Ten Commandments* of Waves and Optics

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A necessary[1] condition for passing the oral exam in Waves and Optics is the knowledge of the following facts.

1. Euler's identity $e^{i\varphi} = \cos\varphi + i\sin\varphi$ and its consequence $\operatorname{Re} e^{i\varphi} = \cos\varphi$.
2. The solution to the LHO equation, $\ddot{x} + \omega^2 x = 0$, can be written in equivalent forms:
$$x(t) = A\cos(\omega t + \varphi) = A\sin(\omega t + \phi) = a\cos\omega t + b\sin\omega t = c_1 e^{i\omega t} + \bar{c}_1 e^{-i\omega t}.$$
3. Average values
$$\langle\cos\omega t\rangle = \langle\sin\omega t\rangle = 0, \qquad \langle\cos^2\omega t\rangle = \langle\sin^2\omega t\rangle = \frac{1}{2}.$$
4. 1D wave equation
$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial z^2},$$
where v represents the phase velocity—the speed of propagation—of travelling waves and $\psi(z,t) : \mathbb{R}^2 \to \mathbb{R}$.
5. Boundary conditions at a fixed and free end at $z = z_0$:
$$\psi(z_0,t) = 0 \quad \text{(fixed)}, \qquad \frac{\partial\psi}{\partial z}(z_0,t) = 0 \quad \text{(free)}.$$
6. Initial conditions for a medium on $z \in \langle 0, L\rangle$ described by the wave equation:
$$\psi(z,0) = f(z) \quad \text{(initial position)}, \qquad \frac{\partial\psi}{\partial t}(z,0) = g(z) \quad \text{(initial velocity)},$$
where $f, g : \langle 0, L\rangle \to \mathbb{R}$.
7. D'Alembert's solution to the 1D wave equation:
$$\psi(z,t) = F(z - vt) + G(z + vt),$$
where $F, G : \mathbb{R} \to \mathbb{R}$ are arbitrary functions (of one variable) that describe the shape of the wave propagating in the positive (for F) and negative (for G) directions along the z-axis at phase velocity v.

*Of twenty-six points, i.e. it is 10 in number base of 26.

[1] But not sufficient...

8. Harmonic travelling wave in real and complex form:

$$\psi(z,t) = A\cos(\omega t - kz + \varphi), \qquad \psi(z,t) = Ae^{i(\omega t - kz + \varphi)},$$

where $\omega \in \mathbb{R}^+$ is the angular frequency and $k \in \mathbb{R}^+$ is the wavenumber. This wave propagates through the medium with phase velocity $v_\varphi = \frac{\omega}{k}$.

9. The dispersion relation gives the allowed combinations of ω and k, for which a travelling wave can propagate in a given medium. The dispersion relation is given by the function $\omega(k)$, or inversely $k(\omega)$ (or implicitly by $f(\omega,k) = 0$). The permissible ω for a given k is obtained as $\omega = \omega(k)$ (and permissible k for a given ω is obtained as $k = k(\omega)$).

10. The group velocity for a wave packet with a central wavenumber k_0 is

$$v_g = \frac{d\omega}{dk}(k_0).$$

This velocity represents the speed at which the wave packet (its amplitude envelope) propagates.

11. At the interface between two media, an incident wave of the form $F(x)$ reflects in the form $RF(-x)$ and transmits in the form $PF(\frac{v_1}{v_2}x)$, where v_i are the phase velocities of the respective media, and R and P are the reflection and transmission coefficients. The shape of the reflected wave is mirrored along "vertical" axis and the shape of the transmitted wave is contracted/elongated along the direction of propagation by a factor $\frac{v_2}{v_1}$.

For harmonic travelling waves, the incident wave is $\psi_d(z,t) = e^{i(\omega t - k_1 z)}$, the reflected wave is $\psi_r(z,t) = Re^{i(\omega t + k_1 z)}$, and the transmitted wave is $\psi_t(z,t) = Pe^{i(\omega t - k_2 z)}$.

12. The spatial wave equation is the equation for a spatial wave $\psi(\vec{r},t)$ of the form:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \Delta \psi,$$

where Δ is the Laplace operator in the respective dimension. In 3D Cartesian coordinates:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The phase velocity v represents the speed at which the wavefronts propagate through the medium.

13. Wavefronts are surfaces of constant phase of a given wave. Specifically, for $\psi(\vec{r},t) = e^{i\varphi(\vec{r},t)}$, the wavefronts are defined by the equation $\varphi(\vec{r},t) = \varphi_0$ for individual values of φ_0.

14.
   - A harmonic travelling plane 3D wave is of the form:

     $$\psi(\vec{r},t) = Ae^{i(\omega t - \vec{k}\cdot\vec{r})},$$

     where $\vec{k} = k\vec{n}$ is the wave vector, $k = |\vec{k}|$ is the wavenumber, and $\vec{n}$, $|\vec{n}| = 1$, is the direction of propagation. The wavefronts are planes perpendicular to $\vec{n}$. The propagation speed is given by the phase velocity $v_\varphi = \frac{\omega}{k}$.
   - A harmonic travelling spherical 3D wave is of the form:

     $$\psi(\vec{r},t) = \frac{A}{r}e^{i(\omega t - kr)}.$$

     The wavefronts are spheres with the centers at the origin. The propagation speed is given by the phase velocity $v_\varphi = \frac{\omega}{k}$. The amplitude decreases as $\frac{1}{r}$.

15. Maxwell's equations in a homogeneous medium (described by permittivity ε and permeability μ – so-called linear medium) without free charges and currents are of the form:

$$\operatorname{div}\vec{E} = 0 \quad \text{(Gauss's law)}, \qquad \operatorname{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday's law)},$$

$$\operatorname{div}\vec{B} = 0, \qquad \operatorname{rot}\vec{B} = \varepsilon\mu\frac{\partial \vec{E}}{\partial t} \quad \text{(Ampère-Maxwell law)}.$$

16. A plane harmonic electromagnetic wave is a solution of the wave equations for $\vec{E}$ and $\vec{B}$ derived from Maxwell's equations,

$$\frac{\partial^2 \vec{E}}{\partial t^2} = v^2 \Delta \vec{E}, \qquad \frac{\partial^2 \vec{B}}{\partial t^2} = v^2 \Delta \vec{B},$$

with $v = \frac{1}{\sqrt{\varepsilon\mu}}$, in the form

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{r})}, \qquad \vec{B}(\vec{r},t) = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{r})},$$

where $\omega = v|\vec{k}|$ (the dispersion relation for EM waves), $\vec{E} \perp \vec{n}$ and $\vec{B} \perp \vec{n}$ (the EM wave is a transverse wave), $\vec{E} \perp \vec{B}$, $E = vB$, and $(\vec{E}, \vec{B}, \vec{n})$ form a right-handed set of vectors.

17. The intensity of an EM wave is given by $I = \langle \vec{S} \rangle$, where $\vec{S}$ is the Poynting vector (energy flux), which for an EM wave has the form $\vec{S} = \sqrt{\frac{\varepsilon}{\mu}} E^2 \, \vec{n}$, where $\vec{n}$ is a direction of propagation.

18.
- A fully (generally elliptically) polarized EM wave traveling in the z-axis direction has the form
$$\vec{E}(\vec{r},t) = E_{x0}\,\vec{x}\,e^{i(\omega t - kz + \varphi_1)} + E_{y0}\,\vec{y}\,e^{i(\omega t - kz + \varphi_2)}.$$
- A linearly polarized EM wave has the form
$$\vec{E}(\vec{r},t) = E_0\,\vec{n}\,e^{i(\omega t - kz + \varphi)},$$
where $\vec{n}$ is the unit vector of the polarization direction. For a given z, the electric field $\vec{E}$ traces a line segment in the xy plane over time.
- A circularly polarized EM wave is a wave where, for a given z, the electric field $\vec{E}$ traces a circle in the xy plane over time. One possible form of a circularly polarized wave is
$$\vec{E}(\vec{r},t) = E_0\,\vec{x}\cos(\omega t) \pm E_0\,\vec{y}\sin(\omega t),$$
where the different signs correspond to different directions of the rotation of the vector $\vec{E}$ in the xy plane.

19.
- A polarizer transmits only the electric field component in the transmission direction $\vec{n}$ according to the relation
$$\vec{E}_{out} = (\vec{E}_{in} \cdot \vec{n})\,\vec{n}.$$
- A wave plate with a phase shift $\Delta\varphi$ and axis $\vec{n}$ alters the electric field in the following manner. If the input field is
$$\vec{E}_{in} = E_1\,\vec{n}\,e^{i(\omega t + \varphi_1)} + E_2\,\vec{n}_\perp e^{i(\omega t + \varphi_2)},$$
then the output field has the form
$$\vec{E}_{out} = E_1\,\vec{n}\,e^{i(\omega t + \varphi_1 + \Delta\varphi)} + E_2\,\vec{n}_\perp e^{i(\omega t + \varphi_2)},$$
where $\vec{n}_\perp$ is the unit vector perpendicular to $\vec{n}$.

20. The refractive index n of a medium is defined as $n = \frac{c}{v}$, where v is the phase velocity in the medium. The corresponding dispersion relation is of the form $\omega = \frac{c}{n}|\vec{k}|$.

21. *The law of reflection and refraction of a plane EM wave at a planar interface.* For the angles of incidence ϑ_d, reflection ϑ_r, and refraction ϑ_t, measured from the normal to the interface, the following holds:
$$\vartheta_d = \vartheta_r, \qquad n_1 \sin\vartheta_d = n_2 \sin\vartheta_t,$$
where n_1 and n_2 are the refractive indices of the "incident" and "transmitted" media. The critical angle ϑ_C is given by $\sin\vartheta_C = \frac{n_2}{n_1}$ for $n_2 < n_1$. For $n_1 < n_2$ there is no critical angle. For incidence angles $\vartheta_d \geq \vartheta_C$, total internal reflection occurs.

22. The diffraction integral
$$\vec{E} = \vec{E}_0 \int_B \frac{1}{r} e^{i(\omega t - kr)}\, dS,$$
represents the superposition of spherical waves with the same but undefined amplitude, emitted from every point of the aperture B in the screen. The simplest approximation is the so-called Fraunhofer diffraction integral:
$$\vec{E} = \frac{\vec{E}_0}{R} e^{i(\omega t - kR)} \int_B e^{i\frac{k}{R}(xX + yY)}\, dS,$$
with the meaning of the individual symbols detailed in the textbook.

23. A diffraction pattern is the spatial distribution of intensity $I(x, y) = \langle \vec{E}(x, y)^2 \rangle$ on the screen in the xy plane.

24. A diffraction pattern typically contains maxima and minima of intensity, observed at an angle θ (the angular deviation from the aperture axis). Qualitatively, we have
$$\sin\theta \propto m\frac{\lambda}{d},$$
where $m \in \mathbb{N}_0$ is the so-called order of the maximum, λ is the wavelength of the light used, and d is the characteristic dimension of the aperture — for example, the distance between adjacent slits, the size of a circular aperture, etc.

25. For the positions of the maxima on the screen close to the aperture axis, we have
$$y_m \propto mL\frac{\lambda}{d},$$
where L is the distance between the screen and the aperture.

26. A diffraction grating with N slits narrows the diffraction maxima according to the relation
$$\Delta(\sin\theta) \propto \frac{1}{N}\frac{\lambda}{d}.$$