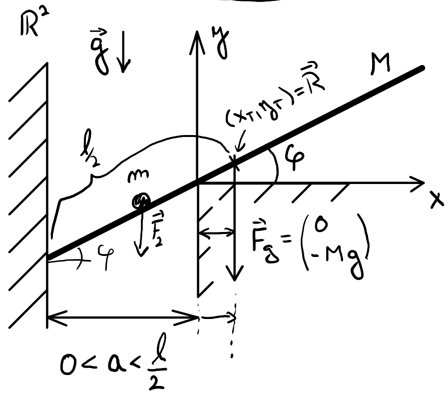
Př. Pomocí prinvipu virtuální práce najděte rovnovážnou polohu pro homogenní tuhou tyč hmotnosti M zatíženou ~~závažím o hmotnosti m~~ v homogenním tíhovém poli a podrobenou ideálním (hladkým vazbám)



P.V.P $\quad 0 = \delta A = \sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \delta \vec{x}_\alpha$

$\rightarrow 0 = \delta A = \vec{F}_g \cdot \delta \vec{R} = 0 \cdot \delta x_T + (-Mg)\delta y_T = -Mg\underbrace{\left(\frac{l}{2}\cos\varphi - \frac{a}{\cos^2\varphi}\right)}_{=0 \ (*)}\delta\varphi = 0$ $\quad$ lib. $\forall \delta\varphi \neq 0$

$\boxed{s = 2 + 1 - 1 - 1 = 1}$ obecná souřadnice φ

$x_T = \frac{l}{2}\cos\varphi - a$

$y_T = x_T \operatorname{tg}\varphi = \left(\frac{l}{2}\cos\varphi - a\right)\frac{\sin\varphi}{\cos\varphi} = \frac{l}{2}\sin\varphi - a\operatorname{tg}\varphi$

$$\delta\vec{R} \begin{cases} \delta x_T = \frac{\partial x_T}{\partial\varphi}\delta\varphi = -\frac{l}{2}\sin\varphi\,\delta\varphi \\ \delta y_T = \frac{l}{2}\cos\varphi\,\delta\varphi - a\frac{1}{\cos^2\varphi}\delta\varphi \end{cases}$$

$(*) \Rightarrow \boxed{\cos^3\varphi = \frac{2a}{l}}$

Př. Jak se může v rovině xy pohybovat částice na kterou nepůsobí žádná síla, je-li podrobená vazbě

$\boxed{\alpha t\dot{x} - \dot{y} = 0}$ (3) $\quad$ $\cancel{m\ddot{x} = 0}$ $\quad$ $\cancel{m\ddot{y} = 0}$

$\alpha t\dot{x}\,dt - \dot{y}\,dt = 0 \quad /dt$ $\quad$ Neholonomní

$\boxed{\alpha t\,dx - dy = 0}$ $\quad \frac{\partial(\alpha t)}{\partial t} \neq 0$, $\quad \frac{\partial(0)}{\partial x} = 0$

$dt = 0 \Rightarrow dx = \delta x$, $dy = \delta y$

$\boxed{\alpha t\,\delta x - \delta y = 0}$ $\quad / \mu = \mu(x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, t)$ Lagrangeův Multiplikátor

$\mu\alpha t\,\delta x - \mu\,\delta y = 0$

d'Alembert $\quad \boxed{s = 2}$

$0 = \delta A_{ef} = \sum_{\alpha=1}^{N}\underbrace{\left(\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha\right)}_{\vec{F}_\alpha^{ef}} \cdot \delta\vec{x}_\alpha$

$\rightarrow 0 = \delta A_{ef} = (0 - m\ddot{x})\delta x + (0 - m\ddot{y})\delta y + 0 = -m\ddot{x}\delta x - m\ddot{y}\delta y + (\mu\alpha t\delta x - \mu\delta y)$

δx, δy nejsou nezávislé

$\Downarrow$

$0 = \underbrace{(\mu\alpha t - m\ddot{x})}_{=0}\delta x + \cancel{(-\mu - m\ddot{y})\delta y}$

lede nezávislé $\forall \delta x$ $\quad$ $\mu = -m\ddot{y}$ Nastavíme

$\boxed{\begin{aligned} m\ddot{x} &= \mu\alpha t \quad (1) \\ m\ddot{y} &= -\mu \quad (2) \end{aligned}}$

Sečtím (1) (2) (3) $\quad$ (1) $\dot{x}$ + (2) $\dot{y}$

$\underbrace{m\ddot{x}\dot{x} + m\ddot{y}\dot{y}} = \mu\alpha t\dot{x} - \mu\dot{y} = \mu(\alpha t\dot{x} - \dot{y}) \overset{(3)}{=} 0$

$\frac{d}{dt}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right) = \frac{dT}{dt} \Rightarrow T = \text{konst}$

$T = \frac{1}{2}mv^2 \Rightarrow v = v_0 = \text{konst.}$

$\dot{x}^2 + \dot{y}^2 = v_0^2$ (3)

$\dot{x}^2 + (\alpha t\dot{x})^2 = v_0^2 \rightarrow \dot{x} = \frac{v_0}{\sqrt{1 + \alpha^2 t^2}}$

(3) $\Rightarrow \dot{y} = \alpha t\dot{x} = \frac{\alpha t v_0}{\sqrt{1 + \alpha^2 t^2}}$

$\int dt \Rightarrow$ D.C.V

$$\boxed{\begin{aligned} x &= x_0 + \frac{v_0}{\alpha}\ln\left(\alpha t + \sqrt{1 + \alpha^2 t^2}\right) \\ y &= \frac{v_0}{\alpha}\sqrt{1 + \alpha^2 t^2} + y_0 \end{aligned}}$$

Malé kmity konzervativních soustav kolem Stabilní Rovnovážné Polohy (SRP) $v\ \vec{q} = 0$

(skleronomní holonomní vazby a konzervativní síly)

BÚNO $\vec{q} = \vec{q}^{(SRP)}$ $\Downarrow$ $\vec{q} = \vec{q} - \vec{q}^{(SRP)}$

$q_i = 0 \ \forall i \in \hat{s}$

$T = \frac{1}{2}T_{ij}(\vec{q})\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\vec{q}}^T\,\mathbb{T}(\vec{q})\,\dot{\vec{q}}$ $\quad$ P.D. kvadratická forma v $\dot{\vec{q}}$

$\mathbb{T} = \mathbb{T}(0) = (T_{ij}(0)) = \left(\frac{\partial^2 T}{\partial\dot{q}_i\partial\dot{q}_j}\right)_{\vec{q}=0}$ (SRP)

Taylor kolem SRP $\vec{q} = 0$

$U = U(\vec{q}) = \underset{\substack{\| \\ \text{konst} \\ \| \\ 0}}{U(0)} + \sum_{i=1}^{s}\underbrace{\left(\frac{\partial U}{\partial q_i}\right)_{\vec{q}=0}}_{\substack{\| \\ 0}} q_i + \frac{1}{2}\sum_{i,j=1}^{s}\underbrace{\left(\frac{\partial^2 U}{\partial q_i\partial q_j}\right)_{\vec{q}=0}q_i q_j}_{\vec{q}^T\mathbb{U}\vec{q}\ \dots\ \text{P.D. kvadratická forma}} + \dots$

$\mathbb{U} = \underbrace{\left(\frac{\partial^2 U}{\partial q_i\partial q_j}\right)_{\vec{q}=0}}_{U_{ij}}$

P.D. Symetrické $\quad \mathbb{U}^T = \mathbb{U}$, $\ \mathbb{T}^T = \mathbb{T}$

$\nabla U = -\vec{F} = 0 \Leftrightarrow$ R.P.

SRP $\Rightarrow$ lokální min $U(\vec{q})$ ostré

$-\vec{q} \longleftarrow \cdot \longrightarrow \vec{q}$

APROXIMACE $\quad L = T - U \doteq \frac{1}{2}\sum_{ij}T_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij}U_{ij}q_i q_j$

$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{1}{2}\left[T_{ij}\delta_{ik}\dot{q}_j + T_{ij}\dot{q}_i\delta_{jk}\right]\right) + \frac{1}{2}\left(U_{ij}\delta_{ik}q_j + U_{ij}q_i\delta_{jk}\right) = \boxed{T_{ki}\ddot{q}_i + U_{ki}q_i = 0}$ $\quad k = 1, 2, \dots, s$

$T_{kj}\dot{q}_j + T_{ik}\dot{q}_i = T_{kj}\dot{q}_j + T_{ki}\dot{q}_i = 2T_{ki}\dot{q}_i$

$\mathbb{T}\ddot{\vec{q}} + \mathbb{U}\vec{q} = 0$

Řešení ve tvaru Módu $\quad \vec q(t) = \vec A\cos(\omega t+\varphi)$

$\vec A \in \mathbb{R}^{s}$ $\quad \ddot{\vec q}(t) = -\omega^2 \vec A\cos(\omega t+\varphi)$

$(\mathbb{T}(-\omega^2\vec A) + \mathbb{U}\vec A)\cos(\cancel{\omega t+\varphi}) = 0 \quad /\forall t$

$(\mathbb{U}-\underbrace{\omega^2\mathbb{T}}_{})\vec A = 0$ Homogenní lin. rce.

singulární $\Rightarrow \det(\mathbb{U}-\omega^2\mathbb{T}) = 0 \Rightarrow \omega_1,\dots,\omega_s$

Frekvence Módů

$\vec A^{(\alpha)} \in \mathrm{Ker}(\mathbb{U}-\omega_\alpha^2\mathbb{T})$ Najít jádro Matic

Obecné řešení

$$\vec q(t) = \sum_{\alpha=1}^{s} \underbrace{a_\alpha\cos(\omega_\alpha t+\varphi_\alpha)}_{\eta_\alpha \dots \text{ Normální souřadnice}} \frac{\vec A^{(\alpha)}}{\|\vec A^{(\alpha)}\|_T}$$

$\|\vec A\|_T = \sqrt{\vec A^T \mathbb{T}\vec A}$

$a_\alpha, \varphi_\alpha \dots$ z Poč. Podmínek

V Normálních souřadnicích

$L = \frac12\sum_{\alpha=1}^{s} \dot\eta_\alpha^2 - \frac12\sum_{\alpha=1}^{s} \omega_\alpha^2\eta_\alpha^2$

$\hookrightarrow \ddot\eta_\alpha + \omega_\alpha^2\eta_\alpha = 0 \quad \forall\alpha=1,\dots,s$

s× Lin. Harm. oscilátor.

65. Určete kmity soustavy dvou lineárních harmonických oscilátorů spojených slabou bilineární vazbou

① $L = \underbrace{\frac12(\dot x_1^2+\dot x_2^2)}_{T} \underbrace{- \frac12\omega_0^2(x_1^2+x_2^2) + \alpha x_1x_2}_{-\,U(\vec x)}$ $\quad 0<\alpha\ll\omega_0^2$ $\quad$ SRP $x_1=x_2=0$ $\quad U=\frac12\omega_0^2(x_1^2+x_2^2)-\alpha x_1x_2$

$\frac{\partial U}{\partial x_1} = \cancel{\frac{1}{2}}\omega_0^2\cancel{2}x_1 - \alpha x_2 = \omega_0^2 x_1 - \alpha x_2$ $\quad \frac{\partial^2 U}{\partial x_1\partial x_1}=\omega_0^2$

② $\mathbb{T} = \left(\frac{\partial^2 T}{\partial\dot x_i\partial\dot x_j}\right)_{SRP} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\quad \mathbb{U} = \left(\frac{\partial^2 U}{\partial x_i\partial x_j}\right)_{SRP} = \begin{pmatrix}\omega_0^2 & -\alpha\\ -\alpha & \omega_0^2\end{pmatrix}$? P.D. ? $\quad \frac{\partial^2 U}{\partial x_1\partial x_2} = -\alpha$

SYMetrická ✓

③ $0 = \det(\underbrace{\mathbb{U}-\omega^2\mathbb{T}}_{\mathbb{M}}) = \begin{vmatrix}\omega_0^2-\omega^2 & -\alpha\\ -\alpha & \omega_0^2-\omega^2\end{vmatrix} = (\omega_0^2-\omega^2)^2-\alpha^2 = \dots = 0$

$(\omega_0^2-\omega^2-\alpha)\cdot(\omega_0^2-\omega^2+\alpha)=0 \Rightarrow \omega_1=\sqrt{\omega_0^2-\alpha}$, $\omega_2=\sqrt{\omega_0^2+\alpha}$ Frekvence Módů

④ $\omega_1^2=\omega_0^2-\alpha$ $\quad \mathbb{M}_1 = \mathbb{U}-\omega_1^2\mathbb{T} = \begin{pmatrix}\omega_0^2-(\omega_0^2-\alpha) & -\alpha\\ -\alpha & +\alpha\end{pmatrix} = \begin{pmatrix}\alpha&-\alpha\\-\alpha&\alpha\end{pmatrix}\sim\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\sim\begin{pmatrix}1&-1\\0&0\end{pmatrix} \Rightarrow \mathrm{Ker}\,\mathbb{M}_1 = \left\langle\begin{pmatrix}1\\1\end{pmatrix}\right\rangle_{\mathbb{R}}$

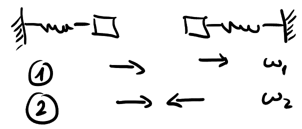
$\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\| = \sqrt{(1,1)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}} = \sqrt2$

$\omega_2^2=\omega_0^2+\alpha$ $\quad \mathbb{M}_2 = \mathbb{U}-\omega_2^2\mathbb{T} = \begin{pmatrix}\omega_0^2-(\omega_0^2+\alpha) & -\alpha\\ -\alpha & -\alpha\end{pmatrix} = \begin{pmatrix}-\alpha&-\alpha\\-\alpha&-\alpha\end{pmatrix}\sim\begin{pmatrix}1&1\\1&1\end{pmatrix}\sim\begin{pmatrix}1&1\\0&0\end{pmatrix} \Rightarrow \mathrm{Ker}\,\mathbb{M}_2 = \left\langle\begin{pmatrix}1\\-1\end{pmatrix}\right\rangle_{\mathbb{R}}$

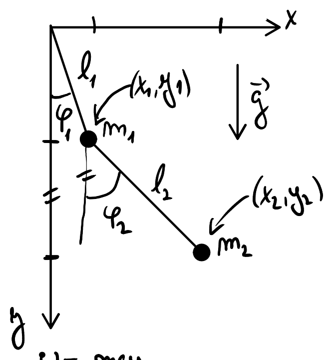
$\left\|\begin{pmatrix}1\\-1\end{pmatrix}\right\| = \sqrt{(1,-1)\mathbb{1}\begin{pmatrix}1\\-1\end{pmatrix}} = \sqrt2$

Obecné řešení

$$\vec x(t) = \begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \underbrace{a_1}_{\wedge\text{ amplituda Módu}}\cos(\overbrace{\sqrt{\omega_0^2-\alpha}}^{\omega_1}\,t+\varphi_1)\begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{pmatrix} + \underbrace{a_2}_{}\cos(\overbrace{\sqrt{\omega_0^2+\alpha}}^{\omega_2}\,t+\varphi_2)\begin{pmatrix}\frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2}\end{pmatrix}$$



66. Určete normální kmity dvojitého rovinného matematického kyvadla



$U=-mgy$

① $L = \underbrace{\frac12 m_1(\dot x_1^2+\dot y_1^2)+\frac12 m_2(\dot x_2^2+\dot y_2^2)}_{} + m_1gy_1+m_2gy_2 = \underbrace{T}_{}-U$

Vazby $x_1^2+y_1^2=l_1^2$ $\quad (x_2-x_1)^2+(y_2-y_1)^2=l_2^2$

$s = 2\cdot2-2 = 2$

OBECNÉ S. φ_1, φ_2 $\quad$ (SRP $\varphi_1=\varphi_2=0$)

$x_1 = l_1\sin\varphi_1$ $\quad \dot x_1 = l_1\dot\varphi_1\cos\varphi_1$

$y_1 = l_1\cos\varphi_1$ $\quad \dot y_1 = -l_1\dot\varphi_1\sin\varphi_1$

$x_2 = l_1\sin\varphi_1 + l_2\sin\varphi_2$ $\quad \dot x_2 = l_1\dot\varphi_1\cos\varphi_1 + l_2\dot\varphi_2\cos\varphi_2$

$y_2 = l_1\cos\varphi_1 + l_2\cos\varphi_2$ $\quad \dot y_2 = -l_1\dot\varphi_1\sin\varphi_1 - l_2\dot\varphi_2\sin\varphi_2$

$T = \frac12 m_1(l_1^2\dot\varphi_1^2) + \frac12 m_2\Big(l_1^2\dot\varphi_1^2 + l_2^2\dot\varphi_2^2 + 2l_1l_2\dot\varphi_1\dot\varphi_2\underbrace{(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)}_{\cos(\varphi_1-\varphi_2)}\Big)$

$T = \frac12(m_1+m_2)l_1^2\dot\varphi_1^2 + \frac12 m_2 l_2^2\dot\varphi_2^2 + m_2l_1l_2\dot\varphi_1\dot\varphi_2\cos(\varphi_1-\varphi_2)$

$\mathbb{T} = \left(\frac{\partial^2 T}{\partial\dot\varphi_i\partial\dot\varphi_j}\right)_{\varphi_1=\varphi_2=0} = \begin{pmatrix}(m_1+m_2)l_1^2 & m_2l_1l_2\cos(\varphi_1-\varphi_2)\\ m_2l_1l_2\cos(\varphi_1-\varphi_2) & m_2l_2^2\end{pmatrix}_{\varphi_1=\varphi_2=0} = \begin{pmatrix}(m_1+m_2)l_1^2 & m_2l_1l_2\\ m_2l_1l_2 & m_2l_2^2\end{pmatrix}$

$U = -m_1gy_1 - m_2gy_2 =$

$= -m_1gl_1\cos\varphi_1 - m_2g(l_1\cos\varphi_1 + l_2\cos\varphi_2)$

$\mathbb{U} = \left(\frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j}\right)_{\varphi_1=\varphi_2=0} = \begin{pmatrix}(m_1+m_2)gl_1\cos\varphi_1 & 0\\ 0 & m_2gl_2\cos\varphi_2\end{pmatrix}_{\varphi_1=\varphi_2=0} = \begin{pmatrix}(m_1+m_2)gl_1 & 0\\ 0 & m_2gl_2\end{pmatrix}$

KONST.

$\det(\mathbb{U}-\omega^2\mathbb{T}) = 0 \Rightarrow \dots =$

D.C.V.