Generalized geometry and effective actions for strings and branes

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Introduction

- Collaboration with Branislav Jurčo (Charles University) and Peter Schupp (JUB).
- Based on two papers:
  1. Branislav Jurčo, Peter Schupp a Jan Vysoký. "On the Generalized Geometry Origin of Noncommutative Gauge Theory". In: *JHEP* 1307 (2013), s. 126. DOI: 10.1007/JHEP07(2013)126. arXiv: 1303.6096 [hep-th]
  2. Branislav Jurčo, Peter Schupp a Jan Vysoký. "Extended generalized geometry and a DBI-type effective action for branes ending on branes". In: *JHEP* 1408 (2014), s. 170. DOI: 10.1007/JHEP08(2014)170. arXiv: 1404.2795 [hep-th]
- To appear in extended form in my PhD. thesis (soon :))

Generalized geometry & string sigma model

- Start with Polyakov sigma model action

$$S_P[X,h] = \frac{1}{2}\int_\Sigma g_{ij}(X) dX^i \wedge *_h dX^j + B_{ij}(X) dX^i \wedge dX^j,$$

- Fixing $h_{\alpha\beta} = \text{diag}(1,-1)$, and calculating the Hamiltonian gives

$$H[X,P] = \frac{1}{2}\int d^2\sigma \begin{pmatrix} \partial_1 X \\ P \end{pmatrix}^T \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \begin{pmatrix} \partial_1 X \\ P \end{pmatrix}.$$

- We are interested in the $2n \times 2n$ matrix in the middle, we have

$$\mathbf{G} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}.$$

- It can be interpreted as positive definite fiberwise metric on vector bundle $E = TM \oplus T^*M \rightarrow$ **Generalized geometry**.

- **G** is called **generalized metric**, with various equivalent definitions.
- It is equivalent to definition of rank n positive definite subbundle $V_+ \subseteq E$ with respect to canonical pairing $\langle \cdot, \cdot \rangle_E$ on $TM \oplus T^*M$.
- Space of all generalized metrices is $O(n,n)/(O(n) \times O(n))$, where $O(n,n)$ is a group of vector bundle morphisms preserving $\langle \cdot, \cdot \rangle_E$.
- This space is invariant with respect to natural $O(n,n)$ action:

$$\mathbf{G}' = \mathcal{O}^T \mathbf{G} \mathcal{O}, \text{ where } \mathcal{O} \in O(n,n).$$

- This can encode various field redefinitions in string theory
  1. $\mathcal{O} = e^{-F} = \begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix}$ for $F \in \Omega^2(M)$. If $\mathbf{G} \approx (g, B)$, then $\mathbf{G}' \approx (g, B+F) \rightarrow$ **Gauge transformations**
  2. $\mathcal{O} = e^{-\theta} = \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix}$ for $\theta \in \mathfrak{X}^2(M)$. For $\mathbf{G}' \approx (G, \Phi)$, this gives

$$\frac{1}{g+B} = \frac{1}{G+\Phi} + \theta,$$

$\rightarrow$ **Seiberg-Witten OC relations**.

- Adding a non-commutative parameter $\theta \Leftrightarrow$ orthogonal transformation of $\mathbf{G}$.
- Generalized metric $\mathbf{G}$ corresponds to subbundle

$$V_+ = \{X + (g+B)(X) \mid X \in TM\} \subseteq E.$$

- The map $g + B : TM \to T^*M$ plays the role in *DBI* action. In particular, the same map corresponding to $e^{-F}\mathbf{G}$ defines the integral density in

$$S_{\text{DBI}}[F] = -\int_D d^d x \frac{1}{g_m} \sqrt{\det(g+B+F)}.$$

- We can now use the generalized metric interpretation to re-derive the correspondence of commutative and non-commutative DBI actions.
- Let $\theta \in \mathfrak{X}^2(M)$, and (G, Φ) be the fields satisfying

$$(g+B)^{-1} = (G+\Phi)^{-1} + \theta.$$

- We add a fluctuation $F \to$ We look for a fluctuation F' of Φ, and possibly new θ', such that

$$(g+B+F)^{-1} = (G+\Phi+F')^{-1} + \theta'.$$

- In fact, we just solve the matrix equation

$$\begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -F' & 1 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & N^{-T} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\theta' & 1 \end{pmatrix}.$$

It has unique solutions for all θ and F such that $\det(1+\theta F) \neq 0$.

$$\theta' = (1+\theta F)^{-1}\theta, \; F' = (1+F\theta)^{-1}F, \; N = 1+\theta F.$$

- Just by formal block matrix multiplications, one obtains non-commutative field strength F' of Seiberg and Witten.
- Presence of non-trivial map N hints that there must be a change of coordinates to compensate in the equivalence of commutative and non-commutative DBI actions. For $F = dA$, and θ a Poisson bivector we can define it as a flow of time dependent vector field $A_t = (1+t\theta F)^{-1}\theta(A)$, which does the job $\rightarrow$ Seiberg-Witten map.

Membrane sigma model

- Consider now a p-brane instead of string, $p \geq 0$.
- We assume that p-brane moves in Eucleidian spacetime (M, g) in a presence of $C \in \Omega^{p+1}(M)$.
- Geometrical Nambu-Goto action is classically equivalent to Polyakov-like action, Hamiltonian of which can be after some gauge fixing written as

$$H[X,P] = \frac{1}{2} \int d^p\sigma \begin{pmatrix} P \\ \widetilde{\partial X} \end{pmatrix}^T \begin{pmatrix} g^{-1} & -g^{-1}C \\ -C^T g^{-1} & \widetilde{g} + Cg^{-1}C \end{pmatrix} \begin{pmatrix} P \\ \widetilde{\partial X} \end{pmatrix},$$

where $\widetilde{\partial X}^I = (dX^{i_1} \wedge \ldots \wedge dX^{i_p})_{1\ldots p}$, and $\widetilde{g}_{IJ} = \delta_I^{k_1 \ldots k_p} g_{k_1 j_1} \ldots g_{k_p j_p}$ defines a fiberwise metric on $\Lambda^p TM$.

- What is the interpration of the matrix in the middle?

- It can be viewed as fiberwise metric on $T^*M \oplus \Lambda^p TM$, an inverse of the following fiberwise metric $\mathbf{G}$ on $E = TM \oplus \Lambda^p T^*M$:

$$\mathbf{G} = \begin{pmatrix} 1 & -C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & \widetilde{g}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -C^T & 1 \end{pmatrix}.$$

- This looks exactly as for $p = 1$, and for $p = 1$ it limits to the original case. However, there is no canonical $O(D, D)$ form on E.
- There is canonical pairing valued in $\Lambda^{p-1} T^*M$, defined as

$$\langle V + \xi, W + \eta \rangle_E = i_V \eta + i_W \xi.$$

- We can however still work formally using the same manipulations.
- For example, let $\Pi \in \mathfrak{X}^{p+1}(M)$, and define $e^{-\Pi} = \begin{pmatrix} 1 & -\Pi \\ 0 & 1 \end{pmatrix}$. Then

$$\mathbf{G}' = \begin{pmatrix} 1 & -\Phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & \widetilde{G}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Phi^T & 1 \end{pmatrix}, \; \mathbf{G}' = (e^{-\Pi})^T \mathbf{G} e^{-\Pi},$$

still has the unique solution $(G, \widetilde{G}, \Phi)$ for arbitrary Π.

- Somehow mysteriously, this could be written also as block matrix equation

$$\begin{pmatrix} g & C \\ -C^T & \widetilde{g} \end{pmatrix}^{-1} = \begin{pmatrix} G & \Phi \\ -\Phi^T & \widetilde{G}^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix}.$$

- There is a simple observation explaining this. Denote

$$\mathcal{G} = \begin{pmatrix} g & 0 \\ 0 & \widetilde{g} \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}, \ \Theta = \begin{pmatrix} 0 & \Pi \\ -\Pi^T & 0 \end{pmatrix},$$
$$\mathcal{H} = \begin{pmatrix} G & 0 \\ 0 & \widetilde{G} \end{pmatrix}, \ \Xi = \begin{pmatrix} 0 & \Phi \\ -\Phi^T & 0 \end{pmatrix}.$$

- We then write the above equation as $(\mathcal{G} + \mathcal{B})^{-1} = (\mathcal{H} + \Xi)^{-1} + \Theta$.
- It has the same form as open-closed relations for $p = 1$.
- This is an equality of two maps from W^* to W, where the vector bundle W is defined as $W = TM \oplus \Lambda^p TM$.
- We can thus embed the objects of membrane sigma models into generalized geometry of $W \oplus W^*$.

- Note that $W \oplus W^*$ has natural pairing $\langle \cdot, \cdot \rangle_W$, and hence an $O(D, D)$ structure (where $D = n + \binom{n}{p}$).
- **G** is then just a restriction of bigger generalized metric on $W \oplus W^*$.
- Every single calculation for $p = 1$ case then carries out in the same way, simply because it is based on the same underying objects.
- The geometrical nature of this "embedding" is still not clear.
- Vector bundle $W \oplus W^*$ can be equipped with a natural Leibniz algebroid structure. Explicitly, define

$$[(X, P, \alpha, \xi), (Y, Q, \beta, \eta)] = ([X, Y], \mathcal{L}_X Q, \mathcal{L}_X \beta + d\xi(Q), \mathcal{L}_X \eta - i_y d\xi).$$

This bracket allows to treat closed $(p+1)$-forms and Nambu-Poisson manifolds $\Pi \in \mathfrak{X}^{p+1}(M)$ on equal footing - as isotropic involutive subbundles of $W \oplus W^*$.

DBI action

- We have used this framework to develop the idea of p-DBI action first proposed in
  Branislav Jurčo a Peter Schupp. "Nambu-Sigma model and effective membrane actions". In: *Phys.Lett.* B713 (2012), s. 313–316. DOI: 10.1016/j.physletb.2012.05.067. arXiv: 1203.2910 [hep-th]
- They considered the action

$$S_{p\text{-DBI}}[F] = -\int d^{p'+1}x \frac{1}{g_m} \det{}^{\frac{p}{2(p+1)}}(g) \det{}^{\frac{1}{2(p+1)}}[g+(C+F)\widetilde{g}^{-1}(C+F)^T].$$

- Note that integrand is of the form $[\det(g)]^{\frac{1}{2}-N}[\det \mathcal{G} + \mathcal{B}]^N$, where the power N was determined by equivalence of commutative and non-commutative action.
- Generalized geometry can be again used by repeating every line of $p = 1$ calculation, in particular to obtain some non-trivial determinant formulas.

- Non-commutative parameter $\theta \in \mathfrak{X}^2(M)$ is replaced by $(p+1)$-vector Π.
- If $F = dA$ for $A \in \Omega^p(M)$, and Π is Nambu-Poisson, we can construct a generalization of Seiberg-Witten map ρ, inducing the crucial change of variables.
- Non-commutative version of p-DBI action is defined as

$$-\int d^{p'+1}x \, \frac{1}{\widehat{G}_m} \frac{|\widehat{\Pi}|^{\frac{1}{p+1}}}{|\Pi|^{\frac{1}{p+1}}} \det{}^{\frac{p}{2(p+1)}} \widehat{G} \cdot \det{}^{\frac{1}{2(p+1)}} \left[\widehat{G} + (\widehat{\Phi} + \widehat{F}') \widehat{\tilde{G}}^{-1} (\widehat{\Phi} + \widehat{F}')^T \right],$$

  where $\hat{F}(x) = F(\rho(x))$, and equivalently for other fields.
- $|\Pi|$ is the Jacobian of the change of coordinates of Π into Weinstein-Darboux coordinates.
- $G_m = g_m(\det G / \det g)^{\frac{p}{2(p+1)}}$

- We can show the advantages of "doubled geometry" approach on the following generalization of $p = 1$ background independent gauge.
- For given $\mathcal{G}$ and $\mathcal{B}$ (in the form above), there always exists Θ, such that $\Theta\mathcal{B} = \mathcal{P}$, $\mathcal{P}\Theta = \Theta$, where $\mathcal{P}$ is an OG projector onto $\ker \mathcal{B}^{\perp}$.
- Now solve open-closed relations for such Θ. One gets

$$\Xi = -\mathcal{B},\ \mathcal{H} = (1 - \mathcal{P})^T \mathcal{G}(1 - \mathcal{P}) - \mathcal{B}\mathcal{G}^{-1}\mathcal{B}.$$

- Now one can simply harvest these results, to find a couple of projectors P and $\widetilde{P}$, projecting onto "non-singular" subspaces of C^T and C, and

$$G = (1 - P)^T g(1 - P) + C\widetilde{g}^{-1}C^T,$$
$$\widetilde{G} = (1 - \widetilde{P})^T \widetilde{g}(1 - \widetilde{P}) + C^T g^{-1} C,\ \Phi = -C.$$

Under certain considerations about C, projectors P and $\widetilde{P}$ give us well-behaved integrable distributions in M - "non-commutative directions".

- Having non-commutative directions, we could introduce the idea of "double scaling limit", generalized the approach of Seiberg-Witten who used it to cut-off the terms in the non-commutative DBI expansion to obtain non-commutative Yang-Mills model.
- It correctly gives background independent gauge as a result of scaling the fields g and C.
- We can now expand the DBI action above up to the first order in double scaling parameter ϵ, to obtain a version of semi-classical "matrix model":

$$S_{p\text{-NCDBI}} = \int d^{p'+1}x \, D(x)(1+\frac{1}{2(p+1)!}\{\hat{X}^a,\ldots,\hat{X}^b\}\{\hat{X}_a,\ldots,\hat{X}_b\}+\ldots.$$

- Here $\{\ldots\}$ denotes the Nambu-Poisson bracket corresponding to Θ, and $\hat{X}^a = \rho^*(x^a)$, where ρ is the Seiberg-Witten map, and x^a are original coordinates on M.

Outlook

- We still have only bosonic part $\rightarrow$ supersymmetrization?
- We would like to understand the underlying geometry on $W \oplus W^*$. In particular, not all $O(D, D)$ transformations and generalized metrics are relevant for membrane theory $\rightarrow$ is there reasonable geometrical restriction (some $O(D, D)$ subgroup?).
- Seiberg-Witten map and non-commutative DBI action induce "Nambu-Poisson Gauge Theory", which can be formulated independently of M-theory, see Branislav Jurčo, Peter Schupp a Jan Vysoký. "Nambu-Poisson Gauge Theory". In: *Physics Letters B* 733C (2014), s. 221–225. eprint: arXiv:1403.6121.
- It seems that dimensional reduction of our p-DBI action works, as was shown recently by J-H. Ho, C-T. Ma.
- We would like to understand the duality rotations of Duff-Lu in the language of generalized geometry.

Thank you for your attention!