Solitons and instantons in gauge theories*

Petr Jizba
FNSPE, Czech Technical University, Prague, Czech Republic
ITP, Freie Universität Berlin, Germany

March 15, 2007

*From: M. Blasone, PJ and G. Vitiello,
Quantum Field Theory and its Macroscopic Manifestations (WS, 2007)
Praha, 15.3.2007
Outlook:
Outlook:

•
Outlook:

- Introduction & motivations
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
Outlook:

• Introduction & motivations
• Korteweg–de Vries solitons
• Topological solitons in 1 + 1 relativistic field theories
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in $1+1$ relativistic field theories
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in 1 + 1 relativistic field theories
  - sine-Gordon solitons
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in $1 + 1$ relativistic field theories
  - sine-Gordon solitons
Outlook:

• Introduction & motivations

• Korteweg–de Vries solitons

• Topological solitons in $1 + 1$ relativistic field theories
  – sine-Gordon solitons
  – $\phi^4$ solitary waves
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in 1 + 1 relativistic field theories
  - sine-Gordon solitons
  - $\phi^4$ solitary waves
- Topological solitons in gauge theories with SSB
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in 1+1 relativistic field theories
  - sine-Gordon solitons
  - $\phi^4$ solitary waves
- Topological solitons in gauge theories with SSB
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in 1 + 1 relativistic field theories
  - sine-Gordon solitons
  - $\phi^4$ solitary waves
- Topological solitons in gauge theories with SSB
  - Nielsen–Olesen vortex solution
Outlook:

• Introduction & motivations

• Korteweg–de Vries solitons

• Topological solitons in $1 + 1$ relativistic field theories
  – sine-Gordon solitons
  – $\phi^4$ solitary waves

• Topological solitons in gauge theories with SSB
  – Nielsen–Olesen vortex solution
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in $1+1$ relativistic field theories
  - sine-Gordon solitons
  - $\phi^4$ solitary waves
- Topological solitons in gauge theories with SSB
  - Nielsen–Olesen vortex solution
  - ’t Hooft–Polyakov magnetic monopole solution
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in $1 + 1$ relativistic field theories
  - sine-Gordon solitons
  - $\phi^4$ solitary waves
- Topological solitons in gauge theories with SSB
  - Nielsen–Olesen vortex solution
  - ’t Hooft–Polyakov magnetic monopole solution
Outlook:

- Introduction & motivations
- Korteweg–de Vries solitons
- Topological solitons in $1+1$ relativistic field theories
  - sine-Gordon solitons
  - $\phi^4$ solitary waves
- Topological solitons in gauge theories with SSB
  - Nielsen–Olesen vortex solution
  - ’t Hooft–Polyakov magnetic monopole solution
- Homotopy theory and soliton classification
Introduction & motivation

In QM bound states describe extended structures (nucleons, atoms, molecules) that are inherently quantum (they are stable against radiation only in the QM setting).
Introduction & motivation

In QM bound states describe extended structures (nucleons, atoms, molecules) that are inherently quantum (they are stable against radiation \textit{only} in the QM setting).

Q.:
Introduction & motivation

In QM bound states describe extended structures (nucleons, atoms, molecules) that are inherently quantum (they are stable against radiation only in the QM setting).

Q.: Can we have stable bound states in (relativistic) QFT which are not inherently quantum?
Introduction & motivation

In QM bound states describe extended structures (nucleons, atoms, molecules) that are inherently quantum (they are stable against radiation *only* in the QM setting).

Q.: Can we have stable bound states in (relativistic) QFT which are not inherently quantum?

A:
In QM bound states describe extended structures (nucleons, atoms, molecules) that are inherently quantum (they are stable against radiation *only* in the QM setting).

**Q.**: Can we have stable bound states in (relativistic) QFT which are not inherently quantum?

**A**: Yes. In a *non-linear* field theory, with an “appropriate” amount of non-linearity, stable bound states can exist on a classical, as well as quantum level. Such bound states are called *solitons*. 
**Introduction & motivation**

In QM bound states describe extended structures (nucleons, atoms, molecules) that are inherently quantum (they are stable against radiation *only* in the QM setting).

**Q.:** Can we have stable bound states in (relativistic) QFT which are not inherently quantum?

**A:** Yes. In a non-linear field theory, with an “appropriate” amount of non-linearity, stable bound states can exist on a classical, as well as quantum level. Such bound states are called **solitons**.

⇒ solitons can be viewed as elementary objects in much the same way as elementary particles are (sol. are stabilized by topological charge).
Other properties
Other properties

- S. have localized energy density (they do not dissipate)
Other properties

- S. have localized energy density (they do not dissipate)
- S. have sharply defined mass and momentum
Other properties

- S. have localized energy density (they do not dissipate)
- S. have sharply defined mass and momentum
- S. dynamics obeys (Poincaré) Galileo symmetry
Other properties

- S. have localized energy density (they do not dissipate)
- S. have sharply defined mass and momentum
- S. dynamics obeys (Poincaré) Galileo symmetry
- After solitons scatter they retain their shape – they are stable against collisions
Other properties

- S. have localized energy density (they do not dissipate)
- S. have sharply defined mass and momentum
- S. dynamics obeys (Poincaré) Galileo symmetry
- After solitons scatter they retain their shape – they are stable against collisions

Technical note: Solitons vs. solitary waves
Other properties

- S. have localized energy density (they do not dissipate)
- S. have sharply defined mass and momentum
- S. dynamics obeys (Poincaré) Galileo symmetry
- After solitons scatter they retain their shape – they are stable against collisions

Technical note: Solitons vs. solitary waves
Other properties

- S. have localized energy density (they do not dissipate)
- S. have sharply defined mass and momentum
- S. dynamics obeys (Poincaré) Galileo symmetry
- After solitons scatter they retain their shape – they are stable against collisions

Technical note: Solitons vs. solitary waves
- Solitons: system has multi-soliton solutions that are stable against collisions
Other properties

- S. have localized energy density (they do not dissipate)
- S. have sharply defined mass and momentum
- S. dynamics obeys (Poincaré) Galileo symmetry
- After solitons scatter they retain their shape – they are stable against collisions

Technical note: Solitons vs. solitary waves

- Solitons: system has multi-soliton solutions that are stable against collisions
- Solitary waves: system does not have multi-soliton solutions or they do not obey cluster decomposition
Korteweg–de Vries solitons
Korteweg–de Vries solitons

Solitary wave was first observed in 1834 by a Scottish engineer John Scott Russell on a canal near Edinburgh.

"the Report of the British Association for the Advancement of Science"
Theoretical explanation by D.J. Korteweg and G. de Vries in 1895

Propagation of waves in the $x$-direction on the surface of shallow water has the form

$$\frac{\partial \eta}{\partial t} + u_0 \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0$$

- non-dispersive evolution
- finite amplitude effects
- dispersive term

$u_0, \alpha$ and $\beta$ are constants $\alpha, \beta > 0$

Passing to the frame $S_{u_0}$ moving with the speed $u_0$

$\Rightarrow$ Galileo transformations is; $x \mapsto x - u_0 t$ and $t \mapsto t$

$\Rightarrow$ “canonical” KdV equation for the surface elevation

$$\frac{\partial \eta}{\partial t} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0$$
By wanting the localized wave solution we must require that \( \eta \to 0, \eta' \to 0 \) and \( \eta'' \to 0 \) as \( |x| \to \infty \).

The solution can be find explicitly:

\[
\eta(t, x) = a \operatorname{sech}^2 \left( \sqrt{\frac{a\alpha}{12\beta}} (x - x_0 - ut) \right), \quad a = \frac{3u_0}{\alpha} \left( \frac{u}{u_0} - 1 \right)
\]

where the soliton velocity is

\[
u = u_0 \left( 1 + \frac{a\alpha}{3u_0} \right)
\]

\( \Rightarrow \) propagation velocity increases with the amplitude of the hump.

As the soliton width is \( (a\alpha/12\beta)^{-1/2} \) \( \Rightarrow \) the smaller the width the larger the height \( \Rightarrow \) taller KdV solitons travel faster.
Note: The localized behavior results from a balance between nonlinear steepening and dispersive spreading and so it cannot be attained in linear equations because an appropriate amount of nonlinearity is necessary.

!!! KdV equation allows for a superposition of solitons and hence it can describe an interaction between solitons.

Numerical investigations of the interaction of two solitons reveal that if a high, narrow soliton is formed behind a low, broad one, it will catch up with the low one, they undergo a non-linear interaction — the high soliton passes through the low one — and both emerge with their shape unchanged.

Note: KdV equation can be generalized. Ensuing equations are called higher-order KdV equations and they have form

\[
\frac{\partial \eta}{\partial t} + u_0 \frac{\partial \eta}{\partial x} + (p + 1) \eta^p \frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial \eta^3} = 0, \quad p \geq 2
\]
They admit single-soliton solutions

\[ \eta(x, t) = a \text{sech}^{2/p}(b(x - x_0 - ut)) \]

The same analysis as for ordinary KdV solitons gives

\[
a = \left[ \frac{p + 2}{2} u_0 \left( \frac{u}{u_0} - 1 \right) \right]^{1/p} \quad \text{and} \quad b = p \sqrt{\frac{a^p}{2(p + 2)}}
\]

⇒ slimmer higher-order KdV solitons travel faster than smaller and thicker ones

Numerical simulations indicate that for \( p > 2 \) the higher-order KdV soliton equations do not allow for a nonlinear superposition of solitons
Profile of the KdV soliton at $t = 0$. Soliton parameters are chosen to be:

$a = 1$, $x_0 = 0$ and the soliton width is 1.

Interaction of two KdV solitons.
Topological solitons in $1 + 1$ relativistic field theories

Important class of systems which often exhibit solitonic solutions are $1 + 1$ dimensional relativistic field theories.

For the single scalar field $\phi$ the Lagrangians reads

$$\mathcal{L} = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\phi')^2 - V(\phi)$$

and the corresponding E–L equation is

$$\ddot{\phi} - \phi'' = -\frac{dV(\phi)}{d\phi}$$

The energy is obtained from the $^{00}$ component of the E-M tensor ⇒

$$E[\phi] = \int dx\ T^{00}(\phi) = \int dx\left(\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\phi')^2 + V(\phi)\right)$$
To find the solitons we solve E-L equation in the soliton’s rest frame and then boosting the solution with a Lorentz transformation. The rest-frame soliton obeys the equation

\[ \phi'' = \frac{dV(\phi)}{d\phi} \Rightarrow \frac{1}{2}((\phi')^2)' = \frac{dV(\phi)}{d\phi} \phi' \Rightarrow \frac{1}{2}(\phi')^2 = V(\phi) + A \]

**Conditions:**
Localization of the solitonic solution \( \Rightarrow \phi'(|x| \to \infty) = 0 \)
Energy localization \( \Rightarrow E[\phi(|x| \to \infty)] = 0 \Rightarrow V(\phi(|x| \to \infty)) = 0. \)

**Note:** \( V(\phi) = 0 \) and \( \phi' = 0 \) at \( |x| \to \infty \) \( \Rightarrow A = 0 \)

Take now the square root and integrate \( \Rightarrow \)

\[ \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2V(\phi)}} = x - x_0 \]

which gives a solitonic solution \( \phi = \phi(\phi(x_0), x - x_0). \)
a) sine-Gordon solitons

The sine-Gordon system* that is described by the Lagrangian

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \frac{m^4}{\lambda} \left[ \cos \left( \frac{\sqrt{\lambda}}{m} \phi \right) - 1 \right] \]

\((\mu = 0, 1)\) The ensuing E–L equations read

\[ \Box \phi(t, x) + \frac{m^3}{\sqrt{\lambda}} \sin \left( \frac{\sqrt{\lambda}}{m} \phi(t, x) \right) = 0 \]

Taking \(x_\mu \mapsto mx_\mu\) and \(\phi \mapsto \sqrt{\lambda} \phi/m\) the equation boils down to

\[ \Box \phi(t, x) + \sin \phi(t, x) = 0 \]

By inserting the sine-Gordon potential to *** we obtain

\[ \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{2\sin(\phi/2)} = x - x_0 \]

By resolving this equation w.r.t. \( \phi \) we have

\[ \phi(x) = \pm 4 \arctan [\exp(x - x_0)] \]

After boosting to the frame that moves with the velocity \( u \)

\[ \phi(t, x) = \pm 4 \arctan \left[ \exp \left( \frac{x - x_0 - ut}{\sqrt{1 - u^2}} \right) \right] \]

Soliton solutions with “+” are called **kinks**, with “−” **antikinks**.
Resulting static kink/antikink energy density is

\[ \varepsilon(x) = \frac{1}{2}(\phi')^2 + V(\phi) = 2V(\phi) = 16\frac{e^{2(x+x_0)}}{(e^{2x} + e^{2x_0})^2} \]

or in the original (unscaled) variables

\[ \varepsilon(x) = \frac{16m^4}{\lambda} \frac{e^{2m(x+x_0)}}{(e^{2mx} + e^{2mx_0})^2} \]

Total energy of the sine-Gordon kink/antikink is

\[ E[\phi] = \int_{-\infty}^{\infty} dx \ T^{00}(t, x) = \frac{1}{\sqrt{1-u^2}} \frac{8m^3}{\lambda} \equiv \frac{M}{\sqrt{1-u^2}} \]

\( M \) is the so called sine-Gordon kink/antikink mass
Stability of s-G solitons can be conveniently characterized by a conserved quantity known as *topological charge*.

**Note:** The kink profile has the asymptotes; \( \phi(t, x \to \infty) = \pi/2 \) and \( \phi(t, x \to -\infty) = 0 \) at any fixed time \( t \).

**Note:** When \( \phi \) is solution of E-L equation, then also \( \phi \pm 2\pi N; \ N \in \mathbb{Z} \)!!! It takes an infinite energy to change kink to one of its constant vacuum configurations \( \phi = 2\pi N \).
As S. are finite-energy solutions they must at \( x \to \pm \infty \) tend towards one of vacua, labeled by \( N \Rightarrow \) define the conserved charge \( Q \) as

\[
Q = \int_{-\infty}^{\infty} dx \ J^0(t, x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ \phi'(t, x)
\]

\[
= \frac{1}{2\pi} [\phi(t, x \to \infty) - \phi(t, x \to -\infty)] = N_2 - N_1
\]

\( N_1 \) and \( N_2 \) are integers corresponding to asymptotic values of the field. 

\( J^\mu \) that satisfies the continuity equation can be defined as

\[
J^\mu(t, x) = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi(t, x)
\]

(\( \epsilon^{\mu\nu} \) is the 2-dimensional Levi–Civita tensor)

\( Q = \text{const.} \Rightarrow \) S. with one \( Q \) cannot evolve into S. with a different \( Q \).

**Note:** Topological charge \( Q \) cannot be derived from Noether's theorem since it is *not* related to any continuous symmetry of the Lagrangian.
b) $\phi^4$ solitary waves

For small field elevations, we can expand in s-G $\sin(\ldots)$ up to the third order. After rescaling $\lambda/3!$ to $\lambda$ the resulting E–L equation is

$$\Box \phi(t, x) + m^2 \phi(t, x) - \lambda \phi^3(t, x) = 0$$

Ensuing Lagrange density can be written as

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{\lambda}{4} \left(\frac{\phi^2 - m^2}{\lambda}\right)^2$$

Inserting the $V(\phi)$ into $\cdots \Rightarrow$ static kink equation

$$\pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{\frac{\lambda}{2} \left(\phi^2 - \frac{m^2}{\lambda}\right)}} = x - x_0$$
\[ \phi(x) = \pm \frac{m}{\sqrt{\lambda}} \tanh \left[ \frac{m}{\sqrt{2}}(x - x_0) \right] \]

The boosted solution then reads

\[ \phi(t, x) = \pm \frac{m}{\sqrt{\lambda}} \tanh \left[ \frac{m}{\sqrt{2}} \frac{(x - x_0 - ut)}{\sqrt{1 - u^2}} \right] \]

One calls solutions with "+" as kinks and "−" sign solutions as antikinks. Resulting kink/antikink energy density \( T^{00} \) is

\[
T^{00}(t, x) = \frac{1}{2} (\phi)^2 + \frac{1}{2} (\phi')^2 + V(\phi)
\]

\[
= \frac{m^4}{2\lambda(1 - u^2)} \text{sech}^4 \left[ \frac{m}{\sqrt{2}} \frac{(x - x_0 - ut)}{\sqrt{1 - u^2}} \right]
\]
Energy density of the *static* kink/antikink reads

\[ \varepsilon(x) = \frac{1}{2} (\phi'(x))^2 + V(\phi(x)) = 2V(\phi(x)) \]

\[ = \frac{m^4}{2\lambda} \text{sech}^4 \left[ \frac{m}{\sqrt{2}} (x - x_0) \right] \]

Energy density *a*) and kink solution *b*) for the $\phi^4$ system. We chose $m = \lambda = 1$ and $x_0 = 0$. 

20
The kink/antikink energy is

\[ E[\phi] = \int_{-\infty}^{\infty} dx \ T^{00}(t, x) = \frac{1}{\sqrt{1 - u^2}} \frac{m^3 2\sqrt{2}}{3\lambda} \equiv \frac{\bar{M}}{\sqrt{1 - u^2}} \]

\( \bar{M} \) is the \( \phi^4 \) kink/antikink mass

**Note:** \( \phi(t, x \to \infty) = m/\sqrt{\lambda} \) and \( \phi(t, x \to -\infty) = -m/\sqrt{\lambda} \) at any fixed \( t \)

⇒ define the topological charge \( Q \) as

\[ Q = \int_{-\infty}^{\infty} dx \ \mathcal{J}^0(t, x) \equiv \frac{\sqrt{\lambda}}{2m} \int_{-\infty}^{\infty} dx \ \phi'(t, x) \]

\[ = \frac{\sqrt{\lambda}}{2m} [\phi(t, x \to \infty) - \phi(t, x \to -\infty)] = \frac{\sqrt{\lambda}}{2m} [\phi_+ - \phi_-] \]

\( \phi_+ \) and \( \phi_- \) correspond to the asymptotics
The associated topological current $J^\mu$ can be defined as

$$ J^\mu(t, x) = \frac{\sqrt{\lambda}}{2m} \varepsilon^{\mu\nu} \partial_\nu \phi(t, x) $$

⇒ possible values of $Q$ are $\{-1, 0, 1\}$

**Note:** Values $|N| > 1$ are not allowed because the corresponding field configurations do not have boundary conditions compatible with the finite energy

⇒ no multi-solitonic solutions with $|N| > 1$ can exist

**Note:** Field configurations containing a finite mixture of kinks and antikinks alternating along the $x$-direction preserving $N = \{-1, 0, 1\}$ can be (numerically) constructed

**but there are no static solutions of this type**
Domain walls/kinks are associated with models in which there is more than one separated minimum.
Topological solitons in gauge theories with SSB

Among prominent systems exhibiting solitonic solutions are the spontaneously broken gauge theories

a) Abelian Higgs model — Nielsen–Olesen soliton (vortex)

Vortex* is the simplest soliton in Yang–Mills gauge theory with scalar fields $\Leftarrow$ Abelian gauge models

Action for the 3 + 1 dimensional Abelian Higgs model is

$$S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - \frac{g}{2} \left( |\phi|^2 - v^2 \right)^2 \right]$$

*Originally found by A.A. Abrikosov (1957) in type II superconductors (flux lines, flux tubes or fluxons). Independently by H.B. Nielsen and P. Olesen (1973) in the context of the 2 + 1 dim. $U(1)$ Higgs model
$A_\mu$ is a $U(1)$ gauge field and $\phi$ a complex Higgs field of charge $e$

The covariant derivative and the field strength are

$$D_\mu \phi = (\partial_\mu + ieA_\mu) \phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$ 

Note: $S$ is invariant under the $U(1)$ gauge transformations

$$\phi(x) \mapsto e^{i\alpha(x)} \phi(x), \quad A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$$

We look for a static string-like soliton configurations (vortices).

!!! Static configuration = configuration in the Weyl (i.e., temporal) gauge $A_0 = 0$ with $A_i$ and $\phi$ not dependent on $t$, i.e.,

$$A_i = A_i(x), \quad \phi = \phi(x)$$
Energy functional has then the form

$$E[F^{\mu\nu}, \phi] = \int d^3 x \left[ \frac{1}{4} F_{ij} F^{ij} + |D_i \phi|^2 + \frac{g}{2} (|\phi|^2 - v^2)^2 \right]$$

Assume that the prospective solitons are straight, static vortices along the $z$-direction ⇒ energy functional per unit length is

$$E[F^{\mu\nu}, \phi] = \int d^2 x \left[ \frac{1}{2} B^2 + |D_x \phi|^2 + |D_y \phi|^2 + \frac{g}{2} (|\phi|^2 - v^2)^2 \right]$$

$B = \partial_x A_y - \partial_y A_x$ is the $z$-component of the magnetic field

**Note:** Necessary conditions for finite-energy configurations are

$$|\phi|^2 |_{|x|\to\infty} = v^2, \quad D_{x,y} \phi |_{|x|\to\infty} = O(1/r), \quad B |_{|x|\to\infty} = O(1/r)$$
Note: (classical) vacuum manifold for the $\phi$ field is a circle $S_{vac}^1$ parametrized by the phase angle $\vartheta$ ($\phi = |\phi|e^{i\vartheta}$) $\Rightarrow$

$$\phi|_{|x|\to\infty} = \phi(n) = \nu e^{i\vartheta(n)} \quad \text{with} \quad n = (x/r, y/r)$$

$\Rightarrow$ with each finite-energy static conf. $\phi$ is associated a map $S_\infty^1 \hookrightarrow S_{vac}^1$. Because the field is single-valued $\vartheta(2\pi) = \vartheta(0) + 2\pi n$ for some $n \in \mathbb{Z}$.

Note: $n$ is known as the winding number, and it is the topological charge/number of the field configuration.

Note: This is equivalent to saying that the map $S_\infty^1 \hookrightarrow S_{vac}^1$ is topologically characterized by the fundamental homotopy group $\pi_1(S^1) = \mathbb{Z}$.

To construct a vortex solution we set an Ansatz for $\phi$ and $A_{x,y}$
Q.: How can I find an appropriate Ansatz?
A.: Note two points:

a) The most symmetric form of $\phi(n)$ with the winding number $n$ is

$$\phi(n) = v e^{in\vartheta}$$

b) The covariant derivative gives

$$D_{x,y}\phi|_{x \to \infty} = 0 \implies A_{x,y}|_{x \to \infty} = -\frac{i}{e} \partial_{x,y} \log(\phi(n))$$

$$\implies A_{x,y}|_{x \to \infty} \text{ has the azimuthal component}$$

$$A_{\vartheta}|_{x \to \infty} = -\frac{i}{e} \frac{1}{r} \frac{d}{d\vartheta} \log(\phi(n)) = \frac{n}{er}$$
Ansatz can be assumed in the form

\[ \phi(r, \vartheta, z) = v f(r) e^{in \vartheta}, \]
\[ A_\vartheta(r, \vartheta, z) = \frac{n}{er} h(r) \quad \Rightarrow \quad B(r, \vartheta, z) = \frac{n}{er} h'(r) \]

\( f \) and \( g \) should be smooth with \( f(0) = g(0) = 0, f(\infty) = g(\infty) = 1 \)

E–L equations for \( f \) and \( h \) read

\[ f'' + \frac{1}{r} f' - \frac{n^2}{r^2} (1 - h)^2 f + gv^2 (1 - f^2) f = 0, \]
\[ h'' - \frac{1}{r} h' + 2e^2 v^2 f^2 (1 - h) = 0 \]

**Note:** Suitability of the Ansatz is justified when E–L eq. yield a non-trivial solution. Although, *no* analytic solution is known so far, the consistency of the Ansatz can be checked numerically.
(Cosmic) strings/vortices are associated with models in which the set of minima are not simply-connected, that is, the vacuum manifold has ‘holes’ in it.

The minimum energy states on the left form a circle and the string corresponds to a non-trivial winding around this.
A non-trivial insight into the structure of the solutions follows from the asymptotic behavior

$$|\phi|_{x \to \infty} = v$$

The second equation can be solved exactly

$$1 - h(r) \left| x \right| \to \infty \propto \sqrt{\frac{2}{\pi \sqrt{2ev}}} r^{1/2} e^{-\sqrt{2evr}}$$

The first equation in can be linearized at large $r$. The asymptotic behavior for $f$ satisfying the condition $f|_{x \to \infty} = 1$ is

$$1 - f(r) \left| x \right| \to \infty \propto \begin{cases} r^{-1} e^{-2\sqrt{2evr}} & \text{if } 2e < \sqrt{g} \\ r^{-1/2} e^{-\sqrt{2gvr}} & \text{if } 2e > \sqrt{g} \end{cases}$$
**Note:** There are two length scales governing the large $r$ behavior of $f$ and $h$, namely the inverse masses of the scalar and vector excitations (Higgs and gauge particles), i.e.,

$$m_s^2 = 2gv^2, \quad m_v^2 = 2e^2v^2$$

**Note:** Asymptotic behavior depends on the ration (Ginzburg–Landau parameter)

$$\kappa = \frac{m_s^2}{m_v^2} = \frac{g}{e^2}$$

for $\kappa < 4$ the behavior of $f$ is controlled by $m_s$

for $\kappa > 4$, $m_v$ controls behaviors of $f$ and $h$
**Note:** In superconductors, the two length scales are known as the correlation length \( \xi = 1/m_s \) and the London penetration depth \( \lambda = 1/m_v \).

Values of \( \kappa \) distinguish type II and type I superconductors.

In type II superconductors \( \xi < \lambda \)

\[ \Rightarrow \text{superconductors can be penetrated by magnetic flux lines} \]

**Note:** Vortices with \(|n| > 1\) are unstable \(\Rightarrow\) repulsive force between parallel \(n = 1\) \(\Rightarrow\) lattice of vortices (Abrikosov vortex lattice)

In type I superconductors \( \xi > \lambda \)

\[ \Rightarrow \text{no penetration by magnetic flux lines (Meissner–Ochsenfeld effect)} \]
The Kibble mechanism for the formation of (cosmic) strings/vortices.

**Note:** There is no immediate meaning of $\kappa$ in in cosmic strings
By utilizing Stoke’s theorem we obtain for a magnetic flux

\[ \Phi = \int d^2xB = \lim_{r \to \infty} \oint_r dA_i = \lim_{r \to \infty} \int_0^{2\pi} rA_\theta d\theta = \frac{2\pi n}{e} \]

⇒ Magnetic flux is “quantized”

**Note:** Similar flux quantization exists in type II superconductors. True flux quantization in superconductors following from quantum theory reads

\[ \Phi = \frac{2\pi n\hbar}{2q} \]

2q corresponds to the charge of a Cooper pair

!!! It is only at the quantum level that the coupling constant \( e \) and the charge of the Cooper pair 2q are related through \( 2q = e\hbar \)
Snapshot of a cosmic string network (B. Allen & E. P. Shellard)
b) Georgi–Glashow $SO(3)$ model
—’t Hooft–Polyakov soliton (magnetic monopole)

In 1974 ’t Hooft and independently Polyakov discovered a topologically nontrivial finite energy solution in the $SO(3)$ Georgi–Glashow model — ’t Hooft–Polyakov monopole or non-abelian magnetic monopole

Georgi–Glashow $SO(3)$ model:

- $SO(3)$ gauge group
- triplet of real Higgs scalar fields (isovector) $\phi$

Lagrangian

$$
\mathcal{L} = -\frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2} (D_\mu \phi) \cdot (D^\mu \phi) - \frac{g}{4} (\phi \cdot \phi - v^2)^2
$$

$$
= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} (D_\mu \phi)^a (D^\mu \phi)^a - \frac{g}{4} (\phi^a \phi^a - v^2)^2
$$
Note: Convention

⊗ $\mu, \nu = 0, 1, 2, 3$
⊗ *(Killing) normalization for $T^a$ s.t. $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$
⊗ $3 \times 3$ matrices $T^a$ are the $SO(3)$ generators
⊗ $F_{\mu \nu} = F^a_{\mu \nu} T^a$, $A_\mu = A^a_\mu T^a$

Note: Some elements of non-Abelian gauge theories

Recall that the algebraic structure of $d$ dim. Lie algebra is given by the commutators

$$ [T_a, T_b] = i C^c_{ab} T^c $$

$C^c_{ab}$ are the *structure constants* and $a = 1, \ldots, d$
Adjoint representation is defined so that

\[(T_a)_b^c = iC_{ba}^c \quad \text{or equivalently} \quad (T_a)_{bc} = -iC_{abc}\]

⇒ \(T_a\) is \(d \times d\) matrix

Example: adjoint representation for \(SU(2) \cong SO(3)\) has \(d = 3\) ⇒ representation space is 3 dim. vector space with matrix elements \((T_a)_{bc}\):

\[(T_a)_{bc} = -i\varepsilon_{abc}\]

Note: Gauge fields correspond to elements of a Lie algebra in its adjoint representation

Fundamental representation corresponds to the defining matrix representation
Example:

FR of $SO(3)$ corresponds to $3 \times 3$ orthogonal matrices of determinant $1 \Rightarrow$ representation space is $3$ dim.

FR of $SU(2)$ has $2$ dim. representation space

The covariant derivative is

$$D_\mu \phi = \partial_\mu \phi - ie A_\mu^a T^a \phi$$

$e$ is a real constant (gauge coupling constant)

Under the gauge transformation

$$\phi(x) \mapsto g(x)\phi(x),$$

$$A_\mu(x) \mapsto g(x)A_\mu(x)g^{-1}(x) + ie^{-1}(\partial_\mu g(x))g^{-1}(x)$$
the covariant derivative transforms covariantly

\[ D_\mu \phi(x) \mapsto g(x)D_\mu \phi(x) \]

The field strength \( F_{\mu\nu} \) transforms under the gauge transformation covariantly

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] = -ie^{-1}[D_\mu, D_\nu] \]

\[ \mapsto gF_{\mu\nu}g^{-1} \]

\[ \Rightarrow \mathcal{L} \text{ is invariant under the } SO(3) \text{ gauge group} \]

To show that topological soliton exist in this model we consider static field configurations with finite energy
Static configuration are configurations in the Weyl gauge $A_0^a = 0$ and $A_i^a$ and $\phi^a$ are $t$ indep., i.e.

$$A_i^a = A_i^a(x), \quad \phi^a = \phi^a(x)$$

⇒ energy functional is

$$E[F^{\mu\nu}, \phi] = \int d^3x \left[ \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} (\mathcal{D}_i \phi)^a (\mathcal{D}_i \phi)^a + \frac{g}{4} \left( \phi^a \phi^a - v^2 \right)^2 \right]$$

Necessary condition for the finiteness of the energy is that

$$\lim_{|x| \to \infty} (\phi^a \phi^a) = v^2$$

The vacuum $\phi^a$ may depend on the direction in the physical space, i.e.

$$\lim_{|x| \to \infty} \phi^a(x) = \phi^a(n) \quad n = x/r$$
each configuration of fields with finite energy is associated with the mapping $S^2_\infty \mapsto S^2_{vac}$

Note:
From algebraic topology the second homotopy group $\pi_2(S^2) = \mathbb{Z}$

$\Rightarrow$ mapping $S^2_\infty \mapsto S^2_{vac}$ is classified by integer topological numbers $n = 0, \pm 1, \pm 2, \ldots$

Q.: How can I choose the soliton Ansatz
A.: Note that the most symmetric form of $\phi^a(n)$ corresponding to a mapping $S^2_\infty \mapsto S^2_{vac}$ is

$$\phi^a|_{|x| \to \infty} = n^a v$$

As for the asymptotic behavior $A^a_i(x)$, note the covariant derivative
\[ D_\mu \phi = \partial_\mu \phi - i e A_\mu^a T^a \phi \]

must vanish at \( r \to \infty \). At the same time

\[ \partial_i \phi^a |_{|x| \to \infty} = \frac{1}{r} (\delta^a i - n^a n^i) v \]

Requirement of zero covariant derivative \( \to \) asymptotic gauge field:

\[ A_i^a (x) |_{|x| \to \infty} = \frac{1}{er} \varepsilon^{aij} n^j \]

Here we have used that \((T_a)^{bc} = i \varepsilon^{bac}\) and

\[ \varepsilon^{ijk} \varepsilon^{ilm} = \delta^{jl} \delta^{km} - \delta^{jm} \delta^{kl} \]
To obtain a smooth soliton solution in $\mathbb{R}^3$ we take the Ansatz

$$\phi^a = n^a v f(r), \quad A_i^a = \frac{1}{e r} \varepsilon^{aij} n^j (1 - h(r))$$

$f(r)$ and $h(r)$ are smooth with $f(\infty) = h(0) = 1$, $f(0) = h(\infty) = 0$

$\Rightarrow$ Static energy functional reads

$$E[f, h] = 4\pi \int_0^\infty dr \left[ \frac{1}{e^2 (h')^2} + \frac{r^2 v^2}{2} (f')^2 + \frac{1}{2 e^2 r^2} (1 - h^2)^2 \right.$$

$$+ \ v^2 f^2 h^2 + \left. \frac{gr^2 v^4}{4} (f^2 - 1)^2 \right]$$

$\Rightarrow$ static equations of motion are

$$f'' + \frac{2}{r} f' = \frac{2 f}{r^2} h^2 + g v^2 f (f^2 - 1),$$

$$h'' = \frac{h}{r^2} (h^2 - 1) + e^2 v^2 f^2 h$$
Note: The exact analytic solutions are difficult to find
Note: Existence of non-trivial solutions is confirmed numerically

Solution for $\phi^a$ is also known as a “hedgehog” to stress that the Higgs isovector $\phi$ at a given point of space is directed along the radius vector.

Only the three-dimensional ‘hedgehog’ configuration on the left corresponds to a monopole.

Note: Presence of gauge fields is essential for hedgehog to be a soliton
Without gauge fields the Higgs field would be linearly divergent corresponding to the infinite rather than finite energy solution

**Q.:** Why is ’t Hooft–Polyakov soliton a magnetic monopole?

**A.:** Notice that we have theory with SSB \((SO(3) \rightarrow SO(2))\). One may then identify the unbroken \(SO(2) \cong U(1)\) symmetry with the electromagnetic field

To be able to identify the (physically observable) electromagnetic field that is embedded in the original \(SO(3)\) theory ’t Hooft proposed the gauge invariant definition of the electromagnetic field in the form

\[
\mathcal{F}_{\mu\nu} = \frac{\phi^a}{|v|} F^a_{\mu\nu} - \frac{1}{e|v|^3} \varepsilon_{abc} \phi^a (D_\mu \phi^b)(D_\nu \phi^c)
\]
In the broken phase one can chose $\phi^a = \delta^3|v|$ then $F_{\mu \nu}$ fulfills the usual Maxwell equations

$$F_{\mu \nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$$

⇒ The massless gauge boson corresponding to the unbroken $U(1)$ group can be identified with the photon i.e., $A^3_\mu = A_\mu$ provided we fix $\phi^a = \delta^3|v|$

⇒ $F_{\mu \nu}$ is a $4 \times 4$ combination of magnetic and electric fields with the components

$$F_{ij} = \varepsilon_{ijk}B^k, \quad F_{0i} = E_i = 0$$

⇒ static soliton configuration carries only magnetic field
Using that $F_{\mu\nu}^a$ has at large distances the asymptotic behavior:

$$F_{ij}^a \bigg|_{|x| \to \infty} = \frac{n^k n^a v}{er^2} \varepsilon_{ijk} = \frac{n^k}{er^2} \varepsilon_{ijk} \phi^a \bigg|_{|x| \to \infty}, \quad F_{0i}^a = 0$$

we have

$$B^k \bigg|_{|x| \to \infty} = \frac{1}{2} \varepsilon^{kij} F_{ij} \bigg|_{|x| \to \infty} = \frac{n^k}{er^2}$$

Applying Gauss's law, the total magnetic-monopole charge is

$$Q = \lim_{r \to \infty} \oint_{S_r^2} \mathbf{B} \cdot d\mathbf{s} = \frac{4\pi}{e}$$

$\Rightarrow$ 't Hooft–Polyakov m. satisfies the Dirac-like quantization condition

$$Qe = 2\pi n, \quad n \in \mathbb{N}_+$$

for $n = 2$
Note: As in the Nielsen–Olesen case, the origin of the “quantization” condition is purely topological and is not related to the dynamics nor quantum theory.

True Dirac’s quantization condition is $Qq = 2\pi\hbar n$ (in the CGS units $Qq = \hbar n/2$) Dirac’s quantization condition emerges only at the quantum level which enforces the Lagrange coupling constant $e$ to be related with the electric charge $q$ via $q = \hbar e$. 
