Kaluza-Klein reduction of Supergravity: Geometric approach

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Motivation

Low-energy effective action coming from string theory:

$$S[g,B,\phi] = \int_P e^{-2\phi}\{\mathcal{R}(g) - \frac{1}{2}\langle H', H'\rangle_g + 4\langle d\phi, d\phi\rangle_g - 2\Lambda\} d\,\mathrm{vol}_g\,. \quad (1)$$

- (P, g) is a (pseudo)Riemannian oriented manifold;
- $B \in \Omega^2(P)$ is a skew-symmetric *Kalb-Ramond* 2-form (B-field);
- $H' = H + dB$, where $H \in \Omega^3(P)$ is a fixed 3-form on M.
- $\phi \in C^\infty(P)$ is a scalar *dilaton field*.
- $\Lambda \in \mathbb{R}$ is a cosmological constant.

- For $H = \Lambda = 0$, neglecting all physical constants, equations of motion correspond to the Weyl invariance of the 2D sigma model in the target space with backgrounds (g, B, ϕ) in the first order of α'.
- Action S constitutes the bosonic sector of type IIB supergravity.

Equations of motion take the form:

$$0 = \beta_\phi \equiv \mathcal{R}(g) - \frac{1}{2}\langle H', H' \rangle_g + 4\Delta_g(\phi) - 4\|\nabla^g \phi\|^2 - 2\Lambda, \tag{2}$$

$$0 = \beta_g \equiv \mathrm{Ric} - \frac{1}{2} H' \circ_g H' + 2(\nabla^{LC}(d\phi))_{sym}, \tag{3}$$

$$0 = \beta_B \equiv \delta_g(e^{-2\phi} H'), \tag{4}$$

where $(H' \circ_g H')(X, Y) = \langle i_X H', i_Y H' \rangle_g$. β_ϕ is scalar, β_g and β_B are symmetric and skew-symmetric $(2, 0)$-tensors, respectively.

- Recently (2013), Garcia-Fernandez obtained these equations of motion in terms of geometrical objects, so called generalized Levi-Civita connections on Courant algebroids.
- We found it (unaware of G.-F.'s work) "independently" circa 2014-2015. We take more tensorial approach.
- Question: Is this approach useful for something?
- Answer: To some extent.

- In the early days of general relativity and quantum field theory, Theodor Kaluza (1921) and Felix Klein (1926) suggested the following idea, which is usually called the **Kaluza-Klein reduction**:

- Consider 5-dimensional metric manifold (P, g) with one "circular" direction, e.g. $P = M_4 \times \mathbb{S}^1$.
- Assume that components of g do not depend on the circular coordinate.
- Consider the theory of gravity on P and observe that under the assumption on g, one obtains a gravity interacting with a Maxwell field A and a scalar field φ on M_4.

- Modern language: P is a principal $U(1)$-bundle $\pi : P \to M_4$, where g is invariant under $U(1)$ action on the total space P.
- Generalization: non-Abelian gauge fields by considering a principal G-bundle, where G is any compact Lie group, e.g. $SU(2)$.

Main question: Is there a way how to reduce the whole (bosonic part) of the supergravity action S? What are the required symmetries?

Example (H-twisted Dorfman bracket)

Consider $E = TM \oplus T^*M \equiv \mathbb{T}M$. Thus $\Gamma(E) = \mathfrak{X}(M) \oplus \Omega^1(M)$. Define the $\mathbb{R}$-bilinear bracket $[\cdot,\cdot]_E$ as

$$[(X,\xi),(Y,\eta)]_E = ([X,Y], \mathcal{L}_X\eta - i_Y d\xi - H(X,Y,\cdot)), \tag{5}$$

where $H \in \Omega^3(M)$ is a given closed 3-form.

- It satisfies the **Jacobi identity** as written above.
- It obeys the **Leibniz rule** in the form

  $$[\psi, f\psi']_E = f[\psi,\psi']_E + \rho(\psi)^i(\partial_i f)\psi', \tag{6}$$

  for all $\psi,\psi' \in \Gamma(E)$, where $\rho(X,\xi) = X$ is called the **anchor map**, forming a $C^\infty(M)$-linear map $\rho : \Gamma(E) \to \mathfrak{X}(M)$.
- The bracket is not **skew-symmetric**. Instead, one has

  $$[\psi,\psi']_E = -[\psi',\psi]_E + (0, d\langle\psi,\psi'\rangle_E), \tag{7}$$

  where $\langle(X,\xi),(Y,\eta)\rangle_E = \eta(X) + \xi(Y)$.
- The bracket $[\cdot,\cdot]_E$ and pairing $\langle\cdot,\cdot\rangle_E$ are **compatible**:

  $$\rho(\psi)^i\partial_i\langle\psi',\psi''\rangle_E = \langle[\psi,\psi']_E,\psi''\rangle_E + \langle\psi',[\psi,\psi'']_E\rangle_E. \tag{8}$$

- This bracket (and its skew-symmetrization) appeared for $H = 0$ in physics papers by Dorfman (1987) and Courant (1990) in order to make sense of Dirac bracket and various integrability conditions.

Definition (Courant algebroid)

Consider a 4-tuple $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$, where

- E is a vector bundle over manifold M.
- $\rho : \Gamma(E) \to \mathfrak{X}(M)$ is a $C^\infty(M)$-linear map.
- $\langle \cdot, \cdot \rangle_E$ is a fiber-wise metric on E.
- $[\cdot, \cdot]_E$ is an $\mathbb{R}$-bilinear bracket on $\Gamma(E)$.

Then $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ is a **Courant algebroid**, if all objects have the same properties as for H-twisted Dorfman bracket.

- Courant algebroids originally appeared as skew-symmetric brackets with much more complicated sets of axioms.
- One should think of Courant algebroids as a generalization of **quadratic Lie algebras**. Indeed, for singleton $M = \{\bullet\}$, Courant algebroid is just a Lie algebra with an invariant symmetric bilinear and non-degenerate form.

Definition

Generalized (Riemannian) metric G on the vector bundle $(E, \langle\cdot,\cdot\rangle_E)$ is a positive-definite fiber-wise metric on E, such that the induced map $\mathbf{G} : E \to E^*$ is orthogonal with respect to $\langle\cdot,\cdot\rangle_E$ and the corresponding dual metric $\langle\cdot,\cdot\rangle_{E^*}$ on E^*.

Example

For $E = \mathbb{T}M$, every **G** us uniquely determined by a pair (g, B), where $g > 0$, $B \in \Omega^2(M)$ and **G** has the block form

$$\mathbf{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}. \tag{9}$$

Properties of generalized metric

Formula $\mathbf{G}(\psi, \psi') = \langle \psi, \tau(\psi')\rangle_E$ determines an orthogonal involution $\tau \in \mathrm{End}(E)$. Its ± 1 eigenbundles give a unique decomposition

$$E = V_+ \oplus V_-, \tag{10}$$

where V_+ is maximal positive subbundle of E, and $V_- = V^\perp$.

- We can try to generalize the other concepts of usual geometry.
- Central point of GR are linear connections on manifolds.

Definition (Courant algebroid connections)

Consider an $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ with properties

$$\nabla_{f\psi}(\psi') = f \nabla_{\psi} \psi', \;\; \nabla_{\psi}(f\psi') = \rho(\psi)^i (\partial_i f) \psi'. \tag{11}$$

We impose the compatibility with $\langle \cdot, \cdot \rangle_E$:

$$\rho(\psi)^i \partial_i \langle \psi', \psi'' \rangle_E = \langle \nabla_{\psi} \psi', \psi'' \rangle_E + \langle \psi', \nabla_{\psi} \psi'' \rangle_E. \tag{12}$$

$\nabla_{\psi} = \nabla(\psi, \cdot)$. Then ∇ is called a **Courant algebroid connection**.

Example

Let $E = \mathbb{T}M$. Define $\nabla_{(X,\xi)}(Y, \eta) = (\nabla^M_X Y, \nabla^M_X \eta)$, where ∇^M is a linear connection on M and $\langle \nabla^M_X \eta, Z \rangle = X^i \partial_i \langle \eta, Z \rangle - \langle \eta, \nabla^M_X Z \rangle$.

- In fact, for any linear and $\langle \cdot, \cdot \rangle_E$-compatible Ehresmann connection ∇' on E, formula $\nabla_{\psi} \psi' = \nabla'_{\rho(\psi)} \psi'$ defines a Courant algebroid connection.

- We can extend ∇ to the whole tensor algebra $\mathcal{T}(E)$. Compatibility with $g_E \equiv \langle \cdot, \cdot \rangle_E$ can be then written as $\nabla g_E = 0$.
- No clear way how to interpret ∇ in the usual geometrical sense (find some frame bundle etc.).
- Unlike Ehresmann connections, ∇ is "balanced", i.e. both inputs are from $\Gamma(E)$. We can thus consider the torsion operator.

Definition (Torsion 3-form)

Let ∇ be a Courant algebroid connection. Define $T_G \in \Omega^3(E)$ as

$$T_G(\psi, \psi', \psi'') = \langle \nabla_\psi \psi' - \nabla_{\psi'} \psi - [\psi, \psi']_E, \psi'' \rangle_E + \langle \nabla_{\psi''} \psi, \psi' \rangle_E, \quad (13)$$

for all $\psi, \psi', \psi'' \in \Gamma(E)$. ∇ **torsion free** if $T_G = 0$.

- Third term is necessary in order to ensure the tensoriality of T_G
- The complete skew-symmetry of T_G is a non-trivial implication of Courant algebroid axioms.
- Discovered independently by Gualtieri (2007) and Alekseev-Xu (200?). Clear geometrical meaning is again unknown.

- One can try to similarly fix the Riemann curvature tensor. Consider the following naive definition (via the Ricci identity):

$$R^{(0)}(\phi', \phi, \psi, \psi') = \langle \phi', [\nabla_{\psi}, \nabla_{\psi'}]\phi - \nabla_{[\psi,\psi']_E}\phi \rangle_E. \quad (14)$$

- $R^{(0)}$ has no reasonable symmetries. It is **even not** a tensor on E!
- Possible fix was discovered by Hohm and Zwiebach in 2013 within the *double field theory*. Once more, there is no evidence yet of a nice geometrical interpretation.

Definition (Generalized Riemann tensor)

Let ∇ be a Courant algebroid connection. Then $R \in \mathcal{T}_4^0(E)$ is defined as

$$\begin{aligned} R(\phi', \phi, \psi, \psi') = &\frac{1}{2}\{R^{(0)}(\phi', \phi, \psi, \psi') + R^{(0)}(\psi', \psi, \phi, \phi')\} \\ &+ \frac{1}{2}\langle \langle \nabla\psi, \psi' \rangle_E, \langle \nabla\phi, \phi' \rangle_E \rangle_{E^*}. \end{aligned} \quad (15)$$

- Amazingly, R has all usual symmetries, including quite involved algebraic Bianchi identity proportional to (covariant derivatives of) the torsion 3-form T_G. Interchange symmetry holds for all ∇.

- Symmetries are important, they allow for unambiguous definition of Ricci tensor, trace is unique up to a multiple.

Definition (Ricci tensor and scalar)

Let $\{\psi_\lambda\}_{\lambda=1}^{rank(E)}$ be a local frame on E. Let $\psi_E^\lambda = g_E^{-1}(\psi^\lambda)$. Then

$$\mathrm{Ric}(\psi, \psi') = R(\psi_E^\lambda, \psi, \psi_\lambda, \psi') \tag{16}$$

is a well-defined and symmetric **Ricci tensor** corresponding to ∇.

$$\mathcal{R}_E = \mathrm{Ric}(\psi_E^\lambda, \psi_\lambda) \tag{17}$$

is associated to any ∇ and called the **Courant-Ricci scalar** of ∇.

Definition (LC connection)

Let $\mathbf{G}$ be a generalized metric on Courant algebroid E. We say that ∇ is a **Levi-Civita connection** on E with respect to $\mathbf{G}$, if

- ∇ is torsion-free.
- ∇ is metric compatible with $\mathbf{G}$, that is $\nabla \mathbf{G} = 0$.

- For any E and $\mathbf{G}$, there always exists a LC connection.

Problem

Unless for very low-rank cases, there are **infinitely many** Levi-Civita connections on every Courant algebroid with **G**. There is not any direct formula to write down ∇ in terms of **G** and $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$.

- Interestingly, this can be used to our advantage!

Definition (The other scalar and Ricci compatibility)

Assume that ∇ is a Levi-Civita connection on E with respect to **G**. We can define a **Ricci scalar of ∇ with respect to G** as

$$\mathcal{R}_{\mathbf{G}} = \mathrm{Ric}(\mathbf{G}^{-1}(\psi^{\lambda}), \psi_{\lambda}). \tag{18}$$

We say that ∇ is **Ricci-compatible with G** if $\mathrm{Ric}(V_+, V_-) = 0$.

- All quantities involved in our definitions transform "covariantly" under Courant algebroid isomorpisms. This is one of the main advantages of our more "tensorial" approach.
- There are some additional covariantly transforming conditions which can serve to further fix ∇. But not completely.

Equations of motion

Levi-Civita connections on $E = \mathbb{T}P$ can be quite useful.

- Consider a generalized metric $\mathbf{G}$ corresponding to a pair (g, B).
- Let $H \in \Omega^3(P)$ be a closed 3-form and let $[\cdot,\cdot]_E$ the corresponding H-twisted Dorfman bracket.
- Consider a Levi-Civita connection ∇ on E with respect to $\mathbf{G}$ subject to two additional conditions:

$$X^i_\nabla(\partial_i f) := \langle \nabla_{\psi_\lambda}(0, df), \psi^\lambda \rangle \tag{19}$$

defines an everywhere vanishing vector field X_∇ on P, and one more complicated involving a scalar function ϕ.

Theorem (B. Jurčo & JV (2015))

There exists such ∇ (not unique).
Background fields (g, B, ϕ) satisfy the equations of motion given by action S, if and only if ∇ is Ricci compatible with $\mathbf{G}$ and $\mathcal{R}_{\mathbf{G}} = 2\Lambda$.

Reduction of Courant algebroids

- Now assume that $\pi : P \to M$ is a principal G-bundle. Let $A \in \Omega^1(P, \mathfrak{g})$ be a fixed principal bundle connection on P.
- G is a compact and semisimple Lie group.
- One says that Courant algebroid on vector bundle $q : E \to P$ is **equivariant**, if there exists $\Re : \mathfrak{g} \to \Gamma(E)$ with properties

$$\rho \circ \Re = \#, \quad \Re([x, y]_{\mathfrak{g}}) = [\Re(x), \Re(y)]_E, \tag{20}$$

where $\# : \mathfrak{g} \to \mathfrak{X}(P)$ is an infinitesimal generator of the principal G-action on P, and $\Re$ integrates to a Lie group action $\mathbf{R}$ acting by vector bundle morphisms over the principal bundle G-action R.

Conditions for $E = \mathbb{T}P$

The question of equivariance for generalized tangent bundle can be answered completely. One example is the choice:

$$\Re(x) = (\#x, -\frac{1}{2}\langle A, x\rangle_{\mathfrak{g}}), \quad H = \pi^*(H_0) + \frac{1}{2} CS_3(A), \tag{21}$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is the Killing form and $CS_3(A)$ is the Chern-Simons of A.

- Chern-Simons 3-form is defined by A and its curvature $\Omega = DA$:

$$CS_3(A) = \langle \Omega \wedge A \rangle_{\mathfrak{g}} - \frac{1}{3!} \langle [A \wedge A]_{\mathfrak{g}} \wedge A \rangle_{\mathfrak{g}} \tag{22}$$

- H is closed iff only there holds a structure equation:

$$dH_0 + \frac{1}{2} \langle F \wedge F \rangle_{\mathfrak{g}} = 0. \tag{23}$$

- Every equivariant Courant algebroid can be reduced to a Courant algebroid E' over the base manifold M.

In the case above $E' = TM \oplus \mathfrak{g}_P \oplus T^*M$, where $\mathfrak{g}_P \equiv P \times_{Ad} \mathfrak{g}$ is a vector bundle associated to the adjoint representation. $\Gamma(\mathfrak{g}_P) = C^{\infty}_{Ad}(P, \mathfrak{g})$.
$\rho'(X, \Phi, \xi) = X$ and $\langle (X, \Phi, \xi), (Y, \Phi', \eta) \rangle_{E'} = \eta(X) + \xi(Y) + \langle \Phi, \Phi' \rangle_{\mathfrak{g}}$.
The bracket becomes quite complicated:

$$\begin{aligned}[\psi, \psi']_{E'} = ([X, Y], D_X \Phi' - D_Y \Phi - [\Phi, \Phi']_{\mathfrak{g}} - F(X, Y), \mathcal{L}_X \eta - i_Y d\xi \\ - H_0(X, Y, \cdot) + \langle D\Phi, \Phi' \rangle_{\mathfrak{g}} - \langle F(X), \Phi' \rangle_{\mathfrak{g}} + \langle F(Y), \Phi \rangle_{\mathfrak{g}})\end{aligned}$$

- This is usually called the **heterotic Courant algebroid**.

- This led us to consider generalized metrics and respective Levi-Civita connections on E'. This is a far more involved question.

Generalized metrics on E'

Every generalized metric $\mathbf{G}'$ on E' is uniquelly parametrized by a triple (g_0, B_0, ϑ), where g_0 is a Riemannian metric on M, $B_0 \in \Omega^2(M)$ and $\vartheta \in \Omega^1(M, \mathfrak{g}_P)$. ϑ represents freedom in the choice of connection on P.

Every $\mathbf{G}'$ can be obtained by reduction from a unique generalized metric $\mathbf{G}$. The corresponing (g, B) take the block form

$$g = \begin{pmatrix} 1 & \vartheta^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_0 & 0 \\ 0 & -\frac{1}{2}c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \vartheta & 1 \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & \frac{1}{2}\vartheta^T c \\ -\frac{1}{2}c\vartheta & 0 \end{pmatrix}, \tag{24}$$

with respect to the decomposition $\mathfrak{X}_G(P) = \mathfrak{X}(M) \oplus \Gamma(\mathfrak{g}_P)$ given by A. Here $c = \langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is the Killing form.

This will be the assumptions on (g, B) for **Kaluza-Klein reduction**.

- We have now considered the simplest possible Levi-Civita connection ∇' on E' with respect to $\mathbf{G}'$. We call it the **minimal connection** for this very reason.
- One can add a dilaton ingredient $\phi_0 \in C^\infty(M)$ into this connection, in a way similar to $E = \mathbb{T}P$ case.
- We have calculated its scalar curvature with respect to $\mathbf{G}'$:

$$\begin{aligned} \mathcal{R}'_{\mathbf{G}'} = \mathcal{R}(g_0) - \frac{1}{2}\langle H'_0, H'_0 \rangle_{g_0} + \frac{1}{2}\langle\langle F', F' \rangle\rangle \\ + 4\Delta_{g_0}(\phi_0) - 4\langle d\phi_0, d\phi_0 \rangle_{g_0} + \frac{1}{6}\dim \mathfrak{g}. \end{aligned} \tag{25}$$

The involved quantities are:

- $H'_0 = H_0 + dB_0 - \frac{1}{2}\tilde{C}_3(\vartheta) - \langle F \wedge \vartheta \rangle_{\mathfrak{g}}$, where $\tilde{C}_3(\vartheta)$ is a "Chern-Simons like" form made out of ϑ.
- $F' = F + D\vartheta + \frac{1}{2}[\vartheta \wedge \vartheta]_{\mathfrak{g}}$.
- $\langle\langle \cdot, \cdot \rangle\rangle$ is a fiber-wise scalar product combining the usual product $\langle \cdot, \cdot \rangle_{g_0}$ of two p-forms and the Killing form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$

Question

Is there an action functional S_0, such that properties of ∇' give the equations of motion in a similar way to S?

Conjecture

Consider the action functional S_0 given by

$$S_0[g_0, B_0, \phi_0, \vartheta] = \int_M e^{-2\phi_0}\{\mathcal{R}(g_0) - \frac{1}{2}\langle H_0', H_0'\rangle_{g_0} + \frac{1}{2}\langle\!\langle F', F'\rangle\!\rangle + 4\langle d\phi_0, d\phi_0\rangle_{g_0} - 2\Lambda_0\} \cdot d\,\mathrm{vol}_{g_0} \,. \tag{26}$$

- It turns out that the first naive guess was indeed true. It is not at all obvious that we will obtain correctly all remaining (rather non-trivial) equations of motion!

Theorem (2016)

Field $(g_0, B_0, \phi_0, \vartheta)$ satisfy the equations of motion given by action S_0, if and only if ∇' is Ricci compatible with $\mathbf{G}'$ and $\mathcal{R}'_{\mathbf{G}'} = 2\Lambda_0 + \frac{1}{6}\dim\mathfrak{g}$.

- The main idea is to relate the connection ∇ describing S to the connection ∇' describing S_0, their scalar curvatures and Ricci compatibility conditions.
- The situation is more complicated than with generalized metrics.
- In particular, not any connection ∇ describing the equations of motion of S reduces to the connection on ∇' (in some sense).
- We did take the opposite approach - the minimal connection ∇' can be extended to a suitable connection ∇ on $E = \mathbb{T}P$. This is where the additional freedom comes in hand.

Theorem

Let ∇ be a Levi-Civita connection on E with respect to $\mathbf{G}$ suitable for reduction (in some sense), defining thus a Levi-Civita connection ∇' on E' with respect to $\mathbf{G}'$. Then

$$\mathcal{R}_{\mathbf{G}} = \mathcal{R}'_{\mathbf{G}'} \circ \pi + \frac{1}{6} \dim \mathfrak{g}, \quad \mathcal{R}_E = \mathcal{R}'_{E'} + \frac{1}{6} \dim \mathfrak{g}. \tag{27}$$

Moreover, ∇ is Ricci compatible with $\mathbf{G}$ iff ∇' is Ricci compatible with $\mathbf{G}'$.

Kaluza-Klein reduction

- We are finally ready to state and prove the main theorem.

Theorem (Kaluza-Klein reduction of supergravity)

Let (g, B, ϕ) be the fields of S on P having the form

$$g = \begin{pmatrix} 1 & \vartheta^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_0 & 0 \\ 0 & -\frac{1}{2}c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \vartheta & 1 \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & \frac{1}{2}\vartheta^T c \\ -\frac{1}{2}c\vartheta & 0 \end{pmatrix}, \tag{28}$$

and $\phi = \phi_0 \circ \pi$. Let $H = \pi^*(H_0) + \frac{1}{2} CS_3(A)$ and $\Lambda = \Lambda_0 + \frac{1}{6} \dim \mathfrak{g}$. Then (g, B, ϕ) satisfy the equations of motion for S, if and only if $(g_0, B_0, \phi_0, \vartheta)$ satisfy the equations of motion for S_0.

- Proof is actually very easy using the theorems above. One only has to fine-tune the relation of cosmological constants to make it fit together.
- Everything works fine for g of arbitrary signature and also for non-compact G. This can be modified for torus bundles.

- What exactly is the action S_0?
- The field ϑ can be used to define a new connection $A' \in \Omega^1(P, \mathfrak{g})$ from A, namely $A'(X^h) = \vartheta(X)$, where X^h is the horizontal lift of the original connection A. Every connection A' has this form.
- $F' \in \Omega^2(M, \mathfrak{g}_P)$ is then a curvature of A'.
- $\frac{1}{2}\langle\langle F', F'\rangle\rangle$ is then a usual kinetic term for a Yang-Mills theory corresponding to the connection A', that is if $\mathcal{A}' = \sigma^*(A')$ and $\mathcal{F}' = \sigma^*(DA')$ are local connection and curvature forms corresponding to the choice of gauge $\sigma : U \to P$, then

$$\langle\langle F', F'\rangle\rangle = \langle\langle \mathcal{F}', \mathcal{F}'\rangle\rangle \tag{29}$$

- 3-form H_0' can be on (contractible) U rewritten as

$$H_0' = d\mathcal{B} - \frac{1}{2}\mathcal{CS}_3(\mathcal{A}'), \tag{30}$$

where $\mathcal{B} \in \Omega^2(M)$ now transforms under gauge transformations non-trivially, and $\mathcal{CS}_3(\mathcal{A}') = \sigma^*(CS_3(A'))$ is a local Chern-Simons form corresponding to A'.

- The action S_0 can be viewed as an action for fields $(g_0, \mathcal{B}, \phi_0, \mathcal{A}')$.
- Assume that M is now a 10-dimensional spin manifold with g of signature $(9,1)$. One can take $P = P_{\mathbf{YM}} \times_M P_{\mathbf{Spin}}$ with structure group $G = K \times \mathrm{Spin}(9,1)$. Usually $K = \mathrm{E}(8) \times \mathrm{E}(8)$ or $\mathrm{SO}(32)$.
- One has $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{so}(9,1)$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = \langle \cdot, \cdot \rangle_{\mathfrak{k}} + \langle \cdot, \cdot \rangle_{\mathbf{so}}$. One can take the liberty to rescale the pairing:

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}} = \alpha' (- \langle \cdot, \cdot \rangle_{\mathfrak{k}} + \langle \cdot, \cdot \rangle_{\mathfrak{so}}) \tag{31}$$

  This only changes the relation of cosmological constants.
- With these choices, S_0' is in this case precisely the bosonic sector of **heterotic supergravity**. The condition on H_0' given by Courant algebroid give the **anomaly cancellation condition**:

$$\langle \mathcal{F}'_{\mathbf{YM}}, \mathcal{F}'_{\mathbf{YM}} \rangle_{\mathfrak{k}} = \langle \mathcal{F}'_{\mathbf{Spin}}, \mathcal{F}'_{\mathbf{Spin}} \rangle_{\mathbf{so}}. \tag{32}$$

- The final relation of constants is $\Lambda = \Lambda_0 + \frac{1}{6\alpha'}(45 - 496)$.

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Thank you for your attention!