# Graded Jet Geometry

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# Motivation

- **Jet geometry** allows for an intrinsic geometrical description of "derivatives of objects". Crucial e.g. for geometrical formulation of classical (Lagrangian) mechanics.
- To any vector bundle $\pi : E \to M$, one can assign a sequence of vector bundles $\{\varpi^k : \mathfrak{J}^k_E \to M\}_{k \in \mathbb{N}_0}$. For each $m \in M$, the fiber is

$$(\mathfrak{J}^k_E)_m = \{[\psi]_m \mid \psi \in \Gamma_E(U) \text{ for some } U \in \mathrm{Op}_m(M)\},$$

  where $\psi \sim \psi'$, iff partial derivates of the components of $\psi$ and $\psi'$ at $m$ agree up to the order $k$ (i.e. the $k$-th order Taylor polynomials of the components at $m$ coincide). $\mathfrak{J}^k_E$ **is $k$-th order jet bundle of $E$.**
- Differential equations are submanifolds of $\mathfrak{J}^k_E$. Solutions are their certain submanifolds (integral submanifolds of Cartan distribution).
- Linear differential operators are bundle maps $\mathfrak{J}^k_E \to E$.
- Can we do this for **graded manifolds and graded vector bundles** (or supermanifolds) as well?

# Graded manifolds

- **Layman's terms**: Graded manifolds are geometrical objects, functions on which locally depend on coordinates which do not necessarily commute. The coordinates commute according to their degree valued in $\mathbb{Z}$. We mean exclusively $\mathbb{Z}$-graded manifolds.

**Definition (Graded manifold)**

The graded manifold $\mathcal{M}$ consists of the following data:

1. $M$ be a second countable Hausdorff space; $M$ is called **the body** of $\mathcal{M}$.
2. Suppose $\mathcal{C}^\infty_\mathcal{M}$ be a sheaf of $\mathbb{Z}$-graded commutative associative algebras with a unit; $\mathcal{C}^\infty_\mathcal{M}$ is a **sheaf of functions on** $\mathcal{M}$.
3. There is a finite sequence $(n_j)_{j\in\mathbb{Z}}$ of non-negative integers, such that $\mathcal{C}^\infty_\mathcal{M}$ is locally isomorphic to the **graded domain** $\mathcal{C}^\infty_{(n_j)}$. The sequence $(n_j)_{j\in\mathbb{Z}}$ is called the **graded dimension** of $\mathcal{M}$. Note that $n := \sum_{j\in\mathbb{Z}} n_j < \infty$.

- To each $U \in \mathrm{Op}(M)$, we assign an algebra $\mathcal{C}^\infty_\mathcal{M}(U)$, and for $V \subseteq U$, we can restrict $f \in \mathcal{C}^\infty_\mathcal{M}(U)$ to $f|_V \in \mathcal{C}^\infty_\mathcal{M}(V)$.
- Functions can be compared locally and glued from locally defined functions, if they agree on overlaps.

## Example (**Graded domain**)

Let $(n_j)_{j \in \mathbb{Z}}$ be a finite sequence in $\mathbb{N}_0$. Let $n_* := \sum_{j \neq 0} n_j$. Consider the variables $\{\xi_1, \dots, \xi_{n_*}\}$, each of them is assigned a **degree** $|\xi_\mu| \in \mathbb{Z}$.

1. Exactly $n_j$ of them has degree $j \in \mathbb{Z}$.
2. Suppose that they commute according to the rule

$$\xi_\mu \xi_\nu = (-1)^{|\xi_\mu||\xi_\nu|} \xi_\nu \xi_\mu .$$

For each $U \in \mathrm{Op}(\mathbb{R}^n)$, $\mathcal{C}^\infty_{(n_j)}(U)$ is declared to be **the algebra of formal power series in** $\{\xi_1, \dots, \xi_n\}$ **with coefficients in** $\mathcal{C}^\infty(U)$, that is

$$f = \sum_{\mathbf{p}} f_{\mathbf{p}}(x^1, \dots, x^{n_0}) \xi^{\mathbf{p}},$$

where $\mathbf{p} = (p_1, \dots, p_{n_*})$ run over certain subset of $(\mathbb{N}_0)^{n_*}$ and $\xi^{\mathbf{p}} := (\xi_1)^{p_1} \dots (\xi_{n_*})^{p_{n_*}}$. Restrictions act by restricting the coefficients. It is a (tautological) example of a graded manifold.

# Graded vector bundles

- They are defined (solely) in terms of sheaves of their sections. Let $\mathcal{M}$ be a graded manifold. A **sheaf of graded $\mathcal{C}^\infty_\mathcal{M}$-modules** $\mathcal{F}$
  1. Assigns to each $U \in \mathrm{Op}(M)$ a graded $\mathcal{C}^\infty_\mathcal{M}(U)$-module $\mathcal{F}(U)$.
  2. For any $V \subseteq U$, we have a restriction $\mathcal{F}(U) \mapsto \mathcal{F}(V)$ compatible with the additional structures.
  3. We can compare locally and glue.

**Definition (Graded vector bundle)**

By a **graded vector bundle** $\mathcal{E}$ over a graded manifold $\mathcal{M}$, we mean a locally freely and finitely generated sheaf $\Gamma_\mathcal{E}$ of graded $\mathcal{C}^\infty_\mathcal{M}$-modules of a constant graded rank $(r_j)_{j\in\mathbb{Z}}$.

- Elements of $\Gamma_\mathcal{E}(U)$ are called **sections of $\mathcal{E}$ over $U$**.
- For each $m \in M$, there exists $U \in \mathrm{Op}_m(M)$ and a **local frame for $\mathcal{E}$ over $U$**: $\{\Phi_\lambda\}_{\lambda=1}^r \subseteq \Gamma_\mathcal{E}(U)$, such that
  1. Exactly $r_j$ of them has degree $j$ and $r = \sum_{j\in\mathbb{Z}} r_j$.
  2. For each $V \in \mathrm{Op}(U)$, every $\psi \in \Gamma_\mathcal{E}(V)$ can be uniquely decomposed as $\psi = \psi^\lambda \cdot \Phi_\lambda|_V$.

The sequence $(r_j)_{j\in\mathbb{Z}}$ is called the **graded rank** of $\mathcal{E}$.

## Example (**Tangent bundle**)

Let $\mathcal{M}$ be a graded manifold. For each $U \in \mathsf{Op}(M)$, let

$$\mathfrak{X}_{\mathcal{M}}(U) := \{X \in \mathrm{End}_{\mathbb{R}}(\mathcal{C}^{\infty}_{\mathcal{M}}(U)) \mid X(fg) = X(f)g + (-1)^{|X||f|} fX(g)\}.$$

1. Elements of $\mathfrak{X}_{\mathcal{M}}(U)$ are called **vector fields on $\mathcal{M}$ over $U$**. $\mathfrak{X}_{\mathcal{M}}$ can be made into a sheaf of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-modules.
2. Around each point there is a local frame $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n_0}}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_{n_*}}\}$. Consequently, $\mathfrak{X}_{\mathcal{M}}$ is locally freely and finitely generated.
3. Set $\Gamma_{T\mathcal{M}} := \mathfrak{X}_{\mathcal{M}}$. $T\mathcal{M}$ is called the **tangent bundle** to $\mathcal{M}$. If $(n_j)_{j\in\mathbb{Z}} := \mathrm{gdim}(\mathcal{M})$, then $\mathrm{grk}(T\mathcal{M}) = (n_{-j})_{j\in\mathbb{Z}}$.

## Example (**Trivial graded vector bundle**)

Let $K$ be a finite-dimensional graded vector space. Let

$$(\mathcal{C}^{\infty}_{\mathcal{M}}[K])(U) := \mathcal{C}^{\infty}_{\mathcal{M}}(U) \otimes_{\mathbb{R}} K.$$

$\mathcal{C}^{\infty}_{\mathcal{M}}[K]$ is a sheaf of sections of a **trivial graded vector bundle**

# Bundle of differential operators

**Definition (Differential operators)**

Let $\mathcal{E}$ be a graded vector bundle over $\mathcal{M}$. Let $U \in \mathrm{Op}(M)$ and $k \in \mathbb{N}_0$. An $\mathbb{R}$-linear map $D : \Gamma_{\mathcal{E}}(U) \to \Gamma_{\mathcal{E}}(U)$ is called a **$k$-th order differential operator on $\mathcal{E}$ over $U$**, if

1. It is $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-linear, if $k = 0$;
2. For each $f \in \mathcal{C}^{\infty}_{\mathcal{M}}(U)$, the operator $D^{(1)}_{(f)}$ defined by the formula

$$D^{(1)}_{(f)}(\psi) = f \cdot D(\psi) - (-1)^{|f||D|} D(f \cdot \psi)$$

   is a $(k-1)$-th order differential opereator, if $k > 0$.

Such operators form a graded $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-module $\mathrm{Dif}^k_{\mathcal{E}}(U)$.

**Theorem (It is nice!)**

*The assignment $U \mapsto \mathrm{Dif}^k_{\mathcal{E}}(U)$ is a locally freely and finitely generated sheaf of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-modules, hence a sheaf of sections of a* **graded vector bundle $\mathfrak{D}^k_{\mathcal{E}}$ of $k$-th order differential operators on $\mathcal{E}$**.

- It is easy to prove (by induction on $k$ and definitions) that it is a sheaf of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-modules.
- To find a local frame, one needs the following local data:
  1. A local coordinates $\{\mathbb{z}^A\}_{A=1}^n$ for $\mathcal{M}$ over $U \subseteq M$;
  2. A local frame $\{\Phi_\lambda\}_{\lambda=1}^r$ for $\mathcal{E}$ over $U$.
  3. Let $\overline{\mathbb{N}}^n(q)$ be certain subset of all $\mathbb{N}_0$-valued $n$-indices:

$$\overline{\mathbb{N}}^n(q) := \{\mathbf{I} = (i_1, \dots, i_n) \in (\mathbb{N}_0)^n \mid w(\mathbf{I}) = q \text{ and } \mathbb{z}^{\mathbf{I}} \neq 0\},$$

  where $w(\mathbf{I}) = \sum_{A=1}^n i_A$ and $z^{\mathbf{I}} = (z^1)^{i_1} \cdots (z^n)^{i_n}$. For $\mathbf{I} \in \overline{\mathbb{N}}^n(q)$ let

$$\partial^{\text{op}}_{\mathbf{I}} := (\frac{\partial}{\partial \mathbb{z}^n})^{i_n} \cdots (\frac{\partial}{\partial \mathbb{z}^1})^{i_1}.$$

  Then every $D \in \mathrm{Dif}^k_{\mathcal{E}}(U)$ can be uniquely decomposed as a $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-linear combination of

$$\mathfrak{P}_{\mathbf{I}}{}^{\lambda}{}_{\mu}(\psi) = \pm \partial^{\text{op}}_{\mathbf{I}}(\psi^\lambda) \cdot \Phi_\mu,$$

  where $\psi = \psi^\lambda \cdot \Phi_\lambda$, and $\mathbf{I} \in \bigcup_{q=0}^k \overline{\mathbb{N}}^n(q)$ and $\lambda, \mu \in \{1, \dots, r\}$.
- One proves this using "combinatorics" for polynomial $\psi^\lambda$, the general case uses Hadamard's lemma (similarly to vector fields!).

- For example, there is a **(principal) symbol map** $\sigma$ fitting into the short exact sequence

$$0 \longrightarrow \mathfrak{D}^{k-1}_{\mathcal{E}} \longrightarrow \mathfrak{D}^{k}_{\mathcal{E}} \xrightarrow{\sigma} \underline{\mathrm{Hom}}(S^k(T^*\mathcal{M}), \mathfrak{D}^0) \longrightarrow 0 \ ,$$

where Hom is the inner hom in the category of vector bundles.
- There is an inclusion $I_{(k)} : S^k(T\mathcal{M}) \to \underline{\mathrm{Hom}}(S^k(T^*\mathcal{M}), \mathfrak{D}^0)$. Set

$$\overline{\mathrm{Dif}}^k_{\mathcal{E}}(U) = \{D \in \mathrm{Dif}^k_{\mathcal{E}}(U) \mid \sigma(D) = I_{(k)}(X) \text{ for some } X \in \widetilde{\mathfrak{X}}^k_{\mathcal{M}}(U)\}$$

Write $\ell_{(k)}(D) = X$ (it is unique).
- $D \in \overline{\mathrm{Dif}}^k_{\mathcal{E}}(U)$ and $D' \in \overline{\mathrm{Dif}}^m_{\mathcal{E}}(U)$. Then $[D, D'] \in \overline{\mathrm{Dif}}^{k+m-1}_{\mathcal{E}}(U)$ and

$$\ell_{(k+m-1)}([D, D']) = [\ell_{(k)}(D), \ell_{(\ell)}(D')]_S,$$

where $[\cdot, \cdot]_S$ is a Schouten-Nijenhuis bracket on $\widetilde{\mathfrak{X}}^{\bullet}_{\mathcal{M}}(U)$.
- In particular, $\mathrm{At}_{\mathcal{E}} := \overline{\mathfrak{D}}^1_{\mathcal{E}}$ together with $\ell_{(1)}$ and $[\cdot, \cdot]$ is a **graded Lie algebroid**.

# Geometric presheaves

- For each $a \in M$ and $U \in \mathrm{Op}(M)$, let

$$\mathcal{J}^a_{\mathcal{M}}(U) = \begin{cases} \{f \in \mathcal{C}^\infty_{\mathcal{M}}(U) \mid f(a) = 0\} & \text{for } a \in U \\ \mathcal{C}^\infty_{\mathcal{M}}(U) & \text{for } a \notin U. \end{cases}$$

  $\mathcal{J}^a_{\mathcal{M}} \subseteq \mathcal{C}^\infty_{\mathcal{M}}$ is a sheaf of ideals. Let $f \in \mathcal{C}^\infty_{\mathcal{M}}(U)$. Then

$$f = 0 \Leftrightarrow f \in \bigcap_{a \in U} \bigcap_{q \in \mathbb{N}} (\mathcal{J}^a_{\mathcal{M}}(U))^q.$$

  This is crucial e.g. in the proof of the above theorem!
- Let $P$ be a graded $\mathcal{C}^\infty_{\mathcal{M}}(U)$-module, $p \in P$. What if

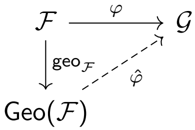
$$p = 0 \Leftrightarrow p \in \bigcap_{a \in U} \bigcap_{q \in \mathbb{N}} \{(\mathcal{J}^a_{\mathcal{M}}(U))^q \triangleright P\} \equiv P^\bullet?$$

  One says that $P$ is **geometric**, iff $P^\bullet = 0$.
- More generally, a presheaf $\mathcal{F}$ of graded $\mathcal{C}^\infty_{\mathcal{M}}$-modules is geometric, iff $\mathcal{F}(U)$ is geometric for all $U \in \mathrm{Op}(M)$.

- For any graded vector bundle $\mathcal{E}$ over $\mathcal{M}$, $\Gamma_{\mathcal{E}}$ is geometric.
- For any presheaf $\mathcal{F}$ of graded $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-modules, the presheaf

$$U \mapsto \mathrm{Geo}(\mathcal{F})(U) := \mathcal{F}(U)/\mathcal{F}(U)^{\bullet}$$

  is geometric. $\mathrm{Geo}(\mathcal{F})$ is called the **geometrization of** $\mathcal{F}$. The quotient map defines a presheaf morphism $\mathrm{geo}_{\mathcal{F}} : \mathcal{F} \to \mathrm{Geo}(\mathcal{F})$.
- Universality: for any geometric presheaf $\mathcal{G}$ and any presheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a *unique* $\hat{\varphi}$ fitting into




- One can make $\mathcal{F} \mapsto \mathrm{Geo}(\mathcal{F})$ into a functor into the category of geometric presheaves of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-modules.
- If $\mathcal{F}(M)$ is finitely generated and $\mathcal{F}$ is a *sheaf*, then so is $\mathrm{Geo}(\mathcal{F})$.
- The dual $\mathcal{F}^*$ of any presheaf $\mathcal{F}$ is a geometric sheaf.

# Jet bundle: construction

- It is not possible to define graded jet bundles by describing their fibers! Ordinary construction is destined to fail.
- There is an construction of "jet modules" associated to any modules over commutative algebras by Krasil'shchik, Lychagin and Vinogradov. We used similar ideas to directly define the sheaf of sections of a graded jet bundle.
- Throughout this section, $\mathcal{E}$ is a fixed graded vector bundle over a fixed graded manifold $\mathcal{M}$.
- For each $U \in \mathrm{Op}(M)$, let us consider a graded vector space

$$\mathcal{X}(U) := \mathcal{C}^{\infty}_{\mathcal{M}}(U) \otimes_{\mathbb{R}} \Gamma_{\mathcal{E}}(U)$$

  There are two different graded $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-module structures on $\mathcal{X}(U)$:

  1. $f \triangleright (h \otimes \psi) := (fh) \otimes \psi$;
  2. $f \blacktriangleright (h \otimes \psi) := (-1)^{|f||h|} h \otimes (f \cdot \psi)$.

  They are not the same, let $\delta_f := (f \triangleright \cdot) - (f \blacktriangleright \cdot)$. This map is $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$-linear (w.r.t. both $\triangleright$ and $\blacktriangleright$) of degree $|f|$.

- Let $k \in \mathbb{N}_0$ and consider a graded subspace

$$\mathcal{V}_{(k)}(U) = \mathbb{R}\{\delta_{f_1} \dots \delta_{f_{k+1}}(h \otimes \psi) \mid f_1, \dots, f_{k+1}, h \in \mathcal{C}^\infty_{\mathcal{M}}(U), \psi \in \Gamma_{\mathcal{E}}(U)\}.$$

In fact, it forms a graded $\mathcal{C}^\infty_{\mathcal{M}}(U)$-submodule (w.r.t both $\triangleright$ and $\blacktriangleright$).

- For any $k \in \mathbb{N}_0$, one can thus form a quotient graded vector space

$$\mathrm{pJet}^k_{\mathcal{E}}(U) := \mathcal{X}(U)/\mathcal{V}_{(k)}(U).$$

In fact, it forms a graded $\mathcal{C}^\infty_{\mathcal{M}}(U)$-module (with two actions $\triangleright$ and $\blacktriangleright$ inherited from $\mathcal{X}(U)$).

- $\mathrm{pJet}^k_{\mathcal{E}}$ forms a **presheaf** of graded $\mathcal{C}^\infty_{\mathcal{M}}(U)$-modules. In *ideal case scenario*, it would be a locally finitely and freely generated sheaf.
- The proof resembles the one for $\mathrm{Dif}^k_{\mathcal{E}}$, except at certain remainder from Hadamard's lemma ends up in $\mathrm{pJet}^k_{\mathcal{E}}(U)^\bullet$. No reason to conclude that it is zero.
- One can avoid this problem by geometrization functor. Let

$$\mathrm{gpJet}^k_{\mathcal{E}} := \mathrm{Geo}(\mathrm{pJet}^k_{\mathcal{E}}).$$

This one is locally freely and finitely generated, but not a sheaf.

## Theorem (**The one and only**)

*Let $\mathrm{Jet}^k_{\mathcal{E}}$ be a sheafification of the presheaf $\mathrm{gpJet}^k_{\mathcal{E}}$. Then it is a locally freely and finitely generated sheaf of graded $\mathcal{C}^\infty_{\mathcal{M}}$-modules of a constant graded rank, hence a sheaf of sections of a graded vector bundle $\mathfrak{J}^k_{\mathcal{E}}$. $\mathfrak{J}^k_{\mathcal{E}}$ is called a $k$-**th order jet bundle** of a graded vector bundle $\mathcal{E}$.*

- The proof explicitly constructs the local frame for $\mathfrak{J}^k_{\mathcal{E}}$ over $U$, where we have some graded local chart $(U, \varphi)$ for $\mathcal{M}$ and a local frame $\{\Phi_\lambda\}^r_{\lambda=1}$ for $\mathcal{E}$ over $U$. Let us explain this a bit:
  1. Let $h \otimes_{(k)} \psi$ denote the equivalence class in $\mathrm{pJet}^k_{\mathcal{E}}(U)$ represented by $h \otimes \psi \in \mathcal{X}(U)$;
  2. Let $[h \otimes_{(k)} \psi]^\bullet$ denote the equivalence class in the quotient $\mathrm{gpJet}^k_{\mathcal{E}}(U)$ represented by $h \otimes_{(k)} \psi$;
  3. Then the local generators for $\mathrm{gpJet}^k_{\mathcal{E}}$ over $U$ are elements of the form

     $$[\delta_{\mathbb{z}^{B_1}} \cdots \delta_{\mathbb{z}^{B_q}}(1 \otimes_{(k)} \Phi_\lambda)]^\bullet,$$

     where $0 \leq q \leq k$, and $\{\mathbb{z}^B\}^n_{B=1}$ are the local coordinate functions for $\mathcal{M}$ corresponding to $(U, \varphi)$.

# Jet bundle: properties

So far, it is not completely clear why $\mathfrak{J}^k_{\mathcal{E}}$ is a good candidate. Let us list some expected properties.

- There is a canonical "jet prolongation" $\mathrm{j}^k : \Gamma_{\mathcal{E}} \to \mathrm{Jet}^k_{\mathcal{E}}$. It is a composition of the sheafification morphism and a presheaf morphism
$$\hat{\mathrm{j}}^k_U(\psi) := [1 \otimes_{(k)} \psi]^{\bullet},$$
for all $U \in \mathrm{Op}(M)$ and $\psi \in \Gamma_{\mathcal{E}}(U)$.
- $\mathrm{j}^0 : \Gamma_{\mathcal{E}} \to \mathrm{Jet}^0_{\mathcal{E}}$ is a $\mathcal{C}^\infty_{\mathcal{M}}$-linear isomorphism, that is $\mathcal{E} \cong \mathfrak{J}^0_{\mathcal{E}}$.
- More generally, one can obtain a graded vector bundle $\mathfrak{D}^k_{\mathcal{E},\mathcal{F}}$ of $k$-th order differential operators from $\mathcal{E}$ to $\mathcal{F}$.
To any $D \in \mathrm{Dif}^k_{\mathcal{E},\mathcal{F}}(U)$, there is a *unique* $\mathcal{C}^\infty_{\mathcal{M}}(U)$-linear map $\hat{D} : \mathrm{Jet}^k_{\mathcal{E}}(U) \to \Gamma_{\mathcal{F}}(U)$ satisfying
$$D = \hat{D} \circ \mathrm{j}^k_U$$
One uses the universality of sheafification and first defines
$$\hat{D}_0[f \otimes_{(k)} \psi]^{\bullet} := (-1)^{|f||D|} f \cdot D(\psi).$$

- This correspondence in fact defines a canonical graded vector bundle isomorphism

$$\mathfrak{D}_{\mathcal{E},\mathcal{F}} \cong \underline{\mathrm{Hom}}(\mathfrak{J}^k_{\mathcal{E}}, \mathcal{F}).$$

  This bijection is natural in $\mathcal{E}$ and $\mathcal{F}$. In particular, for a fixed $\mathcal{E}$, the properties of the "internal Yoneda" functor fix $\mathfrak{J}^k_{\mathcal{E}}$ (having this property) up to the unique isomorphism.

- For each $\ell \leq k$, there is a canonical surjective degree zero $\mathcal{C}^\infty_{\mathcal{M}}$-linear sheaf morphism

$$\pi^{k,\ell} : \mathsf{Jet}^k_{\mathcal{E}} \to \mathsf{Jet}^\ell_{\mathcal{E}}$$

  satisfying $\pi^{k,\ell} \circ \mathrm{j}^k = \mathrm{j}^\ell$. Obtained from the maps

$$\hat{\pi}^{k,\ell}_U [f \otimes_{(k)} \psi]^\bullet = [f \otimes_{(\ell)} \psi]^\bullet.$$

- $\{\mathsf{Jet}^k_{\mathcal{E}}\}_{k \in \mathbb{N}_0}$ together with these morphisms form a inverse system in the category of sheaves of graded $\mathcal{C}^\infty_{\mathcal{M}}$-modules. Thus

$$\mathsf{Jet}^\infty_{\mathcal{E}} := \varprojlim \mathsf{Jet}^k_{\mathcal{E}}$$

  makes sense. This is a sheaf of graded $\mathcal{C}^\infty_{\mathcal{M}}$-modules.

- One can construct a short exact sequence of graded vector bundles

$$0 \longrightarrow \underline{\mathrm{Hom}}(S^k(T\mathcal{M}), \mathcal{E}) \xrightarrow{J_{(k)}} \mathfrak{J}^k_{\mathcal{E}} \xrightarrow{\pi^{k,k-1}} \mathfrak{J}^{k-1}_{\mathcal{E}} \longrightarrow 0,$$

for all $k \in \mathbb{N}_0$ (if we declare $\mathfrak{J}^{-1}_{\mathcal{E}} := 0$). This property also determines the sequence $\{\mathfrak{J}^k_{\mathcal{E}}\}_{k \in \mathbb{N}_0}$ up to isomorphisms.
- Let $a \in M$. Let $\mathcal{C}^\infty_{\mathcal{M},a}$ be the stalk of $\mathcal{C}^\infty_{\mathcal{M}}$ at $a$. This is a local graded ring with the maximal ideal $\mathcal{J}_{\mathcal{M},a}$.
The $k$**-th order jet space at** $a \in M$ is a graded vector space

$$\mathfrak{J}^k_{\mathcal{E},a} := \frac{\Gamma_{\mathcal{E},a}}{(\mathcal{J}_{\mathcal{M},a})^{k+1} \triangleright \Gamma_{\mathcal{E},a}}.$$

Two germs $[\psi]_a$ and $[\psi']_a$ coincide in the quotient, iff their components (w.r.t. to any local frame) have the same Taylor polynomials of order $k$ at $a$ (w.r.t. to any local coordinates).
Then there is a canonical isomorphism $(\mathfrak{J}^k_{\mathcal{E}})_a \cong \mathfrak{J}^k_{\mathcal{E},a}$, i.e. the fibers of $\mathfrak{J}^k_{\mathcal{E}}$ are jet spaces.
- For $\mathcal{E} := \mathbb{R}_{\mathcal{M}} \equiv \mathcal{M} \times \mathbb{R}$, $\mathfrak{J}^k_{[\mathcal{M}]} := \mathfrak{J}^k_{\mathbb{R}_{\mathcal{M}}}$ is called the $k$**-th order jet manifold of** $\mathcal{M}$.

# Remarks and outlooks

- Where is the jet *geometry*? To an arbitrary graded vector bundle $\mathcal{E}$ over $\mathcal{M}$, there is an actual graded manifold $\mathcal{E}$ and a surjective submersion $\pi : \mathcal{E} \to \mathcal{M}$, unique up to the isomorphism (commuting with projections).(Algebraic) graded vector bundle maps are then graded smooth maps of total spaces.
  Consequently, $\mathfrak{D}^k_{\mathcal{E}}$ and $\mathfrak{J}^k_{\mathcal{E}}$ or $\mathfrak{J}^k_{[\mathcal{M}]}$ provide new examples of graded manifolds.
- Every graded vector bundle allows one to construct its total space graded manifold. We can thus talk about submanifolds of $\mathfrak{J}^k_{\mathcal{E}}$. Do they in some cases correspond to "differential equations" similarly to the ordinary case?
- The crucial part of the standard jet geometry is the Cartan distribution, a vital example of a contact structure. Is there a suitable "graded" generalization of contact geometry? This could prove problematic since there are no "top forms" and there is no Frobenius theorem (yet).

**Thank you for your attention!**