Introduction to Strings

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November 20, 2024

Introduction

These are unofficial lecture notes for the course ”Introduction to Strings”. It will be expanded and modified over time. I encourage the reader to look for mistakes and provide a feedback.

This course is based mainly on an excellent book [1]. It assumes an elementary knowledge of classical mechanics, quantum mechanics and special relativity. A basic knowledge of quantum field theory and general relativity is helpful but not necessary.

Question: What is string theory?

Originally, it is a quantum field theory obtained by a quantization of the mechanics of a relativistic string moving in the spacetime. For last fifty years it is still believed to be a promising candidate for a *quantum theory of gravity*. Why is this important? We know four basic forces of nature: electromagnetism, weak interaction (description of β-decay), strong force (interaction of constituents of the atom nuclei) and gravity. There are two major achievements of physics of the twentieth century:

(i) **Standard model of particle physics**: This is a pinnacle of quantum field theory. It describes all known elementary particles and correctly predicts their interactions. It is fully compatible with special relativity. It describes (and to some extent unifies) three of the above forces.

(ii) **General relativity**: This is a description of gravity in terms of geometry of spacetime. It proved to be extremely successful in describing the macroscopic Universe. Modern cosmology is fully based on general relativity and it correctly predicts black holes, gravitational waves, gravitational lenses, etc.

It seemed inevitable that there is also a quantum field theory of gravity - with a new elementary particle “graviton”, an intermediate boson serving as a carrier of the gravitational force. However, it turned out that even in the simplest cases (the spacetime is almost flat) and out of any extremities (black holes), all naive attempts fail. This is mainly because the resulting theory always horribly fails at high energies. Using more clever words, one observes that it is *non-renormalizable*. But we *need* a high-energy sector of gravity to understand early universe and black holes! String theory is one of *many* theories offering partial answers.

What are **successful** features of string theory?

(1) Gravity is discovered *within* string theory - it is an intrinsic feature.

(2) It contains all particles of the Standard model.

(3) By design, it is free of UV divergences, the biggest plague of naive quantum gravity.

(4) String theory fueled tremendous advancements in mathematics.

What are **failures** of string theory?

(1) The original idea was to reduce number of “god given” parameters of the Standard model. However, it turns that there is unfortunately many possibilities ($10^{500} - 10^{272000}$) how the universe can look like. This leads to a not-so-well received *anthropic principle*.

(2) It requires *supersymmetry* to be consistent. In a nutshell, supersymmetry conjectures a symmetry between bosons (carriers of force) and fermions (matter fields). Each particle (e.g. photon) has its superpartner (e.g. photino). No such particles were ever observed. If supersymmetry fails, string theory fails.

(3) Some critics of string theory say that its unfalsifiabile - that is there is no experiment which can be designed to test whether the theory is true or false. It does not give any predictions which can be actually measured.

(4) Many people object that the hype surrounding string theory overshadows other viable quantum gravity theories (e.g. loop quantum gravity) and makes it difficult for people working in those to get funds and job positions.

Despite the listed shortcomings, string theory is still a fascinating achievement of theoretical physics, fueling a unforeseen and fruitful collaboration of mathematicians and physicists. I hope this course will serve well as a glimpse into this tremendously huge theory.

Please be aware that your lecturer is learning with you. Do not hesistate to ask question, but do not always expect well-thought-out answers.

1 Special relativity and extra dimensions

1.1 Conventions and basic notions

As any viable modern physical theory, string theory must be fully compatible with special relativity. This amounts to working with spacetime. Let us thus recall basic conventions we will use throughout this course.

If we want to describe some event happening, we record a time t when it happened, and its coordinates (x, y, z) in some chosen reference *inertial* frame. It is convenient to combine those into an 4-tuple (ct, x, y, z), where c is the speed of light. Note that all four variables have a dimension of length. It is convenient to introduce those on equal footing as

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z). \tag{1}$$

In other words, we fix some coordinate frame x^μ on a **spacetime** $\mathbb{R}^4$.

Now, suppose two events are described by coordinates x^μ and $x^\mu + \Delta x^\mu$. The **invariant interval** Δs^2 separating these two events is defined by

$$-\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2. \tag{2}$$

The main postulate of special relativity is the following: If x'^μ and $x'^\mu + \Delta x'^\mu$ are the coordinates of the same event with respect to any other inertial frame, one has $\Delta s^2 = \Delta s'^2$.

We say that this interval is **timelike**, if $\Delta s^2 > 0$. For example, movement of a massive particle can be described by a curve $x^\mu = x^\mu(\tau)$ caled the **worldline**, any two points on this curve are separated by a time-like interval. In this case, we define

$$\Delta s := \sqrt{\Delta s^2}. \tag{3}$$

We say that the interval is **spacelike**, if $\Delta s^2 < 0$ and lightlike, if $\Delta s^2 = 0$. Any two points on the world-line of photon are separated by a light-like interval.

Now, suppose we have two sets of coordinates x^μ and x'^μ with respect to their inertial frames. Suppose those are related by a linear transformation, that is

$$x'^\mu = L^\mu{}_\nu x^\nu, \tag{4}$$

for some matrix $L = [L^\mu{}_\nu] \in \mathbb{R}^{4,4}$. Note that we will always label rows by the first index and columns by the second index, regardless of its vertical position. Let us record two events in both coordinates:

(i) First event is described by $(0, 0, 0, 0)$ in both frames;

(ii) The second event is described by x^μ and x'^μ, respectively.

The invariance of the spacetime interval forces

$$-(x'^0)^2 + (x'^1)^2 + (x'^2)^2 + (x'^3)^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{5}$$

It is convenient to write this as

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu, \tag{6}$$

where $\eta_{00} = -1$, $\eta_{11} = 1$, $\eta_{22} = 1$, $\eta_{33} = 1$, that is we have a matrix $\eta = [\eta_{\mu\nu}] \in \mathbb{R}^{4,4}$ in the form

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

If we introduce a notation $x_\mu := \eta_{\mu\nu} x^\nu$, we can write the above equation simply as $x'_\mu x'^\mu = x_\mu x^\mu$, where we always utilize the *Einstein summation convention.* If we plug in (4), it is not difficult to work out the relation

$$\eta_{\mu\nu} L^\mu{}_\lambda L^\nu{}_\kappa = \eta_{\lambda\kappa}, \tag{8}$$

for each $\lambda, \kappa \in \{0, 1, 2, 3\}$. In matrix form, this can be written as $L^T \eta L = \eta$. Linear transformations satisfying this constraints are called **Lorentz transformations** and they form the **Lorentz group** $\mathrm{O}(3, 1)$.

Now, suppose we have a quantity $a = (a^\mu)$ labeled by four spacetime indices, with respect to some Lorentz frame coordinates x^μ. One says that a is a **Lorentz vector**, if its version $a' = (a'^\mu)$ in with respect to the frame (4) takes the form

$$a'^\mu = L^\mu{}_\nu a^\nu. \tag{9}$$

It is easy to see that $a^\mu a_\mu$ is independent of the Lorentz frame, it is an example of a **Lorentz scalar**. For any two Lorentz vectors a and b, we write

$$a \cdot b := a_\mu b^\mu = a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu. \tag{10}$$

Now, suppose we have a worldline of a massive particle described as a curve $x^\mu = x^\mu(t)$. Let us fix two times $t_0 < t_1$. The **proper time** elapsed between events $x^\mu(t_0)$ and $x^\mu(t_1)$ is defined as an integral

$$s(t_1, t_0) := \int_{t_0}^{t_1} \sqrt{1 - \frac{v(t)^2}{c^2}} \mathrm{d}t, \tag{11}$$

where $\vec{v}(t)$ is the usual velocity, that is $\vec{v}(t) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$. One often fixes t_0 and writes

$$s(t) := s(t, t_0) = \int_{t_0}^{t} \frac{\mathrm{d}t}{\gamma(t)} \tag{12}$$

It has an important feature. Suppose (ct', x', y', z') is a different Lorentz frame, related to (ct, x, y, z) by (4). Then $s'(t'_1, t'_0) = s(t_1, t_0)$. Moreover, note that

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \frac{1}{\gamma(t)} > 0, \tag{13}$$

which means that we may use s as a convenient parametrization of worldlines, that is write $x^\mu = x^\mu(s)$ as a curve parametrized by the proper time s.

Exercise 1.1. *Show that proper time is indeed a Lorentz invariant.*

Proof. Suppose $x'^\mu = L^\mu{}_\nu x^\nu$. Since we require both $t_0 < t_1$ and $t'_0 < t'_1$, we only consider an ortochronous Lorentz transformation. Equivalently, $L^0{}_0 > 0$. Then

$$s'(t'_1, t'_0) = \int_{t'_0}^{t'_1} \sqrt{1 - \frac{v'^2(t')}{c^2}} \mathrm{d}t' = \int_{t'_0}^{t'_1} \frac{1}{c} \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x'^\mu}{\mathrm{d}t'} \frac{\mathrm{d}x'^\nu}{\mathrm{d}t'}} \mathrm{d}t' \tag{14}$$

We can now plug in the transformation

$$\frac{\mathrm{d}x'^{\mu}}{\mathrm{d}t'} = L^{\mu}{}_{\lambda}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}t'} = L^{\mu}{}_{\lambda}\frac{\mathrm{d}t}{\mathrm{d}t'}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}t} \tag{15}$$

By plugging this in, we can continue and write

$$\begin{aligned}\int_{t'_0}^{t'_1}\frac{1}{c}\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x'^{\mu}}{\mathrm{d}t'}\frac{\mathrm{d}x'^{\nu}}{\mathrm{d}t'}}\mathrm{d}t' &= \int_{t'_0}^{t'_1}\frac{1}{c}\sqrt{-\eta_{\mu\nu}L^{\mu}{}_{\lambda}L^{\nu}{}_{\kappa}(\frac{\mathrm{d}t}{\mathrm{d}t'})^2\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}t}\frac{\mathrm{d}x^{\kappa}}{\mathrm{d}t}}\mathrm{d}t' \\ &= \int_{t'_0}^{t'_1}\frac{1}{c}\sqrt{-\eta_{\kappa\lambda}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}t}\frac{\mathrm{d}x^{\kappa}}{\mathrm{d}t}}\frac{\mathrm{d}t}{\mathrm{d}t'}\mathrm{d}t' \\ &= \int_{t_0}^{t_1}\frac{1}{c}\sqrt{-\eta_{\kappa\lambda}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}t}\frac{\mathrm{d}x^{\kappa}}{\mathrm{d}t}}\mathrm{d}t = s(t_1,t_0).\end{aligned} \tag{16}$$

Note that the fact that L is ortochonous implies that $\frac{\mathrm{d}t}{\mathrm{d}t'} > 0$. ∎

1.2 Light-cone variables

Let $x^{\mu} = (x^0, x^1, x^2, x^3)$ be a fixed set of Cartesian coordinates of the Minkowski spacetime. It is convenient to introduce new coordinates

$$x^+ := \frac{1}{\sqrt{2}}(x^0 + x^1), \quad x^- := \frac{1}{\sqrt{2}}(x^0 - x^1). \tag{17}$$

Remaining coordinates are not modified. It is easy to find the inverse transformations. Coordinates (x^+, x^-, x^2, x^3) are called **light-cone coordinates** corresponding to x^{μ}. For a photon moving in the positive x direction, we have x^- constant and x^+ increases.

Note that x^{μ} and (x^+, x^-, x^2, x^3) are never related by Lorentz transformation. For example, we can calculate the components of the Minkowski metric

$$\eta_{++} = \frac{\partial x^{\mu}}{\partial x^+}\frac{\partial x^{\nu}}{\partial x^+}\eta_{\mu\nu} = -\frac{\partial x^0}{\partial x^+}\frac{\partial x^0}{\partial x^+} + \frac{\partial x^1}{\partial x^+}\frac{\partial x^1}{\partial x^+} = -\frac{1}{2} + \frac{1}{2} = 0, \tag{18}$$

$$\eta_{--} = \frac{\partial x^{\mu}}{\partial x^+}\frac{\partial x^{\nu}}{\partial x^+}\eta_{\mu\nu} = -\frac{\partial x^0}{\partial x^-}\frac{\partial x^0}{\partial x^-} + \frac{\partial x^1}{\partial x^-}\frac{\partial x^1}{\partial x^-} = -\frac{1}{2} + \frac{1}{2} = 0, \tag{19}$$

$$\eta_{+-} = \frac{\partial x^{\mu}}{\partial x^+}\frac{\partial x^{\nu}}{\partial x^-}\eta_{\mu\nu} = -\frac{\partial x^0}{\partial x^+}\frac{\partial x^0}{\partial x^-} + \frac{\partial x^1}{\partial x^+}\frac{\partial x^1}{\partial x^-} = -\frac{1}{2} - \frac{1}{2} = -1. \tag{20}$$

The matrix of the Minkowski metric thus takes the form

$$\hat{\eta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

Obviously $\hat{\eta} \neq \eta$. Note that every Lorentz vector a can be also described by its coordinates $a^{\pm} := \frac{1}{\sqrt{2}}(a^0 \pm a^1)$ together with a^2 and a^3. Note that in light-cone coordinates, one has

$$a \cdot b = \hat{\eta}_{\mu\nu}a^{\mu}b^{\nu} = -a^+b^- - a^-b^+ + a^2b^2 + a^3b^3. \tag{22}$$

We can also lower and raise indices using $\hat{\eta}$, that is define $a_{\mu} := \hat{\eta}_{\mu\nu}a^{\nu}$. Note that then $a_{\pm} = -a^{\mp}$. To avoid confusion, we shall henceforth use light-cone indices $+$ and $-$ only in the upper position.

1.3 Energy and momentum

Having an invariant proper time, let $x^\mu = x^\mu(s)$ be a worldline of a massive particle. Then its **four-velocity** is defined as

$$u^\mu(s) := \frac{\mathrm{d}x^\mu}{\mathrm{d}s}(s), \tag{23}$$

It follows that $u^\mu(s)$ forms a Lorentz vector. One can express it in terms of the coordinate time t. Then, plugging in the inverse to (13), one has

$$u^\mu(t) = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}(t)\frac{\mathrm{d}t}{\mathrm{d}s} = \gamma(t)\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \gamma(t)\cdot(c, \vec{v}(t)). \tag{24}$$

Note that $u^2 = u_\mu u^\mu = -\gamma(t)^2(c^2 - v^2(t)) = -c^2\gamma(t)^2(1 - \frac{v^2(t)}{c^2}) = -c^2$. If m is the rest mass of the particle, we can define its **four-momentum** as

$$p^\mu := mu^\mu = m\gamma \cdot (c, \vec{v}) = (\frac{E}{c}, \vec{p}), \tag{25}$$

where $E := \gamma mc^2$ and $\vec{p} := \gamma m\vec{v}$. It follows that $p^2 = p^\mu p_\mu = -m^2c^2$, which proves the relativistic **energy-momentum constraint**

$$\frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m^2c^2. \tag{26}$$

Note that unlike 4-velocity, 4-momentum is well-defined also for massless particles.

Let us discuss energy and momentum in light-cone variables. Observe that

$$\frac{\mathrm{d}x^\pm}{\mathrm{d}s} = \frac{\partial x^\pm}{\partial x^0}u^0 + \frac{\partial x^\pm}{\partial x^1}u^1 = \frac{1}{\sqrt{2}}(u^0 \pm u^1) > 0. \tag{27}$$

This is because $(u^0)^2 - (u^1)^2 \geq -u\cdot u = c^2$, so $|u^0| > |u^1|$. This means that for massive particles, both x^+ and x^- increase along the worldline. It is easy to show that this is true also for photons, except the case when they move in $\pm x^1$ direction - then either x^- or x^+ "freezes". One can thus declare either of those variables to be the *light-cone time* coordinate. By convention, one chooses this to be x^+. What variable plays the role of energy in light-cone coordinates? Note that

$$p^0 = \frac{E}{c} = \sqrt{\vec{p}\cdot\vec{p} + m^2c^2} > |\vec{p}| \geq |p^1|, \tag{28}$$

so $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1)$ is always positive. Note that $p\cdot x = -p^0x^0 + \vec{x}\cdot\vec{p}$. In light-cone variables, we have

$$p\cdot x = p_+x^+ + p_-x^- + p_2x^2 + p_3x^3. \tag{29}$$

Since we have chosen x^+ to be our "time variable", we propose $p_+ = -\frac{E_{\mathrm{lc}}}{c}$. Since $p_+ = -p^-$, and we have promised to not use the lower light-cone indices, we define

$$E^{\mathrm{lc}} := cp^-. \tag{30}$$

This choice is motivated as follows. Recall that wavefunction of a particle with a momentum $p = (\frac{E}{c}, \vec{p})$ can be written as

$$\psi(t, \vec{x}) = \exp(-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})) = \exp(\frac{i}{\hbar}p\cdot x). \tag{31}$$

The Schrödinger equation can be rewritten as

$$i\hbar \frac{\partial \psi}{\partial x^0} = \frac{E}{c}\psi. \tag{32}$$

It follows that in light-cone variables, we get

$$i\hbar \frac{\partial \psi}{\partial x^+} = i\hbar \frac{i}{\hbar} p_+ \psi = p^- \psi =: \frac{E_{\text{lc}}}{c}\psi. \tag{33}$$

1.4 Extra dimensions

It will turn out that it is not enough to consider a four-dimensional spacetime. Instead, we will have to assume that our spacetime is D-dimensional, where $D > 4$. Mathematically, nothing changes too much, we will consider coordinates x^μ, $\mu \in \{0, \dots, D\}$.

The Minkowski metric tensor on $\mathbb{R}^D$ is given by $\eta = -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \sum_{i=1}^{D} \mathrm{d}x^i \otimes \mathrm{d}x^i$, a by Lorentz transformations, we mean elements of the group $\mathrm{O}(d, 1)$, where $d := D - 1$. Light-cone coordinates are defined in the same way as before. As they involve only x^0 and x^1, all considerations are valid for an arbitrary $D \geq 4$.

However, it will also turn out that the extra dimensions will be in fact **compact**. In strict mathematical sense of the word, space time manifold M will be a fiber bundle over the Minkowski spacetime $\mathbb{R}^4$ with compact fibers. For our purposes, we assume that some of the coordinates describe space with some its points identified.

For example, we can consider a space of points x on the real line, where we impose the identification $x \sim x + 2\pi n R$ for all $n \in \mathbb{Z}$, where $R > 0$. Clearly, this is just a circle of a radius R. This is a convenient how to treat compact spaces which can be constructed in this way, since functions on the circle can be described as $f = f(x)$ of the original "Cartesian" variable, only with the identification $f(x) = f(x + 2\pi n R)$ for all $n \in \mathbb{Z}$.

We will sometimes use the term **fundamental domain**, which is a connected subset of the space "before identification", such that

(i) No its two points are identified;

(ii) Every point is either in the fundamental domain, or is identified with some point in the fundamental domain.

In the above example, the fundamental domain is for example $[0, 2\pi R)$. The resulting compact space can be constructed by adding a boundary to the fundamental domain and applying the identifications on the boundary. In our example, take $[0, 2\pi R]$ and identify $0 \sim 2\pi R$ to obtain the circle of radius R.

Example 1.2 (Extra dimensions affect physics). Let us first consider a well-known example of a one-dimensional "infinite potential well". When looking for the energy spectrum of this problem, one solves the equation

$$-\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(x) = E\psi(x), \tag{34}$$

where the potential $V(x)$ satisfies

$$V(x) = \begin{cases} 0 & \text{if } x \in (0, a) \\ \infty & \text{if } x \notin (0, a) \end{cases} \tag{35}$$

One finds $\psi(x) = 0$ for all $x \notin (0, a)$ and for $x \in (0, a)$, the eigenfunctions satisfying the correct boundary conditions are

$$\psi_k(x) = \sqrt{\frac{2}{a}} \sin(\frac{k\pi x}{a}), \quad k \in \mathbb{N}. \tag{36}$$

The corresponding energies are then given by

$$E_k = \frac{\hbar^2}{2m}(\frac{k\pi}{a})^2. \tag{37}$$

Suppose that we now add an extra dimension y, being curled into a small circle of radius R, that is $y \sim y + 2\pi R$. For simplicity, assume that V does not depend on y. One solves the Schrödinger equation by the separation of variables, that is $\psi(x, y) = \psi(x) \cdot \phi(y)$. The equation for $x \in (0, a)$ turns into

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} - \frac{\hbar^2}{2m}\frac{1}{\phi(y)}\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} = E. \tag{38}$$

Hence both x-dependent and y-dependent parts of this equation have to be separately constant. The solutions are of the form $\psi_{k,\ell}(x, y) = \psi_k(x)\phi_\ell(y)$, where

$$\psi_k(x) = c_k \sin(\frac{k\pi x}{a}), \tag{39}$$

is the solution of the original Schrödinger equation satisfying the boundary conditions at $x \in \{0, a\}$, and $\phi_\ell(y)$ is of the form

$$\phi_\ell(y) = a_\ell \sin(\frac{\ell y}{r}) + b_\ell \cos(\frac{\ell y}{R}), \tag{40}$$

which is a unique solution to the 1-dimensional problem with the periodicity condition $\phi_\ell(y) = \phi_\ell(y + 2\pi R)$. There are no other restrictions in $\phi_\ell(y)$. The corresponding eigenvalues are

$$E_{k,\ell} = \frac{\hbar^2}{2m}[(\frac{k\pi}{a})^2 + (\frac{\ell}{R})^2]. \tag{41}$$

The original energy levels correspond to $\ell = 0$. Let us now consider a lowest non-trivial "new" energy level, that is $E_{1,1}$. One has

$$E_{1,1} = \frac{\hbar^2}{2m}(\frac{\pi^2}{a^2} + \frac{1}{R^2}). \tag{42}$$

Suppose that the compact dimension is very small, that is $R \ll a$. We see that in this case $E_{1,1} \approx \frac{\hbar^2}{2m}\frac{1}{R^2}$. Let us compare this to the original energy levels E_k, that is we ask for which k one has $E_{1,1} \approx E_k$. This gives us

$$\frac{1}{R} \approx \frac{k\pi}{a}, \tag{43}$$

that is $k \approx \pi\frac{a}{R}$. Since $R \ll a$, this means that k is "very large". We start to notice something happening to the spectrum only at very high energy levels.

2 Electromagnetism and gravity

2.1 Covariant electrodynamics

Recall that in Heavyside-Lorentz units, Maxwell equations (in vacuum) take the form

$$\mathrm{rot}(\vec{E}) + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0, \tag{44}$$

$$\mathrm{div}(\vec{B}) = 0, \tag{45}$$

$$\mathrm{div}(\vec{E}) = \rho, \tag{46}$$

$$\mathrm{rot}(\vec{B}) - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c}\vec{j}, \tag{47}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ρ is the charge density and $\vec{j}$ is the current density. Note that $\epsilon_0 = \mu_0 = 1$ in this system and $[\vec{E}] = [\vec{B}]$. First two Maxwell equations are called *homogeneous*.

Suppose $(\vec{E}, \vec{B})$ solve homogeneous Maxwell equations. Since $\mathrm{div}(\vec{B}) = 0$, one can write

$$\vec{B} = \mathrm{rot}(\vec{A}) \tag{48}$$

for some vector field $\vec{A}$. Plugging this into the first equation, we see that $\mathrm{rot}(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}) = \vec{0}$, hence there is a scalar field Φ, such that

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \Phi. \tag{49}$$

Conversely, if we choose any $(\Phi, \vec{A})$, we can solve the homogeneous Maxwell equations by declaring $\vec{B}$ and $\vec{E}$ by (48) and (49), respectively. We can certainly modify $\vec{A}$ by adding a gradient of some scalar field ϵ, that is

$$\vec{A}' := \vec{A} + \nabla \epsilon, \tag{50}$$

without changing $\vec{B}$. If want this transformation to preserve also $\vec{E}$, we must modify the scalar potential Φ as

$$\Phi' := \Phi - \frac{1}{c}\frac{\partial \epsilon}{\partial t} \tag{51}$$

We say that the physics is invariant under the **gauge transformations**. One can combine those fields into a single Lorentz vector $A^\mu := (\Phi, \vec{A})$, called the 4**-potential** of an electromagnetic field. Corresponding Lorentz covector is $A_\mu := \eta_{\mu\nu} A^\nu = (-\Phi, \vec{A})$. One can then construct a electromagnetic **field strength** as

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{52}$$

where we will always use the shorthand notation $\partial_\mu := \frac{\partial}{\partial x^\mu}$. The gauge transformations can be then simply written as

$$A'_\mu := A_\mu + \partial_\mu \epsilon. \tag{53}$$

Exercise 2.1. *Show that $\vec{E}$ and $\vec{B}$ can be obtained from the field strength $F_{\mu\nu}$ as*

$$E_i = F_{i0}, \;\; B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}. \tag{54}$$

Let us construct the completely skew-symmetric tensor:

$$T_{\lambda\mu\nu} := \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda}. \tag{55}$$

Show that this tensor is completely skew-symmetric and homogeneous Maxwell equations are equivalent to $T_{\lambda\mu\nu} = 0$.

Proof. One has

$$\begin{aligned} E_i &= \partial_i A_0 - \partial_0 A_i = -\partial_i \Phi - \frac{1}{c}\partial_t A_i \\ B_i &= \frac{1}{2}\epsilon_{ijk}(\partial_j A_k - \partial_k A_j) = \epsilon_{ijk}\partial_j A_k \equiv \mathrm{rot}(\vec{A})_i \end{aligned} \tag{56}$$

The fact that $T_{\lambda\mu\nu}$ is skew-symmetric follows from a general fact: if $S_{\lambda\mu\nu}$ is skew-symmetric in $\mu\nu$, the symbol $T_{\lambda\mu\nu} := S_{[\lambda\mu\nu]} \equiv S_{\lambda\mu\nu} + S_{\nu\lambda\mu} + S_{\mu\nu\lambda}$ is completely skew-symmetric. The only non-trivial components of $T_{\lambda\mu\nu}$ are

$$\begin{aligned} T_{123} &= \partial_1 F_{23} + \partial_3 F_{12} + \partial_2 F_{31} = \partial_i B_i = \mathrm{div}(\vec{B}), \\ \frac{1}{2}\epsilon_{ijk}T_{0jk} &= \frac{1}{2}\epsilon_{ijk}\{\partial_0 F_{jk} + \partial_k F_{0j} + \partial_j F_{k0}\} \\ &= \frac{1}{c}\partial_t B_i + \epsilon_{ijk}\partial_j E_k = \frac{1}{c}\partial_t B_i + \mathrm{rot}(\vec{E})_i. \end{aligned} \tag{57}$$

This proves the claim. ■

Let us examine the inhomogeneous Maxwell equations. One combines $(\rho, \vec{j})$ into a single quantity, 4-current $j^\mu := (c\rho, \vec{j})$.

Exercise 2.2. *Show that inhomogeneous Maxwell equations can be written as*

$$\partial_\mu F^{\mu\nu} + \frac{1}{c}j^\nu = 0. \tag{58}$$

where $F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\kappa}F_{\lambda\kappa}$.

Proof. This equation has four components. For $\nu = 0$, using the skew-symmetry of $F^{\mu\nu}$ and $F^{i0} = -F_{i0} = -E_i$, one finds

$$0 = \partial_i F^{i0} + \frac{1}{c}c\rho = -\,\mathrm{div}(\vec{E}) + \rho. \tag{59}$$

Next, note that the relation of $F_{\mu\nu}$ to $\vec{B}$ can be inverted, one finds $F^{ij} = F_{ij} = \epsilon_{ijk}B_k$. For each $k \in \{1, 2, 3\}$, one thus has

$$\begin{aligned} 0 &= \partial_\mu F^{\mu k} + \frac{1}{c}j^k = \partial_0 F^{0k} + \partial_j F^{jk} + \frac{1}{c}j^k \\ &= \frac{1}{c}\partial_t E_k + \partial_j(\epsilon_{jk\ell}B_\ell) + \frac{j^k}{c} = \frac{1}{c}\partial_t E_k - \epsilon_{kj\ell}\partial_j B_\ell + \frac{j^k}{c} \\ &= \frac{1}{c}\partial_t E_k - \mathrm{rot}(\vec{B})_k + \frac{j^k}{c}. \end{aligned} \tag{60}$$

This finishes the calculation. ■

Exercise 2.3. *Let $x'^\mu = L^\mu{}_\nu x^\nu$. We impose that 4-potential transforms as Lorentz vector, that is $A'^\mu(x') = L^\mu{}_\nu A^\nu(x)$.*

(i) How does A_μ transform?

(ii) Show that $F_{\mu\nu}$ transforms as a Lorentz covariant 2-tensor.

(iii) Show that Maxwell equations are invariant under Lorentz transformations, if j^μ forms a Lorentz 4-vector.

Proof. Suppose a^μ are components of the Lorentz vector. The relation $L^T \eta L = \eta$ can be also translated as $L^{-1} = \eta^{-1} L^T \eta$, that is $[L^{-1}]^\mu{}_\nu = \eta^{\mu\lambda} L^\kappa{}_\lambda \eta_{\kappa\nu} \equiv L_\nu{}^\mu$. Then one finds

$$a'_\mu = \eta_{\mu\nu} a'^\nu = \eta_{\mu\nu} L^\nu{}_\lambda a^\lambda = \eta^{\kappa\lambda} L^\nu{}_\lambda \eta_{\nu\mu} a_\kappa = L_\mu{}^\kappa a_\kappa. \tag{61}$$

Consequently, we immediately find that

$$A'_\mu(x') = L_\mu{}^\nu A_\nu(x). \tag{62}$$

We can now also easily deduce the transformation rules for partial derivatives:

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = [L^{-1}]^\nu{}_\mu \partial_\nu = L_\mu{}^\nu \partial_\nu. \tag{63}$$

It is now easy to find the transformation rule for $F_{\mu\nu}$, namely

$$\begin{aligned} F'_{\mu\nu}(x') &= \partial'_\mu A'_\nu(x') - \partial'_\nu A'_\mu(x') = L_\mu{}^\kappa L_\nu{}^\lambda \{\partial_\kappa A_\lambda(x) - \partial_\lambda A_\kappa(x)\} \\ &= L_\mu{}^\kappa L_\nu{}^\lambda F_{\kappa\lambda}(x) \end{aligned} \tag{64}$$

It is straightforward to write the transformation rule for $F^{\mu\nu}$, namely $F'^{\mu\nu}(x') = L^\mu{}_\kappa L^\nu{}_\lambda F^{\kappa\lambda}(x)$. We can thus for example verify that the inhomogeneous Maxwell equations transform as Lorentz vector. In particular, they are hold in all inertial frames at once:

$$\partial'_\mu F'^{\mu\nu}(x') = L_\mu{}^\lambda \partial_\lambda \{L^\mu{}_\kappa L^\nu{}_\chi F^{\kappa\chi}(x)\} = [L^{-1}]^\lambda{}_\mu L^\mu{}_\kappa L^\nu{}_\chi \partial_\lambda F^{\kappa\chi}(x) = L^\nu{}_\chi \partial_\lambda F^{\lambda\chi}(x). \tag{65}$$

Similarly, one can prove that $T_{\lambda\mu\nu}$ transforms as a covariant Lorentz 3-tensor. ■

2.2 Electrodynamics in more dimensions

The above description of electromagnetism allows for an easy generalization to an arbitrary dimension. We simply *declare* A^μ to be the potential, $\mu \in \{0, \ldots, D\}$, and the field strength is the corresponding 2-tensor $F_{\mu\nu}$ defined by the same formula.

Hence let $D \geq 4$, and write $d := D - 1$. Observe that one can still define the electric field, namely let $E_i := F_{i0}$ for $i \in \{1, \ldots, D\}$. Note that the magnetic field is no longer a vector field, but rather a (time-dependent) 2-form on $\mathbb{R}^d$.

Note that the zeroth component of the inhomogeneous Maxwell equation (58) still gives the Gauss law:

$$\partial_i E_i = \rho. \tag{66}$$

Suppose we want to find the electric field of a static point charge q at the origin. Let $r := \sqrt{(x^1)^2 + \cdots + (x^d)^2}$ be the radius. We can integrate both sides over the d-dimensional ball $B^d(r)$ around the origin. This gives

$$\int_{B^d(r)} \partial_i E_i \cdot \mathrm{d}vol = q. \tag{67}$$

The Stokes theorem is still valid - we may replace the integral by the flow of $\vec{E}$ over the boundary $\partial B^d(r) = S^{d-1}(r)$. Since we can assume that $\vec{E}$ is radial and depends only on r, one finds

$$\int_{B^d(r)} \partial_i E_i \cdot \mathrm{dvol} = \int_{S^{d-1}(r)} \vec{E}(r) \cdot \mathrm{d}\vec{S} = \mathrm{vol}(S^{d-1}(r)) \cdot E(r) = \frac{2\pi^{d/2} r^{d-1}}{\Gamma(\frac{d}{2})} E(r). \tag{68}$$

We thus find the electric field of a point charge in the form

$$E(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{q}{r^{d-1}}. \tag{69}$$

We see that in more space dimensions, electric field of the point charge falls of faster! In general, suppose that we have time-independent A_μ. Then

$$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i = -\partial_i \Phi, \tag{70}$$

that is $\vec{E} = -\operatorname{grad}(\Phi)$. Plugging this into Gauss law gives the **Poisson equation** $\Delta\Phi = -\rho$, where the Laplacian is defined accordingly as $\Delta = \sum_{i=1}^{d-1} \partial_i^2$.

Exercise 2.4. *Let $d \geq 1$. Prove that the volume of of the $(d-1)$-dimensional unit sphere $S^{d-1} = \{(x^1, \cdots, x^d) \in \mathbb{R}^d \mid (x^1)^2 + \cdots + (x^d)^2 = 1\}$ is given by the formula*

$$\mathrm{vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}, \tag{71}$$

where $\Gamma(x)$ is the function defined for each $x > 0$ by the integral

$$\Gamma(x) = \int_0^\infty \mathrm{d}t \; e^{-t} t^{x-1}. \tag{72}$$

Derive the formula for the general radius r. Prove that the volume of the d-dimensional unit ball is given by

$$\mathrm{vol}(B^d) = \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})}. \tag{73}$$

Proof. One proves the formula by evaluating the integral

$$I_d := \int_{\mathbb{R}^d} \mathrm{d}x^1 \cdots \mathrm{d}x^n e^{-r^2} \tag{74}$$

in two different ways. First, one can write this as a product of one-dimensional Gaussian integrals

$$I_d = \prod_{i=1}^{d} \int_{-\infty}^{\infty} \mathrm{d}x^i e^{-(x^i)^2} = (\sqrt{\pi})^d = \pi^{\frac{d}{2}}. \tag{75}$$

On the other hand, by a simple rescaling argument, one can argue that $\text{vol}(S^{d-1}(r)) = r^{d-1}\,\text{vol}(S^{d-1})$, where $S^{d-1}(r)$ denotes the $(d-1)$-dimensional sphere of radius r. One can calculate the integral I_d by dissecting $\mathbb{R}^d$ into thin spherical shells of radius r and thus

$$\begin{aligned} I_d = & \int_0^\infty \mathrm{d}r\, \text{vol}(S^{d-1}(r)) e^{-r^2} = \text{vol}(S^{d-1}) \cdot \int_0^\infty \mathrm{d}r\; r^{d-1} e^{-r^2} \\ = & \ \text{vol}(S^{d-1}) \cdot \frac{1}{2} \int_0^\infty \mathrm{d}t\; t^{\frac{d}{2}-1} e^{-t} = \text{vol}(S^{d-1}) \cdot \frac{1}{2} \Gamma(\frac{d}{2}). \end{aligned} \tag{76}$$

Comparing the two expressions gives the result. For the volume of the unit ball, one can certainly calculate it as

$$\text{vol}(B^d) = \int_0^1 \text{vol}(S^{d-1}(r)) \mathrm{d}r = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} [\frac{r^d}{d}]_0^1 = \frac{\pi^{\frac{d}{2}}}{\frac{d}{2}\Gamma(\frac{d}{2})} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}, \tag{77}$$

where in the last step, we have used the recurrence relation $\Gamma(x+1) = x \cdot \Gamma(x)$. This can be obtained immediately by using per partes for the evaluation of $\Gamma(x+1)$. It is easy to check that

$$\Gamma(1) = \int_0^\infty \mathrm{d}t\; e^{-t} = 1, \tag{78}$$

$$\Gamma(\frac{1}{2}) = \int_0^\infty \mathrm{d}t t^{\frac{1}{2}} e^{-t} = 2 \int_0^\infty \mathrm{d}r e^{-r^2} = \sqrt{\pi}. \tag{79}$$

This can be now easily used together with the above recurrence relation to prove the required volumes. E.g. for $d = 3$, one finds

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}, \text{ hence } \text{vol}(S^2(r)) = \frac{2\pi^{\frac{3}{2}}}{\frac{1}{2}\sqrt{\pi}} r^2 = 4\pi r^2. \tag{80}$$

This finishes the discussion. ■

2.3 Gravity and Planck's length

Recall that in general relativity, gravity is described by a metric tensor g. In some (local) coordinates, it can be written as

$$g = g_{\mu\nu}(x) \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \tag{81}$$

and the equations for g are given by the Einstein's field equations

$$R_{\mu\nu} + (\Lambda - \frac{1}{2}R) g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{82}$$

For the purposes of quantum theory (and classical limit), one assumes that $g_{\mu\nu}$ can be expanded as a fluctuation around a flat metric, that is

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{83}$$

Plugging this into (82) for $\Lambda = 0$ and $T_{\mu\nu} = 0$ and considering only terms linear in h gives the linearized equation for h, namely

$$\Box h^{\mu\nu} - \partial_\alpha(\partial^\mu h^{\nu\alpha} + \partial^\nu h^{\mu\alpha}) + \partial^\mu \partial^\nu h = 0, \tag{84}$$

where $h^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\kappa}h_{\lambda\kappa}$ and $h = \eta^{\mu\nu}h_{\mu\nu}$. This can be viewed as a gravitational analogue to the Maxwell equations without the presence of sources, since

$$0 = \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial_\mu\partial^\nu A^\mu. \tag{85}$$

Linearized gravity exhibits some similarities to electromagnetism. Indeed, suppose that we consider a coordinate transformation

$$x'^\mu := x^\mu - \epsilon^\mu(x), \tag{86}$$

for small ϵ^μ (and with small derivatives). In new coordinates, the metric tensor field is given by

$$g'_{\mu\nu}(x') = \frac{\partial x^\kappa}{\partial x'^\nu}\frac{\partial x^\lambda}{\partial x'^\mu}g_{\mu\nu}(x). \tag{87}$$

By plugging in the expansions around the flat metric and neglecting second order terms in ϵ, we find the transformation rule

$$h'^{\mu\nu}(x) = h^{\mu\nu}(x) + \partial^\mu\epsilon^\nu(x) + \partial^\nu\epsilon^\mu(x) + O(\epsilon, h) \equiv h^{\mu\nu}(x) + \delta_0 h^{\mu\nu}(x) + O(\epsilon, h). \tag{88}$$

where $O(\epsilon, h)$ contains terms linear in h and ϵ. Instead of viewing this as an expression of the same tensor in different coordinates, one can view this as a gauge transformation of the field $h^{\mu\nu}(x)$. In fact, it turns out that the equations (84) are (exactly!) invariant under the transformation

$$h'^{\mu\nu}(x) = h^{\mu\nu}(x) + \delta_0 h^{\mu\nu}(x) \tag{89}$$

Note that the field $h_{\mu\nu}(x)$ transforms as a covariant Lorentz 2-tensor under Lorentz transformations. This makes the analogy with electromagnetism complete.

Now, let us discuss dimensions. Newton's gravitation law in four dimensions says that the magnitude of force between two masses m_1 and m_2 separated by a distance r is given by

$$|\vec{F}^{(4)}| = \frac{Gm_1m_2}{r^2}. \tag{90}$$

For the dimensions, this gives us

$$[G] = [\text{Force}] \cdot \frac{L^2}{M^2} = \frac{ML}{T^2}\frac{L^2}{M^2} = \frac{L^3}{MT^2}. \tag{91}$$

Numerical values of the three fundamental fundamental constants G, c and $\hbar$ are

$$G = 6.674 \times 10^{-11}\frac{m^3}{kg \cdot s^2}, \quad c = 2.998 \times 10^8\frac{m}{s}, \quad \hbar = 1.055 \times 10^{-34}\frac{kg \cdot m^2}{s}. \tag{92}$$

One can attempt to find new units of length, mass and times, such that the numerical value of those constants in these units is *one*. These are called the Planck length ℓ_P, the Planck time t_P and the Planck mass m_P. We thus require

$$G = 1 \cdot \frac{\ell_P^3}{m_P \cdot t_P^2}, \quad c = 1 \cdot \frac{m_P}{t_P}, \quad \hbar = 1 \cdot \frac{m_P\ell_P^2}{t_P}. \tag{93}$$

To do so, let us plug these expressions into

$$(G)^\alpha(c)^\beta(\hbar)^\gamma = \ell_P^{3\alpha+\beta+2\gamma} \cdot m_P^{\gamma-\alpha} \cdot t_P^{-2\alpha-\beta-\gamma}. \tag{94}$$

There is a unique choice of parameters (α, β, γ) allowing us to express ℓ_P, m_P and t_P:

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.616 \times 10^{-33} cm, \tag{95}$$

$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-5} g, \tag{96}$$

$$t_P = \sqrt{\frac{G\hbar}{c^5}} = 5.391 \times 10^{-44} s. \tag{97}$$

This also shows that ℓ_P is the unique length, which can be expressed as a product of powers of fundamental constants $(G, c, \hbar)$.

Suppose we have a gravitational field $\vec{g}$. Force it exerts on test particle of mass m is given by $\vec{F} = m\vec{g}$. From Newton's gravitational law, one can deduce the **Gauss gravitational law**, in its differential form

$$\operatorname{div} \vec{g} = -4\pi G\rho, \tag{98}$$

where $\rho = \rho(\vec{x})$ is the mass density. Since gravity is a conservative force, one has $\vec{g} = -\operatorname{grad}(V_g)$, and the above equation becomes the Poisson equation for V_g, namely

$$\Delta V_g = 4\pi G\rho. \tag{99}$$

Let us discuss units. Since $[\vec{g}] = [\text{Force}] \cdot M^{-1} = L \cdot T^{-2}$, we have $[V_g] = L \cdot [\vec{g}] = L^2 \cdot T^{-2}$. Consequently, one has $[\Delta V_g] = T^{-2}$. Suppose that we want this to hold in arbitrary number of D spacetime dimensions, that is

$$\Delta V_g^{(D)} = 4\pi G^{(D)} \rho. \tag{100}$$

The left-hand side has the same dimension, but note that then

$$[G^{(D)}] = [\Delta V_g^{(D)}] \cdot [\rho]^{-1} = \frac{L^{D-1}}{MT^2} \tag{101}$$

One can now again define the D-dimensional Planck length $\ell_P^{(D)}$ as the one which can be expressed as a unique product of powers of $G^{(D)}$, c and $\hbar$. One finds

$$(\ell_P^{(D)})^{D-2} = \frac{\hbar G^{(D)}}{c^3} = \frac{\hbar G}{c^3} \cdot \frac{G^{(D)}}{G} = \ell_P^2 \cdot \frac{G^{(D)}}{G}, \tag{102}$$

where $G = G^{(4)}$ and $\ell_P = \ell_P^{(4)}$ are the ordinary quantities defined above.

2.4 Compact extra dimensions

Note that $[G^{(D)}/G] = L^{D-4}$, that is precisely the length to the power of *number of extra dimensions*. Recall that we have considered the idea that the extra dimensions are compact, that is "curled" up. They can thus in principle span the finite volume, the quantity precisely of the dimension L^{D-4}. This suggests that the ratio of gravitational constants may be in theory related to this volume. This can be verified by the following thought experiment.

Suppose we have three usual spatial directions (x^1, x^2, x^3), and one extra dimension x^4, which is assumed to be curled into a circle of radius R, that is we impose the identification $x^4 \sim x^4 + 2\pi R$. We suppose that in five-dimensional spacetime, the total mass M is distributed uniformly at the circle $x^1 = x^2 = x^3 = 0$. We thus have $M = 2\pi R m$, where m is the mass per unit length. For symmetry reasons, the resulting potential $V_g^{(5)}$ cannot depend on the coordinate x_4. The corresponding mass density can be written as

$$\rho^{(5)} = m \cdot \delta(x_1)\delta(x_1)\delta(x_2). \tag{103}$$

This is well normalized and has correct dimensions, since

$$\int_{\mathbb{R}^3} \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \int_0^{2\pi R} \mathrm{d}x^4 \rho^{(5)} = 2\pi R m = M. \tag{104}$$

Now, an effectively four-dimensional observer, this is observed as a point mass M at $(0,0,0)$, hence $\rho^{(4)} = M\delta(x^1)\delta(x^2)\delta(x^3)$. The 5-dimensional Gauss law takes the form

$$\Delta V_g^{(5)}(x^1, x^2, x^3) = 4\pi G^{(5)}\rho^{(5)} = 4\pi \frac{G^{(5)}}{2\pi R}\rho^{(4)}. \tag{105}$$

Since $V_g^{(5)}$ is for all purposes (forces on test masses) the effective potential in the 4-dimensional world, and Δ is effectively just a three-dimensional Laplacian, this has to be the Gauss law for the point mass M. But this shows that $G = \frac{G^{(5)}}{2\pi R} \equiv \frac{G^{(5)}}{\ell_C}$, where ℓ_C is the length of the extra compact dimension. This can be, with some grain of salt, generalized to

$$\frac{G^{(D)}}{G} = V_C, \tag{106}$$

where V_C is the volume of the extra dimensions. Suppose that $V_C = (\ell_C)^{D-4}$. Then one can express ℓ_C, that is the required length of compactified dimensions, in terms of the D-dimensional Planck length $\ell_P^{(D)}$ and our "effective" Planck length ℓ_P as

$$\ell_C = \ell_P^{(D)} (\frac{\ell_P^{(D)}}{\ell_P})^{\frac{2}{D-4}}. \tag{107}$$

It turns out that if ℓ_C is "sufficiently small", the fundamental length scale $\ell_P^{(D)}$ in more dimensions can be a lot bigger. For example, one can suppose that it is only a tiny bit smaller than today scope of experiments, say $\ell_P^{(D)} \approx 10^{-18} cm$. Then $\ell_C \approx 10^{\frac{30}{D-4}-18} cm$. Hence e.g. for $D = 10$, one has $\ell_C \approx 10^{-13} cm$.

3 Non-relativistic string

3.1 Equations of motion

In this section, we will consider a motion of non-relativistic string of length a in (x, y) plane, stretched along the x axis by tension T_0. We consider only infinitesimal transversal oscillations. We assume that the string does not stretch, that is the tension and mass density per unit length do not change, and the motion is fully described as a function $y = y(t, x)$, where $x \in [0, a]$.

By analyzing the forces exerted on infinitesimal pieces of the string, one arrives to the **wave equation**:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T_0}{\mu_0} \frac{\partial^2 y}{\partial x^2}, \tag{108}$$

where T_0 is the string tension and μ_0 is the mass density per unit length. Let $v_0 := \sqrt{T_0/\mu_0}$ be the corresponding phase velocity of the propagating waves.

One usually imposes two different kinds of boundary conditions at string endpoints $x \in \{0, a\}$.

(a) **Dirichlet boundary conditions**: This assumes that the endpoints of the string are fixed and their position is equal to zero, that is

$$y(t,0) = y(t,a) = 0, \tag{109}$$

for all $t \in \mathbb{R}$. In this case the solution can be found as a sum of modes in the form

$$y(x,t) = y_n(x)\sin(\omega_n t + \varphi_n), \tag{110}$$

for each $n \in \mathbb{N}$, where $\omega_n = v_0 \frac{n\pi}{a}$ and the function $y_n(x)$ has the form

$$y_n(x) = A_n \sin(\frac{n\pi x}{a}). \tag{111}$$

(b) **Neumann boundary conditions**: This assumes that the endpoints of the string are massless hoops which can slide along infinite poles. This requires

$$\frac{\partial y}{\partial x}(t,0) = \frac{\partial y}{\partial x}(t,a) = 0, \tag{112}$$

for all $t \in \mathbb{R}$. In this case, the shape of the solution in the n-th mode is given by

$$y_n(x) = A_n \cos(\frac{n\pi x}{a}), \tag{113}$$

and the equation also allows for a uniformly moving string $y(t,x) = a_0 t + y_0$.

Both boundary conditions can be also combined. The constants $\{A_n, \varphi_n\}_{n=1}^{\infty}$ must be determined by initial conditions. The general solution of the wave equation is of the d'Alembert form

$$y(t,x) = h_+(x - v_0 t) + h_-(x + v_0 t), \tag{114}$$

where $h_\pm = h_\pm(u)$ are functions of a single variable. By declaring the initial shape $y(x) = y(x,0)$ and initial velocity $v(x) = \frac{\partial y}{\partial t}(0,x)$, one can fully solve the equation. Indeed, one obtains the system of equations

$$\begin{aligned} y(x) &= h_+(x) + h_-(x), \\ v(x) &= -v_0 h'_+(x) + v_0 h'_-(x). \end{aligned} \tag{115}$$

One can express $h_-(x)$ in terms of $h_+(x)$ and a known function $y(x)$, plug it in the second one and solve the ordinary differential equation for $h_+(x)$. The second function $h_-(x)$ can be then calculated from the first equation.

Exercise 3.1. *Let us consider the Dirichlet string. Find the general solution using the procedure hinted above.*

Proof. First note that the above system in fact determines $h_\pm(x)$ only for $x \in [0,a]$. However, we may try to extend y and v to entire $\mathbb{R}$. Hence suppose that we have done so. Plugging from the first equation to the second equation gives

$$h'_+(x) = \frac{1}{2}y'(x) - \frac{v(x)}{2v_0} \tag{116}$$

This can be integrated uniquely up to an additive constant, which will play no role in the final solution, hence

$$h_+(x) = \frac{1}{2}y(x) - \int_0^x \frac{v(x)}{2v_0}. \tag{117}$$

Consequently, one finds $h_-(x) = \frac{1}{2}y(x) + \int_0^x \frac{v(x)}{2v_0}$. Hence the full solution is given by

$$y(t,x) = \frac{1}{2}y(x - v_0 t) + \frac{1}{2}y(x + v_0 t) + \int_{x-v_0 t}^{x+v_0 t} \frac{v(x)}{2v_0}\mathrm{d}x. \tag{118}$$

Let us now examine the boundary conditions. First, let $x = 0$. This gives us

$$0 = \frac{1}{2}y(-v_0 t) + \frac{1}{2}y(v_0 t) + \int_{-v_0 t}^{v_0 t} \frac{v(x)}{2v_0}\mathrm{d}x \tag{119}$$

We see that the simplest way to solve this condition is to assume that the extension of v to $\mathbb{R}$ is an *odd* function. This makes the integral to vanish and we realize that the extension of y to $\mathbb{R}$ must be *odd*. Now, plugging in the other boundary condition gives

$$0 = \frac{1}{2}y(a - v_0 t) + \frac{1}{2}y(a + v_0 t) + \int_{a-v_0 t}^{a+v_0 t} \frac{v(x)}{2v_0}\mathrm{d}x. \tag{120}$$

Since this has to hold for all $t \in \mathbb{R}$, we can change the variable to $u := v_0 t - a$. This gives us

$$0 = \frac{1}{2}y(-u) + \frac{1}{2}y(u + 2a) + \int_{-u}^{u+2a} \frac{v(x)}{2v_0}\mathrm{d}x. \tag{121}$$

Since we already know that both y and v are odd functions. we can rewrite this as

$$y(u) = y(u + 2a) + \int_{u}^{u+2a} \frac{v(x)}{v_0}\mathrm{d}x. \tag{122}$$

We see that to get rid of the integral, we may assume that $v(x)$ is periodic with period $2a$. Then we get

$$\int_{u}^{u+2a} \frac{v(x)}{v_0}\mathrm{d}x = \int_{-a}^{a} \frac{v(x)}{v_0}\mathrm{d}x = 0. \tag{123}$$

Finally, we see that y simply has to be periodic with period $2a$. Note that this also forces $y(a) = 0$, which is in accordance with the fact that $y(x) = y(0, x)$ should satisfy the boundary conditions. This gives us a general answer:

For any initial conditions $y = y(x)$, $v = v(x)$, where $x \in [0, a]$, find their unique odd extensions periodic with period $2a$. Then one can write the solution using (118). ■

3.2 Lagrangian mechanics of a string

Now, let us suppose we want to write down the Lagrangian for the string defined in the previous section. Note that one should view it as a system with infinitely many degrees of freedom whose coordinates are labeled by $x \in [0, a]$. We expect the Lagrangian for a given string configuration $y = y(t, x)$ be a function of time, given by the difference of the overall kinetic and potential energy at a given time t:,

$$L(t) = T(t) - V(t), \tag{124}$$

The kinetic energy is simply the sum of kinetic energies of inifinitesimal pieces of string:

$$T(t) = \int_0^a \mathrm{d}x\ \mu_0 (\frac{\partial y}{\partial t})^2(t,x) \tag{125}$$

Now, the work which has to be to stretch the infinitesimal piece of string to its configuration is T_0 times the change of length, one finds

$$\begin{aligned} &T_0\ \big(\sqrt{(\mathrm{d}x^2 + y(t,x+\mathrm{d}x) - y(t,x))^2} - \mathrm{d}x\big) = T_0\big(\sqrt{1 + (\frac{\partial y}{\partial x})^2(t,x)} - 1)\mathrm{d}x \\ &= \frac{1}{2} T_0 (\frac{\partial y}{\partial x})^2(t,x). \end{aligned} \tag{126}$$

The overall potential energy of the string is thus the sum of these, that is

$$V(t) = \int_0^a \mathrm{d}x\ T_0 (\frac{\partial y}{\partial x})^2(t,x)\mathrm{d}x \tag{127}$$

We thus propose the string Lagrangian in the form

$$L(t) = \int_0^a [\frac{1}{2}\mu_0(\frac{\partial y}{\partial t})^2 - \frac{1}{2}T_0(\frac{\partial y}{\partial x})^2]\ \mathrm{d}x \equiv \int_0^a \mathcal{L}\ \mathrm{d}x, \tag{128}$$

where we drop the explicit writing of the arguments and define the **Lagrangian density** as

$$\mathcal{L}(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}) := \frac{1}{2}\mu_0(\frac{\partial y}{\partial t})^2 - \frac{1}{2}T_0(\frac{\partial y}{\partial x})^2. \tag{129}$$

Strictly speaking, $\mathcal{L}$ is an ordinary function of two variables, which we compose with the partial derivatives of the field $y = y(t,x)$, and integrate the resulting function of (t,x) over $x \in [0,a]$.

The action functional S has the function $y = y(t,x)$ as its dynamical variable, and

$$S[y] := \int_{t_i}^{t_f} L(t)\mathrm{d}t = \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x\ [\frac{1}{2}\mu_0(\frac{\partial y}{\partial t})^2 - \frac{1}{2}T_0(\frac{\partial y}{\partial x})^2]. \tag{130}$$

Now, we expect to obtain the equations of motion by Hamilton's principle. In other words, the actual motion in the time interval $[t_i, t_f]$ should extremalize the action functional. We find it by performing an infinitesimal variation

$$y'(t,x) = y(t,x) + \epsilon \cdot \delta y(t,x). \tag{131}$$

One finds

$$S[y'] = S[y] + \epsilon \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x\ [\mu_0 \frac{\partial y}{\partial t}\frac{\partial(\delta y)}{\partial t} - T_0 \frac{\partial y}{\partial x}\frac{\partial(\delta y)}{\partial x}] + O(\epsilon^2). \tag{132}$$

Let us denote the term proportional to ϵ as δS. We can get rid of partial derivatives of variations by performing the respective per parts integration. One gets

$$\begin{aligned} \delta S = &-\int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x\ [\mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2}] \cdot \delta y \\ &+ \int_0^a \mathrm{d}x\ [\mu_0 \frac{\partial y}{\partial t}\delta y]_{t=t_i}^{t=t_f} \\ &+ \int_{t_i}^{t_f} \mathrm{d}t\ [-T_0 \frac{\partial y}{\partial x}\delta y]_{x=0}^{x=a}. \end{aligned} \tag{133}$$

Now, since we can choose δy to be a bump function around any point $(t, x) \in (t_i, t_f) \times (0, a)$, the function multiplying δy in the first term must vanish. This gives use the corresponding Lagrange-Euler equation, which indeed happens to be the wave equation. Next, we always consider only the variations which do not change the initial and final configuaration of the string, that is $\delta y(t_i, x) = \delta y(t_f, x) = 0$ for all $x \in [0, a]$. This make the second term to disappear.

We see that $\delta S = 0$ for any $y = y(t, x)$ describing the movement of of the string must satisfy

$$0 = \int_{t_i}^{t_f} \mathrm{d}t[-T_0 \frac{\partial y}{\partial x}(t, a)\delta y(t, a) + T_0 \frac{\partial y}{\partial x}(t, 0)\delta y(t, 0)]. \tag{134}$$

Let $x_* \in \{0, a\}$ be the generic notion for the endpoint of the string.

1. If we allow an arbitrary motion and variation of a given endpoint x_*, we are forced to impose the Neumann boundary condition

$$\frac{\partial y}{\partial x}(t, x_*) = 0, \tag{135}$$

   for all $t \in [t_i, t_f]$.

2. We can impose the Dirichlet boundary condition at a given endpoint x_*, which forces $y(t, x_*) = 0$. In particular, this forces $\delta y(t, x_*) = 0$ and the corresponding term above vanishes. It is convenient to write the Dirichlet condition as

$$\frac{\partial y}{\partial t}(t, x_*) = 0, \tag{136}$$

   although the position of the endpoint has to be specified (e.g. by initial conditions).

Let us demonstrate the physical significance of the boundary conditions. The transversal momentum of the string is given by the sum of the momenta carried by infinitesimal elements of the string, that is

$$p_y(t) = \int_0^a \mu_0 \frac{\partial y}{\partial t}(t, x)\, \mathrm{d}x. \tag{137}$$

We can tackle the question of its conservation. One has

$$\frac{\mathrm{d}}{\mathrm{d}t} p_y(t) = \int_0^a \mu_0 \frac{\partial^2 y}{\partial t^2} \mathrm{d}x = \int_0^a T_0 \frac{\partial^2 y}{\partial x^2} \mathrm{d}x = T_0 \Big(\frac{\partial y}{\partial x}(t, a) - \frac{\partial y}{\partial x}(t, 0) \Big). \tag{138}$$

This means that for a string not satisfying the Neumann boundary condition, the overall momentum of the strung is *not* conserved. This is not unphysical - e.g. for a string with fixed endpoints, the "wall" exerts a force to its endpoints - the momentum flows out of and back in the string. Let us finish this section by rewritting the above calculation slightly differently, mainly for the future purposes. Write $\dot{y} := \frac{\partial y}{\partial t}$ and $y' := \frac{\partial y}{\partial x}$. Then $\mathcal{L} = \mathcal{L}(\dot{y}, y')$. Let

$$\mathcal{P}^t := \frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu_0 \dot{y}, \;\; \mathcal{P}^x := \frac{\partial \mathcal{L}}{\partial y'} = -T_0 y'. \tag{139}$$

$\mathcal{P}^t$ can be viewed as a **momentum density** corresponding to the variable y. The variation of the action can be then written as

$$\begin{aligned}\delta S = & \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x \, [\frac{\partial \mathcal{L}}{\partial \dot{y}} \delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'} \delta y'] = \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x \, [\mathcal{P}^t \delta \dot{y} + \mathcal{P}^x \delta y'] \\ = & - \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x \, [\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x}] \delta y \\ & + \int_0^a \mathrm{d}x \, [\mathcal{P}^t \delta y]_{t=t_i}^{t=t_f} + \int_{t_i}^{t_f} [\mathcal{P}^x \delta y]_{x=0}^{x=a}.\end{aligned} \tag{140}$$

We see that the Lagrange-Euler equation can be written as

$$\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0, \tag{141}$$

the Neumann boundary condition at the given endpoint $x_* \in \{0, a\}$ then reads $\mathcal{P}^x(t, x_*) = 0$, and for a string satisfying the Dirichlet condition at a given endpoint, we have $\mathcal{P}^t(t, x_*) = 0$.

Exercise 3.2. *Let $y(t, x) = A \sin(k_n x) \cdot \cos(\omega_n t + \varphi)$ be the n-th mode of the Dirichlet string. Calculate the corresponding momentum $p_y(t)$.*

Proof. Recall that $k_n = \frac{n\pi}{a}$ and $\omega_n = v_0 k_n$, where $v_0 = \sqrt{T_0/\mu_0}$. Then

$$\mathcal{P}^t(t, x) = \mu_0 \frac{\partial y}{\partial t} = -\mu_0 A \omega_n \cdot \sin(k_n x) \sin(\omega_n t + \varphi). \tag{142}$$

By integrating this over $x \in [0, a]$, we obtain

$$p_y(t) = \frac{\mu_0 A \omega_n}{k_n} \sin(\omega_n t + \varphi) \cdot [\cos(k_n x)]_0^a = A \cdot \sqrt{T_0 \mu_0} [(-1)^{n+1} - 1] \sin(\omega_n t + \varphi). \tag{143}$$

We see that the momentum of even modes is conserved, whereas the momentum of odd modes is not. ∎

Exercise 3.3. *Let us consider a closed string wrapped around an infinite cylinder of radius R, such that it can move without friction along its axis. This amounts to considering the identification $x \sim x + 2\pi R$. The equation of motion for $y = y(t, x)$ is still the wave equation with a general d'Alembert solution*

$$y(t, x) = h_+(x - v_0 t) + h_-(x + v_0 t). \tag{144}$$

(i) What conditions must be imposed on y? What are the corresponding periodicity conditions on derivatives of $h_\pm$?

(ii) Show that one can write $h_+(u) = \alpha u + f(u)$ and $h_-(u) = \beta u + g(u)$, where f and g are periodic functions and $\alpha, \beta \in \mathbb{R}$. What is the relation of α and β?

(iii) Calculate the momentum carried by the string in the y direction. Is it conserved?

Proof. We must obviously impose the periodicity condition $y(t, x) = y(t, x + 2\pi R)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$. But this implies also the periodicity of both its partial derivatives:

$$\dot{y}(t, x) = \dot{y}(t, x + 2\pi R), \quad y'(t, x) = y'(t, x + 2\pi R). \tag{145}$$

One finds

$$\dot{y}(t,x) = -v_0 h'_+(x - v_0 t) + v_0 h'_-(x + v_0 t), \tag{146}$$
$$y'(t,x) = h'_+(x - v_0 t) + h'_-(x + v_0 t). \tag{147}$$

Plugging those into above periodicity conditions at $t = 0$ gives the system

$$-h'_+(x + 2\pi R) + h'_-(x + 2\pi R) = -h'_+(x) + h'_-(x) \tag{148}$$
$$h'_+(x + 2\pi R) + h'_-(x + 2\pi R) = h'_+(x) + h'_-(x). \tag{149}$$

Taking the sum and the difference of these two equations yields

$$h'_\pm(x + 2\pi R) = h'_\pm(x). \tag{150}$$

It is easy to see that this already implies (145). This concludes (i). Next, let $h_+(u)$ be a primitive of a periodic function $h'_+(u)$, hence we can write it as

$$h_+(u) = \int_{u_0}^{u} h'_+(v)\mathrm{d}v, \tag{151}$$

for some $u_0 \in \mathbb{R}$. Let $\alpha := \frac{1}{2\pi R}\int_0^{2\pi R} h'_+(v)\mathrm{d}v$. Then the function

$$f(u) := h_+(u) - \alpha u \tag{152}$$

is easily checked to be periodic in u with period $2\pi R$. This is because α is chosen precisely so that $h_+(u+2\pi R) = h_+(u)+(2\pi R)\alpha$. This shows that we can write $h_\pm(u)$ as in (ii). By plugging in, one finds

$$y(t,x) = \alpha(x - v_0 t) + \beta(x + v_0 t) + f(x - v_0 t) + g(x + v_0 t). \tag{153}$$

By plugging into the periodicity condition at $t = 0$ now immediately gives $\alpha = -\beta$. In conclusion, one can write the most general solution of the motion of the closed string as

$$y(t,x) = vt + f(x - v_0 t) + g(x + v_0 t), \tag{154}$$

where v is an arbitrary constant having the dimension of velocity, and $f = f(u)$ and $g = g(u)$ are completely arbitrary differentiable functions with the period $2\pi R$. The transversal momentum reads

$$\begin{aligned} p_y(t) &= \int_0^{2\pi R} \mu_0 \dot{y}(t,x)\,\mathrm{d}x = \mu_0 \int_0^{2\pi R} [v - v_0 f'(x - v_0 t) + v_0 g'(x + v_0 t)]\mathrm{d}x \\ &= (2\pi R)\mu_0 v + [-v_0 f(x - v_0 t) + v_0 g(x + v_0 t)]_{x=0}^{x=2\pi R} \\ &= Mv, \end{aligned} \tag{155}$$

where $M = (2\pi R)\mu_0$ is the overall mass of the string. And yes, p_y is conserved. ■

4 Relativistic free particle

4.1 Action functional

Suppose we want to find an action function describing the motion of a free massive particle in D-dimensional spacetime. We want the resulting equations of motion to be Lorentz invariant in the following sense. If one Lorentz observer concludes that particle obeys equations of motion in his frame (it is "physical"), every other Lorentz observer must come to the same conclusion.

One idea is to define an action functional which becomes a Lorentz scalar. Since the motion happens along a path which is a stationary point of the action, both observes will agree on that stationary point, regardless of its coordinate description.

Let $x^\mu(t)$ be a worldline of the particle, connecting the initial point $x^\mu(t_i)$ and the final point $x^\mu(t_f)$, where $t_i < t_f$. We have already constructed a Lorentz scalar out of these data, namely the proper time $s(t_i, t_f)$! Note that the dimension of the action is $[S] = [\text{Energy}] \cdot T$, that is

$$[S] = ML^2T^{-1} = [\hbar]. \tag{156}$$

To obtain correct dimensionality, we must multiply $s(t_i, t_f)$ by some constant having the dimension of energy. We thus propose

$$S := -mc^2 s(t_i, t_f) = -mc^2 \int_{t_i}^{t_f} \sqrt{1 - \frac{v^2}{c^2}}\, \mathrm{d}t. \tag{157}$$

We see that the Lagrangian of the theory is given by

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}. \tag{158}$$

Note that in a fixed frame, the dynamical variables are spatial coordinates of the particle. This obscures the Lorentz invariance of the theory. It is much more convenient to consider the *arbitrary* parametrization of the worldline $x^\mu = x^\mu(\tau)$, the only requirement being that the value of the parametr in $[\tau_i, \tau_f]$ strictly increases between as the world-line goes from the initial point x_i^μ to the final point x_f^μ. The action functional S is now written as

$$S[x^\mu] = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}} \mathrm{d}\tau, \tag{159}$$

and its dynamical variables are functions $x^\mu = x^\mu(\tau)$ for $\mu \in \{0, \dots, D\}$.

Exercise 4.1. *(i) Prove that S is reparametrization invariant.*

(ii) Prove that S is a Lorentz scalar.

(iii) Check that if we choose the parameter to be the coordinate time, we obtain the action above.

Proof. The calculation is completely the same as in Exercise 1.1. ■

4.2 Equations of motion

Let us calculate the equations of motion from the Hamilton's principle. Let us consider a variation $x'^\mu(\tau) = x^\mu(\tau) + \epsilon \cdot \delta x^\mu(\tau)$ satisfying $\delta x^\mu(\tau_i) = \delta x^\mu(\tau_f) = 0$.

To do so, let us use the shorthand notation $\dot{x}^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ and note that we implicitly assume that $(\dot{x})^2 = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu < 0$. Then the action takes the form $S[x^\mu] = -mc \int_{\tau_i}^{\tau_f} \sqrt{-(\dot{x})^2} \mathrm{d}\tau$. Then

$$(\dot{x}')^2 = (\dot{x})^2 + 2\epsilon \eta_{\mu\nu} \dot{x}^\mu \delta \dot{x}^\nu + O(\epsilon^2), \tag{160}$$

where we write $\delta\dot{x}^\nu = \frac{\mathrm{d}}{\mathrm{d}\tau}\delta x^\nu$. We can thus write

$$-(\dot{x}')^2 = -(\dot{x})^2(1 - 2\epsilon\frac{\dot{x}_\mu\delta\dot{x}^\mu}{-(\dot{x})^2} + O(\epsilon^2)) \tag{161}$$

Taking the square root of this equation and using $\sqrt{1-y} \approx 1 - \frac{1}{2}y$, we arrive to

$$\sqrt{-(\dot{x}')^2} = \sqrt{-(\dot{x})^2} - \epsilon\frac{\dot{x}_\mu\delta\dot{x}^\mu}{\sqrt{-(\dot{x})^2}} + O(\epsilon^2). \tag{162}$$

Plugging this into the action, we get that $S[x'^\mu] = S[x^\mu] + \epsilon \cdot \delta S + O(\epsilon^2)$, where

$$\delta S = mc\int_{\tau_i}^{\tau_f} \frac{\dot{x}_\mu\delta\dot{x}^\mu}{\sqrt{-(\dot{x})^2}}\mathrm{d}\tau \tag{163}$$

Since we assume $\delta x^\mu(t_i) = \delta x^\mu(t_f) = 0$, we can use the integration by parts to move the τ derivative, obtaining the expression

$$\delta S = -mc\int_{\tau_i}^{\tau_f} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\frac{\dot{x}_\mu}{\sqrt{-(\dot{x})^2}}\Big) \cdot \delta x^\mu \mathrm{d}\tau. \tag{164}$$

From this we already obtain the equations of motion in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(mc\frac{\dot{x}_\mu}{\sqrt{-(\dot{x})^2}}\Big) = 0. \tag{165}$$

Exercise 4.2. *Show that the equations (165) are reparametrization and Lorentz invariant.*

Do the above equations agree with our expectactions? To see this, let us observe how four-velocity looks in the general parametrization of the worldline. Recall that the proper time can be expressed in terms of τ as

$$s(\tau) = \int_{\tau_i}^{\tau} \frac{1}{c}\sqrt{-(\dot{x})^2}\,\mathrm{d}\tau, \text{that is } \frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{1}{c}\sqrt{-(\dot{x})^2}. \tag{166}$$

Consequently, one finds

$$u_\mu = \frac{\mathrm{d}x_\mu}{\mathrm{d}s} = \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{c\dot{x}_\mu}{\sqrt{-(\dot{x})^2}}. \tag{167}$$

But this means that the above equation (165) can be rewritten simply as

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = 0. \tag{168}$$

This is fully with our expectations for the movement of a relativistic free particle.

Exercise 4.3. *What is the canonical momentum associated with the coordinate $x^\mu = x^\mu(\tau)$?*

Proof. The Lagrangian of the theory is given by

$$L(\dot{x}) = -mc\sqrt{-(\dot{x})^2}. \tag{169}$$

The canonical momentum associated with x^μ is

$$p_\mu := \frac{\partial L}{\partial\dot{x}^\mu} = -mc\frac{1}{2\sqrt{-(\dot{x})^2}}\frac{\partial}{\partial\dot{x}^\mu}(-\eta_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda) = \frac{mc}{\sqrt{-(\dot{x})[2}}\eta_{\mu\lambda}\dot{x}^\lambda = \frac{mc\dot{x}_\mu}{\sqrt{-(\dot{x})^2}}. \tag{170}$$

We see that the canonical momentum corresponds to the actual "physical" four-momentum. ■

4.3 Relativistic particle with electric charge

Suppose that our relativistic particle has an electric charge q and moves through the electromagnetic field $F_{\mu\nu}$. First observe that the Lorentz force acting on the particle can be compactly written as a Lorentz covector:

$$F^L_\mu = \frac{q}{c} F_{\mu\nu} u^\nu. \tag{171}$$

The equation of motion for a charged particle moving in the field $F_{\mu\nu}$ can be written as

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}s} = \frac{q}{c} F_{\mu\nu}(x) u^\nu = \frac{q}{mc} F_{\mu\nu}(x) p^\nu. \tag{172}$$

Exercise 4.4. *Examine the components F^L_μ in terms of coordinate velocities and usual Maxwell fields in four dimensions. What is the content of (172) in the zeroth component?*

Proof. Recall that in a Lorentz frame (ct, x, y, z), $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \gamma(t) \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = (c\gamma(t), \gamma(t)\vec{v})$. One has

$$F^L_0 = \frac{q}{c} F_{0i} u^i = -\gamma(t) \frac{q}{c} E_i v_i = -\gamma(t) \frac{q}{c} \vec{E} \cdot \vec{v}. \tag{173}$$

$$F^L_i = \frac{q}{c} F_{i\nu} u^\nu = \frac{q}{c} E_i u^0 + \frac{q}{c} \epsilon_{ijk} B_k u^j = \gamma(t) q (\vec{E} + \frac{1}{c} \vec{v} \times \vec{B})_i \tag{174}$$

Hence, up to the $\gamma(t)$ factor, we recover the standard Lorentz force. Finally, observe that

$$\frac{\mathrm{d}p_0}{\mathrm{d}s} = -\gamma(t) \frac{1}{c} \frac{\mathrm{d}E}{\mathrm{d}t} \tag{175}$$

Ther zeroth component of (172) thus gives the equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = q \vec{E} \cdot \vec{v}, \tag{176}$$

which shows that the energy of the particle changes by the power exerted by the Lorentz force acting on the particle. ∎

Let us now attempt to add the interaction to the action. We again want to produce the Lorentz scalar, the first idea which comes to mind is the contraction of the form $A_\mu u^\mu$, where A_μ is the four-potential for $F_{\mu\nu}$. We thus propose, in an arbitrary parametrization:

$$S_I[x] := \frac{q}{c} \int_{\tau_i}^{\tau_f} A_\mu(x(\tau)) \, \dot{x}^\mu(\tau) \mathrm{d}\tau. \tag{177}$$

The entire action is thus given by $S = S_0 + S_I$, where S_0 is the original free particle action. Let us find the variation of the action:

$$\begin{aligned} S_I[x'] &= \frac{q}{c} \int_{\tau_i}^{\tau_f} A_\mu(x + \epsilon\delta x) \cdot (\dot{x}^\mu + \epsilon\delta\dot{x}^\mu) \mathrm{d}\tau \\ &= S_I[x] + \frac{q\epsilon}{c} \int_{\tau_i}^{\tau_f} [(\partial_\nu A_\mu)(x) \delta x^\nu \dot{x}^\mu + A_\nu(x) \delta\dot{x}^\nu] \, \mathrm{d}\tau + O(\epsilon^2). \end{aligned} \tag{178}$$

Using the integration by parts and $\frac{\mathrm{d}}{\mathrm{d}\tau} A_\nu(x) = (\partial_\mu A_\nu)(x) \dot{x}^\mu$, we find that

$$\delta S_I = \int_{\tau_i}^{\tau_f} \frac{q}{c} F_{\mu\nu}(x) \dot{x}^\nu \delta x^\mu \mathrm{d}\tau = \int_{\tau_i}^{\tau_f} F^L_\mu(x, \dot{x}) \delta x^\mu \mathrm{d}\tau. \tag{179}$$

Since the variation of the free particle action takes the form

$$\delta S_0 = \int_{\tau_i}^{\tau_f} -\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\delta x^\mu \mathrm{d}\tau, \tag{180}$$

we indeed obtain the equation of motion (172).

Exercise 4.5. *We want to promote the electromagnetic field A_μ to the dynamic variable. To do so, we introduce the action*

$$S[x, A] := S_0[x] + S_I[x, A] + S_{EM}[A], \tag{181}$$

where the kinetic term for the action is given by

$$S_{EM}[A] := -\frac{1}{4c}\int \mathrm{d}^D x\ F_{\mu\nu}F^{\mu\nu}. \tag{182}$$

Find the equations of motion of this action. What role does the particle $x^\mu = x^\mu(\tau)$ play?

Proof. One considers the variation $A'_\mu = A_\mu + \epsilon \cdot \delta A_\mu$. It is a straightforward calculation that

$$S_{EM}[A'] = S_{EM}[A] - \frac{\epsilon}{c}\int \mathrm{d}^D x\ F^{\mu\nu}\partial_\mu(\delta A_\nu) + O(\epsilon^2) \tag{183}$$

One can now perform the integration by parts, assuming the δA_ν disappears in the inifinities. This gives the variation of the kinetic term in the action

$$\delta S_{EM} = \frac{1}{c}\int \mathrm{d}^D x\ \partial_\mu F^{\mu\nu} \cdot \delta A_\nu. \tag{184}$$

To calculate the variation of the term $S_I[x, A] = \frac{q}{c}\int_{\tau_i}^{\tau_f} A_\mu(x(\tau))\dot{x}^\mu(\tau)\ \mathrm{d}\tau$, one first has to find a spacetime integral there. We will do this by inserting a delta function:

$$\begin{aligned} S_I[x, A] &= \frac{q}{c}\int_{\tau_i}^{\tau_f} A_\mu(x(\tau))\dot{x}^\mu(\tau)\mathrm{d}\tau \\ &= \frac{q}{c}\int_{\tau_i}^{\tau_f}\int \mathrm{d}^D x\ A_\mu(x)\delta^D(x - x(\tau))\dot{x}^\mu(\tau)\mathrm{d}\tau \\ &= \frac{q}{c}\int \mathrm{d}^D x\ A_\mu(x)\int_{\tau_i}^{\tau_f} \delta^D(x - x(\tau))\dot{x}^\mu(\tau)\mathrm{d}\tau \end{aligned} \tag{185}$$

It is now easy to calculate the variation of the action under the variation of A_μ. One finds

$$\delta S_I = \frac{q}{c}\int \mathrm{d}^D x\ [\int_{\tau_i}^{\tau_f} \delta^D(x - x(\tau))\dot{x}^\nu(\tau)\mathrm{d}\tau] \cdot \delta A_\nu. \tag{186}$$

This suggests to define a four-current $j^\nu(x)$ as

$$j^\nu(x) := qc\int_{\tau_i}^{\tau_f} \delta^D(x - x(\tau))\dot{x}^\nu(\tau)\mathrm{d}\tau \tag{187}$$

The resulting Lagrange-Euler equation for the electromagnetic potential $A_\mu = A_\mu(x)$ is thus precisely the Maxwell equation

$$\partial_\mu F^{\mu\nu} + \frac{1}{c}j^\nu = 0. \tag{188}$$

What is the interpretation of j^μ. Suppose we choose τ to be the coordinate time in a given Lorentz frame. Then $x^\mu(\tau) = (c\tau, \vec{x}(\tau))$. It is convenient to write the D-dimensional delta function as

$$\delta^D(x - x(\tau)) = \delta(x^0 - c\tau) \cdot \delta^d(\vec{x} - \vec{x}(\tau)) = \frac{1}{c}\delta(\frac{x^0}{c} - \tau) \cdot \delta^d(\vec{x} - \vec{x}(\tau)). \tag{189}$$

If we consider $t \in (\tau_i, \tau_f)$, one thus obtains

$$j^0(t, \vec{x}) = cq \cdot \delta^d(\vec{x} - \vec{x}(t)). \tag{190}$$

Since $j^0 = c \cdot \rho$, this shows that $\rho(t, \vec{x}) = q \cdot \delta^d(\vec{x} - \vec{x}(t))$. This is a charge density of a single point charge. For spatial components, one obtains

$$j^k(t, \vec{x}) = q \cdot \delta^d(\vec{x} - \vec{x}(t))\dot{x}^k(t). \tag{191}$$

But this is precisely the current density for a point charge moving with a velocity $\vec{x} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$. ■

Exercise 4.6. *Find a Hamiltonian formulation of the relativistic particle. Verify that Hamilton equations of motion give the correct result.*

Proof. We have to work in a given Lorentz frame (ct, x, y, z). The transition to the Hamiltonian formulation is not Lorentz covariant. We have

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}. \tag{192}$$

The Legandre transformation is done with respect to spatial velocities and momenta, that is

$$H = \frac{\partial L}{\partial v^i}v^i - L, \tag{193}$$

and we have to write v^i as functions of positions and momenta to obtain the Hamiltonian. We have already shown that

$$p_i := \frac{\partial L}{\partial v^i} = \frac{mv^i}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{194}$$

hence one finds the formula

$$H = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + mc^2\sqrt{1 - \frac{v^2}{c^2}} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{195}$$

This is the expression for a relativistic energy of the free particle. Instead of trying to express v^i explicitly on terms of p_i, we just remember the energy-momentum constraint

$$E^2 = m^2c^4 + (\vec{p} \cdot \vec{p})c^2. \tag{196}$$

Consequently, we obtain the Hamiltonian in the form

$$H(\vec{x}, \vec{p}, t) = c\sqrt{m^2c^2 + \vec{p} \cdot \vec{p}} \tag{197}$$

This is not Lorentz covariant in any sense (it is a zero component of 4-momentum). What are the Hamilton equations? Recall that they have the form

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{198}$$

Since the Hamiltonian is independent of the positions, we get immediately $\dot{p}_i = 0$ and the second equation gives the expression of velocities in terms of momenta:

$$\dot{x}^i = \frac{cp^i}{\sqrt{m^2c^2 + \vec{p} \cdot \vec{p}}} = \frac{c^2 p^i}{E}. \tag{199}$$

This is a known fact that spatial velocity can be expressed as a ration $\dot{x}^i = c\frac{p^i}{p^0}$. ∎

Exercise 4.7. *Viewing the relativistic particle as a field theory, one can consider the "field theoretic" Hamiltonian (density), defined by*

$$H = \frac{\partial L}{\partial \dot{x}^\mu} \dot{x}^\mu - L. \tag{200}$$

What does it give for $L = -mc\sqrt{-(\dot{x})^2}$? Why do you think this happens?

5 Relativistic string

5.1 Nambu-Goto action

Observe that the action for the relativistic particle can be interpreted as follows. We have a curve $x^\mu(\tau)$. The action then simply measures its length between two events, $x_i^\mu = x^\mu(\tau_i)$ and $x_f^\nu = x^\nu(\tau_f)$. We can view the curve as a map $x : \mathbb{R} \to \mathbb{R}^D$. One then pulls back the "target space" Minkowski metric η to construct a new metric $g = x^*(\eta)$ on $\mathbb{R}$. Viewing τ as a coordinate function on $\mathbb{R}$, we have simply $g = g(\tau)\, \mathrm{d}\tau \otimes \mathrm{d}\tau$, where

$$g = \eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{201}$$

"Measuring the length" then corresponds to finding the volume of the interval $[\tau_i, \tau_f]$ using the volume form corresponding to g. Note that for general x, g is not a metric. This is where physics comes in - we are interested in a movement of a massive particle, so we only consider curves with *timelike* tangent vectors, since the particle cannot move faster then light. This is equivalent to

$$g(\tau) < 0, \tag{202}$$

for all $\tau \in [\tau_i, \tau_f]$. The corresponding volume form is then $\sqrt{-g(\tau)}\mathrm{d}\tau$, so the volume of $[\tau_i, \tau_f]$ is indeed given by the integral.

$$\int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \mathrm{d}\tau. \tag{203}$$

Now, let us attempt to describe the motion of a string in a D-dimensional spacetime.

It will be described by a D-tuple of functions $X^\mu = X^\mu(\tau, \sigma)$ of two parameters. τ will play the role of a "time parameter" and σ will describe the position on the string. In a more mathematical language, we shall consider embeddings $X : \Sigma \to \mathbb{R}^D$ from a 2-dimensional parameter manifold Σ to the Minkowski spacetime $\mathbb{R}^D$. (τ, σ) are (possibly local) coordinates on Σ and

$$X^\mu(\tau, \sigma) = x^\mu(X(\tau, \sigma)). \tag{204}$$

The image of Σ under X is called the *worldsheet of the string.* To make things confusing, Σ is also usually called a worldsheet.

There are thus some restrictions to X. We assume that each point of the worldsheet is (at least locally) uniquely described by parameters (τ, σ). Recall that one can consider the tangent vectors to the worldsheet, pointing in the "coordinate" directions. From a differential geometry course, you know that their components with respect to the coordinates x^μ on $\mathbb{R}^D$ are given by

$$\frac{\partial X^\mu}{\partial \tau}(\tau, \sigma), \quad \frac{\partial X^\mu}{\partial \sigma}(\tau, \sigma). \tag{205}$$

Since they can be also written as $X_*(\partial_\tau)$ and $X_*(\partial_\sigma)$, respectively, they have to be linearly independent. Since τ is to be a "time parameter", we assume that the time coordinate of the string flows as τ flows, that is

$$\frac{\partial X^0}{\partial \tau}(\tau, \sigma) > 0, \tag{206}$$

for all values of σ.

Similarly to the relativistic particle, we may now define the induced metric g on Σ. Let ξ^α be some general coordinates on Σ, $\alpha \in \{1, 2\}$. Let $g := X^*(\eta)$. Hence

$$g = g_{\alpha\beta}(\xi^1, \xi^2)\, \mathrm{d}\xi^\alpha \otimes \mathrm{d}\xi^\beta, \tag{207}$$

where the functions $g_{\alpha\beta}$ are given by the formula

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta}. \tag{208}$$

Note that even if X is an embedding, this is not necessarily a metric. The reason is η being indefinite. Moreover, to calculate the volume form, we need to take the square root of the *absolute value* of the determinant. We would be happy to avoid this. Let $\xi_1 = \tau$ and $\xi_2 = \sigma$, and write

$$\dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu := \frac{\partial X^\mu}{\partial \sigma}. \tag{209}$$

We will also write $\dot{X}$ and X' for the whole vectors. One can thus write

$$g_{11} = (\dot{X})^2, \quad g_{12} = \dot{X} \cdot X', \quad g_{22} = (X')^2, \tag{210}$$

where by square and $\cdot$ we mean the Lorentz pseudoscalar product. Hence the 2×2 component matrix of g takes the form (where we mildly abuse the notation):

$$g = \begin{pmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{pmatrix}. \tag{211}$$

Consequently, one has

$$\det(g) = (\dot{X})^2 (X')^2 - (\dot{X} \cdot X')^2. \tag{212}$$

We will now find some physical arguments ensuring that this is always strictly negative, except for string endpoints.

Observe that the space tangent to every point of a worldsheet is a two-dimensional vector space spanned by vectors $\dot{X}$ and X'. Now, if we observe string at fixed value of time in any Lorentz frame, the tangent vector to this constant time section of the worldsheet must be spacelike. It thus makes sense to make the following assumption:

We will thus assume that at each point of the worldsheet, except for string endpoints, that there are both spacelike *and* timelike vectors. Note that this assumption is purely geometrical - it does not assume any particular parametrization.

Remark 5.1. Except for string endpoints, there is no significant point on the string. $\tau \mapsto X^{\mu}(\tau, \sigma)$ *is not* a worldline of a "piece of string". In particular, $\dot{X}^{\mu}$ can be a spacelike vector!

Lemma 5.2. *At any point p of the worldsheet, there exist both spacelike and tangent vectors to the worldsheet, iff* $[\det(g)](p)$ *is strictly negative.*

Proof. Up to a multiplicative constant, the most general tangent vector at p takes the form

$$v^{\mu}(\lambda) = \dot{X}^{\mu} + \lambda X'^{\mu}, \tag{213}$$

for some $\lambda \in \mathbb{R}$, since $\dot{X}^{\mu}$ and X'^{μ} are linearly independent. Note that $X'^{\mu}(p)$ is obtained as a limit $\lambda \to \infty$. The character of the vector is preserved under scalar multiplications. Then

$$v^{\mu}(\lambda)v_{\mu}(\lambda) = \lambda^2 (X')^2 + 2\lambda(\dot{X} \cdot X') + (\dot{X})^2. \tag{214}$$

This is a polynomial quadratic in λ. Its graph is a parabola. It attains both positive and negative value, iff it has two distinct roots. In other words, the corresponding discriminant is strictly positive. This gives the condition

$$(\dot{X} \cdot X') - (\dot{X})^2 (X')^2 > 0. \tag{215}$$

By (212), this is equivalent to the strict negativity of the determinant at p. ■

It cannot happen that all of the tangent vectors at a given point of the worldsheet are spacelike. Regardless of parametrization, no point of a string would at some time t be able to reach any other point of the string in next instant $t + \mathrm{d}t$ without moving faster then light. There thus must be at least some lightlike tangent vector. Since there are no timelike vectors, this means that the quadratic equation has precisely one root λ_0, and its discriminant has to vanish. This means that there has to be a unique "lightlike direction", corresponding to a physical movement of the given point with a speed of light. It turns out that we have to allow precisely this to happen at the endpoints of the open string.

Remark 5.3. Observe that the negativity of the determinant ensures that g is an indefinite metric, hence of a signature $(1, 1)$.

Having the sign of the determinant settled, we thus propose the **Nambu-Goto string action** in the form of the area functional of the worldsheet

$$S[X] := -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \, \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}, \tag{216}$$

Let us briefly discuss dimensions. We assume $[\tau] = T$ and $[\sigma] = L$, but dimensions of parameters completely cancel anyway. This is expected, since the area obviously has dimension L^2. The dimension of the action must be energy times time, so the constant in front of it must be the one of force divided by speed. Hence T_0 has a dimension of force. We will show that it is related to the tension of the string.

Being just the scalar multiple of the area functional, the Nambu-Goto action can be rewritten using arbitrary parameters (ξ^1, ξ^2):

$$S[X] = -\frac{T_0}{c} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \sqrt{-\det(g_{\alpha\beta})}, \tag{217}$$

where $g_{\alpha\beta}$ is a matrix of functions defined by (208).

Exercise 5.4. *Verify this claim explicitly.*

5.2 Equations of motion, boundary conditions, D-branes

Let us find the equations of motion defined by S. The corresponding Lagrangian density takes the form

$$\mathcal{L}(\dot{X}^\mu, X'^\mu) = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}. \tag{218}$$

Let us define the canonical momentum densities

$$\mathcal{P}^\tau_\mu := \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \;\; \mathcal{P}^\sigma_\mu := \frac{\partial \mathcal{L}}{\partial X'^\mu}. \tag{219}$$

We let $X'^\mu := X^\mu + \epsilon \delta X^\mu$. We again get $S[X'] = S[X] + \epsilon\delta S + O(\epsilon^2)$, where

$$\begin{aligned}\delta S &= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \, [\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\delta\dot{X}^\mu + \frac{\partial \mathcal{L}}{\partial X'^\mu}\delta X'^\mu] \\ &= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \, [\mathcal{P}^\tau_\mu \delta\dot{X}^\mu + \mathcal{P}^\sigma_\mu \delta X'^\mu].\end{aligned} \tag{220}$$

Using the fact that $\delta\dot{X}^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta X^\mu)$ and $\delta X'^\mu = \frac{\mathrm{d}}{\mathrm{d}\sigma}(\delta X^\mu)$ and integration by parts, this gives

$$\begin{aligned}\delta S = &-\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \, [\frac{\partial \mathcal{P}^\tau_\mu}{\partial\tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial\sigma}]\,\delta X^\mu \\ &+ \int_0^{\sigma_1} \mathrm{d}\sigma \, [\delta X^\mu \mathcal{P}^\tau_\mu]_{\tau=\tau_i}^{\tau=\tau_f} + \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \, [\delta X^\mu \mathcal{P}^\sigma_\mu]_{\sigma=0}^{\sigma=\sigma_1}\end{aligned} \tag{221}$$

We again assume $\delta X^\mu(\tau_i, \sigma) = \delta X^\mu(\tau_f, \sigma) = 0$ for all $\sigma \in [0, \sigma_1]$. The first term must vanish independently of the last one, which gives the Largange-Euler equation:

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial\tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial\sigma} = 0. \tag{222}$$

It remains to deal with the boundary term. At each string endpoint $\sigma_* \in \{0, \sigma_1\}$, we thus have to impose some boundary condition.

1. We can require the endpoint to remain stationary in a given direction. We cannot do so for X^0 since we assume (206). For any given $\mu \in \{1, \ldots, D\}$ and any given $\sigma_* \in \{0, \sigma_1\}$, we can thus impose a Dirichlet boundary condition.

$$\frac{\partial X^\mu}{\partial\tau}(\tau, \sigma_*) = 0. \tag{223}$$

2. We can allow for a free motion of the string endpoint. Then we have to impose a **free endpoint condition**:

$$\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0, \tag{224}$$

for each $\mu \in \{1, \ldots, D\}$ and $\sigma_* \in \{0, \sigma_1\}$. For $\mu = 0$, we *have to impose* the boundary condition

$$\mathcal{P}^\sigma_0(\tau, \sigma_1) = \mathcal{P}^\sigma_0(\tau, 0) = 0. \tag{225}$$

So far, the situation does not look too complicated. The devil is in the details. The momenta densities are ugly. In particular, one has

$$\mathcal{P}^{\tau}_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_{\mu} - (X')^2 \dot{X}_{\mu}}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}, \tag{226}$$

$$\mathcal{P}^{\sigma}_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_{\mu} - (\dot{X})^2 X'_{\mu}}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}. \tag{227}$$

In particular, it is not easy to directly interpret free endpoint conditions.

Let us elaborate on Dirichlet boundary conditions. By declaring some of the spatial string coordinates X^{μ} to have some prescribed value at the string endpoint σ_*, we effectively allow its free motion only along the p-dimensional objects called **D-branes**. Or Dp-branes, if we want to emphasize the dimension. For example, if we require the Dirichlet boundary condition at a given endpoint σ_* for all $\mu \in \{1, \dots, D\}$, we specify a D0-brane and require the string to be attached to it. On the other hand, if we choose the free endpoint boundary condition for every $\mu \in \{1, \dots, D\}$, we define the space-filling D(d-1)-brane. D-branes in string theory are not necessarily hyperplanes and they have their own dynamics and physical properties.

Note that we can also consider the movement of a closed string. The σ direction is now made into a circle, which corresponds to the identification $(\tau, \sigma) \sim (\tau, \sigma + \sigma_c)$. We can however simply parametrize it by $\sigma \in [0, \sigma_c]$ and require $X^{\mu}(\tau, 0) = X^{\mu}(\tau, \sigma_c)$. There are no boundary conditions in this case.

5.3 Static gauge and a static string

The key to understand the motion of the string is a convenient choice of its parametrization. The most obvious one is to fix τ so that it corresponds to a coordinate time in some Lorentz frame. This is possible thanks to our assumption $\frac{\partial X^0}{\partial \tau}(\tau, \sigma) > 0$. Indeed, we can simply do the transformation

$$\tau' := \frac{1}{c} X^0(\tau, \sigma), \;\; \sigma' := \sigma. \tag{228}$$

It follows that in these coordinates, one has $X^0(\tau', \sigma) = c\tau'$. Let us henceforth drop the prime. This simplifies mainly the temporal components of the tangent vectors, namely

$$X' = (\frac{\partial X^0}{\partial \sigma}, \frac{\partial \vec{X}}{\partial \sigma}) = (0, \frac{\partial \vec{X}}{\partial \sigma}), \tag{229}$$

$$\dot{X} = (\frac{\partial X^0}{\partial \tau}, \frac{\partial \vec{X}}{\partial \tau} = (c, \frac{\partial \vec{X}}{\partial \tau}) \tag{230}$$

The expressions for momenta are still incredibly ugly. Let us try to give some meaning to the constant T_0. Let us consider a static string stretched from $x^1 = 0$ to $x^1 = a$. We thus have

$$X^0(\tau, \sigma) = c\tau, \;\; X^1(\tau, \sigma) = f(\sigma), \;\; X^2(\tau, \sigma) = \dots = X^D(\tau, \sigma) = 0, \tag{231}$$

where $f : [0, \sigma_1] \to \mathbb{R}$ is a function satisfying $f(0) = 0$ and $f(\sigma_1) = a$, such that $f'(\sigma) > 0$.

One has $X' = (0, f'(\sigma), 0, \dots, 0)$ and $\dot{X} = (c, \vec{0})$, whence

$$(X')^2 = (f'(\sigma))^2, \; (\dot{X})^2 = -c^2, \;\; X' \cdot X' = 0. \tag{232}$$

The Lagrangian of this string is calculated as an integral of the density $\mathcal{L}$, one finds

$$L(\tau) = \int_0^{\sigma_1} \mathcal{L}(\dot{X}, \dot{X}') \, \mathrm{d}\sigma = -\frac{T_0}{c} \int_0^{\sigma_1} c f'(\sigma) \, \mathrm{d}\sigma = -T_0 [f(\sigma)]_{\sigma=0}^{\sigma=a} = -T_0 a. \tag{233}$$

Since the static string has no kinetic energy and $L(\tau) = T(\tau) - V(\tau)$, we identify $T_0 a$ with the potential energy of the string. Note that it does not depend on the particular function $f = f(\sigma)$. This is in accordance with the reparametrization invariance.

Since the nonrelativistic static string has a zero potential energy, this should be identified with the with the rest mass energy of the string. It is also precisely the energy required for stretching a infinitesimally small string to a finite length a, assuming the tension stays constant throughout the process. If μ_0 is the rest mass of the string per unit length, we get $\mu_0 c^2 = T_0$, so

$$\mu_0 = \frac{T_0}{c^2}. \tag{234}$$

The mass of the relativistic arise only due to it having a tension!

Our interpretations can be invalid, if the proposed solution describing the static string would fail to satisfy the equations of motion! One has

$$\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} = c f'(\sigma). \tag{235}$$

It is easy to see

$$\mathcal{P}_0^\tau = -\frac{T_0}{c} f'(\sigma), \;\; \mathcal{P}_0^\sigma = 0, \;\; \mathcal{P}_1^\tau = 0, \;\; \mathcal{P}_1^\sigma = -T_0, \tag{236}$$

and trivially $\mathcal{P}_i^\tau = \mathcal{P}_i^\sigma = 0$ for all $i \in \{2, \ldots, D\}$. It is easy to check now that the equations of motion are satisfied. Finally, one has to check that in the temporal direction, the free endpoint condition is satisfied. But we have $\mathcal{P}_0^\sigma = 0$ identically.

Note that in this parametrization, $\dot{X}$ is indeed timelike and X' is spacelike. In general, notice that in the static gauge, X' is always spacelike.

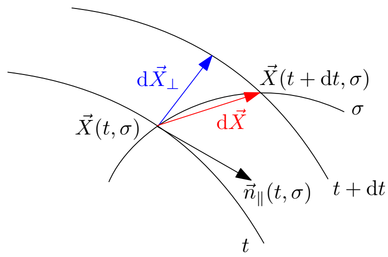
5.4 Action in terms of a transverse velocity

As we have already noted, it does not make any physical sense to interpret $\dot{X}^i(\tau, \sigma)$ as a velocity of a piece of string corresponding to the fixed value of the parameter σ. We assume that we work in the static gauge. There is however a parametrization independent notion of **transverse velocity**.

At each coordinate time t, we may record the position of the string $\vec{X}(t, \sigma)$. At that point, we construct a hyperplane perpendicular to the string at the fixed time t. This is always possible, since the vector $\frac{\partial \vec{X}}{\partial \sigma}$ is nonzero along the worldsheet. An infinitesimal moment later, at $t + \mathrm{d}t$, we record the point where the string intersects the hyperplane. The difference of the two points in the hyperplane defines a space vector $\mathrm{d}\vec{X}_\perp$, and the transverse velocity is obtained as

$$\vec{v}_\perp(t, \sigma) := \frac{\mathrm{d}\vec{X}_\perp}{\mathrm{d}t}. \tag{237}$$

This velocity should be independent of the parametrization σ. Suppose that $\vec{n}_\parallel(t, \sigma)$ is a unit vector tangent to the string at $\vec{X}(t, \sigma)$. Consider the following image:



We see that $\mathrm{d}\vec{X}_\perp$ can be obtained as a projection of the difference $\mathrm{d}\vec{X} := \vec{X}(t+\mathrm{d}t,\sigma) - \vec{X}(t,\sigma)$ into the direction perpendicular to $\vec{n}_\parallel(t,\sigma)$. It follows that

$$\mathrm{d}\vec{X}_\perp = \mathrm{d}\vec{X} - (\mathrm{d}\vec{X} \cdot \vec{n}_\parallel(\tau,\sigma))\, \vec{n}_\parallel(\tau,\sigma). \tag{238}$$

Dividing this by $\mathrm{d}\tau$, we see that the transverse velocity can be obtained (or defined) by

$$\vec{v}_\parallel = \frac{\partial \vec{X}}{\partial t} - (\frac{\partial \vec{X}}{\partial t} \cdot \vec{n}_\parallel)\, \vec{n}_\parallel. \tag{239}$$

We only have to find an explicit formula for $\vec{n}_\parallel$. There are two ways how to define it. First, one can define a function $s = s(t,\sigma)$ measuring the length of a string at a given time t. Explicitly, it has the following form:

$$s(t,\sigma) = \int_0^\sigma \|\frac{\partial \vec{X}}{\partial \sigma}(t,\sigma)\|\, \mathrm{d}\sigma. \tag{240}$$

Note that we can view this as a reparametrization $(t,\sigma) \mapsto (t,s)$. It is well-defined, since the Jacobi matrix of the coordinate transformation is

$$\begin{pmatrix} 1 & 0 \\ \frac{\partial s}{\partial t} & \|\frac{\partial \vec{X}}{\partial \sigma}\| \end{pmatrix}, \tag{241}$$

which is everywhere non-singular. We claim that we can then choose $\vec{n}_\parallel = \frac{\partial \vec{X}}{\partial s}$. One has

$$\frac{\partial \vec{X}}{\partial \sigma} = \frac{\partial \vec{X}}{\partial s}\frac{\partial s}{\partial \sigma} + \frac{\partial \vec{X}}{\partial t}\frac{\partial t}{\partial \sigma} = \frac{\partial \vec{X}}{\partial s}\, \|\frac{\partial \vec{X}}{\partial \sigma}\|. \tag{242}$$

But this shows that $\frac{\partial \vec{X}}{\partial s} = \|\frac{\partial \vec{X}}{\partial \sigma}\|^{-1} \frac{\partial \vec{X}}{\partial \sigma}$ is just a normalization of the vector $\frac{\partial \vec{X}}{\partial \sigma}$, which is tangent to the string. This proves the claim. The final formula is thus

$$\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t} - (\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s})\frac{\partial \vec{X}}{\partial s}. \tag{243}$$

Exercise 5.5. *Verify that $\vec{v}_\perp$ does not depend on the parametrization.*

Proof. We shall consider the reparametrization $t' = t$ and $\sigma' = f(t,\sigma)$. We also require

$$\frac{\partial f}{\partial \sigma} \neq 0. \tag{244}$$

The base tangent vectors to worldsheet are related as

$$\frac{\partial \vec{X}}{\partial t} = \frac{\partial \vec{X}}{\partial t'} + \frac{\partial f}{\partial t}\frac{\partial \vec{X}}{\partial \sigma'}, \tag{245}$$

$$\frac{\partial \vec{X}}{\partial \sigma} = \frac{\partial f}{\partial \sigma}\frac{\partial \vec{X}}{\partial \sigma'}. \tag{246}$$

This explicitly demonstrates that the tangent to the worldsheet along the line of constant σ indeed has no significance. The second equation shows that both tangent vectors along the line of constant $t' = t$ are colinear. But this means

$$\frac{\partial \vec{X}}{\partial s} = \pm \frac{\partial \vec{X}}{\partial s'}, \tag{247}$$

depending on the sign of $\frac{\partial f}{\partial \sigma}$. Since $\frac{\partial \vec{X}}{\partial t}$ and $\frac{\partial \vec{X}}{\partial t'}$ differ by something parallel to the string, their projections to the perpendicular direction are the same. This can be also verified directly by plugging in (245) and (246) to the definition of $\vec{v}_\perp$. ■

We will now argue that the action can be rewritten in a neat way using the transverse velocity. First, notice that

$$v_\perp^2 = (\frac{\partial \vec{X}}{\partial t})^2 - (\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s})^2. \tag{248}$$

In the following, we will write

$$\frac{\mathrm{d}s}{\mathrm{d}\sigma} := \frac{\partial s}{\partial \sigma} = \| \frac{\partial \vec{X}}{\partial \sigma} \| \tag{249}$$

Using this notation, one has

$$(\dot{X})^2 = (\frac{\partial \vec{X}}{\partial t})^2 - c^2, \;\; (X')^2 = (\frac{\mathrm{d}s}{\mathrm{d}\sigma})^2, \;\; \dot{X} \cdot X' = \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} \tag{250}$$

Consequently, one finds

$$\begin{aligned} (\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2 &= (\frac{\mathrm{d}s}{\mathrm{d}\sigma})^2 [(\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s})^2 + c^2 - (\frac{\partial \vec{X}}{\partial t})^2] \\ &= (\frac{\mathrm{d}s}{\mathrm{d}\sigma})^2 \, c^2 (1 - \frac{v_\perp^2}{c^2}). \end{aligned} \tag{251}$$

We see that the Nambu-Goto action can be rewritten in terms of the transverse velocity as

$$S = -T_0 \int_{t_i}^{t_f} \mathrm{d}t \int_0^{\sigma_1} \mathrm{d}\sigma \, \frac{\mathrm{d}s}{\mathrm{d}\sigma} \sqrt{1 - \frac{v_\perp^2}{c^2}}. \tag{252}$$

In particular, we see that at points where the term under the square root is positive, the transverse velocity is strictly smaller then the speed of light!

Exercise 5.6. *Show that one can express the momenta densities as*

$$\mathcal{P}^{\tau 0} = \frac{T_0}{c} \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{1}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}, \;\; \vec{\mathcal{P}}^\tau = \frac{T_0}{c^2} \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{\vec{v}_\perp}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}, \tag{253}$$

$$\mathcal{P}^{\sigma 0} = -\frac{T_0}{c} \frac{\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}, \;\; \vec{\mathcal{P}}^\sigma = -\frac{T_0}{c^2} \frac{(\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s}) \frac{\partial \vec{X}}{\partial t} + (c^2 - (\frac{\partial \vec{X}}{\partial t})^2) \frac{\partial \vec{X}}{\partial s}}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}. \tag{254}$$

Although these expressions are still immensely complicated, they allow for a very important observation. In the next paragraph, we shall assume the free endpoint condition in all directions, that is the string is attached to the space-filling D-brane. This requires $\mathcal{P}^{\sigma\mu}(t, \sigma_*) = 0$ for

$\sigma_* \in \{0, \sigma_1\}$. First, since the expression in the denominator is always in the interval $[0,1]$, we get the following condition at the endpoints:

$$\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial s} = 0. \tag{255}$$

Recall that the endpoints of the open string are physically significant and their velocity $\frac{\partial \vec{X}}{\partial t}(t, \sigma_*)$ is of importance. The condition means the endpoints of the string move transversaly to the string, and thus also $\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t}$ at both endpoints.

Exercise 5.7. *Under which conditions is $\frac{\partial \vec{X}}{\partial t}$ well-behaved at the endpoints.*

Proof. In Exercise 5.5, we have examined the transformations of the two tangent vectors under the most general reparametrization preserving the satic gauge. In particular, we have found

$$\frac{\partial \vec{X}}{\partial t} = \frac{\partial \vec{X}}{\partial t'} + \frac{\partial f}{\partial t}\frac{\partial \vec{X}}{\partial \sigma'} \tag{256}$$

If we want the velocities the endpoints to stay the same, we get the condition

$$\frac{\partial f}{\partial t}(t, \sigma_*) = 0. \tag{257}$$

But this only means that the position of the endpoints in the new parametrization, described by $f(t, \sigma_*)$ are actually independent of time. We secretly always assume this, otherwise e.g. the integration per parts in the double integral leading to the free endpoint conditions would not be possible. ■

The condition (255) in principle still allows for a zero velocity at the endpoints. However, plugging into the expression for $\vec{\mathcal{P}}^\sigma$ and noting the $\vec{v} := \frac{\partial \vec{X}}{\partial t} = \vec{v}_\perp$ at the endpoints, we find the condition

$$\vec{0} = \vec{\mathcal{P}}^\sigma(t, \sigma_*) = -T_0\sqrt{1 - \frac{v^2(t, \sigma_*)}{c^2}}\frac{\partial \vec{X}}{\partial s}(t, \sigma_*) \tag{258}$$

This shows that necessarily

$$v^2(t, \sigma_*) = c^2, \tag{259}$$

that is the **string endpoints move with the speed of light**.

Exercise 5.8. *Consider the relativistic string with endpoints attached at $(0, \vec{0})$ and $(a, \vec{0})$. Find the non-relativistic limit.*

Proof. We assume $X^1(t, 0) = 0$ and $X^1(t, \sigma_1) = a$. Write $\vec{X} = (X^1, \vec{y})$. We assume small oscillations, that is the situation differs only slightly from the static string. In particular, we assume that

$$\frac{\partial X^1}{\partial \sigma} > 0. \tag{260}$$

This allows us to parametrize the string by $x = x^1$, that is we choose the new parameter $x := X^1(t, \sigma)$. Hence $\vec{X}(t, x) = (x, \vec{y}(t, x))$ and consequently

$$\frac{\partial \vec{X}}{\partial x} = (1, \frac{\partial \vec{y}}{\partial x}), \quad \frac{\partial \vec{X}}{\partial t} = (0, \frac{\partial \vec{y}}{\partial t}). \tag{261}$$

and thus

$$\frac{\mathrm{d}s}{\mathrm{d}\sigma} = \sqrt{1 + (\frac{\partial \vec{y}}{\partial x})^2} \approx 1 + \frac{1}{2}(\frac{\partial \vec{y}}{\partial x})^2. \tag{262}$$

For small oscilations, the transverse direction and the direction perpendicular to x are almost the same, hence

$$\vec{v}_\perp \approx \frac{\partial \vec{X}}{\partial t} = (0, \frac{\partial \vec{y}}{\partial t}). \tag{263}$$

We also assume that $v_\perp^2 \ll c^2$, so we can rewrite the action as

$$\begin{aligned} S \approx & -T_0 \int_{t_i}^{t_f} \mathrm{d}t \int_0^a \mathrm{d}x (1 + (\frac{\partial y}{\partial x})^2) \cdot (1 - \frac{1}{2c^2}(\frac{\partial \vec{y}}{\partial t})^2) \\ \approx & \int_{t_i}^{t_f} \mathrm{d}t \,\big(-T_0 a + \int_0^a \mathrm{d}x \, [\frac{1}{2}\frac{T_0}{c^2}(\frac{\partial y}{\partial t})^2 - \frac{1}{2}T_0(\frac{\partial y}{\partial x})^2] \end{aligned} \tag{264}$$

This is indeed an action functional for the non-relativistic string with tension T_0 and a mass density $\mu_0 = T_0/c^2$. The constant term corresponds to the potential energy of the static string - the energy we need to put in the string with tension T_0 to stretch it between its endpoints. ■

Exercise 5.9. *At $t = 0$, a closed string forms a circle of radius R_0 in the (x, y) plane and has zero velocity. Assume that the string remains circular, that is it is described by a single function $R = R(t)$. Find the evolution of R by looking at the Nambu-Goto action.*

Proof. If we make the assumption about its movement, we certainly have $v_\perp = \dot{R}$. The Lagrangian then takes the form

$$L(t) = -T_0 \int_{\mathbb{S}_{R(t)}} \mathrm{d}s \sqrt{1 - \frac{\dot{R}^2(t)}{c^2}} = -T_0 2\pi R(t) \cdot \sqrt{1 - \frac{\dot{R}^2(t)}{c^2}}. \tag{265}$$

One can view this as a Lagrangian for a system with one degree of freedom, described by a function $R(t)$. The canonical momentum P corresponding to R is thus

$$P := \frac{\partial L}{\partial \dot{R}} = \frac{2\pi T_0 R\dot{R}}{c^2\sqrt{1 - \frac{\dot{R}^2}{c^2}}}. \tag{266}$$

We can thus pass to the Hamiltonian function, finding

$$H = P\dot{R} - L = \frac{2\pi T_0 R}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}} \tag{267}$$

Since L does not explicitly depent on time, H corresponds to the conserved energy. By plugging $R(0) = R_0$ and $\dot{R}(0) = 0$, we get that the energy of the string equals to

$$2\pi T_0 R_0. \tag{268}$$

We thus obtain the energy contraint

$$R_0 = \frac{R}{\sqrt{1 - \frac{\dot{R}^2}{c^2}}} \tag{269}$$

This can be rewritten as

$$\dot{R}^2 + (\frac{c}{R_0})^2 R^2 = c^2. \tag{270}$$

One can differentiate this equation to get the equation of motion

$$2\dot{R}(\ddot{R} + (\frac{c}{R_0})^2 R) = 0. \tag{271}$$

Assume for a moment that $\dot{R} \neq 0$, so we get the equation of motion for the harmonic oscillator with the solution (satisfying the initial conditions):

$$R(t) = R_0 \cos(\frac{c}{R_0} t). \tag{272}$$

Note that then $\dot{R}(t) = -c \sin(\frac{c}{R_0} t)$. The string collapses to zero at $t_1 = \frac{\pi R_0}{2c}$, and note that then $\dot{R}(t_1) = -c$. The solution $\dot{R} \equiv 0$ fails to satisfy the original Largrange-Euler equations. ■

Exercise 5.10. *Let $\mathcal{L}$ be the Lagrangian density for the Nambu-Goto action in the static gauge, expressed in terms of $\partial_t \vec{X}$ and $\partial_\sigma \vec{X}$. The canonical conjugate momentum is defined by*

$$\vec{\mathcal{P}}^\tau(t, \sigma) := \frac{\partial \mathcal{L}}{\partial(\partial_t \vec{X})} = \frac{T_0}{c^2} \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{\vec{v}_\perp}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}. \tag{273}$$

Find the Hamiltonian density $\mathcal{H} := \vec{\mathcal{P}}^\tau \cdot \partial_\tau \vec{X} - \mathcal{L}$ and express it in terms of $\vec{v}_\perp$.

Proof. One has $\partial_t \vec{X} = \vec{v}_\perp +$ (something tangent to the string). The scalar product of the coordinate velocity with the transverse velocity thus gives simply the square $v_\perp^2$. Whence

$$\mathcal{H} = \vec{\mathcal{P}}^\tau \cdot \vec{v}_\perp - \mathcal{L} = \frac{T_0}{c^2} \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{v_\perp^2}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} + T_0 \frac{\mathrm{d}s}{\mathrm{d}\sigma} \sqrt{1 - \frac{v_\perp^2}{c^2}} = \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{T_0}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \tag{274}$$

Consequently, we find

$$H = \int_0^{\sigma_1} \mathrm{d}\sigma \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{T_0}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} = \int \mathrm{d}s \frac{\mu_0 c^2}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}, \tag{275}$$

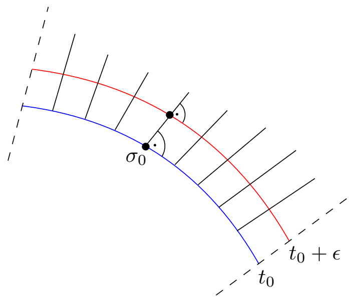
so the energy of the string is the sum over relativistic energies of pieces with mass per unit length μ_0 moving with the velocity $\vec{v}_\perp$. ■

6 String parametrization and classical motion

6.1 Choosing a σ parametrization

So far, working in the static gauge, we have shown that we need to examine the **string surface** described by functions $\vec{X}(t, \sigma)$. We would like to conveniently choose the parametrization of the string surface. The idea is the following:

At some fixed time t_0, the string is parametrized by $\sigma \in [0, \sigma_1]$. For each value σ_0 of the parameter σ, draw a straight line perpendicular to the $t = t_0$ string. For some small $\epsilon > 0$, it intersects some point of the $t = t_0 + \epsilon$ string. This point on the $t = t_0 + \epsilon$ string will be assigned to the same value σ_0 of the parameter. In this way, we will obtain the parametrization of the $t = t_0 + \epsilon$ string. By repeating this procedure, we can find a parametrization of $t = t_0 + 2\epsilon$ string.



By repeating this procedure, we obtain a parametrization σ of the whole string surface, having the property the **lines of constant σ are perpendicular to the strings**, that is

$$\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} = 0. \tag{276}$$

Since the movement of the string endpoints is transverse to strings, the value of the new parameter at the endpoints remains sting equal to the ones at $t = t_0$ string we have started with, say $\sigma_* \in \{0, \sigma_1\}$. For closed string, we have $\sigma \in [0, \sigma_c]$.

Let us attempt to discuss this more rigorously. Suppose that we are given functions $\vec{X}(t, \sigma)$ where $(t, \sigma) \in [t_i, t_f] \times [0, \sigma_1]$ and we assume that $\partial_t \vec{X} \cdot \partial_\sigma \vec{X} = 0$ for all $(t, \sigma_*) \in [t_i, t_f] \times \{0, \sigma_1\}$. We are looking for a new parameter $\sigma' := f(t, \sigma)$. Let us also formally write $t' := t$. To simplify notation, let $t_0 = 0$. We look for the function f, satisfying $\partial_\sigma f \neq 0$ and the condition

$$f(0, \sigma) = \sigma, \tag{277}$$

this is because at $t = 0$ string, we want the new parametrization to coincide with the old one. We have already shown that the tangent vectors transform as in (245, 246). By inverting those relations, we can express primed derivatives in terms of unprimed:

$$\frac{\partial \vec{X}}{\partial \sigma'} = (\frac{\partial f}{\partial \sigma})^{-1} \frac{\partial \vec{X}}{\partial \sigma}, \quad \frac{\partial \vec{X}}{\partial t'} = \frac{\partial \vec{X}}{\partial t} - \frac{\partial f}{\partial t}(\frac{\partial f}{\partial \sigma})^{-1}\frac{\partial \vec{X}}{\partial \sigma}. \tag{278}$$

Our requirement is that the vectors tangent to the strings $t' = t'_0$ are perpendicular to the vectors tangent to the lines of constant σ'. This gives us the equation

$$0 = \frac{\partial \vec{X}}{\partial \sigma'} \cdot \frac{\partial \vec{X}}{\partial t'} = (\frac{\partial f}{\partial \sigma})^{-1}\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} - \frac{\partial f}{\partial t}(\frac{\partial f}{\partial \sigma})^{-2}\|\frac{\partial \vec{X}}{\partial \sigma}\|^2. \tag{279}$$

This can be rewritten as the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \sigma} \frac{\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma}}{\|\frac{\partial \vec{X}}{\partial \sigma}\|^2}. \tag{280}$$

Observe that the fraction multiplying the partial derivative with respect to σ is a known function of (t, σ), that is we are solving the equation

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \sigma} K(t, \sigma). \tag{281}$$

Observe that we have assumed that $K(t, \sigma_*) = 0$. This immediately implies that the solution to this equation will satisfy $\frac{\partial f}{\partial t}(t_*, \sigma) = 0$. This confirms that the numerical value of the endpoints remains constant in time.

Exercise 6.1. *Show that there is a solution to (281) with the initial condition $f(0,\sigma)=\sigma$ defined on entire string surface.*

Proof. The solution to this partial differential equation is obtained by "method of characteristics". It follows that the graph of the solution defines a surface

$$S := \{(t,\sigma,f(t,\sigma)) \mid (t,\sigma)\in[t_i,t_f]\times[0,\sigma_1]\}, \tag{282}$$

such that the vector field $V := (1,-K(x,y),0)$ is tangent to the surface for all $(x,y,z)\in S$.

For future purposes, suppose that we impose a boundary condition $f(0,\sigma)=F(\sigma)$, where $F:[0,\sigma_1]\to[0,\sigma_1]$ is an arbitrary function satisfying

$$F(0)=0,\ \ F(\sigma_1)=\sigma_1,\ \ F'(\sigma)>0. \tag{283}$$

But this means that the surface S must necessarily contain the curve

$$\Gamma := \{(0,r,F(r)) \mid r\in[0,\sigma_1]\}. \tag{284}$$

The idea is to find the integral curve of V starting from $(0,r,F(r))$ for each $r\in[0,\sigma_1]$. We thus look for functions $x=x(s,r)$, $y=y(s,r)$ and $z=z(s,r)$, satisfying the system of equations

$$\frac{\partial x}{\partial s}(s,r) = 1, \tag{285}$$

$$\frac{\partial y}{\partial s}(s,r) = -K(x(s,r),y(s,r)), \tag{286}$$

$$\frac{\partial z}{\partial s}(s,r) = 0, \tag{287}$$

together with the initial conditions at $s=0$, which give

$$x(0,r)=0,\ \ y(0,r)=r,\ \ z(0,r)=F(r). \tag{288}$$

We can fully solve two of the equations, namely $x(s,r)=s$ and $z(s,r)=F(r)$, and it remains to solve a single ordinary differential equation

$$\frac{\partial y}{\partial s}(s,r) = -K(s,y(s,r)), \tag{289}$$

which is possible for all $r\in[0,\sigma_1]$ and $s\in[-a,a]$ for some $a>0$. Let us now consider a map $\phi(s,r) := (x(s,r),y(s,r))$. One can calculate its differential at $(0,r)$:

$$(\mathbf{D}\phi)(0,r) = \begin{pmatrix} \frac{\partial x}{\partial s}(0,r) & \frac{\partial x}{\partial r}(0,r) \\ \frac{\partial y}{\partial s}(0,r) & \frac{\partial y}{\partial r}(0,r) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -K(0,r) & 1 \end{pmatrix} \tag{290}$$

By the inverse function theorem, there is a neighborhood U of each $(0,r)$ and a neighborhood V of $\phi(0,r)=(0,r)$, such that $\phi:U\to V$ is a diffeomorphism. Since we can do this for all $r\in[0,\sigma_1]$, we can find the inverse expressions $s=s(x,y)$ and $r=r(x,y)$ for all $x\in[-b,b]$ and $y\in[0,\sigma_1]$, where $b>0$.

Finally, define $f(t,\sigma) := z(s(t,\sigma),r(t,\sigma)) = F(r(t,\sigma))$, for all $t\in[-b,b]$ and $\sigma\in[0,\sigma_1]$. It follows from the construction that f solves the partial differential equation (281) with the correct boundary condition.

It remains to check that $\frac{\partial f}{\partial \sigma}(t,\sigma) \neq 0$. One has

$$\frac{\partial f}{\partial \sigma}(t,\sigma) = F'(r(t,\sigma))\,\frac{\partial r}{\partial \sigma}(t,\sigma). \tag{291}$$

Since we have assumed that $F'(\sigma) > 0$, this amounts to checking that $\frac{\partial r}{\partial y}(x,y) \neq 0$ for all $x \in [-b,b]$ and $y \in [0,\sigma_1]$. To do so, observe that

$$(\mathbf{D}\phi)(s,r) = \begin{pmatrix} 1 & 0 \\ -K(s,y(s,r)) & \frac{\partial y}{\partial r}(s,r) \end{pmatrix} \tag{292}$$

This matrix is invertible wherever ϕ is invertible. In particular, at these points one has $\frac{\partial y}{\partial r}(s,r) \neq 0$, and it follows that $\frac{\partial r}{\partial y}(x,y)$ is given by the bottom-right corner of the inverse matrix. Explicitly, one has

$$\frac{\partial r}{\partial y}(x,y) = \frac{1}{\frac{\partial y}{\partial r}(s(x,y),r(x,y))} \neq 0 \tag{293}$$

for all $x \in [-b,b]$ and $y \in [0,\sigma_1]$.

Observe that the resulting domain of f does depend only on the function K, not on the function F describing the boundary condition! By only a slight modification, for each $t_0 \in [t_i,t_f]$, we find $b > 0$, such that for any $F : [0,\sigma_1] \to [0,\sigma_1]$ as above, there is a solution to (281) defined on $[t_0-b,t_0+b] \times [0,\sigma_1]$ and satisfying the boundary condition $f(t_0,\sigma) = F(\sigma)$.

We can now cover $[t_i,t_f]$ by finitely many such intervals, we can find a finite subdivision $t_0 = t_i < \cdots < t_n = t_f$, such that for each $i \in \{0,\ldots,n-1\}$ we have a solution f_i on some rectangle containing $[t_i,t_{i+1}] \times [0,\sigma_1]$ for each boundary condition $f_i(t_i,\sigma) = F_i(\sigma)$. We can now inductively construct f. We declare it to be f_0 on $[t_0,t_1] \times [0,\sigma_1]$ with the boundary condition $f_0(0,\sigma) = \sigma$. Then we choose f_1 on $[t_1,t_2] \times [0,\sigma_1]$ with the boundary condition $f_1(t_1,\sigma) = f_0(t_1,\sigma)$. By iterating this procedure, we find the solution to f on the whole $[t_i,t_f] \times [0,\sigma_1]$. It obviously satisfies $\frac{\partial f}{\partial \sigma} \neq 0$. ■

In our new parametrization, the transverse velocity is therefore given simply by

$$\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t}. \tag{294}$$

We shall henceforth denote is simply by $\vec{v}$. We can now rewrite the canonical momenta densities in our new gauge. One finds

$$\mathcal{P}^{\tau 0} = \frac{T_0}{c}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\quad \vec{\mathcal{P}}^{\tau} = \frac{T_0}{c^2}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{295}$$

$$\mathcal{P}^{\sigma 0} = 0,\quad \vec{\mathcal{P}}^{\sigma} = -T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial \vec{X}}{\partial s}. \tag{296}$$

6.2 Physical interpretation of the string equation of motion

Recall that the string equation of motion is simply

$$\frac{\partial \mathcal{P}^{\tau\mu}}{\partial \tau} + \frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \sigma} = 0, \tag{297}$$

for each $\mu \in \{0, \dots, D\}$. In the static gauge, we have $\tau = t$. Plugging the above expressions for $\mu = 0$ thus gives the equation

$$\frac{\partial}{\partial t}\Big(\frac{T_0}{c}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Big) = 0. \tag{298}$$

This is nothing but the energy conservation law. In fact, one can define a function

$$E(\sigma) := \int_0^\sigma \mathrm{d}\sigma\, \frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{299}$$

which is independent of time and satisfies $E'(\sigma) > 0$. This will be an important observation. On the other hand, by plugging into the spatial components of the equation and using (298), one finds

$$\frac{T_0}{c^2}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial^2\vec{X}}{\partial t^2} = \frac{\partial}{\partial\sigma}(T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial\vec{X}}{\partial s}) \tag{300}$$

Finally, rewrite the σ differentiation using the s differentiation and cancel $\frac{\mathrm{d}s}{\mathrm{d}\sigma}$. We find the equation

$$\frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial^2\vec{X}}{\partial t^2} = \frac{\partial}{\partial s}(T_0\sqrt{1-\frac{v^2}{c^2}}\frac{\partial\vec{X}}{\partial s}). \tag{301}$$

This already somewhat resembles the wave equation for a non-relativistic string, which can be written as $\mu_0\frac{\partial^2\vec{y}}{\partial t^2} = \frac{\partial}{\partial x}(T_0\frac{\partial\vec{y}}{\partial x})$. This leads us to define the **effective tension** and the **effective mass density**. They are both functions of both t and σ:

$$T_{\text{eff}} := T_0\sqrt{1-\frac{v^2}{c^2}},\quad \mu_{\text{eff}} := \frac{T_0}{c^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{302}$$

6.3 Wave equation and constraints

Let us suggestively rewrite the equation (300) as

$$\frac{T_0}{c^2}\frac{\partial^2\vec{X}}{\partial t^2} = T_0\frac{\sqrt{1-\frac{v^2}{c^2}}}{\frac{\mathrm{d}s}{\mathrm{d}\sigma}}\frac{\partial}{\partial\sigma}\Big(\frac{\sqrt{1-\frac{v^2}{c^2}}}{\frac{\mathrm{d}s}{\mathrm{d}\sigma}}\frac{\partial\vec{X}}{\partial\sigma}\Big). \tag{303}$$

If only we could choose a new parameter $\sigma' = \sigma'(\sigma)$, such that

$$\frac{\partial}{\partial\sigma} = \frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial}{\partial\sigma'}. \tag{304}$$

But we are in luck, since there is one such parameter - the energy of the string segment from 0 to σ, up to a constant. In other words, let

$$\sigma' := \frac{1}{T_0}E(\sigma) = \int_0^\sigma \frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{305}$$

Note that $[\sigma'] = L$, as required. Importantly, lines of constant σ' still remain transversal to strings. We shall henceforth use this parametrization. Note that the range of this new coordinate is $[0, \sigma_1']$, where $\sigma_1' = \frac{E}{T_0}$, where E is the conserved overall energy of the string. We will henceforth drop the primes and assume that σ corresponds to the string energy.

Note that we still have to remember that σ corresponds to the energy of the string. This gives us a constraint

$$\sigma = \int_0^\sigma \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \mathrm{d}\sigma. \tag{306}$$

Since the values of both sides are the same at $\sigma = 0$, we can instead compare the derivatives of both sides with respect to σ, finding

$$1 = \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{307}$$

Taking the square and recalling that $\frac{\mathrm{d}s}{\mathrm{d}\sigma} = \|\frac{\partial \vec{X}}{\partial \sigma}\|^2$ gives the equation

$$(\frac{\partial \vec{X}}{\partial \sigma})^2 + \frac{1}{c^2}(\frac{\partial \vec{X}}{\partial t})^2 = 1. \tag{308}$$

This equation is a relic of our parametrization condition. When we look for a solution of equations of motion, we impose this as a **constraint**. This is quite a common occurrence in theories with some gauge invariance. Let us also examine the momenta densities and boundary conditions. It follows from (307) that

$$\frac{\mathrm{d}s}{\mathrm{d}\sigma} = \sqrt{1 - \frac{v^2}{c^2}}. \tag{309}$$

Plugging this into (295) and (296), we find

$$\mathcal{P}^{\tau 0} = \frac{T_0}{c}, \;\; \vec{\mathcal{P}}^{\tau} = \frac{T_0}{c^2}\frac{\partial \vec{X}}{\partial t}, \;\; \mathcal{P}^{\sigma 0} = 0, \;\; \vec{\mathcal{P}}^{\sigma} = -T_0 \frac{\partial \vec{X}}{\partial \sigma}. \tag{310}$$

But this shows that the free endpoint boundary condition $\mathcal{P}^{\sigma\mu}(t, \sigma_*) = 0$ turns into the Neumann boundary condition:

$$\frac{\partial X^\mu}{\partial \sigma}(t, \sigma_*) = 0. \tag{311}$$

Let us summarize all the equations and constraints we have to solve to find the solution:

$$\text{wave equation:} \quad \frac{\partial^2 \vec{X}}{\partial t^2} = c^2 \frac{\partial^2 \vec{X}}{\partial \sigma^2}, \tag{312}$$

$$\text{transversality condition:} \quad \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} = 0, \tag{313}$$

$$\text{energy parametrization:} \quad (\frac{\partial \vec{X}}{\partial \sigma})^2 + \frac{1}{c^2}(\frac{\partial \vec{X}}{\partial t})^2 = 1, \tag{314}$$

$$\text{boundary conditions:} \quad \frac{\partial \vec{X}}{\partial \sigma}(t, \sigma_*) = 0. \tag{315}$$

Note that the relic of the choice of the static gauge is that $X^0(t, \sigma) = ct$.

6.4 General motion of an open string

The most general solution to the wave equation (312) can be written as

$$\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{G}(ct-\sigma)), \tag{316}$$

for arbitrary vector functions $\vec{F} = \vec{F}(u)$ and $\vec{G} = \vec{G}(u)$. We impose free endpoint conditions (315) at both endpoints. The $\sigma_* = 0$ condition leads to

$$\vec{F}'(ct) - \vec{G}'(ct) = 0, \tag{317}$$

for all $t \in \mathbb{R}$. Hence $\vec{G}(u) = \vec{F}(u) + \vec{a}_0$ for some constant vector. Plugging this back in gives

$$\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma) + \vec{a}_0). \tag{318}$$

One can absorb the constant vector $\vec{a}_0$ by redefining $\vec{F}(u) \mapsto \vec{F}(u) + \vec{a}_0/2$, and we find

$$\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma)). \tag{319}$$

Next, we can plug this into the boundary condition (315) at $\sigma_* = \sigma_1$. We find

$$\vec{F}'(ct+\sigma_1) - \vec{F}'(ct-\sigma_1) = 0. \tag{320}$$

Since this has to hold for all $t \in \mathbb{R}$, this shows that $\vec{F}'(u) = \vec{F}'(u+2\sigma_1)$. This means that $\vec{F}$ is quasiperiodic in u, that is

$$\vec{F}(u+2\sigma_1) = \vec{F}(u) + 2\sigma_1\frac{\vec{v}_0}{c}, \tag{321}$$

where we have chosen the unknown constant $\vec{v}_0$ to have the dimension of velocity. It remains to plug into the parametrization conditions (313, 314). It is useful to add and subtract $2/c$ multiple of the first condition to the second, finding

$$(\frac{\partial\vec{X}}{\partial\sigma} \pm \frac{1}{c}\frac{\partial\vec{X}}{\partial t})^2 = 1. \tag{322}$$

By plugging in (319), we get

$$\frac{\partial\vec{X}}{\partial\sigma} = \frac{1}{2}(\vec{F}'(ct+\sigma) - \vec{F}'(ct-\sigma)), \tag{323}$$

$$\frac{1}{c}\frac{\partial\vec{X}}{\partial t} = \frac{1}{2}(\vec{F}'(ct+\sigma) + \vec{F}'(ct-\sigma)). \tag{324}$$

Plugging this back gives

$$\frac{\partial\vec{X}}{\partial\sigma} \pm \frac{1}{c}\frac{\partial\vec{X}}{\partial t} = \pm\vec{F}'(ct\pm\sigma). \tag{325}$$

We thus find the simple condition on $\vec{F}$, namely

$$\|\vec{F}'(u)\|^2 = 1. \tag{326}$$

We thus conclude that the general motion of a string satisfying (312 - 315) can be written as

$$\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma)), \quad \sigma \in [0,\sigma_1], \tag{327}$$

where $\sigma_1 = E/T_0$ and the (yet) undetermined vector function $\vec{F} = \vec{F}(u)$ has to satisfy

$$\|\vec{F}'(u)\|^2 = 1, \quad \vec{F}(u + 2\sigma_1) = \vec{F}(u) + 2\sigma_1 \frac{\vec{v}_0}{c}. \tag{328}$$

There is a direct interpretation of the function $\vec{F}$. Indeed, one has $\vec{X}(t, 0) = \vec{F}(ct)$, that is

$$\vec{F}(u) = \vec{X}(\frac{u}{c}, 0). \tag{329}$$

We see that $\vec{F}(u)$ is the **position of the string** $\sigma = 0$ **endpoint at time** $\frac{u}{c}$. We can further interpret the free velocity parameter $\vec{v}_0$.

Exercise 6.2. *Show that*

$$\vec{X}(t + \frac{2\sigma_1}{c}, \sigma) - \vec{X}(t, \sigma) = (\frac{2\sigma_1}{c})\vec{v}_0, \tag{330}$$

$\vec{v}_0$ ***is the average velocity of any point*** σ ***in an arbitrary time interval of length*** $\frac{2\sigma_1}{c}$.

Exercise 6.3. *Show that*

$$\frac{\partial \vec{X}}{\partial t}(t + \frac{2\sigma_1}{c}, \sigma) = \frac{\partial \vec{X}}{\partial t}(t, \sigma), \tag{331}$$

that is the velocity must be periodical with period $\frac{2\sigma_1}{c}$.

Let us now consider the following problem. We assume that we have an open string which rotates rigidly in (x, y) plane around its origin with a constant angular frequency ω. Suppose that it is of length ℓ. The movement of its endpoint will thus be described as

$$\vec{X}(t, 0) = (\frac{\ell}{2}\cos(\omega t), \frac{\ell}{2}\sin(\omega t)), \tag{332}$$

where we explicitly describe only the first two components. This forces $\vec{F}$ to have the form

$$\vec{F}(u) \equiv \vec{X}(\frac{u}{c}, 0) = (\frac{\ell}{2}\cos(\frac{\omega u}{c}), \frac{\ell}{2}\sin(\frac{\omega u}{c})). \tag{333}$$

Since $\vec{v}_0$ is the average velocity during the period of the velocity vector of the endpoint, and this does repeat after a full turn of a string, we have $\vec{v}_0 = \vec{0}$. We thus assume that $\vec{F}$ is periodic with period $2\sigma_1$. This gives the condition

$$\frac{\omega}{c}(2\sigma_1) = 2\pi m \Rightarrow \frac{\omega}{c} = \frac{\pi}{\sigma_1} m. \tag{334}$$

where $m \in \mathbb{Z}$. Now, observe that this gives

$$\vec{X}(0, \sigma) = \frac{1}{2}(F(\sigma) + F(-\sigma)) = (\frac{\ell}{2}\cos(\frac{\pi\sigma}{\sigma_1}m), 0). \tag{335}$$

But $\vec{X}(0, \sigma)$ must be injective for $\sigma \in [0, \sigma_1]$, which forces $m = 1$. We find that

$$\vec{F}(u) = (\frac{\ell}{2}\cos(\frac{\pi}{\sigma_1}u), \frac{\ell}{2}\sin(\frac{\pi}{\sigma_1}u)) \tag{336}$$

Finally, we have the normalization condition on the tangent vector to the curve $\vec{F}(u)$. One has

$$\vec{F}'(u) = (-\frac{\ell\pi}{2\sigma_1}\sin(\frac{\pi}{\sigma_1}u), \frac{\ell\pi}{2\sigma_1}\cos(\frac{\pi}{\sigma_1}u)). \tag{337}$$

We see that

$$\|\vec{F}'(u)\| = 1 \Leftrightarrow \frac{\ell^2\pi^2}{4\sigma_1^2} = 1. \tag{338}$$

If we recall that $\sigma_1 = \frac{E}{T_0}$, this gives a relation between the length ℓ and the overall energy of the string, namely

$$E = \frac{\pi}{2}T_0\ell. \tag{339}$$

In terms of energy, we $\omega = \frac{T_0\pi c}{E}$. The velocity of the endpoint is thus

$$\frac{\ell}{2}\omega = \frac{1}{2}\frac{2E}{\pi T_0}\frac{T_0\pi c}{E} = c. \tag{340}$$

In conclusion, we have found that

$$\vec{F}(u) = \frac{\sigma_1}{\pi}(\cos(\frac{\pi u}{\sigma_1}), \sin(\frac{\pi u}{\sigma_1})). \tag{341}$$

We can now use it to find the final solution to our problem, finding:

$$\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos(\frac{\pi\sigma}{\sigma_1})\cdot(\cos(\frac{\pi ct}{\sigma_1}), \sin(\frac{\pi ct}{\sigma_1})). \tag{342}$$

6.5 Motion of closed strings and cusps

Let us now try to solve the motion of the closed string. We again solve (312 - 314), except there is no boundary condition (315). It is replaced by the periodicity condition in parameter σ. The solution to (312) is given by

$$\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{G}(ct-\sigma)). \tag{343}$$

By plugging into (322), we find the normalization of the tangent vectors in the form

$$\|\vec{F}'(u)\|^2 = \|\vec{G}'(v)\|^2 = 1, \tag{344}$$

for all $u, v \in \mathbb{R}$. We have to impose the condition

$$\vec{X}(t,\sigma+\sigma_1) = \vec{X}(t,\sigma), \tag{345}$$

where $\sigma_1 = E/T_0$. We can pass to new independent variables $u := ct + \sigma$ and $v := ct - \sigma$. By plugging in (343) into (345), we find

$$\frac{1}{2}(\vec{F}(u+\sigma_1) + \vec{G}(v-\sigma_1)) = \frac{1}{2}(\vec{F}(u) + \vec{G}(u)). \tag{346}$$

This can be rewritten as

$$\vec{F}(u+\sigma_1) - \vec{F}(u) = \vec{G}(v) - \vec{G}(v-\sigma_1). \tag{347}$$

Since u and v are independent variables, both sides must be the identical constant vectors. This means that both $\vec{F} = \vec{F}(u)$ and $\vec{G} = \vec{G}(v)$ change by the same constant vector when their argument is increased by σ_1. In particular, one finds

$$\vec{F}'(u+\sigma_1) = \vec{F}'(u), \;\; \vec{G}'(v+\sigma_1) = \vec{G}'(v). \tag{348}$$

Up to some integration constant, the motion of the closed string is thus described by two periodic unit vectors $\vec{F}'(u)$ and $\vec{F}'(v)$. These can be equivalently viewed as two loops on a unit sphere. There are some strange things that can happen when the two loops intersects, that is

$$\vec{F}'(u_0) = \vec{G}'(v_0), \tag{349}$$

for some values $u_0, v_0 \in \mathbb{R}$. Let (t_0, σ_0) be the corresponding values of the parameters, that is $u_0 = ct_0 + \sigma_0$ and $v_0 = ct_0 - \sigma_0$. One finds

$$\frac{1}{c}\frac{\partial \vec{X}}{\partial t}(t_0, \sigma_0) = \frac{1}{2}(\vec{F}'(u_0) + \vec{G}'(v_0)) = \vec{F}'(u_0). \tag{350}$$

Since the right-hand side is a unit vector, we realize that at t_0, the point $\sigma = \sigma_0$ on the string reaches the speed of light! What is even worse, one gets

$$\frac{\partial \vec{X}}{\partial \sigma}(t_0, \sigma_0) = \frac{1}{2}(\vec{F}'(u_0) - \vec{G}'(v_0)) = \vec{0}. \tag{351}$$

But this means that the parametrization of the t_0 string becomes singular at $\sigma = \sigma_0$. By fixing $t = t_0$, we can consider the Taylor expansion:

$$\vec{X}(t_0, \sigma) = \vec{X}(t_0, \sigma_0) + \frac{1}{2}(\sigma - \sigma_0)^2 \frac{\partial^2 \vec{X}}{\partial \sigma^2}(t_0, \sigma_0) + \frac{1}{3!}(\sigma - \sigma_0)^3 \frac{\partial^3 \vec{X}}{\partial \sigma^3}(t_0, \sigma_0) + \cdots \tag{352}$$

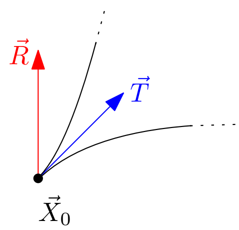
By defining the constant vectors

$$\vec{X}_0 := \vec{X}(t_0, \sigma_0), \quad \vec{T} := \frac{\partial^2 \vec{X}}{\partial \sigma^2}(t_0, \sigma_0), \quad \vec{R} := \frac{\partial^3 \vec{X}}{\partial \sigma^3}(t_0, \sigma_0), \tag{353}$$

we see that

$$\vec{X}(t_0, \sigma) = \vec{X}_0 + \frac{1}{2}(\sigma - \sigma_0)^2 \vec{T} + \frac{1}{3!}(\sigma - \sigma_0)^3 \vec{R} + \cdots . \tag{354}$$

In the general scenario, $\vec{T}$ and $\vec{R}$ are non-zero and non-parallel. As σ reaches σ_0 from below, the string reaches $\vec{X}_0$ along the $\vec{T}$ and then moves away along $\vec{T}$ as σ grows above σ_0. One says that the string forms a **cusp**. For bigger $\sigma - \sigma_0$ the cusp opens thanks to the $\vec{R}$ term.



Note that since $\vec{F}'(u)$ and $\vec{G}'(v)$ are periodic with period σ_1, the cusps on the closed strings do appear and disappear periodically. It can also happen that the two loops on the sphere intersect at several points - each of these intersections comes with its sequence of cusps.

Exercise 6.4. *The movement of the initially static closed string: Suppose we assume*

$$\frac{\partial \vec{X}}{\partial t}(0, \sigma) = 0. \tag{355}$$

How the general discussion changes? Suppose that at $t = 0$, the string traces a closed curve $\vec{\gamma}$ of length ℓ. Describe the procedure of a solution.

Proof. By plugging (343) into (355), we find that

$$\vec{0} = \frac{1}{c}\frac{\partial \vec{X}}{\partial t}(0,\sigma) = \frac{1}{2}(\vec{F}'(\sigma) + \vec{G}'(-\sigma)), \tag{356}$$

This shows that $\vec{G}'(u) = -\vec{F}'(-u)$ for all $u \in \mathbb{R}$, which can be integrated to $\vec{G}(u) = \vec{F}(-u) + \vec{a}_0$. By plugging this back into (343), we see that one can absorb the integration constant by suitably redefining $\vec{F}$ and we can assume that

$$\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(\sigma + ct) + \vec{F}(\sigma - ct)). \tag{357}$$

We can now plug into the periodicity condition $\vec{X}(t, \sigma + \sigma_1) = \vec{X}(t, \sigma)$. By defining the independent variables $u := \sigma + ct$ and $v := \sigma - ct$, one obtains the condition

$$\vec{F}(u + \sigma_1) - \vec{F}(u) = -(\vec{F}(v + \sigma_1) - \vec{F}(v)). \tag{358}$$

Both sides must be equal to a constant vector. But this is only possible if this vector is zero and we conclude that $\vec{F}$ must be a periodic function:

$$\vec{F}(u + \sigma_1) = \vec{F}(u). \tag{359}$$

The interpretation of $\vec{F}$ is obvious - it corresponds to the initial shape of the string, since

$$\vec{F}(u) = \vec{X}(0, u). \tag{360}$$

To fully solve the equations of motion, we thus have to specify the initial shape of the string. However, there are still parametrization constraints. One finds

$$\frac{1}{c}\frac{\partial \vec{X}}{\partial t}(t,\sigma) = \frac{1}{2}(\vec{F}'(u) - \vec{F}'(v)), \quad \frac{\partial \vec{X}}{\partial \sigma}(t,\sigma) = \frac{1}{2}(\vec{F}'(u) + \vec{F}'(v)). \tag{361}$$

We thus obtain a single condition, namely

$$|\vec{F}'(u)|^2 = 1, \tag{362}$$

for all $u \in \mathbb{R}$. Equivalently, for an initially static string, we can examine (314) at $(0, \sigma)$, finding

$$(\frac{\partial \vec{X}}{\partial \sigma})^2(0,\sigma) = 1. \tag{363}$$

But this means that at $t = 0$, the length the initial string must be parametrized by its length! This gives us the relation of the initial length ℓ an the energy of the string, namely

$$E = T_0 \ell. \tag{364}$$

Compare this to (339). Let us now discuss the general solution. Suppose we want the string to trace the initial shape $\vec{\gamma} = \vec{\gamma}(\lambda)$, where $\lambda \in [0, \lambda_0]$. We thus have to parametrize the string by its length, that is introduce a new parameter

$$\sigma(\lambda) := \int_0^\lambda \|\frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}\lambda}\| \,\mathrm{d}\lambda \tag{365}$$

In order for σ to be a well-defined parameter, we must assume that $\|\frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}\lambda}\| > 0$. The length of the string is then just $\ell = \sigma(\lambda_0)$. One then has to invert the relation and write $\lambda = \lambda(\sigma)$. The function F is then defined simply by

$$\vec{F}(u) := \vec{\gamma}(\lambda(u)). \tag{366}$$

■

Exercise 6.5. *Let us consider the situation from Exercise 5.9, where at $t = 0$, the closed string forms a circle of radius R_0 in the (x, y) plane. Show that the assumption that the string remains circular was correct.*

Proof. In light of the previous exercise, we have $\vec{\gamma}(\lambda) = R_0(\cos(\lambda), \sin(\lambda))$, $\lambda \in [0, 2\pi]$. We have $\|\frac{d\vec{\gamma}}{d\lambda}\| = R_0$, whence $\sigma(\lambda) = R_0\lambda$. We thus have to set

$$\vec{F}(u) = R_0(\cos(\frac{u}{R_0}), \sin(\frac{u}{R_0})). \tag{367}$$

We also get the relation of the energy of the string to R_0, that is $E = 2\pi T_0 R_0$, which agrees with Exercise 5.9. Finally, the full movement of the string is given by

$$\vec{X}(t, \sigma) = \frac{1}{2}(\vec{F}(\sigma + ct) + \vec{F}(\sigma - ct)) = R_0 \cos(\frac{ct}{R_0}) \cdot (\cos(\frac{\sigma}{R_0}), \sin(\frac{\sigma}{R_0})). \tag{368}$$

The function multiplying the vector is indeed the previously obtained function $R(t)$ and for each t, the string is a circle of radius $R(t)$. ■

7 Worldsheet currents

7.1 Noether theorem for field theories

Let us consider the following general setting. Suppose we have an action functional

$$S = \int d\xi^1 \cdots d\xi^k \mathcal{L}(\phi^a, \partial_\alpha \phi^a), \tag{369}$$

for fields $\phi^a = \phi^a(\xi^1, \dots, \xi^k)$, $a \in \{1, \dots, n\}$. We write $\partial_\alpha \phi^a = \frac{\partial \phi^a}{\partial \xi^\alpha}$. The Lagrangian density $\mathcal{L}$ is an ordinary function in $n + k \cdot n$ variables. We can consider the variation

$$\phi'^a := \phi^a + \delta\phi^a \tag{370}$$

The derivatives vary accordingly, that is $\partial_\alpha \phi'^a = \partial_\alpha \phi^a + \partial_\alpha(\delta\phi^a)$. We usually consider the infinitesimal variations of the form $\delta\phi^a = \epsilon^i h^a_i(\phi)$, where ϵ^i are independent parameters of the transformation which are assumed to be small. One finds

$$\mathcal{L}(\phi'^a, \partial_\alpha \phi'^a) = \mathcal{L}(\phi^a, \partial_\alpha \phi^a) + \delta\mathcal{L} + O(\epsilon^2), \tag{371}$$

We say that $\mathcal{L}$ is **invariant with respect to** (370), if $\delta\mathcal{L} = 0$. Explicitly, one has

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\partial_\alpha(\delta\phi^a). \tag{372}$$

Beware that there are two Einstein summations involved. Now, suppose that ϕ^a satisfy the equations of motion. This means that they are subject to the Lagrange-Euler equation:

$$\frac{\partial \mathcal{L}}{\partial \phi^a} = \partial_\alpha(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}), \tag{373}$$

for each $a \in \{1, \dots, n\}$. For the solutions of equations of motion, one thus finds

$$0 = \delta\mathcal{L} = \partial_\alpha(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a) = \epsilon^i \partial_\alpha(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}h^a_i(\phi)). \tag{374}$$

This suggests to define the **conserved currents** in the form

$$j_i^\alpha := \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} h_i^a(\phi). \tag{375}$$

Let us emphasize that j_i^α are functions of just $(\xi^1, \dots, \xi^k)$, we assume that we plug in some field configuration to the right-hand side. We have just argued that for a solution of equations of motion, assuming that $\mathcal{L}$ is invariant under (370), these satisfy

$$\partial_\alpha j_i^\alpha = 0. \tag{376}$$

Exercise 7.1. *Suppose j_i^α is a conserved current, and assume that ξ^0 has the role of a "time parameter". Let us consider the quantity*

$$Q_i(\xi^0) := \int \mathrm{d}\xi^1 \cdots \mathrm{d}\xi^k j_i^0. \tag{377}$$

Show that $\frac{\mathrm{d}Q}{\mathrm{d}\xi^0} = 0$, if j_i^α vanish at the boundary (or in infinities) for $\alpha \in \{1, \dots, k\}$. For each infinitesimal parameter, we thus obtain an actual integral of motion Q_i called the ***conserved charge*** *(which is actually conserved).*

7.2 Conserved currents on the worldsheet

Now, recall that relativistic string is a two-dimensional field theory, where $(\xi^1, \xi^2) = (\tau, \sigma)$. The fields of the theory are $X^\mu = X^\mu(\tau, \sigma)$. In the context of the previous subsection, we thus consider the infinitesimal variations

$$\delta X^\mu = \epsilon^i h_i^\mu(X), \tag{378}$$

and the corresponding conserved currents are then have two components:

$$j_i^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} h_i^\mu = \mathcal{P}_\mu^\tau h_i^\mu(X), \;\; j_i^\sigma = \frac{\partial \mathcal{L}}{\partial X'^\mu} h_i^\mu(X) = \mathcal{P}_\mu^\sigma h_i^\mu(X) \tag{379}$$

In particular, the Lagrange density depends only on the derivatives of fields. This means that one can consider transformations

$$\delta X^\mu := \epsilon^\mu = \epsilon^\nu \delta_\mu^\nu. \tag{380}$$

The index labeling the infinitesimal parameters coincides with the spacetime index. The corresponding conserved currents are thus labeled by $\mu \in \{0, \dots, D\}$ and have the components

$$j_\mu^\tau = \mathcal{P}_\mu^\tau, \;\; j_\mu^\sigma = \mathcal{P}_\mu^\sigma. \tag{381}$$

The equation $\partial_\alpha j_\mu^\alpha = 0$ turns into

$$\frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = 0, \tag{382}$$

that is equations of motion for the string. Now, according to Exercise 7.1, the corresponding charge is the total space momentum

$$p_\mu(\tau) := \int_0^{\sigma_1} \mathrm{d}\sigma \; \mathcal{P}_\mu^\tau(\tau, \sigma). \tag{383}$$

Let us examine its derivative with respect to the parameter τ. One finds

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = \int_0^{\sigma_1} \mathrm{d}\sigma \frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} = -\int_0^{\sigma_1} \mathrm{d}\sigma \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = -[\mathcal{P}^\sigma_\mu]^{\sigma_1}_{\sigma=0}. \tag{384}$$

For open string, if the free endpoint condition is imposed on both ends, this makes the quantity p_μ constant in τ. This is true also for a closed string, since there is a periodicity condition imposed on the fields. p_μ is not conserved for Dirichlet endpoints!

7.3 The complete momentum current

There is one issue with the above discussion. Our description of the string was intended to be reparametrization invariant. Our definition of the conserved momentum currents depends on the parametrization. Is there some way to describe a momentum conservation of a string?

First, there is one important observation. Let us consider a general parametrization (ξ^1, ξ^2) of the worldsheet. Let

$$\mathcal{P}^\alpha_\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \tag{385}$$

be the associated momentum current. This can be viewed as a vector field $\mathcal{P}_\mu = \mathcal{P}^\alpha_\mu \partial_\alpha$.

Now, suppose that a Lorentz observer a string at some coordinate time t. He uses a static gauge $\tau = t$ do describe the worldsheet and constructs a a density currents $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$. He then defines a momentum of a $\tau = t$ string by a integral $p_\mu(t) = \int_0^{\sigma_1} \mathrm{d}\sigma \mathcal{P}^\tau_\mu(\sigma, t)$. This can be interpreted as a flux of $\mathcal{P}_\mu$ across the curve γ of constant τ. Indeed, the unit vector $\vec{n}_\perp$ perpendicular to γ is ∂_τ and $\mathcal{P}_\mu \cdot \partial_\tau = \mathcal{P}^\tau_\mu$.

This leads us to the following idea. Let γ be *any* curve in the worlsheet connecting its $\sigma = 0$ boundary to the $\sigma = \sigma_1$ boundary. The flux of the vector field $\mathcal{P}_\mu$ across γ is usually written as

$$p_\mu(\gamma) := \int_\gamma (\mathcal{P}^\tau_\mu \mathrm{d}\sigma - \mathcal{P}^\sigma_\mu \mathrm{d}\tau) \tag{386}$$

In more detail, if $\gamma(\lambda) = (\tau(\lambda), \sigma(\lambda))$ for $\lambda \in [0, \lambda_0]$, the integral is given by

$$p_\mu(\gamma) = \int_0^{\lambda_0} \mathrm{d}\lambda \left(\mathcal{P}^\tau_\mu(\tau(\lambda), \sigma(\lambda)) \frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}(\lambda) - \mathcal{P}^\sigma_\mu(\tau(\lambda), \sigma(\lambda)) \frac{\mathrm{d}\tau}{\mathrm{d}\lambda}(\tau)\right. \tag{387}$$

The crucial observation is the following one:

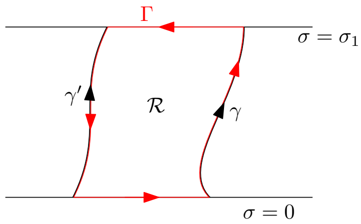
Proposition 7.2. *Suppose γ' is any other curve connecting the $\sigma = 0$ and $\sigma = \sigma_1$ boundary of the worldsheet. Then*

$$p_\mu(\gamma) = p_\mu(\gamma'). \tag{388}$$

Proof. First observe that if Γ is any closed curve in Σ encircling a simply connected region $\mathcal{R}$, then the outgoing flux over Γ is

$$\oint_\Gamma \mathcal{P}^\tau_\mu \mathrm{d}\sigma - \mathcal{P}^\sigma_\mu \mathrm{d}\tau = \int_\mathcal{R} (\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma}{\partial \sigma}) \mathrm{d}\tau \mathrm{d}\sigma = 0. \tag{389}$$

If γ and γ' are two curves connecting the boundaries of the worldsheet, we may consider the curve Γ as in the following figure:

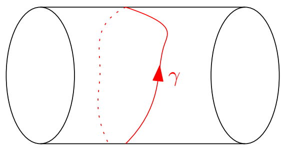


We see that outgoing flux across Γ is $p_\mu(\gamma) - p_\mu(\gamma')$ plus the flux across the $\sigma = \sigma_0$ and $\sigma = \sigma_1$ segments. But those do not contribute thanks to the free endpoint conditions. Indeed, the flux across the lower segment is

$$-\int_{\tau_0}^{\tau_1} \mathcal{P}^\sigma_\mu(\tau, 0)\mathrm{d}\tau = 0. \tag{390}$$

This finishes the proof. ■

For closed strings, the situation is somewhat similar, excepts γ is assumed to be an arbitrary closed curve wrapping once around the worldsheet. The proof of the independence of $p_\mu(\gamma)$ on a particular choice of the curve is analogous, except the boundary Γ of the region $\mathcal{R}$ consists just of the two curves γ and γ'.



How is this useful for the notion of a string momentum? Suppose a given Lorentz observer observes a string at some coordinate time t. Geometrically, this is an intersection of the string worldsheet with the $x^0 = ct$ hyperplane. Suppose he uses an arbitrary parametrization (τ, σ). In this parametrization, string is a general curve γ in the parameter space connecting the two edges of a worldsheet. He then views $p_\mu(\gamma)$ as a momentum of the string at time t.

If he decides to do so at any other coordinate time t', using the same parametrization, he calculates $p_\mu(\gamma')$ along the different curve γ'. By comparing the two numerical values, he realizes that the momentum is preserved.

Note that the numerical value of p_μ can still depend on the parametrization. It turns out that this is not the case

Exercise 7.3. *Consider a general reparametrization $\xi'^\alpha = \xi'^\alpha(\xi^1, \xi^2)$. Let*

$$\mathbf{J} = \begin{pmatrix} \frac{\partial \xi'^1}{\partial \xi^1} & \frac{\partial \xi'^1}{\partial \xi^2} \\ \frac{\partial \xi'^2}{\partial \xi^1} & \frac{\partial \xi'^2}{\partial \xi^2} \end{pmatrix}, \tag{391}$$

be its Jacobi matrix. Show that the corresponding momentum currents $\mathcal{P}^\alpha_\mu$ and $\mathcal{P}'^\alpha_\mu$ are related by the formula

$$\begin{pmatrix} \mathcal{P}'^1_\mu \\ \mathcal{P}'^2_\mu \end{pmatrix} = \frac{1}{|\det \mathbf{J}|}\mathbf{J}\begin{pmatrix} \mathcal{P}^1_\mu \\ \mathcal{P}^2_\mu \end{pmatrix} \tag{392}$$

Show that when $|\det \mathbf{J}| > 0$, the flux (386) in both parametrizations is the same.

Proof. Directly from (226) and (227), using the fact that the Lagrangian density gets multiplied by $|\det \mathbf{J}|$ under reparametrization, one can derive the formulas

$$\mathcal{P}^1_\mu = \frac{\det \mathbf{J}}{|\det \mathbf{J}|}(\mathbf{J}^2_2 \mathcal{P}'^1_\mu - \mathbf{J}^2_1 \mathcal{P}'^2_\mu), \quad \mathcal{P}^2_\mu = \frac{\det \mathbf{J}}{|\det \mathbf{J}|}(-\mathbf{J}^1_2 \mathcal{P}'^1_\mu + \mathbf{J}^1_1 \mathcal{P}'^2_\mu), \tag{393}$$

By rearranging this using the explicit formula for 2×2 matrix inverse, we get (392). Now, the differentials transform as

$$(\mathrm{d}\xi'^1, \mathrm{d}\xi'^2) = (\mathrm{d}\xi^1, \mathrm{d}\xi^2) \cdot \mathbf{J}. \tag{394}$$

Let us write $\mathcal{P}_\mu = \mathcal{P}^\alpha_\mu \frac{\partial}{\partial \xi^\alpha}$ and similarly $\mathcal{P}'_\mu$ is defined with primed variables. Those are now the same vector fields, but one has $\mathcal{P}'_\mu = |\det \mathbf{J}|^{-1} \mathcal{P}_\mu$. However, one can define a 1-form

$$\iota_{\mathcal{P}_\mu}(\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2). \tag{395}$$

The contributions from the determinants cancel, and one finds

$$\iota_{\mathcal{P}'_\mu}(\mathrm{d}\xi'^1 \wedge \mathrm{d}\xi'^2) = \mathrm{sgn}(\det(\mathbf{J})) \cdot \iota_{\mathcal{P}_\mu}(\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2) \tag{396}$$

If $\det(\mathbf{J}) > 0$, those forms coincide. But the flux (386) in two parametrizations is then just an integral of the same 1-form over γ. ∎

Finally, note that the constants p_μ are connected with the choice of Lorentz frame x^μ. It is easy to check that if we choose a different Lorentz frame $x'^\mu = L^\mu{}_\nu x^\nu$, they transform as Lorentz covectors, that is

$$p'_\mu = L_\mu{}^\nu p_\nu. \tag{397}$$

7.4 Currents associated to Lorentz symmetry

We have constructed the Lagrangian density for the string to be invariant under Lorentz transformations. How does this translate in terms of infinitesimal transformations?

Recall that the spacetime coordinates transform as $x'^\mu = L^\mu{}_\nu x^\nu$. To examine the infinitesimal transformations, write $L^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$, for some matrix $\epsilon^\mu{}_\nu$ with very small inputs. Since L is to be a Lorentz transformation, we get the relation

$$\eta_{\lambda\kappa} = \eta_{\mu\nu} L^\mu{}_\lambda L^\nu{}_\kappa = \eta_{\mu\nu}(\delta^\mu_\lambda + \epsilon^\mu{}_\lambda)(\delta^\nu_\kappa + \epsilon^\nu{}_\kappa) = \eta_{\lambda\kappa} + \eta_{\lambda\nu}\epsilon^\nu{}_\kappa + \eta_{\mu\kappa}\epsilon^\mu{}_\lambda + O(\epsilon^2) \tag{398}$$

This imposes the condition $\eta_{\lambda\nu}\epsilon^\nu{}_\kappa + \eta_{\mu\kappa}\epsilon^\mu{}_\lambda = 0$. Invoking the usual convention for raising and lowering indices, this can be rewritten simply as $\epsilon_{\lambda\kappa} + \epsilon_{\kappa\lambda} = 0$. Pluging back into the coordinate transformation then reads

$$x'^\mu = (\delta^\nu_\nu + \epsilon^\mu{}_\nu)x^\nu = x^\mu + \epsilon^{\mu\nu} x_\nu, \tag{399}$$

where the matrix with two indices up is also skew-symmetric, that is $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$.

Going back to the relativistic string, this means that the corresponding infinitesimal transformation of the string fields reads

$$X'^\mu = X^\mu + \epsilon^{\mu\nu} X_\nu, \text{ that is } \delta X^\mu = \epsilon^{\mu\nu} X_\nu. \tag{400}$$

Exercise 7.4. *Check that $\mathcal{L}$ of the Nambu-Goto action is indeed invariant under (400).*

Proof. Since $\mathcal{L}$ is independent of X^μ, one has

$$\begin{aligned}\delta\mathcal{L} = & \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^\mu)}\partial_\alpha(\delta X^\mu) = \mathcal{P}^\tau_\mu \delta\dot{X}^\mu + \mathcal{P}^\sigma_\mu \delta(X'^\mu) \\ = & \,\epsilon^{\mu\nu}(\mathcal{P}^\tau_\mu \dot{X}_\nu + \mathcal{P}^\sigma_\mu X'_\nu) \\ = & -\frac{\epsilon^{\mu\nu}T_0}{c}\Big(\frac{(\dot{X}\cdot X')(X'_\mu\dot{X}_\nu + \dot{X}_\mu X'_\nu) - (X')^2\dot{X}_\mu\dot{X}_\nu - (X')^2 X'_\mu X'_\nu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}\Big) \\ = & \,0,\end{aligned} \tag{401}$$

where we have used the fact that $\epsilon^{\mu\nu}$ is skew-symmetric and the term in the big parentheses is symmetric in (μ, ν). ■

We can rewrite the variation in the following form

$$\delta X^\mu = \epsilon^{\lambda\nu}\delta^\mu_\lambda X_\nu = \frac{1}{2}\epsilon^{\lambda\nu}(\delta^\mu_\lambda X_\nu - \delta^\mu_\nu X_\lambda). \tag{402}$$

We see that, using the notation introduced above, one identifies $h^\mu_{\lambda\nu} = \delta^\mu_\lambda X_\nu - \delta^\mu_\nu X_\lambda$, and the corresponding conserved current is

$$j^\alpha_{\lambda\nu} := \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^\mu)}h^\mu_{\lambda\nu} = \mathcal{P}^\alpha_\mu(\delta^\mu_\lambda X_\nu - \delta^\mu_\nu X_\lambda) = \mathcal{P}^\alpha_\lambda X_\nu - \mathcal{P}^\alpha_\nu X_\lambda. \tag{403}$$

Since the overall factor is irrelevant, one defines the respective currents as:

$$\mathcal{M}^\alpha_{\mu\nu} := X_\mu\mathcal{P}^\alpha_\nu - X_\nu\mathcal{P}^\alpha_\mu. \tag{404}$$

For X^μ solving the equations of motion, one has

$$\frac{\partial\mathcal{M}^\tau_{\mu\nu}}{\partial\tau} + \frac{\partial\mathcal{M}^\sigma_{\mu\nu}}{\partial\sigma} = 0. \tag{405}$$

Similarly to the momentum, the corresponding conserved charges are defined by an integral

$$M_{\mu\nu} := \int_\gamma(\mathcal{M}^\tau_{\mu\nu}\mathrm{d}\sigma - \mathcal{M}^\sigma_{\mu\nu}\mathrm{d}\tau). \tag{406}$$

The fact that the definition does not depend on γ and can be interpreted as a conserved quantity needs to be checked for the open strings. For this to be correct, the integrals along the segments on the worldsheet boundary must vanish. But this happens since $\mathcal{M}^\sigma_{\mu\nu}(\tau,\sigma_*) = 0$ due to free endpoint conditions.

Now, for $D = 4$, we have six independent conserved charges. Three correspond to rotations, and are usually identified with the conserved angular momentum, that is $L_k = \frac{1}{2}\epsilon_{kij}M_{ij}$. The other two are associated to the Lorentz boosts. Explicitly, in the static gauge, one has

$$M^{0i} = \int_0^{\sigma_1}\mathrm{d}\sigma(X^0\mathcal{P}^{\tau i} - X^i\mathcal{P}^{\tau 0}) = ctp^i - \int_0^{\sigma_1}\mathrm{d}\sigma X^i\mathcal{P}^{\tau 0}. \tag{407}$$

By multiplying both sides by c/E, where E is the overall energy of the string, and rearranging, one has

$$\frac{1}{E}\int_0^{\sigma_1}\mathrm{d}\sigma(X^i\cdot\frac{\mathcal{P}^{\tau 0}}{c}) = -\frac{cM^{0i}}{E} + t\frac{c^2p^i}{E}. \tag{408}$$

Since $\mathcal{P}^{\tau 0}/c$ is the energy density, the left-hand side can be viewed as a position of the center of mass, and we have found that

$$X^i_{cm}(t) = -\frac{cM^{0i}}{E} + t\frac{c^2p^i}{E}. \tag{409}$$

7.5 The slope parameter α'

Let us consider the rigid rotating string in the (x, y) plane. We have already solved this problem, finding the solution (342). We sill show that there is a constant of proportionality α' relating the overall angular momentum J of the string to its energy as

$$\frac{J}{\hbar} = \alpha' E^2. \tag{410}$$

Note that $[J] = ML^2T^{-1} = [\hbar]$, so the left-hand side has no dimension and thus $[\alpha'] = [E]^{-2}$. The only non-vanishing spatial component of the angular momentum is M_{12}, given by the formula

$$M_{12} = \int_0^{\sigma_1} (X_1 \mathcal{P}_2^\tau - X_2 \mathcal{P}_1^\tau) \mathrm{d}\sigma, \tag{411}$$

that is $J = |M_{12}|$. In our chosen parametrization (t, σ), we have

$$\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi} \cos(\frac{\pi\sigma}{\sigma_1}) \cdot (\cos(\frac{\pi ct}{\sigma_1}), \sin(\frac{\pi ct}{\sigma_1})), \tag{412}$$

and thus

$$\vec{\mathcal{P}}^\tau(t,\sigma) = \frac{T_0}{c^2}\frac{\partial \vec{X}}{\partial t} = \frac{T_0}{c}\cos(\frac{\pi\sigma}{\sigma_1})(-\sin(\frac{\pi ct}{\sigma_1}), \cos(\frac{\pi ct}{\sigma_1})) \tag{413}$$

Consequently, one finds

$$M_{12} = \frac{\sigma_1}{\pi}\frac{T_0}{c}\int_0^{\sigma_1} \mathrm{d}\sigma \ \cos^2(\frac{\pi\sigma}{\sigma_1})\{\cos^2(\frac{\pi ct}{\sigma_1}) + \sin^2(\frac{\pi ct}{\sigma_1})\} = \frac{\sigma_1^2 T_0}{2\pi c}. \tag{414}$$

Recall that $\sigma_1 = E/T_0$, that is we find

$$J = \frac{1}{2\pi T_0 c}E^2 = \frac{\hbar}{2\pi T_0 \hbar c}E^2, \tag{415}$$

hence finding

$$\alpha' = \frac{1}{2\pi T_0 \hbar c}, \ \ T_0 = \frac{1}{2\pi\alpha' \hbar c}. \tag{416}$$

Note that $[\alpha'] = M^{-2}T^4L^{-4}$. Note that the triple $\{\alpha', \hbar, c\}$ has the same convenient property as $\{G, \hbar, c\}$ we have discussed before. In particular, there is a unique length which can be obtained as a product of powers of those three quantities. This is called a **string length**:

$$\ell_s = \hbar c\sqrt{\alpha'} = \sqrt{\frac{\hbar c}{2\pi T_0}}. \tag{417}$$

Note that if ℓ_s is of the same order as ℓ_P, we find $T_0 \propto 10^{27}N$.

8 Light-cone relativistic string

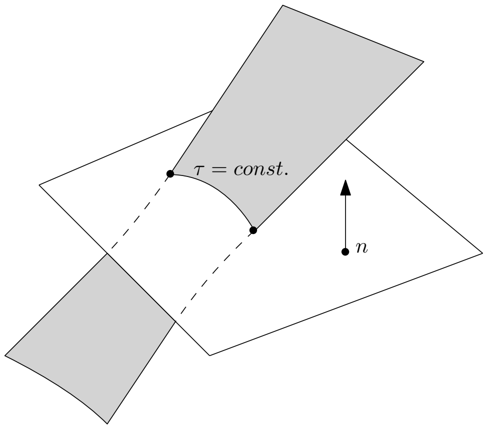
8.1 A class of choices for τ

So far, we have worked in the static gauge, where $X^0(\tau, \sigma) = c\tau$. This can be interpreted as follows - we want the intersection of the worldsheet and hyperplanes of constant t to correspond to the constant value of τ. The normal vector to those hyperplanes is $n = (1, 0, \dots, 0)$. The above condition can be thus rewritten as $n_\mu X^\mu(\tau, \sigma) = c\tau$.

Now, suppose n is an arbitrary non-zero constant vector. We can now require

$$n_\mu X^\mu(\tau, \sigma) = \lambda\tau, \tag{418}$$

where $\lambda \neq 0$ is some constant. This means that the intersection of any hyperplane orthogonal to n with the worldsheet is a line of constant τ.



Let ΔX^μ be a difference of any two points of the worldsheet with the same value of τ. We *do not* want this to be a timelike vector. Since $n_\mu \Delta X^\mu = 0$, it suffices to assume that n is *not spacelike.*

Exercise 8.1. *Let $n \in \mathbb{R}^D$ be a non-zero vector which is not spacelike. Then any Lorentz vector $a \in \mathbb{R}^D$ which satisfies $n \cdot a = 0$ is not timelike.*

Proof. First, suppose that n is timelike. Since $n \cdot a = 0$ is a Lorentz invariant condition, we can choose the frame where $n = (n^0, \vec{0})$. Then $n \cdot a = n^0 a_0$ and thus $a_0 \neq 0$, that is a cannot be timelike. Next, suppose that n is lightlike. Write $n = (n^0, \vec{n})$. We thus have $(n^0)^2 = (\vec{n})^2$. Suppose for a contradiction that a is timelike. We can thus examine everything in the frame where $a = (a^0, \vec{0})$, $a^0 \neq 0$. Hence $0 = a \cdot n = a_0 n^0$ implies $n^0 = 0$ and thus also $(\vec{n})^2 = 0$, which contradicts the assumption $n \neq 0$. ■

We must also argue that we can always consider such a gauge. Starting with a parametrization (τ, σ), we thus pass to a new parameters $\sigma' := \sigma$ and $\tau' := \frac{1}{\lambda} n_\mu X^\mu(\tau, \sigma)$. For this to be a reparametrization, we need to check that $n \cdot \dot{X} \neq 0$. Without the loss of generality, we can assume that (t, σ) are the convenient parameters from the previous sections, where $\dot{X} = (c, \vec{v}_\perp)$. This vector is timelike except for string endpoints. If n is timelike, $\dot{X}$ can never be orthogonal to it by previous exercise. If n is lightlike (which we will consider), it can happen that $n \cdot \dot{X} = 0$ at string endpoints - the parametrization has its limits.

Exercise 8.2. *Suppose $a, b \in \mathbb{R}^D$ are two non-zero lightlike vectors with $a \cdot b = 0$. Then a and b are colinear.*

Proof. We can rotate the spatial coordinates so that $a = (a^0, a^0, \vec{0})$ and $b = (b^0, b^1, \vec{b})$ for $\vec{b} \in \mathbb{R}^{D-2}$ and $a^0, b^0 \neq 0$. The condition $a \cdot b = 0$ implies $a^0(b^0 - b^1) = 0$, that is $b^1 = b^0$. But the assumption $b^2 = 0$ then implies

$$0 = -(b^0)^2 + (b^0)^2 + (\vec{b})^2 \Rightarrow \vec{b} = 0. \tag{419}$$

This proves the claim. ■

This exercise shows that the issue arises for a lightlike n when some of the endpoints moves with the velocity $\vec{v}_\lambda = c \cdot \frac{\vec{n}}{\|\vec{n}\|}$. We thus have to assume that this is not happening. The non-zero constant λ is still for us to choose. Now, we have argued that to each string, there is an associated conserved momentum p^μ. This is a non-zero timelike vector, hence $n \cdot p \neq 0$. We will thus write

$$n_\mu X^\mu(\tau, \sigma) = (n \cdot p)\overline{\lambda}\tau, \tag{420}$$

for some non-zero constant $\overline{\lambda}$. Note that the momentum is conserved for open string with free endpoints. However, we only need to ensure that $n \cdot p$ is a conserved quantity. In order to achieve that, it suffices to impose a weaker condition on the endpoints, namely

$$n^\mu \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0. \tag{421}$$

Let us discuss units. We will henceforth choose (τ, σ) to be dimension-less. The dimension of n plays no role in the discussion, whence

$$[\overline{\lambda}] = L[p]^{-1} = L(MLT^{-1})^{-1} = TM^{-1}. \tag{422}$$

This is velocity divided by force, so the natural choice would be $\overline{\lambda} \propto \frac{c}{T_0} = 2\pi\alpha'\hbar c^2$.

Let us also further work in the **natural units**, where we set $\hbar = 1$ and $c = 1$, as if they were dimension-less. This a great computational help. If we want to physically interpret any resulting quantity, we must keep track of its original dimension - and insert (the suitable) power of $\hbar$ and c to get it. The simiplified relations between α' and T_0 and ℓ_s are then

$$\alpha' = \frac{1}{2\pi T_0}, \quad T_0 = \frac{1}{2\pi\alpha'}, \quad \ell_s = \sqrt{\alpha'}. \tag{423}$$

In the natural units, we fix (for open strings) the constant of proportionality to be $\overline{\lambda} = 2\alpha'$, so our final gauge condition is

$$n_\mu X^\mu(\tau, \sigma) = 2\alpha'(n \cdot p)\tau. \tag{424}$$

Exercise 8.3. *Argue whe p^μ is a non-zero timelike vector.*

Proof. We can work in the static gauge and a convenient parametrization. We have

$$\mathcal{P}^{\tau 0} = \frac{T_0}{c}, \quad \vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\vec{v}_\perp. \tag{425}$$

For a given t, we thus have

$$\begin{aligned} p^\mu p_\mu &= \int_0^{\sigma_1} \mathrm{d}\sigma \int_0^{\sigma_1} \mathrm{d}\sigma' \mathcal{P}^{\tau\mu}(t, \sigma)\mathcal{P}^\tau_\mu(t, \sigma') \\ &= -\frac{T_0^2}{c^4} \int_0^{\sigma_1} \mathrm{d}\sigma \int_0^{\sigma_1} \mathrm{d}\sigma' (c^2 - \vec{v}_\perp(t, \sigma) \cdot \vec{v}_\perp(t, \sigma')) \end{aligned} \tag{426}$$

We claim that the term under integral is strictly positive inside of the $[0, \sigma_1] \times [0, \sigma_1]$ square. Using the Cauchy-Schwarz inequality, we have

$$|\vec{v}_\perp(t, \sigma) \cdot \vec{v}_\perp(t, \sigma')| \leq v_\perp(t, \sigma) \cdot v_\perp(t, \sigma') \tag{427}$$

This expression strictly lesser then c^2 inside of the square. We see that $p^\mu p_\mu < 0$. ■

8.2 The associated σ parametrization

Now, in our choice of σ parametrization, we have chosen σ so that

$$1 = \frac{\mathrm{d}s}{\mathrm{d}\sigma} \frac{1}{\sqrt{1 - \frac{v_\perp^2}{c^2}}} \tag{428}$$

In particular, we require that $\mathcal{P}^{\tau 0}$ is (a very particular) constant along each string. Since $n = (1, \vec{0})$ in that case, we are in fact imposing $n \cdot \mathcal{P}^\tau$ to be constant along each string.

Now, let us consider a general reparametrization $\tau' := \tau$ and $\sigma' = \sigma'(\tau, \sigma)$. Let us examine how the quantity $n \cdot \mathcal{P}^\tau$ transforms under such reparametrization. Note that we are assuming $n \cdot X = \lambda\tau$, so $n \cdot \dot{X} = \lambda$ and $n \cdot X' = 0$. By plugging into (226), we thus have

$$n \cdot \mathcal{P}^\tau = -\frac{1}{2\pi\alpha'} \frac{\lambda T_0 (X')^2}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{429}$$

Now, recall that one has

$$\frac{\partial X^\mu}{\partial \sigma} = \frac{\partial \sigma'}{\partial \sigma} \frac{\partial X^\mu}{\partial \sigma'} \tag{430}$$

The denominator gets multiplied by an absolute value of the determinant of the transformation. But this is $|\frac{\partial \sigma'}{\partial \sigma}|$. We conclude that

$$n \cdot \mathcal{P}^\tau = |\frac{\partial \sigma'}{\partial \sigma}| \, (n \cdot \mathcal{P}'^\tau) \tag{431}$$

Exercise 8.4. *Calculate $\mathcal{P}^\tau_\mu \mathcal{P}^{\tau\mu}$ in an arbitrary parametrization (τ, σ). Show that in the gauge (418), $\mathcal{P}_\tau$ is not spacelike and it is timelike if and only if X' is spacelike.*

Proof. One uses (226). By simply plugging this into the square gives

$$\mathcal{P}^\tau_\mu \mathcal{P}^{\tau\mu} = -(\frac{1}{2\pi\alpha'})^2 (X')^2. \tag{432}$$

Now, differentiating (418) with respect to σ gives $n \cdot X' = 0$. Since n is non-zero and not spacelike, X' cannot be timelike. ■

Looking at (429), we see that $n \cdot \mathcal{P}^\tau$ is non-zero at all points where X' is spacelike. Those are precisely the points where $\mathcal{P}^\tau_\mu$ is timelike.

Exercise 8.5. *Show that X' cannot be lightlike at string endpoints.*

Proof. We know that X' is everywhere non-vanishing. Since $n \cdot X' = 0$, it can only be lightlike, if it is a non-zero multiple of n, say $X' = \alpha n$. Evaluating the term under the square root of a Nambu-Goto action then gives

$$(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2 = (\dot{X} \cdot X')^2 = \alpha^2 (\dot{X} \cdot n) = \alpha^2 \lambda^2 > 0. \tag{433}$$

But we have argued that this quantity has to vanish at string endpoints. ■

Let us further assume that this does not happen, that is $n \cdot \mathcal{P}^\tau \neq 0$. We can then consider a reparametrization

$$\sigma'(\tau,\sigma) := \kappa \cdot \int_0^\sigma (n \cdot \mathcal{P}^\tau(\tau,\sigma)) \mathrm{d}\sigma, \tag{434}$$

for some non-zero constant κ. One finds $\sigma'(\tau, 0) = 0$ and $\sigma'(\tau,\sigma_1) = \kappa \cdot (n \cdot p)$. We want a new parameter to be dimensionless. By convention, we choose $\kappa = \pi(n \cdot p)^{-1}$, so that $\sigma' \in [0,\pi]$. Now, since σ' grows as σ grows, one has

$$\frac{\partial \sigma'}{\partial \sigma} = \frac{\pi(n \cdot \mathcal{P}^\tau)}{n \cdot p} > 0. \tag{435}$$

By plugging into (431), we thus find

$$n \cdot \mathcal{P}^\tau = \frac{\pi(n \cdot \mathcal{P}^\tau)}{n \cdot p} \cdot (n \cdot \mathcal{P}'^{\tau'}). \tag{436}$$

By canceling $n \cdot \mathcal{P}^\tau$ on both sides and rearranging the constants, and finally dropping primes, we have found ourselves a parametrization (τ,σ) satisfying the constraint

$$n \cdot \mathcal{P}^\tau = \frac{n \cdot p}{\pi}. \tag{437}$$

Let us examine consequences of this choice. Multiplying the equations of motion by n, we get

$$\frac{\partial}{\partial \tau}(n \cdot \mathcal{P}^\tau) + \frac{\partial}{\partial \sigma}(n \cdot \mathcal{P}^\sigma) = 0. \tag{438}$$

The first term vanishes and we conclude that $\partial_\sigma(n \cdot \mathcal{P}^\sigma) = 0$. Since for open strings, we have $(n \cdot \mathcal{P}^\sigma)(t, \sigma_*) = 0$, this ensures that

$$(n \cdot \mathcal{P}^\sigma)(\tau,\sigma) = 0, \tag{439}$$

for all τ and $\sigma \in [0,\pi]$.

The discussion for closed strings is a bit more involved (there is no free endpoint condition). First, for convenience, the gauge fix for the closed strings is slightly modified to

$$n \cdot X(\tau,\sigma) = \alpha'(n \cdot p)\tau, \tag{440}$$

that is without a factor 2. The consequent σ parametrization is then defined to satisfy the constraint

$$n \cdot \mathcal{P}^\tau = \frac{n \cdot p}{2\pi}, \tag{441}$$

so that its range for closed strings is $\sigma \in [0, 2\pi]$. In (434) defined a new parameter σ to have value zero wherever the original parameter did. As $n \cdot \mathcal{P}^\tau$ is constant in both (τ,σ), we are free to shift the $\sigma = 0$ line wherever we want to. More precisely, we can consider a further reparametrization

$$\tau' := \tau, \;\; \sigma' := \sigma - f(\tau), \tag{442}$$

where $f(\tau)$ is some arbitrary function of τ. By looking at (431), this does not change $n \cdot \mathcal{P}^\tau$ and thus the gauge condition (434), neither the condition (440). This is defined so that the $\sigma' = 0$ line corresponds to the curve $(\tau, f(\tau))$ in the (τ,σ) parametrization.

Our guiding principle is that the quantity $n \cdot \mathcal{P}'^{\sigma'}$ must vanish along $\sigma' = 0$ line, that is

$$(n \cdot \mathcal{P}'^{\sigma'})(\tau', 0) = 0, \tag{443}$$

for all τ'. Recall that the (τ, σ) satisfying (440), one can plug into (227) to find

$$n \cdot \mathcal{P}^{\sigma} = -\frac{n \cdot p}{2\pi} \frac{\dot{X} \cdot X'}{\sqrt{(\dot{X} \cdot \dot{X}')^2 - (\dot{X})^2 (X')^2}} \tag{444}$$

The derivatives transform as

$$\frac{\partial X^{\mu}}{\partial \tau} = \frac{\partial X^{\mu}}{\partial \tau'} - f' \frac{\partial X^{\mu}}{\partial \sigma'}, \quad \frac{\partial X^{\mu}}{\partial \sigma} = \frac{\partial X^{\mu}}{\partial \sigma'}. \tag{445}$$

Since the Jacobian of the transformation (442) is just 1, one finds the transformation rule

$$n \cdot \mathcal{P}^{\sigma} = n \cdot \mathcal{P}'^{\sigma'} + f' \frac{n \cdot p}{2\pi} \frac{(X')^2}{\sqrt{(\dot{X} \cdot \dot{X}')^2 - (\dot{X})^2 (X')^2}}. \tag{446}$$

In other words, we have just found that

$$n \cdot \mathcal{P}'^{\sigma'} = -\frac{n \cdot p}{2\pi} \frac{\dot{X} \cdot X' - f'(X')^2}{\sqrt{(\dot{X} \cdot \dot{X}')^2 - (\dot{X})^2 (X')^2}} \tag{447}$$

By evaluating both sides at $(t, f(\tau))$, our condition leads to the ordinary differential equation

$$f'(\tau) = (\frac{\dot{X} \cdot X'}{(X')^2})(\tau, f(\tau)), \tag{448}$$

for an unknown function $f(\tau)$. Note that the initial condition can be set arbitrarily. This can be always solved on a compact interval $[\tau_i, \tau_f]$. We can thus drop the primes and declare that in our gauge, one has $(n \cdot \mathcal{P}^{\sigma})(\tau, 0) = 0$ for all τ. Since $\partial_{\sigma}(n \cdot \mathcal{P}^{\sigma}) = 0$ by equations of motion, we find that $n \cdot \mathcal{P}^{\sigma} = 0$ also for a closed string.

Let us summarize our choice of gauge:

$$n \cdot X(\tau, \sigma) = \beta \alpha' (n \cdot p) \tau, \tag{449}$$

$$\frac{2\pi}{\beta} n \cdot \mathcal{P}^{\tau} = n \cdot p, \tag{450}$$

$$n \cdot \mathcal{P}^{\sigma} = 0, \tag{451}$$

where $\beta = 2$ for open strings and $\beta = 1$ for closed strings, and $\sigma \in [0, (3 - \beta)\pi]$.

8.3 Constraints and wave equations

Let us examine the consequences of (449-451). By differentiating (449) with respect to τ and plugging the result for the expression for $n \cdot \mathcal{P}^{\sigma}$ obtained from (227), the condition (451) gives

$$\dot{X} \cdot X' = 0. \tag{452}$$

Using this in (226) and plugging this into (450) leads to together with (449) gives

$$1 = \frac{(X')^2}{\sqrt{-X^2 \dot{X}^2}} \tag{453}$$

Using the assumption $(X')^2 > 0$, this can be squared and rearranged to

$$(\dot{X})^2 + (X')^2 = 0. \tag{454}$$

These two constraints can be added and subtracted to obtain a convenient form

$$(\dot{X} \pm X')^2 = 0. \tag{455}$$

Exercise 8.6. *Check that for $n = (1, \vec{0})$, we obtain the static gauge situation.*

Proof. The only difference should be in the constants. First, (449) gives

$$X^0(\tau, \sigma) = (\beta\alpha' p^0)\tau. \tag{456}$$

Note that in natural units, one has $[\alpha'] = L^2$, whereas $[p^0] = M = L^{-1}$, so τ is indeed dimensionless. On thus has

$$\dot{X} = (\beta\alpha' p^0, \dot{\vec{X}}), \quad X' = (0, \vec{X}'). \tag{457}$$

The condition (452) thus gives simply $\dot{\vec{X}} \cdot \vec{X}' = 0$, and (454) reads

$$(\dot{\vec{X}})^2 + (\vec{X}')^2 = \beta^2\alpha'^2(p^0)^2. \tag{458}$$

■

Using (452) and (454) significantly simplifies the expressions for the momenta. Indeed, e.g. for $\mathcal{P}^{\tau\mu}$, one finds

$$\mathcal{P}^\tau_\mu = -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} = \frac{1}{2\pi\alpha'}\dot{X}_\mu. \tag{459}$$

Similarly, one gets

$$\mathcal{P}^\sigma_\mu = -\frac{1}{2\pi\alpha'}X'_\mu. \tag{460}$$

Observe that $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$ indeed satisfy (450) and (451) thanks to (449). Finally, plugging into the equations of motion, one obtains just the wave equations!

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0. \tag{461}$$

8.4 Wave equation and mode expansions

Let us now solve the equation of motion (461) satisfying the constraint (455) together with the free endpoint boundary conditions. The most general solution to (461) has the form

$$X^\mu(\tau, \sigma) = \frac{1}{2}(f^\mu(\tau + \sigma) + g^\mu(\tau - \sigma)). \tag{462}$$

The free endpoint boundary condition turns into a Neumann boundary condition $X^\mu(\tau, \sigma_*) = 0$. At $\sigma_\alpha = 0$, this gives

$$f'^\mu(\tau) - g'^\mu(\tau) = 0. \tag{463}$$

Hence f^μ and g^μ differ only by a constant, which can be absorbed into a definition of f^μ, so

$$X^\mu(\tau, \sigma) = \frac{1}{2}(f^\mu(\tau + \sigma) + f^\mu(\tau - \sigma)). \tag{464}$$

Plugging into the boundary condition at $\sigma_* = \pi$ then gives

$$f'^{\mu}(\tau + \pi) - f'^{\mu}(\tau - \pi) = 0, \tag{465}$$

that is f'^{μ} is periodic with period 2π. We can thus expand it using Fourier series as

$$f'^{\mu}(u) = f_1^{\mu} + \sum_{n=1}^{\infty} \big(a_n^{\mu} \cos(nu) + b_n^{\mu} \sin(nu)\big). \tag{466}$$

We can integrate this (and absorb the integration constants into new coefficients) to find

$$f^{\mu}(u) = f_0^{\mu} + f_1^{\mu} u + \sum_{n=1}^{\infty} \big(A_n^{\mu} \cos(nu) + B_n^{\mu} \sin(nu)\big). \tag{467}$$

Plugging this into the formula for X^{μ} and using some formulas for sums of (co)sines, this gives

$$X^{\mu}(\tau, \sigma) = f_0^{\mu} + f_1^{\mu}\tau + \sum_{n=1}^{\infty} \big(A_n^{\mu} \cos(n\tau) + B_n^{\mu} \sin(n\tau)\big) \cos(n\sigma). \tag{468}$$

Now, the idea is to replace pairs of real constants A_n^{μ} and B_n^{μ} with a complex constants with better physical interpretation. One has

$$\begin{aligned} A_n^{\mu} \cos(n\tau) + B_n^{\mu} \sin(n\tau) = & -\frac{i}{2}\Big((B_n^{\mu} + iA_n^{\mu})e^{in\tau} - (B_n^{\mu} - iA_n^{\mu})e^{-in\tau}\Big) \\ = & -i\frac{\sqrt{2\alpha'}}{\sqrt{n}}\Big(a_n^{\mu *} e^{in\tau} - a_n^{\mu} e^{-in\tau}\Big). \end{aligned} \tag{469}$$

Note that $[A_n^{\mu}] = [B_n^{\mu}] = L$, so we have chosen a normalization by a constant of dimension $[\sqrt{\alpha'}] = L$, so that new coefficients are dimension-less. Next, note that

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} f_1^{\mu} + \sum_{n=1}^{\infty} (\cdots) \cos(n\sigma). \tag{470}$$

Integrating this over $[0, \pi]$, this gives us the overall momentum of a string. Since $\int_0^{\pi} \cos(n\sigma) = 0$ for every $n \in \mathbb{N}$, one has

$$p^{\mu} = \int_0^{\pi} \frac{1}{2\pi\alpha'} f_1^{\mu} = \frac{1}{2\alpha'} f_1^{\mu} \Rightarrow f_1^{\mu} = 2\alpha' p^{\mu}. \tag{471}$$

By writing $f_0^{\mu} = x_0^{\mu}$, we can thus write

$$X^{\mu}(\tau, \sigma) = x_0^{\mu} + 2\alpha' p^{\mu}\tau - i\sqrt{2\alpha'} \sum_{n=1}^{\infty} (a_n^{\mu *} e^{in\tau} - a_n^{\mu} e^{-in\tau}) \frac{\cos(n\sigma)}{\sqrt{n}}. \tag{472}$$

Finally, this can further rewritten as follows. One introduces new complex constants $\{\alpha_n\}_{n\in\mathbb{Z}}$ as

$$\alpha_0^{\mu} := \sqrt{2\alpha'} p^{\mu}, \quad \alpha_n^{\mu} := a_n^{\mu} \sqrt{n}, \quad \alpha_{-n}^{\mu} = a_n^{\mu *} \sqrt{n}, \text{ for } n \geq 1. \tag{473}$$

This allows us to write the above expression as

$$X^{\mu}(\tau, \sigma) = x_0^{\mu} + \sqrt{2\alpha'} \alpha_0^{\mu} \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-in\tau} \cos(n\sigma) \tag{474}$$

Why is this parametrization convenient? Because partial derivatives od X^μ look particularly nice! One finds

$$\dot{X}^\mu(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in\tau}\cos(n\sigma), \tag{475}$$

$$X'^\mu(\tau,\sigma) = -i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in\tau}\sin(n\sigma). \tag{476}$$

In particular, one finds

$$\dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in(\tau\pm\sigma)}. \tag{477}$$

Remark 8.7. Note that the solution using mode expansions has its drawbacks (usually ignored). First, $X = X(\tau,\sigma)$ may happen to be *not* injective. Moreover, it can happen that in isolated points, the tangent vector X' vanishes, i.e. the string has cusps.

8.5 Light-cone solution of equations of motion

In this subsection, we will finally fix our choice of the direction vector n. Set

$$n_\mu = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0). \tag{478}$$

With this choice, we thus have

$$n \cdot X = \frac{1}{\sqrt{2}}(X^0 + X^1) \equiv X^+, \;\; n \cdot p = \frac{1}{\sqrt{2}}(p^0 + p^1) \equiv p^+. \tag{479}$$

The gauge conditions (449 - 451) can be thus written as

$$X^+(\tau,\sigma) = \beta\alpha' p^+ \tau, \;\; \mathcal{P}^{\tau +} = \frac{\beta}{2\pi}p^+, \;\; \mathcal{P}^{\sigma +} = 0. \tag{480}$$

This is what is usually called the **light-cone gauge**. Let us write $X^I = (X^2, \dots, X^d)$, that is $X = (X^+, X^-, X^I)$ in light-cone coordinates. The string coordinates X^I are called **transverse**. Note that $p^+ = n \cdot p > 0$ for a physical string. The crucial observation is that the constraint (455) can be now solved in a very easy way. Indeed, one has

$$0 = (\dot{X} \pm X')^2 = -2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2, \tag{481}$$

where we write $(a^I)^2 := \sum_{I=2}^d a^I a^I$. But one has $\dot{X}^+ \pm X'^+ = \beta\alpha' p^+ \neq 0$. Hence

$$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^I \pm X^{I'})^2. \tag{482}$$

This means that $\dot{X}^-$ and X'^- can be fully expressed purely in terms of the transversal string coordinates. This is the reason why the light-cone gauge is so useful. More precisely, the transversal string coordinates fully determine a 1-form

$$\mathrm{d}X^- = \dot{X}^- \mathrm{d}\tau + X'^- \mathrm{d}\sigma. \tag{483}$$

To find X^- we need to fix its value $X^-(p)$ at some point of the worldsheet. For any other point q, one connects it to p by a curve γ and defines $X^-(q) := X^-(p) + \int_\gamma \mathrm{d}X^-$. This is always possible for open strings. For closed strings, there is an obstruction (to be discussed later).

Let us try to fully solve the open string case using the mode expansion above. Since $X^\pm$ and X^I satisfy the wave equations and free endpoint conditions, we can use completely the same notation. For the + coordinate, we find

$$x_0^+ = 0, \;\; \alpha_0^+ = \sqrt{2\alpha'}p^+, \;\; \alpha_n^+ = 0 \text{ for all } n \neq 0. \tag{484}$$

Moreover, one has

$$\dot{X}^- \pm X'^- = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^- e^{-in(\tau\pm\sigma)}, \tag{485}$$

$$\dot{X}^I \pm X'^I = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^I e^{-in(\tau\pm\sigma)}. \tag{486}$$

Plugging this into (482) gives

$$\begin{aligned} \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^- e^{-in(\tau\pm\sigma)} &= \frac{1}{2\alpha'}\frac{1}{2p^+} 2\alpha' \sum_{p,q\in\mathbb{Z}} \alpha_p^I \alpha_q^I e^{-i(p+q)(\tau\pm\sigma)} \\ &= \frac{1}{2p^+} \sum_{n\in\mathbb{Z}} (\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I) e^{-in(\tau\pm\sigma)}. \end{aligned} \tag{487}$$

By comparing the coefficients, we find that

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+} \sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I. \tag{488}$$

The combination of the transverse oscillators on the right-hand side has its own name, called the **transverse Virasoso** mode $L_n^\perp$. One has

$$L_n^\perp := \frac{1}{2} \sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I \Rightarrow \sqrt{2\alpha'}\alpha_n^- = \frac{L_n^\perp}{p^+}. \tag{489}$$

Note that $(L_n^\perp)^* = L_{-n}^\perp$. In particular, for $n = 0$, this gives $2p^+p^- = \frac{1}{\alpha'} L_0^\perp$. It follows that X^- can be expressed using the transverse Virasoro modes as

$$X^-(\tau, \sigma) = x_0^- + \frac{1}{p^+} L_0^\perp \tau + \frac{i}{p^+} \sum_{n\neq 0} \frac{1}{n} L_n^\perp e^{-in\tau} \cos(n\sigma). \tag{490}$$

The mass of the string is defined as $M^2 = -p^2 = 2p^+p^- - p^I p^I$. Using the above expression for $2p^+p^-$, one finds the expression

$$2p^+p^- = \frac{1}{\alpha'} L_0^\perp = \frac{1}{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_{-n}^I \alpha_n^I = \frac{1}{2\alpha'} \alpha_0^I \alpha_0^I + \frac{1}{\alpha'} \sum_{n=1}^\infty n a_n^{I*} a_n^I = p^I p^I + \frac{1}{\alpha'} \sum_{n=1}^\infty n a_n^{I*} a_n^I. \tag{491}$$

We conclude that

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^\infty n a_n^{I*} a_n^I. \tag{492}$$

We observe that $M^2 \geq 0$, which is in accordance with our expectations. Moreover, one has $M = 0$, iff $a_n^I = 0$ for all $n \in \mathbb{N}$ and $I \in \{2, \ldots, d\}$. What is the string then? It is easy to see that it collapses to a point massless particle (hence moving with the speed of light). To conclude, the motion of the open string is fully determined by the following constants:

$$x_0^-, \;\; p^+, x_0^I \;\; \alpha_n^I \text{ for all } I \in \{2, \ldots, d\} \text{ and } n \in \mathbb{N}_0. \tag{493}$$

Note that for negative $n < 0$, one has $\alpha_n^I = \alpha_{-n}^{I*}$.

Exercise 8.8. *Consistency checks on the solution for X^-.*

Suppose $\dot{X}^-$ and X'^- are defined by the formula (482). It is not clear that this gives a solution to the wave equation with appropriate boundary conditions.

(i) Check that the 1-form $\mathrm{d}X^-$ defined by (483) is exact, if X^I satisfies the wave equation for every $I \in \{2, \dots, d\}$.

(ii) Show that X^- satisfies the wave equation, if X^I does so for every $I \in \{2, \dots, d\}$.

(iii) Suppose that for each $I \in \{2, \dots, n\}$, X^I satisfies either Dirichlet or Neumann boundary condition at either endpoint. Then X^- satisfies the Neumann boundary condition.

Proof. It follows from (482) that

$$\dot{X}^- = k\big((\dot{X}^I + X'^I)^2 + (\dot{X}^I - X'^I)^2\big) = 2k\big(((\dot{X}^I)^2 + (X'^I)^2)\big), \tag{494}$$

$$X'^- = k\big((\dot{X}^I + X'^I)^2 + (\dot{X}^I - X'^I)^2\big) = 2k\big(2\dot{X}^I X'^I\big), \tag{495}$$

where $k > 0$ is some constant. It is not relevant for this exercise, so we write simply

$$\dot{X}^- = (\dot{X}^I)^2 + (X'^I)^2, \ X'^- = 2\dot{X}^I X'^I \tag{496}$$

Consequently, we find

$$\partial_\sigma \dot{X}^- = 2\dot{X}^I(\partial_\sigma \dot{X}^I) + 2X'^I(\partial_\sigma X'^I), \tag{497}$$

$$\partial_\tau X'^- = 2(\partial_\tau \dot{X}^I)X'^I + 2\dot{X}^I(\partial_\tau X'^I). \tag{498}$$

Subtracting both sides thus gives

$$\partial_\sigma X'^- - \partial_\tau X'^- = 2\dot{X}^I(\partial_\sigma \dot{X}^I - \partial_\tau X'^I) + 2X'^I(\partial_\sigma X'^I - \partial_\tau \dot{X}^I). \tag{499}$$

Now, the term proportional to $\dot{X}^I$ vanishes since X^I have continous partial derivatives. The term proportional to X'^I vanishes if X^I satisfies the wave equation. This proves (i). Also

$$\partial_\tau \dot{X}^- = 2\dot{X}^I(\partial_\tau \dot{X}^I) + 2X'^I(\partial_\tau X'^I), \tag{500}$$

$$\partial_\sigma X'^- = 2(\partial_\sigma \dot{X}^I)X'^I + 2\dot{X}^I(\partial_\sigma X'^I).. \tag{501}$$

Subtracting both sides thus gives

$$\partial_\tau \dot{X}^- - \partial_\sigma X'^- = 2\dot{X}^I(\partial_\tau \dot{X}^I - \partial_\sigma X'^I) + 2X'^I(\partial_\tau X'^I - \partial_\sigma \dot{X}^I). \tag{502}$$

The term proportional to $\dot{X}^I$ is precisely the wave equation for X^I, the term proportional to X'^I vanishes since each X^I has continuous partial derivatives. This proves (ii).

Finally, we see that $X'^-(\tau, \sigma_*) \propto \dot{X}^I(\tau, \sigma_*)X'^I(\tau, \sigma_*)$. This vanishes whenever for each I, one has either a Neumann boundary condition $X'^I(\tau, \sigma_*) = 0$ or a Dirichlet boundary condition $\dot{X}^I(\tau, \sigma_*) = 0$. This proves (iii). ∎

Exercise 8.9. *Consider the open string described by $x_0^- = x_0^I := 0$ and $\alpha_1^{(2)} := a$, $\alpha_1^{(3)} = ia$, where $a > 0$. Other coefficients vanish.*

(i) What is the mass M of this string;

(ii) Write down $X^2(\tau,\sigma)$ and $X^3(\tau,\sigma)$. What is the length of the string?

(iii) Calculate $L_n^\perp$ for all $n \in \mathbb{N}_0$. Use this to write $X^-(\tau,\sigma)$;

(iv) Choose p^+ to ensure that the string moves in (x^2, x^3) plane, that is $X^1(\tau,\sigma) = 0$; Find the relation between t and τ;

(v) Find the relation between energy and length.

Proof. One has $M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} n a_n^{I*} a_n^I = \frac{1}{\alpha'}(a^2 + a^2) = \frac{2a^2}{\alpha'}$. Next, one has

$$\begin{aligned} X^2(\tau,\sigma) &= i\sqrt{2\alpha'}(\alpha_1^{(2)} e^{-i\tau} - \alpha_{-1}^{(2)*} e^{i\tau})\cos(\sigma) = -i\sqrt{2\alpha'}a(e^{i\tau} - e^{-i\tau})\cos(\sigma) \\ &= \sqrt{2\alpha'}2a\sin(\tau)\cos(\sigma), \\ X^3(\tau,\sigma) &= \sqrt{2\alpha'}2a\sin(\tau)\cos(\sigma). \end{aligned} \tag{503}$$

This is a rigidly rotating string in (x^2, x^3) plane of length $\ell = \sqrt{2\alpha'}4a$. Let us calculate the transverse virasoro modes. One has

$$L_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I = \frac{1}{2}(\alpha_{n-1}^I \alpha_1^I + \alpha_{n+1}^I \alpha_{-1}^I). \tag{504}$$

The only non-zero situation can happen for $n = 0$ and $n = \pm 2$, and one finds

$$L_0^\perp = \alpha_{-1}^I \alpha_1^I = |\alpha_1^I|^2 = 2a^2, \tag{505}$$

$$L_2^\perp = \frac{1}{2}\alpha_1^I \alpha_1^I = \frac{1}{2}(a^2 - a^2) = 0. \tag{506}$$

Plugging into (490) thus gives

$$X^-(\tau,\sigma) = \frac{1}{p^+}L_0^\perp \tau = \frac{2a^2}{p^+}\tau. \tag{507}$$

Recall that $X^+(\tau,\sigma) = 2\alpha' p^+ \tau$. One gets

$$X^1(\tau,\sigma) = \frac{1}{\sqrt{2}}(X^+(\tau,\sigma) - X^-(\tau,\sigma)) = \frac{1}{\sqrt{2}}(2\alpha' p^+ - \frac{2a^2}{p^+})\tau. \tag{508}$$

The requirement $X^1(\tau,\sigma) = 0$ thus uniquely determines $p^+ = \sqrt{a^2/\alpha'}$. Consequently, one has

$$X^0(\tau,\sigma) = \frac{4}{\sqrt{2}}\sqrt{\alpha'}a\tau. \tag{509}$$

The relation to the coordinate time t (in natural units) is therefore

$$t = \frac{4}{\sqrt{2}}\sqrt{\alpha'}a\tau. \tag{510}$$

Finally, one has $2p^+p^- = \frac{1}{\alpha'}L_0^\perp = \frac{2a^2}{\alpha'}$, so

$$p^- = \frac{1}{2p^+}\frac{2a^2}{\alpha'} = \frac{1}{2}\frac{\sqrt{\alpha'}}{a}\frac{2a^2}{\alpha'} = \frac{a}{\sqrt{\alpha'}}. \tag{511}$$

We can use this to calculate the energy of the string, namely

$$E = p^0 = \frac{1}{\sqrt{2}}(p^+ + p^-) = \frac{\sqrt{2}a}{\sqrt{\alpha'}} \tag{512}$$

We find that $a = \frac{\ell}{4\sqrt{2\alpha'}}$ and thus $E = \frac{\ell}{4\alpha'} = 2\pi T_0 \frac{\ell}{4} = \frac{\pi}{2} T_0 \ell$. This precisely the relation (339). ■

Exercise 8.10. *Study the solution for the open string with $x_0^- = x_0^I = 0$ and $\alpha_n^I = 0$ except for*

$$\alpha_1^{(2)} = \alpha_{-1}^{(2)*} = a. \tag{513}$$

Fix constants so that strings moves in (x^1, x^2) having a zero momentum in this plane.

(i) Find explicit expressions for X^0, X^1 and X^2;

(ii) Confirm that τ flows as t flows.

(iii) At $\tau = 0$ the string has a zero length. Study a motion for $\tau \ll 1$. Calculate $\tau = \tau(t, \sigma)$ and find $X^1(t, \sigma)$ and $X^2(t, \sigma)$.

Proof. The only non-zero Virasoro operators are

$$L_0^\perp = \alpha_{-1}^I \alpha_1^I = a^2, \tag{514}$$

$$L_2^\perp = \frac{1}{2}\alpha_1^I \alpha_1^I = \frac{1}{2}a^2. \tag{515}$$

Plugging this into the formula for X^-, one gets

$$X^+(\tau, \sigma) = 2\alpha' p^+ \tau, \tag{516}$$

$$X^-(\tau, \sigma) = \frac{1}{p_+} L_0^\perp + \frac{1}{p^+} L_2^\perp \sin(2\tau)\cos(2\sigma). \tag{517}$$

The momentum in the $-$ direction is

$$p^- = \frac{1}{2p^+\alpha'} L_0^\perp = \frac{a^2}{2p^+\alpha'}. \tag{518}$$

Since $p^2 \propto \alpha_0^{(2)} = 0$, we need $0 = p^1 \propto (p^+ - p^-)$. This gives the condition

$$p^+ = \frac{a}{\sqrt{2\alpha'}}. \tag{519}$$

By plugging in into the above formulas give

$$\frac{1}{\sqrt{2\alpha'}a} X^+(\tau, \sigma) = \tau, \tag{520}$$

$$\frac{1}{\sqrt{2\alpha'}a} X^-(\tau, \sigma) = \tau + \frac{1}{2}\sin(2\tau)\cos(2\sigma) \tag{521}$$

Combining those into X^0 and X^1, one gets the final list

$$\frac{1}{\sqrt{2\alpha'}a} X^0(\tau, \sigma) = \sqrt{2}(\tau + \frac{1}{4}\sin(2\tau)\cos(2\sigma)), \tag{522}$$

$$\frac{1}{\sqrt{2\alpha'}a} X^1(\tau, \sigma) = -\frac{1}{2\sqrt{2}}\sin(2\tau)\cos(2\sigma), \tag{523}$$

$$\frac{1}{\sqrt{2\alpha'}a} X^2(\tau, \sigma) = 2\sin(\tau)\cos(\sigma). \tag{524}$$

This finishes the part (i). We will further ignore the constants. One has

$$\frac{\partial X^0}{\partial \tau} = \sqrt{2}(1 + \frac{1}{2}\sin(2\tau)\cos(2\sigma)) > 0, \tag{525}$$

This shows that the "time flows" as τ flows, that is part (ii). Now, $\tau = 0$ corresponds to $t = 0$ string. This string has zero length, since $X^1(0, \sigma) = X^2(0, \sigma) = 0$. Now, if $\tau \ll 1$, we have

$$X^0(\tau, \sigma) \approx \sqrt{2}(\tau + \frac{1}{4}2\tau\cos(2\sigma)) = \sqrt{2}(1 + \frac{1}{2}\cos(2\sigma))\tau. \tag{526}$$

We can now introduce a new parameter t, so that $X^0(t, \sigma) = t$. This amounts to solving the equation

$$t = \sqrt{2}(1 + \frac{1}{2}\cos(2\sigma))\tau, \tag{527}$$

which can be done to get the formula

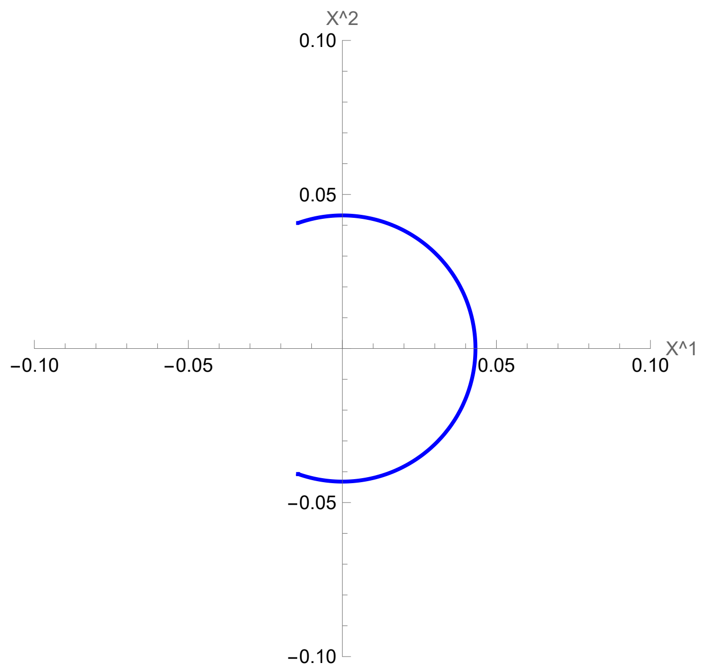
$$\tau = \frac{\sqrt{2}}{2 + \cos(2\sigma)}t. \tag{528}$$

For each $t \ll 1$, we thus get a parametrized string:

$$X^1(t, \sigma) = -\frac{\cos(2\sigma)}{2 + \cos(2\sigma)}t, \tag{529}$$

$$X^2(t, \sigma) = \frac{2\sqrt{2}\cos(\sigma)}{2 + \cos(2\sigma)}t. \tag{530}$$

This is the following constant shape expanding linearly in time:



■

9 Light-cone fields and particles

Let us now briefly recall some classical fields appearing in physics. We will also discuss their quantized version. This is to later identify those as states of quantum strings.

9.1 Real scalar field

Scalar field is given by a single real function $\phi = \phi(x^0, \dots, x^D)$. Its action is given by

$$S[\phi] = \int \mathrm{d}^D x (-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2), \tag{531}$$

where $m > 0$ is its rest mass and $\partial_\mu \phi \equiv \frac{\partial \phi}{\partial x^\mu}$. The corresponding Lagrange-Euler equation (for variations vanishing at infinity) is the **Klein-Gordon equation**:

$$\partial^\mu (\partial_\mu \phi) - m^2 \phi = 0. \tag{532}$$

The canonical momentum associated to ϕ is

$$\Pi := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi, \tag{533}$$

and the corresponding Hamiltonian is then obtained as

$$H = \int \mathrm{d}^d x (\Pi \partial_0 \phi - \mathcal{L}) = \int \mathrm{d}^d x \frac{1}{2} (\Pi^2 + \|\nabla \phi\|^2 + m^2 \phi^2). \tag{534}$$

One usually solves the Klein-Gordon equation by performing a D-dimensional continuous Fourier transformation. One defines a function $\hat{\phi}(p)$ by the formula

$$\phi(x) = \int \mathrm{d}^D p \frac{1}{(2\pi)^D} e^{ip\cdot x} \hat{\phi}(p). \tag{535}$$

By plugging this into the Klein-Gordon equation (532), one obtains

$$-\int \mathrm{d}^D p \frac{1}{(2\pi)^D} (p^2 + m^2) \hat{\phi}(p) = 0, \tag{536}$$

which is equivalent to the equation

$$(p^2 + m^2) \hat{\phi}(p) = 0. \tag{537}$$

We see that on the **mass-shell** $p^2 + m^2 = 0$, the value $\hat{\phi}(p)$ can be arbitrary, and it has to vanish elsewhere. The mass shell is a hyperboloid $(\vec{p})^2 - (p^0)^2 = m^2$. Its two components are described by $p^0 = \pm\sqrt{(\vec{p})^2 + m^2} \equiv \pm E_p$. We can thus write

$$\hat{\phi}(p) = (2\pi)^D a(\vec{p}) \cdot \delta(p^0 - E_p) + (2\pi)^D b(\vec{p}) \cdot \delta(p^0 + E_p). \tag{538}$$

Now note that $\phi(x)$ has to be a real function. Plugging this into (535), we obtin the condition

$$\hat{\phi}^*(p) = \hat{\phi}(-p). \tag{539}$$

Plugging the above expression, we find that this forces $b(\vec{p}) = a^*(-\vec{p})$. We can plug this back into (535), finding

$$\begin{aligned}\phi(x) &= \int \mathrm{d}^D p\big(a(\vec{p})\delta(p^0 - E_p) + a^*(-\vec{p})\delta(p^0 + E_p)\big)e^{ip\cdot x} \\ &= \int \mathrm{d}^d\vec{p}\,\left(a(\vec{p})e^{-iE_p t} + a^*(-\vec{p})e^{iE_p t})e^{i\vec{p}\cdot\vec{x}}\right) \\ &= \int \mathrm{d}^d\vec{p}\,\left(a(\vec{p})e^{-iE_p t+\vec{p}\cdot\vec{x}} + a^*(\vec{p})e^{iE_p t - i\vec{p}\cdot\vec{x}}\right).\end{aligned} \tag{540}$$

This is a standard solution as a superposition of plane-waves. Note that in the light-cone gauge, the procedure is similar. Write $x = (x^+, x^-, \vec{x}_T)$ and $p = (p^+, p^-, \vec{p}_T)$. This time, one considers the Fourier transform

$$\phi(x^+, x^-, \vec{x}_T) = \int \frac{\mathrm{d}p_+}{(2\pi)} \int \frac{\mathrm{d}^{d-1}\vec{p}_T}{(2\pi)^{d-1}} e^{-ix^-p^+ + i\vec{x}_T\cdot\vec{p}_T}\hat{\phi}(x^+, p^+, \vec{p}_T). \tag{541}$$

The Klein-Gordon equation takes the form

$$-2\partial_+(\partial_-\phi) + \Delta_T\phi - m^2\phi = 0, \tag{542}$$

where $\Delta_T := \partial_I\partial_I$ is the "transverse" Laplacian. Plugging the above expansion gives

$$2ip^+\partial_+\hat{\phi} - ((\vec{p}_T)^2 + m^2)\hat{\phi} = 0. \tag{543}$$

We claim that this has a non-trivial solutions only for $p^+ \neq 0$. Indeed, suppose that $p^+ = 0$. But then $p^2 = -2p^+p^- + (\vec{p}_T)^2 = (\vec{p}_T)^2 \geq 0$. But for $p^+ = 0$, the equation would give $(\vec{p}_T)^2 + m^2)\hat{\phi} = 0$ and thus $\hat{\phi} = 0$. To get non-trivial solutions, we can thus divide both sides of (543) by $2p^+$ to get the "Schrödinger type equation":

$$i\partial_+\hat{\phi} = \frac{1}{2p^+}((\vec{p}_T)^2 + m^2))\hat{\phi} \tag{544}$$

Note that the right-hand side is precisely the solution for p^- of the mass-shell condition. We will make use of this equation later.

9.2 Quantum scalar fields and particle states

Suppose that our classical field is restricted to a finite d-dimensional volume V, e.g. a box with sides $L_1, \ldots, L_d$. In this case $V = L_1 \cdots L_d$. Let us consider a plane-wave solution as above, just with a different normalization:

$$\phi_p(t, \vec{x}) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E_p}}(a_p e^{-iE_p t + i\vec{p}\cdot\vec{x}} + a_p^* e^{iE_p t - i\vec{p}\cdot\vec{x}}), \tag{545}$$

One usually requires the field ϕ_p to be periodic in each its variable with period L_i. This can be ensured by by requiring

$$p_i L_i = 2\pi n_i, \tag{546}$$

for each $i \in \{1, \ldots, d\}$, where $n_i \in \mathbb{Z}$. We thus assume that the momenta p_i become "quantized". Let us evaluate the Klein-Gordon action (531) for ϕ_d. Let us move chose the spatial coordinates so that the the box correspond to $x^i \in [0, L_i]$. Recall that

$$\mathcal{L} = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_i\phi)(\partial_i\phi) - \frac{1}{2}m^2\phi^2. \tag{547}$$

By plugging in (545), observe that no term which contains $e^{\pm 2i\vec{p}\cdot\vec{x}}$ survives the integration. This is because

$$\int_0^{L_1} \mathrm{d}x^1 \cdots \int_0^{L_d} \mathrm{d}x^d e^{\pm 2i\vec{p}\cdot\vec{x}} = 0. \tag{548}$$

By plugging into the action and Hamiltonian (534), we find

$$S[\phi_p] = 0, \tag{549}$$

$$H[\phi_p] = E_p a_p^* a_p. \tag{550}$$

Exercise 9.1. *Let us consider a Lagrangian density $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\alpha \phi^a)$, and let us consider an infinitesimal transformation*

$$\phi'^a = \phi^a + \epsilon^\beta \partial_\beta \phi^a, \tag{551}$$

that is $\delta\phi^a = \epsilon^\beta \partial_\beta \phi^a$. Show that

$$\delta\mathcal{L} = \epsilon^\beta \partial_\alpha (\delta^\alpha_\beta \mathcal{L}). \tag{552}$$

Show that in this case, there is a conserved current

$$T^\alpha{}_\beta = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \partial_\beta \phi^a - \delta^\alpha_\beta \mathcal{L}. \tag{553}$$

This quantity is called the ***energy-momentum tensor****.*

The momentum density $\vec{\mathcal{P}}$ of the Klein-Gordon field is obtained from components $T^i{}_0$, $i \in \{1, \ldots, d\}$, and the conserved momentum $\vec{P}$ is thus given by

$$P_i = \int \mathrm{d}^d x \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \partial_0 \phi = -\int \mathrm{d}^d x (\partial_0 \phi)(\partial_i \phi). \tag{554}$$

Plugging in (545) gives

$$\vec{P} = \vec{p}\, a_p^* a_p. \tag{555}$$

These observations suggest how to "quantize" things. We will declare a_p to be the annihilation operator and a_p^* to be the creation operator $a_p^\dagger$. We impose the canonical commutation relations

$$[a_p, a_p^\perp] = 1, \tag{556}$$

with other combinations vanishing. In full generality, the full quantum field $\phi(x)$ is a sum over all spatial momenta:

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-iE_p t + i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{iE_p t - i\vec{p}\cdot\vec{x}} \right), \tag{557}$$

where one imposes the (only non-vanishing) commutation relations

$$[a_p, a_q^\perp] = \delta_{p,q}. \tag{558}$$

The quantum version of the Hamiltonian H and $\vec{P}$ take the form

$$H = \sum_{\vec{p}} E_p a_p^\dagger a_p, \quad \vec{P} = \sum_{\vec{p}} \vec{p}\, a_p^\dagger a_p. \tag{559}$$

One assumes the existence of a vacuum state $|\Omega\rangle$, having the property that $a_p|\Omega\rangle = 0$ for all momenta p. It follows that $H|\Omega\rangle = 0$ and $\vec{P}|\Omega\rangle = 0$.

One defines the k-particle states by acting by k creation operators, that is

$$a^\dagger_{p_1} \cdots a^\dagger_{p_k}|\Omega\rangle. \tag{560}$$

It is easy to see that (560) are eigenvectors of H and $\vec{P}$ with eigenvalues $E_{p_1} + \cdots + E_{p_k}$ and $\vec{p}_1 + \cdots + \vec{p}_k$, respectively.

Exercise 9.2. *Solve 544 and plug it into (541) to find a plane-wave expansion in the light-cone gauge. How do you quantize such a field?*

Proof. The solution of this equation is simple, just

$$\hat{\phi}(x^+, p^+, \vec{p}_T) = a_{p^+,\vec{p}_T} \exp(-\frac{i}{2p^+}((\vec{p}_T)^2 + m^2)x^+), \tag{561}$$

for an arbitrary complex number $a_{p^+,\vec{p}_T}$. Since $p^+ \neq 0$, it is convenient to write the solution using two independent constants which depend only on the absolute value $|p^+|$, that is write

$$\begin{aligned}\hat{\phi}(x^+, p^+, \vec{p}_T) = (2\pi)^d \vartheta(p^+) \cdot b_{|p^+|,\vec{p}_T} \exp(-\frac{i}{2p^+}((\vec{p}_T)^2 + m^2)x^+) \\ + (2\pi)^d \vartheta(-p^+) \cdot b'_{|p^+|,\vec{p}_T} \exp(-\frac{i}{2p^+}((\vec{p}_T)^2 + m^2)x^+),\end{aligned} \tag{562}$$

where $\vartheta(x)$ is the Heaviside function. The reason why we use this strange parametrization is the following. We want $\phi(x^+, x^-, \vec{x}_T)$ to be real. This implies

$$\hat{\phi}(x^+, p^+, \vec{p}_T)^* = \hat{\phi}(x^+, -p^+, -\vec{p}_T). \tag{563}$$

Plugging the above expression into this condition implies $b'_{|p^+|,\vec{p}_T} = b^*_{|p^+|,-\vec{p}_T}$, that is

$$\begin{aligned}\hat{\phi}(x^+, p^+, \vec{p}_T) = (2\pi)^d \vartheta(p^+) \cdot b_{|p^+|,\vec{p}_T} \exp(-\frac{i}{2p^+}((\vec{p}_T)^2 + m^2)x^+) \\ + (2\pi)^d \vartheta(-p^+) \cdot b^*_{|p^+|,-\vec{p}_T} \exp(-\frac{i}{2p^+}((\vec{p}_T)^2 + m^2)x^+),\end{aligned} \tag{564}$$

Plugging this into the expansion (541) gives the expression

$$\begin{aligned}\phi(x) = \int \mathrm{d}p_+ \int \mathrm{d}^{d-1}\vec{p}_T \, \big(\vartheta(p^+) b_{|p^+|,\vec{p}_T} e^{-i|p^-|x^+ - ix^-p^+ + \vec{x}_T\cdot\vec{p}_T} \\ + \vartheta(-p^+) b^*_{|p^+|,-\vec{p}_T} e^{i|p^-|x^+ - ix^1p^+ + \vec{x}_T\cdot\vec{p}_T}\big),\end{aligned} \tag{565}$$

where $|p^-| := \frac{1}{2|p^+|}((\vec{p}_T)^2 + m^2)$. Finally, we can change the integration variables from $(p^+, \vec{p}_T)$ to $(-p^+, -\vec{p}^T)$ in the second term, change the range of the p_+ integral to get

$$\phi(x) = \int_0^\infty \mathrm{d}p_+ \int \mathrm{d}^{d-1}\vec{p}_T \big(b_{p^+,\vec{p}_T} e^{ip\cdot x} + b^*_{p^+,\vec{p}_T} e^{-ip\cdot x}\big). \tag{566}$$

Note that $p \cdot x = -p^-x^+ - p^+x^- + \vec{p}_T\vec{x}_T$, where $p^- = \frac{1}{2p^+}((\vec{p}_T)^2 + m^2)$. ■

Lagrangian of the action (531) in light-cone coordinates reads

$$\mathcal{L} = \partial_+\phi\partial_-\phi - \frac{1}{2}(\nabla^T\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{567}$$

Now, x^+ plays the role of the light-cone time, so the corresponding conjugate momentum Π_+ takes the form

$$\Pi_+ = \frac{\partial\mathcal{L}}{\partial(\partial_+\phi)} = \partial_-\phi. \tag{568}$$

But this means that the light-cone Hamiltonian density is just

$$\mathcal{H}^{lc} = \Pi_+\partial_+\phi - \mathcal{L} = \frac{1}{2}(\nabla^T\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{569}$$

The light-cone Hamiltonian is then

$$H^{\rm lc} = \int \mathrm{d}x_- \int \mathrm{d}\vec{x}_T\ \mathcal{H}^{\rm lc}. \tag{570}$$

It follows that the convenient parametrization of the single plane-wave solution is now

$$\phi_p(x) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2p^+}}\big(b_{p^+,\vec{p}_T}e^{ip\cdot x} + b^*_{p^+,\vec{p}_T}e^{-ip\cdot x}\big). \tag{571}$$

We again assume the integration only over a finite box in $(x^-,\vec{x}_T)$ space of volume V and quantizing the momenta p^+ and $\vec{p}_T$, so that the exponentials $e^{ip^+x_-}$ and $e^{ip^kx^k}$ are killed in the integration process. One finds

$$H^{\rm lc} = \frac{1}{2p^+}((\vec{p}_T)^2 + m^2)b^*_{p^+,\vec{p}_T}b_{p^+,\vec{p}_T} \tag{572}$$

But this is fully in accordance with our expectation that the light-cone energy should be p^-! The other conserved momenta are given by

$$P^+ = \int \mathrm{d}x_- \int \mathrm{d}\vec{x}_T(\partial_-\phi)^2,\ \ P^I = -\int \mathrm{d}x_- \int \mathrm{d}\vec{x}_T\partial_I\phi\partial_i\phi, \tag{573}$$

and one finds that $P^+[\phi_p] = p^+b^*_{p^+,\vec{p}_T}b_{p^+,\vec{p}_T}$, $P^I[\phi_p] = p^Ib^*_{p^+,\vec{p}_T}b_{p^+,\vec{p}_T}$. This again suggests to define creation and annihilation operators $b^\dagger_{p^+,\vec{p}_T}$ and $b_{p^+,\vec{p}_T}$ parametrized by $p^+ > 0$ and $\vec{p}_T \in \mathbb{R}^{d-1}$. We impose the commutation relations

$$[b_{p^+,\vec{p}_T}, b^\dagger_{q^+,\vec{q}_T}] = \delta_{p^+,q^+}\delta_{\vec{p}_T,\vec{q}_T}. \tag{574}$$

9.3 Maxwell fields and photon states

Recall that the classical Maxwell field is given by Lorentz covector $A_\mu = A_\mu(x)$. In vacuum, it is subject to the Maxwell equations in the form

$$\Box A^\mu - \partial^\mu(\partial_\nu A_\nu) = 0. \tag{575}$$

These equations are invariant under the gauge transformations $A'_\mu = A_\mu + \partial_\mu\epsilon$, where $\epsilon = \epsilon(x)$ is an arbitrary function. We may try to solve the Maxwell equations using the Fourier transform:

$$A_\mu(x) = \int \frac{\mathrm{d}^Dp}{(2\pi)^D}e^{ip\cdot x}\hat{A}_\mu(p). \tag{576}$$

Plugging this into (575) gives the equation for $\hat{A}_\mu(p)$ in the form

$$p^2 \hat{A}^\mu(p) - p^\mu(p \cdot \hat{A}(p)) = 0. \tag{577}$$

One can Fourier-transform also the gauge parameter, that is

$$\epsilon(x) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} e^{ip\cdot x} \hat{\epsilon}(p) \tag{578}$$

It follows that in the momentum representation, the gauge transformation takes the form

$$\hat{A}'_\mu(p) = A_\mu(p) + ip_\mu \hat{\epsilon}(p). \tag{579}$$

Now, since both $A_\mu(x)$ and $\epsilon(x)$ must be real functions, $\hat{\epsilon}(p)$ must satisfy the condition

$$\hat{A}^*_\mu(p) = \hat{A}_\mu(-p), \;\; \hat{\epsilon}^*(p) = \hat{\epsilon}(-p). \tag{580}$$

We can look on the gauge transformation of the + light-cone component of the field $\hat{A}(p)$, finding

$$\hat{A}'^+(p) = \hat{A}^+(p) + ip^+ \hat{\epsilon}(p). \tag{581}$$

Assuming the $p^+ > 0$ (we always do this in light-cone gauge situations), one can choose $\hat{\epsilon}(p) = \frac{i}{p^+}\hat{A}^+(p)$ to make the + component of $\hat{A}(p)$ to vanish. This is called the **light-cone gauge**:

$$\hat{A}^+_\mu(p) = 0. \tag{582}$$

Looking at the + component of (577) gives

$$0 = p^+ \hat{A}^+(p) - p^+(p \cdot \hat{A}) = p^+(p \cdot \hat{A}(p)), \tag{583}$$

that is $p \cdot \hat{A}(p) = 0$. Expanding this condition gives

$$-p^+ \hat{A}^-(p) - p^- \hat{A}^+(p) + p^I \hat{A}^I(p) = 0. \tag{584}$$

This allows us to express $\hat{A}^-(p)$ as

$$\hat{A}^-(p) = \frac{1}{p^+}(p^I A^I). \tag{585}$$

We can now use $p \cdot \hat{A}(p) = 0$ in (577) to get

$$p^2 \hat{A}^\mu(p) = 0, \tag{586}$$

for all $\mu \in \{+, -, I\}$. If we impose it for $\mu = I$, the $-$ component already follows from (585). For each $I \in \{2, \dots, d\}$, one thus finds the equation

$$p^2 \hat{A}^I(p) = 0. \tag{587}$$

For $p^2 \neq 0$, one necessarily has $\hat{A}^I(p) = 0$. For $p^2 = 0$, there is no constraint on $\hat{A}^I(p)$. Using the similar procedure as in Exercise 9.2, one can obtain annihilation and creation operators $a^I_{p^+,\vec{p}_T}$ and $a^{I\dagger}_{p^+,\vec{p}_T}$, where $p^+ > 0$. The one-photon states are then given by

$$a^{I\dagger}_{p^+,\vec{p}_T} |\Omega\rangle. \tag{588}$$

I labels *polarizations.* For each point on the physical ($p^+ > 0$) part of mass-shell described by $(p^+, \vec{p}_T)$, we have $(D - 2$ linearly independent one-photon states. The general one-photon state of the space-time momentum $p = (p^+, \vec{p}_T)$ is given by

$$\sum_{I=2}^{d} \xi_I a^{I\dagger}_{p^+,\vec{p}_T} |\Omega\rangle. \tag{589}$$

Exercise 9.3. *Show that in arbitrary gauge, every solution of (577) for $p^2 \neq 0$ is a pure gauge, hence defines a zero electromagnetic field.*

Proof. Recall that A_μ is called a **pure gauge**, if it is gauge-equivalent to the zero field. In other words, one has $A_\mu(x) = \partial_\mu \epsilon(x)$ for some function ϵ. In the momentum representation, this translates as

$$\hat{A}_\mu(p) = ip_\mu \hat{\epsilon}(p). \tag{590}$$

Now, suppose that $\hat{A}_\mu(p)$ solves (577) for $p^2 \neq 0$. Then we can express it as

$$\hat{A}_\mu(p) = \frac{p_\mu(p \cdot \hat{A}(p)}{p^2} = ip_\mu \frac{-i(p \cdot \hat{A}(p))}{p^2}. \tag{591}$$

Note that in momentum representation, for pure gauge one has

$$\hat{F}_{\mu\nu}(p) = ip_\mu \hat{A}_\nu(p) - ip_\nu \hat{A}_\mu(p) = ip_\mu(ip_\nu \hat{\epsilon(p)}) - ip_\nu(ip_\mu \hat{\epsilon}(p)) = 0. \tag{592}$$

This shows that for $p^2 \neq 0$, there is no contribution to the electromagnetic field. ■

9.4 Gravitational fields and graviton fields

Now, recall that we have considered a linearized gravity action for a metric fluctuation $h = h_{\mu\nu}(x)$. The corresponding equation of motion was given by (84), that is

$$\Box h^{\mu\nu} - \partial_\alpha(\partial^\mu h^{\nu\alpha} + \partial^\nu h^{\mu\alpha}) + \partial^\mu \partial^\nu h = 0, \tag{593}$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$. We have also claimed that these equations are invariant under the gauge transformation

$$h'^{\mu\nu}(x) = h^{\mu\nu}(x) + \delta_0 h^{\mu\nu}(x), \;\; \delta_0 h^{\mu\nu}(x) = \partial^\mu \epsilon^\nu(x) + \partial^\nu \epsilon^\mu(x). \tag{594}$$

We will now examine those in the momentum representation, that is write

$$h^{\mu\nu}(x) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} e^{ip\cdot x} \hat{h}^{\mu\nu}(p). \tag{595}$$

Plugging this into (593) gives

$$S^{\mu\nu}(p) := p^2 \hat{h}^{\mu\nu}(p) - p_\alpha(p^\mu \hat{h}^{\nu\alpha}(p) + p^\nu \hat{h}^{\mu\alpha}(p)) + p^\mu p^\nu \hat{h}(p) = 0. \tag{596}$$

Now, in momentum representation, the gauge transformation takes the form

$$\delta \hat{h}^{\mu\nu}(p) = ip^\mu \hat{\epsilon}^\nu(p) + ip^\nu \hat{\epsilon}^\mu(p). \tag{597}$$

Exercise 9.4. *Check that (596) is indeed invariant under such a transformation.*

Proof. First, observe that

$$\delta \hat{h}(p) = \eta_{\mu\nu} \delta \hat{h}^\mu(p) = 2ip \cdot \hat{\epsilon}(p). \tag{598}$$

Consequently, one finds

$$
\begin{aligned}
\frac{1}{i}\delta_0 S^{\mu\nu}(p) &= p^2\big(p^\mu\hat{\epsilon}^\nu(p)+p^\nu\hat{\epsilon}^\mu(p)\big)\\
&\quad - p_\alpha\big(p^\mu(p^\nu\hat{\epsilon}^\alpha(p)+p^\alpha\hat{\epsilon}^\nu(p))\big) - p_\alpha\big(p^\nu(p^\mu\hat{\epsilon}^\alpha(p)+p^\alpha\hat{\epsilon}^\mu(p))\big)\\
&\quad + 2p^\mu p^\nu p\cdot\hat{\epsilon}(p)\\
&= p^2\big(p^\mu\hat{\epsilon}^\nu(p)+p^\nu\hat{\epsilon}^\mu(p)\big)\\
&\quad - p^\mu p^\nu p\cdot\hat{\epsilon}(p) - p^2p^\mu\hat{\epsilon}^\nu(p) - p^\mu p^\nu p\cdot\hat{\epsilon}(p) - p^2p^\nu\hat{\epsilon}^\mu(p)\\
&\quad + 2p^\mu p^\nu p\cdot\hat{\epsilon}(p) = 0.
\end{aligned}
\tag{599}
$$

This finishes the check. ■

Now, we shall write the components of $\hat{h}^{\mu\nu}(p)$ with respect to the light-cone coordinates. We would like to choose the gauge so that all components containing the + direction are zero. One finds

$$\delta\hat{h}^{++}(p) = 2ip^+\hat{\epsilon}^+(p), \tag{600}$$

$$\delta\hat{h}^{+-}(p) = ip^+\hat{\epsilon}^-(p) + ip^-\hat{\epsilon}^+(p), \tag{601}$$

$$\delta\hat{h}^{+I}(p) = ip^+\hat{\epsilon}^I(p) + ip^I\hat{\epsilon}^+(p). \tag{602}$$

We see that we can choose $\hat{\epsilon}^+(p)$ to make the right-hand side of $\delta\hat{h}^{++}$ equal to $-\hat{h}^{++}(p)$, namely

$$\hat{\epsilon}^+(p) = \frac{i}{2p^+}\hat{h}^{++}(p). \tag{603}$$

Similarly, we can choose $\hat{\epsilon}^-$ and $\hat{\epsilon}^I$ to force $\delta\hat{h}^{+-}(p) = -\hat{h}^{+-}(p)$ and $\delta\hat{h}^{+I}(p) = -\hat{h}^{+I}(p)$:

$$\hat{\epsilon}^-(p) = \frac{i}{p^+}(\hat{h}^{+-}(p) - \frac{p^-}{2}\hat{h}^{++}(p)),\quad \hat{\epsilon}^I(p) = \frac{i}{p^+}(\hat{h}^{+I}(p) - \frac{p^I}{2}\hat{h}^{++}(p)). \tag{604}$$

We again assume that $p^+\neq 0$, so this is possible. We conclude that we can choose a **light-cone gauge**, where $\hat{h}^{++} = \hat{h}^{+-} = \hat{h}^{+I} \equiv 0$. Let us now try to solve the equations of motion. First, the ++ component of (596) gives

$$(p^+)^2\hat{h}(p) = 0, \tag{605}$$

that is $\hat{h}(p) = 0$. Since $\hat{h}(p) = -2\hat{h}^{+-}(p) + \hat{h}^{II} = \hat{h}^{II}$, we conclude that the matrix $\hat{h}^{IJ}$ of the transversal components is traceless. We thus remain with

$$p^2\hat{h}^{\mu\nu}(p) = p_\alpha(p^\mu\hat{h}^{\nu\alpha}(p) + p^\nu\hat{h}^{\mu\alpha}(p)). \tag{606}$$

If we choose $\mu = +$, this forces

$$p_\alpha\hat{h}^{\alpha\nu}(p) = 0, \tag{607}$$

for all $\nu\in\{+,-,I\}$. By plugging this back into the remaining equation, we get

$$p^2\hat{h}^{\mu\nu}(p) = 0. \tag{608}$$

First, let us examine the implications of (607). Its only non-trivial components are for $\nu\in\{I,-\}$:

$$0 = p_\alpha\hat{h}^{\alpha I}(p) = -p^+\hat{h}^{-I}(p) + p_J\hat{h}^{JI}(p) \Rightarrow \hat{h}^{-I}(p) = \frac{1}{p^+}p_J\hat{h}^{JI}(p), \tag{609}$$

$$h0 = p_\alpha\hat{h}^{\alpha -}(p) = -p^+\hat{h}^{--}(p) + p_J\hat{h}^{J-}(p) \Rightarrow \hat{h}^{--}(p) = \frac{1}{p^+}p_J\hat{h}^{J-}(p) = \frac{1}{(p^+)^2}p_Ip_J\hat{h}^{IJ}(p). \tag{610}$$

This means that we express $\hat{h}^{-I}(p)$ and $\hat{h}^{--}(p)$ in terms of the transverse directions! It remains to examine the equation (608) for transversal indices. We find

$$p^2 \hat{h}^{IJ}(p) = 0. \tag{611}$$

Note that the remaining equations $p^2 \hat{h}^{-I}(p) = 0$ and $p^2 \hat{h}^{--}(p) = 0$ follow automatically from (611) and (609, 610).

It follows that for each physical value of momentum p, that is $p^2 = 0$ with $p^+ > 0$, we can choose arbitrary symmetric trace-less $(D-2) \times (D-2)$ matrix $\hat{h}^{IJ}(p)$. By repeating the procedure of Example 9.2, one arrives to annihilation and creation operators $a^{IJ}_{p^+,\vec{p}_T}$ and $a^{IJ\dagger}_{p^+,\vec{p}_T}$, which give raise to general **one-graviton states** with momentum $(p^+, \vec{p}_T)$:

$$\sum_{I,J=2}^{d} \xi_{IJ} a^{IJ\dagger}_{p^+,\vec{p}_T} |\Omega\rangle. \tag{612}$$

where ξ_{IJ} is the symmetric and *trace-less* **polarization tensor**. The number $n(D)$ of independent polarizations of graviton is thus equal to the dimension of the space of symmetric trace-less $(D-2) \times (D-2)$ matrices, i.e.

$$n(D) = \frac{(D-1)(D-2)}{2} - 1 = \frac{1}{2} D(D-3). \tag{613}$$

We see that $n(4) = 2$, $n(10) = 35$ and $n(26) = 299$.

References

[1] B. Zwiebach, *A First Course in String Theory.* Cambridge University Press, 2004.