Plurality of effective actions: Quasi-Poisson case

Jan Vysoký (in collaboration with Branislav Jurčo)

Fields and Strings Seminar, LMU Munich, 26th July 2018

Poisson-Lie T-duality

Two-dimensional σ-model

- (Σ, h) an oriented 2-dimensional Lorentzian manifold;
- G a Lie group with a metric g and a 2-form B, $\mathfrak{g} = \mathrm{Lie}(G)$;
- An action functional for the field $\ell : \Sigma \to G$ (smooth):

$$S_\sigma[\ell] = \int_\Sigma \langle h, \ell^*(g) \rangle_h \cdot d\,\mathrm{vol}_h + \int_\Sigma \ell^*(B). \tag{1}$$

- Writing $S_\sigma[\ell]$ using some local coordinates (τ, σ), where h is the Minkowski metric, and writing $z = \sigma + \tau$ and $\bar{z} = \sigma - \tau$, one has

$$S_\sigma[\ell] = \int_\Sigma dz d\bar{z}\, (\ell^* \mathbb{E})(\partial_z, \partial_{\bar{z}}) \equiv \int_\Sigma dz d\bar{z} \frac{\partial \ell^i}{\partial z} \frac{\partial \ell^j}{\partial \bar{z}} \mathbb{E}_{ij}(\ell(z, \bar{z})), \tag{2}$$

where $\mathbb{E} = g + B$, a single map $\mathbb{E} \in \mathrm{Hom}(TG, T^*G)$.
- To a solution ℓ, one assigns a "Noetherian" 1-form $J \in \Omega^1(\Sigma, \mathfrak{g}^*)$.

- One wants J to satisfy the Maurer-Cartan equation for some Lie algebra structure $[\cdot,\cdot]_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$: $dJ = \frac{1}{2}[J \wedge J]_{\mathfrak{g}^*}$.
- This is not possible in general. However, there is a solution.

Definition

- Let D be a $2n$-dimensional Lie group, with a quadratic Lie algebra $(\mathfrak{d}, \langle\cdot,\cdot\rangle_{\mathfrak{d}})$, the signature of $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ is (n,n);
- There exist Lie subgroups $G, G^* \subset D$, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^* = \mathrm{Lie}(G^*)$;
- $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is a Lagrangian decomposition of $\mathfrak{d}$ w.r.t. $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$.
- D is called the **Drinfel'd double**, $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ the **Manin triple**

- We find the restrictions on $\mathbb{E}$, it is constructed using the structures on D and a **fixed maximal positive subspace** $\mathcal{E} \subset \mathfrak{d}$.
- M-C equation ensures: there is $\widetilde{h} : \Sigma \to G^*$, such that $J = \widetilde{h}^*(\theta_{R'})$.
- Form a single map $d : \Sigma \to D$, $d(z,\bar{z}) = \ell(z,\bar{z}) \cdot \widetilde{h}(z,\bar{z})$

- Sometimes, one can decompose d the other way round:

$$d(z,\bar{z}) = h(z,\bar{z}) \cdot \widetilde{\ell}(z,\bar{z}). \tag{3}$$

- Construct the background $\widetilde{\mathbb{E}} = \widetilde{g} + \widetilde{B}$ on G^* using the similar (dual) procedure and **the same** subspace $\mathcal{E} \subset \mathfrak{d}$.

Theorem (Poisson-Lie T-duality)

Let $\ell : \Sigma \to G$ be a solution of EOM for $S_\sigma[\ell]$ with background $\mathbb{E}$.

Then $\widetilde{\ell} : \Sigma \to G^*$ solves EOM for $\widetilde{S}_\sigma[\widetilde{\ell}]$ with background $\widetilde{\mathbb{E}}$, and the corresponding Noetherian 1-forms $\widetilde{J}$ are obtained as $\widetilde{J} = h^*(\theta_R)$.

- One can use also a scalar field $\phi \in C^\infty(G)$ and modify $S_\sigma[\ell]$.
- Consistence of the quantization of such a theory leads to the EOM of the effective field theory given by the action:

$$S[g,B,\phi] = \int_G e^{-2\phi}\{\mathcal{R}(g) - \frac{1}{2}\langle dB, dB\rangle_g + 4\|\nabla^g\phi\|_g^2\} \cdot d\,\mathrm{vol}_g\,. \tag{4}$$

- How does PLT duality work on the level of effective actions?

Quasi-Poisson Lie Groups

Definition

A **Manin pair** $(\mathfrak{d}, \mathfrak{g})$ is a quadratic Lie algebra $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}})$ with its Lagrangian subalgebra $\mathfrak{g}$, that is $\mathfrak{g} = \mathfrak{g}^{\perp}$. In particular $\dim(\mathfrak{g}) = \frac{1}{2} \dim(\mathfrak{d})$.

- We assume that it integrates to a (connected Lie) **group pair** (D, G), where $G \subset D$ is a closed Lie subgroup, such that $\mathfrak{d} = \mathrm{Lie}(D)$ and $\mathfrak{g} = \mathrm{Lie}(G)$.
- As $\mathfrak{g}$ is Lagrangian, we have a canonical short exact sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\ i\ } \mathfrak{d} \underset{j}{\overset{i'}{\rightleftarrows}} \mathfrak{g}^* \longrightarrow 0. \tag{5}$$

- Choose its **isotropic splitting** $j \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$. The triple $(\mathfrak{d}, \mathfrak{g}, j)$ is called the **split Manin pair**.
- One can define $\delta \in \mathrm{Hom}(\mathfrak{g}, \Lambda^2 \mathfrak{g})$ and $\mu \in \Lambda^3 \mathfrak{g}$ by setting

$$\delta(x)(\xi, \eta) = \langle [j(\xi), j(\eta)]_{\mathfrak{d}}, i(x) \rangle_{\mathfrak{d}}, \tag{6}$$

$$\mu(\xi, \eta, \zeta) = \langle [j(\xi), j(\eta)]_{\mathfrak{d}}, j(\zeta) \rangle_{\mathfrak{d}}. \tag{7}$$

- A triple $(\mathfrak{g}, \delta, \mu)$ satisfies certain axioms, forming so called **Lie quasi-bialgebra**. They are 1-1 with split Manin pairs.
- If $j(\mathfrak{g}^*) \subset \mathfrak{d}$ is a Lie subalgebra, one has $\mu = 0$, $(\mathfrak{g}, \delta)$ is called a **Lie bialgebra** and $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ a **Manin triple**. This is exactly the PLT duality scenario.
- Every $(\mathfrak{d}, \mathfrak{g}, j)$ induces some structure on the group pair (D, G). One can define a **standard** r**-matrix** $r \in \mathfrak{d} \otimes \mathfrak{d}$ as

$$r(\xi, \eta) := \langle i^T(\xi), j^T(\eta) \rangle. \quad (8)$$

Then $\Pi_D := r^L - r^R$ is a **multiplicative bivector** on D:

$$(\Pi_D)_{dd'} = L_{d*}(\Pi_D)_{d'} + R_{d'*}(\Pi_D)_d. \quad (9)$$

It is not (in general) a Poisson structure, as it satisfies the equations

$$\frac{1}{2}[\Pi_D, \Pi_D] = i(\mu)^L - i(\mu)^R, \;\; [\Pi_D, i(\mu)^L] = 0. \quad (10)$$

- Π_D restricts naturally to a bivector $\Pi_G \in \mathfrak{X}^2(G)$, which inherits the multiplicativity and the two properties

$$\frac{1}{2}[\Pi_G, \Pi_G] = \mu^L - \mu^R, \ \ [\Pi_G, \mu^L] = 0. \tag{11}$$

  Any triple (G, Π_G, μ) having these these properties is called the **quasi-Poisson Lie group**. It contains full information about Lie quasi-bialgebra $(\mathfrak{g}, \delta, \mu)$ and vice versa.

- Consider now the **left coset space** $S = D/G$. This is a smooth manifold making the quotient map $\pi : D \to S$ into a smooth surjective submersion.
- D acts transitively on S using the **dressing action** $\triangleright$ given by

$$d \triangleright \pi(d') := \pi(dd') \tag{12}$$

  let $\#^{\triangleright} : \mathfrak{d} \to \mathfrak{X}(S)$ denote its infinitesimal generator.

- Note that also $(\mathfrak{d}, \mathrm{ad}^{(2)}(r), i(\mu))$ is a Lie quasi-bialgebra, and $(D, \Pi_D, i(\mu))$ is a quasi-Poisson Lie group.

- There is a unique bivector $\Pi_S \in \mathfrak{X}^2(S)$ such that $\Pi_S = \pi_*(\Pi_D)$. Once more, it may not be a Poisson structure, one has

$$\frac{1}{2}[\Pi_S, \Pi_S] = -\#^{\triangleright}(i(\mu)), \ \ [\Pi_S, \#^{\triangleright}(i(\mu)] = 0. \tag{13}$$

- It also behaves in a certain way with respect to the action $\triangleright$, namely

$$\mathcal{L}_{\#^{\triangleright}(x)}(\Pi_S) = \#^{\triangleright}(\mathfrak{D}_x(\Pi_S)), \tag{14}$$

  where $\mathfrak{D}_x(\Pi_S) = (\mathcal{L}_{x^L}(\Pi_S))_e \in \Lambda^2\mathfrak{d}$ is the **intrinsic derivative** of Π_S, and the map $\triangleright : (D \times S, \Pi_D + \Pi_S) \to (S, \Pi_S)$ is a bivector map.
- (S, Π_S) is thus an example of **quasi-Poisson** D**-space**, and $\triangleright$ is called a **quasi-Poisson action** of $(D, \Pi_D, i(\mu))$ on (S, Π_S).
- Via restriction, (S, Π_S) is also a quasi-Poisson G-space where the quasi-Poisson action of (G, Π_G, μ) on (S, π_S) is the restriction of $\triangleright$ onto its Lie subgroup G.
- For $\mu = 0$, one can identify $S \cong G^*$, where $\mathfrak{g}^* = \mathrm{Lie}(G^*)$ and $\Pi_{G^*} = -\Pi_S$ is the **Poisson-Lie group** integrating the dual Lie bialgebra $(\mathfrak{g}^*, \delta^*)$.

Courant algebroids

CA should be thought of as a generalization of quadratic LA.

Definition (Courant algebroid)

CA is a 4-tuple $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ of interplaying objects:

1. Vector bundle $q : E \to M$;
2. Bundle map $\rho \in \mathrm{Hom}(E, TM)$, called the **anchor** of CA;
3. $C^\infty(M)$-bilinear metric $g_E = \langle\cdot,\cdot\rangle_E$ on $\Gamma(E)$;
4. $\mathbb{R}$-bilinear bracket $[\cdot,\cdot]_E$ on $\Gamma(E)$.

These are subject to four axioms:

- **Leibniz rule**: $[\psi, f\psi']_E = f[\psi,\psi']_E + (\mathcal{L}_{\rho(\psi)} f)\psi'$.
- g_E **is invariant**: $\mathcal{L}_{\rho(\psi)}\langle\psi',\psi''\rangle_E = \langle[\psi,\psi']_E,\psi''\rangle_E + \langle\psi',[\psi,\psi'']_E\rangle_E$.
- **Leibniz identity**: $[\psi,\cdot]_E$ is a derivation of $[\cdot,\cdot]_E$.
- **Symmetric part**: $[\psi,\psi]_E = \frac{1}{2}\mathcal{D}\langle\psi,\psi\rangle_E$, where $\mathcal{D} = g_E^{-1}\circ\rho^T\circ d$.

Example (H-twisted Dorfman bracket)

Let $E = \mathbb{T}M = (T \oplus T^*)M$. Set $\rho = pr_{TM}$ and

$$\langle (X,\xi),(Y,\eta)\rangle_E = \eta(X) + \xi(Y) \tag{15}$$

For any $H \in \Omega^3_{cl}(M)$, define the H-**twisted Dorfman bracket**:

$$[(X,\xi),(Y,\eta)]_E = ([X,Y], \mathcal{L}_X\eta - i_Y(d\xi) - H(X,Y,\cdot)) \tag{16}$$

Definition (Generalized metric)

Let $(E, \langle\cdot,\cdot\rangle_E)$ be a VB equipped with a fiber-wise metric. **Generalized metric** (GM) is a *maximal* positive subbundle $V_+ \subseteq E$. E decomposes as

$$E = V_+ \oplus V_-, \tag{17}$$

where $V_- = V_+^\perp$ is a maximal negative subbundle of E.

Example

For $E = \mathbb{T}M$, one has $\Gamma(V_+) = \{(X,(g+B)(X)) \mid X \in \mathfrak{X}(M)\}$, where $g > 0$ is a Riemannian metric and $B \in \Omega^2(M)$.

Definition (Courant algebroid connections)

Courant algebroid connection ∇ on $(E, \rho, \langle \cdot, \cdot \rangle_E)$ is a $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying

$$\nabla_\psi(f\psi') = f\nabla_\psi\psi' + (\mathcal{L}_{\rho(\psi)}f)\psi', \quad \nabla_{f\psi}\psi' = f\nabla_\psi\psi', \tag{18}$$

where $\nabla_\psi = \nabla(\psi, \cdot)$, and $\nabla_\psi g_E = 0$ for all $\psi \in \Gamma(E)$.

Proposition (Gualtieri, Jurčo & Vysoký)

To any CA connection ∇, there exists a well-defined **torsion** 3-**form** $T_\nabla \in \Omega^3(E)$ *and a* **generalized Riemann tensor** $R_\nabla \in \mathcal{T}_4^0(E)$.

Definition (LC connections)

We say that ∇ is **compatible with the generalized metric** V_+, if

$$\nabla_\psi(V_+) \subseteq V_+. \tag{19}$$

One says that ∇ is **torsion-free**, if $T_\nabla = 0$. **Levi-Civita connection** on E with respect to V_+ has both properties. We write $\nabla \in \mathrm{LC}(E, V_+)$. For any E and V_+, one has $\mathrm{LC}(E, V_+) \neq \emptyset$. However, also $\mathrm{LC}(E, V_+) \neq \{\nabla\}$, there are infinitely many of them.

Definition

- **Generalized Ricci tensor** Ric_{∇} is a unique symmetric tensor (modulo sign) given as a partial trace of R_{∇}.
- There is a **divergence operator** $\mathrm{div}_{\nabla} : \Gamma(E) \to C^{\infty}(M)$.
- The connection ∇ is **Ricci compatible** with V_+, if $\mathrm{Ric}_{\nabla}(V_+, V_-) = 0$.
- Generalized metric V_+ induces a **scalar curvature** $\mathcal{R}^{+}_{\nabla} \in C^{\infty}(M)$.

Low-energy effective action

Let M be any manifold, g be a **Riemannian metric**, $B \in \Omega^2(M)$, $\phi \in C^{\infty}(M)$ and let $H' = H + dB$ for a closed 3-form $H \in \Omega^3(M)$. We consider the action functional:

$$S[g, B, \phi] = \int_M e^{-2\phi} \{ \mathcal{R}(g) - \frac{1}{2} \langle H', H' \rangle_g + 4 \| \nabla^g \phi \|^2_g \} \cdot d\,\mathrm{vol}_g, \quad (20)$$

In the following theorem $E = \mathbb{T}M$ with the H-twisted Dorfman bracket.

Theorem (**EOM in terms of LC connections**)

Let $\nabla \in \mathrm{LC}(\mathbb{T}M, V_+)$, where V_+ corresponds to fields (g, B). Suppose there exists a scalar function $\phi \in C^\infty(M)$, such that

$$\mathrm{div}_\nabla(\psi) = \mathrm{div}_{\nabla_g^{LC}} \rho(\psi) - \mathcal{L}_{\rho(\psi)}(\phi). \tag{21}$$

Then (g, B, ϕ) solve EOM for low-energy effective action (20) iff ∇ is Ricci compatible with V_+ and $\mathcal{R}^+_\nabla = 0$.

- The condition (21) restricts us to a certain subset $\mathrm{LC}(\mathbb{T}M, V_+, \phi)$ of the space $\mathrm{LC}(\mathbb{T}M, V_+)$. This subset is non-empty.
- Observe that whenever $\nabla, \nabla' \in \mathrm{LC}(\mathbb{T}M, V_+)$ have the same divergence operator, one has $\mathcal{R}^+_{\nabla'} = \mathcal{R}^+_\nabla$ and $\mathrm{Ric}^{+-}_{\nabla'} = \mathrm{Ric}^{+-}_\nabla$.
- In particular, we can choose any $\nabla \in \mathrm{LC}(\mathbb{T}M, V_+, \phi)$ to geometrically describe the EOM.

Geometry of q-PLT plurality

- We consider the group double (D, G).
- If E is a Courant algebroid over some principal G-bundle $\pi : P \to G$, there exists a notion of a G-equivariant Courant algebroid, where G acts also on the total space E, preserving all of the involved structures.
- For any G-equivariant Courant algebroid, there is a reduction procedure defining the **reduced Courant algebroid** E' over the base P/G. It involves the reduction of the fiber-wise metric $\langle \cdot, \cdot \rangle_E$. It resembles a symplectic reduction.
- One can equip the bundle $\mathbb{T}D$ with the structure of twisted Dorfman bracket with $H = \frac{1}{2} CS_3(\theta_L)$, θ_L is the left Maurer-Cartan form.
- It turns out that this makes $\mathbb{T}D$ into the D-invariant and G-invariant Courant algebroid at once. We thus have two options to obtain a reduced Courant algebroid.

Reduction by the whole group D

- View D as a principal D-bundle $\pi_D : D \to \{*\}$.
- The action on $\mathbb{T}D$ is induced by the right multiplication of D.
- The reduced CA $E'_{\mathfrak{d}}$ over $D/D = \{*\}$ is just $(\mathfrak{d}, 0, \langle\cdot,\cdot\rangle_{\mathfrak{d}}, -[\cdot,\cdot]_{\mathfrak{d}})$.

Reduction by the Lagrangian subgroup $G \subset D$

- View D as a principal G-bundle $\pi : D \to S \equiv D/G$ (left cosets).
- We have the corresponding dressing action $\triangleright : D \times S \to S$.
- Then $E'_{\mathfrak{g}} = S \times \mathfrak{d}$, where on constant sections $\psi_x \in \Gamma(E'_{\mathfrak{g}})$:

$$\rho'_{\mathfrak{g}}(\psi_x) = \#^{\triangleright}(x), \;\; \langle \psi_x, \psi_y \rangle_{E'_{\mathfrak{g}}} = \langle x, y \rangle_{\mathfrak{d}}, \tag{22}$$

$$[\psi_x, \psi_y]_{E'_{\mathfrak{g}}} = -\psi_{[x,y]_{\mathfrak{d}}}. \tag{23}$$

Moving the **generalized metric**

- Pick a generalized metric, maximal positive subspace $\mathcal{E} \subseteq \mathfrak{d} \equiv E'_{\mathfrak{d}}$.
- Lift it to the D-invariant subbundle of $E = \mathbb{T}D$.
- Push it downwards to the subbundle $\mathcal{E}'_{\mathfrak{g}}$ of $E'_{\mathfrak{g}} = S \times \mathfrak{d}$.
- This subbundle is positive and maximal - generalized metric.

- One has to make certain technical assumptions on the split Manin pair $(\mathfrak{d}, \mathfrak{g}, j)$. One says that j is an **admissible splitting** at $s \in S$ if

$$\mathrm{Ad}_d(i(\mathfrak{g})) \cap j(\mathfrak{g}^*) = 0, \tag{24}$$

  for all $d \in \pi^{-1}(s)$. Every splitting is admissible at $s_0 = \pi(e)$.
- If j is admissible at $s \in S$, it is admissible on some its neighborhood.
- For $s \in S$, there is a splitting admissible at that point.
- One says that the (D, G) is a **complete group pair** if there exists an everywhere admissible splitting. There are many convenient consequences of the assumption.

1. The map $\xi \mapsto \#_s^{\triangleright}(j(\xi))$ is a linear isomorphism of $\mathfrak{g}^*$ and $T_s(S)$. $\mathfrak{X}(S)$ is thus globally generated by vector fields

$$\xi^{\triangleright} := \#^{\triangleright}(j(\xi)). \tag{25}$$

2. There are global generators $x^{\triangleright} \in \Omega^1(S)$ for $x \in \mathfrak{g}$, and one has

$$\#^{\triangleright}(i(x) + j(\xi)) = \xi^{\triangleright} - \Pi_S(x^{\triangleright}). \tag{26}$$

3. The choice of admissible splitting chooses a convenient vector bundle isomorphism $\mathbf{\Psi} : \mathbb{T}S \to E'_{\mathfrak{g}}$. The CA structure on $E'_{\mathfrak{g}}$ then corresponds to the H-twisted Dorfman on $\mathbb{T}S$ with

$$H(\xi^{\triangleright}, \eta^{\triangleright}, \zeta^{\triangleright}) = -\frac{1}{2}\mu(\xi, \eta, \zeta). \tag{27}$$

4. The adjoint representation Ad on $\mathfrak{g} \oplus j(\mathfrak{g}^*)$ can be decomposed as

$$\mathrm{Ad}_d = \begin{pmatrix} \mathbf{k}(d) & \mathbf{b}(d) \\ \mathbf{c}(d) & \mathbf{a}(d) \end{pmatrix}, \tag{28}$$

where $\mathbf{k}(d) \in \mathrm{End}(\mathfrak{g})$ is **invertible** for all $d \in D$.

- We can this use the isomorphism $\mathbf{\Psi} : \mathbb{T}S \rightarrow E'_{\mathfrak{g}}$ to induce a generalized metric $V'_{+} \subset \mathbb{T}S$.
- This provides us with a unique Riemannian metric $g > 0$ on S and a 2-form $B \in \Omega^2(S)$.
- One can find an explicit formula using the special frame above. Let

$$\mathbf{\Pi}(x, y) := -\Pi_S(x^{\triangleright}, y^{\triangleright}), \tag{29}$$

thus defining $\mathbf{\Pi} \in C^{\infty}(S, \Lambda^2 \mathfrak{g}^*)$. The subspace $\mathcal{E} \subset \mathfrak{d}$ can be written as a graph of a map $x \mapsto j(E_0(x))$. The tensor $\mathbb{E} = g + B$ can be then for all $\xi, \eta \in \mathfrak{g}^*$ written as

$$\mathbb{E}(\xi^{\triangleright}, \eta^{\triangleright}) \equiv \mathbf{E}(\xi, \eta) = (E_0 + \mathbf{\Pi})^{-1}(\xi, \eta). \tag{30}$$

- PLT scenario fits into this procedure. One usually assumes the **completeness of the dressing fields**, which implies that the inclusion $j \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{d})$ of the dual Lie algebra is everywhere admissible isotropic splitting. There $S = D/G \cong G^*$ and $\mathbb{E}$ forms precisely the background of the *dual* sigma model on G^*.

The Idea

- We will consider $\nabla^0 \in \mathsf{LC}(\mathfrak{d}, \mathcal{E})$.
- Lift it upstairs and push it downwards to obtain $\nabla^{\mathfrak{g}} \in \mathsf{LC}(E', \mathcal{E}'_{\mathfrak{g}})$.
- Under the same isomorphism, we obtain $\nabla \in \mathsf{LC}(\mathbb{T}S, V'_+)$.

Theorem (Jurčo & Vysoký)

- *This works and does not depend on any intermediate steps.*
- ∇^0 *is Ricci compatible with respect to* $\mathcal{E}$ *if and only if* ∇ *is Ricci compatible with respect to* V'_+.
- $\mathcal{R}^+_{\nabla^0} = \mathcal{R}^+_{\nabla}$ *(in parcirular,* $\mathcal{R}^+_{\nabla}$ *is always constant).*

A Tiny Little Catch

∇ **does not** in general satisfy the assumptions of the theorem interpreting properties of LC connections as EOM.

In other words, there is no scalar function which satisfies the divergence condition (21). Finding the sufficient assumptions was the most complicated part of the entire procedure.

Proposition

Suppose ∇^0 *is* **divergence-free** *and the Lie algebra* $\mathfrak{g}$ *is unimodular. Then the connection* ∇ *satisfies the assumptions of the EOM theorem for a scalar function* $\phi \in C^\infty(S)$ *unique up to an additive constant.*

- Fitting the connection to our theorems thus provides a final background for the effective theory on S up to an additive constant.
- Explicitly, the formula for ϕ is

$$\phi = -\frac{1}{2} \ln(1 + g_0^{-1}(\mathbf{\Pi} + B_0)) + \frac{1}{2}\nu(s), \tag{31}$$

  where $\nu(\pi(d)) = \ln(\det(\mathbf{k}(d)))$, and $E_0 = g_0 + B_0$ is a decomposition onto its symmetric and skew-symmetric part.
- Note that the function ν is well-defined as $\det(\mathbf{k}(g)) = 1$ for all $g \in G$, as $\mathfrak{g}$ is unimodular, and $\det(\mathbf{k}(d)) \neq 0$ for all $d \in D$.

quasi-PLT plurality

- The EOM for the (particular) backgrounds (g, B, ϕ) and the low energy effective action on $S = D/G$ are equivalent to the *algebraic conditions* for the connection ∇^0 and consequently for the positive subspace $\mathcal{E} \subset \mathfrak{d}$.
- These conditions depend only on $\mathfrak{d}$ and the subspace $\mathcal{E}$.
- One can thus choose another Manin triple $(\mathfrak{d}, \mathfrak{g}')$ integrating to a complete group pair (D, G') and consider the coset space $S' = D/G$. The algebra $\mathfrak{g}'$ must be unimodular.
- One can use **the same** subspace $\mathcal{E} \subset \mathfrak{d}$ to construct a background (g', B', ϕ') on S', together with the closed 3-from H'.

The main statement

The backgrounds (g', B', ϕ') satisfy the EOM of the effective action on S' with H', if and only if (g, B, ϕ) satisfy the EOM of the effective action on S with H.

Some concluding remarks & outlooks

- By fixing the divergence of ∇^0, the whole statement does not depend on the remaining freedom. Note that there always exists a divergence-free ∇^0.
- The choice of the admissible splitting is in fact not very relevant, as (g, ϕ) do not change at all, whereas B and H change so that $H + dB$ remains constant.
- By solving the algebraic equations for $\mathcal{E} \subset \mathfrak{d}$, we obtain a machine for producing backgrounds (g, B, ϕ) and H on any of the coset spaces $S = D/G$ solving the EOM for the respective low-energy effective actions.
- If $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{d}, \mathfrak{g}', \mathfrak{g}'^*)$ are Manin triples integrating to their respective Poisson-Lie groups, one has $H' = H = 0$ and obtains a pair of sigma model backgrounds on G^* and G'^*, respectively. These sigma models are equivalent due to the **Poisson-Lie T-plurality**.

- The same formula (in the Poisson-Lie plurality case) for the dilaton was obtained in the literature via a careful analysis of the path integral densities, whereas we have used the effective actions (invented to describe the β-functions for the sigma models).
- In fact, the corresponding sigma models (with WZW term) targeted in S with the background (g, B, H) have been quite recently (in some sense) proved equivalent, see

  Pavol Ševera: **On integrability of 2-dimensional σ-models of Poisson-Lie type**, JHEP 1711 (2017) 015, arXiv:1709.02213

  Our result supports this observation on the level of effective actions.
- It should be possible to generalize everything to the case where P is a general principal D-bundle. This leads to an interesting relation of the orbit spaces P/G and P/D and their (generalized) geometry. It also includes the Poisson-Lie T-duality with spectators.

Branislav Jurčo, Jan Vysoký: **Poisson-Lie T-duality of String Effective Actions: A New Approach to the Dilaton Puzzle**, arXiv:1708.04079,

Branislav Jurčo, Jan Vysoký: **Courant Algebroid Connections and String Effective Actions**, arXiv:1612.0154.

Thank you for your attention!