34. Ukažte, že rotace bezsilového setrvačníku je stabilní kolem os s maximálním a minimálním momentem setrvačnosti a nestabilní kolem osy se středním momentem setrvačnosti.

$\dot{\vec{\Omega}} = X(\vec{\Omega})$ ① Rovnovážný stav $\vec{\Omega}_e$ (def. $X(\vec{\Omega}_e)=0$) je stabilní pokud $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \vec{\Omega}(t)$ řešení ① platí $|\vec{\Omega}(0) - \vec{\Omega}_e| < \delta \Rightarrow |\vec{\Omega}(t) - \vec{\Omega}_e| < \varepsilon\ \forall t$

Linearizace $\vec{\Omega} = \vec{\Omega}_e + \delta\vec{\Omega}$

Taylor do 1. řádu kolem $\vec{\Omega}_e$

$$\frac{d(\vec{\Omega})}{dt} = \frac{d(\vec{\Omega}_e + \delta\vec{\Omega})}{dt} = \frac{d(\delta\vec{\Omega})}{dt} = X(\vec{\Omega}_e + \delta\vec{\Omega}) = \underbrace{X(\vec{\Omega}_e)}_{=0} + X'(\vec{\Omega}_e)\delta\vec{\Omega} + \cancel{\delta^2(\delta\vec{\Omega})}$$

$$\frac{d(\delta\vec{\Omega})}{dt} = X'(\vec{\Omega}_e)\delta\vec{\Omega} \quad ②$$

Rovnovážný stav $\vec{\Omega}_e$ je lineárně stabilní pokud stav $\delta\vec{\Omega} = 0$ je stabilní pro systém ②.

Stabilita $\Rightarrow$ Lineární Stabilita (Lin. nestabilita $\Rightarrow$ nestabilita).

Euler. rovnice $\quad X_1(\vec{\Omega})$

$$\dot{\Omega}_1 = \overbrace{\frac{(I_2 - I_3)}{I_1}\Omega_2\Omega_3}^{X_1(\vec{\Omega})}$$
$$\dot{\Omega}_2 = \frac{(I_3 - I_1)}{I_2}\Omega_3\Omega_1$$
$$\dot{\Omega}_3 = \frac{(I_1 - I_2)}{I_3}\Omega_1\Omega_2$$

bezsilový $\vec{N}^{(e)} = 0$ asymetrický $I_1 \neq I_2 \neq I_3 \neq I_1$

$\vec{\Omega}_e = \begin{pmatrix} 0 \\ 0 \\ \Omega_3 \end{pmatrix}$ rovnovážný stav $\qquad \vec{\Omega} = \vec{\Omega}_e + \delta\vec{\Omega}$ — malé

$$\frac{d\delta\Omega_1}{dt} = \delta\dot{\Omega}_1 = X_1(\vec{\Omega}_e) + \left.\frac{\partial X_1}{\partial \Omega_1}\right|_{\vec{\Omega}_e}\delta\Omega_1 + \left.\frac{\partial X_1}{\partial \Omega_2}\right|_{\vec{\Omega}_e}\cdot\delta\Omega_2 + \left.\frac{\partial X_1}{\partial \Omega_3}\right|_{\vec{\Omega}_e}\delta\Omega_3$$

$$\delta\dot{\Omega}_1 = 0 + \left[\frac{(I_2 - I_3)}{I_1}\underbrace{\Omega_3}_{=\Omega_3}\right]_{\vec{\Omega}_e}\delta\Omega_2 + \left[\frac{(I_2 - I_3)}{I_1}\underbrace{\Omega_2}_{=0}\right]_{\vec{\Omega}_e}\delta\Omega_3 = \underline{\underline{\frac{(I_2 - I_3)}{I_1}\Omega_3\delta\Omega_2}} \quad \Big/\frac{d}{dt}$$

$$\delta\dot{\Omega}_2 = \left[\frac{(I_3 - I_1)}{I_2}\Omega_3\right]_{\vec{\Omega}_e}\delta\Omega_1 + \left[\frac{(I_3 - I_1)}{I_2}\underbrace{\Omega_1}_{=0}\right]_{\vec{\Omega}_e}\cdot\delta\Omega_3 \quad = \frac{I_3 - I_1}{I_2}\Omega_3\delta\Omega_1$$

$$\delta\dot{\Omega}_3 = 0 \quad \Rightarrow \quad \underline{\delta\Omega_3 = \text{konst}}$$

$$\delta\ddot{\Omega}_1 = \frac{(I_2 - I_3)}{I_1}\Omega_3\delta\dot{\Omega}_2 = \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}\Omega_3^2\delta\Omega_1 \quad \Rightarrow \quad \delta\ddot{\Omega}_1 + \underbrace{\frac{(I_3 - I_2)}{I_1}\frac{(I_3 - I_1)}{I_2}\Omega_3^2}_{k^2 \neq 0}\delta\Omega_1 = 0$$

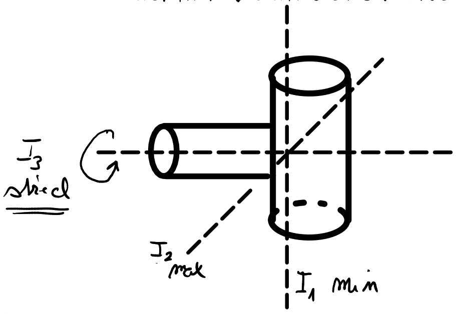
dané poč. podm. (malé) $\qquad$ lineárně

$k^2 > 0 \quad \delta\Omega_1 = A\cos(|k|t + B)$, $\quad \delta\Omega_2 = \tilde{A}\sin(|k|t + B)$ } stabilní rotace kolem I_3 $\Leftrightarrow$ $I_3 > I_1, I_2$; $\quad I_3 < I_1, I_2$ $\quad I_3 = \max\{I_1, I_2, I_3\}$, $\min\{ \text{—''—} \}$

$k^2 < 0 \quad \delta\Omega_1 = Ae^{|k|t} + Be^{-|k|t}$, $\quad \delta\Omega_2 = \ldots$ } labilní rotace kolem I_3 $\Leftrightarrow$ $I_1 < I_3 < I_2$; $\quad I_1 > I_3 > I_2$ $\quad I_3$ střední

$\Downarrow$ nestabilita (obecně)

Vladimir Džanibekov 1985



35. Nejděte matici otáčení $\mathbb{S}(t)$ ($\in SO(3)$) homogenní kouli v bezsiloém prostředí.

$\tilde{I}_1 = \tilde{I}_2 = \tilde{I}_3 \qquad \vec{N}^{(e)} = 0$

① $\tilde{I}_1\dot{\tilde{\Omega}}_1 = 0 \Rightarrow \tilde{\Omega}_1 = \text{konst}$
$\tilde{I}_2\dot{\tilde{\Omega}}_2 = 0 \Rightarrow \tilde{\Omega}_2 = \text{konst}$ $\qquad \vec{\tilde{\Omega}} = \text{konst}$
$\tilde{I}_3\dot{\tilde{\Omega}}_3 = 0 \Rightarrow \tilde{\Omega}_3 = \text{konst}$

③ $\tilde{I} = \begin{pmatrix} \tilde{I}_1 & 0 & 0 \\ 0 & \tilde{I}_2 & 0 \\ 0 & 0 & \tilde{I}_3 \end{pmatrix} = \tilde{I}_1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

④ $\vec{L} = I\cdot\vec{\Omega}$

$\uparrow$ pro nás konstanta v lab. soustavě

② $\overbrace{(-\mathbb{S}^T\dot{\mathbb{S}})_{ij}}^{\tilde{\omega}} = \varepsilon_{ijk}\tilde{\Omega}_k$

$-S_{ki}\dot{S}_{kj} = \varepsilon_{ijk}\tilde{\Omega}_k \quad / S_{li}$

$-\delta_{kl}\dot{S}_{kj} = \varepsilon_{ijk}\tilde{\Omega}_k S_{li}$

$$\boxed{\dot{S}_{lj} = -\varepsilon_{ijk}\tilde{\Omega}_k S_{li}} \quad \forall l = 1,2,3, \quad \forall j = 1,2,3$$

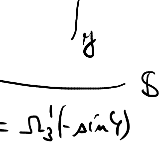
od $\tilde{S}$ $\rightarrow$ přejdeme v tělese k soustavě S' kde $\vec{\Omega}' = \begin{pmatrix} 0 \\ 0 \\ \Omega_3' \end{pmatrix}$

$\dot{S}_{l1} = -\varepsilon_{i1k}\Omega_k' S_{li} = +\Omega_3' S_{l2}$

$\dot{S}_{l2} = -\Omega_3' S_{l1} \qquad \forall l = 1,2,3$

$\dot{S}_{l3} = 0$

zvolíme Lab. soustavu S tak aby $\vec{L} = \begin{pmatrix} 0 \\ 0 \\ L_3 \end{pmatrix}$



$S' \xrightarrow{\mathbb{S}} \tilde{S}$

$\mathbb{S} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\boxed{l=1}$ $\quad -\dot{\varphi}\sin\varphi = \Omega_3'(-\sin\varphi)$

$\Rightarrow \Omega_3' = \dot{\varphi}$

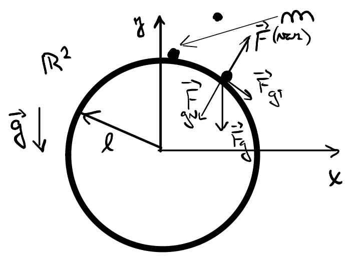
$$\boxed{\varphi(t) = \Omega_3' t + \varphi_0}$$

$\forall l$ rce pro l-tý řádek

$l = 1 \quad \dot{S}_{1j} = -\underbrace{\varepsilon_{ijk}\tilde{\Omega}_k}\underline{\underline{S_{1i}}} \qquad \vec{S}_1^T = (S_{11}, S_{12}, S_{13})$

$\dot{\vec{S}}_1^T = -\vec{S}_1^T\mathbb{M} \qquad \mathbb{M} = \begin{pmatrix} 0 & \tilde{\Omega}_3 & -\tilde{\Omega}_2 \\ -\tilde{\Omega}_3 & 0 & \tilde{\Omega}_1 \\ \tilde{\Omega}_2 & -\tilde{\Omega}_1 & 0 \end{pmatrix}$

37. Neudržující vazba. Hmotný bod je vypuštěn s nulovou počáteční rychlostí z těsné blízkosti nejvyššího bodu svislé kružnice, odkud začne klouzat (vlivem tíže) bez tření. Kdy tento bod kružnici opustí?



$f(x,y) = x^2+y^2-l^2 \geq 0$

$= 0$ pokud je bod na kružnici

neholonomní
ideální
skleronomní

$m\ddot x = 0 + 2\lambda x$

$m\ddot y = -mg + 2\lambda y$

$\vec F^{(vaz)} = \lambda \nabla f = \lambda\begin{pmatrix}2x\\2y\end{pmatrix}$

$\vec F_g = \begin{pmatrix}0\\-mg\end{pmatrix}$

$\boxed{LR1.D}$

$\dot f = 2x\dot x + 2y\dot y = 0 \;/\frac{d}{dt}$

$2(\dot x^2 + x\ddot x + \dot y^2 + y\ddot y) = 0$

ZZE

$\frac12 mv^2 + mgy = mgl$

$\boxed{v^2 = 2g(l-y)}$

$\underbrace{(\dot x^2+\dot y^2)}_{v^2} + x\ddot x + y\ddot y = 0 \Rightarrow 2g(l-y) + x\frac{2\lambda x}{m} + y\left(-g+\frac{2\lambda y}{m}\right) = 0$

$2gl - 2gy + \frac{2\lambda}{m}\underbrace{(x^2+y^2)}_{l^2 \leftarrow \text{vazba}} - yg = 0$

$0 \overset{?}{=} \lambda = (3gy-2gl)\frac{m}{2l^2}$

bod opustí vazbu

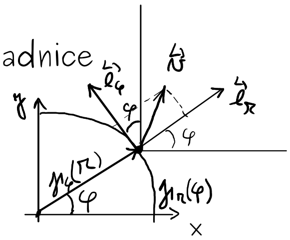
$3y - 2l = 0 \quad \boxed{y = \frac{2l}{3}}$

38. + 39. Křivočaré souřadnice

Polární

$x = r\cos\varphi \quad r\in(0,+\infty)$

$y = r\sin\varphi \quad \varphi\in\langle 0,2\pi)$



Báze $B = (\vec e_x, \vec e_y)$

$\tilde B = (\vec e_r, \vec e_\varphi)$

$\tilde B^{ON} = (\vec e_r^{\,ON}, \vec e_\varphi^{\,ON})$

$\vec e_i^{\,ON} = \vec e_j S^{ON}_{ji}$

$S^{ON} = {}^{\tilde B^{ON}}(Id)^B = \left((\vec e_r^{\,ON})_B, (\vec e_\varphi^{\,ON})_B\right)$

$S^{ON} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} \in SO(3)$

$v_x = \dot x = \dot r\cos\varphi + r(-\dot\varphi\sin\varphi)$

$v_y = \dot y = \dot r\sin\varphi + r\dot\varphi\cos\varphi$

Souřadnicové křivky

$\gamma_\varphi : r \to \begin{pmatrix}r\cos\varphi\\ r\sin\varphi\end{pmatrix}$

$\gamma_r : \varphi \to \begin{pmatrix}r\cos\varphi\\ r\sin\varphi\end{pmatrix}$

$\vec e_r = \gamma_\varphi' = \frac{d\gamma_\varphi(r)}{dr} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} = (\vec e_r)_B$

$\| - \| - \| = 1$

$\vec e_\varphi = \gamma_r' = \frac{d\gamma_r(\varphi)}{d\varphi} = \begin{pmatrix}-r\sin\varphi\\ r\cos\varphi\end{pmatrix} = (\vec e_\varphi)_B$

$\| - \| - \| = r$

$\vec e_r^{\,ON} = \frac{\gamma_\varphi'}{\|\gamma_\varphi'\|} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$

$\vec e_\varphi^{\,ON} = \frac{\gamma_r'}{\|\gamma_r'\|} = \frac1r\begin{pmatrix}-r\sin\varphi\\ r\cos\varphi\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$

$S = {}^{\tilde B}(Id)^B = \left((\vec e_r)_B, (\vec e_\varphi)_B\right) = \begin{pmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{pmatrix}$

$\vec{\tilde v} = S^{-1}\vec v \quad (S^{ON})^{-1} = (S^{ON})^T$

$\vec{\tilde v}^{ON} = \begin{pmatrix}v_r^{ON}\\ v_\varphi^{ON}\end{pmatrix} = \overbrace{\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \begin{pmatrix}\dot r\\ r\dot\varphi\end{pmatrix}$

$\boxed{\begin{matrix}v_r^{ON} = \dot r\\ v_\varphi^{ON} = r\dot\varphi\end{matrix}}$

$\begin{pmatrix}a_r^{ON}\\ a_\varphi^{ON}\end{pmatrix} = (S^{ON})^T\begin{pmatrix}\ddot x\\ \ddot y\end{pmatrix} \Rightarrow \dots$