Stefan-Sussmann integrability

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These are little notes serving as a guide to the famous paper by Héctor J. Sussmann [1]. In the second part, we will look on the paper on the same topic by Peter Štefan [2]. Notably, both papers contain some significant mistakes, which we mean to clarify in detail.

1 Sussmann story

1.1 Vector fields and their flows

Let M be a smooth manifold. By $\mathfrak{X}(M)$, we denote the set of global smooth vector fields. For an open subset $U \subseteq M$, we write $\mathfrak{X}(U)$ of $\mathfrak{X}_U(M)$ for smooth vector fields defined on U. By $\hat{\mathfrak{X}}(M)$ we denote the set of all locally defined vector fields on all open subsets of M.

Let $X \in \mathfrak{X}(M)$ be a globally well-defined vector field. It is a fundamental theorem of differential geometry proving that there exists a unique **maximal local flow** $\phi^X : \mathcal{D} \to M$, where

1. $\mathcal{D} \subseteq M \times \mathbb{R}$ is an open subset called the **flow domain of** X, such that for each $m \in M$
$$I_m = \{t \in \mathbb{R} \mid (m,t) \in \mathcal{D}\} \tag{1}$$
is an open interval containing 0. Moreover, for any $s \in I_m$, one has
$$I_{\phi^X(m,s)} = \{t - s \mid t \in I_m\}. \tag{2}$$

2. For each $m \in M$, the map $\phi^X_{(m)} : I_m \to M$ obtained as $\phi^X_{(m)}(t) = \phi^X(m,t)$ is the unique maximal integral curve of the vector field X starting at m.

3. For all $m \in M$, one has $\phi^X(m,0) = m$ and for all $t \in I_m$, $s \in I_{\phi^X(m,t)}$ we have
$$\phi^X(\phi^X(m,t), s) = \phi^X(m, t+s). \tag{3}$$
This usually called the 1**-parameter subgroup** property.

4. For each $t \in \mathbb{R}$, let $M_t = \{m \in M \mid (m,t) \in \mathcal{D}\}$. Then $\phi^X_t : M_t \to M_{-t}$ defined as $\phi^X_t(m) = \phi^X(m,t)$ is a diffeomorphism whose inverse is ϕ^X_{-t}.

Remark 1.1. (i) First, note that I_m is always an open subset of $\mathbb{R}$. Indeed, its complement I^c_m consists of points $t \in \mathbb{R}$, such that $(m,t) \in \mathcal{D}^c$ (a closed complement of $\mathcal{D}$). If $\{t_n\} \subseteq I^c_m$ converges to $t \in \mathbb{R}$, then $\{(m,t_n)\} \subseteq \mathcal{D}^c$ has to converge to $(m,t) \in \mathcal{D}^c$, whence $t \in I^c_m$. For similar reasons, M_t form open subsets of M, with $M_0 = M$.

(ii) For the future reference, note that if $m \in M_t$, there is $\epsilon > 0$, such that $m \in M_s$ for all $s \in (-\epsilon, t+\epsilon)$. Indeed, if $m \in M_t$, we have $(m,t) \in \mathcal{D}$ and thus $t \in I_m$. This is an open interval containing 0, so it has to contain $(-\epsilon, t+\epsilon)$ for some $\epsilon > 0$. But then $(m,s) \in \mathcal{D}$ for all $s \in (-\epsilon, t+\epsilon)$, which proves the claim.

(iii) Next, observe that (2) ensures that the right-hand side of (3) makes sense. Indeed, for any $t \in I_m$ and $s \in I_{\phi^X(m,t)}$, one has $t+s \in I_m$.

(iv) Finally, note that the fourth property can be in fact deduced from the previous ones. Indeed, we can deduce that $\phi^X_t(M_t) = M_{-t}$: Let $m' \in \phi^X_t(M_t)$. There is thus some $m \in M_t$, such that $m' = \phi^X(m,t)$. In particular, $t \in I_m$. From (2) it follows that $-t \in I_{m'}$, which proves that $m' \in M_{-t}$. This proves the inclusion $\phi^X_t(M_t) \subseteq M_{-t}$. But we can now apply ϕ^X_{-t} on both sides and use (3) to find $M_t \subseteq \phi^X_{-t}(M_{-t})$. But we could have started with $-t$ instead of t, which gives us $M_{-t} \subseteq \phi^X_t(M_t)$. Whence $\phi^X_t(M_t) = M_{-t}$.

If $X \in \mathfrak{X}(U)$ is defined locally on some open subset $U \subseteq M$, everything remains valid, except that for the flow domain, we have $\mathcal{D} \subseteq U \times \mathbb{R}$. If $X, Y \in \hat{\mathfrak{X}}(M)$, their commutator $[X,Y]$ is well-defined on the intersection of their domains. We formally include the empty vector field on empty domain, so we do not have to discuss whether the domains of X and Y intersect.

Let ξ denote the m-tuple of vector fields $\xi = (X_1, \dots, X_m)$, where $X_i \in \hat{\mathfrak{X}}(M)$. Let $\mathcal{D}_i \subseteq M \times \mathbb{R}$ be their flow domains. Let $T = (t_1, \dots, t_m) \in \mathbb{R}^m$. Then the composition

$$\xi_T(m) := (\phi^{X_1}_{t_1} \circ \dots \circ \phi^{X_m}_{t_m})(m) \tag{4}$$

is defined for $(m,T) \in M \times \mathbb{R}^m$ in some open subset $\mathcal{D}(\xi)$. Let $\mathcal{D}_T(\xi) \subseteq M$ be a subset of all $m \in M$, such that $(m,T) \in \mathcal{D}(\xi)$.

Finally, let us clarify the following technicalities:

Lemma 1.2. *Let M be a smooth manifold and $i : S \to M$ any its immersed submanifold. Suppose $X \in \hat{\mathfrak{X}}(M)$ is some (possibly local) smooth vector field. Suppose that $X(s) \in T_sS$ for all $s \in \mathrm{Dom}(X) \cap S$.*

Then there is a unique smooth vector field $X' \in \hat{\mathfrak{X}}(S)$, such that $(T_s i)(X'(s)) = X(i(s))$ for all $s \in \mathrm{Dom}(X) \cap S$. In other words, X' and X are i-related, $X' \sim_i X$. In particular, if $Y \in \hat{\mathfrak{X}}(M)$ is another smooth vector field also tangent to S, then $[X,Y]$ is tangent to S.

Finally, suppose $\gamma : I \to M$ is some integral curve of X, such that $\gamma(0) \in S$. Then there is an open subinterval $I' \subseteq I$ containing 0, such that $\gamma(I') \subseteq S$.

Proof. Write $U := \mathrm{Dom}(X)$. By definition $X \in \Gamma_U(TM)$. We can always define a smooth section $X^! \in \Gamma_{U'}(TM_S)$ by composing it with i. Here $U' = U \cap S$. Equivalently, the restricted vector bundle TM_S is nothing but a pullback vector bundle $i^!(TM)$. The tangent map $T(i) : TS \to TM$ defines a vector bundle morphism, which uniquely factorizes through the pullback bundle:

$$\begin{array}{ccccc} TS & \xrightarrow{T^!(i)} & TM_S & \xrightarrow{i^!} & TM \\ \downarrow{\scriptstyle \tau_S} & & \downarrow{\scriptstyle \tau^!_M} & & \downarrow{\scriptstyle \tau_M} \\ S & \xrightarrow{1_S} & S & \xrightarrow{i} & M \end{array} \tag{5}$$

The tangent bundle TS is usually identified with its image under the fiber-wise injective vector bundle morphism $T^!(i)$ in TM_S. Now, the smooth pullback section $X^! : U' \to TM_S$ has by assumption values in the subbundle $\mathrm{im}(T^!(i)) \subseteq TM_S$. Every subbundle is a closed embedded submanifold, whence $X^!$ defines a smooth local section of $\mathrm{im}(T^!(i))$. This subbundle is isomorphic to TS, and we define $X' \in \Gamma_{U'}(TS)$ using $X^!$ and this isomorphism. It is clear from the construction that $(T_s i)(X'(s)) = X(i(s))$ for all $s \in U'$.

If $Y \in \hat{\mathfrak{X}}(M)$ is another local vector field defined on $V \subseteq M$, such that $Y(s) \in T_s S$ for all $s \in V \cap S$, we find $Y' \in \Gamma_{V'}(TS)$ defined on $V' = V \cap S$ with $(T_s i)(V'(s)) = V(i(s))$. We have $[X, Y] \in \Gamma_{U \cap V}(TM)$ and $[X', Y'] \in \Gamma_{U' \cap V'}(TS)$. It is a well-known fact that

$$(T_s i)([X', Y'](s)) = [X, Y](i(s)) \tag{6}$$

for all $s \in U' \cap V' = (U \cap V) \cap S$. As the left-hand side is tangent to S, then so is the right-hand side. This proves the second claim.

Finally, let $\gamma : I \to M$ be any integral curve for X with $\gamma(0) \in S$. Write $\gamma(0) = i(s_0)$ for $s_0 \in S$. Find the vector field X' as above. Let $\gamma' : J \to S$ be any its integral curve, such that $\gamma'(0) = s_0$. Then $\widetilde{\gamma} := i \circ \gamma' : J \to M$ is an integral curve of X, such that $\widetilde{\gamma}(0) = i(s_0)$. But any two integral curves of a vector field have to coincide on the intersection of their domain. For any $t \in I \cap J$, we thus have $\gamma(t) = \widetilde{\gamma}(t) = i(\gamma'(t))$. Choosing $I' = I \cap J$, we have the result. ■

1.2 Families of vector fields and their orbits

A local diffeomorphism of M is a smooth diffeomorphism from an open subset $U \subseteq M$ to an open subset $U' \subseteq M$. If $\delta_i : U_i \to U'_i$, $i \in \{1, 2\}$ are two local diffeomorphism, their composition $\delta_1 \circ \delta_2$ is a local diffeomorphism with the domain $\delta_2^{-1}(U_1)$ and the image $\delta_1(U_1 \cap U'_2)$. Compositions, if they are defined, satisfy the formal laws

$$(\delta_1 \circ \delta_2) \circ \delta_3 = \delta_1 \circ (\delta_2 \circ \delta_3), \quad (\delta_1 \circ \delta_2)^{-1} = \delta_2^{-1} \circ \delta_1^{-1}. \tag{7}$$

A **group of local diffeomorphisms** is a set G of local diffeomorphisms closed under compositions and inverses. For any vector field $X \in \hat{\mathfrak{X}}(M)$, each map $\phi^X_t : M_t \to M_{-t}$ forms a local diffeomorphism. A collection $G_X = \{\phi^X_t\}_{t \in \mathbb{R}}$ is clearly closed under compositions and inverses, and it is called the **group of local diffeomorphisms generated by** X.

More generally, let $D \subseteq \hat{\mathfrak{X}}(M)$. Then there exists the smallest group of local diffeomorphisms containing the union $\cup_{X \in D} G_X$. It is denoted by G_D and called the **group of local diffeomorphisms generated by** X. Let $\xi * \eta$ denote the concatenation of sequences. G_D then consists of mappings ξ_T for all possible $\xi \in D^n$ and $T \in \mathbb{R}^n$ for all integers $n \geq 1$. If $T = (t_1, \dots, t_n)$, let $\hat{T}$ denote the sequence $\hat{T} = (t_n, \dots, t_1)$. We then obtain the rules:

$$\xi_T \circ \eta_{T'} = (\xi * \eta)_{T * T'}, \quad (\xi_T)^{-1} = \xi_{-\hat{T}}. \tag{8}$$

We say that the subset $D \subseteq \hat{\mathfrak{X}}(M)$ is **everywhere defined**, if the union of domains of elements of D is M. Similarly, a group of local diffeomorphisms G is **everywhere defined** if each $m \in M$ belongs to a domain of some $g \in G$. Clearly, if D is everywhere defined, then so is G_D.

Let G be an everywhere defined group of local diffeomorphisms. We say that two elements m and m' of M are G-**equivalent**, if there is $g \in G$, such that $g(m_1) = m_2$. This determines an equivalence relation on M. The equivalence classes are called the **orbits of** G (or G-orbits). For an everywhere defined $D \subseteq \hat{\mathfrak{X}}(M)$, the orbits of G_D are called just the orbits of D. Two points m_1 and m_2 belong to the same D-orbit, if and only if there exists $\xi \in D^n$ and $T \in \mathbb{R}^n$, such that $\xi_T(m_1) = m_2$. There exists a following alternative description:

Lemma 1.3. *The two points m_1 and m_2 belong to the same D-orbit, if there exists a curve $\gamma : [a, b] \to M$ such that $\gamma(a) = m_1$ and $\gamma(b) = m_2$, which has the property:*

(PI) There exist numbers t_i, such that $a = t_0 < t_1 < \cdots < t_r = b$ and vector fields $X_i \in D$, $i \in \{1, \dots, r\}$, such that for each $i \in \{1, \dots, r\}$, the restriction of γ to $[t_{i-1}, t_i]$ is a restriction of some integral curve of X_i of of $-X_i$.

A curve γ satisfying (PI) is called a ***piecewise integral curve*** *of D.*

Proof. Let m_1 and m_2 be D-equivalent. There is thus $\xi = (Y_1, \dots, Y_r) \in D^r$ and $T = (s_1, \dots, s_r) \in \mathbb{R}^r$, such that $m_2 = \xi_T(m_1)$. This means that

$$m_2 = (\phi_{s_1}^{Y_1} \circ \phi_{s_2}^{Y_2} \circ \cdots \circ \phi_{s_r}^{Y_r})(m_1). \tag{9}$$

We can assume that $s_i \neq 0$, otherwise we could have left them out from the sequence.

Suppose that $s_r > 0$. Set $X_1 = Y_r$, set $t_0 = 0$ and $t_1 = s_r$, and define $\gamma_1 : [t_0, t_1] \to M$ as $\gamma_1(t) = \phi_t^{Y_r}(m_1)$. Clearly γ_1 comes from a restriction of some integral curve for X_1 on $[t_0, t_1]$. If $s_r < 0$, one sets $t_1 = -s_r$ and define $\gamma_1(t) = \phi_{-t}^{Y_r}(m_1)$ on $[t_0, t_1]$. In this case $\gamma_1(t)$ is a restriction of the integral curve for $-X_1$.

Next, suppose $s_{r-1} > 0$. Let $X_2 = Y_r$. Let $t_2 = t_1 + s_{r-1}$, and define $\gamma_2 : [t_1, t_2] \to M$ as $\gamma_2(t) = \phi_{t-t_1}^{Y_{r-1}}(\phi_{s_r}^{Y_r}(m_1))$. For $s_{r-1} < 0$, we set $t_2 = t_1 - s_{r-1}$ and $\gamma_2(t) = \phi_{-(t-t_1)}^{Y_{r-1}}(\phi_{s_r}^{Y_r}(m_1))$.

By repeating this procedure, we obtain a collection of curves $\{\gamma_i\}_{i=1}^r$, where $\gamma_i : [t_i, t_{i-1}] \to M$ is a restriction of some integral curve of X_i or $-X_i$, based on the sign of s_i. Using them to define a single curve γ, we obtain the curve γ having the property (PI).

For the converse, let $\gamma : [a, b] \to M$ be the smooth curve having the property (PI). Let $\gamma_1 : [t_0, t_1] \to M$ be the restriction of γ. By definition, γ_1 is a restriction of some integral curve $\hat{\gamma}_1 : (q_0, q_1) \to M$, so that $[t_0, t_1] \subseteq (q_0, q_1)$. Define $\widetilde{\gamma}_1(t) = \hat{\gamma}_1(t_0 + t)$. Then $\widetilde{\gamma}_1$ is the integral curve for X_1 defined on an open interval $(q_0 - t_0, q_1 - t_0)$ starting (at the zero time) at $\gamma(t_0) = m_0$. In particular, we have $\gamma(t_1) = \widetilde{\gamma}_1(t_1 - t_0) = \phi_{t_1 - t_0}^{X_1}(m_0)$. Set $s_r = t_1 - t_0$ and $Y_r = X_1$. This procedure is then simply repeated, until we find a collection of vector fields in D and time parameters fitting into (9). ∎

One can define a topology on each D-orbit. Let $m \in M$ and $\xi \in D^n$. Define the map $\rho_{\xi,m}$ by $\rho_{\xi,m}(T) = \xi_T(m)$. Let $U_{\xi,m} \subseteq \mathbb{R}^n$ be its domain. Now, let S be any orbit of D through m. Then S can be written as union of images of all $\rho_{\xi,m}$. We declare $U \subseteq S$ to be open, if and only if $\rho_{\xi,m}^{-1}(U) \subseteq U_{\xi,m}$ is open for all $\xi \in D^n$ and all $n \geq 1$.

Clearly S and $\emptyset$ are open. Let $\{U_\alpha\}_{\alpha \in I}$ be any union of open sets of S. Then

$$\begin{aligned} \rho_{\xi,m}^{-1}(\bigcup_{\alpha \in I} U_\alpha) &= \{T \in U_{\xi,m} \mid \rho_{\xi,m}(T) \in \bigcup_{\alpha \in I} U_\alpha\} \\ &= \{T \in U_{\xi,m} \mid \exists \alpha \in I \text{ such that } \rho_{\xi,m}(T) \in U_\alpha\} \\ &= \bigcup_{\alpha \in I} \rho_{\xi,m}^{-1}(U_\alpha). \end{aligned} \tag{10}$$

But this is an open subset of $U_{\xi,m}$, whence $\bigcup_{\alpha \in I} U_\alpha$ is open in S. Similarly, one proves that any finite intersection of open sets is open. Whence we got ourselves a topology on S.

Importantly, let $U \subseteq S$ be a set open in the *subset topology* of S, that is $U = S \cap V$ for an open subset $V \subseteq M$. We have $\rho_{\xi,m}^{-1}(U) = \rho_{\xi,m}^{-1}(V) \subseteq U_{\xi,m}$, which is open. Whence U is open in the D-orbit topology of S. This proves that the inclusion map $i : S \to M$ is continuous. In particular, this proves that S is Hausdorff, as we can certainly separate two points of S by two sets open in the subspace topology, hence in the D-orbit topology.

It remains to argue that the topology on S does not depend on the choice of the base point $m \in S$. By definition, there exists $\eta \in D^k$ and $T_0 \in \mathbb{R}^k$, such that $m = \eta_{T_0}(m')$. Then for any $\xi \in D^n$ and $T \in U_{\xi,m}$, one has $\rho_{\xi,m}(T) = \xi_T(\eta_{T_0}(m')) = \rho_{\xi*\eta,m'} \circ R_{T_0}(T)$, where $R_{T_0}(T) = T * T_0$. This is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^{n+k}$. Whence if $\rho_{\xi,m'}$ are continuous for all $\xi \in D^n$ and $n \geq 1$, then so are $\rho_{\xi,m}$ for all $\xi \in D^n$ and $n \geq 1$. But this proves that two topologies coincide.

1.3 Distributions

A **distribution** on a manifold M is a mapping Δ which assigns to every $m \in M$ a linear subspace $\Delta(m) \subseteq T_m M$. A set of vector fields is said to **span** Δ, if for every $m \in M$, $\Delta(m)$ is a linear hull of values of the vector fields from this set, that is if $A \subseteq \hat{\mathfrak{X}}(M)$, one has $\Delta(m) = \mathbb{R}\{X(m) \mid X \in A_{(m)}\}$, where $A_{(m)} = \{X \in A \mid m \in \mathrm{Dom}(X)\}$

Let $D \subseteq \hat{\mathfrak{X}}(M)$ be an everywhere defined set, there is a distribution Δ_D spanned by D:

$$\Delta_D(m) := \mathbb{R}\{X(m) \mid X \in D_{(m)}\} \tag{11}$$

Δ is called the **smooth distribution**, if there exists an everywhere defined family $D \subseteq \hat{\mathfrak{X}}(M)$, such that $\Delta = \Delta_D$. A vector field $X \in \hat{\mathfrak{X}}(M)$ **belongs to the distribution**, if $X(m) \in \Delta(m)$ for all $m \in \mathrm{Dom}(X)$. For a given distribution Δ, let $D_\Delta \subseteq \hat{\mathfrak{X}}(M)$ be the family of all vector fields which belong to Δ. Note that D_Δ is everywhere defined. There is a simple observation:

Proposition 1.4. *A distribution Δ is smooth if and only if it is spanned by D_Δ.*

Proof. Let Δ be smooth. The inclusion $\Delta_{D_\Delta}(m) \subseteq \Delta(m)$ for all $m \in M$ is trivial and holds for any distribution. If $\Delta = \Delta_D$ for some everywhere defined family $D \subseteq \hat{\mathfrak{X}}(M)$, every $X \in D$ is in $D_{(m)}$ for all $m \in \mathrm{Dom}(X)$ and thus $X(m) \in \Delta_D(m) = \Delta(m)$ for all $m \in \mathrm{Dom}(X)$. But this shows that $D \subseteq D_\Delta$ and the inclusion $\Delta(m) \subseteq \Delta_{D_\Delta}(m)$ follows. ■

The following lemma gives a good criterion for the smoothness of the distribution:

Lemma 1.5. *Let Δ be a distribution. Then Δ is smooth, if and only if for every $m \in M$ and every $x \in \Delta(m)$, there is a vector field $X \in D_\Delta$, such that $X(m) = x$.*

Proof. If Δ is smooth, it is by previous proposition spanned by D_Δ. There is thus some k-tuple $(X_i)_{i=1}^k$ of vector fields in D_Δ, such that $x = \alpha^i X_i(m)$ for some constants $\alpha_i \in \mathbb{R}$. Taking $U = \cap_{i=1}^k \mathrm{Dom}(X_i)$, $X = \alpha^i X_i$ is a smooth vector field on U. Clearly $X \in D_\Delta$ and $X(m) = x$. Conversely, the statement immediately implies the non-trivial inclusion $\Delta(m) \subseteq \Delta_{D_\Delta}(m)$. ■

Definition 1.6. If $\dim(\Delta(m)) = k$ for all $m \in M$, one says that Δ is a **regular distribution**. Smooth regular distributions can be easily shown to be in one-to-one correspondence with subbundles of the tangent bundle TM.

Define the function $\mathrm{rk}(\Delta) : M \to \mathbb{R}$ as $\mathrm{rk}(\Delta)(m) := \dim(\Delta(m))$. For smooth distribution, quite a lot can be said about $\mathrm{rk}(\Delta)$.

Lemma 1.7. *For each point $m \in M$, there exists an open neighborhood U, such that $\mathrm{rk}(\Delta)(m') \geq \mathrm{rk}(\Delta)(m)$ for all $m' \in U$. In particular, $\mathrm{rk}(\Delta)$ is lower semi-continuous.*

Proof. Let $m \in M$. Choose any basis $(v_1, \ldots, v_k)$ of $\Delta(m)$. Here $k = \mathrm{rk}(\Delta)(m)$. By Lemma 1.5, there are some vector fields $V_1, \ldots, V_k \in D_\Delta$, such that $V_i(m) = v_i$ for all $i \in \{1, \ldots, k\}$. The subset of $n \times n$ matrices with rank greater then k is open, which proves that the k-tuple $(V_1(m'), \ldots, V_k(m'))$ is formed of linearly-independent vectors for all $m' \in U$ for some neighborhood U of m. But by definition, $V_i(m') \in \Delta(m')$ for all $i \in \{1, \ldots, k\}$. Whence $\mathrm{rk}(\Delta)(m') \geq rk(\Delta(m)$ for all $m' \in U$.

Now, recall that $f : M \to \mathbb{R}$ is lower semi-continuous, if for any $\lambda \in \mathbb{R}$, the set $U_\lambda = \{m \in M \mid f(m) > \lambda\}$ is open in M. Let $\lambda \in \mathbb{R}$ and $m \in U_\lambda$. In particular, $\mathrm{rk}(\Delta)(m) > \lambda$. We have an open neighborhood U of m with the property $\mathrm{rk}(\Delta)(m') \geq \mathrm{rk}(\Delta)(m) > \lambda$ for all $m' \in U$. Clearly $U \subseteq U_\lambda$ and we see that U_λ is indeed open. Note that lower semi-continuity does not imply the first property. ■

Let G be any group of local diffeomorphisms on M. The distribution Δ is said to be **G-invariant**, if $T_m(g)(\Delta(m)) \subseteq \Delta(g(m))$ for any $g \in G$ and $m \in \mathrm{Dom}(g)$. In fact, for any G-invariant distribution, we have $T_m(g)(\Delta(m)) = \Delta(g(m))$. In particular, the dimension of $\Delta(m)$ is constant along any given G-orbit.

For any distributions Δ_1 and Δ_2, we say that Δ_1 is **contained** in Δ_2, if $\Delta_1(m) \subseteq \Delta_2(m)$ for all $m \in M$. We write $\Delta_1 \subseteq \Delta_2$. For any distribution Δ and any group G of local diffeomorphisms, there is the smallest distribution Δ^G which contains Δ and is G-invariant. Explicitly, at all $m \in M$, $\Delta^G(m)$ must be a linear hull of $\Delta(m)$ together with all $w \in T_m M$, such that $w = T_{m'}(g)(v)$ for some $v \in \Delta(m')$ and $g \in G$, such that $g(m') = m$.

For any $D \subseteq \hat{\mathfrak{X}}(M)$, we may form a family $D^G \subseteq \hat{\mathfrak{X}}(M)$ defined as the union of D together with all vector fields obtained as $T(g)$-images of vector fields in D, for all $g \in G$. If Δ is spanned by D, then Δ^G is panned by D^G. In particular, if Δ is smooth, then so is Δ^G. If $D \subseteq \hat{\mathfrak{X}}(M)$ is any subset, we say that distribution Δ is **D-invariant**, if it is G_D-invariant. The smallest D-invariant distribution which contains Δ is denoted as Δ^D.

For an everywhere defined set $D \subseteq \hat{\mathfrak{X}}(M)$ spans a smooth distribution Δ_D. We will be interested in the smallest D-invariant distribution Δ_D^D which contains Δ_D. We denote this distribution as P_D. It is a smooth distribution. Moreover, the dimension of $P_D(m)$ is constant along any D-orbit; if S is any D-orbit, we may define $\mathrm{rk}(S) = \dim(P_D(m))$ for any $m \in M$. The number $\mathrm{rk}(S)$ is called the **rank of the orbit** S.

A set $D \subseteq \hat{\mathfrak{X}}(M)$ is **involutive** if for any $X \in D$ and $Y \in D$ we have $[X, Y] \in D$. For any D, there is the smallest involutive subset $D^* \subseteq \hat{\mathfrak{X}}(M)$ which contains D. A smooth distribution Δ is **involutive**, if the set D_Δ is involutive.

In the following, we will show that for everywhere defined $D \subseteq \hat{\mathfrak{X}}(M)$, the distribution P_D is involutive. We claim that we have three inclusions

$$\Delta_D \subseteq \Delta_{D^*} \subseteq P_D. \tag{12}$$

The first one is obvious. For the second inclusion, by definition every $X, Y \in D$ belong to P_D. As P_D is involutive (to be shown), then $[X, Y]$ must also belong to P_D. But this shows that any vector field in D^* belongs to P_D, which proves the inclusion. The left inclusion may be proper, but so may be the rightmost one:

Example 1.8. Consider $M = \mathbb{R}^2$ with standard coordinates (x, y). Consider the family D consisting of

$$X_1 = \frac{\partial}{\partial x}, \;\; X_2 = \psi(x) \cdot \frac{\partial}{\partial y}, \tag{13}$$

where $\psi \in C^\infty(\mathbb{R})$, such that $\psi(x) = 0$ for all $x \leq 0$ and $\psi(x) > 0$ for all $x > 0$. One example of such function is $\psi(x) = e^{-x}$ for all $x > 0$. Now, observe that any point $(x, y) \in \mathbb{R}^2$ can be joined with the point say $(1, 1)$ by a piece-wise integral curve of D. Simply flow along X_1 to reach the $x = 1$ line and then along X_2 to reach the point $(1, 1)$. By Lemma 1.3, this means that $\mathbb{R}^2$ is the single D-orbit.

According to our remark above, the dimension of $P_D(x, y)$ is thus constant and equal to 2. However, for any $x \leq 0$, we have $D^*(x, y) = \mathbb{R}\{X_1(x, y)\}$, which has a dimension 1.

A submanifold S of M is called the **integral submanifold of the distribution** Δ, if for every $s \in S$, the tangent space T_sS is exactly $\Delta(s)$. A smooth distribution Δ has the **integral manifolds property** if for every $m \in M$ there exists an integral submanifold S of Δ, such that $m \in S$. If Δ has the integral manifolds property, every vector field X which belongs to Δ must be tangent to every its integral submanifold. This proves the following:

Proposition 1.9. *Every smooth distribution with integral manifolds property is involutive.*

The converse is not true, see the previous example. None of the points $(0, y)$ to some integral submanifold of Δ_D spanned by $D = \{X_1, X_2\}$.

Definition 1.10. Let Δ be a smooth distribution. A **maximal integral submanifold of** Δ is a connected submanifold S of M such that

(a) S is an integral submanifold of Δ

(b) Every connected integral submanifold of Δ which intersects S is its open submanifold.

Clearly, two maximal integral submanifolds through a point $m \in M$ must coincide. We say that Δ has a **maximal integral manifolds property** if through every $m \in M$ there passes a maximal integral submanifold of M.

Remark 1.11. Let D be an everywhere defined family of vector fields. We want to find a smooth distribution Δ, such that the D-orbits are precisely the maximal integral submanifolds of Δ. It makes sense to define $\Delta(m)$ to be the space of all vectors in T_mM which are at m tangent to some smooth curve $\gamma : I \to M$ entirely contained in some D-orbit. Denote the set of such smooth curves as Γ_m.

First, as for each $X \in D$, $X(m)$ is tangent to the integral curve $\phi_t^X(m)$ which is obviously contained in the single D-orbit. Whence $\Delta_D \subseteq \Delta$. Moreover, for any $X, Y \in D$, the curve $\gamma(t) := \phi_t^X \circ \phi_t^Y \circ \phi_{-t}^X \circ \phi_{-t}^Y$ is also in Γ_m. But $\dot{\gamma}(0) = [X, Y](m)$. This proves that necessarily $\Delta_{D^*} \subseteq \Delta$. Finally, let $X \in D$ and $s \in \mathbb{R}$ be arbitrary. Let $m' = \phi_{-s}^X(m)$, and let $\gamma \in \Gamma_{m'}$ be arbitrary. Then the curve $\delta(t) = \phi_s^X(\gamma(t))$ is the curve in Γ_m. Whence $\dot{\delta}(0) \in \Delta(m)$. But

$$\dot{\delta}(0) = \frac{d}{dt}\bigg|_{t=0} \phi_s^X(\gamma(t)) = T_{m'}(\phi_s^X)(\dot{\gamma}(0)). \tag{14}$$

By definition, $\dot{\gamma}(0) \in \Delta(m')$. This shows that Δ must be D-invariant. It is thus suggestive that the natural candidate is $\Delta = P_D$. In particular, this will show that P_D is involutive, proving the inclusion $\Delta_{D^*} \subseteq P_D$.

1.4 Main theorems

Theorem 1.12. *Let M be a smooth manifold, and let D be an everywhere defined set of smooth vector fields.*

(a) If S is an orbit of D, then S (with the topology introduced in Section 1.2) admits a unique smooth structure, such that S is an immersed submanifold of M. The dimension of S is equal to its rank.

(b) With the topology and smooth structure of (a), every orbit of D is a maximal integral submanifold of P_D.

(c) P_D has the maximal integral manifolds property.

(d) P_D is involutive.

This theorem emphasizes the pivotal role of the family D and its D-orbits. Clearly, the statements (c) and (d) immediately follow from (a) and (b). For the second theorem, we write it in its original version. It turned out that one implication is in fact not correct, as we will comment below.

Theorem 1.13. *Let M be a smooth manifold, and let Δ be a smooth distribution on M. Let D be a set of smooth vector fields spanning Δ. Then the following conditions are equivalent:*

(a) Δ has the integral manifolds property.

(b) Δ has the maximal integral manifolds property.

(c) Δ is D-invariant.

(d) For every $X \in D$, $t \in \mathbb{R}$ and $m \in M$, such that ϕ_t^X is defined, the map $T_m(\phi_t^X)$ maps $\Delta(m)$ into $\Delta(\phi_t^X(m))$.

(e) For every $m \in M$, there exist elements $X_1, \dots, X_k$ of D, such that

(1) $\Delta(m) = \mathbb{R}\{X_1(m), \dots, X_k(m)\}$.

(2) For every $X \in D$, there exists $\epsilon > 0$ and smooth functions $f_{ij} \in C^\infty(-\epsilon, \epsilon)$, such that

$$[X, X_i](\phi_t^X(m)) = \sum_{j=1}^{k} f_{ij}(t) X^j(\phi_t^X(m)), \tag{15}$$

for all $t \in (-\epsilon, \epsilon)$

(f) $\Delta = P_D$.

Remark 1.14. In fact, it was shown by Balan in [3] that the implication $(e) \Rightarrow (d)$ in the Sussmann's proof is not correct. However, all remaining equivalences in fact hold. We will comment on that later, providing a modification to the condition

1.5 Proof of the first theorem

Let $D \in \hat{\mathfrak{X}}(M)$ be an everywhere defined set of smooth vector fields. Let S be an orbit of D. Let $k = \mathrm{rk}(S)$. We will use the maps $\rho_{\xi,m} : U_{\xi,m} \to M$ as defined in Section 1.2. Let $D^\infty = \cup_{n=1}^\infty D^n$, where D^n is just the n-th Cartesian power of D. Recall that for $\xi \in D^n$, we have an open subset $U_{\xi,m} \subseteq \mathbb{R}^n$. We have defined the topology on S, such that $\rho_{\xi,m} : U_{\xi,m} \to S$ is continuous. For any $\xi \in D^\infty$, $m \in M$ and $T \in \mathcal{U}_{\xi,m}$, let $V(\xi, m, T)$ denote the vector subspace

$$V(\xi, m, T) = \mathrm{im}(T_T(\rho_{\xi,m})) \subseteq T_{\xi_T(m)}(M). \tag{16}$$

We will now state the sequence of four lemmas, the proof which (except for the first one) we postpone.

Lemma 1.15. *Let N and S be any smooth manifolds. Suppose there is a continuous map $\varphi : N \to N'$ and two smooth maps $f : N \to M$ and $i : S \to M$ into some smooth manifold M, such that the following diagram commutes:*

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & S \\ & {}_f\searrow \quad \swarrow_i & \\ & M & \end{array} . \tag{17}$$

If i is an injective immersion, then φ is smooth.

Proof. This relies on the rank theorem. Let $n \in N$ be an arbitrary point. Write $s = \varphi(n)$ and $m = i(s)$. As i is an injective immersion, there are neighborhoods U of s and V of m, such that $i(U) \subseteq V$, together with the coordinates $(x^1, \dots, x^k)$ on U and $(y^1, \dots, y^\mu)$ on V, such that i locally looks like

$$i(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0). \tag{18}$$

On the other hand, we can find a neighborhood $W \subseteq N$ of n, together with some local coordinates $(z^1, \dots, z^q)$ on W, such that $\varphi(W) \subseteq U$. This is where we have to use that φ is continuous, as we need to know that $\varphi^{-1}(U)$ is open in N. Write $\varphi^i(z) := x^i(\varphi(z^1, \dots, z^q))$ for all $i \in \{1, \dots, k\}$ and $f^\nu(z) := y^\nu(f(z^1, \dots, z^q))$ for all $\nu \in \{1, \dots, \mu\}$. Then

$$(i \circ \varphi)(z^1, \dots, z^q) = (\varphi^1(z), \dots, \varphi^k(z), 0, \dots, 0). \tag{19}$$

On the other hand, from the commutativity of the diagram, one has

$$(i \circ \varphi)(z^1, \dots, z^q) = (f^1(z), \dots, f^k(z), f^{k+1}(z), \dots, f^\mu(z)). \tag{20}$$

Note that in particular, we have $f^\nu(z) = 0$ for $\nu > k$. By assumption, all functions $f^\mu(z)$ are smooth. Then $\varphi^i(z) = f^i(z)$ are smooth for all $i \in \{1, \dots, k\}$. But this proves that for each point $n \in N$, there is an open neighborhood W of n, an open neighborhood U of $\varphi(n)$, together with local coordinates, such that the local representation of φ is smooth. Whence φ is smooth. ■

Corollary 1.16. *Let $i : S \to M$ be a continuous injective map, where S is a given topological space and M a smooth manifold. If there exists a smooth structure on S making i into the injective immersion, it is unique.*

Proof. Let S' denote the topological space S equipped with another smooth structure, such that $i' : S' \to M$ is an injective immersion based on the same map i. We then simply apply the previous lemma on the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{1_S} & S' \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle i'} & \\ & M & \end{array} \tag{21}$$

This proves that $1_S : S \to S'$ is smooth. Exchanging the role of S and S', we prove that 1_S is a diffeomorphism. Both smooth structures on S coincide. ■

Lemma 1.17. *Let $\xi \in D^\infty$, $m \in S$, $T \in U_{\xi,m}$, $m_0 = \xi_T(m)$. Then $V(\xi, m, T) \subseteq P_D(m_0)$.*

Lemma 1.18. *Let $m_0 \in S$. Then there exist $\xi \in D^\infty$, $m \in S$, $T \in U_{\xi,m}$, such that $\xi_T(m) = m_0$ and $V(\xi, m, T) = P_D(m_0)$.*

Lemma 1.19. *Let N be a connected integral submanifold of P_D and let U be its underlying set of points. If U intersects S then U is an open subset of S.*

Proof of the theorem based on lemmas. Let $m_0 \in S$. By Lemma 1.18 there exist $m \in S$, $\xi \in D^\infty$ and $T \in U_{\xi,m}$, such that $\xi_T(m) = m_0$ and $V(\xi, m, T) = P_D(m_0)$. In particular, the differential $T(\rho_{\xi,m})$ has the rank $k = \text{rk}(S)$ at T. In fact, by Lemma 1.17 the rank of $T(\rho_{\xi,m})$ cannot be larger then k on entire $U_{\xi,m}$. The set of of points where $T(\rho_{\xi,m})$ has the rank greater or equal to k is open in $U_{\xi,m}$. These two facts imply that $\rho_{\xi,m}$ has a locally constant rank. By usual rank theorem, there exist open neighborhoods $U \subseteq U_{\xi,m}$ of T and $V \subseteq M$ of m_0, and diffeomorphisms ϕ, ψ fitting into the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\rho_{\xi,m}} & V \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \psi} \\ C^n & \overset{E_{n,\mu,k}}{\dashrightarrow} & C^\mu, \end{array} \tag{22}$$

where $C^q = \{(t_1, \dots, t_q) \in \mathbb{R}^q \mid -1 < t_i < 1 \text{ for all } i \in \{1, \dots, q\}\}$ are "open cubes" in $\mathbb{R}^q$ centered at 0 and $E_{n,\mu,k}(t^1, \dots, t^k, \dots, t^n) = (t_1, \dots, t_k, 0, \dots, 0)$. Here $\mu = \dim(M)$.

Now, let $\Lambda = \rho_{\xi,m}(U)$. This is a submanifold of M as it is an inverse image by ψ of the submanifold $E_{n,\mu,k}(C^n)$. For any $T' \in U$, the tangent space to Λ at $m' = \rho_{\xi,m}(T')$ is $V(\xi, m, T')$. By Lemma 1.17, we have $V(\xi, m, T') \subseteq P_D(m')$. both spaces are k-dimensional, whence $T_{m'}\Lambda = V(\xi, m, T') = P_D(m')$. This proves that Λ is an integral submanifold of P_D, which is clearly contained in S. Let $I : \Lambda \to S$ be the inclusion map.

By Lemma 1.19, the set of points of Λ is open in S. Clearly, one can use it also to any connected open subset W of Λ, and such open connected sets constitute a basis for the topology of Λ. Whence the inclusion map $I : \Lambda \to S$ is open. On the other hand, we can write

$$\Lambda \xrightarrow{I'} V \xrightarrow{\psi} C^\mu \xrightarrow{E_{\mu,n,k}} C^n \xrightarrow{\phi^{-1}} U \xrightarrow{\rho_{\xi,m}} S \ , \qquad (\Lambda \overset{I}{\dashrightarrow} S) \tag{23}$$

where $I' : \Lambda \to V$ is the inclusion. All maps are continuous, especially the last one, which follows from the definition of topology on S. This shows that I is continuous open map, that is $I(\Lambda)$ is an open subset of S homeomorphic to Λ.

Let Σ denote the set of all manifolds Λ which are obtained by the construction in the previous paragraph. For each $\Lambda \in \Sigma$, let $U(\Lambda) \subseteq S$ denote the underlying set of Λ which is open in S. It follows that $U(\Lambda)$ form an open cover of S, where the inclusion of Λ into S is a homeomorphism onto its image.

Declaring them into diffeomorphisms, we obtain a smooth atlas on each element of an open cover $U(\Lambda)$. We want to define a smooth structure on S, such that the elements of Σ will be the open submanifolds of S.

It suffices to show that for any $\Lambda_1, \Lambda_2 \in \Sigma$, the $U(\Lambda_1) \cap U(\Lambda_2)$ has the same smooth structure viewed as an open submanifold of Λ_1 or Λ_2. Let $W_1 \subseteq \Lambda_1$ and $W_2 \subseteq \Lambda_2$ denote these smooth structures. We have to show that the identity map $J : W_1 \to W_2$ is smooth. But J fits into the commutative diagram:

$$\begin{array}{ccc} W_1 & \xrightarrow{J} & W_2 \\ & \searrow^{I_1} \quad \swarrow_{I_2} & \\ & M & \end{array}, \tag{24}$$

where I_1 and I_2 are smooth injective immersions. If J is continuous, one can use Lemma 1.15 to show that J has to be smooth. But J is continuous as W_1 and W_2 are homeomorphic images of the same open subset $U(\Lambda_1) \cap U(\Lambda_2)$ in S.

By construction, the inclusion $i : S \to M$ is a smooth injective immersion, making S into an integral submanifold of P_D. As the topology on S is fixed (it is the one in Section 1.2), such smooth structure is unique by Corollary 1.16. Note that $\dim(S) = k = \mathrm{rk}(S)$.

We have to argue that S is a maximal integral submanifold of P_D. It is clearly connected, as it is a D-orbit, where all points are connected by piecewise integral curve of D. Let Γ be some connected integral submanifold of P_D which intersects S. Let $U(\Gamma)$ be the underlying set of points. By Lemma 1.19, the set $U(\Gamma)$ is the open subset of S. Moreover, we can apply the same lemma on any open connected subset of Γ (we now know that it automatically intersects S). As open connected subsets form the basis for topology of Γ, we prove that the inclusion map $I : \Gamma \to S$ is open.

Let Γ' be the open submanifold[1] with the same underlying set $U(\Gamma)$. Let $1 : \Gamma' \to \Gamma$ be the identity map. By previous remarks it is continuous. It fits into the commutative diagram

$$\begin{array}{ccc} \Gamma' & \xrightarrow{1} & \Gamma \\ \big\downarrow e & & \big\downarrow j \\ S & \xrightarrow{i} & M, \end{array} \tag{25}$$

where $j : \Gamma \to M$ is the injective immersion from the definition. As 1 is continuous and the composition of the remaining two arrows is smooth, it follows from 1.15 that $1 : \Gamma' \to \Gamma$ is smooth. Moreover, it is an immersion, which follows from the commutativity of the above diagram. Any smooth bijective immersion has to be a diffeomorphism, which proves that Γ is an open submanifold of S.

This proves that S is the maximal integral submanifold of P_D. We have thus proved the claims (a) and (b) of Theorem 1.12. The claims (c) and (d) follow immediately. ■

Proof of Lemma 1.17. Let $\xi \in D^n$. We will prove the claim by induction on n.

For $n = 1$, we have $\xi = (X)$ for some $X \in D$. The set $U_{\xi,m}$ is the interval $I_m \subseteq \mathbb{R}$ where the integral curve $t \mapsto \phi_t^X(m)$ is defined. In fact, $\rho_{\xi,m}(t) = \phi_t^X(m)$. Let $t_0 \in I$. We identify $T_{t_0}(I_m)$

[1] With smooth structure inherited from S.

with $\mathbb{R}$. For any $\lambda \in \mathbb{R}$, we have $\lambda = \dot{\sigma}(0)$ for some smooth curve $\sigma : I \to I_m$ with $\sigma(0) = t_0$. Then

$$T_{t_0}(\rho_{\xi,m})(\lambda) = \frac{d}{dt}\bigg|_{t=0} \phi^X_{\sigma(t)}(m) = \lambda \cdot \frac{d}{dt}\bigg|_{t=t_0} \phi^X_t(m) = \lambda \cdot X(\phi^X_{t_0}(m)) = \lambda \cdot X(m_0). \qquad (26)$$

But in particular, we have $\Delta_D(m_0) \subseteq P_D(m_0)$, that is $\lambda \cdot X(m_0) \in P_D(m_0)$. This proves the $n = 1$ statement.

Now, let $\xi \in D^n$, and let $T \in U_{\xi,m}$. We have $\xi = X * \eta$ for $\eta \in D^n$ and $T = t_0 * T'$, where $T' \in U_{\eta,m}$ and $t_0 \in \mathbb{R}$. Let $v \in T_T(U_{\xi,m})$ be any tangent vector. Then $v = \dot{\sigma}(0)$, where $\sigma : I \to U_{\xi,m}$ decomposes as $\sigma(t) = (\sigma_0(t), \sigma'(t))$, such that $\sigma_0 : I \to \mathbb{R}$ satisfies $\sigma_0(0) = t_0$ and $\sigma' : I \to U_{\eta,m}$ satisfies $\sigma'(0) = T'$. Then

$$T_T(\rho_{\xi,m})(v) = \frac{d}{dt}\bigg|_{t=0} \phi^X_{\sigma_0(t)}(\rho_{\eta,m}(\sigma_1(t)) = T_{\eta_{T'}(m)}(\phi^X_t)(T_{T'}(\rho_{\eta,m})(w)) + \lambda \cdot X(\rho_{\xi,m}(T)), \qquad (27)$$

where $w = \dot{\sigma}'(0)$ and $\lambda = \dot{\sigma}_0(0)$. This proves that $V(\xi, m, T)$ is spanned by $X(\xi_T(m))$ and the image of $V(\eta, m, T')$ under the linear map $T_{\eta_{T'}(m)}(\phi^X_{t_0})$. The span of $X(\xi_T(m))$ is in $P_D(\xi_T(m))$ as in $n = 1$ case. By induction hypothesis $V(\eta, m, T') \in P_D(\eta_{T'}(m))$ and the result follows from the D-invariance of P_D. ■

Proof of Lemma 1.18. We will prove the two assertions:

(a) If $(\xi, m, T) \in D^\infty \times S \times U_{\xi,m}$ and $(\eta, m', T') \in D^\infty \times S \times U_{\eta,m'}$ are such that $\xi_T(m) = \eta_{T'}(m') = m_0$, then there exists $(\sigma, m'', T'') \in D^\infty \times S \times \Omega_{\sigma,m''}$, such that both $V(\xi, m, T)$ and $V(\eta, m', T')$ are contained in $V(\sigma, m'', T'')$.

(b) There is a subset A of $P_D(m_0)$ which spans $P_D(m_0)$ and is such that for every $v \in A$, there exists $\xi \in D^\infty$, $m \in S$, $T \in U_{\xi,m}$, such that $\xi_T(m) = m_0$ and $v \in V(\xi, m, T)$.

Let us first argue why this already implies the main statement of the lemma. As one inclusion is true is true for all (ξ, m, T) by Lemma 1.17 , we only have to find a particular (ξ, m, T), such that $m_0 = \xi_T(m)$ and $P(m_0) \subseteq V(\xi, m, T)$. From (b) we obtain the set generating set A. As $P_D(m_0)$ is a finite-dimensional vector space, we may choose its basis $(v_1, \ldots, v_k)$ of consisting of the elements of A and find (ξ_i, m_i, T_i) such that $v_i \in V(\xi_i, m_i, T_i)$ for all $i \in \{1, \ldots, k\}$. Clearly, one can use the induction to generalize (a) to any finite collection $(\xi_i, m_i, T_i)_{i=1}^k$ to find (ξ, m, T), such that $V(\xi_i, m_i, T_i) \subseteq V(\xi, m, T)$ for all $i \in \{1, \ldots, k\}$. By construction $P_D(m_0) \subseteq V(\xi, m, T)$ and the proof is finished.

Let us thus prove the claim (a): Take $m'' = m'$, $\sigma = \xi * \hat{\xi} * \eta$ and $T'' = T * (-\hat{T}) * T'$. Then $\sigma''_{T''}(m'') = \xi_T(\hat{\xi}_{-\hat{T}}(\eta_{T'}(m'))) = \eta_{T'}(m') = m_0$. Let $\xi \in D^n$ and $\eta \in D^{n'}$. We can choose the tangent vector $v \in \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}^{n'}$ at T'' to have first two components zero. Image of such vectors under $T_{T''}(\rho_{\sigma,m''})$ is precisely $V(\eta, m', T')$. Similarly, choosing the last two components zero, we obtain $V(\xi, m, T)$. This proves that both of the subspaces are contained in $V(\sigma, m'', T'')$. This proves the claim (a).

We can finish by proving (b). As already noted, $P_D(m_0)$ is spanned by values $X(m_0)$, where X is a $T(g)$-image of some vector field in $Y \in D$, where $g \in G_D$. Take A to be such set. Every element $v \in A$ is thus of the form $v = T(\xi_T)(Y(m))$ where $Y \in D$ and $\xi_T(m) = m_0$. Consider $\eta = \xi * Y$ and $T' = (T, 0)$. Clearly $\eta_{T'}(m) = m_0$. Finally, consider the tangent vector $w = (0, 1) \in \mathbb{R}^n \oplus \mathbb{R}$ at T'. It is easy to see that $T_{T'}(\rho_{\eta,m})(w) = T(\xi_T)(Y(m)) = v$, and thus $v \in V(\eta, m, T')$. This concludes the proof. ■

Proof of Lemma 1.19. Let $\mathcal{D}$ be the set of all vector fields that are of the form $T(g)(X)$ for some $X \in D$ and $g \in G_D$. Let $Y \in \mathcal{D}$, that is $Y = T(g)(X)$ for some $X \in D$. If γ is an integral curve of Y, it is the image under g of an integral curve of X. In particular, all points of γ are in the same D-orbit.

Now, let N be a connected integral submanifold of P_D, and let $m \in N$. Let $(X_1, \ldots, X_p)$ be elements of $\mathcal{D}$, such that $(X_1(m), \ldots, X_p(m))$ is a basis of $P_D(m)$. Consider the mapping

$$(t_1, \ldots, t_p) \mapsto (\phi_{t_1}^{X_1} \circ \cdots \circ \phi_{t_p}^{X_p})(m) \tag{28}$$

Its differential at 0 is an isomorphism of $T_0(\mathbb{R}^p)$ with $T_m N = P_D(m)$. It thus defines a diffeomorphism of some neighborhood of 0 in $\mathbb{R}^p$ onto a neighborhood of m in N. As noted above, images of the above map are contained in the single D-orbit S containing m. Every point of N thus has an open neighborhood entirely contained in one D-orbit.

Now, we make use of the fact that N is connected. Let $m_1, m_2 \in N$ be two distinct points. There is thus a continuous path $\sigma : [0, 1] \to N$ connecting these two points. We can cover this path by finitely many open subsets of N, each of them contained in the single D-orbit. As any point cannot be contained in two orbits simultaneously, it follows that all of them must be in the single D-orbit. In particular, this is true for m_1 and m_2.

We have thus proved that the underlying set U (of N) is contained entirely in the single D-orbit. For any orbit S, if $U \cap S$ is non-empty, then $U \subseteq S$.It remains to prove that U is open in S. We must show that, for any $m \in S$ and $\xi \in D^n$, the set $\rho_{\xi,m}^{-1}(U)$ is open in $\mathbb{R}^n$. Let $T \in U_{\xi,m}$ be in this set, that is $\rho_{\xi,m}(T) \in U$. Clearly, for any $i \in \{1, \ldots, n\}$, any curve $\gamma(t) := \rho_{\xi,m}(\tau_1, \ldots, \tau_{i-1}, t, \tau_{i+1}, \ldots, \tau_n)$ is an integral curve of some $X \in \mathcal{D}$. If $\gamma(t_0) \in U$ for some t_0, this must be true for all t in some open neighborhood of t_0. We can thus inductively find an open neighborhood of T, such that its image under $\rho_{\xi,m}$ is in U. Whence U is open in S and the proof of the Lemma is finished. ■

1.6 Proof of the second theorem

Implications (d) $\Rightarrow$ (c) $\Rightarrow$ (f) are trivial, as is (b) $\Rightarrow$ (a). The implication (f) $\Rightarrow$ (b) is precisely Theorem 1.12.

One can now prove (a) $\Rightarrow$ (e). Assume that Δ has the integral manifolds property. Let $m \in M$ and let S be the integral manifold of Δ through m. Let $X_1, \ldots, X_k$ be elements of D, such that $(X_1(m), \ldots, X_k(m))$ forms a basis of $\Delta(m)$. For any $X \in D$, the vector fields $[X, X_i]$ are tangent to S. Therefore, their restrictions to S are linear combinations of X_i with smooth coefficients in some neighborhood U of m (open in S). For small enough $\epsilon > 0$, the curve $t \mapsto \phi_t^X(m)$ is contained in U (for $|t| < \epsilon$). This implies (a) $\Rightarrow$ (e).

However, the final implication (e) $\Rightarrow$ (d) required to complete the implication snake is not valid. This was noted in [3]. Fortunately, not everything is lost. Sussmann used the following simple observation to proceed:

Lemma 1.20. *Let $X, Y \in \hat{\mathfrak{X}}(M)$, let $m \in M$ and $\epsilon > 0$, such that $\phi_t^X(m)$ is defined for all $t \in (-\epsilon, \epsilon)$. Define $W(t) \in T_m M$ as*

$$W(t) = \phi_{-t*}^X(Y(\phi_t^X(m))) \tag{29}$$

for all $t \in (-\epsilon, \epsilon)$. Then $W(t)$ satisfies the ordinary differential equation

$$\frac{d}{dt} W(t) = \phi_{-t*}^X([X, Y](\phi_t^X(m))) \tag{30}$$

with the initial condition $W(0) = Y(m)$.

Proof. This is just a definition of the Lie derivative (or Lie bracket of vector fields). Indeed:

$$\begin{aligned} \frac{d}{dt}W(t) = & \left.\frac{d}{ds}\right|_{s=0} W(t+s) = \phi^X_{-t*}(\left.\frac{d}{ds}\right|_{s=0} \phi^X_{-s*}(Y(\phi^X_s(m')))) \\ & = \phi^X_{-t*}([X,Y](m')) = \phi^X_{-t*}([X,Y](\phi^X_t(m))), \end{aligned} \tag{31}$$

where we have denoted $m' = \phi^X_t(m)$. This finishes the proof of this simple lemma. ■

To prove (d), it suffices to show[2] it for every $X \in D$, every m and $t \in (-\epsilon, \epsilon)$ for arbitrarily small $\epsilon > 0$. Suppose that (e) holds. In particular, we get $\epsilon > 0$ and a collection $X_1, \ldots, X_k$ forming a basis of $\Delta(m)$ when evaluated at m. One defines

$$W_i(t) = \phi^X_{-t*}(X_i(\phi^X_t(m))) \tag{32}$$

Clearly, $W_i(0) = X_i(m)$. From (30), we find the equation

$$\frac{d}{dt}W_i(t) = \phi^X_{-t*}([X, X_i](\phi^X_t(m))). = \sum_{j=1}^{k} f_{ij}(t) W_j(t), \tag{33}$$

where we have plugged in from the assumption (15).

Lemma 1.21. *The set $(W_1(t), \ldots, W_k(t))$ forms a basis of $\Delta(m)$ for any $t \in (-\epsilon, \epsilon)$.*

Proof. First, as the initial conditions live in $\Delta(m)$, the existence and uniqueness theorem for linear systems of first-order differential equations ensures that $W_i(t) \in \Delta(m)$ for all $t \in (-\epsilon, \epsilon)$. The next step is to form a matrix (without the loss of generality, we may now work in $\mathbb{R}^k$) $(\Phi(t))_{ij} = (W_i(t))_j$. Let $\mathbf{A}(t)$ be the $k \times k$ matrix $(\mathbf{A}(t))_{ij} = f_{ij}(t)$. The system (30) can be then rewritten as a matrix equation:

$$\frac{d}{dt}\Phi(t) = \mathbf{A}(t) \cdot \Phi(t). \tag{34}$$

We can now look for a differential equation satisfied by a single function $\det(\Phi(t))$. Recall that the time derivative of the determinant of any t-dependent matrix can be written as

$$\frac{d}{dt}\det(\Phi(t)) = \mathrm{Tr}(\mathrm{Adj}(\Phi(t)) \cdot \frac{d}{dt}\Phi(t)), \tag{35}$$

where $\mathrm{Adj}(\Phi(t))$ is the adjugate matrix, defined as $(\mathrm{Adj}(\Phi(t))_{ij} = (-1)^{i+j}\det(\Phi(t)_{(ji)})$. Here $\Phi(t)_{(ji)}$ is the $(k-1) \times (k-1)$ matrix obtained from Φ by erasing j-th row and i-th column. Equivalently, it is given by formula

$$\Phi(t) \cdot \mathrm{Adj}(\Phi(t)) = \det(\Phi) \cdot \mathbf{1} \tag{36}$$

Plugging in from (34), we find that

$$\begin{aligned} \frac{d}{dt}\det(\Phi(t)) = & \ \mathrm{Tr}(\mathrm{Adj}(\Phi(t)) \cdot \mathbf{A}(t) \cdot \Phi(t)) \\ = & \ \mathrm{Tr}(\Phi(t) \cdot \mathrm{Adj}(\Phi(t)) \cdot \mathbf{A}(t)) \\ = & \ \det(\Phi(t)) \cdot \mathrm{Tr}(\mathbf{A}(t)) \end{aligned} \tag{37}$$

[2]Mark this exact point - this is where the error was committed.

This resulting ordinary differential equation is easily solvable, one finds

$$\det(\Phi(t)) = \det(\Phi(0)) \cdot \exp\Big(\int_0^t \mathrm{Tr}(\mathbf{A}(t'))dt'\Big). \tag{38}$$

This a famous result known as the **Liouville's formula**. In particular, it shows that $\det(\Phi(0)) = 0$ if and only if $\det(\Phi(t)) = 0$. We have assumed that $(W_1(0), \dots, W_k(0))$ form the basis of $\Delta(m)$, whence $\det(\Phi(0)) \neq 0$. But this proves that $(W_1(t), \dots, W_k(t))$ form a basis for every $t \in (-\epsilon, \epsilon)$. This finishes the proof of the lemma. ■

Using the definition of $W_i(t)$, we immediately obtain the equation

$$\phi^X_{t*}(W_i(t)) = X_i(\phi^X_t(m)). \tag{39}$$

But this proves that $\phi^X_{t*}(\Delta(m)) \subseteq \Delta(\phi^X_t(m))$. Where is the error hidden? As always, the devil lies hidden in the tiniest of details. Note that although Balan [3] found a counter-example, he found the gap in the Sussmann's proof where there was none, and missed the actual one. Actually, Sussmann just modified the erroneous statement of Lobry [4]. The gap in the proof of therein Lemma 1.2.1. was noted by Štefan e.g. in [2]. It is in fact a quite general problem, so it is worth noting it here.

Let $X \in \hat{\mathfrak{X}}(M)$ be any smooth vector field, and let Δ be any distribution. We say that X **preserves** Δ, if for all $(m,t) \in \mathcal{D}_X$, one has $(T_m\phi^X_t)(\Delta(m)) = \Delta(\phi^X_t(m))$. Here $\mathcal{D}_X$ denotes the maximal flow domain of X. We now prove the following theorem:

Proposition 1.22. *Let Δ be a smooth distribution and $X \in \hat{\mathfrak{X}}(M)$ be a smooth vector field. Then the following statements are equivalent:*

(i) X preserves Δ.

(ii) For every $(m,t) \in \mathcal{D}_X$, one has $(T_m\phi^X_t)(\Delta(m)) \subseteq \Delta(\phi^X_t(m))$.

(iii) For each $m \in M$, there exists a real number $\epsilon(m) > 0$ such that for all $t \in (-\epsilon(m), \epsilon(m))$, one has $(T_m\phi^X_t)(\Delta(m)) = \Delta(\phi^X_t(m))$.

Proof. Implications (i) ⇒ (ii) and (i) ⇒ (iii) are trivial. The implication (ii) ⇒ (i) follows from the fact that for $(m,t) \in \mathcal{D}_X$, one has $(\phi^X_t(m), -t) \in \mathcal{D}_X$. An application of (i) for this pair then yields the converse inclusion.

To show (iii) ⇒ (i), let I_m be the domain for maximal integral curve $t \mapsto \phi^X_t(m)$, and let $J_m \subseteq I_m$ be the set $J_m = \{t \in I_m \mid (T_m\phi^X_t)(\Delta(m)) = \Delta(\phi^X_t(m))\}$. Suppose $t_0 \in J_m$. Let $m' = \phi^X_{t_0}(m)$. Then for all $t \in (t_0 - \epsilon(m'), t_0 + \epsilon(m'))$, one has

$$T_m(\phi^X_t) = T_{m'}(\phi^X_{t-t_0}) \circ T_m(\phi^X_{t_0}) \tag{40}$$

The first map maps $\Delta(m)$ bijectively onto $\Delta(m')$, the second one maps $\Delta(m')$ bijectively onto $\Delta(\phi^X_t(m))$ as $t - t_0 \in (-\epsilon(m'), \epsilon(m'))$. We thus have an open neighborhood of t_0 which is entirely contained in J_m. Whence J_m is open.

We will show that it is also closed. Suppose there is a convergent sequence $\{t_n\}_{n=1}^{\infty}$ of points in J_m, that is $T_m(\phi^X_{t_n})(\Delta(m)) = \Delta(\phi^X_{t_n}(m))$. Let $m_n = \phi^X_{t_n}(m)$. We have $t_n \to t'$ and $m_n \to m'$. We have to show that $T_m(\phi^X_{t'}) = \Delta(m')$. But note that each of the points m_n is reachable from m' using the integral curve of X, that is

$$m_n = \phi^X_{t_n}(m) = \phi^X_{t_n}(\phi^X_{-t'}(m')) = \phi^X_{t_n - t'}(m'). \tag{41}$$

As $t_n \to t'$, there exists n, such that $|t_n - t'| < \epsilon(m')$. We can write

$$T_m(\phi^X_{t'}) = T_{m_n}(\phi^X_{t'-t_n}) \circ T_m(\phi^X_{t_n}) = T_{m'}(\phi^X_{t_n-t'})^{-1} \circ T_m(\phi^X_{t_n}). \tag{42}$$

The first map maps $\Delta(m)$ isomorphically onto $\Delta(\phi^X_{t_m}(m))$, whereas the second map is the inverse of the isomorphism which maps $\Delta(m')$ onto $\Delta(\phi^X_{t_m}(m))$. This proves the claim above, that is $t' \in J_m$. Whence J_m is closed in I_m.

As I_m is a connected space, this proves that $J_m = I_m$ and (i) is proved. ■

Example 1.23. We will now show that in Proposition 1.22, we cannot replace the equality in (iii) by inclusion (as the equivalence (i) $\Leftrightarrow$ (ii) suggests).

Let $M = \mathbb{R}^2$ and consider Δ spanned by the family of vector fields $D = \{X, Y\}$, where $X = \partial_x$, $Y = x \cdot \partial_y$. Clearly $\text{rk}(\Delta)(x, y) = 2$ whenever $x \neq 0$ and $\text{rk}(\Delta)(0, y) = 1$. We will now show that although X satisfies the local version of (ii), it does not preserve Δ.

Its (globally defined) flow is $\phi^X_t(x, y) = (x + t, y)$ and its action on the generators of Δ can be trivially calculated:

$$\begin{aligned} (T_{(x,y)}\phi^X_t)(X_{(x,y)}) &= X_{\phi^X_t(x,y)}, \\ (T_{(x,y)}\phi^X_t)(Y_{(x,y)}) &= x \cdot (\partial_y)_{\phi^X_t(x,y)} = \frac{x}{x+t} \cdot Y_{\phi^X_t(x,y)}. \end{aligned} \tag{43}$$

Let $J'_{(x,y)}$ denote the subset of $I_{(x,y)}$, where (ii) is satisfied. We will distinguish two cases, namely $x \neq 0$ and $x = 0$ case.

For $x \neq 0$, we see the only problem arises for $t = -x$. This is clearly the point where the curve $\phi^X_t(x, y)$ crosses the $x = 0$ line. We thus have $J'_{(x,y)} = \mathbb{R} - \{-x\}$. This is an open set, but it is not closed. For $x = 0$, observe that $Y_{(0,y)} = 0$. The linear map $T_{(0,y)}\phi^X_t$ maps the single generator $X_{(0,y)}$ onto $X_{(t,y)}$. This is well-defined for all $t \in \mathbb{R}$, and we find $J'_{(0,y)} = \mathbb{R}$. We see that problem arises only with the closedness of $J'_{(x,y)}$ and exactly at the points of M, where the rank of the distribution Δ jumps down.

We see on this example that it is not only the proof of the previous proposition which fails. This also shows that X simply does not preserve Δ, whence the implication "local (ii)" $\Rightarrow$ (i) is plainly wrong. This is where Lobry made the crucial mistake. He used the "compactness" argument for the integral curve $\phi^X_t(m)$ - as the assumption of local (ii) gives one the cover of the image of the interval I_m. When I_m is compact, we can choose a finite subcover and inductively deduce (ii) from its local version. However, as shows the above $I_m = \mathbb{R}$ example, this does not work for non-compact intervals.

Example 1.24 (The implication (e) $\Rightarrow$ (d) of Theorem 1.13 is wrong). We can use the same example as above. One can easily show that Δ satisfies (e), but not (d).

Luckily, the other five equivalences of Theorem 1.13 are correct. One just has to leave out (e) (unfortunately obviously the most useful one). We thus have

Theorem 1.25 (Sussmann modified). *Statements in Theorem 1.13 except for (e) are equivalent.*

Proof. Based on the original proof, we will prove that (a) implies (d). We have already shown that (a) implies (e). It remains to show that (a) together with (e) imply (d). The statement (d) says that every $X \in D$ satisfies the condition (ii) of Proposition 1.22. It thus suffices to show that every $X \in D$ satisfies the condition (iii) of Proposition 1.22.

We have already proved that for each $m \in M$, there is $\epsilon'(m) > 0$, such that the inclusion $T_m(\phi_t^X)(\Delta_D(m)) \subseteq \Delta_D(\phi_t^X(m))$. holds for all $t \in (-\epsilon'(m), \epsilon'(m))$. If we assume (a), there is an integral submanifold S containing m. By Lemma 1.2, there is an open subinterval $K_m \subseteq I_m$ containing zero, such that $\phi_t^X(m) \in S$ for all $t \in K_m$. But for $t \in K_m \cap (-\epsilon'(m), \epsilon'(m))$, we certainly have $\dim(\Delta(m)) = \dim(\Delta(\phi_t^X(m))$, which proves that both subspaces have to be equal. This finishes the proof. ■

1.7 Hermann integrability

Let us start with the following definition.

Definition 1.26. Let D be a family of smooth vector fields. We say that D is **locally finitely generated**, if for each $m \in M$, there exists a finite tuple $(X_1, \dots, X_k)$ of vector fields in D defined on some neighborhood U of m, such that every $X \in D$ can be on $U' = \mathrm{Dom}(X) \cap U$ decomposed as

$$X = \sum_{j=1}^{k} f_j \cdot X_k \tag{44}$$

for some smooth functions $f_j \in C^\infty(U')$.

First, let us note an obvious but important observation.

Lemma 1.27. *Let Δ be a distribution spanned by a locally finitely generated family D. Then to each point $m \in M$, there exists a finite tuple $(X_1, \dots, X_k)$ of vector fields in D defined on some neighborhood U of M, such that $\Delta(\widetilde{m}) = \mathbb{R}\{X_1(\widetilde{m}), \dots, X_k(\widetilde{m})\}$ for all $\widetilde{m} \in U$.*

Proof. For every $m \in M$, find $(X_1, \dots, X_k)$ and U from the definition of locally finitely generated distributions. Let $\widetilde{m} \in U$, and $v \in \Delta(\widetilde{m})$. By definition, $v = \alpha^q V_q(\widetilde{m})$ for some finite collection $V_1, \dots, V_p \in D$. By assumption, each of $V_q(\widetilde{m})$ is a $\mathbb{R}$-linear combination of $(X_1(\widetilde{m}), \dots, X_k(\widetilde{m}))$. Whence clearly $v \in \mathbb{R}\{X_1(\widetilde{m}), \dots, X_k(\widetilde{m})\}$. The converse inclusion is trivial. ■

We will now prove the crucial property of locally finitely generated families of vector fields, which will in fact close the gap in the Lobry's proof.

Lemma 1.28 (Hermann [5]). *Let D be a locally finitely generated family of vector fields. Let $m \in M$ and $(X_1, \dots, X_k)$ be the generators of D on some neighborhood U of m as by definition. Suppose $X \in \hat{\mathfrak{X}}(M)$ is some vector field which on $U' := U \cap \mathrm{Dom}(X)$ satisfies the condition*

$$[X, X_i] = \sum_{j=1}^{k} f_{ij} \cdot X_j, \tag{45}$$

for some smooth functions $f_{ij} \in C^\infty(U')$. Then there exists $\epsilon > 0$, such that for all $t \in (-\epsilon, \epsilon)$

$$\dim(\Delta_D(\phi_t^X(m))) = \dim(\Delta_D(m)). \tag{46}$$

Proof. We can safely assume that $X(m) \neq 0$, otherwise one has $\phi_t^X(m) = m$ and the result holds trivially. There are thus some local coordinates $(x^1, \dots, x^\mu)$ on some neighborhood $W \subseteq U \cap \mathrm{Dom}(X)$ of m, such that $X = \partial_1$ on W. The generators X_i can be on W decomposed:

$$X_i(m') = \sum_{\nu=1}^{\mu} \mathbf{A}_{i\nu}(m') \cdot \partial_\nu(m'). \tag{47}$$

for $k \times \mu$ matrix $\mathbf{A}(m')$ at each point m'. Clearly, $\dim(\Delta_D(m')) = \mathrm{rank}(\mathbf{A}(m'))$. To prove our claim, it suffices to show that this rank does not depend on x^1. By assumption, we have

$$[\partial_1, X_i] = \sum_{j=1}^{k} f_{ij} \cdot X_j \tag{48}$$

Plugging into this equation from (47) provides the matrix equation

$$\partial_1 \mathbf{A}_{i\nu} = f_{ij} \mathbf{A}_{j\nu}. \tag{49}$$

Fixing the values of all other coordinates, this is a linear homogeneous system of ordinary differential equations (in x^1). Using the arguments similar to those of Lieouville's formula (see proof of Lemma 1.21), one can show that the rank of the matrix $\mathbf{A}$ does not depend on x^1. ■

The main assumption of the previous lemma suggests that one could impose it on all vector fields from the family generating a given smooth distribution.

Definition 1.29. Let D be a locally finitely generated family of vector fields. We say that D is **locally of a finite type** if each $X \in D$ satisfies the condition (45).

Remark 1.30. Note that originally, in Lobry's paper [4], one only assumed that there are vector fields $(X_1, \dots, X_k)$ generating $\Delta_D(m)$ at a given point $m \in M$, not in some its neighborhood. As already discussed in this section, this is not good enough definition.

Theorem 1.31. *Let D be of a locally finite type. Then Δ_D is integrable.*

Proof. By assumption, Δ_D satisfies the condition (e) in Theorem 1.13. This was enough to prove that for each $m \in M$ and $X \in D$, there exists $\epsilon > 0$, such that

$$(T_m \phi_t^X)(\Delta_D(m)) \subseteq \Delta_D(\phi_t^X(m)). \tag{50}$$

However, by Lemma 1.28, the dimension of $\Delta_D(\phi_t^X(m))$ is constant in t on some neighborhood of 0, whence we have $\epsilon' > 0$, such that $(T_m \phi_t^X)(\Delta_D(m)) = \Delta_D(\phi_t^X(m))$ for all $t \in (-\epsilon', \epsilon')$. By Proposition 1.22, this is enough to prove (d) of Theorem 1.13, whence Δ_D is integrable. ■

We can now quite easily prove the following theorem.

Theorem 1.32 (Hermann integrability [5]). *Suppose D is a Lie subalgebra of $\mathfrak{X}(M)$, such that for every $m \in M$, there exists its neighborhood U and a finite-dimensional subspace $D_U \subseteq D$, such that each $X \in D$ can be on U written as a $C^\infty(U)$-linear combination of elements in D_U.*

Then Δ_D is an integrable smooth distribution.

Proof. One just has to prove that D is of a locally finite type. For each $m \in M$, we find U from the assumptions of the theorem. Let $(X_1, \dots, X_k)$ be any basis of D_U. This clearly makes D into a locally finitely generated family. Finally, for any $X \in D$ and X_i as above, we have $[X, X_i] \in D$ as D is assumed to be a Lie subalgebra. As it is locally finitely generated, $[X, X_i]$ can be on U decomposed as

$$[X, X_i] = \sum_{j=1}^{k} f_{ij} \cdot X_j. \tag{51}$$

Whence D is of a locally finite type, thus integrable. ■

In fact, one can easily deduce the other well-known integrability condition.

Theorem 1.33 (**Frobenius**). *Let Δ be an involutive smooth distribution, such that* $\dim(\Delta(m)) = k$ *for all $m \in M$. Then Δ is integrable.*

Proof. First, we know that Δ is spanned by the family D_Δ. Any constant rank distribution forms a subbundle $\Delta \subseteq TM$. In particular, for each $m \in M$, there is a neighborhood U and some vector fields $(X_1, \dots, X_k)$ on U which form the local frame for the subbundle Δ over U. In particular, they make D_Δ into a locally finitely generated family. Moreover, as D_Δ is involutive, it can be easily seen to be locally of a finite type. By Theorem 1.31, it is integrable. ■

Example 1.34. To conclude this section, it to examine the distribution Δ from Example 1.23 and see which of the assumptions of Theorem 1.31 fail.

Recall that $M = \mathbb{R}^2$ and Δ is spanned by $D = \{X, Y\}$, where $X = \partial_x$ and $Y = x \cdot \partial_y$. Obviously, D is locally (in fact globally) finitely generated. However, one has

$$[X, Y] = \partial_y. \tag{52}$$

This shows that any point on the $x = 0$ line is in trouble, as the right-hand side must be a smooth combination of X and Y on whole its neighborhood. This is not possible and we conclude that D is not of a locally finite type according to our Definition 1.29.

1.8 Leibniz algebroids and induced distributions

We will now introduce a rather large class of integrable distributions. In fact, this class includes a well-known characteristic distribution of a Poisson manifold.

Definition 1.35. Let $q : E \to M$ be a vector bundle. Let $\rho \in \mathrm{Hom}(E, TM)$ be a vector bundle map and $[\cdot, \cdot]_E$ a $\mathbb{R}$-bilinear bracket on $\Gamma(E)$. We say that $(E, \rho, [\cdot, \cdot]_E)$ is a **Leibniz algebroid**, if the following two conditions are satisfied:

(i) The bracket $[\cdot, \cdot]_E$ and the **anchor map** ρ satisfy the **Leibniz rule**:

$$[\psi, f\psi']_E = f \cdot [\psi, \psi']_E + \mathcal{L}_{\rho(\psi)}(f) \cdot \psi', \tag{53}$$

for all $\psi, \psi' \in \Gamma(E)$ and $f \in C^\infty(M)$.

(ii) The bracket $[\cdot, \cdot]_E$ makes $\Gamma(E)$ into a Leibniz algebra, that is the **Leibniz identity**

$$[\psi, [\psi', \psi'']_E]_E = [[\psi, \psi']_E, \psi'']_E + [\psi', [\psi, \psi'']_E]_E \tag{54}$$

holds for all $\psi, \psi', \psi'' \in \Gamma(E)$.

From the definition of Leibniz algebroid, one can immediately deduce the following property:

Lemma 1.36. *For any Leibniz algebroid $(E, \rho, [\cdot,\cdot]_E)$, the map $\rho : \Gamma(E) \to \mathfrak{X}(M)$ is a bracket homomorphism, that is for all $\psi, \psi' \in \Gamma(E)$, one has*

$$\rho([\psi, \psi']_E) = [\rho(\psi), \rho(\psi')]. \tag{55}$$

This observation naturally leads to the following statement:

Proposition 1.37. *Let $(E, \rho, [\cdot,\cdot]_E)$ be a Leibniz algebroid. Then the family $D = \rho(\Gamma(E)) = \{\rho(\psi) \mid \psi \in \Gamma(E)\}$ spans the integrable distribution Δ_D, called the* ***characteristic distribution*** *of the Leibniz algebroid $(E, \rho, [\cdot,\cdot]_E)$.*

Proof. Instead of D, we can take the family $\hat{D}$ of ρ-images of all local sections of E. We will show that $\hat{D}$ satisfies the assumptions of Theorem 1.31.

Let $m \in M$ be any point. As E is a vector bundle, there exists some local frame $(\psi_\mu)_{\mu=1}^{\mathrm{rk}(E)}$ over some neighborhood U of m. Then $X_\mu = \rho(\psi_\mu)$ are smooth vector fields in $\hat{D}$ defined on U for all $\mu \in \{1, \dots, \mathrm{rk}(E)\}$. Now, let $X \in \hat{D}$. Let $V = \mathrm{Dom}(X)$. By definition, there is some local smooth section $\psi \in \Gamma_V(E)$, such that $X = \rho(\psi)$. By definition of the local frame, there are unique smooth functions $f^\mu \in C^\infty(U \cap V)$, such that on $U \cap V$, one has $\psi = f^\mu \psi_\mu$. Thus

$$X = \rho(\psi) = f^\mu \rho(\psi_\mu) = f^\mu X_\mu, \tag{56}$$

on $U \cap V$. This proves that $\hat{D}$ is locally finitely generated. Using the same notation, we have

$$[X, X_\mu] = [\rho(\psi), \rho(\psi_\mu)] = \rho([\psi, \psi_\mu]_E) = \langle \psi^\nu, [\psi, \psi_\mu]_E \rangle \cdot X_\nu. \tag{57}$$

Here $(\psi^\nu)_{\nu=1}^{\mathrm{rk}(E)}$ is the dual frame for E^* over U. The whole equations makes sense on $U \cap \mathrm{Dom}(X)$, which proves that $\hat{D}$ is locally of a finite type.

Using Theorem 1.31, we conclude that Δ_D is integrable. ∎

Example 1.38 (Lie algebra actions). First, one can do it the pedestrian way.

Let $\# : \mathfrak{g} \to \mathfrak{X}(M)$ be the infinitesimal Lie algebra action. Define $D = \#(\mathfrak{g})$. By definition, D is a Lie subalgebra of $\mathfrak{X}(M)$ and it is globally generated by $X_\mu = \#(t_\mu)$, where $(t_\mu)_{\mu=1}^{\dim(\mathfrak{g})}$ is some fixed basis for $\mathfrak{g}$. Whence by Theorem 1.32, it is integrable. Maximal integral submanifolds are orbits of the local flows of the vector fields $\#x$. If there is a Lie group G integrating $\mathfrak{g}$ and a Lie group action $\triangleleft : M \times G \to M$ integrating $\#$, the maximal integral submanifolds of D are precisely the orbits of this action.

Second, one can do it in a more fancy way. Set $E = M \times \mathfrak{g}$. Let $\rho : E \to TM$ be the fiber-wise extension of $\#$ and let $[\cdot,\cdot]_E$ be the Leibniz-rule extension of $[\cdot,\cdot]_{\mathfrak{g}}$. Then $(E, \rho, [\cdot,\cdot]_E)$ is a Leibniz algebroid (in fact a Lie algebroid). Its characteristic distribution is precisely Δ_D.

Example 1.39 (Poisson manifolds). Let (M, Π) be a Poisson manifold. The Hamiltonian vector field X_f corresponding to $f \in C^\infty(M)$ is defined via its action on $g \in C^\infty(M)$:

$$\mathcal{L}_{X_f}(g) := \{f, g\}_\Pi, \tag{58}$$

where $\{\cdot,\cdot\}_\Pi$ is the Poisson bracket corresponding to Π. Setting D to be the set of all Hamiltonian vector fields, the corresponding smooth distribution Δ_D is called the **characteristic distribution of** (M, Π). First, note that one of the basic features of Hamiltonian vector fields is the property $\mathcal{L}_X \Pi = 0$ for all $X \in D$. Indeed, let $\alpha \in \Omega^1(M)$ be arbitrary, and let $\Pi(\alpha) \in \mathfrak{X}(M)$ be the induced vector field. One can show that there holds the equation

$$\frac{1}{2}[\Pi, \Pi]_S(\alpha, \beta, \cdot) = (\mathcal{L}_{\Pi(\alpha)}\Pi)(\beta) + \Pi(i_{\Pi(\beta)}(d\alpha)) \tag{59}$$

for any $\Pi \in \mathfrak{X}^2(M)$ and all $\alpha, \beta \in \Omega^1(M)$. However, if Π is Poisson and α closed, this gives the equation $\mathcal{L}_{\Pi(\alpha)}\Pi = 0$. In particular, we have $X_f = -\Pi(df)$, what proves our claim. Thus for any $X \in D$, this can be integrated to the equation $\phi^X_{t*}(\Pi(m)) = \Pi(\phi^X_t(m))$ for any $(m, t) \in \mathcal{D}_X$.In view of the induced linear maps, this can be written as

$$(T_m \phi^X_t) \circ \Pi(m) = \Pi(\phi^X_t(m)) \circ T_{\phi^X_t(m)}(\phi^X_{-t}). \tag{60}$$

By definition, one can clearly write $\Delta_D(m) = \Pi(m)(T^*_m M)$. The above equation shows that $(T_m \phi^X_t)(\Delta_D(m)) \subseteq \Delta_D(\phi^X_t(m))$. By Theorem 1.13, this implies the integrability.

In fact, this example is another consequence of Proposition 1.37. Indeed, for a Poisson manifold (M, Π), one can consider the Lie algebroid $(T^*M, \Pi, [\cdot, \cdot]_\Pi)$, where the bracket is given by

$$[\xi, \eta]_\Pi = \mathcal{L}_{\Pi(\xi)}\eta - i_{\Pi(\eta)}(d\xi), \tag{61}$$

for all $\xi, \eta \in \Omega^1(M)$. After some effort, this can be shown to be a Lie algebroid, and clearly Δ_D is its characteristic distribution.

2 Štefan story

We will now recall the content of two of the Štefan's papers [6, 2]. It requires a lot of definitions, as Štefan introduced a lot of names (more or less common) for the assumptions he made. A stellar example is the first subsection.

2.1 Neat submanifolds and singular foliations

Let $(E, \|\cdot\|)$ be a normed finite-dimensional real vector space. One writes E^1 for an open unit ball $E^1 = \{x \in E \mid \|x\| < 1\}$. A **box** on the differentiable manifold M is a triple (ψ, E, F), where E and F are normed finite-dimensional real vector spaces and $\psi : E^1 \times F^1 \to U$ is a diffeomorphism onto some open subset $U \subseteq M$. Equivalently, $\phi = \psi^{-1}$ defines a coordinate chart (U, ϕ) on M, which maps U onto the product $E^1 \times F^1$ of open unit balls.

Now, let S be an immersed submanifold of M, and let $i : S \to M$ be the corresponding smooth immersion. As usual, we will identify S with its image $i(S) \subseteq M$.

Remark 2.1. It will be obvious in the following that Štefan allows (sub)manifolds to be a disjoint union of connected components, each of a possibly different dimension.

We will now distinguish a special class of immersed submanifolds:

Definition 2.2. An immersed submanifold $S \subseteq M$ is called **tame** if for each $s \in S$, there exists a box (ψ, E, F) on M, such that

(a) $\psi(0, 0) = s$ and $S \cap \psi(E^1 \times F^1) = \psi(E^1 \times A)$ for some subset $A \subseteq F^1$.

(b) $\psi(\cdot, 0) : E^1 \to S$ is a diffeomorphism of E^1 onto some open subset of S.

(c) $\psi(\cdot, t) : E^1 \to S$ is a smooth map for any $t \in A$.

Moreover, one says that S is **neat**, if each point $s \in S$ has a box (ψ, E, F) satisfying (a) and

(d) $\psi(\cdot, t) : E^1 \to S$ is a diffeomorphism of E^1 onto some open subset of S for any $t \in A$.

Remark 2.3. Again, it is quite convenient to reformulate this definition in terms of the local coordinates on M. The immersed submanifold S is **tame**, if for each $s \in S$, there exists a coordinate chart (U, φ) on some neighborhood, such that $\varphi(U) = E^1 \times F^1$, and

(a) $\varphi(U \cap S) = E^1 \times A$ for some subset $A \subseteq F^1$.

(b) The set $V = \varphi^{-1}(E^1 \times \{0\})$ is open in S, and the restriction of $\varphi' = \varphi \circ i$ onto V defines a diffeomorphism $\varphi' : V \to E^1$. This implies that (V, φ') is in fact a coordinate chart for S. Note that necessarily $s \in V$.

(c) For any $t \in A$, we can restrict ϕ^{-1} to an embedded submanifold $E^1 \times \{t\}$ of $E^1 \times F^1$. This defines a map from E^1 to M which happens to have values in the immersed submanifold S. This maps in general fails to be continuous from E^1 to S (this is because topology of S is finer than the subspace topology). However, one assumes that *it is continuous.*

Let $j_t : E^1 \to E^1 \times F^1$ denote the embedding of E^1 onto its image $E^1 \times \{t\}$. The axiom (c) says that the unique map $\psi'_t : E^1 \to S$ fitting into the diagram

$$\begin{array}{ccc} E^1 & \xrightarrow{j_t} & E^1 \times F^1 \\ \downarrow{\scriptstyle \psi'_t} & & \downarrow{\scriptstyle \psi} \\ S & \xrightarrow{i} & M \end{array} \tag{62}$$

is smooth. By taking the tangents of this diagram, we find that $\psi'_t : E^1 \to S$ has to be an injective immersion.

In fact, neat submanifolds and tame submanifolds are almost the same definition, as every every tame submanifold S is always just a disjoint union of neat components.

Lemma 2.4. *Every connected component of a tame submanifold is neat.*

Proof. Let S be a neat submanifold. Let $s \in S$ and let $S_0 \subseteq S$ be the connected component containing s. Let (ψ, E, F) be the box for S on some neighborhood containing s. By assumption, $\psi'_t : E^1 \to S$ are smooth for all $t \in A$. In particular, each $\psi'_t(E^1) \subseteq S$ is contained in a single connected component of S. It thus makes sense to define

$$A_0 = \{t \in A \mid \psi'_t(E^1) \subseteq S_0\}. \tag{63}$$

We claim that A_0 is a set which "parametrizes" S_0, that is one has

$$S_0 \cap \psi(E^1 \times F^1) = \psi(E^1 \times A_0). \tag{64}$$

The inclusion of the right-hand side into the left-hand side set is clear. On the other hand, if $s \in S_0 \cap \psi(E^1 \times F^1)$, from Definition 2.2 (a), $s \in \psi'_t(E)$ for some $t \in A$. But $\psi'_t(E)$ is then a connected set non-trivially intersecting S_0. Whence $t \in A_0$. Whence $s \in \psi(E^1 \times A_0)$.

To show that S_0 is neat, we have to verify that $\psi'_t : E^1 \to S_0$ forms a diffeomorphism onto an open subset of S_0 for each $t \in A_0$. We have already argued that for each $t \in A$, $\psi'_t : E^1 \to S$ is an injective immersion. However, it follows from Definition 2.2 (b) that $\dim(S_0) = \dim(E^1)$, whence for each $t \in A_0$, $\psi'_t : E^1 \to S_0$ is an injective local diffeomorphism. In particular, it is an open map and thus a diffeomorphism onto its open image $\psi'_t(E^1) \subseteq S_0$. ■

In fact, the proof of this lemma can be easily generalized to show the following:

Corollary 2.5. *Any disjoint union of connected components of a tame submanifold, where all are of the same dimension, is neat.*

Proof. Let S' be the union of connected components in question. For each $s \in S'$, one again starts with a box (ψ, E, F) for S around s. Instead of A_0, define

$$A' = \{t \in A \mid \psi'_t(E^1) \subseteq S'\}. \tag{65}$$

For the same reasons as above, one can write $S' \cap \psi(E^1 \times F^1) = \psi(E^1 \times A')$. By assumption, all components forming S' have the same dimension, which is by Definition 2.2 (b) equal to the dimension of E^1. In particular, for each $t \in A'$, the map $\psi'_t : E^1 \to S'$ is an injective local diffeomorphism and the conclusion follows as above. ■

We see that every tame submanifold is a disjoint union of its neat components. The notion of tame immersed submanifolds with multiple components of varying dimension allows for the following neat (and very compact) definition of foliations:

Definition 2.6. Let M be a smooth manifold. The **foliation of** M is a tame submanifold $i : \mathcal{F} \to M$, such that $i(\mathcal{F}) = M$. $\mathcal{F}$ is called the **regular foliation** of M, if $\mathcal{F}$ is neat. The connected components of $\mathcal{F}$ are called the **leaves of the foliation**.

Before continuing onward, let us bring up some examples.

Example 2.7 (**Embedded submanifolds are neat**)**.** Let $S \subseteq M$ be an embedded submanifold. For each $s \in S$, there exists local chart (U, φ) onto some open set $\varphi(U) \subseteq \mathbb{R}^n$, such that $\varphi(U \cap S) = \{(x_1, \dots, x_n) \in \varphi(U) \mid x_{k+1} = c_{k+1}, \dots, x_n = c_n\}$. One can shrink U and modify the coordinate functions so that $\varphi(U) = E^1 \times F^1$ for $E = \mathbb{R}^k$ and $F = \mathbb{R}^{n-k}$, and $\varphi(U \cap S) = E^1 \times \{0\}$. Taking $\psi = \varphi^{-1}$ defines a box (ψ, E, F) for S around s. As S has a subspace topology, Definition 2.2 (b) is easily satisfied and (c) is trivial (in this case $A = \{0\}$).

Remark 2.8 (**The set** A). Let $i : S \to M$ be a tame submanifold. For each $s \in S$, we thus have the corresponding box (ψ, E, F). By definition, we have $\psi(E^1 \times F^1) \cap S = \psi(E^1 \times A)$ for some subset $A \subseteq F^1$. How to characterize this set? We claim that

$$A = \{t \in F^1 \mid \psi(0, t) \in S\}. \tag{66}$$

Let A' denote the set on the right-hand side of this equation. If $t \in A$, we have $\psi(e, t) \in S$ for all $e \in E^1$, in particular for $e = 0$. Whence $t \in A'$. Conversely, if $t \in A'$, we have $\psi(0, t) \in \psi(E^1 \times A)$ and in particular, $t \in A$. Let $\psi_0 : F_1 \to M$ be the smooth map $\psi_0(f) = \psi(0, f)$. We can thus write the set A as smooth preimage of S (strictly speaking of $i(S)$)

$$A = \psi_0^{-1}(S). \tag{67}$$

This is important in the case of a singular foliation, as then $A = \psi_0^{-1}(M) = F_1$.

We can now prove a certain implication of the definition of (singular) foliations. It is sometimes given as a *definition* of the foliation. However, it is not equivalent to the original one.

Proposition 2.9. *Let $\mathcal{F}$ be the foliation of M. Then M can be written as a disjoint union of the leaves of $\mathcal{F}$. Let $m \in M$ and $F_m \subseteq \mathcal{F}$ be the leaf through m. Let $k = \dim(F_m)$.*

Then there exists a local coordinate chart (U, φ) around m, such that $\varphi(U)$ is an open cube $(-\epsilon, \epsilon)^n$ in $\mathbb{R}^n$ and the following is true:

(i) Let $(F_m \cap U)_0$ be the connected component of $F_m \cap U$ which contains m. Then

$$\varphi(F_m \cap U)_0 = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^{k+1} = \cdots = x^n = 0\}. \tag{68}$$

(ii) For any $(c^{k+1}, \dots, c^n) \in (-\epsilon, \epsilon)^{n-k}$, the φ^{-1}-image of the slice

$$\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^{k+1} = c^{k+1}, \dots, x^n = c^n\} \tag{69}$$

is entirely contained in a single leaf of $\mathcal{F}$.

Proof. By definition, for each $m \in M$, we have a diffeomorphism $\psi : E^1 \times F^1 \to U$, such that $\psi'_0 : E^1 \to \mathcal{F}$ is a diffeomorphism onto an open subset and $\psi'_t : E^1 \to \mathcal{F}$ is smooth for each $t \in F^1$. Recall that $i \circ \psi'_t(e) = \psi(e, t)$ for all $(e, t) \in E^1 \times F^1$. We claim that

$$\psi'_0(E^1) = (F_m \cap U)_0. \tag{70}$$

By assumption, $\psi'_0(E^1)$ is a connected open subset of $\mathcal{F}$. It is thus contained in a single leaf. As $\psi'_0(0) = m$, it must be F_m. Clearly it is also contained in U, whence in $F_m \cap U$. As it is connected and contains m, it must be in $(F_m \cap U)_0$. Thus $\psi'_0(E^1) \subseteq (F_m \cap U)_0$. We will now prove that $\psi'_0(E^1)$ is closed in $(F_m \cap U)_0$.

As $\psi : E^1 \times F^1 \to U$ is a diffeomorphism, $\psi(E^1 \times \{0\}) \subseteq U$ is closed. Let $i_m : F_m \to M$ be the inclusion of the leaf F_m. Consequently $\psi'_0(E_1) = i_m^{-1}(\psi(E^1 \times \{0\}))$ is closed in $F_m \cap U$ and as it is connected, it is closed in the connected component $(F_m \cap U)_0$.

This proves that $\psi'_0(E^1)$ is both closed and open subset of a connected space $(F_m \cap U)_0$ containing m, whence the equality (70) must hold. Finally, for each point $t \in F_1$, the map $\psi'_t : E_1 \to \mathcal{F}$ is smooth, whence $\psi'_t(E_1) \subseteq \mathcal{F}$ is connected, and thus contained in a single leaf of $\mathcal{F}$. Replacing the open balls E^1 and F^1 with the (diffeomorphic) open cubes $(-\epsilon, \epsilon)^k$ and $(-\epsilon, \epsilon)^{n-k}$, we already obtain the statement of the proposition. ∎

In fact, connected neat submanifolds have a very useful property which distinguishes them among the general immersed submanifolds. We start with the following observation:

Lemma 2.10. *Let $i : S \to M$ be a connected neat submanifold. Let $\psi : E^1 \times F^1 \to U$ be the box for S as given by the definition of neat submanifolds. Then $S \cap U$ consists of countably many connected components, each of them is open in S and embedded in M.*

Proof. By definition, we have $U \cap S = \psi(E^1 \times A)$, where $A = \{t \in F^1 \mid \psi(0, t) \in S\}$. For each $t \in A$, we have $S_t := \psi'_t(E^1) \subseteq S$. We will now claim that S_t are exactly the connected components of $U \cap S$.

By assumption, each S_t is an open connected subset of $U \cap S$. It is thus entirely contained in the connected component $W \subseteq U \cap S$ containing the point $\psi(0, t)$. It suffices to show that it is closed in $U \cap S$. As ψ is a diffeomorphism onto U, $S'_t = \psi(E^1 \times \{t\}) \subseteq U$ is closed in U: $S'_t = U \cap C$ for $C \subseteq M$ closed. Then $S_t = S'_t \cap S = (U \cap S) \cap (C \cap S)$. As $C \cap S$ is closed in S, we have just proved that S_t is closed in $U \cap S$. But this shows that $S_t = W$.

Conversely, let W be a connected component of $U \cap S$. Let $w \in W$ be arbitrary. There is thus $(e, t) \in E^1 \times A$, such that $w = \psi(e, t)$. But by the preceding paragraph, necessarily $W = S_t$.

We can thus write $(S \cap U) = \sqcup_{t \in A} S_t$. We have assumed that S is connected, whence second countable. As $S \cap U$ is its open submanifold, it is also second countable. In particular, it has at most countably many connected components. In fact, we have just shown that A itself is at most countable.

By definition each S_t is open in S. It remains to prove that it is embedded in M. But $\varphi = \psi^{-1} : U \to E^1 \times F^1$ defines a local coordinate chart on U, such that S_t corresponds exactly to the slice $E^1 \times \{t\}$. This proves that each of the connected components of $U \cap S$ is embedded in M. This finishes the proof. ■

We can now use this lemma to prove the following important observation.

Proposition 2.11. *Connected neat submanifolds are weakly embedded.*

Proof. Let $\varphi : N \to M$ be any smooth map, such that $\varphi(N) \subseteq S$. To show that S is weakly embedded, we must prove that the unique map $\varphi' : N \to S$ defined by

$$\varphi = i \circ \varphi' \tag{71}$$

is smooth. Let $n \in N$ and set $s = \varphi(n)$. Let $\psi : E^1 \times F^1 \to U$ be the box centered at s. Find arbitrary connected neighborhood B of n, such that $\varphi(B) \subseteq U$. We can thus form a proposition $\mu := \pi_2 \circ \psi^{-1} \circ \varphi$, where $\pi_2 : E^1 \times F^1 \to F^1$ is the projection. Clearly $\mu : B \to F^1$ is smooth and $\mu(B)$ is a connected subset of F^1. As $\varphi(N) \subseteq S$, we have $\mu(B) \subseteq A$. From the previous lemma, A is at most countable. But the only at most countable connected subset of F^1 is a single point: $\mu(B) = \{t_0\}$. In fact, as $\varphi(n) = s$, we have $t_0 = 0$.

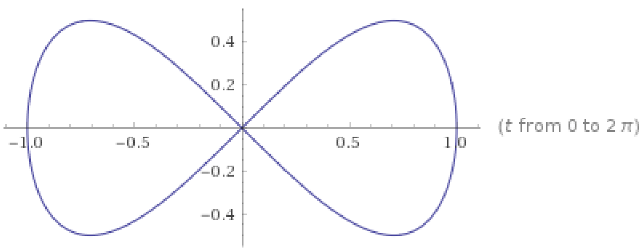
We have just proved that $\varphi(B) \subseteq S_0 \equiv \psi_0'(E^1)$. By previous lemma, S_0 is an open subset of S and an embedded submanifold of M. It follows that $\varphi|_B : B \to M$ is a smooth map taking values in the embedded submanifold S_0. It follows that $\varphi'|_B : B \to S_0$ is smooth. As S_0 is open submanifold of S, it $\varphi'|_B$ is smooth as a map from B to S.

We have just shown that each point $n \in N$ has a neighborhood B, such that $\varphi'|_B$ is smooth. Whence $\varphi' : N \to S$ is smooth. ■

Example 2.12 (Some submanifolds are feral). Consider $i : (0, 2\pi) \to \mathbb{R}^2$ given by

$$i(t) = (\sin(t), \cos(t)\sin(t)) \tag{72}$$

This is the infamous *figure eight curve.* It looks like this:



It is easy to see that i is the injective immersion. Clearly, i is not an embedding as an image of an open subinterval of $(0, 2\pi)$ does not need to be open in the subspace topology. The submanifold S is connected.

We will now show that it is not weakly embedded, hence not neat. Consider the smooth map $\varphi : \mathbb{R} \to \mathbb{R}^2$ defined by $\varphi(t) = (\sin(t), \cos(t)\sin(t))$ for all $t \in \mathbb{R}$. Clearly $\varphi(\mathbb{R}) \subseteq S$. However, the induced map $\varphi' : \mathbb{R} \to (0, 2\pi)$ is not even continuous. Indeed, consider the connected set $B = (\frac{3\pi}{2}, \frac{5\pi}{2})$. Then $\varphi'(B) = (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$, which is not connected.

Remark 2.13. The previous example can be used to demonstrate a funny non-pleasant property of feral immersed submanifolds. Let $i : S \to M$ be an immersed submanifold and let $\varphi : N \to M$ be a smooth map, such that $\varphi(M) \subseteq S$. Let $X \in T_n N$ be a vector tangent to a curve $\gamma : I \to N$, $X = \dot{\gamma}(0)$. Then $\varphi_*(X) \in T_{\varphi(n)} M$ is a vector tangent to the curve $\varphi \circ \gamma$. As $\varphi(\gamma(t)) \in S$ for all $t \in I$, one would naively expect $\varphi_*(X)$ to be tangent to the submanifold S: $\varphi_*(X) \in T_{\varphi(x)} S$.

However, this is simply not true. Consider the example above. Consider the tangent vector $X = \partial_t|_{2\pi} \in T_{2\pi}\mathbb{R}$. It is tangent to the smooth curve $\gamma(s) = 2\pi + s$, where $s \in (-1, 1)$. We have $\varphi(2\pi) = (0, 0)$ and one easily finds that (using the pedestrian notation for tangent vectors):

$$\varphi_*(X) = (1, 1) \tag{73}$$

But the tangent space $T_{(0,0)} S$ is one-dimensional and spanned by the vector tangent vector $(-1, 1)$. This shows that $\varphi_*(X) \notin T_{(0,0)} S$. The reason is of course that the curve $\varphi(\gamma(s))$ is not smooth as viewed as a map from $(-1, 1)$ to $(0, 2\pi)$.

Weakly embedded submanifolds have one more advantage. Unlike for general immersed submanifolds, *both* their topology and smooth structure is uniquely determined.

Proposition 2.14. *Let $i : S \to M$ be a weakly embedded submanifold. Then there is a unique topology and smooth structure such that S has this property.*

Proof. Let $\widetilde{S}$ denote the same set S with a possible different smooth structure and topology, and let $\widetilde{i} : \widetilde{S} \to M$ be the map i. As S is weakly embedded, it follows that the identity $1 : \widetilde{S} \to S$ is smooth. Replacing the role of S and $\widetilde{S}$, the identity $1 : \widetilde{S} \to S$ is smooth. It follows that $1 : \widetilde{S} \to S$ is a diffeomorphism and both smooth structures (and topologies) coincide. ∎

2.2 Arrows and accessible sets

We will now consider the special class of local diffeomorphism of a manifold. It is motivated by properties of vector field flows - however, it is slightly more general.

Definition 2.15. Let M be a smooth manifold. A map $a : \mathcal{D} \to M$ is called the **arrow**, if

(i) $\mathcal{D} \subseteq M \times \mathbb{R}$ is open subset, such that for each $m \in M$, the open subset

$$I_m = \{t \in \mathbb{R} \mid (m, t) \in \mathcal{D}\} \tag{74}$$

is either empty or an interval containing 0.

(ii) For every $t \in \mathbb{R}$, let $M_t = \{m \in M \mid (m, t) \in \mathcal{D}\}$. Then the map $a_t : M_t \to M$ defined by $a_t(m) = a(m, t)$ for all $m \in M_t$ is a diffeomorphism onto an open subset of M.

Finally, $a_0 : M_0 \to M$ is an identity on M_0.

We see that arrows provide a generalization of local flows of vector fields, where we do not impose any composition rules. Clearly, for any $X \in \hat{\mathfrak{X}}(M)$, $\phi^X : \mathcal{D}_X \to M$ is an example of an arrow. Clearly, for each $m \in M$, we have a curve $t \mapsto a_t(m)$ whose tangent vector at $a_t(m)$ is denoted as $\dot{a}_t(m)$. For each $(m, t) \in \mathcal{D}$, we also have a linear isomorphism of the respective tangent spaces: $T_m(a_t) : T_m M \to T_{a_t(m)} M$.

Now, let A be a collection of arrows. We form the following sets.

1. Let $\theta(A)$ be the set of all local diffeomorphisms induced by all arrows of A.
2. Let $\Psi(A)$ be the set containing the identity on M and all local diffeomorphisms of the form $\varphi_1 \circ \cdots \circ \varphi_p$ where p is an arbitrary positive integer and for all $i \in \{1, \ldots, p\}$ either φ_i or its inverse belong to $\theta(A)$.

We can now define a new equivalence relation on M. We say that $x \sim_A y$ if there is an element $\varphi \in \Psi(A)$, such that $y = \varphi(x)$. Equivalence classes of $\sim_A$ are called the **accessible sets of** A**.** For each $x \in M$, we can form two linear subspaces of T_xM, namely

$$A(x) = \mathbb{R}\{\dot{a}_t(y) \mid a \in A \text{ and } x = a_t(y)\}, \tag{75}$$

$$\bar{A}(x) = \mathbb{R}\{T_y(\varphi)(X) \mid \varphi \in \Psi(A),\ x = \varphi(y) \text{ and } X \in A(y)\}. \tag{76}$$

Let $\varphi \in \Psi(A)$, let $y = \varphi(x)$. Then clearly $(T_x\varphi)(\bar{A}(x)) = \bar{A}(y)$. It follows that whenever $x \sim_A y$, one has $\dim(\bar{A}(y)) = \dim(\bar{A}(x))$, that is the dimension of $\bar{A}(x)$ is constant along the accessible sets. Let us now formulate two main theorems of Štefan from [6].

Theorem 2.16. *Let $\mathcal{F}$ be a partition of M into the accessible sets of A. Then $\mathcal{F}$ is a singular foliation of M and for every $x \in M$, $T_x(\mathcal{F}) = \bar{A}(x)$. In particular, every accessible set of A is a leaf of $\mathcal{F}$, thus having a unique topology and a smooth structure making it into a connected neat submanifold of M.*

Let $\sim$ be any equivalence relation on M. We say that a local diffeomorphism φ **preserves** $\sim$, if $\varphi(x) \sim x$ whenever $x \in \mathrm{Dom}(\varphi)$. We say that φ **respects** $\sim$ if $\varphi(x) \sim \varphi(y)$ whenever $x \sim y$ and $x, y \in \mathrm{Dom}(\varphi)$. One says that an arrow a preserves (or respects) $\sim$ if any $\varphi \in \theta(\{a\})$ preserves (or respects) $\sim$. For example, given a family of arrows A, every $a \in A$ both preserves and respects the equivalence relation $\sim_A$.

Theorem 2.17. *Let $\sim$ be an equivalence relation on M and let A be the collection of all the arrows of M which preserve $\sim$. If φ is a local diffeomorphism of M which respects $\sim$, then φ is a local diffeomorphism of $\mathcal{F}$ given by the foliation of M into the accessible sets of A.*

2.3 Homogeneous and symmetric envelopes

A collection A of arrows on M is called to be **homogeneous** if $A(x) = \bar{A}(x)$ for every $x \in M$. In other words, we have $(T_x\varphi)(A(x)) \subseteq A(y)$ whenever φ or φ^{-1} belong to $\theta(A)$ and $y = \varphi(x)$. We say that A is **symmetric** if $\varphi \in \theta(A)$ implies that φ^{-1} is a composition of finitely many members of $\theta(A)$. Let $A_0(x)$ denote the subspace of T_xM spanned by the tangents to the curves $a_t(x)$ at $t = 0$, of all arrows $a \in A$ where $(x, 0) \in \mathrm{Dom}(a)$.

Proposition 2.18. *If A is any collection of arrows on M, there exists a symmetric homogeneous collection of arrows B, such that the accessible sets of A and B coincide and $\bar{A}(x) = B(x) = B_0(x)$ for every $x \in M$.*

Lemma 2.19. *Let $A^{\#}$ be the collection of all arrows a', such that $a'_t = a_{s+t} \circ (a_s)^{-1}$ or $a'_t = a_{s-t} \circ (a_s)^{-1}$ for some arrow $a \in A$ and some $s \in \mathbb{R}$. Then*

(a) $\Psi(A^{\#}) = \Psi(A)$*;*

(b) $A^{\#}$ *is symmetric;*

(c) $A_0^{\#}(x) = A^{\#}(x) = A(x)$ *for every* $x \in M$*;*

(d) *The accessible sets of A and $A^{\#}$ are the same;*

(e) *If A is homogeneous, then so is $A^{\#}$.*

Proof. Obviously, (d) follows at once from (a), whereas (e) follows from the combination of (a) and (c). We thus only have to prove (a) - (c). Let $b_t = a_{s+t} \circ (a_s)^{-1}$ and $c_t = a_{s-t} \circ (a_s)^{-1}$ be the samples of two elements of $A^{\#}$. We have to prove that b and c are indeed arrows.

First, we must analyze the set $\mathcal{D}^b \subseteq M \times \mathbb{R}$ where b is defined. For any given $s \in \mathbb{R}$, we compose $a : \mathcal{D} \to M$ with the following map. By definition $a_s : M_s^a \to U_s^a$ is a diffeomorphism onto an open subset U_s^a. We define a smooth map $\chi_s : U_s^a \times \mathbb{R} \to M \times \mathbb{R}$ by setting

$$\chi_s(m, t) = (a_s^{-1}(m), s + t) \tag{77}$$

We can thus write $b = a \circ \chi_s$. Indeed, we have $b_t(m) = a(a_s^{-1}(m), t+s) = a_{t+s}(a_s^{-1}(m))$. Then the domain $\mathcal{D}^b$ can be written as a preimage $\mathcal{D}^b = \chi_s^{-1}(\mathcal{D}^a)$ which is clearly open in $U_s^a \times \mathbb{R}$, whence in $M \times \mathbb{R}$. It remains to prove that for each $m \in M$, the open set $J_m = \{t \in \mathbb{R} \mid (m, t) \in \mathcal{D}^b\}$ is empty or an open interval containing 0. But this is clear, as for each $t \in J_m$, there is $\epsilon > 0$, such that $(-\epsilon, t + \epsilon) \subseteq J_m$, where we use the definition of the arrow a. Clearly $b_0 : U_s^a \to M$ is the identity and thus b is an arrow. The proof for c is analogous.

By considering $s = 0$, we obtain $b_t = a_t$, and by considering $t = s$, this gives us $c_s = (a_s)^{-1}$. This gives us the inclusions $\theta(A) \subseteq \theta(A^\sharp) \subseteq \Psi(A)$ and $\theta(A)^{-1} \subseteq \theta(A^\sharp)$.

We want to prove (a). As $\theta(A) \subseteq \theta(A^\sharp)$, we easily obtain the inclusion $\Psi(A) \subseteq \Psi(A^\sharp)$. But the other inclusion follows from the observation that $\theta(A^{\#})^{-1} \subseteq \Psi(A)$. Whence $\Psi(A) = \Psi(A^{\#})$.

It is easy to see that $A^{\#}$ is symmetric, as inverse of $\varphi \in \theta(A^{\#})$ is product of elements in $\theta(A)$ and their inverses - both of them are in $\theta(A^{\#})$. Whence (b) is proved.

Now, let $x \in M$ be arbitrary. Suppose $x = b_t(y)$ for $(y, t) \in \mathcal{D}^b$ of some arrow $b \in A^{\#}$. By definition, there is $s \in \mathbb{R}$ and $a \in A$, such that $b_t(y) = a_{t+s}(a_s^{-1}(y))$. One has

$$\dot{b}_t(y) = \dot{a}_{t+s}(a_s^{-1}(y)). \tag{78}$$

But this proves that the generator of $A^{\#}(x)$ in the form $\dot{b}_t(y)$ is also in $A(x)$. Similar proof shows that the other generator $\dot{c}_t(y)$ too. Whence $A^{\#}(x) \subseteq A(x)$. The other inclusion follows from the fact that $\theta(A) \subseteq \theta(A^{\#})$ and we conclude that $A(x) = A^{\#}(x)$ for any $x \in M$. Moreover, we have $\dot{b}_0(y) = \dot{a}_s(a_s^{-1}(y))$. This proves the inclusion $A(x) \subseteq A_0^{\#}(x)$. Whence $A^{\#}(x) \subseteq A_0^{\#}(x)$. The other inclusion is trivial and (c) follows. ■

Lemma 2.20. *Let A^* be a collection of all the arrows a of M such that*

(i) *the domain of a is of the form $V \times J$, where $J \subseteq \mathbb{R}$ is an open interval and $V \subseteq M$ is an open subset of M;*

(ii) *there exists $b \in A^{\#}$ and $\varphi \in \Psi(A)$ such that for $(x, t) \in V \times J$, one has*

$$a(x, t) = (\varphi \circ b_t \circ \varphi^{-1})(x). \tag{79}$$

Then the following properties stand true:

(a) *every $\varphi \in \Psi(A^*)$ is a restriction of some map in $\Psi(A)$;*

(b) *the accessible sets of A^* and A are the same;*

(c) $A^*(x) = \bar{A}(x)$ *for all* $x \in M$*;*

(d) A^* *is an homogeneous set of arrows.*

Proof. The assertion (d) immediately follows from (a) and (c). Note that arrows A^* are just the various composition of the original arrows and their inverses with their domain eventually restricted to be the product $V \times J$. This proves (a). As we can always consider $\varphi = 1$, we see also (b). Next, the inclusion $A^*(x) \subseteq \bar{A}(x)$ is obvious. It thus remains to show that $\bar{A}(x) \subseteq A^*(x)$.

Let $W \in A(y)$, $x = \varphi(y)$ and $\varphi \in \Psi(A)$. We have to show that $(T_y\varphi)(W) \in A^*(x)$. But by Lemma 2.19, we may write W as $\dot{b}_0(y)$ for some $b \in A^\#$. It follows that there exists a neighborhood $(-\delta, \delta)$ for some $\delta > 0$, such that $a_t = \varphi \circ b_t \circ \varphi^{-1}$ is for all $t \in (-\delta, \delta)$ defined on some open neighborhood of x. Then $(T_y\varphi)(W) = \dot{a}_0(x)$. As $a \in A^*$, we have $\dot{a}_0(x) \in A^*(x)$. ■

Proof of Proposition 2.18. Simply take $B = (A^*)^\#$. ■

This proposition is quite important, as for a given family of arrows, we can always switch to the one with the same accessible sets, which has much better properties (it is homogeneous and symmetric). The only cost is that we have to enlarge the tangent space from $A(x)$ to $\bar{A}(x)$.

2.4 Lemmas and proofs

The proof of the main theorem 2.16 is based on the following lemma.

Lemma 2.21. *Let* L *be a subset of* M*. For any* $x \in L$*, let* $L(x)$ *be a vector subspace of* T_xM*. Suppose* $\dim(L(x)) = k$ *for all* $x \in L$ *and for each* $x \in L$*, there exists a diffeomorphism* $\psi : V \times W \to U$ *onto an open subset* $U \subseteq M$*, such that:*

(a) V *and* W *are connected open neighborhoods of the origin in* $\mathbb{R}^k$ *and* $\mathbb{R}^{n-k}$*, respectively;*

(b) $\psi(0, 0) = x$*;*

(c) $L \cap U = \psi(U \times A)$*, where* $A = \{s \in W \mid \psi(0, s) \in L\}$*.*

(d) $T_{(t,s)}\psi$ *maps the subspace* $\mathbb{R}^k \times \{0\} \subseteq T_{(t,s)}(V \times W)$ *into* $L(\psi(t, s))$ *for all* $(t, s) \in \psi^{-1}(L)$*.*

Then there exists a unique topology and smooth structure on L*, such that*

(i) L *becomes a neat submanifold of* M *and for all* $x \in L$*, one has* $T_xL = L(x)$*;*

(ii) Every smooth map $\varphi : N \to M$ *satisfying* $\varphi(N) \subseteq L$ *and* $\varphi_*(T_nN) \subseteq L(\varphi(n))$ *is smooth as a map* $\varphi : N \to L$*.*

Proof. First, observe that connected neat submanifolds are weakly embedded, see Proposition 2.11. However, in general they can have, according to Štefan's definition of a submanifold, uncountably many connected components. In this case, some strange things can happen. Fortunately, this is saved by a weaker version of the weak embedding (ii). In particular, it proves the uniqueness of a smooth structure on L.

Glancing back, it is clear that it only remains to show that ψ satisfies the condition (d) of Definition 2.2. Any map $\psi : V \times W \to U$ satisfying (a), (c) and (d) is called the **privileged chart of** M. We will prove the Lemma in stages:

(**A**) Let $\psi : V \times W \to U$ be a privileged chart of M and let $\varphi : N \to M$ be a smooth map from a connected manifold N into M, such that $\varphi(N) \subseteq L \cap U$. Moreover, assume that $\varphi_*(T_n N) \subseteq L(\varphi(n))$ for all $n \in N$. Then there exists a constant $w \in W$, such that $f(N) \subseteq \psi(V \times \{w\})$.

To prove this, let $\pi_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the projection onto the second factor, and set $g = \pi_2 \circ \psi^{-1}$. We have $(g \circ \psi)(t,s) = s$. Let $e_i \in \mathbb{R}^k$ be the element of a standard basis, viewed as a vector in $\mathbb{R}^k \times \{0\} \subseteq T_{(t,s)}(V \times W)$. We thus have

$$0 = (T_{(t,s)}(g \circ \psi))(e_i) = (T_{\psi(t,s)} g) \circ (T_{(t,s)} \psi)(e_i). \tag{80}$$

For $\psi(t,s) \in L$, we have by assumption $(T_{(t,s)}\psi)(e_i) \in L(\psi(t,s))$. As ψ is a diffeomorphism, this set of vectors generates $L(\psi(t,s))$. But this proves that $(T_x g)(L(x)) = 0$ for any $x \in U \cap L$. In particular, the assumptions on φ give us that $T_n(g \circ \varphi) = 0$ for all $n \in N$. As N is connected, this can happen if and only if $g \circ \varphi$ is constant. This proves the claim (A).

(**B**) Let $\mathbf{\Psi}$ be the collection of all the functions of the form $\psi_w = \psi(\cdot, w) : V_\psi \to L$, where $\psi : V_\psi \times W_\psi \to U_\psi$ is some privileged chart on M, and $w \in A_\psi = \{w \in W_\psi \mid \psi(0,w) \in L\}$. Let $\varphi : N \to M$ be a smooth map, such that $\varphi(N) \subseteq L$ and for all $n \in N$, one has $\varphi_*(T_n N) \subseteq L(\varphi(n))$. If $\psi_w \in \mathbf{\Psi}$, then $G = \varphi^{-1}(\psi_w(V_\psi))$ is an open subset of N, and $(\psi_w)^{-1} \circ \varphi : G \to \mathbb{R}^k$ is smooth.

Indeed, let us consider the open subset $B = \varphi^{-1}(U_\psi)$. Then each its connected component B_0 together with the restriction of $\varphi : B_0 \to M$ satisfies the assumptions of (A) and it is thus mapped into a single slice $\psi(V_\psi \times \{w_0\})$. It follows that G is a collection of connected components of B, whence an open subset of N. Moreover, we can write $(\psi_w)^{-1} \circ \varphi = \pi_1 \circ \psi^{-1} \circ \varphi|_G$, where $\pi_1 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ is the projection.

(**C**) We declare $\mathbf{\Psi}$ to be the smooth atlas for L.

First, we have to check that it is indeed an atlas. Let $\psi'_z \in \mathbf{\Psi}$ be arbitrary, such that $\psi'_z(V_{\psi'}) \cap \psi_w(V_\psi) \neq \emptyset$. Then $\psi'_z : V_{\psi'} \to M$ satisfies the assumptions of (B), based on the requirement (d) of this lemma. But then $G = \psi'^{-1}_z(\psi_w(V_\psi))$ is open in $V_{\psi'}$ and $(\psi_w)^{-1} \circ \psi'_z : G \to \mathbb{R}^k$ is smooth. Whence the transition maps are smooth. Moreover, let $i : L \to M$ be the inclusion. By construction, we can compose $\psi^{-1} \circ i \circ \psi_w$, which is just an inclusion of $V_\psi \times \{w\}$ into $V_\psi \times W_\psi$, whence i is a smooth injective immersion.

To fit into the usual definitions of manifold, we have to argue that each connected component L_0 of L is Hausdorff and second countable. The Hausdorff property is easy to see. If $l \neq l'$ are two distinct points of L, we find two privileged charts ψ and ψ' around l and l', respectively, by the main assumption of this lemma. However, as l and l' are distinct in the Hausdorff space M, we can separate them in M. By intersecting the ranges of ψ and ψ' with these separating neighborhoods, we obtain two privileged charts $\bar{\psi}$ and $\bar{\psi}'$ with disjoint images around l and l', respectively. It follows that there are some $\bar{\psi}_w \in \mathbf{\Psi}$ and $\bar{\psi}'_{w'} \in \mathbf{\Psi}$ with disjoint images containing l and l', respectively.

But in fact, there is a beautiful little observation (obviously known to Štefan, who comments on it in [2]), which we state here as a lemma:

Lemma 2.22. *Every connected immersed submanifold of a second countable manifold is necessarily second countable.*

Proof. Let $i : S \to M$ be the smooth injective immersion. As M is assumed second countable, there exists a Riemannian metric g on M. We can pullback g using i to form a Riemannian metric $h = i^*(g)$ on S. This makes S into a metrizable space, that is there is a topological metric $\rho_h : S \times S \to [0,\infty)$ making (S, ρ_h) into the metric space, where the usual open ball topology induced by ρ_h coincides with the original topology on S. Note that S is then also automatically Hausdorff.

It is a well-known fact that metric spaces are paracompact, see e.g. [7]. On the other hand, every connected locally Euclidean Hausdorff space is paracompact if and only if it is second countable, see the remark under Theorem 1.15 in [8]. Whence S is second countable. ■

(**D**) L is a neat submanifold with the property $T_x L = L(x)$ for all $x \in L$. The latter statement is clear and follows from the discussion below (80). Clearly, the maps $\psi : V \times W \to U$ form the box charts for L, possibly after modification making V and W into open balls.

(**E**) L satisfies the property (ii). This follows from the fact that $\mathbf{\Psi}$ is the smooth atlas for L and observations made in part (B) of the proof. ■

The proof of Theorem 2.16. Without the loss of generality, by Proposition 2.18, we may assume that A is symmetric, homogeneous and $A(x) = A_0(x)$ for all $x \in M$.

We may thus choose $a^i \in A$, so that $(\dot{a}^i_0(x))^k_{i=1}$ forms a basis of $A(x)$. Let

$$\mathbf{\Phi}(t_1, \dots, t_k, y) = a^1_{t_1} \circ a^2_{t_2} \circ \dots \circ a^k_{t_k}(y). \tag{81}$$

We may possibly restrict the domain of this map to be of the form $V \times W$, where V is a connected open neighborhood of the origin in $\mathbb{R}^k$ and W is an open neighborhood of x. By definition, $\mathbf{\Phi}$ is a smooth map. First, note the following two obvious properties:

(i) For every $t \in V$ and $y \in W$, one has $\mathbf{\Phi}(t, y) \sim_A y$ and $\mathbf{\Phi}(0, y) = y$.

(ii) $(T_{(0,x)}\mathbf{\Phi})(e_i) = \dot{a}^i_0(x)$ for all $i \in \{1, \dots, k\}$.

Naturally, the property *(ii)* is not enough, we need the following:

(iii) For every $t \in V$ and $y \in W$, one has $(T_{(t,y)}\mathbf{\Phi})(e_i) \in A(\mathbf{\Phi}(t, y))$.

But this in fact follows from the fact that A is homogeneous. Indeed, consider an example of $i = 2$. One has $(T_{(t,y)}\Phi)(e_2) = (T_z a^1_{t_1})(\dot{a}^2_{t_2}(w))$, where $w = a^3_{t_3} \circ \dots \circ a^k_{t_k}(y)$ and $z = a^2_{t_2}(w)$ and $\Phi(t, y) = a^1_{t_1}(z)$. By definition, one has $\dot{a}^2_{t_2}(w) \in A(z)$ and thus $(T_{(t,y)}\mathbf{\Phi})(e_2) \in \bar{A}(\mathbf{\Phi}(t, y)) = A(\mathbf{\Phi}(t, y))$. This proves the claim (iii).

Finally, let $n = \dim(M)$. Find an arbitrary $(n-k)$-dimensional (embedded) submanifold Q, such that $T_x M = T_x Q \oplus A(x)$. Find any local coordinate chart $\mu : \bar{W} \to Q$, such that $\mu(0) = x$ and $\mu(\bar{W}) \subseteq W$. Finally, define $\psi : V \times \bar{W} \to M$ as $\psi(t, s) = \Phi(t, f(s))$ for all $(t, s) \in V \times \bar{W}$. Clearly, the tangent map to ψ at $(0, 0)$ is a linear isomorphism, whence a local diffeomorphism. By sufficiently shrinking V and $\bar{W}$ we obtain a chart for M satisfying the conditions (a) – (d) of Lemma 2.21 for $L(x) = A(x)$ and L by *arbitrary accessible set* of A.

There is thus a unique structure of a neat submanifold on any accessible set L. One has to prove that it is connected. But this is clear, as L is clearly path connected as a subset of M (by paths made out arrow maps). Each of this paths has values in L and by property (ii) of smooth structure on L constructed in Lemma 2.21 it is continuous as a map into L. Whence also L is path connected.

Finally, we have to argue why $\mathcal{F}$ is a singular foliation, that is the collection of all accessible sets is a tame submanifold of M, such that $i(\mathcal{F}) = M$. Let $\psi : V \times \bar{W} \to U$ be the box around $x \in M$ constructed as above. As noted in Remark 2.8, the set $A \subseteq \bar{W}$ in the definition of the tame submanifold is in this case entire $\bar{W}$. We thus have to prove that the map $\psi'_s = \psi(\cdot, s) : V \to M$ is smooth as a map into $\mathcal{F}$ for all $s \in \bar{W}$. However, it is clear from the construction of ψ, that $\psi'_s(V)$ is contained in an accessible set L' of A through the point $\psi(0, s)$. Moreover, by the property (iii) above, one has $\psi'_{s*}(T_t V) \subseteq A(\psi'_s(t))$ for all $t \in V$. But L' satisfies the property (ii) in Lemma 2.21, which proves that $\psi'_s : V \to L'$ is smooth. L' being a connected component of $\mathcal{F}$, we have just proved the final statement. ∎

Lemma 2.23. *Let $\sim$ be an arbitrary equivalence relation on M and let A be the set of all arrows which preserve $\sim$. Then A is symmetric and homogeneous and $A(x) = A_0(x)$ for every $x \in M$.*

Proof. In view of notation in Subsection 2.3, we already know that $A \subseteq A^{\#}$. We thus have to argue that $A^{\#} \subseteq A$. By construction, any $\varphi \in \Psi(A)$ preserves $\sim$. By Lemma 2.19 we have $\Psi(A^{\#}) = \Psi(A)$. This means that for any arrow $a \in A^{\#}$ also preserves $\sim$. Whence $A^{\#} \subseteq A$. Whence $A = A^{\#}$ and A is symmetric.

Next, by Lemma 2.20, every $\varphi \in \Psi(A^*)$ is a restriction of a map in $\Psi(A)$. As all maps in $\Psi(A)$ preserve $\sim$, it follows that $A^* \subseteq A$. Note that although in general $A \subsetneq A^*$ as not every arrow in A has a domain in the form of a Cartesian product. However, we have the equality $A^*(x) = \bar{A}(x)$. This is enough, as we then have $A(x) \subseteq \bar{A}(x) = A^*(x) \subseteq A(x)$, where the leftmost inclusion is trivial (by definition), the second one is Lemma 2.20 and the last one is the consequence of $A^* \subseteq A$. Whence $A(x) = \bar{A}(x)$ and A is homogeneous.

The last assertion follows from Lemma 2.19 as $A_0(x) = A^{\#}(x) = A(x)$. ∎

We can proceed to the proof of the second of the Štefan's theorem.

The proof of Theorem 2.17. Let φ be a local diffeomorphism which respects $\sim$. Let $x \in \mathrm{Dom}(\varphi)$. Let L be the accessible set of A through x, and let L' be the accessible set of A through $\varphi(x)$. Let $k = \dim(L)$ and let $a^1, \dots, a^k$ be the arrows of A such that

$$A(x) = \mathbb{R}\{\dot{a}^1_0(x), \dots, \dot{a}^k_0(x)\}. \tag{82}$$

By assumption, the arrows b^i defined by $b^i_t = \varphi \circ a^i_t \circ \varphi^{-1}$ are again in A. This is obvious. Let $\mathbf{\Phi}$ be the smooth map defined as in (81). Let $\Phi_x = \Phi(\cdot, x)$. This is a local diffeomorphism from $\mathbb{R}^k$ to L. For all $(t_1, \dots, t_k) \in \mathrm{Dom}(\Phi_x)$, we may write

$$\varphi(y) = (b^1_{t_1} \circ \cdots \circ b^k_{t_k})(\varphi(x)). \tag{83}$$

We have thus found an open neighborhood of x in L, such that $\varphi(U) \subseteq L'$. As L' is a leaf of a foliation, whence a weakly embedded submanifold of M, it follows that $\varphi : U \to L'$ is smooth. ∎

3 Gluing together Sussmann and Štefan

Clearly, the work done by Štefan is far more general, as it deals with the families of more general arrows. Moreover, it provides much more detailed properties of the leaves forming the singular foliation. For these reasons, we reformulate the Štefan's definition of a singular foliation into a more conventional language, which will exclude the nowadays uncommon notion of manifolds with possibly uncountably many connected components.

Definition 3.1. Let M be a smooth n-dimensional manifold. We say that a collection $\mathcal{F}$ of connected immersed submanifolds of M forms the **singular foliation** of M if

(i) M can be written as a disjoint union of elements of $\mathcal{F}$, that is every $x \in M$ is contained in exactly one connected immersed submanifold $F_x \in \mathcal{F}$ which is called the **leaf of the foliation $\mathcal{F}$ trough** x.

(ii) For each $x \in M$, there exists a coordinate chart (U, φ), where U is a neighborhood of x and $\varphi : U \to V \times W$, where $V \subseteq \mathbb{R}^k$ and $W \subseteq \mathbb{R}^{n-k}$ are open balls containing their respective origins, such that $\varphi(x) = (0, 0)$ and

   (a) For each $t \in W$, the restriction $\psi_t = \varphi^{-1}|_{V \times \{t\}}$ defines a smooth map $\psi_t : V \to F$, where $F \in \mathcal{F}$ is some leaf of the foliation $\mathcal{F}$.

   (b) For $t = 0$, the smooth map $\psi_0 : V \to F_x$ is a diffeomorphism onto an open subset $\psi_0(V) \subseteq F_x$. In particular, it defines a local coordinate chart for F_x around x.

Lemma 3.2. *Štefan's definition agrees with Definition 3.1.*

Proof. Let $i : \mathcal{F} \to M$ be the foliation according to Štefan, that is a tame submanifold, whose underlying set $i(\mathcal{F}) = M$. Connected components of $\mathcal{F}$ form a collection of connected immersed submanifolds, such that M is their disjoint union. This proves (i) of the above definition.

Next, let $\psi : E^1 \times F^1 \to U$ be the box for M around x. As noted in Remark 2.8, the set A parametrizing the intersection of $\mathcal{F}$ with U is now the entire open ball F^1. Clearly, it suffices to take $\varphi = \psi^{-1}$. By the definition of box, $\psi_t : E^1 \to \mathcal{F}$ is smooth. In particular, as E^1 is connected, it must take values in a single leaf $F \in \mathcal{F}$. For similar reasons, $\psi_0 : V \to F_x$ is a diffeomorphism onto an open subset of F_x. The converse is shown by considering the collection $\mathcal{F}$ as a single immersed submanifold of M with the property $i(\mathcal{F}) = M$, and reversing all the steps to show that it is a tame submanifold of M. ■

We can now specialize the theorem of Štefan and compare it to the theorem of Sussmann. First, we will examine the special set of arrows.

Lemma 3.3. *Let $D \subseteq \hat{\mathfrak{X}}(M)$ be an everywhere defined family of smooth vector fields. Then the collection of maximal flows of D forms a symmetric collection of arrows denoted as A_D. One has $A_D(x) = (A_D)_0(x) = \Delta_D(x)$ for all $x \in M$, and the accessible sets of A_D are precisely the D-orbits. Moreover, A_D is homogeneous if and only if the smooth distribution Δ_D is D-invariant.*

Proof. As already noted, if $\phi^X : \mathcal{D}^X \to M$ is a maximal flow of a vector field $X \in \hat{\mathfrak{X}}(D)$, it is an arrow. As $(\phi^X_t)^{-1} = \phi^X_{-t}$, it follows that A_D is symmetric. Moreover, glancing at the definition of the symmetric envelope $A^{\#}_D$ in Lemma 2.19, we immediately obtain $A_D = A^{\#}_D$ thanks to the composition rule for the flows. In particular $(A_D)_0(x) = A_D(x)$ for all $x \in M$.

Next, $A_D(x) = (A_D)_0(x)$ is a linear vector space spanned by vectors in the form $\dot{\phi}^X_0(x) = X(x)$, where $X \in D_{(x)} \equiv \{Y \in D \mid x \in \mathrm{Dom}(X)\}$. Whence

$$A_D(x) = \mathbb{R}\{X(x) \mid X \in D_{(x)}\} = \Delta_D(x). \tag{84}$$

Moreover, we have that A_D is homogeneous if and only if $(T_x\varphi)(A_D(x)) \subseteq A_D(\varphi(x))$ whenever φ or φ^{-1} belong to $\theta(A_D)$, for all $x \in \mathrm{Dom}(x)$. But this is translated as $(T_x\phi^X_t)(\Delta_D(x)) \subseteq \Delta_D(\phi^X_t(x))$ for any $X \in D$ and $(x,t) \in \mathcal{D}^X$. In other words, A_D is homogeneous if and only if Δ_D is D-invariant. Finally, it is clear from the definitions that the accessible sets of A_D are precisely the D-orbits. ■

Note that $P_D(x) = \bar{A}_D(x)$. We can now reformulate main Theorem 2.16 of Štefan:

Theorem 3.4. *Let $D \subseteq \hat{\mathfrak{X}}(M)$ be an everywhere defined family of vector fields. Let $\mathcal{F}_D$ be a partition of M into the D-orbits of M. Then $\mathcal{F}_D$ is a singular foliation. For any leaf $F \in \mathcal{F}$ and any $x \in F$, one has $T_x F = P_D(x)$. In particular, any D-orbit is a leaf of $\mathcal{F}_D$, thus having a unique topology and a smooth structure.*

Conversely, we may start with M together with a given foliation $\mathcal{F}$. We may ask whether it comes from some smooth distribution via the theorem above.

Proposition 3.5. *Let $\mathcal{F}$ be a smooth singular foliation of M. Then there is a unique smooth distribution $\Delta_{\mathcal{F}}$ such that for any leaf $F \in \mathcal{F}$ and any $x \in F$, one has $T_x F = \Delta_{\mathcal{F}}(x)$. Moreover, $\Delta_{\mathcal{F}}$ has the maximal integral manifolds property. $\Delta_{\mathcal{F}}$ is called the* ***tangent distribution to $\mathcal{F}$****.*

Proof. As for any $x \in M$, there is a unique leaf $F \in \mathcal{F}$ through x, the definition $\Delta_{\mathcal{F}}(x) = T_x F$ is unambiguous and uniquely defines a distribution on M. One only has to prove that it is smooth. By definition, for each $x \in M$, there is a coordinate chart (U, φ) around x with the properties as in (ii) of Definition 3.1. Write the corresponding local coordinates as $(x^1, \dots, x^k, y^1, \dots, y^{n-k})$. Any $v \in \Delta_{\mathcal{F}}(x)$ is a unique linear combination $v = \alpha^i \frac{\partial}{\partial x^i}(x)$. It follows from (ii) and (a) in Definition 3.1 that $V = \alpha^i \frac{\partial}{\partial x^i}$ is a vector field defined on U, such that $V(y) \in \Delta_{\mathcal{F}}(y)$ for all $y \in U$ and $V(x) = v$. Whence $\Delta_{\mathcal{F}}$ is smooth.

Finally, we will argue that leaves of $\mathcal{F}$ are precisely the maximal connected integral submanifolds for $\Delta_{\mathcal{F}}$. By construction, each $F \in \mathcal{F}$ is an integral submanifold for $\Delta_{\mathcal{F}}$. It remains to prove the maximality.

Suppose S is some connected integral submanifold of $\Delta_{\mathcal{F}}$ intersecting the leaf of $\mathcal{F}$ at $x \in M$. Whence $S \cap F_x \neq \emptyset$. We have to show that $S \subseteq F_x$ and it is an open submanifold of F_x. Let $i : F_x \to M$ and $i_S : S \to M$ denote both inclusions.

If we show the first statement, we know that F_x is a weakly embedded submanifold, whence i_S factorizes uniquely as $i_S = i \circ i'_S$, where $i'_S : S \to F_x$ is smooth. It follows that i'_S is an immersion. Moreover, by assumption $T_s S = T_s F_x$ for all $s \in S$, whence it is a local diffeomorphism. But local diffeomorphisms are open maps and $S \equiv i'_S(S) \subseteq F_x$ is an open subset.

It thus remain to show that $S \subseteq F_x$. The strategy is to show that each $s \in S$ has an open neighborhood $B \subseteq S$ which is entirely contained in a single leaf $F_s \in \mathcal{F}$. Let (U, φ) be the foliation chart for $\mathcal{F}$ centered at s. Let $B = U \cap S$. By taking its connected component containing s, we may assume that B is a connected open neighborhood of s in S. Let $\varphi' = \pi_2 \circ \varphi \circ i_S|_B$, where $\pi_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is the projection. Calculating the differential at any $t \in B$, we find the composition

$$T_t(\varphi') = T_t(\pi_2 \circ \varphi) \circ T_t(i_S). \tag{85}$$

By assumption, $T_t(i_S)$ is precisely the subspace $T_t(F_t)$. In particular, note that $\dim(F_t) = \dim(F_s)$ for all $t \in B$. Let $\varphi(t) = (v, w)$. By assumption, ψ_w is a smooth map from $V \subseteq \mathbb{R}^k$ into some leaf of $\mathcal{F}$. As $\psi_w(v) = t$, this must be the leaf F_t. As already noted above Lemma 2.4, ψ_w is an injective immersion, and due to $\dim(F_t) = \dim(V) = k$, it is an injective local diffeomorphism, whence a diffeomorphism onto its image $Z := \psi_w(V) \subseteq F_t$. Note that Z is always an embedded submanifold of M open in F_t, diffeomorphic to the open ball $V \subseteq \mathbb{R}^k$, and $T_t(Z) = T_t(F_t)$. It now follows easily that $\ker(T_t(\pi_2 \circ \varphi)) = T_t(Z)$, and finally, $T_t(\varphi') = 0$. As $\varphi' : B \to \mathbb{R}^{n-k}$ is a smooth map defined on a connected set B and $t \in B$ was arbitrary, we find that $\varphi'(t) = \varphi'(s) = 0$ for all $t \in B$. We see that $\varphi(B) \subseteq V \times \{0\}$. But $V \times \{0\}$ is the φ-image of a connected component of the leaf F_s containing s. Whence $B \subseteq F_s$.

Finally, suppose $x \in S \cap F_x$ and consider arbitrary $y \in S$. As S is assumed connected, there is a continuous path $\gamma : [0, 1] \to S$ connecting x to y. As shown in the previous paragraph, for each $t \in [0,1]$, there is a connected open neighborhood $B_t \subseteq S$ of $\gamma(t)$, such that $B_t \subseteq F_{\gamma(t)}$. By a usual compactness argument, we find a finite subdivision $0 = t_0 < \cdots < t_q = 1$, such that $\gamma(t_i)$ and $\gamma(t_{i+1})$ are in the same leaf of $\mathcal{F}$ for all $i \in \{0, \ldots, q-1\}$. From the transitivity of the relation, we find that $x = \gamma(0)$ and $y = \gamma(1)$ are in the same leaf of $\mathcal{F}$, whence $y \in F_x$. We have just shown that $S \subseteq F_x$ and F_x is a maximal integral submanifold for $\Delta_{\mathcal{F}}$. ■

We can finally prove the main theorem of this section.

Theorem 3.6 (**Global Štefan-Sussmann theorem**). *Let M be a smooth manifold. Then there is a bijection between smooth singular distributions Δ with the maximal integral property and smooth singular foliations $\mathcal{F}$ of M.*

Proof. Let Δ be a smooth distribution with the maximal integral property. There is thus a family D of smooth vector fields spanning Δ, and $\Delta = P_D$ by Theorem 1.13 (f). Its maximal integral submanifolds are precisely the D-orbits. By theorem 3.4, they form a leaves of a singular foliation $\mathcal{F}_D$. Maximal integral submanifolds are uniquely determined by Δ, hence we obtain the same singular foliation for any everywhere defined family D spanning Δ and one can write simply $\mathcal{F}_\Delta$. Conversely, given any foliation $\mathcal{F}$, we obtain its tangent distribution $\Delta_{\mathcal{F}}$ which has a maximal integral property. Finally, we will argue that the maps $\Delta \mapsto \mathcal{F}_\Delta$ and $\mathcal{F} \mapsto \Delta_{\mathcal{F}}$ are in fact inverse to each other.

Let Δ be a smooth distribution with the maximal integral property. Let D be any everywhere defined family of smooth vector fields spanning Δ. Let $x \in M$ be arbitrary and let $F \in \mathcal{F}_\Delta$ be the leaf through x. F is a D-orbit containing x, and one has $T_xF = P_D(x)$. By Theorem 1.13 (f), one has $P_D(x) = \Delta(x)$. But by definition, we have $\Delta_{\mathcal{F}_\Delta}(x) = T_xF$ and thus $\Delta(x) = \Delta_{F_\Delta}(x)$ for all $x \in M$. This finishes the discussion in one direction.

Conversely, let $\mathcal{F}$ be any smooth singular foliation. By Proposition 3.5, the tangent distribution Δ_F is a smooth distribution with the maximal integral property, whose maximal integral submanifolds are precisely the leaves of $\mathcal{F}$. But maximal integral submanifolds are uniquely determined by the distribution; in particular they are D-orbits of any everywhere defined family of smooth vector fields D spanning Δ_F, whence precisely the leaves of a distribution $\mathcal{F}_D = \mathcal{F}_{\Delta_{\mathcal{F}}}$. Two smooth singular distribution with the same leaves are necessarily identical, whence $\mathcal{F} = \mathcal{F}_{\Delta_{\mathcal{F}}}$. This concludes the second direction. ■

Example 3.7. In fact, the construction in Section 2 provides an explicit way to construct the coordinate chart required for the foliation. Let $M = \mathbb{R}^2$

Consider the family D consisting of a single vector field $X = x \cdot \partial_y - y \cdot \partial_x$. It is easy to find its local flow, namely $\phi^X_t(x, y) = (x\cos(t) - y\sin(t), y\cos(t) + x\sin(t))$. The corresponding flow domain $\mathcal{D}^X$ is thus entire $\mathbb{R}^2 \times \mathbb{R}$.

Now, fix a point $m_0 := (x_0, y_0) \in \mathbb{R}^2$. Assume $m_0 \neq (0,0)$. By definition, one has $\Delta(m_0) = \mathbb{R}\{X(m_0)\}$. First, we are supposed to define a map $\mathbf{\Phi} : \mathbb{R} \times \mathbb{R}^2 \to M$ by $\mathbf{\Phi}(t, m) = \phi^X_t(m)$. Let $\mathbb{B}^q_r(m)$ denote the open ball in $\mathbb{R}^q$ of radius r around the point $m \in M$. We restrict the domain of $\mathbf{\Phi}$ to $\mathbb{B}^1_{r'}(0) \times \mathbb{B}^2_r(m_0)$.

Next, we are supposed to choose a submanifold Q containing m_0, such that $T_{m_0}Q \oplus \Delta(m_0) = T_{m_0}M$. As $m_0 \neq (0,0)$, we take Q to be the line through the origin $(0,0)$ and m_0, that is

$$Q = \{\lambda \cdot (x_0, y_0) \mid \lambda \in \mathbb{R}\}. \tag{86}$$

We can now consider the smooth map $\mu : \mathbb{B}^1_q(0) \to Q$ be the map defined as $\mu(s) = (1+s)\cdot(x_0, y_0)$. Clearly $\mu(0) = m_0$. For small enough $q > 0$, one has $\mu(\mathbb{B}^1_q(0)) \subseteq \mathbb{B}^2_r(m_0)$. We then define

$$\psi(t, s) = \mathbf{\Phi}(t, \mu(s)) = \phi^X_t(\mu(s)) = (1 + s) \cdot (x_0 \cos(t) - y_0 \sin(t), y_0 \cos(t) + x_0 \sin(t)). \tag{87}$$

This is a map smooth map from $\mathbb{B}^1_{r'}(0) \times \mathbb{B}^1_q(0)$ to $\mathbb{R}^2$, and we find

$$T_{(0,0)}\psi = \begin{pmatrix} -y_0 & x_0 \\ x_0 & y_0 \end{pmatrix}. \tag{88}$$

This map is supposed to be a linear isomorphism and indeed, we have $\det(T_{(0,0)}\psi) = -(x_0^2 + y_0^2)$ which is non-zero as $m_0 \neq (0, 0)$. It thus suffices to consider $r' > 0$ and $q > 0$ small enough so that ψ is a diffeomorphism onto an open subset in $\mathbb{R}^2$.

References

[1] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, *Transactions of the American Mathematical Society* **180** (1973) 171–188.

[2] P. Štefan, *Integrability of systems of vectorfields*, *Journal of the London Mathematical Society* **2** (1980), no. 3 544–556.

[3] R. Balan, *A note about integrability of distributions with singularities*, *Bollettino della Unione Matematica Italiana-A* (1994), no. 3 335–344.

[4] C. Lobry, *Contro labilite des syste mes non lineaires*, *SIAM Journal on Control and Optimization* **8** (1970), no. 4 573.

[5] R. Hermann, *The differential geometry of foliations, ii*, *Journal of Mathematics and Mechanics* **11** (1962), no. 2 303–315.

[6] P. Štefan, *Accessible sets, orbits, and foliations with singularities*, *Proceedings of the London Mathematical Society* **3** (1974), no. 4 699–713.

[7] M. E. Rudin, *A new proof that metric spaces are paracompact*, *Proceedings of the American Mathematical Society* **20** (1969), no. 2 603.

[8] J. Lee, *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics. Springer, 2012.