Integrály pohybu a Poissonovy závorky

$$F \text{ integrál pohybu} \iff 0 = \frac{dF}{dt}\bigg|_{H.R.} = \frac{\partial F}{\partial q_i}\underbrace{\dot q_i} + \frac{\partial F}{\partial p_i}\underbrace{\dot p_i} + \frac{\partial F}{\partial t} = \overbrace{\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}}^{\{F,H\}} + \frac{\partial F}{\partial t} = \boxed{\{F,H\} + \frac{\partial F}{\partial t} = 0}$$

Vlastnosti $\{F,G\} = \sum_{k=1}^{s}\left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right)$

1) $\{F,G\} = -\{G,F\} \Rightarrow \{F,F\} = 0$ antisymetrie
2) $\{aF_1 + bF_2, G\} = a\{F_1,G\} + b\{F_2,G\}$ $a,b\in\mathbb{R}$ bilinearita
3) $\{\{F_1,F_2\},F_3\} + \{\{F_2,F_3\},F_1\} + \{\{F_3,F_1\},F_2\} = 0$ Jacobiho identita

Lie-alg.

4) $\{F_1\cdot F_2, G\} = \{F_1,G\}\cdot F_2 + F_1\cdot\{F_2,G\}$
5) $\frac{\partial}{\partial t}\{F,G\} = \{\frac{\partial F}{\partial t}, G\} + \{F, \frac{\partial G}{\partial t}\}$

Integrály Pohybu

1) $\frac{\partial H}{\partial t} = 0 \Rightarrow \underbrace{\{H,H\}}_{0} + \overbrace{\frac{\partial H}{\partial t}}^{0} = 0$ H je I.P.

2) $\underbrace{\frac{\partial H}{\partial q_j}}_{\text{cyklická}} = 0$ $\dot p_j = -\frac{\partial H}{\partial q_j} = 0 \Rightarrow p_j = \text{konst}$ I.P.

3) $\frac{\partial H}{\partial p_j} = 0$ $\dot q_j = \frac{\partial H}{\partial p_j} = 0 \Rightarrow \underbrace{q_j = \text{konst}}_{\text{I.P.}}$

Ham. rc.
$$\dot p_j = \{p_j, H\}$$
$$\dot q_j = \{q_j, H\}$$

11) Spočtěte $\{e^{\alpha q}, e^{\beta p}\} = \frac{\partial e^{\alpha q}}{\partial q}\frac{\partial e^{\beta p}}{\partial p} - \cancel{\underbrace{\frac{\partial e^{\alpha q}}{\partial p}}_{0}\underbrace{\frac{\partial e^{\beta p}}{\partial q}}_{0}} = \alpha e^{\alpha q}\beta e^{\beta p}$

$\alpha,\beta\in\mathbb{R}$ $s=1$

12) Spočtěte

Fundamentální P.Z.

$\{q_i,q_j\} = 0$

$\{p_i,p_j\} = 0$

$$\{q_i,p_j\} = \underbrace{\frac{\partial q_i}{\partial q_k}}\frac{\partial p_j}{\partial p_k} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{0}\underbrace{\frac{\partial p_j}{\partial q_k}}_{0} = \delta_{ik}\delta_{jk} - 0 = \delta_{ij}$$

13) Spočtěte Poissonovy závorky pro složky hybností a momentů hybností částice. Budou stejné vztahy platit i pro celkovou hybnost a celkový moment hybnosti soustavy částic?

$$L_i = \varepsilon_{ijk} q_j p_k \qquad \{L_i, p_l\} = \{\varepsilon_{ijk} q_j p_k, p_l\} = \varepsilon_{ijk}\{q_j p_k, p_l\} = \varepsilon_{ijk}(\underbrace{\{q_j,p_l\}}_{=\delta_{jl}}p_k + q_j\underbrace{\{p_k,p_l\}}_{=0}) = \varepsilon_{ilk}p_k$$

$$\boxed{\{L_1,p_2\} = p_3}$$

$$\{L_i,L_j\} = \{\varepsilon_{ikl}q_k p_l, \varepsilon_{jmn}q_m p_n\} = \varepsilon_{ikl}\varepsilon_{jmn}\{q_k p_l, q_m p_n\} = \varepsilon_{ikl}\varepsilon_{jmn}(\{q_k, q_m p_n\}p_l + q_k\{p_l, q_m p_n\}) =$$

$$= \varepsilon_{ikl}\varepsilon_{jmn}(\underbrace{\{q_k,q_m\}}_{=0}p_n p_l + q_m\underbrace{\{q_k,p_n\}}_{=\delta_{kn}}p_l + q_k\underbrace{\{p_l,q_m\}}_{=-\delta_{lm}}p_n + q_k q_m\underbrace{\{p_l,p_n\}}_{=0})$$

$$\{q_k, q_m p_n\} = -\{q_m p_n, q_k\} = -(\{q_m,q_k\}p_n + q_m\{p_n,q_k\}) = \{q_k,q_m\}p_n + q_m\{q_k,p_n\}$$

$$= \varepsilon_{ikl}\varepsilon_{jmn}(q_m p_l\delta_{kn} - \delta_{lm}q_k p_n) = \varepsilon_{ikl}\varepsilon_{jmk}q_m p_l - \varepsilon_{ikl}\varepsilon_{jlm}q_k p_m =$$

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$$= -(\delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj})q_m p_l + (\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj})q_k p_m = \delta_{ij}(-\cancel{\delta_{lm}q_m p_l} + \cancel{\delta_{km}q_k p_m}) + \underbrace{\delta_{im}\delta_{lj}q_m p_l - \delta_{im}\delta_{kj}q_k p_m}_{q_i p_j - q_j p_i} =$$

$$= (\delta_{ik}\delta_{mj} - \delta_{im}\delta_{kj})q_k p_m = \varepsilon_{ijl}\varepsilon_{kml}q_k p_m = \varepsilon_{ijl}\underbrace{\varepsilon_{lkm}q_k p_m}_{L_l} = \varepsilon_{ijl}L_l$$

Pro soustavu částic

$$P_i = \sum_{\alpha=1}^{N} p_{\alpha i} \quad i\in\hat 3 \qquad L_i = \varepsilon_{ijk}\sum_{\alpha=1}^{N} q_{\alpha j}p_{\alpha k}$$

$$\{L_i, P_l\} = \{\varepsilon_{ijk}q_{\alpha j}p_{\alpha k}, \sum' p_{\beta l}\} =$$

$$= \sum_{\alpha,\beta,j,k}' \varepsilon_{ijk}(\{q_{\alpha j}, p_{\beta l}\}p_{\alpha k} + q_{\alpha j}\underbrace{\{p_{\alpha k}, p_{\beta l}\}}_{0}) = \sum' \varepsilon_{ijk}\delta_{\alpha\beta}\delta_{jl}p_{\alpha k}$$

$$\sum_{\gamma,r}' \underbrace{\frac{\partial p_{\alpha k}}{\partial q_{\gamma r}}}_{0}\frac{\partial p_{\beta l}}{\partial p_{\gamma r}} - \underbrace{\frac{\partial p_{\alpha k}}{\partial p_{\gamma r}}}_{\delta_{\alpha\gamma}\delta_{kr}}\underbrace{\frac{\partial p_{\beta l}}{\partial q_{\gamma r}}}_{0} =$$

$$\delta_{\alpha\gamma}\delta_{jr}\delta_{\beta\gamma}\delta_{lr} = \delta_{\alpha\beta}\delta_{jl}$$

$$= \sum_{\beta}' \varepsilon_{ilk}p_{\beta k} = \varepsilon_{ilk}P_k$$

14) Dokažte, že pokud jsou L_1 a L_2 integrály pohybu, pak i L_3 je integrál pohybu. $\boxed{L_i = \varepsilon_{ijk}\, x_j\, p_k}$

13) $\Rightarrow$ $\boxed{\{L_1,L_2\} = \varepsilon_{12k} L_k = \varepsilon_{123} L_3 = L_3}$ $\qquad$ $\boxed{\frac{\partial L_i}{\partial t}=0 \ \ \forall i\in\hat{3} \qquad L_i \text{ je I.P.} \Leftrightarrow \{L_i,H\}=0}$

$$0 \overset{?}{=} \{L_3,H\} = \{\{L_1,L_2\},H\} \overset{3)\ \text{Jacobi}}{=} -\left(\{\underbrace{\{L_2,H\}}_{=0},L_1\} + \{\{H,L_1\},L_2\}\right) = +\{\underbrace{\{L_1,H\}}_{=0},L_2\} = 0 \ ✓$$

15) Dokažte Poissonovu větu. $\quad$ Pokud $\{F_i,H\}+\frac{\partial F_i}{\partial t}=0 \quad i=1,2 \quad \Rightarrow \{F_1,F_2\}$ je I.P.

F_1, F_2 jsou I.P. $\qquad \frac{\partial F_i}{\partial t} = -\{F_i,H\}$

$$\text{Dk.}\quad 0 \overset{?}{=} \overbrace{\{\{F_1,F_2\},H\} + \frac{\partial\{F_1,F_2\}}{\partial t}}^{\frac{d}{dt}\{F_1,F_2\}\big|_{H.R.}} = \{\{F_1,F_2\},H\} + \left\{\frac{\partial F_1}{\partial t},F_2\right\} + \left\{F_1,\frac{\partial F_2}{\partial t}\right\} = \{\{F_1,F_2\},H\} - \{\{F_1,H\},F_2\} - \{F_1,\{F_2,H\}\}$$

(-1) $\qquad$ konst.

$$= \{\{F_1,F_2\},H\} + \{\{H,F_1\},F_2\} + \{\{F_2,H\},F_1\} = 0 \qquad \text{3) Jacobiho I.d.}$$

16) Pomocí Poissonovy věty odvoďte další první integrál Hamiltonových pohybových rovnic, znáte-li první integrály:

$$u = \sum_{i=1}^{3}\frac{p_i^2}{2m} - \Omega\cdot(x_1p_2 - x_2p_1) + U(r) \qquad v = \sum_{i=1}^{3}\frac{p_i^2}{2m} + U(r) \qquad r = \sqrt{\sum x_j^2}$$

$$\{u,v\} = \{v - \Omega\cdot(x_1p_2-x_2p_1), v\} = \underbrace{\{v,v\}}_{=0} - \Omega\{x_1p_2-x_2p_1, v\} = -\Omega\left\{x_1p_2-x_2p_1, \sum\frac{p_i^2}{2m}+U(r)\right\} =$$

$$= -\Omega\left(\left\{x_1p_2-x_2p_1,\sum\frac{p_i^2}{2m}\right\} + \{x_1p_2-x_2p_1,U(r)\}\right) = -\Omega\left(\cancel{p_2\frac{p_1}{m}} - \cancel{p_1\frac{p_2}{m}} - \left(-x_2\frac{\partial U}{\partial x_1}+x_1\frac{\partial U}{\partial x_2}\right)\right) =$$

$$= \Omega\left(\cancel{-x_2U'\frac{x_1}{r}} + \cancel{x_1U'\frac{x_2}{r}}\right) = 0 \ ✓ \qquad \frac{\partial}{\partial q_k}\frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k}\frac{\partial}{\partial q_k}$$

$$\frac{\partial U}{\partial x_1} = \frac{dU}{dr}\cdot\frac{\partial r}{\partial x_1} = U'\frac{x_1}{r}$$

17) Ukažte, že $\{L_3,F\}=0$ kde $F=F(\vec{q}\cdot\vec{p})$ je libovolná skalární funkce. $\qquad F:\mathbb{R}\to\mathbb{R}$

$$\{L_3,F\} = \{q_1p_2 - q_2p_1, F\} = \{q_1p_2,F\} - \{q_2p_1,F\} = \{q_1,F\}p_2 + q_1\{p_2,F\} - \{q_2,F\}p_1 - q_2\{p_1,F\} =$$

$$= 1\cdot\frac{\partial F}{\partial p_1}p_2 + q_1\left(-1\frac{\partial F}{\partial q_2}\right) - 1\cdot\frac{\partial F}{\partial p_2}p_1 - q_2\left(-1\frac{\partial F}{\partial q_1}\right) = \cancel{F'q_1p_2} - \cancel{q_1F'p_2} - \cancel{F'q_2p_1} + \cancel{q_2F'p_1} = 0$$

$$\frac{\partial F}{\partial p_1} = F'\cdot\frac{\partial}{\partial p_1}(\vec{q}\cdot\vec{p}) = F'q_1$$

q_ip_i