Graded Generalized Geometry

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Generalized geometry = geometry of $E := TM \oplus T^*M$

- M is an arbitrary smooth manifold, $\mathcal{C}^\infty_M$ its structure sheaf of smooth functions.
- Sections $\Gamma_E = \mathfrak{X}_M \oplus \Omega^1_M$ is a sheaf of $\mathcal{C}^\infty_M$-modules.
- We have a canonical pairing $\langle \cdot, \cdot \rangle_E : \Gamma_E(M) \times \Gamma_E(M) \to \mathcal{C}^\infty_M(M)$

$$\langle (X, \xi), (Y, \eta) \rangle_E = \xi(Y) + \eta(X).$$

- There is a canonical **Dorfman bracket**

$$[(X, \xi), (Y, \eta)]_E = ([X, Y], \mathcal{L}_X \eta - \mathrm{d}\xi(Y, \cdot))$$

  making $(E, \mathrm{pr}_{TM}, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ into a **Courant algebroid**.
- Various geometries arise as sub-structures of E. Poisson manifods are involutive Lagrangian subbundles, generalized Riemannian metrics are maximal positive definite subbundles, etc.

Idea: consider $\mathcal{E} = T\mathcal{M} \oplus T^*\mathcal{M}$, where $\mathcal{M}$ is a $\mathbb{Z}$-**graded manifold**.

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Step 1: what is a graded manifold?

It is a pair $\mathcal{M} = (M, \mathcal{C}^\infty_\mathcal{M})$, having the properties:

1. M a second countable Hausdorff space;
2. $\mathcal{C}^\infty_\mathcal{M}$ is a sheaf of graded commutative associative algebras, i.e.
   - For $U \in \mathbf{Op}(M)$, $\mathcal{C}^\infty_\mathcal{M}(U) \in \mathbf{gcAs}$;
   - For $V \subseteq U$, we can restrict from $\mathcal{C}^\infty_\mathcal{M}(U)$ to $\mathcal{C}^\infty_\mathcal{M}(V)$;
   - For every open cover $\{U_\alpha\}_{\alpha \in I}$ of any $U \in \mathbf{Op}(M)$, we may compare functions locally and glue local functions which agree on the overlaps.
3. $\mathcal{C}^\infty_\mathcal{M}$ is locally isomorphic to the **graded domain** $\mathcal{C}^\infty_{(n_j)}$, where $(n_j)_{j \in \mathbb{Z}}$ is a sequence of non-negative integers (called the **graded dimension** of $\mathcal{M}$) such that $\sum_{j \in \mathbb{Z}} n_j < \infty$.
4. Some technical requirements (graded locally ringed space, etc.).

M becomes an ordinary n_0-dimensional manifold. Each $f \in \mathcal{C}^\infty_\mathcal{M}(U)$ has its **body** $\underline{f} \in \mathcal{C}^\infty_M(U)$. Surjective graded algebra morphism.

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Example (Graded domain)

- $(n_j)_{j\in\mathbb{Z}}$ a sequence of non-negative integers with $\sum_{j\in\mathbb{Z}} n_j < \infty$.
- Let $n_* := \sum_{j\neq 0} n_j$ and consider variables $\{\xi_\mu\}_{\mu=1}^{n_*}$ with $|\xi_\mu| \in \mathbb{Z}$ and

$$n_j = \#\{\mu \in \{1,\dots,n_*\} \mid |\xi_\mu| = j\}.$$

- These variables commute as $\xi_\mu \xi_\nu = (-1)^{|\xi_\mu||\xi_\nu|}\xi_\nu\xi_\mu$.
- For each $U \in \mathbf{Op}(\mathbb{R}^{n_0})$, $f \in \mathcal{C}^\infty_{(n_j)}(U)$ of degree $|f| = k$ is the formal power series in $(\xi_\mu)_{\mu=1}^{n_*}$ with coefficients in $\mathcal{C}^\infty_{\mathbb{R}^{n_0}}(U)$ of degree k, i.e. each summand has the form

$$f(x^1,\dots,x^{n_0}) \cdot (\xi_1)^{p_1} \dots (\xi_{n_*})^{p_{n_*}},$$

where $f \in \mathcal{C}^\infty_{\mathbb{R}^{n_0}}(U)$, $\sum_{\mu=1}^{n_*} p_\mu |\xi_\mu| = k$ and $p_\mu \in \{0,1\}$ for $|\xi_\mu|$ odd.
- Multiplication is a product of formal power series + reordering using the graded commutativity of variables.
- Sheaf restrictions restrict coefficient functions.

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Step 2: what is a graded vector bundle?

By a **graded vector bundle** $\mathcal{E}$ over a graded manifold $\mathcal{M}$, we mean a locally freely and finitely generated sheaf $\Gamma_{\mathcal{E}}$ of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-modules of a constant graded rank. In other words:

- For each $U \in \mathbf{Op}(M)$, $\Gamma_{\mathcal{E}}(U)$ is a graded vector space.
- For each $\psi \in \Gamma_{\mathcal{E}}(U)$ and $f \in \mathcal{C}^{\infty}_{\mathcal{M}}(U)$, we have

$$f\psi \in \Gamma_{\mathcal{E}}(U), \text{ such that } |f\psi| = |f| + |\psi|,$$

  the action is $\mathbb{R}$-bilinear and compatible with the multiplication. $\Gamma_{\mathcal{E}}(U)$ is a **graded** $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$**-module**.
- $(f\psi)|_V = f|_V \psi|_V$ for any $V \subseteq U$.
- There is a finite-dimensional $K \in \mathbf{gVect}$, such that $\Gamma_{\mathcal{E}}$ is locally isomorphic to the sheaf

$$U \mapsto \mathcal{C}^{\infty}_{\mathcal{M}}(U) \otimes_{\mathbb{R}} K.$$

$(r_j)_{j \in \mathbb{Z}} := \text{gdim}(K)$ is called a **graded rank** of $\mathcal{E}$.

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Equivalently, for each $m \in M$, there exists a **local frame** $\{\Phi_\mu\}_{\mu=1}^r$ **over** $U \ni m$ **for** $\mathcal{E}$, that is

1. $\Phi_\mu \in \Gamma_{\mathcal{E}}(U)$, $r_j = \#\{\mu \in \{1, \dots, r\} \mid |\Phi_\mu| = j\}$.
2. Each $\psi \in \Gamma_{\mathcal{E}}(U)$ can be written as

$$\psi = f^\mu \Phi_\mu,$$

for unique functions $f^\mu \in \mathcal{C}^\infty_{\mathcal{M}}(U)$ with $|f^\mu| = |\psi| - |\Phi_\mu|$.

Example (Dual vector bundle)

Let $\mathcal{E}$ be a graded vector bundle over M. For each $U \in \mathbf{Op}(M)$, set

$$(\Gamma_{\mathcal{E}^*}(U))_k := \{\xi : \Gamma_{\mathcal{E}}(U) \to \mathcal{C}^\infty_{\mathcal{M}}(U) \mid |\xi(\psi)| = |\psi| + k, \; \xi \text{ is } \mathbb{R}\text{-linear}, \; \xi(f\psi) = (-1)^{|f|k} f\xi(\psi)\}.$$

Then $\Gamma_{\mathcal{E}^*}$ defines a graded vector bundle $\mathcal{E}^*$ called **the dual to** $\mathcal{E}$. If $(r_j)_{j\in\mathbb{Z}} = \mathrm{grk}(\mathcal{E})$, the $\mathrm{grk}(\mathcal{E}^*) = (r_{-j})_{j\in\mathbb{Z}}$. $(\mathcal{E}^*)^* \cong \mathcal{E}$.

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Example (**Tangent bundle**)

For every graded manifold $\mathcal{M}$ and $U \in \mathbf{Op}(M)$, let

$$\mathfrak{X}_{\mathcal{M}}(U) := \mathrm{gDer}(\mathcal{C}^{\infty}_{\mathcal{M}}(U)).$$

Section $X \in \mathfrak{X}_{\mathcal{M}}(U)$ of degree $|X|$ is called a **vector field on $\mathcal{M}$ of degree** $|X|$ satisfying

$$X(fg) = X(f)g + (-1)^{|X||f|} fX(g).$$

By setting $\Gamma_{T\mathcal{M}} := \mathfrak{X}_{\mathcal{M}}$ we obtain the **tangent bundle to $\mathcal{M}$**. If $(n_j)_{j\in\mathbb{Z}} = \mathrm{gdim}(\mathcal{M})$, then $\mathrm{grk}(T\mathcal{M}) = (n_{-j})_{j\in\mathbb{Z}}$.

- **Cotangent bundle** is $T^*\mathcal{M} := (T\mathcal{M})^*$. $\Omega^1_{\mathcal{M}} := \Gamma_{T^*\mathcal{M}}$.
- For any graded vector bundle $\mathcal{E}$ and any $\ell \in \mathbb{Z}$, we set

$$\Gamma_{\mathcal{E}[\ell]}(U) := (\Gamma_{\mathcal{E}}(U))[\ell].$$

Modify $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-module structure: $f \triangleright' \psi := (-1)^{|f|\ell} f\psi$. $\mathcal{E}[\ell]$ is called the **degree shift of $\mathcal{E}$**.

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Modify $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-module structure: $f \triangleright' \psi := (-1)^{|f|\ell} f\psi$. $\mathcal{E}[\ell]$ is called the **degree shift of $\mathcal{E}$**.

Example (**Tangent bundle**)

For every graded manifold $\mathcal{M}$ and $U \in \mathbf{Op}(M)$, let

$$\mathfrak{X}_{\mathcal{M}}(U) := \text{gDer}(\mathcal{C}^{\infty}_{\mathcal{M}}(U)).$$

Section $X \in \mathfrak{X}_{\mathcal{M}}(U)$ of degree $|X|$ is called a **vector field on $\mathcal{M}$ of degree $|X|$** satisfying

$$X(fg) = X(f)g + (-1)^{|X||f|} fX(g).$$

By setting $\Gamma_{T\mathcal{M}} := \mathfrak{X}_{\mathcal{M}}$ we obtain the **tangent bundle to $\mathcal{M}$**. If $(n_j)_{j\in\mathbb{Z}} = \text{gdim}(\mathcal{M})$, then $\text{grk}(T\mathcal{M}) = (n_{-j})_{j\in\mathbb{Z}}$.

- **Cotangent bundle** is $T^*\mathcal{M} := (T\mathcal{M})^*$. $\Omega^1_{\mathcal{M}} := \Gamma_{T^*\mathcal{M}}$.
- For any graded vector bundle $\mathcal{E}$ and any $\ell \in \mathbb{Z}$, we set

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Modify $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-module structure: $f \triangleright' \psi := (-1)^{|f|\ell} f\psi$. $\mathcal{E}[\ell]$ is called the **degree shift of $\mathcal{E}$**.

$\mathcal{M}$ a graded manifold and $p \in \mathbb{N}$. We say that ω is a ***p*-form on $\mathcal{M}$ of degree** $|\omega|$ and write $\omega \in \Omega^p_{\mathcal{M}}(M)$, if

- $\omega : \mathfrak{X}_{\mathcal{M}}(M) \times \cdots \times \mathfrak{X}_{\mathcal{M}}(M) \to \mathcal{C}^\infty_{\mathcal{M}}(M)$ is p-linear of degree $|\omega|$.
- $\omega(fX_1, \dots, X_p) = (-1)^{|f||\omega|} f \omega(X_1, \dots, X_p)$.
- $\omega(\dots, X_i, X_{i+1}, \dots) = (-1)^{|X_i||X_{i+1}|} \omega(\dots, X_{i+1}, X_i, \dots)$.

There is a way to make it into a sheaf $\Omega^p_{\mathcal{M}}$ of graded $\mathcal{C}^\infty_{\mathcal{M}}$-modules.

Basic facts:

1. We identify $\Omega^0_{\mathcal{M}} \equiv \mathcal{C}^\infty_{\mathcal{M}}$;
2. There is $\wedge : \Omega^p_{\mathcal{M}}(M) \times \Omega^q_{\mathcal{M}}(M) \to \Omega^{p+q}_{\mathcal{M}}(M)$.
3. **Lie derivative** $\mathcal{L}_X : \Omega^p_{\mathcal{M}}(M) \to \Omega^p_{\mathcal{M}}(M)$, $|\mathcal{L}_X(\omega)| = |X| + |\omega|$.
4. **Differential** $\mathrm{d} : \Omega^p_{\mathcal{M}}(M) \to \Omega^{p+1}_{\mathcal{M}}(M)$, $|\mathrm{d}\omega| = |\omega|$.
5. **Interior product** + full set of Cartan relations.

$\Omega^p_{\mathcal{M}}$ can be equivalently obtained as a subsheaf of $\Omega_{\mathcal{M}} := \mathcal{C}^\infty_{T[1+s]\mathcal{M}}$, where $s \in \mathbb{N}_0$ is large enough even, and $T[1+s]\mathcal{M}$ is the total space of shifted tangent bundle.

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Step 3: what is a fiber-wise metric?

Let $\mathcal{E}$ be a graded vector bundle over $\mathcal{M}$ and $\ell \in \mathbb{Z}$.
$g_{\mathcal{E}} : \Gamma_{\mathcal{E}}(M) \to \Gamma_{\mathcal{E}^*}(M)$ is a **fiber-wise metric on $\mathcal{E}$ of degree ℓ**, if

- $g_{\mathcal{E}}$ is a $\mathcal{C}^\infty_{\mathcal{M}}(M)$-linear isomorphism of degree ℓ, i.e. $|g_{\mathcal{E}}(\psi)| = |\psi| + \ell$, $g_{\mathcal{E}}(f\psi) = (-1)^{|f|\ell} f g_{\mathcal{E}}(\psi)$.
- $\langle \psi, \psi' \rangle_{\mathcal{E}} := (-1)^{(|\psi|+\ell)\ell} [g_{\mathcal{E}}(\psi)](\psi')$ satisfies

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It *does not* exist on every $\mathcal{E}$, even for $\ell = 0$.

Example

Let $\mathcal{E} := T\mathcal{M}[\ell] \oplus T^*\mathcal{M}$. One has $\mathcal{E}^* \cong T^*\mathcal{M}[-\ell] \oplus T\mathcal{M}$. Set

$$g_{\mathcal{E}}(X, \xi) := (\xi, X). \;\; |g_{\mathcal{E}}(X,\xi)| = |X| = |(X,\xi)| + \ell.$$

Obvious isomorphism, the corresponding form is

$$\langle (X,\xi), (Y,\eta) \rangle_{\mathcal{E}} = \xi(Y) + (-1)^{|X||Y|} \eta(X).$$

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Step 4: what is a graded Courant algebroid?

A graded Courant algebroid of degree ℓ is $(\mathcal{E}, \rho, g_{\mathcal{E}}, [\cdot,\cdot]_{\mathcal{E}})$, where

- $\mathcal{E}$ is a vector bundle over $\mathcal{M}$.
- $\rho : \Gamma_{\mathcal{E}}(M) \to \mathfrak{X}_{\mathcal{M}}(M)$ is $\mathcal{C}^{\infty}_{\mathcal{M}}(M)$-linear of degree ℓ.
- $g_{\mathcal{E}}$ is a fiber-wise metric on $\mathcal{E}$ of degree ℓ.
- $[\cdot,\cdot]_{\mathcal{E}}$ is an $\mathbb{R}$-bilinear bracket of degree ℓ, that is

$$|[\psi, \psi']_{\mathcal{E}}| = |\psi| + |\psi'| + \ell.$$

- There holds a bunch of axioms:
  1. **Leibniz rule**: $[\psi, f\psi']_{\mathcal{E}} = \pm(\rho(\psi)f)\psi' \pm f[\psi, \psi']_{\mathcal{E}}$;
  2. **Metric compatibility**: $\rho(\psi)\langle\psi', \psi''\rangle_{\mathcal{E}} = \pm\langle[\psi, \psi']_{\mathcal{E}}, \psi''\rangle_{\mathcal{E}} \pm \langle\psi', [\psi, \psi'']_{\mathcal{E}}\rangle_{\mathcal{E}}$;
  3. **Jacobi identity**: $[\psi, [\psi', \psi'']_{\mathcal{E}}]_{\mathcal{E}} = [[\psi, \psi']_{\mathcal{E}}, \psi'']_{\mathcal{E}} \pm [\psi', [\psi, \psi'']_{\mathcal{E}}]_{\mathcal{E}}$.
  4. **Almost skew-symmetry**: $[\psi, \psi']_{\mathcal{E}} \pm [\psi', \psi]_{\mathcal{E}} = \pm(g_{\mathcal{E}}^{-1} \circ \rho^T \circ \mathrm{d})(\langle\psi, \psi'\rangle_{\mathcal{E}})$

  All $\pm$ are signs which have to be carefully determined.

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Example (Degree ℓ graded Dorfman bracket)

Consider $\mathcal{E} := T\mathcal{M}[\ell] \oplus T^*\mathcal{M}$ and $H \in \Omega^3_{\mathcal{M}}(M)$, $|H| = -\ell$.

- Set $\rho(X,\xi) = X$. Note that $|\rho(X,\xi)| = |X| = |(X,\xi)| + \ell$.
- Choose $g_{\mathcal{E}}$ as in the previous example.
- The **degree ℓ graded Dorfman bracket** takes the form

$$[(X,\xi),(Y,\eta)]^H_D = ([X,Y], (-1)^{|X|\ell}\mathcal{L}_X\eta - (-1)^{|X|+\ell}(\mathrm{d}\xi)(Y,\cdot) + H(X,Y,\cdot))$$

- This defines a GCA of degree ℓ, iff $\mathrm{d}H = 0$.

For any $\omega \in \Omega^2_{\mathcal{M}}(M)$ with $|\omega| = -\ell$, we have $\omega^\flat : \mathfrak{X}_{\mathcal{M}}(M)[\ell] \to \Omega^1_{\mathcal{M}}(M)$ of degree zero. Let $e^\omega(X,\xi) = (X, \xi + \omega^\flat(X))$. Then

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The above example represents the equivalence class of **exact GCA's** of degree ℓ corresponding to the Ševera class $[H] \in H^3_{-\ell}(\mathcal{M})$

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Step 5: what are Dirac structures?

- A **subbundle** $\mathcal{L} \subseteq \mathcal{E}$ is a subsheaf $\Gamma_{\mathcal{L}} \subseteq \Gamma_{\mathcal{E}}$ of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-modules, compatible with a local trivialization of $\mathcal{E}$.
- For each $m \in M$, there is a fiber $\mathcal{E}_m \in$ **gVect** of $\mathcal{E}$, $\mathcal{E}_m \cong K$.
- $g_{\mathcal{E}}$ induces a bilinear form $\langle \cdot, \cdot \rangle_m : \mathcal{E}_m \times \mathcal{E}_m \to \mathbb{R}$ of degree ℓ.
- Each subbundle $\mathcal{L}$ has an orthogonal complement $\mathcal{L}^{\perp} \subseteq \mathcal{E}$.

Definition (Dirac structure)

A subbundle $\mathcal{L} \subseteq \mathcal{E}$ of GCA is called a **Dirac structure**, if

1. $\mathcal{L} \subseteq \mathcal{L}^{\perp}$;
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- If $\ell \pmod 4 \neq 0$, first two conditions are $\mathcal{L} = \mathcal{L}^{\perp}$. Maximality is equivalent to conditions on $\mathrm{grk}(\mathcal{L})$.
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Example

Let $\mathcal{E} = T\mathcal{M}[\ell] \oplus T^*\mathcal{M}$ with the graded Dorfman bracket of degree ℓ.

$$\Gamma_{\mathcal{L}}(M) := \{(\Pi^\sharp(\xi), \xi) \mid \xi \in \Omega^1_{\mathcal{M}}(M)\},$$

$\Pi^\sharp : \Omega^1_{\mathcal{M}}(M) \to \mathfrak{X}_{\mathcal{M}}(M)[\ell]$ is $\mathcal{C}^\infty_{\mathcal{M}}(M)$-linear. $\Pi(\xi, \eta) := [\Pi^\sharp(\xi)](\eta)$.

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$$\Gamma_{\mathcal{L}}(M) \text{ involutive} \Leftrightarrow \frac{1}{2}[\Pi, \Pi]_S = \pm H \circ \wedge^3 \Pi^\sharp.$$

Π defines an H-**twisted graded Poisson structure** on $\mathcal{M}$ of degree ℓ.

Step 6: what are generalized complex structures?

Definition (**Generalized complex structure**)

A subbundle $\mathcal{L} \subseteq \mathcal{E}_{\mathbb{C}}$ of *GCA* is a **generalized complex structure**, if

1. $\mathcal{L}$ is isotropic w.r.t $(g_{\mathcal{E}})_{\mathbb{C}}$ and involutive w.r.t $[\cdot, \cdot]_{\mathcal{E}_{\mathbb{C}}}$;
2. $\mathcal{E}_{\mathbb{C}} = \mathcal{L} \oplus \overline{\mathcal{L}}$.

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Π defines an H-**twisted graded Poisson structure** on $\mathcal{M}$ of degree ℓ.

Step 6: what are generalized complex structures?

Definition (**Generalized complex structure**)

A subbundle $\mathcal{L} \subseteq \mathcal{E}_{\mathbb{C}}$ of *GCA* is a **generalized complex structure**, if

1. $\mathcal{L}$ is isotropic w.r.t $(g_{\mathcal{E}})_{\mathbb{C}}$ and involutive w.r.t $[\cdot, \cdot]_{\mathcal{E}_{\mathbb{C}}}$;
2. $\mathcal{E}_{\mathbb{C}} = \mathcal{L} \oplus \overline{\mathcal{L}}$.

Example

Let $\mathcal{E} = T\mathcal{M}[\ell] \oplus T^*\mathcal{M}$ with the graded Dorfman bracket of degree ℓ.

$$\Gamma_{\mathcal{L}}(M) := \{(\Pi^\sharp(\xi), \xi) \mid \xi \in \Omega^1_{\mathcal{M}}(M)\},$$

$\Pi^\sharp : \Omega^1_{\mathcal{M}}(M) \to \mathfrak{X}_{\mathcal{M}}(M)[\ell]$ is $\mathcal{C}^\infty_{\mathcal{M}}(M)$-linear. $\Pi(\xi, \eta) := [\Pi^\sharp(\xi)](\eta)$.

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- The converse is not true. Eigenbundles of $\mathcal{J}^2 = -1$ are not necessarily subbundles.

Step 7: Is there actually something new?

Definition (Differential GCA)

Let $\mathcal{E}$ be a GCA of degree ℓ. A degree 1 map $\Delta : \Gamma_{\mathcal{E}}(M) \to \Gamma_{\mathcal{E}}(M)$ is called a **differential** on $\mathcal{E}$ and $(\mathcal{E}, \Delta)$ a **differential GCA**, if

1. $\Delta^2 = 0$.
2. $\exists \underline{\Delta} \in \mathfrak{X}_{\mathcal{M}}(M)$, s.t. $\Delta(f\psi) = \underline{\Delta}(f)\psi \pm f\Delta(\psi)$;
3. $\Delta g_{\mathcal{E}} = 0$, $\rho \circ \Delta = \pm[\underline{\Delta}, \rho(\cdot)]$
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Example

$\Delta = [\phi, \cdot]_{\mathcal{E}}$ for $\phi \in \Gamma_{\mathcal{E}}(M)$ with $|\phi| = 1 - \ell$ makes $(\mathcal{E}, \Delta)$ into *dGCA* iff $[\phi, \phi]_{\mathcal{E}} = 0$. $\underline{\Delta} = \pm\rho(\phi)$ and one employs GCA axioms.

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Definition

Let $(\mathcal{E}, \Delta)$ be a dGCA of degree ℓ. A Dirac structure $\mathcal{L} \subseteq \mathcal{E}$ is called **Δ-compatible**, if $\Delta(\Gamma_{\mathcal{L}}(M)) \subseteq \Gamma_{\mathcal{L}}(M)$.

Example

Let $\mathcal{E} = T\mathcal{M}[\ell] \oplus T^*\mathcal{M}$. $\mathcal{L} = \mathrm{gr}(\Pi^\sharp)$.

- The most general Δ corresponds to (Q, θ), where $Q \in \mathfrak{X}_{\mathcal{M}}(M)$ with $|Q| = 1$ and $\theta \in \Omega^2_{\mathcal{M}}(M)$ with $|\theta| = 1 - \ell$ satisfies $\mathrm{d}\theta = 0$, and

$$[Q, Q] = 0, \quad \mathcal{L}_Q(\theta + i_Q H) = 0.$$

- The Δ-compatibility of $\mathcal{L}$ takes the form

$$\mathcal{L}_Q(\Pi) \pm (\theta \pm H(Q, \cdot, \cdot)) \circ \Lambda^2 \Pi^\sharp = 0.$$

For $H = 0$ and $\theta = 0$, this gives a **QP manifold**. dGCA together with Δ-compatible Dirac structures provide generalizations.

Δ-compatible GCS are defined analogously - they generalize differential graded symplectic manifolds.

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Outlooks

- More interesting examples of GCA's: transitive ones (using graded principal bundles), graded Lie bialgebroids.
- Examples of GCS encoding some interesting geometries.
- Morphisms of GCA's using Lagrangian relations - needs better understanding of graded vector bundles (some well known theorems do not work!).

Thank you for your attention!

Jan Vysoký: **Global Theory of Graded Manifolds**, arXiv:2105.02534.