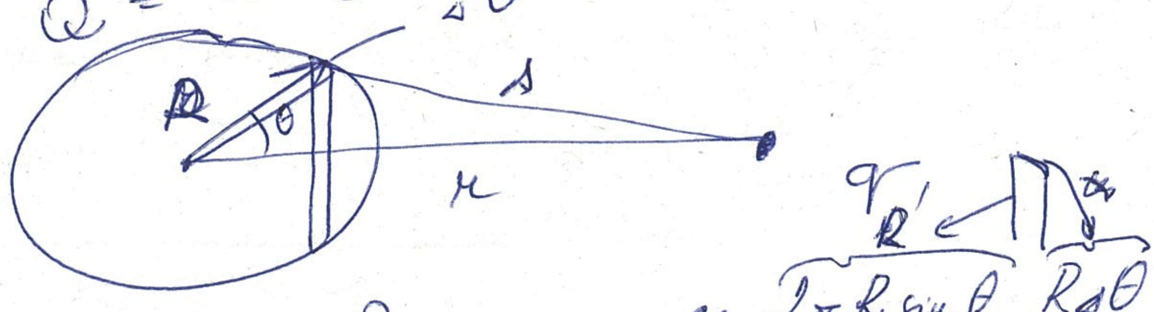
$$W \gtreqless 0$$

Cvičení, skripta – elstat energie krystalů

Homogenně nabitá koule (fikce)

A) Kulová slupka – sféra – jak se chová

$Q = 4\pi R^2 \sigma$



$$\Delta W = k \cdot \frac{q \cdot \Delta Q}{s} \approx k \frac{q \cdot 2\pi R \sin\theta \; R\Delta\theta \; \sigma}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}$$

$$W = \int_0^{\pi} 2\pi k q R^2 \sigma \frac{\sin\theta \; d\theta}{\sqrt{}} =$$

vezmeme $\frac{d}{d\theta}\left[\underbrace{\overbrace{\sqrt{R^2+r^2-2Rr\cos\theta}}^{s}}_{s'}\left(\frac{1}{Rr}\right)\right] =$

$\frac{1}{2}\,\frac{+2Rr\sin\theta\, d\theta}{\sqrt{}}\,\frac{1}{Rr}$!!!

$$= 2\pi k q \sigma R^{2} \int_0^{\pi} \frac{s'(\theta)}{rR}\, d\theta =$$

$$= \frac{2\pi k q \sigma R}{r}\left[\sqrt{R^2+r^2-2Rr\cos\theta}\right]_0^{\pi} =$$

$$= \frac{2\pi k q \sigma R}{r} \begin{cases} r>R: & R+r-(r-R) = k\frac{4\pi R^2\sigma\, q}{r} = k\frac{Qq}{r} \\ r<R: & R+r-(R-r) = k4\pi R\sigma\, q = k\frac{Qq}{R} \end{cases}$$

Z vně jako by byl náboj uprostřed.

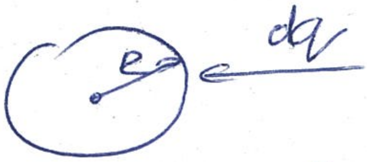
Uvnitř $W =$ konst $\Rightarrow \vec{F} = 0$

Koule? - Doskládám slupky - sféry

$\sigma = \rho \cdot \Delta r$ tloušťky Δr

$\Delta Q = 4\pi r^2 \Delta r\, \rho$; $q_{uprostřed} = \frac{4}{3}\pi r^3 \rho$

$$W = k\int_0^R \frac{4}{3}\pi r^3 \rho\, \frac{4\pi r^2 \rho\, dr}{r} = \frac{16}{15} k\pi\rho^2 R^5 = \frac{3}{5} k \frac{Q^2}{R}$$

Energie slupky 

$$W = k\int_0^Q \frac{q\,dq}{R} = k\frac{Q^2}{2R}$$

Pokud by $|\vec{F}| \sim \frac{1}{r^{2+\varepsilon}}$ $\qquad \int k\frac{q_1 q_2}{r^{2+\varepsilon}}\,dr = k'\frac{q_1 q_2}{r^{1+\varepsilon}}$

$$W = K\int_0^\pi \frac{\sin\theta\, d\theta}{s^{1+\varepsilon}} = K\int_0^\pi \frac{\sin\theta\, d\theta}{(R^2+r^2-2Rr\cos\theta)^{\frac{1+\varepsilon}{2}}}$$

$$= K\frac{1}{Rr}\int_0^\pi \left[\left(\sqrt{R^2+r^2-\ldots}\right)^{1-\varepsilon}\right]' d\theta =$$

$$= K\frac{1}{r}\left[-|R-r|^{1-\varepsilon} + |R+r|^{1-\varepsilon}\right]$$

není konstanta uvnitř.

Elektrostatické pole

$\int_a^b \vec{F}\cdot d\vec{l}$ nezávisí na dráze (superpozice C. z.)

$$\Rightarrow \oint_l \vec{F}\cdot d\vec{l} = 0 = \int_S \operatorname{rot}\vec{F}\, d\vec{S}$$

platí pro $\forall\, l \Longleftrightarrow \operatorname{rot}\vec{F} = 0$

ale $\vec{F}$ budou různá pro různá Q, na něž pole působí

Jak lépe popsat?

Silou na jednotkový náboj

<u>Intenzita el. pole</u> def $\boxed{\vec{E}(\vec{r}_0) = \frac{\vec{F}_{q_0}(\vec{r}_0)}{q_0}} =$

$$= \frac{1}{q_0} \sum_{\alpha=1}^{N} k \cdot \frac{q_0 q_\alpha}{r_{0\alpha}^3} \vec{r}_{0\alpha} = \sum_{\alpha=1}^{N} k \cdot \frac{q_\alpha}{r_{0\alpha}^3} \vec{r}_{0\alpha}$$

Pro spojitý náboj

$$\vec{E}(\vec{r}_0) = k \int \frac{dq}{|\vec{r} - \vec{r}_0|} (\vec{r}_0 - \vec{r}) = k \int_V \frac{\rho \, dV}{|\vec{r}_0 - \vec{r}|^3} (\vec{r}_0 - \vec{r})$$

$$\boxed{[E] = N/C \, ; \; \overset{J}{Nm}/Cm = J/Cm \, , \; V/m}$$

<u>Siločáry a tok intenzity</u>

Siločára – čára k níž je v každém bodě $\vec{E}$ tečné,

– čára po níž by se pohyboval nabitý bod téměř nulovou rychlostí

Každým bodem, kde $\vec{E} \neq 0$, prochází! siločára

Tok intenzity el. pole

$$\Phi = \int_S \vec{E} \cdot d\vec{S}$$

(na okraj: $\frac{N}{S}$ - hustota toku siločar $\frac{N}{S} = E$

$N = \left(\vec{E} \cdot \vec{S}\right)_{max}$)

Gaussův zákon elstatický

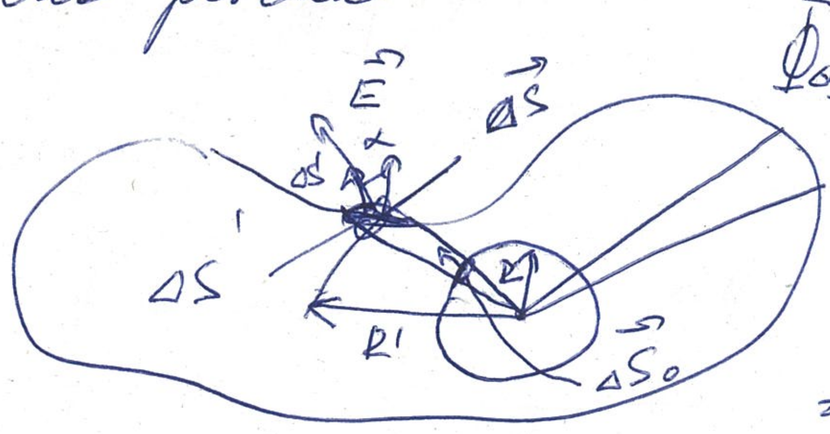
Tok $\vec{E}$ uzavřenou plochou?

a) Sféra, q uprostřed

$$\Phi = \oint_S \vec{E} \cdot d\vec{S} = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \oint_S dS = \frac{q}{\varepsilon_0}$$

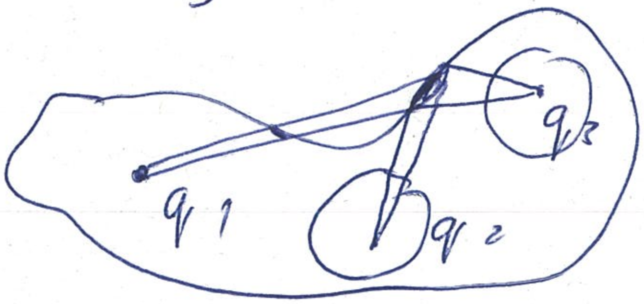
$$\vec{E} \cdot d\vec{S} = E \cdot dS \qquad ! \oint_S d\vec{S} = 0$$

b) obecná plocha



$$\Phi_{\Delta S} = \vec{E} \cdot \Delta\vec{S} = E \cdot \Delta S \cdot \cos\alpha = E \cdot \Delta S' = \Phi_{\Delta S_0}$$

$\Rightarrow \quad \Phi = \oint_S \vec{E}\, d\vec{S} = \frac{Q}{\varepsilon_0}$ ← uvnitř



$$\Phi = \frac{\sum q_i}{\varepsilon_0}$$

$$\Phi = \oint_S \vec{E}\, d\vec{S} = \oint_S \left(\vec{E}(q_1) + \vec{E}(q_2) + \vec{E}(q_3)\right) d\vec{S} =$$

$$= \oint_S \vec{E}(q_1)\, d\vec{S} + \oint_S \vec{E}(q_2)\, d\vec{S} + \oint_S \vec{E}(q_3)\, d\vec{S}$$

pro spojitě rozložený náboj:

$$\Phi = \oint_S \vec{E}\, d\vec{S} = \frac{Q_{celk}}{\varepsilon_0} =$$

$$= \frac{1}{\varepsilon_0} \overbrace{\int_V \rho\, dV}^{Q}$$

Maxwell. rovnice A) $\boxed{\oint_S \vec{E}\, d\vec{S} = \frac{Q}{\varepsilon_0}}$

a z Gaussovy věty o divergenci

$$\oint_S \vec{E}\, d\vec{S} = \int_V \mathrm{div}\, \vec{E}\, dV = \int_V \frac{\rho}{\varepsilon_0}\, dV$$

platí pro $\forall$ S a V $\Rightarrow$ A) $\boxed{\mathrm{div}\, \vec{E} = \frac{\rho}{\varepsilon_0}}$

Elstat.

$$\Gamma = \oint_l \vec{E} \cdot d\vec{l} = \oint \frac{\vec{F}}{q_0} d\vec{l} = 0$$

pro ∀ smyčky $\Rightarrow$ B) $\boxed{\begin{array}{l} \oint_l \vec{E} \cdot d\vec{l} = 0 \\ \mathrm{rot}\,\vec{E} = 0 \end{array}} = \int_S \mathrm{rot}\,\vec{E} \cdot d\vec{S}$

$\boxed{\mathrm{rot}\,\vec{E} = 0} \Rightarrow \exists\, \varphi$ tak, že $E = \mathrm{grad}\,\varphi$.

Víme $\vec{F} = -\mathrm{grad}\, W$; $\vec{E} = \dfrac{\vec{F}}{q_0}$; můžeme ztotožnit

" $\varphi = \dfrac{W}{q_0}$ "

potom $\vec{E} = -\mathrm{grad}\,\varphi$ — φ — potenciál

(obecnější než $\dfrac{W}{q_0}$)

po potenciál $\varphi = \dfrac{W}{q_0}$

rozdíl potenciálů ~ vykonaná práce na náboji "jednotky" proti poli z bodu ① do ②

$\varphi_{21} = \varphi_2 - \varphi_1$ ① φ_1 ② φ_2

$$\varphi_{21} = -\int^2 \vec{E}\, d\vec{l} = \int_1^2 d\varphi$$

Def: napětí U_{21} je práce vykonaná polem z bodu 2 do bodu 1 $\quad U_{21} = \int_2^1 \vec{E}\, d\vec{l}$

napětí a rozdíl potenciálů mají obrácená znaménka

$$\varphi(\vec{r}_0) = \int_V k \frac{\rho(\vec{r})}{|\vec{r} - \vec{r}_0|} dV = k \sum_{i=1}^{N} \frac{q_i}{|\vec{r}_0 - \vec{r}_i|}$$

gauss: div $\vec{E} = \frac{\rho}{\varepsilon_0}$ $\Rightarrow$

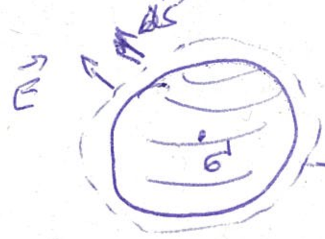
Poissonova r.: div grad $\varphi = -\frac{\rho}{\varepsilon_0}$; $\Delta\varphi = -\frac{\rho}{\varepsilon_0}$

Laplaceova r.: div grad $\varphi = 0 = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ $\Rightarrow$

buď $\forall$ členy 0 (inflexe), nebo pokud jeden > 0 druhý musí být < 0 $\Rightarrow$ $\nexists$ lokální extrém $\Rightarrow$ $\nexists$ místo, kde by F na náboj byla $F=0$ v nejbližším okolí.

(aby bylo minimum musí být $\forall$ $\frac{\partial^2}{\partial z^2} > 0$)

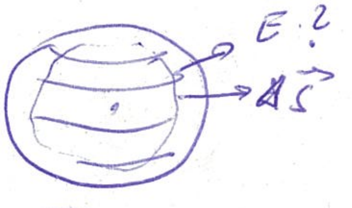
Důsledky Gauss. z. — symetrie!!



nabitá sféra, $\sigma = \text{konst} = \frac{Q}{4\pi R^2}$

$$\Phi = \oint_S \vec{E}\, d\vec{S} = E \cdot S = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$



ze symetrie $\vec{E} \parallel \Delta\vec{S}$

$$\Phi = 0 = E \cdot S \Rightarrow \vec{E} = 0$$

nekonečná deska homogenně nabitá $\sigma = \text{konst}$

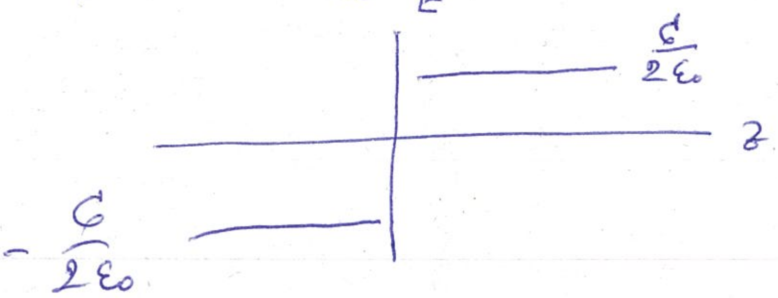
ze symetrie

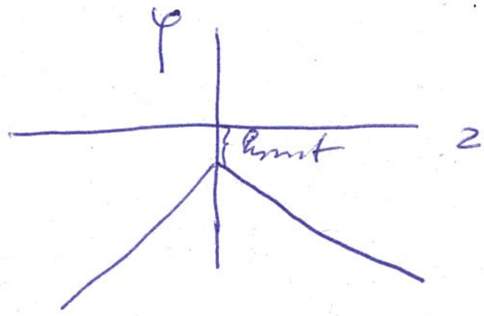
$$\Phi = \underbrace{E \cdot 2S = \frac{Q}{\varepsilon_0}}_{\text{gauss}}$$

$Q = \sigma \cdot S$ $\Rightarrow E = \frac{\sigma}{2\varepsilon_0}$

nebo počítat $\oint E(\vec{r}) = \int \dots$;

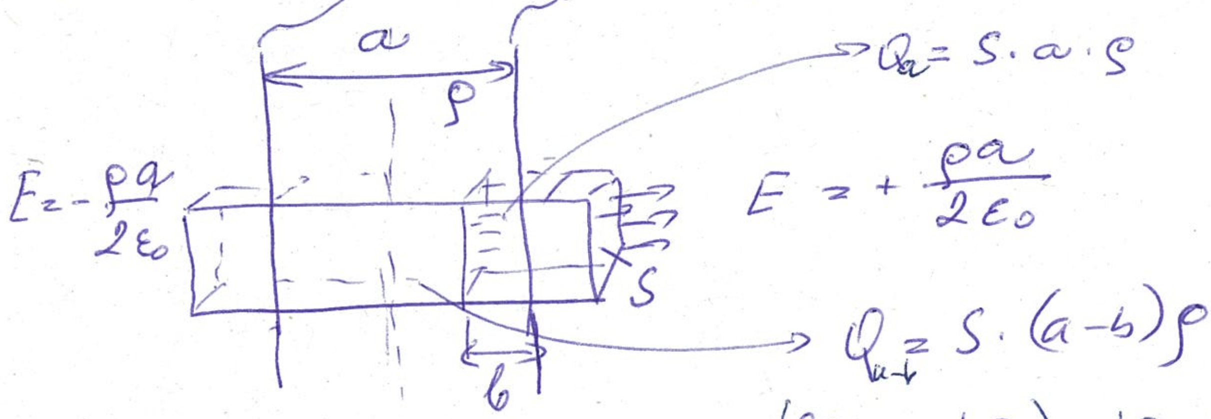
$$\varphi = -\frac{\sigma}{2\varepsilon_0}\cdot z + \text{konst} \qquad E = -\operatorname{grad}\varphi$$





Těsně nad povrchem i pro konečnou desku.

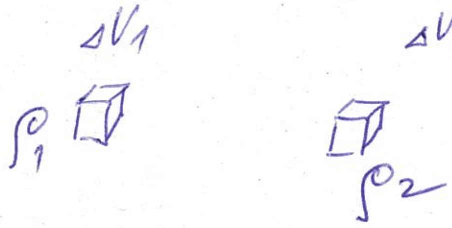
Deska konečné tloušťky – $\rho = \text{konst}$



$$\oint \vec{E}\,d\vec{S} = E(b)S + \frac{\rho a}{2\varepsilon_0}S = \frac{Q_b}{\varepsilon_0} \qquad E(b) = -\left(\frac{\rho a}{2\varepsilon_0} - \frac{b\rho}{\varepsilon_0}\right) = \frac{b\rho}{\varepsilon_0} - \frac{a\rho}{2\varepsilon_0}$$

pro $b = \frac{a}{2}$ $E = 0$

Spojitě rozložený náboj – energie



$$W = k \lim_{\Delta V\to 0} \sum_{i<j} \frac{\rho_i \Delta V_i\, \rho_j \Delta V_j}{r_{ij}} = \frac{1}{2} k \lim_{\Delta V\to 0} \sum_{i,j} \frac{\rho_i \Delta V_i\, \rho_j \Delta V_j}{r_{ij}} =$$

$$= k\,\frac{1}{2}\int_V\int_V \frac{\rho(\vec{r}_1)\,\rho(\vec{r}_2)}{r_{12}}\,dV_1\,dV_2 = \frac{1}{2}\int_V \rho(\vec{r}_1)\,\varphi(\vec{r}_1)\,dV$$

Otázka lokalizace energie (z. z. E. platí lokálně)

Energie ≈ hmotnost

$$\Delta\varphi = -\frac{\rho}{\varepsilon_0} \qquad \rho \Longleftrightarrow \text{změna } \operatorname{grad}\varphi \ (\text{zdroj})$$

$$W = -\frac{\varepsilon_0}{2}\int_V \varphi(\Delta\varphi)\,dV = -\frac{\varepsilon_0}{2}\int_V \varphi\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\right)dV =$$

$$= -\frac{\varepsilon_0}{2}\int_V\left[\frac{\partial}{\partial x}\left(\varphi\frac{\partial\varphi}{\partial x}\right)-\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial y}\right)-\left(\frac{\partial\varphi}{\partial y}\right)^2+\frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial z}\right)-\left(\frac{\partial\varphi}{\partial z}\right)^2\right]dV$$

$$= -\frac{\varepsilon_0}{2}\int_V \operatorname{div}(\varphi\operatorname{grad}\varphi)\,dV + \frac{\varepsilon_0}{2}\int_V(\operatorname{grad}\varphi)^2\,dV =$$

$$= -\frac{\varepsilon_0}{2}\oint_S \varphi(\operatorname{grad}\varphi)\,d\vec{S} + \frac{\varepsilon_0}{2}\int_V \vec{E}\cdot\vec{E}\,dV$$

$$\varphi \sim \frac{1}{r}, \quad \operatorname{grad}\varphi \sim \frac{1}{r^2}$$

$$\lim_{r\to\infty}\oint_{Q_r} k\cdot\frac{1}{r^3}\,dS + \frac{\varepsilon_0}{2}\int_\infty \vec{E}\cdot\vec{E}\,dV$$

$$= 0$$

$$W = \frac{\varepsilon_0}{2}\int_{V(E^2\neq 0)} E^2\,dV = \int_V w\,dV$$

$$w = \frac{\varepsilon_0 E^2}{2}$$ – hustota elstat. energie

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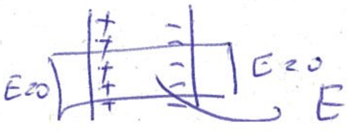
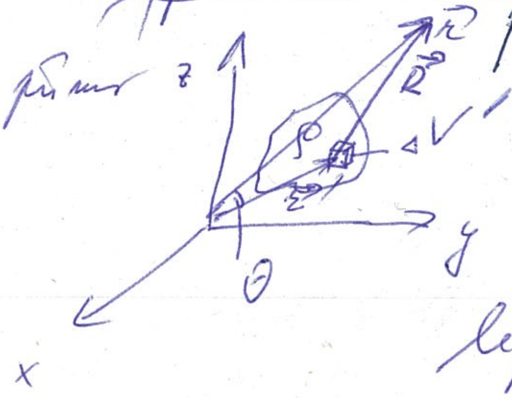
Kondenzátor: 

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 S}$$

$$\Delta W = \Delta q\cdot E(q)\cdot l = \frac{q}{\varepsilon_0 S}\cdot\Delta q\cdot l$$

$$W = \int dW = \frac{1}{2}\frac{Q^2}{\varepsilon_0 S}l = \frac{1}{2}\varepsilon_0 E^2 V$$

Výpočet $\vec{E}, \varphi$ Gauss + symetrie



pro $R >> r'$, $r >> r'$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\vec{r}')\,\vec{R}}{R^3}\, dV'$$

lépe: $$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\vec{r}')}{R}\, dV'$$

$$E = -\operatorname{grad}\varphi \quad (\varphi \in C^1)$$

Pro $|\vec{r}| \approx |\vec{R}| >> |\vec{r}'|$ – multipólový rozvoj (Taylor

$$|\vec{R}| = |\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} = r\sqrt{1 + \underbrace{\left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta}_{\alpha << 1}}$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}\, \rho(\vec{r}')\, dV' =$$

$$\frac{1}{R} = \frac{1}{r}(1+\alpha)^{-\frac{1}{2}} = \frac{1}{r}\left(1 - \frac{1}{2}\alpha + \frac{3}{8}\alpha^2 + \frac{15}{48}\alpha^3 \dots\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \frac{3}{8}\left(\frac{2r'}{r}\cos\theta\right)^2 + O\left(\left(\frac{r'}{r}\right)^3\right)\right) =$$

$$= \frac{1}{r}\left(1 + \frac{r'\cos\theta}{r} + \frac{1}{2r^2}\left(3r'^2\cos^2\theta - r'^2\right) + \dots\right) \Rightarrow$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\int_{V'} \rho(\vec{r}')\, dV'}{r} + \frac{1}{4\pi\varepsilon_0} \frac{\int_{V'} \rho(\vec{r}')\, r'\cos\theta\, dV'}{r^2} +$$

$$+ \frac{1}{4\pi\varepsilon_0} \frac{\int_{V'} \rho(\vec{r}')\left(3r'^2\cos^2\theta - r'^2\right) dV'}{r^3} + \dots =$$

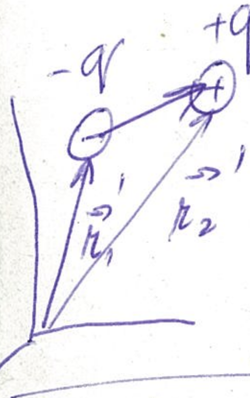
$$= \underbrace{\frac{1}{4\pi\varepsilon} \frac{Q}{r}}_{\varphi^{(0)}} + \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{K_1}{r^2}}_{\varphi^{(1)}} + \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{K_2}{r^3}}_{\varphi^{(2)}} + \dots$$

$$K_1 = \int_{V'} r'\cos\theta\,\rho\,dV' = \int_{V'} \frac{\vec r'\cdot\vec r}{r}\,\rho\,dV' =$$

$$= \frac{\vec r}{r}\int_{V'} \vec r'\,\rho\,dV' = \frac{\vec p\cdot\vec r}{r} = \vec p\cdot\vec e_r$$

$$\vec p = \int_{V'} \vec r'\,\rho(\vec r')\,dV' \quad \left(= \sum_i \vec r_i'\,q_i\right)$$

$$\varphi^{(1)} = \frac{1}{4\pi\varepsilon_0}\frac{K_1}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec p\cdot\vec r}{r^3}$$



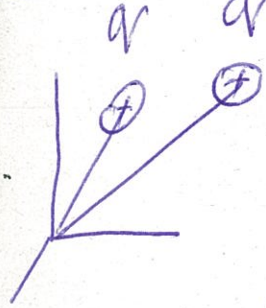
$Q = \sum_i q_i = 0$

$$\vec p = -q\vec r_1' + q\vec r_2' = q(\vec r_2' - \vec r_1') = q\cdot\vec r_{21} = q\cdot\vec l$$

$\sum_i q = 0$

posunutí počátku $\vec r_s' = \vec r_s'' + \vec a$, $\vec p_s = \sum_i \vec r_{si}' q_i + \vec a \sum q_i$

$\boxed{\text{pro } Q = 0\ \vec p \text{ nezávisí na počátku}}$



$Q = 2q$

„$\vec p \neq 0 \Rightarrow K_1 \neq 0$" „$\vec p = q(\vec r_1 + \vec r_2)$"

$K_2 \neq 0$

Koule se Q

pokud

$\vec p = 0$ $\quad \vec p = q(\vec r - \vec r)$ ← kvadrupól

$$K_2 = \int_{V'} \frac12\rho\,r'^2(3\cos^2\theta - 1)\,dV' =$$

$\boxed{\text{příspěvky od nehomogenity kolem „středu"}}$

$$= \frac12\int_{V'}\left(\frac{3(\vec r'\cdot\vec r)^2}{r^2} - \frac{r'^2\cdot r^2}{r^2}\right)\rho\,dV' =$$

$3(xx'+yy'+zz')^2 - r'^2(x^2+y^2+z^2)$ $\qquad x = x_1,\ y = x_2,\ z = x_3$

$$= \frac{1}{2r^2}\sum_{ij}\underbrace{\left[\int_{V'}(3x_i'x_j' - \delta_{ij}r'^2)\,\rho\,dV'\right]}_{Q_{ij}} x_i x_j = \frac{1}{2r^2}(x,y,z)\,\mathbb{Q}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

pro dipól

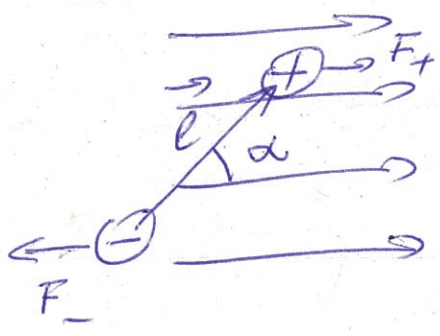
$$\varphi = k\,\frac{\vec{p}\cdot\vec{r}}{r^3} \qquad Q=0$$

$$\vec{E} = -\operatorname{grad}\varphi = -k\left[(\vec{p}\cdot\vec{r})\operatorname{grad}\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\operatorname{grad}(\vec{p}\cdot\vec{r})\right]$$

$$= -k\left[(\vec{p}\cdot\vec{r})\left(-\frac{3\vec{r}}{r^5}\right) + \frac{\vec{p}}{r^3}\right] =$$

$$= k\left[\frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right]$$

Dipól v homogenním el. poli ($\vec{E}=$ konst)



$$\vec{E} \qquad \vec{F}_+ + \vec{F}_- = 0$$

$$\vec{N} = \vec{l}\times\vec{F}_+ = q_+\cdot\vec{l}\times\vec{E} =$$

$$= \vec{p}\times\vec{E}$$

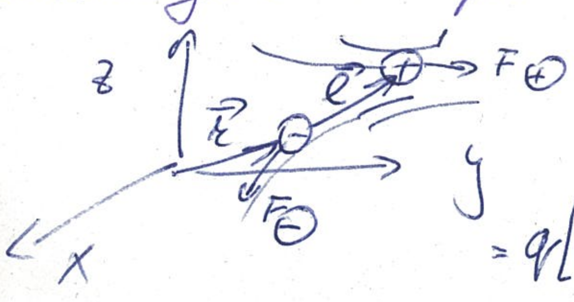
Potenciální energie – proti síle pole

$$W = \int_0^{\alpha} N\cdot d\alpha' = \int_0^{\alpha} pE\sin\alpha'\,d\alpha' =$$

$$= p\cdot E(-\cos\alpha + 1) = -\vec{p}\cdot\vec{E} + \underbrace{pE}_{\text{konst}}$$

---

nehomogenní pole? $\vec{E} \neq$ konst, $\vec{l} = (\Delta x, \Delta y, \Delta z)$



$$F_x = F_{\oplus x} - F_{\ominus x} = q\left(E_x(\vec{r}+\vec{l}) - E_x(\vec{r})\right) =$$

$$= q\Big[\underbrace{\left(E_x(x+\Delta x, y+\Delta y, z+\Delta z) - E_x(x+\Delta x, y+\Delta y, z\right)}_{\Delta z} + \ldots$$

$$+ \frac{E_x(x+\Delta x, y+\Delta y, z) - E_x(x+\Delta x, y, z)}{\Delta y} +$$

$$+ \left. \frac{E_x(x+\Delta x, y, z) - E_x(x, y, z)}{\Delta x} \right] \approx$$

$$\approx q\left(\frac{\partial E_x}{\partial z}\Delta z + \frac{\partial E_x}{\partial y}\Delta y + \frac{\partial E_x}{\partial x}\Delta x\right) =$$

symbolicky

$$= (\operatorname{grad} E_x) \cdot \vec{p} = \vec{p} \cdot \operatorname{grad} E_x = (\vec{p} \cdot \operatorname{grad}) E_x$$

algoritmus k napsání vektorů

Stejně pro F_y a F_z $\Rightarrow$

$$\underline{\vec{F} = (\vec{p} \cdot \operatorname{grad}) \vec{E}}$$

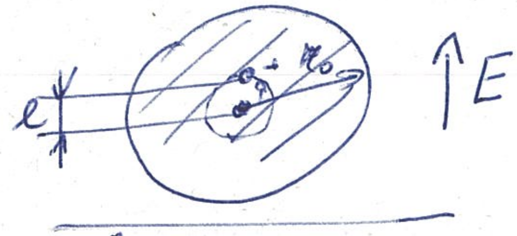
Síla mezi dipóly $\vec{p}_1$ a $\vec{p}_2$

$$\vec{F} = (\vec{p}_1 \cdot \operatorname{grad}) \vec{E} = (\vec{p}_1 \cdot \operatorname{grad}) \left(-k \operatorname{grad} \frac{\vec{p}_2 \cdot \vec{r}}{r^3}\right) \sim$$

$$\sim \frac{1}{r^4}$$

Dipólové momenty atomů a molekul

Ilustrace – indukované – atom



$$q \cdot E = -F_{coulomb}$$

$$\rho = \frac{q}{\frac{4}{3}\pi r_0^3}$$

$$F_c = \frac{1}{4\pi\varepsilon_0} \frac{\overbrace{\frac{4}{3}\pi l^3 \rho}^{q_1}\, q}{l^2} = \quad +$$

$$\boxed{= \frac{1}{4\pi\varepsilon_0} \frac{q^2 l}{r_0^3} = qE} \quad \rightarrow l = \frac{4\pi\varepsilon_0 r_0^3}{q} E$$

$$\vec{p} = q \cdot \vec{l} = \underbrace{4\pi\varepsilon_0 r_0^3}_{\alpha} \vec{E} = \alpha \cdot \vec{E}$$

polarizovatelnost

Experimentální $\vec{p} = \alpha \vec{E}$

	H	He	Li	Na	K	
α:	1,65	0,55	30	70	90	$[\cdot 10^{-41}\ \mathrm{Cm^2/V}]$

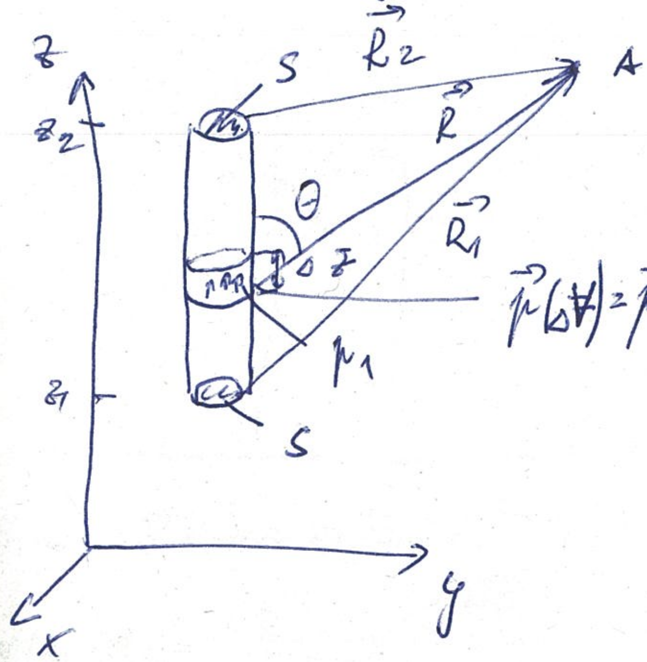
obecně – tenzor polarizovatelnosti $\vec{p} = \alpha \cdot \vec{E}$

Vlastní dipólové momenty molekul

	H_2O	HCl	NH_3	
p:	6,1	3,4	4,7	$\boxed{10^{-30}\ \mathrm{Cm}}$

Plošné a objemové rozložení el. dipólů

Homogenní materiál – dipóly



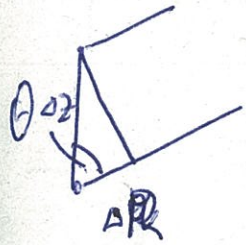
$\vec{p}_1$

$$\vec{p}(\Delta V) = \vec{p}_1 \cdot N \cdot \Delta V = \vec{P} \cdot \Delta V = \vec{P}\,\Delta z \cdot S$$

$\vec{p}_1 \cdot N = \vec{P}$ – vektor polarizace hustota el. dipólů

$$\boxed{\vec{P} = \frac{\vec{p}(\Delta V)}{\Delta V}}$$

nezávisle na tom, jak to vzniklo



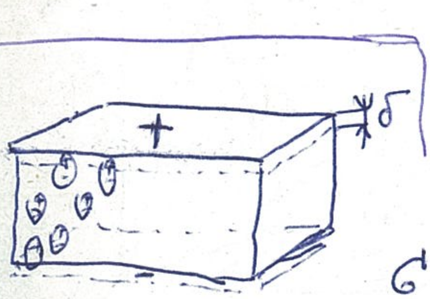
příspěvek k φ

$$\text{„}\Delta\text{"}\varphi_A(\Delta z) = k\,\frac{\vec{p}(\Delta V)\cdot\vec{R}}{R^3} = k\,\frac{P\cdot\cos\theta\cdot S\cdot\Delta z}{R^2}$$

$-\Delta z \cdot \cos\theta \approx \Delta R$

$$\varphi_A \underset{\approx \sum \text{„}\Delta\text{"}\varphi_A(\Delta z)}{=} kPS\int_{z_1}^{z_2}\frac{dz\cos\theta}{R^2} = kPS\int_{R_1}^{R_2} -\frac{dR}{R^2} =$$

$$= kPS\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$



$$\sigma = \frac{q\cdot\delta\cdot S\cdot N}{S} = \underbrace{q\cdot\delta}_{p}\cdot N = P$$

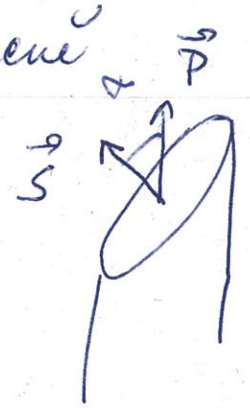
$$\Rightarrow \varphi_A = k\left(\frac{\sigma S}{R_2} - \frac{\sigma S}{R_2}\right) = kQ\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

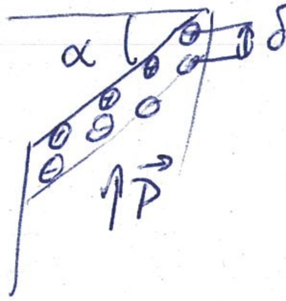
tj. přesně

$$\vec{p} = q\cdot\vec{\delta} \;; \quad Q = P\cdot S = \sigma\cdot S$$

$$\vec{P} \parallel \vec{S}$$

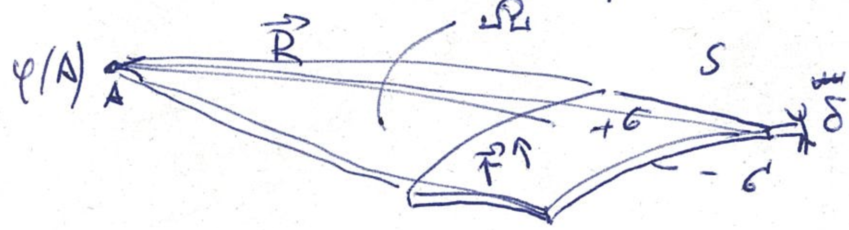
obecně





$$Q \approx q\,\delta N S\cos\alpha = $$
$$= qN\vec{\delta}\cdot\vec{S} = \vec{P}\cdot\vec{S}$$
$$\Rightarrow \boxed{\sigma = \vec{P}\,\vec{e}_\perp}$$

Homogenně polarizovaná „plocha" – dvojvrstva



$$\vec{P}_s \parallel \Delta\vec{S}$$

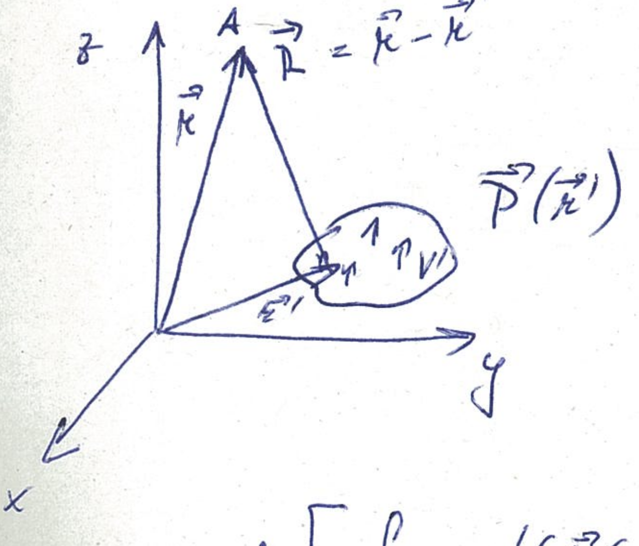
$\vec{P}_s$ – dipol. moment plošný

$$\varphi(A) = k\int_S \frac{\vec{P}_s\cdot\vec{R}}{R^3}\,dS = k\,P_s\int_S \frac{\overbrace{\vec{e}_R\cdot d\vec{S}}^{d\Omega}}{R^2} = k\,P_s\,\Omega$$

Polarizovaný objem

Pomocný vztah

$$\mathrm{div}\left(\vec{P}\cdot\frac{1}{R}\right) = \frac{\partial}{\partial x}\left(\frac{P_x}{R}\right) + \frac{\partial}{\partial y}\left(\frac{P_y}{R}\right) + \dots$$
$$= \frac{1}{R}\,\mathrm{div}\,\vec{P} + \vec{P}\,\mathrm{grad}\,\frac{1}{R}$$



$$\vec{P}(\vec{r}')$$

$$\varphi(\vec{r}) = k\int_{V'} \frac{\vec{P}(\vec{r}')\,\vec{R}}{R^3}\,dV' =$$
$$\mathrm{grad}'\frac{1}{R}\;(\text{! znam. – grad})$$

$$= k\int_{V'} \vec{P}(\vec{r}')\,\mathrm{grad}'\,\frac{1}{R}\,dV' =$$

$$= k\left[\int_{V'} \mathrm{div}'\left(\vec{P}(\vec{r}')\cdot\frac{1}{R}\right)dV' - \int_{V'}\frac{\mathrm{div}'\,\vec{P}(\vec{r}')}{R}\,dV'\right] =$$

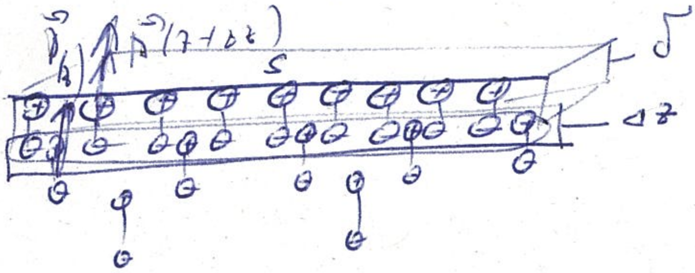
$$= k\oint_{S'} \frac{\vec P(\vec r\,')\cdot d\vec S\,'}{R} - k\int_{V'} \frac{\mathrm{div}'\,\vec P(\vec r\,')}{R}\,dV' =$$

$$= k\oint_{S'} \frac{\sigma_{vaz}(\vec r\,')\,dS'}{R} + k\int_{V'} \frac{\rho_{vaz}(\vec r\,')}{R}\,dV'$$

$\sigma_{vaz} = \vec P\,\vec e_n$ $\qquad$ $\rho_{vaz} = -\,\mathrm{div}\,\vec P$

náboj vázaný na povrchu $\qquad$ vázaný uvnitř

$\boxed{\text{rozložení}}$

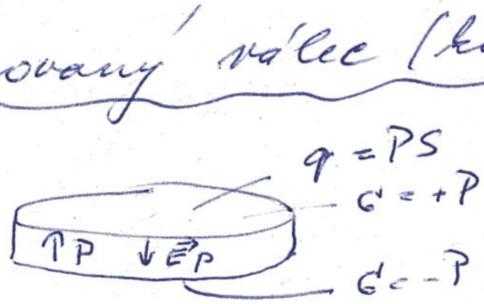


$$\Delta V = S\cdot\Delta z$$

$$Q(\Delta V) = Q(S\Delta z) = \boxed{-\big(P(z+\Delta z) - P(z)\big)\,S} = \boxed{-\big(\sigma(z+\Delta z) - \sigma(z)\big)\,S}$$

$$\rho = \frac{Q}{\Delta V} = -\,\frac{P(z+\Delta z) - P(z)}{\Delta z} \approx -\frac{\partial P}{\partial z}$$

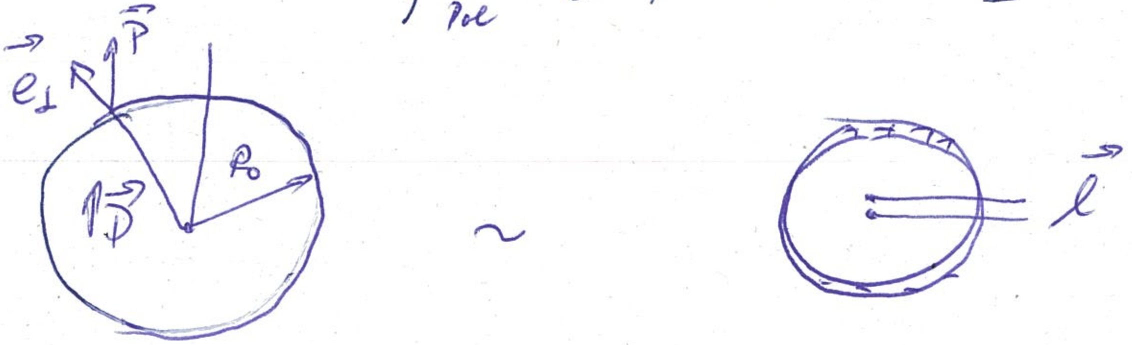
Polarizovaný válec (kotouč) $\qquad$ homogenně $\vec P = \text{konst}$, $\mathrm{div}\,\vec P = 0$



$$\vec E_{P\,\infty\,deska} = \frac{\sigma}{\varepsilon_0} = -\frac{\vec P}{\varepsilon_0}$$ nebo u konečné uprostřed

Polarizované diel. koule (homogenně)

$\vec{P} = konst$; $\rho^{v}_{pol} = 0$; $\sigma_0 = \vec{P} \cdot \vec{e}_\perp = P\cos\theta$



Vně: $\vec{p} = Q \cdot \vec{l} = V \cdot \vec{P} = \frac{4}{3}\pi R_0^3 \vec{P}$

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{p \cdot \cos\theta}{r^2}$$

Uvnitř: $\rho_v = 0$ tj $\Delta\varphi = 0$ a $\varphi_{r=R_0} = \frac{1}{4\pi\varepsilon_0} \frac{\frac{4}{3}\pi R_0^3 P\cos\theta}{R_0^2}$

$$= \frac{1}{3\varepsilon_0} P R_0 \cos\theta = \frac{1}{3\varepsilon_0} P \cdot z_0$$

ale bude-li i uvnitř $\varphi = \frac{1}{3\varepsilon_0} P \cdot z$

potom platí $\Delta\varphi = 0 \Rightarrow \exists!$ řešení

a to je právě ono.

$$\vec{E}_{uvnitř} = -\operatorname{grad}\varphi = -\frac{\vec{P}}{3\varepsilon_0}$$

~~Elstat~~ pole v dielektriku

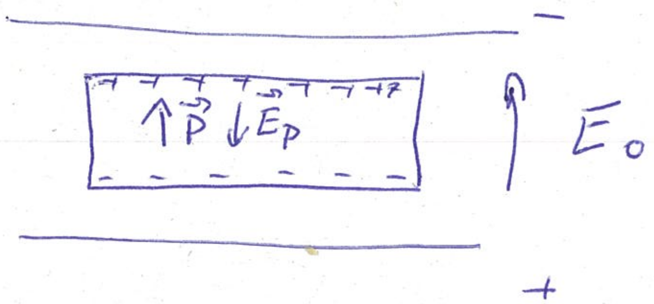
polarizace + volné náboje ~~- tuto prostředí~~

Dielektrika a vodiče v elstat. poli

Dielektrika – ∄ volné náboje ⇒ polarizace

Řídký plyn

iont. krystal.



u indukce

$$\vec{p} = \alpha \vec{E}$$

$$\vec{P} = N\vec{p} = N\alpha\vec{E}$$

$$\vec{E}_p = -\frac{\vec{P}}{\varepsilon_0}$$

$$\vec{E}_p = -\chi\vec{E}$$

el. susceptibilita

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

pro všechny mechanismy polarizace $\varepsilon_0\chi\,(= N\alpha)$

$$\vec{E}_{\text{uvnitř}} = \vec{E}_0 + \vec{E}_p = \vec{E}_0 - \frac{\vec{P}}{\varepsilon_0}$$

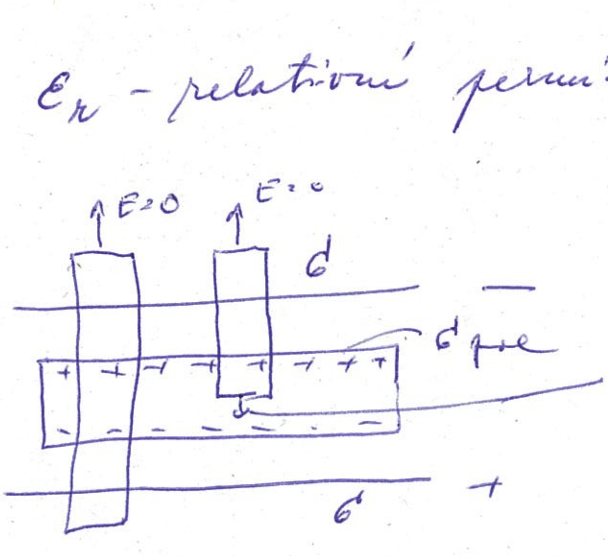
$$\vec{P} = \varepsilon_0 \chi \vec{E}_{\text{uvnitř}} = \varepsilon_0 \chi \vec{E}_0 - \chi \vec{P}$$

$$\vec{P} = \vec{E}_0 \frac{\varepsilon_0 \chi}{1+\chi}$$

$$\vec{E}_{\text{uvnitř}} = \vec{E}_0\left(1 - \frac{\chi}{1+\chi}\right) = \frac{\vec{E}_0}{1+\chi} = \frac{\vec{E}_0}{\varepsilon_r}$$

$\chi > 0$; ε_r – relativní permitivita $=$ „$1 + \frac{N\alpha}{\varepsilon_0}$"

a z Gausse



$$E = \frac{\sigma - \sigma_{pol}}{\varepsilon_0} = \frac{\sigma - P}{\varepsilon_0} = \frac{\sigma - \chi\varepsilon_0 E}{\varepsilon_0}$$

$$\Rightarrow E = \frac{\sigma}{\varepsilon_0}\frac{1}{1+\chi} = \frac{\sigma}{\varepsilon_0 \varepsilon_r}$$

ε_r krát slabší

$$\vec{F} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_1 q_2}{r^3}\vec{r}$$

Elstat. pole v dielektriku

polarizace + volné náboje (třeba nastřelené)

$$\rho_{celk} = \rho + \rho_{Pol} \left(= -\operatorname{div} \vec{P}\right)$$

$$\operatorname{div} \vec{E} = \frac{\rho + \rho_{Pol}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\rho - \operatorname{div} \vec{P}\right)$$

$$\operatorname{div} \underbrace{\left(\varepsilon_0 \vec{E} + \vec{P}\right)}_{\vec{D}} = \rho$$

$\vec{D}$ - vektor el. indukce

$$\vec{D} = \varepsilon_0 \left(\vec{E} + \vec{E}_P\right)$$

$$\operatorname{div} \vec{D} = \rho \qquad \oint_S \vec{D}\, d\vec{S} = \int_V \rho\, dV$$

ρ - volný zdroj => D odpovídá poli bez polarizace (až na ε_0)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} = \varepsilon_0 (1+\chi) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

neizotropní prostředí $(\vec{p} = \alpha \cdot \vec{E})$, $\vec{P} \nparallel \vec{E}$

$$D_i = \varepsilon_{ik} E_k \qquad i,k = 1,2,3$$

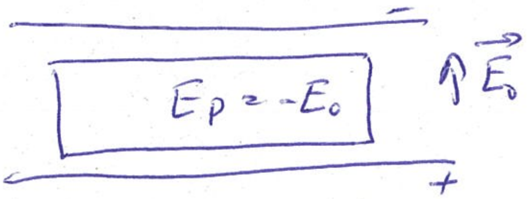
Dielektrikum v kondenzátoru

$$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{\varepsilon_0} = \frac{\vec{E}_0}{\varepsilon_r}$$

bude.

Vodiče v elstat poli – volné náboje

$\vec{E} = 0$ uvnitř

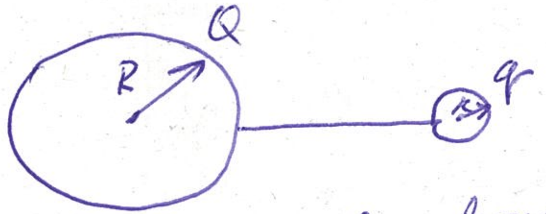


⇒ 1) náboj na povrchu (další výstupní práce)

2) $\langle \vec{E} \rangle = 0$ $\varphi = konst$

3) $\vec{E}$ kolmé k povrchu

4) E velké na hrotech



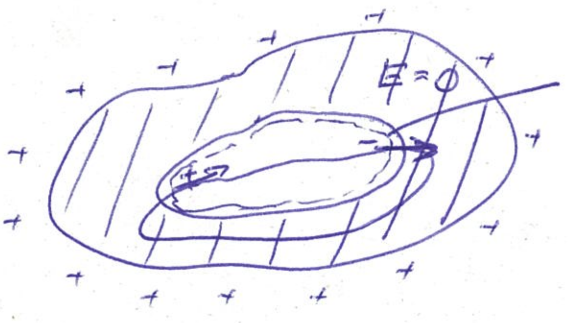
bez indukce

$\varphi =$ konst na povrchu

$$\varphi = \frac{Q}{R} = \frac{q}{r}; \quad E_R = k\frac{Q}{R^2} = \frac{\varphi}{R}; E_r = \frac{\varphi}{r}$$

$$\frac{E_R}{E_r} = \frac{r}{R}$$

Dutina uvnitř vodiče



$$\Phi = 0 \quad \oint_S \vec{E}\,d\vec{S} = 0$$

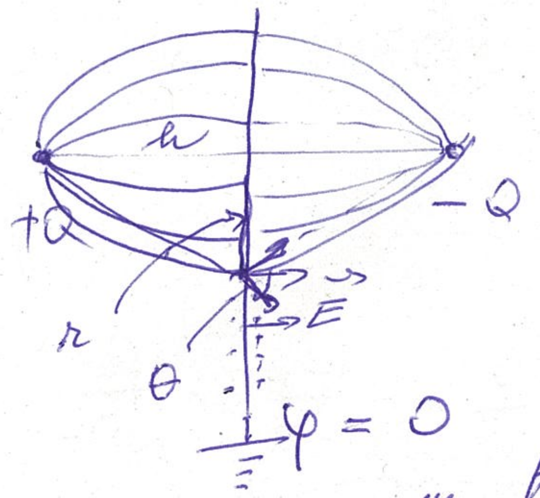
$$\oint_l \vec{E}\,d\vec{l} = 0$$

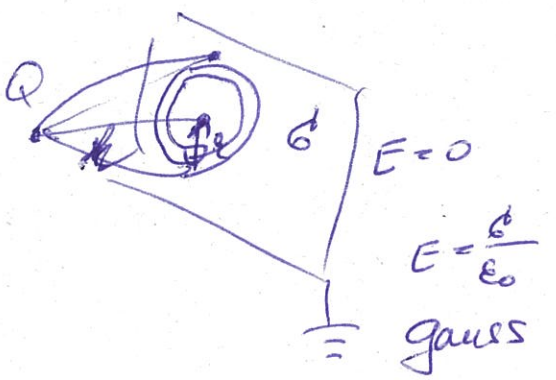
Pole nabitého vodiče

Q – rozloží se, aby $\varphi = \text{konst}$

Jak to počítat? něco se dá numericky

něco umíme – <u>elstat. zobrazení</u> – rozumné





ve $\forall$ bodech

$$E = -\frac{1}{4\pi\varepsilon_0}\frac{2Qh}{(r^2+h^2)^{3/2}} = -\frac{1}{4\pi\varepsilon_0}\frac{2Q}{(r^2+h^2)}\cos\theta$$

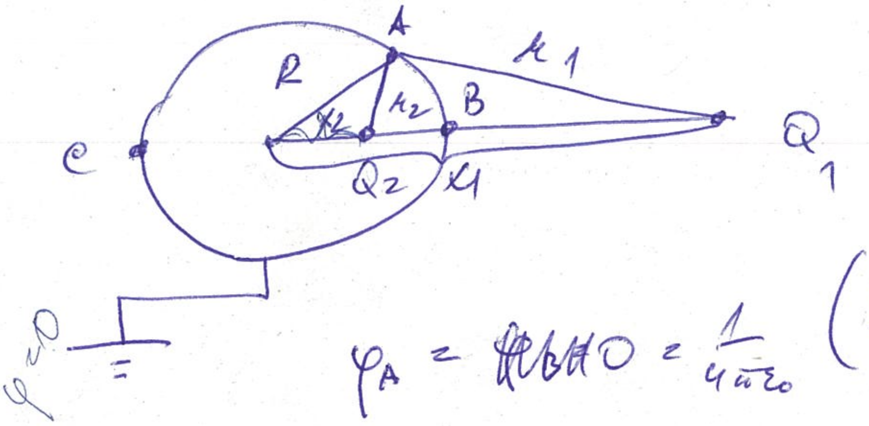
$$\sigma(r) = \varepsilon_0 E(r) = -\frac{2Qh}{4\pi(r^2+h^2)^{3/2}}$$

$$Q = \int_0^\infty \sigma(r)\cdot 2\pi r\, dr = -\frac{2Q\,2\pi h}{4\pi}\int_0^\infty \frac{r\, dr}{(r^2+h^2)^{3/2}} =$$

$$= -Qh\int_0^\infty \left[-\frac{1}{(h^2+r^2)^{1/2}}\right]' dr = Q$$

Bodový náboj u vodivé koule

kulové zobrazení



$$\varphi_A = 0 = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2}\right) = \frac{Q_1}{r_1} = -\frac{Q_2}{r_2}$$

$$\varphi(B) = \varphi(C) = 0 \Rightarrow \frac{Q_1}{x_1 - R} = \frac{-Q_2}{R - x_2} \; ; \; \frac{Q_1}{x_1 + R} = -\frac{Q_2}{x_2 + R}$$

$$\frac{Q_1}{-Q_2} = \underbrace{\frac{x_1 - R}{R - x_2} = \frac{x_1 + R}{x_2 + R}}$$

$$\Rightarrow x_1 x_2 = R^2$$

$$\frac{Q_2^2}{Q_1^2} = \frac{R^2 + x_2^2 - 2Rx_2}{x_1^2 + R^2 - 2Rx_1} = \frac{x_1 x_2 + x_2^2 - 2Rx_2}{x_1^2 + x_1 x_2 - 2Rx_1} = \frac{x_2}{x_1} = \frac{R^2}{x_1^2}$$

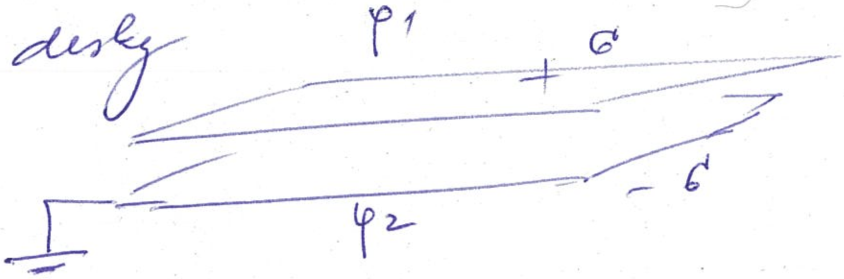
$$\boxed{Q_2 = -\frac{x_2}{R} Q_1}$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{(x_1 - x_2)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{Q_1^2 R}{x_1}\frac{1}{\left(x_1 - \frac{R^2}{x_1}\right)^2} =$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{Q_1^2 x_1 R}{(x_1^2 - R^2)^2}$$

Neuzemněná $\varphi = \text{konst} \neq 0$ – dáme doprostřed náboj $-Q_2$ – celk. náboj je 0.

2∞ desky



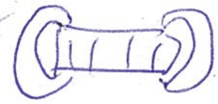
$$E = \frac{\sigma}{\varepsilon_0}$$

$$\varphi_1 - \varphi_2 = U \qquad U = E \cdot z = \frac{\sigma}{\varepsilon_0} z$$

blízko u sebe, konečné – plocha S

$$U = \frac{Q}{\varepsilon_0 S} z$$

Def: kapacita $C = \frac{Q}{U} = \varepsilon_0 \frac{S}{z}$ $z \ll S$

Obecně $S = a \times b$ + okrajové efekty 

$$S = \left(a + \frac{3}{8} z\right)\left(b + \frac{3}{8} z\right)$$

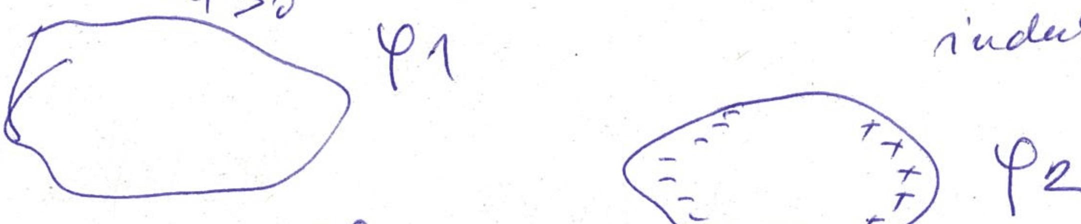
Energie $W = S z w = \frac{\varepsilon_0 E^2}{2} S z =$

$$= S z \frac{\varepsilon_0}{2} \frac{U^2}{z^2} = \frac{1}{2} C U^2 = \frac{1}{2} \frac{Q^2}{C}$$

Síla mezi deskami

$$F = -\frac{dW}{dz} = \frac{1}{2} \frac{Q^2}{C^2} \frac{dC}{dz} = -\frac{1}{2} U^2 \varepsilon_0 \frac{S}{z^2}$$

Obecně – $Q_1 \neq 0$, > 0



dáme-li $2Q_1$ → $2\varphi_2$

1) $Q_2 = 0$

$$\varphi_1 = B_{11} Q_1$$

$$\varphi_2 = B_{21} Q_1$$

B - potenciálové koeficienty

2) $Q_2 \neq 0$

$$\varphi_1 = B_{11} Q_1 + B_{12} Q_2$$

$$\varphi_2 = B_{21} Q_1 + B_{22} Q_2 = \mathbb{B}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

řešení je jednoznačné

$$\mathbb{C} = \mathbb{B}^{-1}$$

$$Q_1 = C_{11}\varphi_1 + C_{12}\varphi_2 + \dots$$

$$Q_2 = C_{21}\varphi_1 + C_{22}\varphi_2 + \dots$$

$$Q_3 = C_{31}\varphi_1 + C_{32}\varphi_2 + C_{33}\varphi_3 + \dots$$

C_{ij} - kapacity - vlastní a vzájemné

Energie soustavy:

$$W = \sum_i A_i$$

postupné nabíjení

$$Q_i'(t) = Q_i t$$

$$dQ_i' = Q_i\, dt$$

$$\varphi_i'(t) = \varphi_i t$$

$$t \in (0,1)$$

$$W = \sum_i \int_0^{Q_i} \varphi_i'(t)\, dQ_i'(t) = \sum_i \int_0^1 \varphi_i Q_i t\, dt =$$

$$= \frac{1}{2} \sum_i \varphi_i Q_i = \frac{1}{2} \sum_i \varphi_i \sum_k c_{ik}\varphi_k = \frac{1}{2} \sum_{ik} c_{ik}\varphi_k \varphi_i$$

$c_{ik} = c_{ki}$