Courant algebroid morphisms revisited

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Definition (Courant algebroid)

It is a vector bundle E over M equipped with additional structures:

1. fiber-wise metric $\langle \cdot, \cdot \rangle$ on E;
2. **anchor**, vector bundle map $\rho : E \to TM$;
3. $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]$ on $\Gamma(E)$;

Then there are four axioms, summarized as:

- bracket is not C^∞-linear $\Rightarrow$ **Leibniz rule** using ρ;
- metric and bracket are compatible à la quadratic Lie algebras;
- some Jacobi-like identity for $[\cdot, \cdot]$;
- symmetric part of $[\cdot, \cdot]$ is non-trivial, but determined by ρ and $\langle \cdot, \cdot \rangle$.

CA's appear naturally in string theory, e.g.

1. current algebras of non-linear σ-models;
2. various aspects of (Poisson–Lie) T-duality
3. geometrical description of (exceptional, heterotic) supergravity;
4. DFT and its relation with para-Hermitian geometry.

Looking for a category

The main question

How to define a **category of Courant algebroids**?

1. **objects**: Courant algebroids
2. **morphisms**: ??

For two CA's E_1 and E_2 over the same base M, this is easy.

Definition (Naive CA morphism)

Let $\mathcal{F} : E_1 \to E_2$ be vector bundle map over 1_M. For every $\psi_1 \in \Gamma(E_1)$, we have $\mathcal{F}(\psi_1) \in \Gamma(E_2)$.

- $\mathcal{F}$ preserves metrics: $\langle \mathcal{F}(\psi_1), \mathcal{F}(\psi_1') \rangle_2 = \langle \psi_1, \psi_1' \rangle_1$.
- $\mathcal{F}$ intertwines the anchor maps: $\rho_2 \circ \mathcal{F} = \rho_1$.
- $\mathcal{F}$ is a bracket morphism: $[\mathcal{F}(\psi_1), \mathcal{F}(\psi_1')]_2 = [\psi_1, \psi_1']_1$.

We need $\mathcal{F} : E_1 \to E_2$ over any smooth map $\varphi : M_1 \to M_2$.
Immediate challenge: there is no section $\mathcal{F}(\psi_1)$!!

Desired properties

- For $M_1 = M_2 = M$ and $\varphi = 1_M$, it reduces to the naive case;
- Sections $\psi_1 \in \Gamma(E_1)$ and $\psi_2 \in \Gamma(E_2)$ are $\mathcal{F}$-**related**, $\psi_1 \sim_{\mathcal{F}} \psi_2$, if

$$\mathcal{F}(\psi_1(m_1)) = \psi_2(\varphi(m_1)), \text{ for all } m_1 \in M_1. \tag{1}$$

  We want $\psi_1 \sim_{\mathcal{F}} \psi_2$, $\psi_1' \sim_{\mathcal{F}} \psi_2'$ to imply $[\psi_1, \psi_1']_1 \sim_{\mathcal{F}} [\psi_2, \psi_2']_2$.
- $\mathcal{F}$ is fiber-wise bijective and φ a diffeomorphism: $(\mathcal{F}^{-1}, \varphi^{-1})$ is automatically a CA morphism.

Question: Is there such a definition?
Answer: Yes, there is one 23 years old, but no one knows it.

P. Popescu, *On generalized algebroids*, in New Developments in Differential Geometry, Budapest 1996, pp. 329-342. Springer, 1999.

It appears there as an example, the modern definition of CA was not even born yet (Roytenberg 1999).

Category of CA's, not good enough

Definition (Classical CA morphism)

$\mathcal{F} : E_1 \to E_2$ over $\varphi : M_1 \to M_2$ satisfying:

- $\mathcal{F}$ fiber-wise preserves metrics.
- $\mathcal{F}$ intertwines the anchor maps $\rho_2 \circ \mathcal{F} = T(\varphi) \circ \rho_1$.
- For every $f_2 \in C^\infty(M_2)$, we have $\mathrm{D}_1(f_2 \circ \varphi) \sim_{\mathcal{F}} \mathrm{D}_2 f_2$, where

  $$\langle \mathrm{D}_2 f_2, \psi_2 \rangle_2 := \mathcal{L}_{\rho_2(\psi_2)} f_2, \text{ for all } f_2 \in C^\infty(M_2), \tag{2}$$

  and D_1 defined similarly.
- $\mathcal{F}$ satisfies a complicated relation of the two brackets, which can be explicitly written only locally. Similar to Lie algebroid morphism.

We obtain a nice category **CAlg**.

Immediate drawback: $\mathcal{F}$ must be fiber-wise inejctive.

Burning question

Can one throw in more morphisms?

Weinstein's symplectic "category"

Definition

Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A map $\varphi : M_1 \to M_2$ is called **symplectic**, if $\varphi^*(\omega_2) = \omega_1$.

- φ must always be an immersion.
- In order to preserve the respective Poisson brackets, φ must be a local diffeomorphism. Hence in **symplectic category**, only diffeomorphic symplectic maps (symplectomorphisms) are considered.

Symplectic "category", A. Weinstein (1982)

Consider **canonical relations** $R : M_1 \dashrightarrow M_2$, where $R \subseteq M_1 \times \overline{M}_2$ is a Lagrangian submanifold of the product symplectic manifold.

- $\overline{M}_2$ denotes the symplectic manifold $(M_2, -\omega_2)$;
- A submanifold S of (M, ω) is Lagrangian, iff $TS = TS^{\perp}$ in TM.

There is a natural composition operation $\circ$. It **does not work** for all canonical relations, hence the quotes.

Involutive structures in CA's

Definition (Supported subbundles)

Let E be a vector bundle over M, $S \subseteq M$ a submanifold.
We say that $L \subseteq E$ is a **subbundle of E supported on S**, if L is a subbundle of the restricted vector bundle E_S.

Let $(E, \rho, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ be a CA. We can impose various conditions on L:

Definition (Properties of subbundles)

1. L is **isotropic**, if $L \subseteq L^{\perp}$, $L^{\perp} \subseteq E_S$ OG complement w.r.t. $\langle \cdot, \cdot \rangle$;
2. It is **maximally isotropic**, if it is isotropic and $\text{rk}(L) = \min\{p, q\}$, where (p, q) is the signature of $\langle \cdot, \cdot \rangle$;
3. L is **compatible with the anchor**, if $\rho(L) \subseteq TS$;

Note that if $p = q$, L is maximally isotropic, iff $L = L^{\perp}$, i.e. L is **Lagrangian**. This condition has *no sense* for $p \neq q$!

Definition (Sections taking values in L)

We write $\psi \in \Gamma(E; L)$, if $\psi \in \Gamma(E)$ and $\psi|_S \in \Gamma(L)$.

Definition (**Involutive subbundles**)

We say that L is an **involutive subbundle**, if for $\psi, \psi' \in \Gamma(E; L)$, one has also $[\psi, \psi'] \in \Gamma(E; L)$.

If L is involutive, it must compatible with the anchor (for $L \neq E_S$), $L^\perp$ also (for $L \neq 0_S$). If it is not isotropic, some bad things happen.

Definition (**Involutive structures**)

L is an **almost involutive structure in E supported on S**, if

- L is isotropic;
- L and $L^\perp$ are compatible with the anchor.

Delete "almost" when L is involutive. If L is maximally isotropic, one says that L is an **(almost) Dirac structure in E supported on S**.

For general S, there is *no induced algebroid structure* on L.

Example

$L = TS \oplus \mathrm{an}(TS)$ is a Dirac structure in $TM \oplus T^*M$ equipped with the standard CA (Dorfman bracket).

CA relations

Let E_1 and E_2 be a pair of CA's over M_1 and M_2. By $\overline{E}_2$, one denotes the CA $(E_2, \rho_2, -\langle \cdot, \cdot \rangle_2, [\cdot, \cdot]_2)$ with the flipped pairing.

Definition

- Involutive structure $R \subseteq E_1 \times \overline{E}_2$ is called a **CA relation from E_1 to E_2**. One writes $R : E_1 \dashrightarrow E_2$.
- If R is supported on $\mathrm{gr}(\varphi)$ for a smooth map $\varphi : M_1 \to M_2$, one says that R is a **CA morphism over φ**.

Remark

- CA morphisms were introduced by Alekseev and Xu (date unknown), CA relations by Li-Bland and Meinrenken (2014).
- They only consider Lagrangian $R = R^{\perp}$. This makes sense *only for* split signature!
- We *do not assume* that R is a Dirac structure (maximally isotropic). Otherwise the composition fails already on the linear algebra level!

Composing CA relations

Let $R : E_1 \dashrightarrow E_2$ and $R' : E_2 \dashrightarrow E_3$ be a pair of CA relations supported on submanifolds $S \subseteq M_1 \times M_2$ and $S' \subseteq M_2 \times M_3$, respectively. Set

$$R' \circ R = \{(e_1, e_3) \in E_1 \times \overline{E}_3 \mid \exists e_2 \in E_2 \text{ s.t. } (e_1, e_2) \in R, (e_2, e_3) \in R'\}.$$

Theorem (Li-Bland, Meinrenken (2014))

- *$R' \circ R : E_1 \dashrightarrow E_3$ is not always a CA relation.*
- *However, there exist reasonable sufficient conditions (manifold-ish for the supports, constant rank-ish for the total spaces).*
- *Support of $R' \circ R$ is $S' \circ S$ given by the same formula.*

The diagonal embedding $\Delta(E) \equiv \mathrm{gr}(1_E) \subseteq E \times \overline{E}$ plays the role of the identity at E. $\circ$ is associative.

We get the Courant algebroids "category" $\overline{\textbf{Calg}}$.

Future endeavor: reformulate using symplectic NQ manifolds.

Examples

Example (Classical CA morphisms)

- Let $\mathcal{F} : E_1 \to E_2$ be a vector bundle map over $\varphi : M_1 \to M_2$;
- Set $R = \mathrm{gr}(\mathcal{F}) \subseteq E_1 \times \overline{E}_2$, supported on $\mathrm{gr}(\varphi)$;

Then $R : E_1 \dashrightarrow E_2$ is a CA relation, iff $\mathcal{F}$ is a classical CA morphism à la Popescu. Hence **Calg** $\subseteq$ $\overline{\textbf{Calg}}$.

One can easily deduce all the "desired" properties of classical CA morphisms from general statements.

Example (Pull & push)

- Let $L \subseteq E$ be an involutive structure supported on $S \subseteq M$.
- One can view them as $L \times \{0\} : E \dashrightarrow \{0\}$ or $\{0\} \times L : \{0\} \dashrightarrow E$.
- Let $R : E_1 \dashrightarrow E_2$ be a CA relation, and let $L_1 \subseteq E_1$, $L_2 \subseteq E_2$ be involutive structures.

Pullback involutive structure: $R^*(L_2) \times \{0\} := (L_2 \times \{0\}) \circ R$.
Pushforward involutive structure: $\{0\} \times R_*(L_1) := (\{0\} \times L_1) \circ R$.

Example (**Dorfman functor**)

- Let $(A, a, [\cdot,\cdot]_A)$ be a Lie algebroid. There is an induced differential $d^A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ and a Lie derivative $\mathcal{L}^A = \{d^A, i\}$.
- There is an induced CA structure on $\mathrm{Df}(A) := A \oplus A^*$, with the anchor $\rho(X,\xi) = a(X)$, canonical pairing $\langle\cdot,\cdot\rangle$, and the bracket

$$[(X,\xi),(Y,\eta)] = ([X,Y]_A, \mathcal{L}^A_X \eta - i_Y(d^A \xi)). \tag{3}$$

- Let $\mathcal{F} : A_1 \to A_2$ be a Lie algebroid morphism.

Then there exists a canonical CA relation $R_{\mathcal{F}} : \mathrm{Df}(A_1) \to \mathrm{Df}(A_2)$. One has $R_{\mathcal{F}'} \circ R_{\mathcal{F}} = R_{\mathcal{F}' \circ \mathcal{F}}$ and $R_{1_A} = \mathrm{gr}(1_{\mathrm{Df}(A)})$.
In other words, we have a **Dorfman "functor"** Df : **Lalg** $\to$ $\overline{\textbf{Calg}}$.

The overline cannot be deleted, we need the bigger category here!

Remark

The standard Courant algebroid on $TM \oplus T^*M$ can be viewed as a composition of Df with the tangent functor T : $\mathbf{Man}^\infty \to$ **Lalg**.

Example (**Para-Hermitian geometry, a.k.a. global DFT**)

- An almost para-Hermitian manifold (P, η, K) is
  1. $2n$-manifold P;
  2. metric η on P of a signature (n, n);
  3. vector bundle map $K : TP \to TP$ such that $K^2 = 1$ and $\eta(K(X), K(Y)) = -\eta(X, Y)$.
- One deletes almost, if the ± 1 eigenbundles $T_\pm$ of K are involutive.
- $T_\pm$ form Lie algebroids (restricted from TP). $\mathrm{Df}(T_\pm)$ are CA's!
- There are canonical v.b. isomorphisms $\rho_\pm$ of $\mathrm{Df}(T_\pm)$ with TP. Declaring them into CA isomorphisms, we have two CA's on TP.

Let $\mathcal{F}_+$ be a foliation corresponding to T_+ and pick its leaf $F \in \mathcal{F}_+$. By definition, $T(i_F) : TF \to T_+$ is a Lie algebroid morphism.

- There is thus $R_{T(i_F)} : \mathrm{Df}(F) \dashrightarrow \mathrm{Df}(T_+)$.
- In this case, $R_{T(i_F)}$ is a graph of a classical CA morphism.

Composing it with ρ_+ gives a classical CA morphism $\mathbf{\Psi}^+_F : \mathrm{Df}(F) \to TP$. *This is how generalized geometry on F "injects" into the ordinary geometry on the "doubled space" P.*

Example (**Reduction of CA's**)

- Let $\varpi : P \to M$ be a principal G-bundle, G connected Lie group. Write $\mathfrak{g} = \mathrm{Lie}(G)$ for its Lie algebra.
- **Equivariant CA** over P: $(E, \rho, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \Re)$ where $\Re : \mathfrak{g} \to \Gamma(E)$ is linear and satisfies ($\# : \mathfrak{g} \to \mathfrak{X}(P)$ generator of G-action on P)

$$\Re([x, y]_{\mathfrak{g}}) = [\Re(x), \Re(y)], \quad \rho \circ \Re = \#. \tag{4}$$

  Moreover, the induced action $x \blacktriangleright \psi := [\Re(x), \psi]$ integrates to a Lie group action $\mathfrak{R}$ on E, making it into G-equviariant vector bundle.
- For technical reasons, ρ has to be fiber-wise surjective.

There is a procedure to obtain a reduced CA E' over M:

1. Let $K = \Re(P \times \mathfrak{g})$. This is a G-invariant subbundle of E. So is $K^{\perp}$.
2. There is a canonical CA structure on E' given as the quotient:

$$E' = \frac{K^{\perp}/G}{(K \cap K^{\perp})/G}. \tag{5}$$

Statement: There is a canonical CA relation $Q(\Re) : E \dashrightarrow E'$.

Remark

- $Q(\Re)$ is supported on $\mathrm{gr}(\varpi)$ and *is not* a graph of a bundle map.
- For compact G, this can serve as a geometrical framework for Kaluza-Klein-like reduction of supergravity (see my paper!).

Example (**Poisson–Lie T-duality, preparation**)

- Consider the above reduction where $K \cap K^{\perp} = 0$.
- Let $H \subseteq G$ be any closed connected Lie subgroup.
- By restricting $\Re$, one obtains H-equivariant CA $(E, \rho, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \Re_0)$.
- There is thus a reduced CA E'_0 over $N \equiv P/H$.

One can always construct a canonical CA relation $R(H) : E'_0 \dashrightarrow E'$ over $\mathrm{gr}(\varphi)$, where $\varphi : P/H \to P/G \equiv M$.

Remark

- There is a non-degenerate form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ induced by $\Re$ on $\mathfrak{g} = Lie(G)$.
- $R(H)$ is a graph of a classical CA morphism, iff $\mathfrak{h} = \mathrm{Lie}(H)$ is coisotropic in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, $\mathfrak{h}^{\perp} \subseteq \mathfrak{h}$.

Example (**Poisson–Lie T-duality, conclusion**)

- Let $H, H' \subseteq G$ be two subgroups with coisotropic Lie algebras $\mathfrak{h}, \mathfrak{h}'$.
- For any CA relation $R : E_1 \dashrightarrow E_2$, we can view it (trivially) as a CA relation $R^T : E_2 \dashrightarrow E_1$. Distinguished for composition purposes.

We thus have $R(H) : E'_0 \dashrightarrow E'$ and $R(H') : E'_1 \dashrightarrow E'$. Define

$$R_{H,H'} := R(H')^T \circ R(H) : E'_0 \dashrightarrow E'_1. \tag{6}$$

This is a CA relation supported on the fibered product $N \times_M N'$. Its existence is the reason why **Poisson–Lie T-duality works**.

Some concluding remarks

- CA relations interplay very well with other notions, e.g. generalized metrics, CA connections and their induced torsion & curvature.
- It should be easy (bachelor's thesis?) to examine the relation of relations and generalized complex structures.
- Graded manifolds perspective (I have to learn it first). Translating the manifold-ish obstructions may be a little complicated.

Thank you for your attention!