Čech Cohomology

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Contents

1 Presheaves and sheaves

These notes are based on my reinterpretation of excellent lecture notes [2]. Let X be given topological spaces. By $\mathbf{Op}(X)$, we denote the category of open subsets of X. Its objects are open subsets of X and morphisms are inclusions. Let $U \subseteq V \subseteq X$ be two open subsets. The unique inclusion morphism is denoted as $i^V_U : U \to V$.

Definition 1.1. Let $\mathbf{C}$ be any category. By a **presheaf** on X with values in $\mathbf{C}$, we mean a contravariant functor $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{C}$. In other words, to each $U \in \mathbf{Op}(X)$, we have an object

$\mathcal{F}(U) \in \mathbf{C}$ and for each inclusion $i_U^V : U \to V$ we have a **restriction morphism** $\rho_U^V = \mathcal{F}(i_U^V) \in \mathbf{C}(\mathcal{F}(V), \mathcal{F}(U))$ satisfying the usual composition relations

$$\rho_U^W = \rho_U^V \circ \rho_V^W \,, \quad \rho_U^U = 1_{\mathcal{F}(U)} \,, \tag{1}$$

where $\circ$ is the composition rule in $\mathbf{C}$ and $1_{\mathcal{F}(U)}$ the identity morphism at $\mathcal{F}(U) \in \mathbf{C}$.

As presheaves are just a special case of contravariant functors, there is an obvious way to define their morphisms.

Definition 1.2. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X with values in $\mathbf{C}$. Then by **presheaf map (or morphism)** $\varphi : \mathcal{F} \to \mathcal{G}$, we mean a natural transormation of the two functors. In other words, for every $U \in \mathbf{Op}(X)$, we have a morphism $\varphi_U \in \mathbf{C}(\mathcal{F}(U), \mathcal{G}(U))$, such that for every $U \subseteq V$, the following diagram of morphisms commutes:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \\ \downarrow{\scriptstyle \mathcal{F}(i_U^V)} & & \downarrow{\scriptstyle \mathcal{G}(i_U^V)} \\ \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \end{array} \tag{2}$$

Usually $\mathbf{C}$ is taken to be some category where it makes sense to talk about the elements of a given object $\mathcal{F}(U)$, that is there is some underlying set $\Gamma(U, \mathcal{F})$. An element $s \in \Gamma(U, \mathcal{F})$ is called a **section of $\mathcal{F}$ above** U. Elements of $\Gamma(X, \mathcal{F})$ are called **global sections** of $\mathcal{F}$. Examples of such categories are Abelian groups **Ab**, Groups **Grp**, sets **Set**, topological spaces **Top**, R-modules R-**Mod**, where R is some fixed commutative ring.

Example 1.3. (i) For any category $\mathbf{C}$, fix an object $G \in \mathbf{C}$. The **constant presheaf G_X on X with values in $G \in \mathbf{C}$** is defined as $\mathcal{F}(U) = G$ for all $U \in \mathbf{Op}(X)$ with $\rho_U^V = 1_G$ for all $U \subseteq V$. Although it may seem stupid, it will play an important role in the following.

(ii) For any topological space $Y \in$ **Top**, one can define a **presheaf C_Y^0) of continuous Y-valued functions** on X. For general Y, one has $C_Y^0(X) : \mathbf{Op}(X) \to \mathbf{Set}$. Set

$$C_Y^0(U) = C^0(U, Y) \equiv \{ f : U \to Y \mid f \text{ is continous} \}. \tag{3}$$

For $U \subseteq V$, the restriction morphism $\rho_U^V : C^0(V, Y) \to C^0(U, Y)$ is an actual restriction of the maps. By imposing further conditions on Y, one can replace **Set** by a more convenient category.

(iii) If X is a smooth manifold, one can define a presheaf C^∞ of smooth $\mathbb{R}$-valued functions as $C^\infty(U) = C^\infty(U, \mathbb{R})$, and restrictions again just restrictions. One can view C^∞ as a functor on X valued in the commutative rings **CRing**.

The presheaf is by its definition very inbalanced, that is we can always go just "more local". One can be sometimes interested in deducing some global properties from the local one. This leads to the following notion:

Definition 1.4. Let $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{C}$ be a presheaf on X with values in a concrete category $\mathbf{C}$. In particular, the objects of $\mathbf{C}$ form actual sets and the notion of local sections makes sense. We say that $\mathcal{F}$ is a **sheaf on X with values in $\mathbf{C}$**, if for any $U \in \mathbf{Op}(X)$ and any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U, that is $U_i \subseteq U$ and $U = \cup_{i \in I} U_i$, we have

(A) **Gluing axiom**: Suppose we have a collection $\{s_i\}_{i\in I}$ of local sections, where $s_i \in \Gamma(U_i, \mathcal{F})$, such that they coincide on the overlaps, that is

$$\rho^{U_i}_{U_{ij}}(s_i) = \rho^{U_j}_{U_{ij}}(s_j) \tag{4}$$

for all $i, j \in I$, where $U_{ij} = U_i \cap U_j$. Note that this does include also the case $U_i \cap U_j = \emptyset$. Then there exists some section $s \in \Gamma(U, \mathcal{F})$ such that $s_i = \rho^X_{U_i}(s)$.

(B) **Monopresheaf axiom**: Suppose $s, t \in \Gamma(U, \mathcal{F})$ are two local sections above U, such that $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$ for all $i \in I$. Then $s = t$.

For any $s \in \Gamma(U, \mathcal{G})$, one often writes $s|_{U_i}$ for $\rho^U_{U_i}(s)$.

There are several remarks in order.

Remark 1.5. (i) For any sheaf $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{C}$, we have a special open subset $\emptyset \subseteq X$. There is thus a particular object $\mathcal{F}(\emptyset) \in \mathbf{C}$. Now, let $s \in \Gamma(\emptyset, \mathcal{F})$. Moreover, an empty set has one distinguished open cover - namely the one containing no sets at all (that is we have $I = \emptyset$). Let $s, t \in \Gamma(\emptyset, \mathcal{F})$ be two local sections above $\emptyset$. The requirement in monopresheaf axiom concerns all $i \in \emptyset$ and is thus always true. Hence $s = t$ for any two local sections. It follows that $\mathcal{F}(\emptyset) = \{*\}$ is a singleton in $\mathbf{C}$. This is terminal object in every concrete category.

(ii) For sheaves valued in R-**Mod** or **CRing**, where each object has a unique zero element 0 and addition and subtraction makes sense, one can replace the condition (B) with the following: Suppose $s \in \Gamma(\mathcal{F}, U)$ satisfies $s|_{U_i} = 0$ for all $i \in I$. Then $s = 0$.

(iii) Again, suppose $\mathcal{F}$ is valued in R-**Mod** or **CRing**. Then both axioms (A) and (B) can be stated as follows. Suppose $U \subseteq X$ is a non-empty subset, and let $U = \cup_{i\in I} U_i$. Define a pair of morphisms $\varphi : \mathcal{F}(U) \to \Pi_{i\in I}\mathcal{F}(U_i)$ and $\psi : \Pi_{i\in I}\mathcal{F}(U_i) \to \Pi_{i,j\in I}\mathcal{F}(U_{ij})$ as

$$\varphi(s) = (\rho^U_{U_i}(s))_{i\in I}, \tag{5}$$

$$\psi((s_i)_{i\in I}) = (\rho^{U_i}_{U_{ij}}(s_i) - \rho^{U_j}_{U_{ij}}(s_j))_{i,j\in I}. \tag{6}$$

Then (A) and (B) are true if and only if the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi} \Pi_{i\in I}\mathcal{F}(U_i) \xrightarrow{\psi} \Pi_{i,j\in I}\mathcal{F}(U_{ij}) \tag{7}$$

is exact. Indeed, the injectivity of φ is precisely the axiom (B). The gluing axiom (A) is exactly the exactness in the in the second term.

Example 1.6. (1) Let us start with the most important example. Suppose $\pi : E \to X$ be a surjective continous map. The sheaf $\Gamma[E, \pi]$ on X is defined as

$$\Gamma[E, \pi](U) = \{s : U \to E \mid \pi \circ s = 1_U,\ s \text{ is continuous}\}. \tag{8}$$

The restriction morphism is just the restriction. It is easy to see that $\Gamma[E, \pi]$ is a **sheaf of sections of** E. Its importance lies in the fact that every sheaf $\mathcal{F}$ on X is isomorphic to this example for a certain $\pi : E \to X$.

(2) Suppose $E = X \times Y$, and let $\pi : X \times Y \to X$ be the canonical projection. Every section $s \in \Gamma[E, \pi](U)$ thus must take the form $s(x) = (x, f(x))$, where $f : U \to Y$ is continous. The assignment $s \mapsto f$ defines a morphism $\varphi_U : \Gamma[E, \pi](U) \to C^0_Y(U)$. It is easy to see that φ_U is natural in U and thus $\Gamma[E, \pi]$ is a presheaf isomorphic to C^0_Y. In particular, C^0_Y is a sheaf.

(3) Suppose $E = X \times Y$, and let $\pi : X \times Y \to Y$ be the canonical projection. This time, Y is an arbitrary set which we can equip with a discrete topology. Again, every section $s \in \Gamma[E, \pi](U)$ must be of the form $s(x) = (x, f(x))$ for some function $f : U \to Y$. It must be continuous with respect to the discrete topology on Y. Let $x_0 \in U$, and $y_0 = f(x_0)$. Then $\{y_0\}$ is an open subset of Y and thus $V = f^{-1}(y_0) \subseteq U$ must be open. Every point x_0 thus has an open neighborhood V, such that $f(V) = \{f(x_0)\}$. These are called **locally constant functions on** U.

    We can thus form a **sheaf** $\widetilde{Y}_X$ **of locally constant functions on** X **with values in** Y:

$$\widetilde{Y}_X(U) = \{f : U \to Y \mid f \text{ is locally constant}\}, \tag{9}$$

    where restrictions morphisms are just restrictions. To confuse everybody, $\widetilde{Y}_X$ is usually called a **constant sheaf on** X **with values in** Y.

    Beware! the constant sheaf $\widetilde{Y}_X$ is *not the constant presheaf* Y_X! In fact, the presheaf Y_X is not a sheaf unless $Y = \{*\}$. This follows from Remark 1.5-(i), as $Y_X(\emptyset)$ must be the singleton and at the same time $Y_X(\emptyset) = Y$.

Definition 1.7. Let $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{C}$ be a presheaf (sheaf). Then for any open set $U \subseteq X$, one may define a **restricted presheaf (sheaf)** $\mathcal{F}|U$ by simply restricting the functor $\mathcal{F}$ to the full subcategory $\mathbf{Op}(U) \subseteq \mathbf{Op}(X)$.

2 Directed limits

Definition 2.1. Let I be a set with a preorder $\leq$ (any transitive and reflexive binary relation). We say I is a **directed set**, if for every $i, j \in I$, there exists $k \in I$, such that $i \leq k$ and $j \leq k$.

We say that the subset $J \subseteq I$ is **cofinal in** I, if for every $i \in I$, there exists some $j \in J$, such that $i \leq j$. Note that J is automatically a directed set. Indeed, let $i, j \in J$. As I is directed, there must be $k \in I$, such that $i, j \leq k$. But as J is cofinal, there must be some $l \in J$, such that $k \leq l$. Whence $i, j \leq l$ for some $j \in J$ and J is directed.

Remark 2.2. It is useful to view I as a category $\mathbf{I}$ whose objects are elements of I and for each $i, j \in I$, the morphism set $\mathbf{I}(i, j)$ consists of a single arrow if $i \leq j$ and is empty otherwise. Identity arrow in $\mathbf{I}(i, i)$ exists due to the reflexivity of $\leq$ and arrows can be composed due to the transitivity of $\leq$.

This viewpoint on the directed sets is useful in the following definition:

Definition 2.3. A **direct mapping family** indexed by I is a covariant functor $F : \mathbf{I} \to \mathbf{C}$, where $\mathbf{I}$ is the category corresponding to the directed set I and $\mathbf{C}$ is an arbitrary category.

Equivalently, a direct mapping family is a collection $\{F_i\}_{i \in I}$ of objects in $\mathbf{C}$ and of morphisms $\{\rho^i_j\}_{i \leq j}$, where $\rho^i_j \in \mathbf{C}(F_i, F_j)$, such that $\rho^i_i = 1_{F_i}$ for all $i \in I$ and $\rho^i_k = \rho^j_k \circ \rho^i_j$ for all $i, j, k \in I$ such that $i \leq j \leq k$. Clearly $F_i = F(i)$ for all $i \in I$ and ρ^i_j is the functor F evaluated on the unique morphism in $\mathbf{I}(i, j)$ for $i \leq j$.

Example 2.4. Let X be a topological space and let $x \in X$. Let $\mathbf{Op}_x(X)$ denote the full subcategory of open sets containing x. Take $\mathbf{I} = (\mathbf{Op}_x(X))^{op}$. For any two open sets $U, V \subseteq X$ containing x, the morphism set $\mathbf{I}(U, V)$ consists of single arrow whenever $V \subseteq U$.

The directed set I corresponding to $\mathbf{I}$ is thus a set of all open subsets containing x with a preorder $U \prec V$ whenever $V \subseteq U$.

Now, let $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{C}$ be any presheaf on X with values in the category $\mathbf{C}$. One can restrict it to a contravariant functor $\mathcal{F}_x : \mathbf{Op}_x(X) \to \mathbf{C}$, and thus define the *covariant* functor $\mathcal{F}_x : (\mathbf{Op}_x(X))^{op} \to \mathbf{C}$. This defines a direct mapping family indexed by I.

Definition 2.5. Let $F : \mathbf{I} \to \mathbf{C}$ be a direct mapping family. We assume that $\mathbf{C}$ is some well-behaved category like R-**Mod** or **CRing** or **Set**, so that the following operations make sense. A **direct limit** $\varinjlim_{i \in I} F_i$ is an object in $\mathbf{C}$ defined by formula

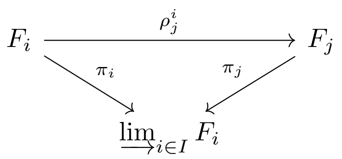
$$\varinjlim_{i \in I} F_i = \bigsqcup_{i \in I} F_i / \sim, \tag{10}$$

where $\bigsqcup$ denotes the disjoint union of the (sets) F_i and the relation $\sim$ is defined as follows. Suppose $f_i \in F_i$ and $f_j \in F_j$. Then $f_i \sim f_j$ if and only if there exists $k \in I$, such that $i \leq k$ and $j \leq k$, such that $\rho^i_k(f_i) = \rho^j_k(f_j)$.

The relation $\sim$ is clearly reflexive and symmetric, one only has to think a bit about transitivity. If $f_i \sim f_j$ and $f_j \sim f_k$, there are thus $m, l \in I$ such that $i, j \leq m$ and $j, k \leq l$. But as I is directed, there is also $r \in I$, such that $m, l \leq r$. We will show that $\rho^i_r(f_i) = \rho^k_r(f_k)$, which would prove the desired claim $f_i \sim f_k$. But one has

$$\rho^i_r(f_i) = \rho^m_r(\rho^i_m(f_i)) = \rho^m_r(\rho^j_m(f_j)) = \rho^j_r(f_j) = \rho^n_r(\rho^j_n(f_j)) = \rho^n_r(\rho^k_n(f_k)) = \rho^k_r(f_k). \tag{11}$$

This proves the transitivity. Let $\epsilon_i : F_i \to \bigsqcup_{i \in I} F_i$ be the canonical inclusion, and let $\pi_i : F_i \to \varinjlim_{i \in I} F_i$ be its composition with the quotient map. Suppose $i \leq j$. We claim that the diagram



(12)

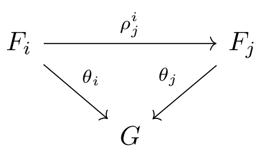
commutes. This amounts to showing $\rho^i_j(f_i) \sim f_i$ for all $f_i \in F_i$. But this is obvious. If we would draw this diagram for all $i \in I$ and connected everything with appropriate arrows, the resulting picture is called a **colimiting cone**, and $\varinjlim_{i \in I} F_i$ is in fact a **colimit of the functor** $F : \mathbf{I} \to \mathbf{C}$.

We have not shown that $\varinjlim_{i \in I} F_i$ is actually an object in $\mathbf{C}$. This is true for certain categories. Let us demonstrate that this is true for $\mathbf{C} =$ R-**Mod**. Let $[f_i]$ and $[f_j]$ be two elements of $\varinjlim_{i \in I} F_i$. As I is directed, there is $k \in I$, such that $i, j \leq k$. Then $\rho^i_k(f_i), \rho^j_k(f_j) \in F_k$ and we may define

$$[f_i] + [f_j] = [\rho^i_k(f_i) + \rho^j_k(f_j)] \tag{13}$$

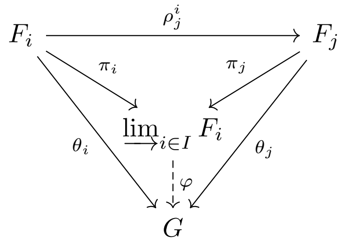
Using the same trick as for the above proof of transitivity, one shows that this is well-defined. Zero element is $[0_i]$ for any $i \in I$. This makes sense as $[0_Y ourlocationi] = [0_j]$ for any $i, j \in I$. Finally, $-[f_i] = [-f_i]$. This gives us a structure of Abelian group on $\varinjlim_{i \in I} F_i$. The R-multiplication is defined as $r \cdot [f_i] = [r \cdot f_i]$ for all $r \in R$. Similar proof works for $\mathbf{C} \in \{\mathbf{CRing}, \mathbf{Grp}, \mathbf{Ab}\}$, etc. However, the most important property of the direct limit is its characteristic universal property, stated as a proposition:

Proposition 2.6 (**Universality property**). *Suppose we have an object $G \in \mathbf{C}$ and a collection of morphisms $\theta_i \in \mathbf{C}(F_i, G)$ for each $i \in I$, such that for every $i \leq j$, the diagram*



(14)

commutes. Then there is a unique morphism $\varphi \in \boldsymbol{C}(\varinjlim_{i\in I} F_i, G)$, *such that* $\varphi \circ \pi_i = \theta_i$ *for all* $i \in I$. *In other words, the following diagram is commutative for every* $i \le j$:



. (15)

In particular, $\varinjlim_{i\in I} F_i$ *is the unique colimit of the functor* F*, up to an isomorphism.*

Proof. Suppose $G \in \mathbf{C}$ and $\{\theta_i\}_{i\in I}$ satisfy the above properties. Define $\varphi([f_i]) = \theta_i(f_i)$. This is the only φ satisfying $\varphi \circ \pi_i = \theta_i$. We only have to show that it is well-defined. Suppose $f_i \sim f_j$. Then there exists $k \in I$, such that $i, j \le k$, and $\rho_k^i(f_i) = \rho_k^j(f_j)$. Then

$$\theta_i(f_i) = \theta_k(\rho_k^i(f_i)) = \theta_k(\rho_k^j(f_j)) = \theta_j(f_j). \tag{16}$$

For the uniqueness of a colimit - if G and a collection of $\theta_i : F_i \to G$ would have the same universality property, then we can first construct a unique map $\varphi : \varinjlim_{i\in I} F_i \to G$ by the universality proved in this proposition, but also a map $\psi : G \to \varinjlim_{i\in I} F_i$ by the universality of G. But the uniqueness then also forces $\varphi \circ \psi = 1 = \psi \circ \varphi$. ■

To show that something is (isomorphic to) the direct limit, one only has to prove the universal property. We obtain the following useful criterion:

Proposition 2.7. *Let* $F : \mathbf{I} \to \boldsymbol{C}$ *be a direct mapping family in a suitable category* $\boldsymbol{C}$. *Suppose* $G \in \boldsymbol{C}$ *and there is a family* $\{\theta_i\}_{i\in I}$ *of morphisms* $\theta_i : F_i \to G$, *such that* $\theta_i = \theta_j \circ \rho_j^i$ *whenever* $i \le j$. *Suppose the following two properties hold:*

(i) For any $g \in G$, *there is some* $i \in I$ *and some* $f_i \in F_i$, *such that* $g = \theta_i(f_i)$;

(ii) For all $i, j \in I$ *and for any* $f_i \in F_i$ *and any* $f_j \in F_j$, *we have*

$$\theta_i(f_i) = \theta_j(f_j), \text{ iff there exists } k \in I, \text{ such that } i, j \le k \text{ and } \rho_k^i(f_i) = \rho_k^j(f_j). \tag{17}$$

Then G *is (isomorphic) to the direct limit of the direct mapping family* F.

Proof. We only have to prove the universal property. Whence suppose, there is an object $H \in \mathbf{C}$ together with the family of morphisms $\{\eta_i\}_{i\in I}$, such that $\eta_i \in \mathbf{C}(F_i, H)$ and $\eta_i = \eta_j \circ \rho_j^i$ whenever $i \le j$. We must construct a unique map $\varphi : G \to H$ satisfying $\eta_i = \varphi \circ \theta_i$ for all $i \in I$.

By property (i) above, to any $g \in G$, there is some $i \in I$ and $f_i \in F_i$, such that $g = \theta_i(f_i)$. The only way (this proves its uniqueness) to define φ is $\varphi(g) = (\varphi \circ \theta_i)(f_i) := \eta_i(f_i)$. Only has to show that it is well-defined. If there are $i, j \in I$, such that $g = \theta_i(f_i) = \theta_j(f_j)$, we can use (ii) to find $k \in I$, such that $i, j \le k$ and $\rho_k^i(f_i) = \rho_k^j(f_j)$. But then

$$\eta_i(f_i) = \eta_k(\rho_k^i(f_i)) = \eta_k(\rho_k^j(f_j)) = \eta_j(f_j). \tag{18}$$

Note that "if" part of (ii) does not have to be checked - it works for any G and θ_i satisfying $\theta_i = \theta_j \circ \rho_j^i$ whenever $i \le j$. ■

Example 2.8. Suppose $\mathbf{C} = \text{R-}\mathbf{Mod}$. There is another common way to obtain the direct limit. First, construct a direct sum $H = \bigoplus_{i \in I} F_i$. This can be viewed as a space of formal *finite* sums of elements from the R-modules F_i. One than constructs the ideal $\mathcal{I}$ (with respect to the Abelian group structure on H), generated by the set

$$S = \{ f_i - \rho^i_j(f_i) \mid f_i \in F_i, i \leq j \} \tag{19}$$

Fefine $G = H/\mathcal{I}$. For each $i \in I$, let $\theta_i : F_i \to G$ denote the composition of the canonical inclusion $\nu_i : F_i \to H$ with the canonical quotient map $H \to H/\mathcal{I}$.

First, let us show that for $i \leq j$, one has $\theta_i = \rho^i_j \circ \theta_j$. For any $f_i \in F_i$, one has, from its definition, $\nu_i(f_i) - \nu_j(\rho^i_j(f_i)) \in \mathcal{I}$. But this means that $\theta_i(f_i) - \theta_j(\rho^i_j(f_i)) = 0$. Now, let us prove the universal property of G together with the collection $\theta_i : F_i \to G$.

Let $K \in \text{R-}\mathbf{Mod}$ and let $\eta_i : F_i \to K$ be the morphisms, such that $\eta_i = \rho^i_j \circ \eta_j$ whenever $i \leq j$. We must prove the existence of a unique R-module morphism $\varphi : G \to K$. Let us define a map $\hat{\varphi} : H \to K$ by a requirement $\hat{\varphi} \circ \nu_i = \eta_i$ for all $i \in I$. As the direct sum is in fact a coproduct in the category $\text{R-}\mathbf{Mod}$, this defines a unique R-module morphism $\hat{\varphi} : H \to K$. Now, note that $\mathcal{I} \subseteq \ker(\hat{\varphi})$. It suffices to show that $S \subseteq \ker(\hat{\varphi})$. But that is clear, as for every $f_i \in F_i$ and $i \leq j$:

$$\hat{\varphi}(f_i - \rho^i_j(f_i)) \equiv \hat{\varphi}(\nu_i(f_i) - \nu_j(\rho^i_j(f_i)) = \eta_i(f_i) - \eta_j(\rho^i_j(f_i)) = 0. \tag{20}$$

It follows that $\hat{\varphi}$ induces a unique R-module morphism $\varphi : G \equiv H/\mathcal{I} \to K$ fitting into the diagram

$$\begin{array}{ccc} H & & \\ \downarrow & \overset{\hat{\varphi}}{\searrow} & \\ G & \overset{\varphi}{\dashrightarrow} & K \end{array} \quad . \tag{21}$$

In particular, it follows that $\varphi \circ \theta_i = \eta_i$ and φ is a unique such morphism.

Note that one can easily construct the explicit form of the isomorphism $\varphi : \varinjlim_{i \in I} F_i \to G$, we simply map the class $[f_i] \in \bigsqcup_{i \in I} F_i / \sim$ onto the class $f_i + \mathcal{I} \in \bigoplus_{i \in I} F_i / \mathcal{I}$. Conversely, each class in $\bigoplus_{i \in I} F_i / \mathcal{I}$ is represented by a formal finite sum of $f_{i_1} + \cdots + f_{i_N}$, where $f_{i_\mu} \in F_{i_k}$ for every $\mu \in \{1, \dots, N\}$. Its image under the inverse $\varphi^{-1} : G \to \varinjlim_{i \in I} F_i$ is the finite sum $[f_{i_1}] + \cdots + [f_{i_N}]$.

The construction works for every category where direct sum and quotients by ideals make sense, in particular in **Ab** or **CRing** or **Vect**.

To finish this section, we need to study what happens with direct limits with respect to the natural transformations of the involved functors.

Definition 2.9. Let I and J be two directed sets, viewed as categories $\mathbf{I}$ and $\mathbf{J}$. An **order preserving map** $\tau : I \to J$ is an object function of a functor $\tau : \mathbf{I} \to \mathbf{J}$. Let $F : \mathbf{I} \to \mathbf{C}$ and $G : \mathbf{J} \to \mathbf{C}$ be two direct mapping families indexed by I and J, respectively.

By **morphism of direct mapping families**, we mean a pair (φ, τ), where τ is an order preserving map and φ is a natural transformation from the functor F to the functor $G \circ \tau$.

Let us decipher this rather compact definition. An order preserving map satisfies $\tau(i) \leq \tau(j)$ whenever $i \leq j$. This corresponds to the fact that $i \leq j$ iff there is an arrow in $\mathbf{I}(i,j)$ and τ as a functor must map it to the (unique) arrow in $\mathbf{J}(\tau(i), \tau(j))$. Now, a natural transformation φ from

F to $G \circ \tau$ consists of a morphism $\varphi_i : F_i \to (G \circ \tau)_i = G_{\tau(i)}$ for each $i \in I$, such that the following diagram commutes for any $i \leq j$:

$$\begin{array}{ccc} F_i & \xrightarrow{\varphi_i} & G_{\tau(i)} \\ \big\downarrow (\rho_F)^i_j & & \big\downarrow (\rho_G)^{\tau(i)}_{\tau(j)} \\ F_j & \xrightarrow{\varphi_j} & G_{\tau(j)}. \end{array} \tag{22}$$

We expect that the direct limit construction somehow reflects this. And it does.

Proposition 2.10. *Let (φ, τ) be a morphism of direct mapping families $F : \mathbf{I} \to \boldsymbol{C}$ and $G : \mathbf{J} \to \boldsymbol{C}$. Then there is a unique morphism $\Phi : \varinjlim_{i \in I} F_i \to \varinjlim_{j \in J} G_j$, such that the diagram*

$$\begin{array}{ccc} F_i & \xrightarrow{\varphi_i} & G_{\tau(i)} \\ \big\downarrow \pi^F_i & & \big\downarrow \pi^G_{\tau(i)} \\ \varinjlim_{i \in I} F_i & \overset{\Phi}{\dashrightarrow} & \varinjlim_{j \in J} G_j \end{array} \tag{23}$$

commutes for all $i \in I$. The vertical arrows are the inclusions from the definition of $\varinjlim$. We say that Φ is a ***direct limit of the family*** *$\{\varphi_i\}_{i \in I}$ and write $\Phi = \varinjlim_{i \in I} \varphi_i$.*

Proof. We will use the universality property or $\varinjlim_{i \in I} F_i$ to construct (the unique) Φ. Let $\theta_i = \pi^G_{\tau(i)} \circ \varphi_i$. We only have to show that whenever $i \leq j$, one has $\theta_i = \theta_j \circ (\rho_F)^i_j$. Then

$$\theta_j \circ (\rho_F)^i_j = \pi^G_{\tau(j)} \circ \varphi_j \circ (\rho_F)^i_j = \pi^G_{\tau(j)} \circ (\rho_G)^{\tau(i)}_{\tau(j)} \circ \varphi_i = \pi^G_{\tau(i)} \circ \varphi_i = \theta_i. \tag{24}$$

The rest of the proof is just the universality - there must exist a unique $\Phi : \varinjlim_{i \in I} F_i \to \varinjlim_{j \in J} G_j$ satisfying $\Phi \circ \pi^F_i = \theta_i = \pi^G_{\tau(i)} \circ \varphi_i$. This is the commutativity of (23). ∎

We will now prove one very useful tool for a calculation of limits. We will prove it in a slightly more general setting. Note that we modify a definition of final functor so that it will fit into the framework of cofinal sets, calling it a filtered functor. The original statement and definition of a final functor are different, see e.g. [3]. We have replaced a requirement of non-empty and connected comma category by filtered.

Definition 2.11. Let $L : \mathbf{J} \to \mathbf{I}$ be a functor. One says that L is a **filtered functor**, if the comma category $(i \downarrow L)$ is filtered for every object $i \in \mathbf{I}$. Let us unfold this definition:

(i) For any $i \in I$, there must exist an object $j \in \mathbf{J}$ together with an arrow $u \in \mathbf{C}(i, L(j))$. This correspond to the fact that the filtered category $(i \downarrow L)$ must be non-empty.

(ii) Suppose (j, u) and (j', u') be two such pairs. There must exist an object $k \in \mathbf{J}$ and an arrow $m \in \mathbf{I}(i, L(k))$ together with a pair of morphisms $h \in \mathbf{J}(j, k)$ and $h' \in \mathbf{J}(j', k)$ such that the following diagram commutes:

$$\begin{array}{ccccc} i & \xrightarrow{1_i} & i & \xleftarrow{1_i} & i \\ \big\downarrow u & & \big\downarrow m & & \big\downarrow u' \\ L(j) & \xrightarrow{L(h)} & L(k) & \xleftarrow{L(h')} & L(j'). \end{array} \tag{25}$$

This is precisely the filtration property - pairs (j, u) and (j', u') are objects in $(i \downarrow L)$. There must exist a third object (k, m) together with a pair of morphisms $(j, u) \to (k, m)$ and $(j', u') \to (k, m)$. But in $(i \downarrow L)$, those are exactly h and h' fitting into the above diagram.

(iii) Suppose (j, u) and (j', u') be two such pairs, and let $h, h' \in \mathbf{J}(j, j')$ be two morphisms both satisfying $L(h) \circ u = u'$ and $L(h') \circ u = u'$. Then there is an object $k \in \mathbf{J}$ together with morphisms $m : i \to L(k)$ and $n : L(j') \to L(k)$, such that $L(n) \circ u' = m$ and $n \circ h = n \circ h'$.

This corresponds to the fact that to any objects (j, u) and (j', u') any pair of parallel arrows between them (corresponding to h and h' and their interplay with u and u'), there is an object (k, m) together with a morphism from (j', u') to (k, m), such that it coequalizes the parallel arrows.

Example 2.12. Suppose I is preodered subset, and let J be a directed subset inheriting the preorder. We can view them as two categories $\mathbf{I}$ and $\mathbf{J}$ together with the inclusion functor $L : \mathbf{J} \to \mathbf{I}$. Then L is a filtered functor if and only if J is a cofinal subset of I.

Indeed, the property (i) corresponds to the fact that to any $i \in I$, there exists $j \in J$, such that $i \leq L(j)$. This forces J to be cofinal in I. The property (ii) says that if we find another such $j' \in J$, there must be some $k \in J$, which also satisfies $i \leq L(k)$ and moreover $j \leq k$ and $j' \leq k$. For this we use the fact that J is directed and k is a common upper bound for j and j'. Then certainly $L(j) \leq L(k)$ and $L(j') \leq L(k)$ as L is the functor and from transitivity of $\leq$ also $i \leq L(k)$.

The property (iii) is in this case trivial, as by construction every two parallel arrows in $\mathbf{J}(j, j')$ are already equal. It thus suffices to choose $k = j'$ and $n = 1_{j'}$.

Theorem 2.13. *Suppose $L : \mathbf{J} \to \mathbf{I}$ is a filtered functor, and let $F : \mathbf{I} \to \boldsymbol{C}$ be any functor. Let $F_L = F \circ L$ be the composed functor $F_L : \mathbf{J} \to \boldsymbol{C}$ and suppose there exists $c = \operatorname{colim}(F_L) \in \boldsymbol{C}$.*

Then there exists also $\operatorname{colim}(F)$ *and as an object of $\boldsymbol{C}$ it equal to c.*

Proof. Recall that a colimit $c = \operatorname{colim}(F_L)$ is an object together with a coliming cone $\pi_j : F_L(j) \to c$ for all $j \in J$, such that $\pi : F_L \to \Delta(c)$ is a natural transformation from F_L to a constant functor $\Delta(c)$, together with a universal property.

For every $i \in \mathbf{I}$, there must by assumption on L exist an object $j \in \mathbf{J}$ together with an arrow $u \in \mathbf{I}(i, L(j))$. We thus have an arrow $F(u) \in \mathbf{C}(F(i), F_L(j))$. By composing it with a cone map $\pi_j : F_L(j) \to c$, we obtain a morphism $\theta_i = \pi_j \circ F(u) \in \mathbf{C}(F(i), c)$. We claim that θ is a colimiting cone for $F : \mathbf{I} \to \mathbf{C}$.

First, we have to argue that θ_i is well-defined. Let $j' \in \mathbf{J}$ and $u' \in \mathbf{I}(i, L(j'))$ be another such combination. We have to show that $\pi_j \circ F(u) = \pi_{j'} \circ F(u')$. By property (ii), there is an object $k \in \mathbf{J}$ together with an arrow $m \in \mathbf{I}(i, L(k))$, and a pair of morphisms $h \in \mathbf{J}(j, k)$ and $\mathbf{J}(j', k)$ such that $m = L(h) \circ u = L(h') \circ u'$.

As $\pi : F_L \to \Delta(c)$ is a limiting cone, its naturality implies $\pi_j = \pi_k \circ F_L(h)$. We thus have

$$\pi_j \circ F(u) = \pi_k \circ F_L(h) \circ F(u) = \pi_k \circ F(L(h) \circ u) = \pi_k \circ F(m) = \cdots = \pi_{j'} \circ F(u'). \tag{26}$$

This proves that the definition of θ_i does not depend on the choice of j. We must prove that $\theta : F \to \Delta(c)$ is a natural transformation. Suppose $q \in \mathbf{I}(i, i')$. We must show that $\theta_i = \theta_{i'} \circ F(q)$. Find $j' \in \mathbf{J}$ and $u' \in \mathbf{I}(i', L(j'))$ by property (i) for the object $i' \in \mathbf{I}$. It follows that the object $j' \in \mathbf{J}$ and $u' \circ q \in \mathbf{I}(i, L(j'))$ is also fine for the definition of θ_i. Then

$$\theta_i = \pi_{j'} \circ F(u' \circ q) = (\pi_{j'} \circ F(u')) \circ F(q) = \theta_{i'} \circ F(q). \tag{27}$$

Whence $\theta : F \to \Delta(c)$ is a natural transformation and thus a cone for F. It suffices to prove its universality. Let $\eta : F \to \Delta(d)$ be any cone over $d \in \mathbf{C}$ with the base F. In other words, we have natural maps $\eta_i : F(i) \to d$. As L is a functor, we may define another cone $\chi_j = \eta_{L(j)} : F_L(j) \to d$, which is by construction natural in j. By universality of $\pi : F_L \to \Delta(c)$, there is a unique arrow

$k \in \mathbf{C}(c,d)$, such that $k \circ \pi_j = \chi_j$ for all $j \in \mathbf{J}$. Now, for any $i \in \mathbf{I}$, there is $j \in \mathbf{J}$ together with an arrow $u \in \mathbf{I}(i, L(j))$. Then

$$k \circ \theta_i = k \circ (\pi_j \circ F(u)) = (k \circ \pi_j) \circ F(u) = \chi_j \circ F(u) = \eta_{L(j)} \circ F(u) = \eta_i. \tag{28}$$

Whence $k \circ \theta_i = \eta_i$. Now, k obtained in this way is unique. Indeed, if this property hold for any $i \in \mathbf{I}$, it must hold for $i = L(j)$ and any $j \in \mathbf{J}$. But $\theta_{L(j)} = \pi_j$ and $\eta_{L(j)} = \chi_j$. A different k would then contradict the uniqueness of k in the equation $k \circ \pi_j = \chi_j$. Hence, there is a unique arrow $k \in \mathbf{C}(c,d)$, such that $k \circ \theta_i = \eta_i$ for all $i \in \mathbf{I}$. This proves the universal property and we conclude that $c = \operatorname{colim}(F)$. ■

Remark 2.14. Note that we have not used the property (iii) of a filtered functor. The assumptions on L can be taken to be just (i) and (ii).

Corollary 2.15. *Let I be any collection with a preorder $\leq$ which is directed. Note that I does not have to be a set, which is important. Let $J \subseteq I$ be a cofinal subset of I. Let $\mathbf{I}$ and $\mathbf{J}$ be the corresponding categories, and let $F : \mathbf{I} \to \boldsymbol{C}$ be a functor valued in some suitable category (e.g.* R-***Mod***, ***CRing*** *or* ***Grp****). Then there exists a directed limit $\varinjlim_{i \in I} F_i$ and can be taken equal to the direct limit $\varinjlim_{j \in J} F_j$ taken over the subset $J \subseteq I$.*

Moreover, the coliming cone $\pi_i : F_i \to \varinjlim_{j \in J} F_j$ can be constructed as follows. As J is cofinal, there exists $j \in J$, such that $i \leq j$. There is thus a map

$$\pi_j^{\mathbf{J}} : F_j \to \varinjlim_{j \in J} F_j, \tag{29}$$

and a map $\rho^i_j : F_i \to F_j$. Then $\pi_i = \pi_j^{\mathbf{J}} \circ \rho^i_j$. This expression is independent of j used. The rest is a trivial application of the above theorem.

Proof. Recall that a directed limit $\varinjlim_{i \in I} F_i$ in the case when I is not a set is just some object $G \in \mathbf{C}$ together with a bunch of maps $\pi_i : F_i \to G$, together with the universal property. We cannot define G by the formulas above as e.g. the disjoint union over a proper class I would not define a set. However, if $J \subseteq I$ is cofinal in I, the inclusion functor $L : \mathbf{J} \to \mathbf{I}$ is filtered, as we have discussed in Example 2.12. ■

3 Open covers of a topological space

Let us first talk a little bit about open covers of topological spaces. By open cover $\mathcal{U}$ of a topological space X, we mean a collection $\{U_\alpha\}_{\alpha \in A}$, such that $X = \bigcup_{\alpha \in A} U_\alpha$. It is useful to thing of open covers as maps $A \to \tau(X) \subseteq 2^X$, where $\tau(X)$ is the topology of X viewed as a subset of the power set 2^X of all subsets of X. The collection $\{U_\alpha\}_{\alpha \in A} \subseteq \tau(X)$ is then just an image of this map.

Definition 3.1. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ be two open covers. We say that $\mathcal{V}$ is a **refinement** of $\mathcal{U}$, if there is a map $\tau : B \to A$, such that $V_\beta \subseteq U_{\tau(\beta)}$ for all $\beta \in B$. We will use the notation $\mathcal{U} \prec \mathcal{V}$.

Proposition 3.2. *The relation $\prec$ is reflexive and transitive. For any open covers $\mathcal{U}$ and $\mathcal{V}$, there is an upper bound $\mathcal{W}$, that is an open cover such that $\mathcal{U}, \mathcal{V} \prec \mathcal{W}$.*

Proof. The reflexivity and transitivity are obvious. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$. Define their **common refinement** as $\mathcal{W} = \{W_{(\alpha,\beta)}\}_{(\alpha,\beta) \in A \times B}$, where $W_{(\alpha,\beta)} = U_\alpha \cap V_\beta$. Let $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ be the two projections. Certainly $W_{(\alpha,\beta)} \subseteq U_{\pi_A(\alpha,\beta)}$ and $W_{(\alpha,\beta)} \subseteq V_{\pi_B(\alpha,\beta)}$. This proves that $\mathcal{U}, \mathcal{V} \prec \mathcal{W}$. ■

Remark 3.3. The common refinement can be constructed for any (including infinite) set of open covers. Indeed, suppose S is any set and we have a collection $\{\mathcal{U}^\sigma\}_{\sigma\in S}$ of open covers. We have $\mathcal{U}^\sigma = \{U^\sigma_{\alpha_\sigma}\}_{\alpha_\sigma\in A_\sigma}$ for some indexing sets A_σ. One can define $A = \prod_{\sigma\in S} A_\sigma$. A consists of sequences $(\alpha_\sigma)_{\sigma\in S}$. Define $V_{(\alpha_\sigma)} = \bigcap_{\sigma\in S} U^\sigma_{\alpha_\sigma}$ and form an open cover $\mathcal{W} = \{V_{(\alpha_\sigma)}\}_{(\alpha_\sigma)\in A}$. For each $\sigma \in S$, there is a projection $\pi_\sigma : A \to A_\sigma$ and $V_{(\alpha_{\sigma'})} \subseteq U_{\pi_\sigma((\alpha_{\sigma'}))}$ for every $\sigma \in S$. In orther words, $\mathcal{U}^\sigma \prec \mathcal{W}$ for all $\sigma \in S$.

As for any preorder, we may induce an equivalence relation: $\mathcal{U} \asymp \mathcal{V}$, iff $\mathcal{U} \prec \mathcal{V}$ and $\mathcal{V} \prec \mathcal{U}$. Note that is not true that if $\mathcal{U} \asymp \mathcal{V}$, then the collections $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$ coincide as subsets of $\tau(X)$. Indeed, any two open covers containing the whole space X are equivalent, but otherwise they can consist of different open sets. However, the converse is clearly true.

There is a certain set-theoretic problem with open covers we have just defined. Indeed, let $\mathbf{OpC}(X)$ be the collection of all open covers of X. We claim that *it is not a set.* In fact, we may use the preorder $\prec$ to make it into a category (hence the bold letters). We may then define a contravariant functor $\mathcal{S} : \mathbf{OpC}(X) \to \mathbf{Set}$ which assigns to each open cover $\{U_\alpha\}_{\alpha\in A}$ the corresponding indexing set, that is $\mathcal{S}(\mathcal{U}) = A$. For $\mathcal{U} \prec \mathcal{V}$ there is a unique arrow from $\mathcal{U}$ to $\mathcal{V}$, to which $\mathcal{S}$ assigns a map $\tau : \mathcal{S}(\mathcal{V}) \to \mathcal{S}(\mathcal{U})$ from the definition of refinement.

If $\mathbf{OpC}(X)$ would be a set, then the image $\mathcal{S}(\mathbf{OpC}(X))$ must be a set. The map $\mathcal{S}$ is surjective, as to any $A \in \mathbf{Set}$, we may assign an open cover $U_\alpha = X$ for all $\alpha \in A$. Thus $\mathcal{S}(\mathbf{OpC}(X)) = \mathbf{Set}$, which is not a set. It follows that $\mathbf{OpC}(X)$ must be a proper class.

We find that one cannot just take the direct limit of a direct mapping family $F : \mathbf{OpC}(X) \to \mathbf{C}$ because it is not a set. There is a workaround, where we consider only a certain subset of $\mathbf{OpC}(X)$. This is an idea of Serre from [4].

Proposition 3.4 (Serre). *Every open cover $\mathcal{U}$ is equivalent to some open cover whose indexing set is some subset of 2^X. Let $\mathbf{OpC}_S(X) \subseteq \mathbf{OpC}(X)$ denote this subset.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be any open cover. Let $N = \{U \in \tau(X) \mid U = U_\alpha \text{ for some } \alpha \in I\}$. Then $N \subseteq \tau(X) \subseteq 2^X$. Then define a collection $\mathcal{V} = \{V_U\}_{U\in N}$, where $V_U = U$ (this is stupid, i know). Let us show that $\mathcal{U} \asymp \mathcal{V}$.

First, for every $U \in N$, there is some $\alpha_U \in I$, such that $U = U_{\alpha_U}$. If we define $\tau : N \to I$ as $\tau(U) = \alpha_U$, we have $V_U = U = U_{\tau(U)}$. This proves $\mathcal{U} \prec \mathcal{V}$. Conversely, define $\tau' : I \to N$ as $\tau'(\alpha) = U_\alpha \in N$. Then, obviously $U_\alpha = V_{\tau'(\alpha)}$. This proves $\mathcal{V} \prec \mathcal{U}$ and thus $\mathcal{U} \asymp \mathcal{V}$. ∎

Note that in particular, the set $\mathbf{OpC}_S(X)$ is cofinal in $\mathbf{OpC}(X)$. We can thus use Corollary 2.15 and calculate all direct limits of functors $F : \mathbf{OpC}(X) \to \mathbf{C}$ by calculating the direct limit of its restriction to $\mathbf{OpC}_S(X)$.

However, for a nice topological spaces, there are even more useful cofinal subsets. For example, if X is so called Lindelöf space, every open cover has a countable subcover. Although it seems strange under the usual perception of refinement, every subcover is a refinement of the original cover. To see this, it suffices to properly state what is a subcover.

Definition 3.5. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of X. By a **subcover** $\mathcal{V}$ of $\mathcal{U}$, we mean a choice of a subset $K \subseteq A$, such that $\mathcal{V} = \{U_\alpha\}_{\alpha\in K}$ is still an open cover of X. One sometimes writes just $\mathcal{V} \subseteq \mathcal{U}$ as it makes sense when viewing $\mathcal{U}$ and $\mathcal{V}$ as subsets of $\tau(X)$.

Nor, let $\tau : K \to A$ be the inclusion of the subset K into A. Then for any $\alpha \in K$, one has $U_\alpha = U_{\tau(\alpha)}$. In other words, $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $\mathcal{U} \prec \mathcal{V}$.

Proposition 3.6. *If $\mathcal{V} \subseteq \mathcal{U}$, then $\mathcal{U} \prec \mathcal{V}$.*

Proposition 3.7. *Let X be a Lindelöf space. Then the set $\boldsymbol{OpC}_{\mathbb{N}}(X)$ of countable covers is cofinal in $\boldsymbol{OpC}(X)$. In particular, the direct limit of any functor $F : \boldsymbol{OpC}(X) \to \boldsymbol{C}$ can be calculated over the subset of countable covers.*

Proof. In a Lindelöf space, every open cover $\mathcal{U}$ has a countable subcover $\mathcal{V} \subseteq \mathcal{U}$. Whence $\mathcal{U} \prec \mathcal{V}$ by previous proposition. The rest is Corollary 2.15. ■

Let A be any indexing set. One can always assume that there is a well total order $\leq$ on A. This is a consequence of the famous **well-ordering theorem**. Just for the sake of fun, let us recall the definitions and its proof here.

Definition 3.8. Let A be any set. A binary relation $\leq$ is said to be a **well total order** on A, if

(i) $\leq$ is a total order, that is it has the following properties:

- antisymmetry: if $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$;
- transitivity: if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$;
- connex property: either $\alpha \leq \beta$ or $\beta \leq \alpha$.

(ii) $\leq$ is a well order, every non-empty subset $K \subseteq A$ has a least element, that is there is $\kappa \in K$, such that $\kappa \leq \alpha$ for all $\alpha \in K$.

Note that a least element in any non-empty subset K is unique. Indeed, if there is some other least element $\kappa' \in K$, we have either $\kappa \leq \kappa'$ or $\kappa' \leq \kappa$ be the connex property. But as both κ and κ' are least elements, we have both $\kappa \leq \kappa'$ and $\kappa' \leq \kappa$, whence $\kappa' = \kappa$ from the antisymmetry property of the relation $\leq$. Moreover, each element α has a unique "larger neighbor" β, that $\alpha \leq \beta$ and whenever $\alpha \leq \gamma$ for some γ, then $\beta \leq \gamma$. Indeed, simply take $K = \{\gamma \in A \mid \alpha \leq \gamma\}$ and take β to be its unique least element.

Theorem 3.9 (Well-ordering theorem). *On every set A, there exists a well order $\leq$.*

Proof. This is an application of Zorn's lemma. Let A be a given set. Set $\mathcal{A}$ be the set of pairs $(K, \leq_K)$, where $K \subseteq A$ and $\leq_K$ is a well order on K. Then $\mathcal{A}$ is non-empty as one-point set $\{\alpha\}$ can be trivially well-ordered.

Moreover, there is a partial order $\preceq$ on $\mathcal{A}$. We say that $(K, \leq_K) \preceq (L, \leq_L)$ iff $K \subseteq L$, $\leq_K$ coincides with $\leq_L$ on K and K is an initial segment of L, that is whenever $\alpha \in K$ and $\beta \leq_L \alpha$, then $\beta \in K$. Clearly, it is reflexive. Suppose $(K, \leq_K) \preceq (L, \leq_L)$ and $(L, \leq_L) \preceq (M, \leq_M)$. The transitivity of the set inclusion relation shows that $K \subseteq M$ and obviously, $\leq_K$ coincides with $\leq_M$ on K. Let $\alpha \in K$ and let $\beta \in M$ satisfy $\beta \leq \alpha$. As $K \subseteq L$, we have $\alpha \in L$ and as L is an initial segment of $(M, \leq_M)$, we have $\beta \in L$. As K is an initial sgment of $(L, \leq_L)$, we have $\beta \in K$. This shows that K is an initial segment of $(M, \leq_M)$ and thus $(K, \leq_K) \preceq (M, \leq_M)$.

Now, let $\mathcal{Z} \subseteq \mathcal{A}$ be a chain in A. Let $R = \cup_{K \in \mathcal{Z}} K$ be the union of the sets in $\mathcal{Z}$. For all $\alpha, \beta \in R$, we say that $\alpha \leq_R \beta$ if there is $K \in \mathcal{Z}$, such that $\alpha, \beta \in K$ and $\alpha \leq_K \beta$. First, we claim that $\leq_R$ is a total order on R.

- Antisymmetry: let $\alpha \leq_R \beta$ and $\beta \leq_R \alpha$. There are thus $K, L \in \mathcal{Z}$, such that $\alpha, \beta \in K$ and $\alpha, \beta \in K$ and $\alpha \leq_K \beta$ and $\alpha, \beta \in L$ and $\beta \leq_L \alpha$. As $\mathcal{Z}$ is a chain, either $K \subseteq L$ or $L \subseteq K$. Without the loss of generality, suppose $K \subseteq L$. Then $\alpha \leq_K \beta$ coincides with $\leq_L$ on K and thus also $\alpha \leq_L \beta$. By antisymmetry of the total order $\leq_L$, we have $\alpha = \beta$.

- Transitivity: let $\alpha \leq_R \beta$ and $\beta \leq_R \gamma$. There are thus $K, L \in \mathcal{Z}$, such that $\alpha, \beta \in K$ and $\beta, \gamma \in L$, such that $\alpha \leq_K \beta$ and $\beta \leq_L \gamma$. Using the same reasoning as above, either $K \subseteq L$ or $L \subseteq K$. Again, suppose the first case is true, whence $\alpha \leq_K \beta$ implies $\alpha \leq_L \beta$ and the transitivity of $\leq_L$ implies $\alpha \leq_L \gamma$. We have thus found $L \in \mathcal{Z}$, such that $\alpha, \gamma \in L$ and $\alpha \leq_L \gamma$. Hence $\alpha \leq_R \gamma$.
- Connex property: Let $\alpha, \beta \in R$. As R is a union of the sets in $\mathcal{Z}$, we have $\alpha \in K$ and $\beta \in L$ for some $K, L \in \mathcal{Z}$. Again, either $K \subseteq L$ or $L \subseteq K$. Suppose $K \subseteq L$. But then $\alpha, \beta \in L$ and as $\leq_L$ is a total order, either $\alpha \leq_L \beta$ or $\beta \leq_L \alpha$. But this means that either $\alpha \leq_R \beta$ or $\beta \leq_R \alpha$. The option $L \subseteq K$ is analogous.

Whence $(R, \leq_R)$ is a total order. We have to show that it is well-ordered. Let $S \subseteq R$ be a non-empty subset. There is thus some $K \in \mathcal{Z}$, such that $S \cap K \neq \emptyset$. As $S \cap K \subseteq K$ is a non-empty subset of the well order $(K, \leq_K)$, there is some element $\alpha_0 \in S \cap K$, such that $\alpha_0 \leq_K \beta$ for all $\beta \in S \cap K$. Now, let $\beta \in S$ be an arbitrary element.

There is thus some $L \in \mathcal{Z}$, such that $\beta \in S \cap L$. Either $K \subseteq L$ or $L \subseteq K$. If $L \subseteq K$, then $\beta \in S \cap K$ and thus $\alpha_0 \leq_K \beta$ and consequently $\alpha_0 \leq_R \beta$. When $K \subseteq L$, we can compare α_0 with β in L. If $\alpha_0 \leq_L \beta$, we are finished, as then $\alpha_0 \leq_R \beta$. Suppose $\beta \leq_L \alpha_0$. But as K is an initial segment of L and $\alpha_0 \in K$, one has $\beta \in K$. As α_0 is the least element in $S \cap K$, we have $\alpha_0 \leq_K \beta$. In fact, this proves that $\alpha_0 = \beta$ from the antisymmetry of $\leq_K$.

This proves that $(R, \leq_R)$ is a well total ordered subset of A, that is $(R, \leq_R) \in \mathcal{A}$.

Moreover, we claim that $(K, \leq_K) \preceq (R, \leq_R)$ for all $K \in \mathcal{Z}$. Clearly $K \subseteq R$ and both orderings coincide on K. We only have to prove that K is an initial segment of R. Let $\alpha \in K$ and let $\beta \leq_R \alpha$ for some $\beta \in R$. There is thus some $L \in \mathcal{Z}$, such that $\alpha, \beta \in L$ and $\beta \leq_L \alpha$. If $L \subseteq K$, we have $\beta \in K$, as was to be proved. If $K \subseteq L$, we use the fact that K is an initial segment of L and thus, again, $\beta \in K$.

We have just prove that every chain $\mathcal{Z}$ has an upper bound $(R, \leq_R) \in \mathcal{A}$. From Zorn's lemma, there is thus some maximal element $(B, \leq_B) \in \mathcal{A}$. We claim that $B = A$. Suppose there is $x \in A$, such that $x \notin B$. Let $K = B \cup \{x\}$, and declare $b \leq_K x$ for all $b \in B$, and $x \leq_K x$. It is easy to see that $(K, \leq_K)$ is a well totally ordered subset of A, such that $(B, \leq_B) \preceq (K, \leq_K)$. But this contradicts the maximality of B, whence $B = A$ and $(B, \leq_B)$ is the required well total ordering on A. The proof is finished. ■

4 Čech cohomology

In this section, assume that $\mathbf{C} = \text{R-}\mathbf{Mod}$, where R is a fixed unital commutative unital ring. Let $\mathcal{F} : \mathbf{Op}(X) \to \text{R-}\mathbf{Mod}$ be any presheaf.

To avoid some unpleasant difficulties, we always assume that $\mathcal{F}(\emptyset) = 0$. In fact, for these reasons, one redefines the constant presheaf G_X to give $G_X(U) = G$ for all *non-empty* $U \in \mathbf{Op}(X)$ and $G_X(\emptyset) = 0$, where the restriction maps $\rho^U_\emptyset$ are trivial maps from G to 0.

Now, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a given open cover. We write

$$U_{\alpha_0 \dots \alpha_p} := U_{\alpha_0} \cap \dots \cap U_{\alpha_p}, \tag{30}$$

for any $(p+1)$-tuple $(\alpha_0, \dots, \alpha_p) \in A^{p+1}$. We allow for repeated indices. Next, one writes

$$U_{\alpha_0 \dots \hat{\alpha}_j \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap \widehat{U}_{\alpha_j} \cap \dots \cap U_{\alpha_p}, \tag{31}$$

where the hat denotes the omission of the j-th set from the intersection. As a convention, we declare the intersection of "no sets" to be X, that is for example $U_{\hat{\alpha}_0} = X$ for any $\alpha_0 \in A$.

Now, for any $(\alpha_0, \ldots, \alpha_p) \in A^{p+1}$, there is a $(p+1)$-tuple of inclusion maps

$$\delta^p_j : U_{\alpha_0 \ldots \alpha_p} \to U_{\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_p}, \quad j \in \{0, \ldots, p\}. \tag{32}$$

In particular, for $p = 0$, there is a single map $\delta^0_0 : U_{\alpha_0} \to X$. For $p = 1$, there are two maps:

$$\delta^1_0 : U_{\alpha_0\alpha_1} \to U_{\alpha_1}, \quad \delta^1_1 : U_{\alpha_0\alpha_1} \to U_{\alpha_0}. \tag{33}$$

Definition 4.1. Let X be a topological space together with its open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$. Let $\mathcal{F} : \mathbf{Op}(X) \to$ R-**Mod** be a presheaf. The R-module $C^p(\mathcal{U}, \mathcal{F})$ of **Čech p-cochains** with values in the presheaf $\mathcal{F}$ corresponding to the open cover $\mathcal{U}$ is defined as

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(\alpha_0,\ldots,\alpha_p)\in A^{p+1}} \mathcal{F}(U_{\alpha_0\ldots\alpha_p}). \tag{34}$$

Every element $\omega \in C^p(\mathcal{U}, \mathcal{F})$ is thus a sequence $(\omega_{\alpha_0\ldots\alpha_p})_{(\alpha_0,\ldots,\alpha_0)\in A^{p+1}}$.

As $\mathcal{F}$ is a presheaf, to each inclusion map $\delta^p_j : U_{\alpha_0\ldots\alpha_p} \to U_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_p}$, there is a restriction morphism $\mathcal{F}(\delta^p_j) : \mathcal{F}(U_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_p}) \to \mathcal{F}(U_{\alpha_0\ldots\alpha_p})$. Let us write $\rho^j_{\alpha_0\ldots\alpha_p} \equiv \mathcal{F}(\delta^p_j)$. We can use those map to define the **coboundary operators** $\delta^p_{\mathcal{F}} : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ as

$$(\delta^p_{\mathcal{F}}(\omega))_{\alpha_0\ldots\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho^j_{\alpha_0\ldots\alpha_{p+1}}(\omega_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_p}), \tag{35}$$

for every $\omega \in C^p(\mathcal{U}, \mathcal{F})$. It is an easy combinatorics to show that $\delta^{p+1}_{\mathcal{F}} \circ \delta^p_{\mathcal{F}} = 0$. Note that the explicit writting of the restriction maps is sometimes omitted. However, note that this practice can be dangerous as it certainly does happen that $U_{\alpha_0\ldots\alpha_p} = \emptyset$, and consequently $\mathcal{F}(U_{\alpha_0\ldots\alpha_p}) = 0$. We have thus obtain the **Čech cochain complex** $(C^\bullet(\mathcal{U}, \mathcal{F}), \delta^\bullet_{\mathcal{F}})$.

Definition 4.2. Given a topological space X and an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$. The p-th **Čech cohomology group** $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F})$ with values in the presheaf $\mathcal{F}$ is the cohomology group corresponding to the Čech cochain complex, that is

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) = Z^p(\mathcal{U}, \mathcal{F})/B^p(\mathcal{U}, \mathcal{F}), \tag{36}$$

where $Z^p(\mathcal{U}, \mathcal{F}) = \ker(\delta^p_{\mathcal{F}})$ and $B^p(\mathcal{U}, \mathcal{F}) = \mathrm{im}(\delta^{p-1}_{\mathcal{F}})$. Let $G \in$ R-**Mod** be a fixed R-module. **Classical Čech cohomology groups** $\check{\mathrm{H}}^p(\mathcal{U}, G)$ are defined as Čech cohomology of the constant presheaf G_X, where we assume $G_X(\emptyset) = 0$.

Let us examine a little bit the 0-th Czech cohomology $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$. Recall that one has

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{\alpha\in A} \mathcal{F}(U_\alpha), \tag{37}$$

and $\delta^0_{\mathcal{F}} : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ is defined for $\omega \in C^0(\mathcal{U}, \mathcal{F})$ as

$$(\delta^0_{\mathcal{F}}(\omega))_{\alpha\beta} = \rho^0_{\alpha\beta}(\omega_\beta) - \rho^1_{\alpha\beta}(\omega_\alpha) \equiv \rho^{U_\beta}_{U_{\alpha\beta}}(\omega_\beta) - \rho^{U_\alpha}_{U_{\alpha\beta}}(\omega_\beta). \tag{38}$$

By definition, one has $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F}) = Z^0(\mathcal{U},\mathcal{F})$. We have just shown that every $\omega \in \check{\mathrm{H}}^0(\mathcal{U},\mathcal{F})$ consists of a $\{\omega_\alpha\}_{\alpha\in A}$, where $\omega_\alpha \in \Gamma(U_\alpha,\mathcal{F})$ are local sections of $\mathcal{F}$ above U_α, such that ω_α and ω_β coincide when restricted to the intersection $U_\alpha \cap U_\beta$ for any $\alpha,\beta \in A$. Recall that we have a sequence

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \xrightarrow{\ r\ } C^0(\mathcal{U},\mathcal{F}) \xrightarrow{\delta^0_{\mathcal{F}}} C^1(\mathcal{U},\mathcal{F}), \tag{39}$$

where $r : \Gamma(X,\mathcal{F}) \to C^0(\mathcal{U},\mathcal{F})$ is the restriction map included by inclusions $\delta^0_0 : U_{\alpha_0} \to X$. We have already argued that if $\mathcal{F}$ forms a sheaf, this sequence is exact.

Proposition 4.3. *If $\mathcal{F}$ is a sheaf, then $r : \Gamma(X,\mathcal{F}) \to C^0(\mathcal{U},\mathcal{F})$ defines an isomorphism of $\Gamma(X,\mathcal{F})$ with $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F})$. In particular, the 0-th Čech cohomology $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{F})$ does not depend on the open cover $\mathcal{U}$.*

Note that presented definition of Čech cochains has some significant inconvenience. One assumes arbitrary p-tuples $(\alpha_0,\dots,\alpha_p)$ with possible repetition. The space of cochains thus grows significantly. In particular, even in the trivial case $X = \bigsqcup_{\alpha\in A} U_\alpha$, with empty two-fold intersections, we still have non-zero $C^p(\mathcal{U},\mathcal{F})$ for any p. For this reason it is useful to consider a certain subcomplex.

Definition 4.4. We say that $\omega \in C^p(\mathcal{U},\mathcal{F})$ is an **alternating cochain** if $\omega_{\alpha_0\dots\alpha_p} = 0$ whenever two of the indices are equal, and one has $\omega_{\alpha_0\dots\alpha_p} = \mathrm{sgn}(\sigma)\cdot\omega_{\alpha_{\sigma(0)}\dots\alpha_{\sigma(p)}}$ for any permutation σ of the set $\{0,\dots,p\}$. Note that $\mathrm{sgn}(\sigma)$ has sense as R is assumed unital and thus contains an additive inverse -1 of the multiplicative unit 1. The submodule of alternating cochains is denoted as $C'^p(\mathcal{U},\mathcal{F})$.

It is readily checked that $\delta^p_{\mathcal{F}}$ maps alternating cochains into alternating cochains, whence $(C'^\bullet(\mathcal{U},\mathcal{F}),\delta^\bullet_{\mathcal{F}})$ forms a subcomplex whose cohomology is denoted as $\check{\mathrm{H}}'^p(\mathcal{U},\mathcal{F})$. It would be great if one could calculate the full Čech cohomology groups in this way. Lucky enough, this is true. Before proving so, let us reformulate some definitions here slightly differently. We have to devote a full section to its proof.

5 Abstract simplicial homology

Let K be any set. By a **simplicial complex** with vertices K, we mean a collection $S \subseteq 2^K$ of *finite* subsets of K, called **simplices**, such that

(i) For every vertex $v \in K$, one has $\{v\} \in S$. In other words, every vertex is a simplex.

(ii) For every simplex $s \in S$, every its non-empty subset $s' \subseteq s$ is also in S. Subsets s' are called faces of s.

Every simplex is uniquely determined by its vertices. If it is formed out of $(q+1)$-vertices, it is called a q-simpled. As K sits inside S as 0-simplices, every simplicial complex is uniquely determined by a set of its simplices. We will often write just K.

(a) Let K be any set. Then the set of all non-empty finite subsets of K forms a simplicial complex.

(b) For any simplex $s \in K$, the set of all its faces (including s itself) forms a simplicial complex denoted as $\bar{s}$.

(c) Let K_1 and K_2 be two simplicial complexes. We define their join $K_1 * K_2$ as a collection

$$K_1 \sqcup K_2 \sqcup \{s_1 \sqcup s_2 \mid s_1 \in K_1, s_2 \in K_2\}. \tag{40}$$

The most common join is $K_1 * \{w\}$, where $\{w\}$ is viewed as a simplicial complex with a single 0-simplex. As an example, let $K_1 = \{\{v_0\}, \{v_1\}, \{v_0, v_1\}\}$ consist of two 0-simplices and one 1-simplex. Then $K_1 * \{w\}$ is the collection

$$K_1 * \{w\} = \{\{v_0\}, \{v_1\}, \{w\}, \{v_0, v_1\}, \{v_0, w\}, \{v_1, w\}, \{v_0, v_1, w\}\}. \tag{41}$$

In particular, 0-simplices of $K_1 * K_2$ are formed out of the disjoint union of 0-simplices of K_1 and K_2.

(d) A **simplicial map** is a map $\varphi : K \to L$ on the sets of vertices, such that whenever $s \in K$ is a simplex, its image $\varphi(s) \in L$ is a simplex in L. The identity map $1_K : K \to K$ is simplicial and composition of simplicial maps is simplicial. In other words, we got ourselves a category of **simplicial complexes** denoted as **Simp**.

(e) We say that $L \subseteq K$ is a **simplicial subcomplex**, if L itself is a simplicial complex. We thus choose some simplices in K and ensure that all their faces are also in L. L is called a **full simplicial subcomplex**, if any simplex in K which has all vertices in L has to be already in L. In other words, a full simplicial subcomplex is completely specified by giving a subset of the vertices of K.

(f) For each simplicial complex K, one can define a category denoted as **Simp**(K) which corresponds to the partially ordered set of all subcomplexes of K.

Now, suppose K is a given simplicial complexe. An **ordered q-simplex** is a sequence $(v_0, \dots, v_q)$ of vertices in K which belong to some simplex. Note that the repetition *is* allowed. An ordered chain complex $(\Delta_\bullet(K), \partial_\bullet)$ is defined as follows. For each $q \geq 0$, define $\Delta_q(K)$ to be the free R-module generated by ordered q-simplices of K. In other words, each element of $\Delta_q(K)$ is a formal R-linear combination of finitely many ordered q-simplices of K. We formally

The boundary operator $\partial_q : \Delta_q(K) \to \Delta_{q-1}(K)$ is defined for every $q > 0$ on the generators by

$$\partial_q(v_0, \dots, v_q) = \sum_{j=0}^{q} (-1)^j (v_0, \dots, \hat{v}_j, \dots, v_q). \tag{42}$$

It is easy to see that $\partial_q \circ \partial_{q+1} = 0$ for all $q > 0$. A sequence $H_q(K)$ of the homology groups is called the **ordered homology of** K.

In fact, if K is non-empty, one can define an augmentation map $\epsilon : \Delta_0(K) \to R$. For every ordered 0-simplex $(v) \in \Delta_0(K)$, define $\epsilon((v)) := 1$. This makes sense as R is assumed to be a unital ring. This map is surjective and $\epsilon \circ \partial_1 = 0$. A cochain complex equipped with such map is called **augmented**.

Equivalently, define another chain complex $(\Delta^\epsilon_\bullet(K), \partial^\epsilon_\bullet)$ where we set $\Delta^\epsilon_{-1}(K) = R$ and $\Delta^\epsilon_q(K) = \Delta_q(K)$ for all $q \geq 0$. The differential is defined as $\partial^\epsilon_0 = \epsilon$ and $\partial^\epsilon_q = \partial_q$ for $q \geq 0$. The homology groups corresponding to this complex are usually denoted as $\widetilde{H}_q(K)$ and called the **reduced ordered homology of** K. Note that as ϵ is surjective, one has $\widetilde{H}_{-1}(K) = 0$. Obviously $\widetilde{H}_q(K) = H_q(K)$ for all $q > 0$. We thus only have to understand $\widetilde{H}_0(K)$. As $\epsilon \circ \partial_1 = 0$, there is an induced epimorphism $\epsilon_* : H_0(K) \to R$. Its kernel is precisely $\widetilde{H}_0(K)$. We thus have a short exact sequence of R-modules

$$0 \longrightarrow \widetilde{H}_0(K) \longrightarrow H_0(K) \xrightarrow{\epsilon_*} R \longrightarrow 0 \tag{43}$$

This proves that there is an isomorphism $H_0(K) \cong \widetilde{H}_0(K) \oplus R$. Explicitly, define a splitting $\sigma : R \to H_0(K)$ of the sequence. Fix some vertex $v \in K$ and define $\sigma(1) = [(v)]$, where $[(v)]$ denotes the class represented by $(v) \in \Delta_0(K)$. Any class in $H_0(K)$ is represented by a finite linear combination $\omega_0 := \sum_k r_k \cdot (v_k)$, where $v_k \in K$ are some vertices of K. Let r be an element of R defined by $r = \epsilon(\omega_0) = \sum_k r_k$. Then $\omega_0 - r \cdot (v) \in \ker(\epsilon)$ and it thus represents a class $[\omega_0 - r \cdot (v)] \in \widetilde{H}_0(K)$. The class $[\omega_0]$ is thus mapped onto the pair $([\omega_0 - r \cdot (v)], r) \in \widetilde{H}_0(K) \oplus R$.

Remark 5.1. There exists a slightly different approach to augmented chain complexes, see e.g. [5]. If $(C_\bullet, \partial_\bullet)$ is any chain complex and $\epsilon : C_0 \to R$ is an augmentation map, then define a chain complex $(\widetilde{C}_\bullet, \widetilde{\partial}_\bullet)$ by setting $\widetilde{C}_0 = \ker(\epsilon) \subseteq C_0$ and $\widetilde{C}_q = C_q$ for $q > 0$. The corresponding cohomology $H_0(\widetilde{C}_\bullet)$ is the same as the one defined in the above diagram.

Let $\mathbf{Ch}_R$ denote the category of chain complexes, where q-chains are R-modules. For every simplicial map $\varphi : K \to L$, define an induced map $\Delta_q(\varphi) : \Delta_q(K) \to \Delta_q(L)$ as

$$\Delta_q(\varphi)((v_0, \dots, v_q)) = (\varphi(v_0), \dots, \varphi(v_q)). \tag{44}$$

This makes sense as if $v_0, \dots, v_q$ belong to some simplex $s \in K$, then $\varphi(v_0), \dots, \varphi(v_q)$ belong to a simplex $\varphi(s) \in L$. It is clear that $\Delta_q(\psi \circ \varphi) = \Delta_q(\psi) \circ \Delta_q(\varphi)$ and $\Delta_q(1_K) = 1_{\Delta_q(K)}$ for every $K \in \mathbf{Simp}$.

Definition 5.2. A chain complex in $\mathbf{Ch}_R$ is called **free**, it its R-module of q-chains is a free R-module for every $q \in \mathbb{Z}$. A functor $\Delta : \mathbf{C} \to \mathbf{Ch}_R$ from any category $\mathbf{C}$ is called **free**, if every $\Delta(c) \in \mathbf{Ch}_R$ is a free chain complex for every $c \in \mathbf{C}$.

Proposition 5.3. *The map $K \mapsto \Delta(K)$ defines a free functor $\Delta : \boldsymbol{Simp} \to \boldsymbol{Ch}_R$.*

Definition 5.4. Let $(C_\bullet, \partial_\bullet) \in \mathbf{Ch}_R$ be a chain complex. We say that that it is **acyclic** if $H_\bullet(C) = 0$. A chain complex is called **chain contractible**, if the identity chain map is chain homotopic to the trivial one.

Those two definitions are closely related. Actually, in the case of a free chain complex, they are equivalent.

Proposition 5.5. *Every chain contractible chain complex in $\boldsymbol{Ch}_R$ is acyclic. Every free acyclic chain complex in $\boldsymbol{Ch}_R$, where R is a principal ideal domain, is contractible.*

Proof. Let $(C_\bullet, \partial_\bullet) \in \mathbf{Ch}_R$ be chain contractible. The identity chain map $1 : C_\bullet \to C_\bullet$ and the trivial map $0 : C_\bullet \to C_\bullet$ are chain homotopic. But then the induced maps 1_* and 0_* on homology groups must coincide. This is possible only if $H_\bullet(C) = 0$.

Conversely, suppose $H_\bullet(C) = 0$. We will explicitly construct the chain homotopy $k : C_\bullet \to C_\bullet$. Every map $\partial_q : C_q \to B_{q-1}(C)$ is an epimorphism. But $B_{q-1}(C) = Z_{q-1}(C)$. We assume that $C_\bullet$ is free and R is a principal ideal domain. This ensures that $Z_{q-1}(C)$ is a free R-module. This allows one to construct a right inverse $s_{q-1} : Z_{q-1}(C) \to C_q$ to ∂_q. The map $1_q - s_{q-1} \circ \partial_q$ then maps C_q into $Z_q(C)$. It thus makes sense to define $k_q : C_q \to C_{q+1}$ as

$$k_q = s_q \circ (1_q - s_{q-1} \circ \partial_q). \tag{45}$$

One only has to check that this is the chain homotopy of the identity $1 : C_\bullet \to C_\bullet$ and the trivial map $0 : C_\bullet \to C_\bullet$. One has

$$\partial_{q+1} \circ k_q + k_{q-1} \circ \partial_q = (\partial_{q+1} \circ s_q) \circ (1_q - s_{q-1} \circ \partial_q) + s_{q-1} \circ (1_{q-1} - s_{q-2} \circ \partial_{q-1}) \circ \partial_q = 1_q. \quad (46)$$

This proves the point. ■

Remark 5.6. Note that in [5] they consider only chain complexes in the category of Abelian groups. Every Abelian group is a $\mathbb{Z}$-module and $\mathbb{Z}$ is a principal ideal domain (and unital ring).

The above statement does not hold for non-free chain complexes, there is a nice counterexample in the same reference.

Definition 5.7. Let $\mathbf{C}$ be any category and let $\mathcal{M}$ be some set of the objects in $\mathbf{C}$. Suppose $F : \mathbf{C} \to \text{R-}\mathbf{Mod}$ is a functor. One says that the collection $\{g_j \in F(m_j)\}_{j \in J}$, where $m_j \in \mathcal{M}$, is a **basis** for F, if for every $c \in \mathbf{C}$, the collection $\{F(h)(g_j)\}_{j \in J, h \in \mathbf{C}(m_j, c)}$ forms a basis for the R-module $F(x)$.

If a functor F has a basis, we say that F is **free functor on C with models** $\mathcal{M}$. A functor $\Delta : \mathbf{C} \to \mathbf{Ch}_R$ is caled a **free functor on C with models** $\mathcal{M}$ if $\Delta_q : \mathbf{C} \to \text{R-}\mathbf{Mod}$ is free on $\mathbf{C}$ with models $\mathcal{M}$ for every $q \in \mathbb{Z}$.

Example 5.8. Let K be simplicial complex. Let $\mathbf{Simp}(K)$ be the category of its subcomplexes. Let us define the subset of models $\mathcal{M}$ as

$$\mathcal{M} = \{\bar{s} \mid s \in K \text{ is a simplex}\} \quad (47)$$

We claim that the functor $\Delta : \mathbf{Simp}(K) \to \mathbf{Ch}_R$ assigning to each subcomplex $L \subseteq K$ the ordered chain complex $(\Delta_\bullet(L), \partial_\bullet)$ is free on $\mathbf{Simp}(K)$ with models $\mathcal{M}$. This requires some work. We have to find the basis for the functor $\Delta_q : \mathbf{Simp}(K) \to \text{R-}\mathbf{Mod}$.

For each simplex $s \in K$, pick the following elements of $\Delta_q(\bar{s})$. If s is a p-simplex, it has vertices $\{w_0, \dots, w_p\}$. Then collect all ordered q-simplices $(v_0, \dots, v_q) \in \Delta_q(\bar{s})$, such that they contain all vertices of s, that is $\{v_0, \dots, v_q\} = \{w_0, \dots, w_p\}$. We claim that this collection forms a basis for Δ_q. Let $L \subseteq K$ be an object in $\mathbf{Simp}(K)$. Then there is a unique arrow $i : \bar{s} \to L$ if and only if the simplex s is in L. Then for any generator $(v_0, \dots, v_q) \in \Delta_q(L)$ the set $\{v_0, \dots, v_p\}$ determines a unique p-simplex $s = \{w_0, \dots, w_p\}$ and $(v_0, \dots, v_q) = \Delta_q(i)(v_0, \dots, v_q)$, where the ordered q-simplex on the right-hand side is in $\Delta_q(\bar{s})$. This shows that the above collection indeed forms a basis. Let us demonstrate this on an example. Let K be a simplicial complex consisting of three vertices and all edges among them:

$$K = \{\{w_0\}, \{w_1\}, \{w_2\}, \{w_0, w_1\}, \{w_0, w_2\}, \{w_1, w_2\}\}. \quad (48)$$

We will now show how the generators of $\Delta_1(K)$ look like. Every generator (v_0, v_1) has its corresponding simplex obtained as a set $\{v_0, v_1\}$. We thus have the following table:

simplex s	$\{w_0\}$	$\{w_1\}$	$\{w_2\}$	$\{w_0, w_1\}$	$\{w_0, w_2\}$	$\{w_1, w_2\}$
generator of $\Delta_1(K)$	(w_0, w_0)	(w_1, w_1)	(w_2, w_2)	(w_0, w_1)	(w_0, w_2)	(w_1, w_2)
				(w_1, w_0)	(w_2, w_0)	(w_2, w_1)

We can thus summarize that the basis for the free functors Δ_q with models $\mathcal{M}$ is a collection

$$\bigcup_{s \in K} \{(v_0, \dots, v_q) \in \Delta_q(\bar{s}) \mid \{v_0, \dots, v_q\} = s\} \quad (49)$$

Note that in general s, the collection $\{(v_0, \dots, v_q) \in \Delta_q(\bar{s}) \mid \{v_0, \dots, v_q\} = s\}$ does not form a basis of $\Delta_q(\bar{s})$. This concludes this important example.

One says the functor $\Delta : \mathbf{C} \to \mathbf{Ch}_R$ is **non-negative** if $\Delta_q(c) = 0$ for all $q < 0$ and all $c \in \mathbf{C}$. Suppose $\Delta : \mathbf{C} \to \mathbf{Ch}_R$ is a functor from the category $\mathbf{C}$ with models $\mathcal{M}$. One says that Δ is **$\mathcal{M}$-acyclic in positive dimensions** if $H_q(\Delta(m)) = 0$ for all $m \in \mathcal{M}$ and $q > 0$. We will now state the most important theorem of this section:

Theorem 5.9. *Let $\Delta : \boldsymbol{C} \to \boldsymbol{Ch}_R$ and $\Delta' : \boldsymbol{C} \to \boldsymbol{Ch}_R$ be a pair of functors, where Δ is non-negative and free on $\boldsymbol{C}$ with models $\mathcal{M}$ and Δ' is $\mathcal{M}$-acyclic in positive dimensions. Then*

(i) any natural transformation of the composed functors $\eta : H_0(\Delta) \to H_0(\Delta')$ is induced by some natural transformation $\tau : \Delta \to \Delta'$;

(ii) any two natural transformations $\tau, \tau' : \Delta \to \Delta'$ inducing the same natural transformation $\tau_* : H_0(\Delta) \to H_0(\Delta')$ are naturally chain homotopic.

Proof. For any object $c \in \mathbf{C}$, we have to define a chain map $\tau(c) : \Delta(c) \to \Delta'(c)$, such that for any $h \in \mathbf{C}(c, d)$, the following diagram commutes:

$$\begin{array}{ccc} \Delta(c) & \xrightarrow{\tau(c)} & \Delta'(c) \\ \downarrow{\scriptstyle \Delta(h)} & & \downarrow{\scriptstyle \Delta'(h)} \\ \Delta(d) & \xrightarrow{\tau(d)} & \Delta'(d) \end{array} . \tag{50}$$

For every $q \geq 0$, let $\{g_j \in \Delta_q(m_j)\}_{j \in J}$ be the basis for the functor Δ_q on $\mathbf{C}$ with models $\mathcal{M}$. One only has to specify $\tau(m_j)(g_j)$. Indeed, for a general object $c \in C$, one has a basis $\{\Delta(h)(g_j)\}_{j \in J, h \in \mathbf{C}(m_j, c)}$ of the group $\Delta(c)$, and the naturality forces

$$\tau(c)(\Delta(h)(g_j)) = \Delta'(h)(\tau(m_j)(g_j)), \tag{51}$$

which specifies τ_c uniquely. Such defined τ_c will be automatically natural in c. We will now inductively define $\tau_q(c)$ for each $q \geq 0$.

Now, let $\eta : H_0(\Delta) \to H_0(\Delta')$ be a natural transformation. For every $j \in J$, let $\tau_0(m_j)(g_j)$ be any representative of the class $\eta(m_j)[g_j]$, that is $[\tau_0(m_j)(g_j)]' = \eta(m_j)[g_j]$. Then define $\tau_0(c)$ using the naturality as described above. This finishes a zeroth induction step. It is easy to see that the construction also ensures the equality

$$[\tau_0(c)(g)]' = \eta(c)[g], \tag{52}$$

for all $c \in \mathbf{C}$ and $g \in \Delta_0(c)$. In particular, η is indeed induced by τ.

Note that g_j are different for each induction step q, we just use the same symbol to denote them - they form a basis for the functor Δ_q for a specific q. Next, we need to take care of the step $q = 1$. For every $j \in J$, we have to define $\tau_1(m_j)(g_j)$, such that

$$(\partial'_1 \circ \tau_1(m_j)(g_j) = (\tau_0(m_j) \circ \partial_1)(g_j) \tag{53}$$

The equation (52) shows that $[\tau_0(m_j)(\partial_1 g_j)]' = \eta(m_j)[\partial_1 g_j] = [0]'$. This proves that $\tau_0(m_j)(\partial_1 g_1) \in B_0(\Delta'(m_j))$ and it is thus a boundary of some element. We set $\tau_1(m_j)(g_j)$ to be this element. Using (51), we define $\tau_1(c)$ for every $c \in \mathbf{C}$ and the induction step $q = 1$ is finished.

Now, suppose $q > 1$ and we have a collection $\tau_r(c) : \Delta_r(c) \to \Delta_r(c')$ for every $0 \leq r < q$, such that $\partial'_r \circ \tau_r(c) = \tau_{r-1}(c) \circ \partial_r$. We will now define $\tau_q(m_j)$ on $g_j \in \Delta_q(m_j)$. We will define $\tau_q(m_j)(g_j)$ so that

$$(\partial'_q \circ \tau_q(m_j))(g_j) = (\tau_{q-1}(m_j) \circ \partial_q)(g_j). \tag{54}$$

First, note that the right-hand side is in $Z_{q-1}(\Delta'(m_j))$. Indeed, one has

$$\partial'_{q-1} \circ (\tau_{q-1}(m_j) \circ \partial_q) = \tau_{q-2}(m_j) \circ (\partial_{q-1} \circ \partial_q) = 0. \tag{55}$$

As $q > 1$, we have $H_{q-1}(\Delta'(m_j)) = 0$ and the right-hand side is thus in $B_{q-1}(\Delta'(m_j))$. It is thus a boundary of some element in $\Delta'_q(m_j)$ which we set equal to $\tau_q(m_j)(g_j)$. Extending it to $\tau_q(c)$ for any $c \in \mathbf{C}$ using (51) finishes the induction step. This finishes the proof of part (i).

We use a similar approach to construct the chain homotopy $k_q(c) : \Delta_q(c) \to \Delta'_{q+1}(c)$ for every $q \geq 0$ and natural in c. For each $q \geq 0$, it must satisfy the condition

$$\tau'_q(c) - \tau_q(c) = \partial'_{q+1} \circ k_q(c) + k_{q-1}(c) \circ \partial_q \tag{56}$$

Note that by convention $k_{-1}(c) = 0$. We do it in a completely analogous manner as before, that is by induction on q. For $q = 0$, let us first define $k_0(m_j)(g_j)$, where $g_j \in \Delta_0(m_j)$. We want it to satisfy the equation

$$(\tau'_0(m_j) - \tau_0(m_j))(g_j) = (\partial'_1 \circ k_0(m_j))(g_j). \tag{57}$$

By assumption, one has $\tau'_{0*} = \tau_{0*}$. This implies

$$[(\tau'_0(m_j) - \tau_0(m_j))(g_j)]' = (\tau'_{0*}(m_j) - \tau_{0*}(m_j))[g_j] = [0]' \tag{58}$$

The left-hand side of the above exaction is thus in $B_0(\Delta'(m_j)$, whence it is a boundary of some element in $\Delta'_1(m_j)$ which we declare to be $k_0(m_j)(g_j)$. We can extend it to $k_0(c) : \Delta_0(c) \to \Delta'(c)$ in the same way as in (51) making it natural in c and satisfying the cochain homotopy equation. This finishes the zeroth induction step.

Now, suppose $q > 0$ and we have a family of natural maps $k_r(c) : \Delta_r(c) \to \Delta_{r+1}(c)$, such that

$$\tau'_r(c) - \tau_r(c) = \partial'_{r+1} \circ k_r(c) + k_{r-1}(c) \circ \partial_r \tag{59}$$

for every $r < q$. Let $\{g_j\}_{j \in J}$ be the basis for the functor Δ_q. We will define $k_q(m_j)$ on g_j to satisfy the condition

$$(\partial'_{q+1} \circ k_q(m_j))(g_j) = (\tau'_q(m_j) - \tau_q(m_j) - k_{q-1}(m_j) \circ \partial_q)(g_j). \tag{60}$$

Using the induction hypothesis, it is easy to see that the right-hand side is in $Z_q(\Delta'(m_j))$. As $q > 0$, it is by assumption on $H_q(\Delta')$ in $B_q(\Delta'(m_j))$ and it is thus a boundary of some element in $\Delta'_{q+1}(m_j)$. We declare $k_q(m_j)(g_j)$ to be this element. One can now extend k_q to all $c \in \mathbf{C}$ by the analogue of (51). This finishes the proof. ■

Definition 5.10. Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes in $\mathbf{Ch}_R$. Let $\tau : C_\bullet \to C'_\bullet$ be a chain map. It's **mapping cone** $C^\tau_\bullet$ is a chain complex, where $C^\tau_q = C_{q-1} \oplus C'_q$, and one defines

$$\partial^\tau_q(c, c') = (-\partial_{q-1}(c), \tau_{q-1}(c) + \partial'_q(c')). \tag{61}$$

It is easy to verify that $\partial^\tau_{q-1} \circ \partial^\tau_q = 0$. One can encode certain properties for the map τ into its mapping cone. Indeed, behold the following proposition:

Proposition 5.11. *The chain map τ is a chain equivalence, if and only if its mapping cone $C^\tau_\bullet$ is chain contractible.*

Proof. Let $\tau : C_\bullet \to C'_\bullet$ be a chain equivalence. By assumption, there is a chain map $\tau' : C'_\bullet \to C_\bullet$ together with a pair of chain homotopies $k : C_\bullet \to C_\bullet$ and $k' : C'_\bullet \to C'_\bullet$, such that $\tau' \circ \tau \sim_k 1_C$ and $\tau \circ \tau' \sim_{k'} 1_{C'}$. We have to define a chain homotopy $\ell : C^\tau_\bullet \to C^\tau_\bullet$, such that $1_{C^\tau} \sim_\ell 0$.

$\ell_q : C^\tau_q \to C^\tau_{q+1}$ is given by a rather complicated formula. For these reasons we stop writing the degree of each map at this moment. We have

$$\ell(c,c') = (\ell^1(c,c'), \ell^2(c,c')), \tag{62}$$

where the component functions ℓ^1 and ℓ^2 are given by

$$\ell^1(c,c') = k(c) + \tau' k' \tau(c) - \tau' \tau k(c) + \tau'(c'), \tag{63}$$
$$\ell^2(c,c') = k' \tau k(c) - k' k' \tau(c) - k'(c'). \tag{64}$$

Recall that we have $\partial^\tau(c,c') = (-\partial(c), \tau(c) + \partial'(c'))$, and also the equations

$$\tau'\tau - 1 = \partial k + k\partial, \tag{65}$$
$$\tau\tau' - 1 = \partial' k' + k'\partial'. \tag{66}$$

We have to verify the equation

$$1 = \partial^\tau \circ \ell + \ell \circ \partial^\tau \tag{67}$$

The first component of this equation applied onto $(c,0)$ boils down to the equation

$$1 = -\partial(k + \tau' k' \tau - \tau'\tau k) - (k + \tau' k' \tau - \tau'\tau k)\partial + \tau'\tau \tag{68}$$

By rearranging the right-hand side and using the chain map property for τ and τ', one finds

$$1 = \tau'\tau - \partial k - k\partial - \tau'(\partial' k' + k'\partial')\tau + \tau'\tau(k\partial + \partial k). \tag{69}$$

Now, this can be rewritten using the definition of k and k' as

$$1 = 1 - \tau'(\tau\tau' - 1)\tau + \tau'\tau(\tau'\tau - 1) \tag{70}$$

One can already see that this is indeed true. The first component applied on $(0,c')$ gives

$$0 = -\partial\tau' + \tau'\partial'. \tag{71}$$

This is precisely the chain map property for τ'. For the second component applied on $(c,0)$:

$$0 = \tau(k + \tau' k' \tau - \tau'\tau k) + \partial'(k'\tau k - k' k' \tau) - (k'\tau k - k' k' \tau)\partial - k'\tau \tag{72}$$

This can be reordered to give the equation

$$0 = (1 - \tau\tau' + \partial' k')\tau k - (1 - \tau\tau' + \partial' k')k'\tau + k' k' \tau\partial - k'\tau k\partial. \tag{73}$$

Using the chain homotopy equation for k' now gives

$$\begin{aligned} 0 = & -k'\partial'\tau k + k'\partial' k'\tau + k' k'\tau\partial - k'\tau k\partial = k'(-\partial'\tau k + \partial' k'\tau + k'\tau\partial - \tau k\partial) \\ &= k'\{(\partial' k' + k'\partial')\tau - \tau(k\partial + \partial k)\} \\ &= k'\{(\tau\tau' - 1)\tau - \tau(\tau'\tau - 1)\}. \end{aligned} \tag{74}$$

We already see that this is true. The second component applied on $(0,c')$ leads to

$$1 = \tau\tau' - \partial' k' - k'\partial'. \tag{75}$$

This is certainly true, and we conclude the only if part. For the if part, suppose $\ell : C^\tau_\bullet \to C^\tau_\bullet$ is a chain contraction. For every $q \geq 0$, define τ', k and k' via the equations

$$\ell_q(0,c') = (\tau'_q(c'), -k'_q(c')), \tag{76}$$
$$\ell_{q+1}(c,0) = (k_q(c), \dots), \tag{77}$$

for all $c \in C_q$ and $c' \in C'_q$. One finds

$$(\partial^{\tau}_{q+1} \circ \ell_q)(0, c') = (-\partial_{q-1}(\tau'_q(c')), \tau_q(\tau'_q(c')) - \partial'_{q+1}(k'_q(c'))) \tag{78}$$
$$(\ell_{q-1} \circ \partial^{\tau}_q)(0, c') = (\tau'_{q-1}(\partial'_q(c')), -k'_{q-1}(\partial'_q(c'))). \tag{79}$$

As ℓ is a chain contraction, the sum of these two terms has to be equal to $(0, c')$. This proves that τ' defines a chain map and k' is a chain homotopy from $\tau \circ \tau'$ to the identity on C'. ■

Now, let us turn our attention to augmented chain complexes. In order to save some space, we make the following definition:

Definition 5.12. By **category of augmented chain complexes $\mathbf{Ch}^{\epsilon}_R$**, we mean a category whose objects are non-negative chain complexes $C_\bullet$ with augmentation ϵ, and morphisms are chain maps $\tau : C_\bullet \to C'_\bullet$ **compatible with the augmentation**, that is

$$\begin{array}{ccc} C_0 & \xrightarrow{\epsilon} & R \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle 1} \\ C'_0 & \xrightarrow{\epsilon'} & R \end{array} \tag{80}$$

commutes. Note that a single chain complex with two different augmentations are considered as two different objects in $\mathbf{Ch}^{\epsilon}_R$.

Note that for any augmented chain complex, one has $H_0(C_\bullet) = \widetilde{H}_0(C_\bullet) \oplus R$, whence an augmented chain complex is never acyclic. Let $C_\bullet$ be an augmented chain complex. By $C^{\epsilon}_\bullet$, we will denote the induced chain complex with added term $C^{\epsilon}_{-1} := R$, that is

$$0 \longleftarrow R \xleftarrow{\epsilon} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} \cdots \tag{81}$$

Finally, the augmentation map ϵ may be viewed as a chain map from $C_\bullet$ to the chain complex R consiting of R in the zero degree and 0 otherwise, that is

$$\begin{array}{ccccccc} 0 & \longleftarrow & C_0 & \xleftarrow{\partial_1} & C_1 & \xleftarrow{\partial_2} & \cdots \\ & & \downarrow{\scriptstyle \epsilon} & & \downarrow & & \\ 0 & \longleftarrow & R & \longleftarrow & 0 & \longleftarrow & \cdots \end{array} \tag{82}$$

commutes. Some properties can be restated in terms of ϵ.

Proposition 5.13. *Let $(C_\bullet, \partial_\bullet) \in \mathbf{Ch}^{\epsilon}_R$. Then $C^{\epsilon}_\bullet$ is chain contractible, if and only if the chain map $\epsilon : C_\bullet \to R$ is a chain equivalence.*

Proof. We will use the mapping cone complex $\bar{C}_\bullet$ for the chain map ϵ. We have $\bar{C}_0 = 0 \oplus R$ and $\bar{C}_q = C_{q-1} \oplus 0$ for $q > 0$. Now, by Proposition 5.11, the chain map ϵ is a chain equivalence, if and only if there is a chain contraction $\ell : \bar{C}_\bullet \to \bar{C}_\bullet$. Parametrize it as

$$\ell_0(0, r) = (k_{-1}(r), 0), \;\; \ell_q(c_{q-1}, 0) = (-k_{q-1}(c_{q-1}), 0), \text{ for } q > 0. \tag{83}$$

Now, let us rewrite the condition $\bar{1}_q = \bar{\partial}_{q+1} \circ \ell_q + \ell_{q-1} \circ \bar{\partial}_q$. For $q = 0$, this gives

$$(0, r) = \bar{1}_0(0, r) = \bar{\partial}_1(k_{-1}(r), 0) = (0, (\epsilon \circ k_{-1})(r)). \tag{84}$$

For $q = 1$, one obtains the condition

$$\begin{aligned}(c_0, 0) = \bar{1}_1(c_0, 0) = \bar{\partial}_2(-k_0(c_0), 0) + \ell_0(0, \epsilon(c_0)) \\ =((\partial_1 \circ k_0 + k_{-1} \circ \epsilon)(c_0), 0).\end{aligned} \tag{85}$$

Finally, for $q > 1$, we find the equation

$$\begin{aligned}(c_{q-1}, 0) = \bar{1}_q(c_{q-1}, 0) = \bar{\partial}_{q+1}(-k_{q-1}(c_{q-1}), 0) + \ell_{q-1}(-\partial_{q-1}(c_{q-1}), 0) \\ = ((\partial_q \circ k_{q-1} + k_{q-2} \circ \partial_{q-1})(c_{q-1}), 0).\end{aligned} \tag{86}$$

We have just proved that ℓ is a chain contraction if and only if k is a chain contraction. ■

Now, suppose $\Delta : \mathbf{C} \to \mathbf{Ch}_R^{\epsilon}$ is a functor from the category $\mathbf{C}$ with models $\mathcal{M}$ to the category of augmented chain complexes. We say that Δ is **$\mathcal{M}$-acyclic**, if the corresponding induced functor, $\Delta^{\epsilon} : \mathbf{C} \to \mathbf{Ch}_R$, is $\mathcal{M}$-acyclic. Note that in particular, Δ is $\mathcal{M}$-acyclic in positive dimensions, but the zeroth cohomology is additionally forced to be R on $\mathcal{M}$. One can now formulate the following version of Theorem 5.9:

Theorem 5.14. *Suppose $\Delta, \Delta' : \boldsymbol{C} \to \boldsymbol{Ch}_R^{\epsilon}$ are two functors,such that Δ is free on $\boldsymbol{C}$ with models $\mathcal{M}$ and Δ' is $\mathcal{M}$-acyclic. Then there exists a natural transformation $\tau : \Delta \to \Delta'$ and any two such natural transformations are chain homotopic.*

Proof. The key is to prove that there is a unique natural transformation $\eta : H_0(\Delta) \to H_0(\Delta')$ commuting with augmentation maps, that is fitting into the diagram

$$\begin{array}{ccc} R & \xleftarrow{\epsilon(c)_*} & H_0(\Delta(c)) \\ \big\downarrow 1 & & \big\downarrow \eta(c) \\ R & \xleftarrow{\epsilon'(c)_*} & H_0(\Delta'(c)) \end{array} . \tag{87}$$

First, by assumption $\widetilde{H}_0(\Delta'(m)) = 0$ for all models $m \in \mathcal{M}$. Recall that in general, there is a short exact sequence of R-modules, for every $c \in \mathbf{C}$ having the form

$$0 \longrightarrow \widetilde{H}_0(\Delta'(c)) \longrightarrow H_0(\Delta'(c)) \xrightarrow{\epsilon'(c)_*} R \longrightarrow 0 \tag{88}$$

For $c = m \in \mathcal{M}$, the map $\epsilon'(m)_* : H_0(\Delta'(m)) \to R$ is an isomorphism. Let $g_j \in \Delta_0(m_j)$ be the basis for the functor Δ_0. We have $\epsilon(m_j)_*[g_j] = \epsilon(m_j)(g_j) = r_j \in R$ for every $j \in J$. By the above assumption, for each $j \in J$, there is a unique class $[z_j]' \in H_0(\Delta_0'(m_j))$, such that $\epsilon'(m_j)_*[z_j]' = r_j$. To satisfy (87), we thus have to set $\eta(m_j)[g_j] = [z_j]'$ for every $j \in J$.

Next, let $c \in \mathbf{C}$. By definition, $\{\Delta_0(h)(g_j)\}_{j \in J, h \in \mathbf{C}(g_j, c)}$ forms a basis of the R-module $\Delta_0(c)$. The naturality of η then forces the equation

$$\begin{aligned}\eta(c)[\Delta(h)(g_j)] = (\eta(c) \circ \Delta(h)_*)[g_j] = (\Delta'(h)_* \circ \eta(m_j))[g_j] \\ = \Delta'(h)_*[z_j]'.\end{aligned} \tag{89}$$

This uniquely determines $\eta(c)$ for all $c \in \mathbf{C}$. It is easy to see that it is natural in c. We have thus constructed a natural transformation $\eta : H_0(\Delta) \to H_0(\Delta')$.

Finally, as Δ is non-negative and free on $\mathbf{C}$ with models $\mathcal{M}$ and Δ' is $\mathcal{M}$-acyclic in positive dimensions, the rest of the theorem statement follows from Theorem 5.9. ■

Corollary 5.15. *Let $\Delta, \Delta' : \boldsymbol{C} \to \boldsymbol{Ch}^{\epsilon}_R$ be pair of two functors valued in augmented chain complexes. Suppose both are free on $\boldsymbol{C}$ with models $\mathcal{M}$ and $\mathcal{M}$-acyclic. Then Δ and Δ' are naturally chain equivalent. In fact, any natural chain map $\tau : \Delta \to \Delta'$ commuting with augmentations is a chain equivalence.*

Proof. By the above theorem, there exist chain maps $\tau : \Delta \to \Delta'$ and $\tau' : \Delta' \to \Delta$ commuting with augmentations. Then $\tau' \circ \tau : \Delta \to \Delta$ and 1_Δ are both natural and commute with augmentations, whence, by the above theorem, they are chain homotopic, $\tau' \circ \tau \sim 1_\Delta$. Similarly, $\tau \circ \tau' \sim 1_{\Delta'}$. The last statement follows from the fact that we have not used any special property of τ and any natural chain map commuting with augmentations will do. ■

Recall that we have already constructed a functor $\Delta : \mathbf{Simp}(K) \to \mathbf{Ch}^{\epsilon}_R$ which was free on $\mathbf{Simp}(K)$ with models $\mathcal{M}$. We will now define another such functor, let us denote it as Δ^a. Suppose the set of vertices of K is totally ordered.

An **oriented q-simplex** is a q-tuple $[v_0, \dots, v_q]$, where $\{v_0, \dots, v_q\}$ is a set of vertices of a simplex in K, and $v_0 < \dots < v_q$. We then define $\Delta'_q(K)$ to be the free R-module generated by oriented q-simplices. For technical purposes, one defines $[v_0, \dots, v_q]$ for any combination of vertices belonging to a simplex in K, where we define

$$[v_0, \dots, v_q] = 0, \tag{90}$$

if any of the vertices repeat, and if they do not repeat, we set

$$[v_0, \dots, v_q] = \operatorname{sgn}(\sigma) \cdot [v_{\sigma(0)}, \dots, v_{\sigma(q)}], \tag{91}$$

where $\sigma \in S_{q+1}$ is a permutation ordering the vertices, that is $v_{\sigma(0)} < \dots < v_{\sigma(q)}$. We define a differential $\partial'_q : \Delta'_q(K) \to \Delta'_{q-1}(K)$ using a similar formula as for the oriented case, that is

$$\partial'_q[v_0, \dots, v_q] = \sum_{j=1}^{0} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_q]. \tag{92}$$

Clearly $\partial'_q \circ \partial'_{q+1} = 0$. A sequence $H'_q(K)$ of the homology groups is called the **oriented homology of K**.

For any simplicial map $\varphi : K \to L$, one can define $\Delta'_q(\varphi) : \Delta'_q(K) \to \Delta'_q(L)$ by

$$\Delta'_q(\varphi)[v_0, \dots, v_q] = [\varphi(v_0), \dots, \varphi(v_q)]. \tag{93}$$

Note that we had to introduce an above convention to deal with the case when φ is not injective and order-preserving. We thus got ourselves a functor $\Delta' : \mathbf{Simp} \to \mathbf{Ch}^{\epsilon}_R$. In fact, we may again view it as a functor from the category $\mathbf{Simp}(K)$ with models $\mathcal{M}$. The subset $\mathcal{M}$ is defined as in Example 5.8, and Δ' is free on $\mathbf{Simp}(K)$ with models $\mathcal{M}$.

We will now observe the main property of the both functors $\Delta, \Delta' : \mathbf{Simp}(K) \to \mathbf{Ch}^{\epsilon}_R$. Note that we need a total ordering on the set of vertices of a komplex K in order to define oriented q-simplices. For these reason, one should think about a total ordering on the set $K \sqcup L$ of vertices of the join $K * L$. We simply declare all vertices of K strictly smaller then those of L.

Proposition 5.16. *Let K be any simplicial complex, and let w be a simplicial complex consisting of a single vertex. Then $\Delta^{\epsilon}_{\bullet}(K * w)$ and $\Delta'^{\epsilon}_{\bullet}(K * w)$ are chain contractible.*

Proof. The proof relies on the Proposition 5.13. As both cases are analogous, we will prove only the Δ case. It suffices to prove that the augmentation map $\epsilon : \Delta_\bullet(K * w) \to R$ is a chain equivalence. We define the inverse $\tau : R \to \Delta_\bullet(K * w)$ as

$$\tau_0(r) = r \cdot (w), \tag{94}$$

for all $r \in R$, and $\tau_q = 0$ for $q > 0$. Obviously it is a chain map and $\epsilon \circ \tau = 1_R$. We only have to prove that $1_{\Delta_\bullet(K*w)} \sim_k \tau \circ \epsilon$ for some chain homotopy $k : \Delta_\bullet(K * w) \to \Delta_\bullet(K * w)$. For any generator $(v_0, \ldots, v_q) \in \Delta_q(K * w)$, set

$$k_q(v_0, \ldots, v_q) = (w, v_0, \ldots, v_q). \tag{95}$$

For $q = 0$, one has $(\partial_1 \circ k_0)(v_0) = \partial_1(w, v_0) = (v_0) - (w) = (1_{\Delta_0(K*w)} - \tau_0 \circ \epsilon)(v_0)$. For $q > 0$, one has $(\tau \circ \epsilon)_q = 0$. For every $(v_0, \ldots, v_q) \in \Delta_q(K * w)$, one has

$$\begin{aligned}
(\partial_{q+1} \circ k_q)(v_0, \ldots, v_q) &= \partial_{q+1}(w, v_0, \ldots, v_q) \\
&= (v_0, \ldots, v_q) - \sum_{j=0}^{q} (-1)^j (w, v_0, \ldots, \hat{v}_j, \ldots, v_q) \\
&= (v_0, \ldots, v_q) - \sum_{j=0}^{q} (-1)^j k_{q-1}(v_0, \ldots, \hat{v}_j, \ldots v_q) \\
&= (v_0, \ldots, v_q) - (k_{q-1} \circ \partial_q)(v_0, \ldots, v_q).
\end{aligned} \tag{96}$$

This proves that k is the chain homotopy and the proof is finished. ■

Corollary 5.17. *The chain complexes $\Delta^\epsilon_\bullet(K * w)$ and $\Delta'^\epsilon_\bullet(K * w)$ are acyclic.*

Corollary 5.18. *For any simplex $s \in K$, the chain complexes $\Delta^\epsilon_\bullet(\bar{s})$ and $\Delta'^\epsilon_\bullet(\bar{s})$ are acyclic. In particular, the functors $\Delta, \Delta' : \boldsymbol{Simp}(K) \to \boldsymbol{Ch}^\epsilon_R$ are $\mathcal{M}$-acyclic.*

Proof. Let s be a q-simplex given by a set of vertices $\{v_0, \ldots, v_q\}$, where we assume $v_0 < \cdots < v_q$. Let s_0 be its face consisting of vertices $\{v_0, \ldots, v_{q-1}\}$. Then we can write $\bar{s} = \bar{s}_0 * v_q$, including the total ordering on the set of vertices. The rest follows from the previous proposition. ■

We assume that K is non-empty so that we have augmented simplicial complexes. First, define the map $\mu(K) : \Delta_\bullet(K) \to \Delta'_\bullet(K)$ as

$$\mu_q(K)(v_0, \ldots, v_q) := [v_0, \ldots, v_q]. \tag{97}$$

This map is natural in K, and it commutes with augmentations. One can define the map $\tau(K) : \Delta'_\bullet(K) \to \Delta_\bullet(K)$ the other way round, which is defined by

$$\tau_q(K)[v_0, \ldots, v_q] := (v_0, \ldots, v_q), \tag{98}$$

where $v_0 < \cdots < v_q$. Again, it is a chain map natural in K, commuting with augmentations. One can easily see that $\mu(K) \circ \tau(K) = 1_{\Delta'_\bullet(K)}$. In fact, from Corollary 5.15, it immediately follows that μ and τ are chain-inverse to each other, that is $\tau(K) \circ \mu(K) \sim 1_{\Delta_\bullet(K)}$.

Let $\nu(K) : \Delta_\bullet(K) \to \Delta_\bullet(K)$ denote the composition $\tau(K) \circ \mu(K)$. Explicitly, one has

$$\nu_q(K)(v_0, \ldots, v_q) = 0, \tag{99}$$

whenever any two vertices repeat. If they are distinct, one has

$$\nu_q(K)(v_0, \ldots, v_q) = \mathrm{sgn}(\sigma) \cdot (v_{\sigma(0)}, \ldots, v_{\sigma(q)}), \tag{100}$$

where $\sigma \in S_{q+1}$ is a permutation so that $v_{\sigma(0)} < \cdots < v_{\sigma(q)}$. By above theorems, we know that there is a chain homotopy $m(K) : \Delta_\bullet(K) \to \Delta_\bullet(K)$, such that $\nu(K) \sim_{m(K)} 1_{\Delta_\bullet(K)}$.

Definition 5.19. Let $m : \Delta_q(K) \to \Delta_r(K)$ be any R-module homomorphism. We say that m **preserves simplices**, if for any $(v_0, \ldots, v_q) \in \Delta_q(K)$, one has

$$m(v_0, \ldots, v_q) = \sum_{\alpha=1} r_\alpha \cdot (w_0^\alpha, \ldots, w_r^\alpha), \tag{101}$$

where $(w_0^\alpha, \ldots, w_r^\alpha) \in \Delta_r(K)$ satisfy $\{w_0^\alpha, \ldots, w_r^\alpha\} \subseteq \{v_0, \ldots, v_q\}$. Equivalently, if $s = \{v_0, \ldots, v_q\}$ is the simplex in K whose vertices form the generator $(v_0, \ldots, v_q)$, the map m stays in the subcomplex generated by $\bar{s}$, that is $m(v_0, \ldots, v_q) \in \Delta_r(\bar{s}) \subseteq \Delta_r(K)$.

We will now trace back the proofs of the theorems to show that for each $q \geq 0$ and $L \subseteq K$, the map $m_q(L) : \Delta_q(L) \to \Delta_{q+1}(L)$ preserves simplices. This will be important for the main application of this section.

First, let $s \in K$ be a given p-simplex consisting of vertices $\{w_0, \ldots, w_p\}$. We will now explicitly construct the chain homotopy $h : \Delta_\bullet^\epsilon(\bar{s}) \to \Delta_\bullet^\epsilon(\bar{s})$. Then we will repeat the proof of Theorem 5.9 finding the chain homotopy $\nu(K)$. Suppose there is an order on K and $w_0 < \cdots < w_p$.

For any ordered q-simplex $(v_0, \ldots, v_q)$, the map k_q is defined as

$$k_q(v_0, \ldots, v_q) = (w_p, v_0, \ldots, v_q). \tag{102}$$

Note that $v_i \in \{w_0, \ldots, w_p\}$ for all $i \in \{0, \ldots, q\}$. We have $\epsilon \circ \tau = 1_R$ and $1_{\Delta_\bullet(\bar{s})} \sim_k \tau \circ \epsilon$.

We then have to use k to construct a chain homotopy $\ell : \bar{C}_\bullet \to \bar{C}_\bullet$ is a mapping cone of $\epsilon : \Delta_\bullet(\bar{s}) \to R$ as in the proof of Proposition 5.11. Finally, the proof of Proposition 5.13 will give us the chain contraction $h : \Delta_\bullet^\epsilon(\bar{s}) \to \Delta_\bullet^\epsilon(\bar{s})$.

Recall that $\bar{C}_0 = 0 \oplus R$ and $\bar{C}_q = \Delta_{q-1}(\bar{s}) \oplus 0$ for $q > 0$. The map ℓ is actually very simple, one finds the expressions

$$\ell_0(0, r) = (\tau_0(r), 0), \quad \ell_q((v_0, \ldots, v_{q-1}), 0) = (-k_{q-1}(v_0, \ldots, v_{q-1}), 0). \tag{103}$$

It is then easy to read out the chain contraction $h : \Delta_\bullet^\epsilon(\bar{s}) \to \Delta_\bullet^\epsilon(\bar{s})$, see the proof of Proposition 5.13. One finds the expressions

$$h_{-1}(r) = \tau_0(r) = r \cdot (w_p), \quad h_q(v_0, \ldots, v_q) = k_q(v_0, \ldots, v_q) = (w_p, v_0, \ldots, v_q) \text{ for } q \geq 0. \tag{104}$$

We can explicitly verify that this is a chain contraction. Indeed, one has

$$(\epsilon \circ h_{-1})(r) = \epsilon(r \cdot (w_p)) = r. \tag{105}$$

In the next degree, one finds

$$(\partial_1 \circ h_0 + h_{-1} \circ \epsilon)(v_0) = \partial_1(w_p, v_0) + h_{-1}(1) = (v_0) - (w_p) + (w_p) = (v_0). \tag{106}$$

For $q > 0$, one has the expression

$$\partial_{q+1} \circ h_q + h_{q-1} \circ \partial_q = \partial_{q+1} \circ k_q + k_{q-1} \circ \partial_q = 1_q, \tag{107}$$

where we have used the fact that $(\tau \circ \epsilon)_q = 0$ for $q > 0$. We have thus found an explicit form of the chain contraction $h : \Delta^\epsilon_\bullet(\bar{s}) \to \Delta^\epsilon_\bullet(\bar{s})$.

Now, we will track the proof of Theorem 5.9 part (*ii*). We have an identical natural transformation $1_\Delta : \Delta_\bullet \to \Delta_\bullet$ and a natural transformation $\nu : \Delta_\bullet \to \Delta_\bullet$. For any $L \subseteq K$ and any $(v_0) \in \Delta_0(L)$, we have $\nu_0(L)(v_0) = 1_0(v_0) = (v_0)$, that is both chain maps induce the same natural transformation $H_0(\Delta) \to H_0(\Delta)$. We will now prove by induction on q that there exists a natural chain homotopy $m_q(L) : \Delta_q(L) \to \Delta_{q+1}(L)$ which **preserves simplices**.

We are thus looking for a sequence of maps $m_q(L)$ satisfying the relation

$$1_q - \nu_q(L) = \partial_{q+1} \circ m_q(L) + m_{q-1}(L) \circ \partial_q \tag{108}$$

for every $q \geq 0$, where we declare $m_{-q}(L) = 0$. As $\nu_0(L) = 1_0$, the good choice for $m_0(L)$ natural in L is obvious, namely $m_0(L) = 0$. Moreover, $m_0(L)$ clearly preserves simplices. This finishes the zeroth induction step.

Suppose $q > 0$ and that for every $0 \leq r < q$, one has a map $m_r(L) : \Delta_r(L) \to \Delta_{r+1}(L)$ which is natural in L and preserves simplices. We have to define $m_q(L)$ satisfying the relation

$$\partial_{q+1} \circ m_q(L) = 1_q - \nu_q(L) - m_{q-1}(L) \circ \partial_q \tag{109}$$

First, one defines $m_q(\bar{s})$, where $s = \{w_0, \dots, w_p\}$ is some p-simplex in K, and $w_0 < \cdots < w_p$ with respect to some given total order on K. The basis elements for Δ_q in $\Delta_q(\bar{s})$ are the generators $(v_0, \dots, v_q)$ satisfying the property $\{v_0, \dots, v_q\} = \{w_0, \dots, w_p\}$. It follows from the induction hypothesis that $1_q - \nu_q(\bar{s}) - m_{q-1}(\bar{s}) \circ \partial_q$ maps $\Delta_q(\bar{s})$ into the q-cochains $Z_q(\Delta_\bullet(\bar{s}))$ and it preserves simplices.

Now, recall that we have a chain contraction $h_q : \Delta^\epsilon_q(\bar{s}) \to \Delta^\epsilon_{q+1}(\bar{s})$, constructed above. As $q > 0$, it is a map $h_q : \Delta_q(\bar{s}) \to \Delta_{q+1}(\bar{s})$ satisfying the condition

$$1_q = \partial_{q+1} \circ h_q + h_{q-1} \circ \partial_q. \tag{110}$$

It follows that h_q restricted onto $Z_q(\Delta_\bullet(\bar{s}))$ is the right inverse to the boundary map ∂_{q+1}. In other words, we find that $m_q(\bar{s})$ must fit into the relation

$$\partial_{q+1} \circ m_q(\bar{s}) = \partial_{q+1} \circ \{h_q \circ (1_q - \nu_q(\bar{s}) - m_{q-1}(\bar{s}) \circ \partial_q)\}. \tag{111}$$

In other words, we may define $m_q(\bar{s})$ using the formula

$$m_q(\bar{s}) = h_q \circ (1_q - \nu_q(\bar{s}) - m_{q-1}(\bar{s}) \circ \partial_q). \tag{112}$$

The right-hand side is a R-module homomorphism which preserves simplices. Indeed, $\nu_q(\bar{s})$, 1_q and ∂_q preserve simplices by definition, $m_{q-1}(\bar{s})$ by induction hypothesis and h_q does so by formula (104). The composition of maps which preserve indices preserves simplices. This proves that $m_q(\bar{s})$ preserves simplices. If $s \in L$ is a p-simplex contained in L, there is a unique morphism $i : \bar{s} \to L$ and the induced map $\Delta_q(\bar{s}) \to \Delta_q(L)$ is just the canonical inclusion. It follows that $m_q(L)$ is defined on each generator $(v_0, \dots, v_q)$, where $\{v_0, \dots, v_q\} = s$, as

$$m_q(L)(v_0, \dots, v_q) = m_q(L)(\Delta_q(i)(v_0, \dots, v_q)) = \Delta_{q+1}(i)\{m_q(\bar{s})(v_0, \dots, v_q)\}, \tag{113}$$

where in the leftmost term, $(v_0, \dots, v_q)$ is viewed as a generator in $\Delta_q(L)$, whereas in the rest of the terms as the generator of $\Delta_q(\bar{s})$. Clearly $m_q(L)$ preserves simplices.

Proposition 5.20. *Let $\nu : \Delta_\bullet \to \Delta_\bullet$ be the natural transformation of the fuctor $\Delta_\bullet : \boldsymbol{Simp}(K) \to \boldsymbol{Ch}_R$ defined above. Then for each $L \in \boldsymbol{Simp}(K)$, there exists a natural chain homotopy $m(L) : \Delta_\bullet(L) \to \Delta_\bullet(L)$ from the identity to $\nu(L)$, which preserves simplices.*

Example 5.21. Consider the simplicial complex K from Example 5.8, that is

$$K = \{\{w_0\}, \{w_1\}, \{w_2\}, \{w_0, w_1\}, \{w_0, w_2\}, \{w_1, w_2\}\}. \tag{114}$$

We will now explicitly construct the map $m(K)$ in lowest degrees. Observe that we can always choose $m_0(K) = 0$. In the next step, we use the naturality and the map

$$m_1(\bar{s}) = h_1 \circ (1_1 - \nu_1(\bar{s})). \tag{115}$$

We are to define $m_1(K)$ on the generators of $\Delta_1(K)$. They are subdivided into their respective simplices according to the table already derived in Example 5.8:

simplex s	$\{w_0\}$	$\{w_1\}$	$\{w_2\}$	$\{w_0, w_1\}$	$\{w_0, w_2\}$	$\{w_1, w_2\}$
generator of $\Delta_1(K)$	(w_0, w_0)	(w_1, w_1)	(w_2, w_2)	(w_0, w_1)	(w_0, w_2)	(w_1, w_2)
				(w_1, w_0)	(w_2, w_0)	(w_2, w_1)

We assume the total ordering $w_0 < w_1 < w_2$ of the vertices of K. As an example, we have for $s = \{w_0, w_1\}$ the expression

$$m_1(\bar{s})(w_1, w_0) = h_1((w_1, w_0) + (w_0, w_1)) = (w_1, w_1, w_0) + (w_1, w_0, w_0) \tag{116}$$

Here (w_1, w_0) is viewed as the generator of $\Delta_1(\bar{s})$. Clearly, the map $m_1(K)$ applied on (w_1, w_0) viewed as a generator in $\Delta_1(K)$ has completely the same form as above, that is

$$m_1(K)(w_1, w_0) = (w_1, w_1, w_0) + (w_1, w_0, w_0). \tag{117}$$

We can now write down the values on all generators, the results are written down in the table:

generator	the value of $m_1(K)$
(w_0, w_0)	(w_0, w_0, w_0)
(w_1, w_1)	(w_1, w_1, w_1)
(w_2, w_2)	(w_2, w_2, w_2)
(w_0, w_1)	0
(w_1, w_0)	$(w_1, w_0, w_1) + (w_1, w_1, w_0)$
(w_1, w_2)	0
(w_2, w_1)	$(w_2, w_1, w_2) + (w_2, w_2, w_1)$
(w_0, w_2)	0
(w_2, w_0)	$(w_2, w_0, w_2) + (w_2, w_2, w_0)$

In particular, we see that $m_1(K)$ indeed preserves simplices, as the value of $m_1(K)$ in the second column contains the vertices of the generator to its left. Finally, as an example, let us calculate $m_2(K)$ on the single generator (w_2, w_2, w_0). One has

$$\begin{aligned} m_2(\bar{s})(w_2, w_2, w_0) &= h_2 \circ (1_2 - \nu_2(\bar{s}) - m_1(\bar{s}) \circ \partial_2)(w_2, w_2, w_0) \\ &= h_2((w_2, w_2, w_0) - 0 - m_1(\bar{s})\{(w_2, w_0) - (w_2, w_0) + (w_2, w_2)\}) \\ &= h_2((w_2, w_2, w_0) - (w_2, w_2, w_2)) \\ &= (w_2, w_2, w_2, w_0) - (w_2, w_2, w_2, w_2). \end{aligned} \tag{118}$$

One can now explicitly test the chain homotopy condition on this particular generator. We have

$$\begin{aligned} (\partial_3 \circ m_2(K) + m_1(K) \circ \partial_2)(w_2, w_2, w_0) &= \partial_3\{(w_2, w_2, w_2, w_0) - (w_2, w_2, w_2, w_2)\} \\ &\quad + m_1(K)\{(w_2, w_0) - (w_2, w_0) + (w_2, w_2)\} \\ &= (w_2, w_2, w_0) - (w_2, w_2, w_0) \\ &\quad + (w_2, w_2, w_0) - (w_2, w_2, w_2) \\ &\quad + (w_2, w_2, w_2) \\ &= (w_2, w_2, w_0) = (1_2 - \nu_2(\bar{s}))(w_2, w_2, w_0). \end{aligned} \tag{119}$$

We see that everything works as it should.

6 Back to Čech

Now, let us return to the Čech cohomology. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a topological space X. Let K_A denote the simplicial complex consisting of all non-empty finite subsets of the indexing (assumed totally ordered) set A. Now, let $\tau : \Delta_q(K_A) \to \Delta_r(K_A)$ be a R-module homomorphism which *preserves simplices*. In other words, we have

$$\tau(\alpha_0, \dots, \alpha_q) = \sum_j r_j \cdot (\alpha_0^j, \dots, \alpha_r^j), \tag{120}$$

where for each j, one has $\{\alpha_0^j, \dots, \alpha_r^j\} \subseteq \{\alpha_0, \dots, \alpha_q\}$. But this ensures that there exists an inclusion map $i^{(\alpha_0^j, \dots, \alpha_r^j)}_{(\alpha_0, \dots, \alpha_q)} : U_{\alpha_0, \dots, \alpha_q} \to U_{\alpha_0^j, \dots, \alpha_r^j}$, and the corresponding restriction map

$$\rho^{(\alpha_0^j, \dots, \alpha_r^j)}_{(\alpha_0, \dots, \alpha_q)} : \mathcal{F}(U_{\alpha_0^j, \dots, \alpha_r^j}) \to \mathcal{F}(U_{\alpha_0, \dots, \alpha_q}) \tag{121}$$

We can thus use τ to induce an R-module homomorphism $\tau^* : C^r(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F})$ defined by

$$(\tau^*(\omega))_{\alpha_0, \dots, \alpha_q} = \sum_j r_j \cdot \rho^{(\alpha_0^j, \dots, \alpha_r^j)}_{(\alpha_0, \dots, \alpha_q)}(\omega_{\alpha_0^j, \dots, \alpha_r^k}). \tag{122}$$

It is not difficult to see that if $\tau' : \Delta_p(K_A) \to \Delta_q(K_A)$ is another R-module morphism which preserves simplices, then $\tau \circ \tau'$ also preserves simplices and

$$(\tau \circ \tau')^* = \tau'^* \circ \tau^*, \quad 1^* = 1. \tag{123}$$

There are two examples of such induced maps.

Example 6.1. Let $\mathcal{F} : \mathbf{Op}(X) \to \text{R-}\mathbf{Mod}$ be a presheaf. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover. Then the Čech differential $\delta^p_{\mathcal{F}} : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ can be written as $\delta^p_{\mathcal{F}} = \partial^*_{p+1}$, where $\partial_{p+1} : \Delta_{p+1}(K_A) \to \Delta_p(K_A)$ is the boundary operator. Indeed, let $\omega \in C^p(\mathcal{U}, \mathcal{F})$. Recall that:

$$\partial_{p+1}(\alpha_0, \dots, \alpha_{p+1}) = \sum_{i=0}^{p+1} (-1)^i (\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{p+1}). \tag{124}$$

We have denoted the restriction morphisms from $U_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}$ to $U_{\alpha_0 \dots \alpha_{p+1}}$ as $\rho^i_{\alpha_0 \dots \alpha_{p+1}}$. Hence

$$(\partial^*_{p+1}(\omega))_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \rho^i_{\alpha_0 \dots \alpha_{p+1}}(\omega_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}). \tag{125}$$

But this is precisely the formula defining the Čech differential $\delta^p_{\mathcal{F}}$. In particular, one may prove the property $\delta^{p+1}_{\mathcal{F}} \circ \delta^p_{\mathcal{F}} = 0$ using the property $\partial_{p-1} \circ \partial_p = 0$ and (123).

Example 6.2. Now, consider another map, namely the chain map $\nu_q(K_A) : \Delta_q(K_A) \to \Delta_q(K_A)$. We now assume that A is totally ordered. Write $\nu_q \equiv \nu_q(K_A)$. We have

$$\nu_q(\alpha_0, \dots, \alpha_q) = 0, \tag{126}$$

whenever some indices repeat. If they do not repeat, we have $\nu_q(\alpha_0, \ldots, \alpha_q) = \mathrm{sgn}(\sigma)\cdot(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(q)})$, where $\sigma \in S_{q+1}$ is the permutation such that $\alpha_{\sigma(0)} < \cdots < \alpha_{\sigma(q)}$. For any $\omega \in C^p(\mathcal{U}, \mathcal{F})$, we then have, for the induced map $\nu_p^* : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$:

$$(\nu_p^*(\omega))_{\alpha_0 \ldots \alpha_p} = 0, \tag{127}$$

whenever any of the two indices repeat. If they do not repeat, one has

$$(\nu_p^*(\omega))_{\alpha_0 \ldots \alpha_p} = \mathrm{sgn}(\sigma) \cdot \omega_{\alpha_{\sigma(0)} \ldots \alpha_{\sigma(p)}}, \tag{128}$$

where $\sigma \in S_{p+1}$ is the permutation, such that $\alpha_{\sigma(0)} < \cdots < \alpha_{\sigma(p)}$.

Now, recall that we have a subcomplex $C'^p(\mathcal{U}, \mathcal{F}) \subseteq C^p(\mathcal{U}, \mathcal{F})$ of alternating cochains. Let $i^p : C'^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$ denote the inclusion. Moreover, there is also a canonical (up to an ordering on the indexing set A of $\mathcal{U}$) projection $\pi^p : C^p(\mathcal{U}, \mathcal{F}) \to C'^p(\mathcal{U}, \mathcal{F})$ defined as

$$(\pi^p(\omega))_{\alpha_0 \ldots \alpha_p} = 0, \tag{129}$$

if any two indices repeat. If they not, one sets $(\pi^p(\omega))_{\alpha_0 \ldots \alpha_p} = \mathrm{sgn}(\sigma) \cdot \omega_{\alpha_{\sigma(0)} \ldots \alpha_{\sigma(p)}}$, where σ is a permutation such that $\alpha_{\sigma(0)} < \cdots < \alpha_{\sigma(p)}$. One has

$$\pi^p \circ i^p = 1_p, \;\; i^p \circ \pi^p = \nu_p^*, \tag{130}$$

where $\nu_p^* : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$ is the map constructed in the previous example. We can now formulate the main proposition of this section.

Theorem 6.3. *The map $i^p : C'^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$ induces an R-module isomorphism $i^p_* : \check{\mathrm{H}}'^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F})$. Its inverse is π^p_* induced by $\pi^p : C^p(\mathcal{U}, \mathcal{F}) \to C'^p(\mathcal{U}, \mathcal{F})$.*

Proof. Clearly $\pi^p_* \circ i^p_* = 1_p$. We only have to prove the second equation, that is $(\nu_p^*)_* = 1_p$. Recall that in Proposition, we have constructed a chain homotopy $m(K) : \Delta_\bullet(K) \to \Delta_\bullet(K)$. Write $m_p = m_p(K)$. The R-module morphism $m_p : \Delta_p(K) \to \Delta_{p+1}(K)$ preserves simplices. Moreover, it fits into the equation

$$\partial_{p+1} \circ m_p + m_{p-1} \circ \partial_p = 1_p - \nu_p \tag{131}$$

This equation works for every $p \geq 0$ and we assume $m_{-1} = 0$. It follows from (123) that

$$m_p^* \circ \partial_{p+1}^* + \partial_p^* \circ m_{p-1}^* = 1_p - \nu_p^*. \tag{132}$$

Recall that by Example 6.1, one has $\partial_{p+1}^* = \delta_{\mathcal{F}}^p$, whence we get

$$m_p^* \circ \delta_{\mathcal{F}}^p + \delta_{\mathcal{F}}^{p-1} \circ m_{p-1}^* = 1_p - \nu_p^* \tag{133}$$

If we write $h^{p+1} := m_p^*$ for every $p \geq 0$, we find the cochain homotopy relation

$$\delta_{\mathcal{F}}^{p-1} \circ h^p + h^{p+1} \circ \delta_{\mathcal{F}}^p = 1_p - \nu_p^* \tag{134}$$

But this proves that the induced maps of cohomology groups on both sides vanish, that is $0 = 1_p - (\nu_p^*)_*$. This proves the claim. ■

This theorem shows that one can use the subcomplex of alternating cochains to calculate the Čech cohomology valued in the presheaf $\mathcal{F}$ corresponding to the open cover $\mathcal{U}$. However, we will stick to the "ordered" case in order to discuss the behavior of the Čech cochains with respect to the refinements. Let $\mathcal{U} \prec \mathcal{V}$. Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta\in B}$, and let $\phi : B \to A$ be the map from the definition of the refinement, that is $V_\beta \subseteq U_{\phi(\beta)}$ for all $\beta \in B$. Note that in the definition of the relation $\prec$, one assumes that there *exists some map* ϕ.

Now, let us construct an R-module morphism $\phi^p : C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{V},\mathcal{F})$ as follows. For any $\omega \in C^p(\mathcal{U},\mathcal{F})$, define

$$(\phi^p(\omega))_{\beta_0\dots\beta_p} = \rho^{U_{\phi(\beta_0)\dots\phi(\beta_p)}}_{V_{\beta_0\dots\beta_p}}(\omega_{\phi(\beta_0)\dots\phi(\beta_p)}). \tag{135}$$

This makes sense as $V_{\beta_0\dots\beta_p} \subseteq U_{\phi(\beta_0)\dots\phi(\beta_p)}$. Moreover, it is easy to see that it commutes with the Čech differential and thus defines a cochain map. From the functoriality of the presheaf $\mathcal{F}$, it follows that if $\mathcal{V} \prec \mathcal{W}$ is another refinement using some map $\phi' : C \to B$, where $\mathcal{W} = \{W_\gamma\}_{\gamma\in C}$, then $\phi'^p \circ \phi^p = (\phi' \circ \phi)^p$. This suggests that the assignment $\mathcal{U} \mapsto C^p(\mathcal{U},\mathcal{F})$ might define a covariant functor from $\mathbf{OpC}(X)$ to R-$\mathbf{Mod}$. However, this is not true. When $\mathcal{U} \prec \mathcal{V}$, there may be another map $\psi : B \to A$ such that $V_\beta \subseteq U_{\psi(\beta)}$ for all $\beta \in B$. In general $\phi^p \neq \psi^p$. On the level of cohomology though, the statement is true.

Proposition 6.4. *Let $\mathcal{U} \prec \mathcal{V}$ be two open covers , where $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta\in B}$, and suppose $\phi, \psi : B \to A$ are two maps from the definition of refinement.*

*Let $\phi^p_*, \psi^p_* : \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{V},\mathcal{F})$ be R-module morphisms induced by the chain maps defined above. Then $\phi^p_* = \psi^p_*$. In other words, the assignment $\mathcal{U} \mapsto \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F})$ defines a covariant functor from **OpC**(X) to* R-***Mod***. *In particular, if $\mathcal{U} \asymp \mathcal{V}$, one has an isomorphism $\check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) \cong \check{\mathrm{H}}^p(\mathcal{V},\mathcal{F})$.*

Proof. The key idea is, as usual, to construct a cochain homotopy of ϕ^p_* and ψ^p_*. Define

$$(k^p(\omega))_{\beta_0\dots\beta_{p-1}} = \sum_{r=0}^{p-1} (-1)^r \rho^{(r)}(\omega_{\phi(\beta_0)\dots\phi(\beta_r)\psi(\beta_r)\dots\psi(\beta_{p-1})}), \tag{136}$$

for all $\omega \in C^p(\mathcal{U},\mathcal{F})$. Here $\rho^{(r)}$ denotes the restriction induced by the inclusion

$$V_{\beta_0\dots\beta_{p-1}} \subseteq U_{\phi(\beta_0),\dots,\phi(\beta_r),\psi(\beta_r),\dots,\psi(\beta_{p-1})}. \tag{137}$$

We claim that $k^p : C^p(\mathcal{U},\mathcal{F}) \to C^{p-1}(\mathcal{V},\mathcal{F})$ fits into the equation

$$\psi^p - \phi^p = \delta^{p-1}_{\mathcal{F}} \circ k^p + k^{p+1} \circ \delta^p_{\mathcal{F}}. \tag{138}$$

To show this, let us drop an explicit writing of the restrictions, as everything is functorial and we do not have to take care of compositions. For any $\omega \in C^p(\mathcal{U},\mathcal{F})$, one has

$$\begin{aligned} ((\delta^{p-1}_{\mathcal{F}} \circ k^p)(\omega))_{\beta_0\dots\beta_p} &= \sum_{j=0}^{p} (-1)^j (k^p(\omega))_{\beta_0\dots\hat{\beta}_j\dots\beta_p} \\ &= \sum_{0\leq r<j\leq p} (-1)^{j+r} \omega_{\phi(\beta_0)\dots\phi(\beta_r)\psi(\beta_r)\dots\psi(\hat{\beta}_j)\dots\psi(\beta_p)} \\ &\quad + \sum_{0\leq j<r\leq p} (-1)^{j+r+1} \omega_{\phi(\beta_0)\dots\phi(\hat{\beta}_j)\dots\phi(\beta_r)\psi(\beta_r)\dots\psi(\beta_p)} \end{aligned} \tag{139}$$

For the other term, one has

$$\begin{aligned}((k^{p+1}\circ\delta^p_{\mathcal{F}})(\omega))_{\beta_0\dots\beta_p} &= \sum_{r=0}^{p}(-1)^r(\delta^p_{\mathcal{F}}(\omega))_{\phi(\beta_0)\dots\phi(\beta_r)\psi(\beta_r)\dots\psi(\beta_p)} \\ &= \sum_{0\leq j<r\leq p}(-1)^j\omega_{\phi(\beta_0)\dots\phi(\hat{\beta}_j)\dots\phi(\beta_r)\psi(\beta_r)\dots\psi(\beta_p)} \\ &\quad+\sum_{r=0}^{p}\omega_{\phi(\beta_0)\dots\phi(\beta_{r-1})\psi(\beta_r)\dots\psi(\beta_p)} \\ &\quad-\sum_{r=0}^{p}\omega_{\phi(\beta_0)\dots\phi(\beta_r)\psi(\beta_{r+1})\dots\psi(\beta_p)} \\ &\quad+\sum_{0\leq r<j\leq p}(-1)^{j+1}\omega_{\phi(\beta_0)\dots\phi(\beta_r)\psi(\beta_r)\dots\psi(\hat{\beta}_j)\dots\psi(\beta_p)}\end{aligned} \tag{140}$$

If we sum the two expressions, the double sums cancel each other. In the sum of two single sums over r, all terms cancel except for the case $r = 0$ in the first sum and $r = p$ in the second sum. These remaining two terms are

$$\omega_{\psi(\beta_0)\dots\psi(\beta_p)} - \omega_{\phi(\beta_0)\dots\phi(\beta_p)} = ((\psi^p - \phi^p)(\omega))_{\beta_0\dots\beta_p}. \tag{141}$$

This finishes the proof as every two chochain homotopic chain maps induce the same maps on cohomology. The rest of statements is obvious. ■

We have thus constructed a direct mapping family $\check{\mathrm{H}}^p(\cdot,\mathcal{F}) : \mathbf{OpC}(X) \to$ R-**Mod** for each $p \geq 0$. If $\mathcal{U} \prec \mathcal{V}$, denote by $\rho^{\mathcal{U}}_{\mathcal{V}} : \check{\mathrm{H}}^\bullet(\mathcal{U},\mathcal{F}) \to \check{\mathrm{H}}^\bullet(\mathcal{V},\mathcal{F})$ the graded R-module homomorphism induced for each $p \geq 0$ by the chain map $\phi^p : C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{V},\mathcal{F})$, as discussed in the previous paragraphs. We know that although $\mathbf{OpC}(X)$ usually forms a large category, the directed limit of any direct mapping family exists and can be calculated over any cofinal subset. We can consider either the subset $\mathbf{OpC}_S(X)$ of open covers indexed by subsets of 2^X, or countable $\mathbf{OpC}_{\mathbb{N}}(X)$. For compact X, we may even consider the subset of finite open covers. The following definition thus makes sense:

Definition 6.5. Let X be a topological space and $\mathcal{F}$ a presheaf of R-modules on X. Then Čech cohomology groups $\check{\mathrm{H}}^p(X,\mathcal{F})$ with values in $\mathcal{F}$ are defined by

$$\check{\mathrm{H}}^p(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}). \tag{142}$$

The classical Čech cohomology groups $\check{\mathrm{H}}^p(X,G)$ are the groups $\check{\mathrm{H}}^p(X,G_X)$, where G_X is the constant presheaf with value $G \in$ R-**Mod**.

Remark 6.6. Recall that for a given $G \in$ R-**Mod**, there is also a **constant sheaf**, denoted as $\widetilde{G}_X$, consisting of locally constant functions on X with values in G, that is

$$\widetilde{G}_X(U) = \{f : U \to G \mid f \text{ is locally constant}\} \tag{143}$$

The classical cohomology groups are sometimes (see e.g. [1]) defined to be the Čech cohomology groups with values in the sheaf $\widetilde{G}_X$. One can show that if X is paracompact, both definitions are equivalent in a sense that $\check{\mathrm{H}}^p(X,G_X) \cong \check{\mathrm{H}}^p(X,\widetilde{G}_X)$. However, the actual proof requires a huge amount of work which is far beoynd the scope of these little nots.

7 Stalks and the sheafification of a presheaf

To a given presheaf $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{C}$ on a topological space X with values in $\mathbf{C}$ and a given point $x \in X$, one may assign a particular object in $\mathbf{C}$. It is constructed as a direct limit, we thus have to assume that colimits exists in $\mathbf{C}$. This will be the case e.g. for **Grp**, **Set**, R-**Mod**, **Ab** etc. For simplicity, assume implicitly that $\mathbf{C} = $ R-$\mathbf{Mod}$.

For each $x \in X$, one may consider the subset $\mathbf{Op}_x(X) = \{U \in \mathbf{Op}(X) \mid x \in X\}$. Define a partial order $\sqsubseteq$ on $\mathbf{Op}_x(X)$ as follows. We say that $U \sqsubseteq V$ if $U \supseteq V$. It is thus opposite to the partial order given by inclusion. Then $(\mathbf{Op}_x(X), \sqsubseteq)$ is a directed set. Indeed, for any $U, V \in \mathbf{Op}_x(X)$, we have $U \cap V \in \mathbf{Op}_x(X)$ and $U, V \supseteq U \cap V$, whence $U, V \sqsubseteq U \cap V$.

One can view $(\mathbf{Op}_x(X), \sqsubseteq)$ as a category, obtained as a opposite to the subcategory of $\mathbf{Op}(X)$ with inclusion $\subseteq$. In the following, by $\mathbf{Op}_x(X)$ we will always denote this category, that is there is a unique arrow from U to V, if $U \sqsubseteq V$. Let $\mathcal{F}_{(x)} : \mathbf{Op}_x(X) \to \mathbf{C}$ be the restriction of $\mathcal{F}$ to *this* category. In particular, $\mathcal{F}_{(x)}$ is a *covariant functor*. As $\mathbf{Op}_x(X)$ is a directed set, it follows that $\mathcal{F}_{(x)}$ defines a direct mapping family.

Definition 7.1. Let $\mathcal{F} : \mathbf{Op}(X) \to$ R-$\mathbf{Mod}$ be a presheaf on X. Then a **stalk of $\mathcal{F}$ at x** is defined as the direct limit

$$\mathcal{F}_x := \varinjlim_{U \in \mathbf{Op}_x(X)} \mathcal{F}_{(x)}. \tag{144}$$

In particular, for every $U \in \mathbf{Op}_x(X)$, there is a R-module morphism $\rho_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x$. For a given local section $f \in \mathcal{F}(U)$, the element $f_x := \rho_{U,x}(f) \in \mathcal{F}_x$ is called the **germ of f at x**.

Example 7.2. Let $\mathcal{F} = C^0$ be the sheaf of continous functions, that is $\mathcal{F}(U) = C^0(U)$. Recall the construction of the direct limit. We have

$$\mathcal{F}_x = \bigsqcup_{U \ni x} C^0(U)/\sim, \tag{145}$$

where the locally defined functions $f \in C^\infty(U)$ and $g \in C^\infty(V)$ are declared equivalent, if there exists an open neighborhood W of x, such that $W \subseteq U \cap V$ and $f|_W = g|_W$.

This example is in fact quite general. Every element of the stalk $\mathcal{F}_x$ is represented by some local section $s \in \Gamma(U, \mathcal{F})$ of $\mathcal{F}$ over $U \ni x$. If $t \in \Gamma(V, \mathcal{F})$ over $V \ni x$ represents the same element, that is $s_x = f_x$, then $s \sim t$ and there is thus some $W \subseteq U \cap V$, such that $s|_W = t|_W$.

So far there was nothing special about the sheaves and their stalks. We fix this in the following proposition:

Proposition 7.3. *Suppose $\mathcal{F}$ is a presheaf satisfying the monopresheaf axiom (B) of Definition 1.4. Then for any open subset $U \subseteq X$, and any local sections $s, t \in \Gamma(U, \mathcal{F})$, one has $s = t$ if and only if $s_x = t_x$ for all $x \in U$.*

Proof. Let $\sim_x$ denote the equivalence relation in the definition of $\mathcal{F}_x$. If $s = t$ then clearly $s \sim_x t$ and thus $s_x = t_x$ for all $x \in U$. Conversely, suppose $s_x = t_x$ for all $x \in U$. For every $x \in U$, we thus have $s \sim_x t$. There thus exists an open neighborhood $W_x \ni x$, such that $W_x \subseteq U$ and $s|_{W_x} = t|_{W_x}$. But then $\mathcal{U} = \{W_x\}_{x \in U}$ forms an open cover of U. By monopresheaf axiom, we have $s = t$. This finishes the proof. ■

Now, suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of two presheaves. In other words, φ is a natural transformation of the two functors. For each $x \in X$, the universality of colimits induces a unique map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ of the corresponding stalks. This is a general statement, which we can formulate as a lemma.

Lemma 7.4. *Suppose* $\mathbf{J}$ *is a directed set and* $F, G : \mathbf{J} \to \boldsymbol{C}$ *two direct mapping families (two covariant functors). Suppose* $\boldsymbol{C}$ *has colimits.*

Let $\eta : F \to G$ *be a natural transfomation. Then there exists a unique map* $\eta' : \varinjlim_{j \in \mathbf{J}} F_j \to \varinjlim_{j \in \mathbf{J}} G_j$ *fitting into the commutative diagram*

$$\begin{array}{ccc} F_j & \xrightarrow{\eta_j} & G_j \\ \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle \theta_j} \\ \varinjlim_{j \in \mathbf{J}} F_j & \xrightarrow{\eta'} & \varinjlim_{j \in \mathbf{J}} G_j \end{array} \tag{146}$$

for each $j \in \mathbf{J}$, where π_j and θ_j are the maps from the definition of a direct limit. Moreover, if $H : \mathbf{J} \to \mathbf{C}$ is another functor and $\nu : G \to H$ an another natural transformation, one has $(\nu \circ \eta)' = \nu' \circ \eta'$ and the identity natural transformation induces an identity on direct limits.

Proof. This is a simple consequence of universal property. Indeed, consider a collection of maps $\chi_j = \theta_j \circ \eta_j$ from F_j to $\varinjlim_{j \in \mathbf{J}} G_j$. The naturality of η ensures that for every $h \in \mathbf{J}(j, j')$, one has $\chi_j = \chi_{j'} \circ F(h)$. Indeed, we can write

$$\chi_{j'} \circ F(h) = (\theta_{j'} \circ \eta_{j'}) \circ F(h) = \theta_{j'} \circ G(h) \circ \eta_j = \theta_j \circ \eta_j = \chi_j. \tag{147}$$

But the universality then ensures that there is a unique $\eta' : \varinjlim_{j \in \mathbf{J}} F_j \to \varinjlim_{j \in \mathbf{J}} G_j$ making the above diagram commutative. The rest of the statements follows immediately from the uniqueness. ∎

Corollary 7.5. *For every map of sheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ *and every* $x \in X$*, there is a unique stalk map* $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$*, such that for every local section* $s \in \Gamma(U, \mathcal{F})$*, one has* $\varphi_x(s_x) = \{\varphi_U(s)\}_x$*.*

Proof. For each $x \in X$, the map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ defines a natural transformation $\varphi_{(x)} : \mathcal{F}_{(x)} \to \mathcal{G}_{(x)}$. The rest follows from the lemma, and the property $\varphi_x(s_x) = \{\varphi_U(s)\}_x$ is precisely the commutativity of (146). ∎

Again, for sheaves, we expect something more to be said about stalk maps.

Proposition 7.6. *Suppose* $\mathcal{F}$ *and* $\mathcal{G}$ *are two presheaves on* X*, and let* $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ *be two maps of presheaves. Suppose* $\mathcal{G}$ *satisfies a monopreshaf axiom (B) of Definition 1.4. Then* $\varphi = \psi$ *if and only if* $\varphi_x = \psi_x$ *for all* $x \in X$*.*

Proof. One direction is obvious. Conversely, let $\varphi_x = \psi_x$ for all $x \in X$. We have to show that $\varphi_U = \psi_U$ for all $U \in \mathbf{Op}(X)$. For any section $s \in \Gamma(U, \mathcal{F})$, one has $\{\varphi_U(s)\}_x = \varphi_x(s_x) = \psi_x(s_x) = \{\psi_U(s)\}_x$ for every $x \in U$. As $\mathcal{G}$ satisfies the monopresheaf axiom, it follows from Proposition 7.3 that $\varphi_U(s) = \psi_U(s)$. This concludes the proof. ∎

Next, we will now follow on the remark in Example 1.6. To each presheaf $\mathcal{F}$, we will now assign a sheaf $\widetilde{\mathcal{F}}$ of sections of certain topological space, which will be isomorphic to $\mathcal{F}$ in the case it is a sheaf. First, let us construct a topological space $S\mathcal{F}$ as a disjoint union of stalks:

$$S\mathcal{F} = \bigsqcup_{x \in X} \mathcal{F}_x. \tag{148}$$

Let $p : S\mathcal{F} \to X$ be a surjective projection map which assigns x to each element of $\mathcal{F}_x$. To each local section $s \in \mathcal{F}(U)$, we may assign an actual section of the above projection, that is a map $\widetilde{s} : U \to S\mathcal{F}$ satisfying $p \circ \widetilde{s} = 1_U$. Set $\widetilde{s}(x) := s_x$ for all $x \in U$. We define topology on $S\mathcal{F}$ to make the functions $\widetilde{s}$ continuous for all $s \in \mathcal{F}(U)$ and $U \in \mathbf{Op}(X)$. Equivalently, the subset Ω is open, if for every $U \subseteq X$ and every $s \in \Gamma(U, \mathcal{F})$, the set $\{x \in U \mid s_x \in \Omega\}$ is open in X.

Definition 7.7. The space $S\mathcal{F}$ defined by (148), together with the described topology, is called the **stalk space of the presheaf $\mathcal{F}$**.

Let us examine a little bit more the topology on $S\mathcal{F}$. It is useful to find some its basis.

Proposition 7.8. *The sets $\widetilde{s}(U)$ are open in $S\mathcal{F}$. Moreover, the collection of such sets over all $U \in \boldsymbol{Op}(X)$ and $s \in \Gamma(U, \mathcal{F})$ forms a basis $\mathcal{B}$ for the topology on $S\mathcal{F}$.*

Proof. Let $\Omega = \widetilde{s}(U)$. Let $V \subseteq X$ and $t \in \Gamma(V, \mathcal{F})$ be arbitrary section. We have to check that

$$\widetilde{t}^{-1}(\Omega) = \{x \in V \mid t_x \in \Omega\} \tag{149}$$

is an open subset. If it is empty, the statement is trivial. Hence suppose it is not empty. Let $x \in \widetilde{t}^{-1}(\Omega)$. This implies $x \in U \cap V$ and $t_x = s_x$. There is thus an open set $W \ni x$, such that $W \subseteq U \cap V$ and $t|_W = s|_W$. In particular, $s_y = t_y$ for all $y \in W$. Whence $W \subseteq \widetilde{t}^{-1}(\Omega)$ and $\widetilde{t}^{-1}(\Omega)$ is open. As V and t were arbitrary, we have proved that Ω is open.

To check that $\mathcal{B}$ forms the basis of the topology, we have to show that any open set Ω can be written as a union of sets in $\mathcal{B}$. Let $e \in \Omega$ be a given point. By definition $e = s_x$ for some $x \in X$ and $s \in \Gamma(V, \mathcal{F})$ where $U \ni x$. We know that the set $V = \widetilde{s}^{-1}(\Omega) \subseteq U \subseteq X$ is open. Let $t = s|_V$. We claim that $\widetilde{t}(V) \subseteq \Omega$. Every element in $\widetilde{t}(V)$ is of the form t_y for some $y \in V$. By construction $t_y = s_y$. But V was defined as a set of $y \in U$, such that $s_y \in \Omega$. For each $e \in \Omega$, we have found its neighborhood in $\mathcal{B}$ fully contained in Ω. This proves the claim.

Finally, for $\Omega, \Omega' \in \mathcal{B}$, and every $x \in \Omega \cap \Omega'$, there must be $\Omega_0 \in \mathcal{B}$, such that $x \in \Omega_0$ and $\Omega_0 \subseteq \Omega \cap \Omega'$. Let $\Omega = \widetilde{s}(U)$ and $\Omega' = \widetilde{t}(V)$. If they have a non-empty intersection, there must be $x \in U \cap V$ such that $s_x = t_x$. But then there is an open set $W \subseteq U \cap V$, such that $r = s|_W = t|_W$. It follows that $\Omega_0 = \widetilde{r}(W)$ satisfies the claim. ■

Finally, we may prove the main statement about the projection map:

Proposition 7.9. *The map $p : S\mathcal{F} \to X$ is a continuous map. In fact, it is a local homeomorphism.*

Proof. To show that p is continuous, we will show that for any $U \subseteq X$, one has

$$p^{-1}(U) = \bigcup_{V \subseteq U} \bigcup_{s \in \mathcal{F}(V)} \widetilde{s}(V), \tag{150}$$

where all V in the first union are open. The inclusion $\supseteq$ is obvious. For the converse statement, let $e \in p^{-1}(U)$. By definition, there is $W \subseteq X$ and $t \in \mathcal{F}(W)$, such that $e = t_x$, where $x = p(e)$. By assumption $x \in U$ and one can set $V = U \cap W$ and $s = t|_V$. We can then write $e = t_x = s_x$, whence $e \in \widetilde{s}(V)$. This shows the above claim and p is continuous by previous proposition.

In fact, p restricted onto an open subset $\widetilde{s}(U)$ is a homeomorphism onto U with continuous inverse $\widetilde{s} : U \to \widetilde{s}(U)$. This proves the second claim. ■

Having the stalk space $S\mathcal{F}$ equipped with a topology, one can define $\widetilde{\mathcal{F}} := \Gamma[S\mathcal{F}, p]$ to be the the sheaf of sections of the projection $p : S\mathcal{F} \to X$. Let us now construct a certain natural transformation of the two functors.

Proposition 7.10. *There exists a canonical natural transformation $\eta : \mathcal{F} \to \widetilde{\mathcal{F}}$, that is a presheaf map from $\mathcal{F}$ to $\widetilde{\mathcal{F}}$. For every $U \in$ **Op**(X) and every $s \in \Gamma(U, \mathcal{F})$, define*

$$\eta_U(s) := \widetilde{s}. \tag{151}$$

Recall that $\widetilde{s} : U \to S\mathcal{F}$ is defined by $\widetilde{s}(x) = s_x$.

Proof. Let $V \subseteq U$. Let $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ denote the restriction morphism. We have to check that for any $x \in V$ and any $s \in \Gamma(U, \mathcal{F})$, one has $\{\rho^U_V(s)\}_x = \{s\}_x$. But this follows from the definition of a stalk. This makes η into a natural transformation. ■

Now, we will show that the "sheaf properties" of $\mathcal{F}$ can be encoded into the properties of η.

Proposition 7.11. *$\mathcal{F}$ satisfies the monopresheaf axiom, if and only if η is injective.*

Proof. First, suppose $\mathcal{F}$ satisfies the axiom (B) of Definition 1.4. Let $U \in \mathbf{Op}(X)$ and let $s, t \in \Gamma(U, \mathcal{F})$ be two sections, such that $\eta_U(s) = \eta_U(t)$. This implies that $s_x = t_x$ for all $x \in U$. From Proposition 7.3 it follows that $s = t$. Hence η_U is injective.

Conversely, let $U \in \mathbf{Op}(X)$, let $\mathcal{U} = \{U_i\}_{i \in I}$ be any open cover of U. Let $s, t \in \Gamma(U, \mathcal{F})$ be two local sections, such that $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$. We have to show that $s = t$. To prove this, it suffices to argue why $\eta_U(s) = \eta_U(t)$. Let $x \in U$ be arbitrary. There is thus $i \in I$, such that $x \in U_i$. By assumption $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$. But this already implies that $s_x = t_x$, and thus $(\eta_U(t))(x) = (\eta_U(s))(x)$. This holds for any x, whence $\eta_U(t) = \eta_U(s)$. ■

Proposition 7.12. *Suppose $\mathcal{F}$ satisfies the monopresheaf axiom. Then $\mathcal{F}$ satisfies the gluing axiom if and only if $\eta : \mathcal{F} \to \widetilde{\mathcal{F}}$ is surjective.*

Proof. First, suppose that $\mathcal{F}$ satisfies the axiom (A) of Definition 1.4. Let $\sigma \in \widetilde{\mathcal{F}}(U)$ be any section.

Let $x \in U$ be arbitrary. We have $\sigma(x) \in \mathcal{F}_x$. There is thus some open neighborhood $U_x \subseteq U$ of x together with a section $s^{(x)} \in \mathcal{F}(U_x)$, such that $(s^{(x)})_x = \sigma(x)$. Let $\widetilde{s}^{(x)} : U_x \to S\mathcal{F}$ be the induced local section. Now, we will use the following lemma:

Lemma 7.13. *Suppose $\sigma, \tau : U \to S\mathcal{F}$ are two local sections on U. Then the set*

$$W = \{x \in U \mid \sigma(x) = \tau(x)\} \tag{152}$$

is open in U.

Proof. We may assume that W is non-empty. Let $x \in W$. Then $\sigma(x) = \tau(x) = s_x$ for some local section $s \in \mathcal{F}(Z)$, where $Z \ni x$ is an open neighborhood of x in U. We know that $\widetilde{s}(Z)$ is open in $S\mathcal{F}$. Consider the set $V = \sigma^{-1}(\widetilde{s}(Z)) \cap \tau^{-1}(\widetilde{s}(Z))$. By assumption, V is an non-empty open subset of U. It follows that $\tau(V), \sigma(V) \subseteq \widetilde{s}(Z)$. For any $v \in V$, we have $(p \circ \sigma)(v) = v$. Thus

$$\sigma(v) = (\widetilde{s} \circ p)(\sigma(v)) = \widetilde{s}((p \circ \sigma)(v)) = \widetilde{s}(v) = \cdots = \tau(v). \tag{153}$$

We have used the fact that $\widetilde{s} \circ p$ is identity on $\widetilde{s}(Z)$ and $\sigma(v) \in \widetilde{s}(Z)$ by construction, and $\cdots$ repeats the arguments using τ.

We have thus found a neighborhood of V of each point $x \in W$, such that $V \subseteq W$. We conclude that W is open. ■

Let us go back to the proof. We have two sections $\widetilde{s}^{(x)}$ and $\sigma|_{U_x}$ on U_x. As they coincide at x, the set $W_x = \{y \in U_x \mid \sigma(y) = \widetilde{s}^{(x)}(y)\}$ is non-empty and open by the above Lemma. Note that the assignment $s^{(x)} \mapsto \widetilde{s}^{(x)}$ is natural, hence we may restrict s to W_x and the natural restriction of $\widetilde{s}$ corresponds to this restriction. We may thus conclude that to a given local section $\sigma : U \to S\mathcal{F}$, to a given point $x \in U$, there exists an open subset $W_x \in \mathbf{Op}_x(U)$ and a local section $s^{(x)} \in \mathcal{F}(W_x)$, such that $(s^{(x)})_y = \sigma(y)$ for all $y \in W_x$.

In particular, for every $z \in Z := W_x \cap W_y$ and any two points $x, y \in U$, we have $(s^{(x)})_z = \sigma(z) = (s^{(y)})_z$. This can be translated as the equation

$$\eta_Z(\rho^{W_x}_Z(s^{(x)})) = \eta_Z(\rho^{W_y}_Z(s^{(y)})). \tag{154}$$

As η_Z is assumed injective, this imples $\rho^{W_x}_Z(s^{(x)}) = \rho^{W_y}_Z(s^{(y)})$. Using the gluing axiom for $\mathcal{F}$, there exists a (unique by monopresheaf axiom) section $s \in \mathcal{F}(U)$, such that $s^{(x)} = \rho^U_{W_x}(s)$. In particular, one has $\widetilde{s}(x) = s_x = (s^{(x)})_x = \sigma(x)$, and we have proved η_U surjective.

Conversely, suppose that η is surjective (and thus bijective). Suppose $U \in \mathbf{Op}(X)$ and let $\mathcal{U} = \{U_i\}_{i \in I}$ be any open cover. Suppose we are given a collection $\{s_i\}_{i \in I}$ of local sections, where $s_i \in \mathcal{F}(U_i)$ and $\rho^{U_i}_{U_{ij}}(s_i) = \rho^{U_j}_{U_{ij}}(s_j)$. They are maped by η to the respective sections $\widetilde{s}_i \in \Gamma(U_i, \widetilde{\mathcal{F}})$. The naturality of η ensures the condition $\widetilde{\rho}^{U_i}_{U_{ij}}(\widetilde{s}_i) = \widetilde{\rho}^{U_j}_{U_{ij}}(\widetilde{s}_j)$. But $\widetilde{F}$ is a sheaf, which implies there exists a unique section $\sigma \in \Gamma(U, \widetilde{\mathcal{F}})$, such that $\widetilde{s}_i = \widetilde{\rho}^U_{U_i}(\sigma)$. Now, as η_U is assumed surjective, there exists (a unique, as it is also injective) section $s \in \mathcal{F}(U)$, such that $\sigma = \widetilde{s}$. Then

$$\eta_{U_i}(\rho^U_{U_i}(s)) = \widetilde{\rho}^U_{U_i}(\widetilde{s}) = \widetilde{s}_i = \eta_{U_i}(s_i). \tag{155}$$

Now, as η_{U_i} is injective, this proves that $\rho^U_{U_i}(s) = s_i$. This is the gluing axiom. ■

Definition 7.14. The sheaf $\widetilde{\mathcal{F}}$ assigned to a presheaf $\mathcal{F}$ is called the **sheafification of $\mathcal{F}$.**

Corollary 7.15. *Let $\mathcal{F}$ be a presheaf. Then $\eta : \mathcal{F} \to \widetilde{\mathcal{F}}$ is a sheaf isomorphism.*

We will examine some properties of this map in the following section.

8 Stalk spaces

In the previous section, starting from a given presheaf $\mathcal{F}$, we have constructed the topological space $S\mathcal{F}$ together with a local difeomorphism $p : S\mathcal{F} \to X$. Then, starting from such a pair (E, p), one constructs a sheaf $\Gamma(E, p)$. It will be useful to view $\mathcal{F} \mapsto S\mathcal{F}$ and $(E, p) \mapsto \Gamma(E, p)$ as functors between certain categories. In this section, we assume that all sheaves all valued in the category **Set**. The more interesting cases are briefly mentioned at its end.

Definition 8.1. Let X be a given topological space. A pair (E, p) of topological space and a surjective local homeomorphism $p : E \to X$ is called the **stalk space** over X. A **morphism of stalk spaces** (E_1, p_1) and (E_2, p_2) over X is a continuous map $\varphi : E_1 \to E_2$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array} \tag{156}$$

commutes. Stalk spaces over X form a category $\mathbf{StalkS}(X)$.

Both presheaves and sheaves over a given topological space X form a category, where morphisms are natural transformations of the respective functors. Let us denote those categories as $\mathbf{PSh}(X)$ and $\mathbf{Sh}(X)$. We suppose that the target category $\mathbf{C}$ is fixed all the time. Note that $\mathbf{PSh}(X)$ is by definition just a functor category $\mathbf{C}^{(\mathbf{Op}(X))^{op}}$. Moreover, $\mathbf{Sh}(X)$ can be viewed as a full subcategory of $\mathbf{PSh}(X)$. First, let us talk about the functor Γ.

Proposition 8.2. *Define the covariant functor $\Gamma : \boldsymbol{StalkS}(X) \to \boldsymbol{Sh}(X)$ by declaring*

$$\Gamma(E,p)(U) = \{\sigma : U \to E \mid \sigma \text{ is continuous and } p \circ \sigma = 1_U\} \equiv \Gamma_U(E) \tag{157}$$

for every $U \in \boldsymbol{Op}(X)$. To every stalk space morphism $\varphi : E_1 \to E_2$, the corresponding natural map $\Gamma(\varphi) : \Gamma(E_1, p_1) \to \Gamma(E_2, p_2)$ is defined by composition. More exactly, if $U \in \boldsymbol{Op}(X)$ and $\sigma \in \Gamma_U(E)$, we set $\Gamma(\varphi)_U(\sigma) = \varphi \circ \sigma$.

Proof. Everything is kind of clear, except one has to prove that $\varphi \circ \sigma \in \Gamma_U(E_2)$. This follows from the assumption (156) made on φ. It is obvious that $\Gamma(1_E)_U = 1_{\Gamma_U(E)}$ and $\Gamma(\varphi \circ \psi) = \Gamma(\varphi) \circ \Gamma(\psi)$. ∎

Before an examination of the functor $S : \mathbf{PSh}(X) \to \mathbf{StalkS}(X)$, we will study some general properties of the stalk spaces following from their definition.

Proposition 8.3. *Let (E, p) be the stalk space. Then the following facts are true:*

(a) The map p is open.

(b) For any $U \in \boldsymbol{Op}(X)$ and any $\sigma \in \Gamma_U(E)$, the subset $\sigma(U)$ is open in E. Such open subsets form a basis for the topology of E.

(c) Let $\varphi : E_1 \to E_2$ be any map making the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \varphi\ } & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array} \tag{158}$$

commutative. Suppose (E_1, φ_1) and (E_2, φ_2) are stalk spaces. Then the map φ is continuous iff it is an open map iff it is a local homeomorphism.

Proof. Ad *(a)*: Let $U \subseteq E$ be open and non-empty. We have to show that $p(U)$ is open. Let $x \in p(U)$ and let $e \in E$ be any point such that $p(e) = x$. On the other hand, p is a local homeomorphism. There is thus some open subset $W \subseteq E$ containing x, such that $p(W) \subseteq X$ is open and $p : W \to p(W)$ is a homeomorphism. Let $Z = p^{-1}(W) \cap U$. This is na open subset of W, whence $p(Z)$ is open in $p(W)$ and thus also in X. By construction $p(Z) \subseteq p(U)$ and $x \in p(Z)$. This proves that $p(U)$ is open. As U was arbitrary non-empty, we conclude that p is open.

Ad *(b)*: Let $U \subseteq X$ be open, and let $\sigma \in \Gamma_U(E)$. Let $e \in \sigma(U)$ and write $x := p(e)$. As p is a local homeomorphism, there is an open neighborhood W of e, such that $p : W \to p(W)$ is a homeomorphism and $p(W) \subseteq X$ is open.

Next, one can consider the set $\sigma^{-1}(W) \subseteq U$. It is non-empty as $\sigma(x) = e \in W$. Finally, consider the intersection $V = p(W) \cap \sigma^{-1}(W)$. It is open and non-empty as $p(e) = x$ and both open sets thus contain x.

Now, set $Z = p^{-1}(V)$. As $V \subseteq p(W)$, we have $Z \subseteq W$. We claim that for every $z \in Z$, one has $z = \sigma(p(z))$. Since $p(z) \in U$, this would prove the inclusion $Z \subseteq \sigma(U)$. By definition, we have

$p(z) \in V = p(W) \cap \sigma^{-1}(W)$. In particular, $\sigma(p(z)) \in W$. As p is injective on W, it sufices to verify that $p(z) = p(\sigma(p(z)))$. But this is clear. Hence Z is an open neighborhood of x inside $\sigma(U)$. Thus $\sigma(U)$ is open, as was to be proved.

To finish this part, we must prove $\mathcal{B} = \{\sigma(U) \mid U \in \mathbf{Op}(X),\ \sigma \in \Gamma_U(E)\}$ forms the basis of the topology on E. Let $V \subseteq E$ be any open set. Let $e \in V$ be arbitrary. As p is a local homeomorphism, there exists an open neighborhood W of e, such that $p(W) \subseteq X$ is open and $p : W \to U$ is a homeomorphism. The set $U = p(V \cap W)$ is open in X. Let $\sigma : U \to E$ be the inverse of p restricted onto U and then composed with the inclusion $W \hookrightarrow E$. This is clearly a continuous local section of E and $\sigma(U) = V \cap W \subseteq V$. This shows that every open subset $V \subseteq E$ can be written as a union of some sets in $\mathcal{B}$.

Next, let $\sigma \in \Gamma_U(E)$ and $\sigma' \in \Gamma_{U'}(E)$ for some open sets $U, U' \in \mathbf{Op}(X)$. Suppose there exists a point $e \in \sigma(U) \cap \sigma'(U')$. We have to find an open subset $V \subseteq X$ and a section $\tau \in \Gamma_V(E)$, such that $\tau(V) \subseteq \sigma(U) \cap \sigma'(U')$. To prove this, we can in fact repeat the proof of Lemma 7.13 to find the similar statement for the general stalk space:

Lemma 8.4. *Let (E, p) be a stalk space. Suppose $\sigma, \tau \in \Gamma_U(E)$ are two its local sections. Then*

$$Z = \{x \in U \mid \sigma(x) = \tau(x)\} \tag{159}$$

is open in U (and thus also in X).

Proof. Without the loss of generality, we may assume that Z is non-empty. Let $x \in Z$. Let $e = \sigma(x) = \tau(x)$. As p is a local homeomorphism, there exists an open neighborhood W of e, such that $p : W \to p(W)$ is a homeomorphism. Let $V = \sigma^{-1}(W) \cap \tau^{-1}(W)$. This set is non-empty as $x \in V$. Moreover, clearly $V \subseteq p(W)$. We claim that $V \subseteq Z$. For every $v \in V$, one has $\sigma(v), \tau(v) \in W$. As p is injective on W, the equation $(p \circ \sigma)(v) = v = (p \circ \tau)(v)$ implies $\sigma(v) = \tau(v)$. Whence $v \in Z$. V is thus an open neighborhood of a given $x \in Z$, such that $V \subseteq Z$. Whence Z is open in U, as was to be proved. ■

Now, we can finally finish the proof of part *(b)*. Let $Z \subseteq U \cap U'$ be the open (by the previous lemma) subset where σ and σ' coincide. It is non-empty as $p(e) \in Z$. Moreover, find an open neighborhood W of e, such that $p : W \to p(W)$ is a homeomorphism. Let $V = Z \cap p(W)$, and define $\tau : V \to E$ to be the inverse of p restricted onto V, composed with the inclusion $W \hookrightarrow E$. Now, for any $e' \in \tau(V)$, one has $e' = \sigma(p(e')) = \sigma'(p(e'))$. As $\tau(V) \subseteq W$ and p is injective on W, this is clear. Hence $\tau(V) \subseteq \sigma(U) \cap \sigma'(U')$ and $e \in \tau(V)$. We conclude that $\mathcal{B}$ indeed forms a basis for the toplogy on E.

Ad *(c)*: This part contains three equivalent claims, namely

(i) φ is continuous;

(ii) φ is open;

(iii) φ is a local homeomorphism.

We will prove a usual chain of implications.

First for (i) $\Rightarrow$ (ii): Suppose φ is continuous. To prove that it is open, it suffices to show that an image of every element in the basis $\mathcal{B}_1$ for the topology of E_1 is open in E_2. We have already shown that the basis is formed by images of continous local sections. Hence suppose $U \in \mathbf{Op}(X)$ and $\sigma \in \Gamma_U(E)$. We have $\varphi(\sigma(U)) = (\varphi \circ \sigma)(U)$. But φ is continous and thus $\varphi \circ \sigma \in \Gamma_U(E_2)$. Consequently, $(\varphi \circ \sigma)(U) \in \mathcal{B}_2$. In particular, it is open. Thus φ is open.

Next, for (ii) $\Rightarrow$ (iii): Suppose φ is open. Let $e_1 \in E_1$ be an arbitrary point. Let $x = p_1(e_1)$. There is an open set $U \in \mathbf{Op}_x(X)$ and $\sigma \in \Gamma_U(E_1)$ be such $e_1 \in \sigma(U)$. By assumption, the set $(\varphi \circ \sigma)(U)$ is open in E_2. We will show that $\varphi : \sigma(U) \to (\varphi \circ \sigma)(U)$ is a homeomorphism. As it is an open map, it suffices to show that (the restriction of) φ is a continuous bijection.

Obviously, it is surjective. Every $f_1 \in \sigma(U)$ can be written as $f_1 = \sigma(y_1)$ for a unique $y_1 \in U$. If $(\varphi \circ \sigma)(y_1) = (\varphi \circ \sigma)(y_1')$, one can apply p_2 and use (158) to show $y_1' = y_1$. Hence φ restricted on $\sigma(U)$ is injective. It remains to prove that it is continuous.

It suffices to consider the open sets in the form $\tau(W)$, where $W \subseteq U$ and $\tau \in \Gamma_W(E_2)$ satisfy $\tau(W) \subseteq (\varphi \circ \sigma)(U)$. Such sets form the basis of the subspace topology on $(\varphi \circ \sigma)(U)$. We have to argue that the set $\varphi^{-1}(\tau(W)) \cap \sigma(U)$ is open in $\sigma(U)$. Let $f_1 \in \varphi^{-1}(\tau(W)) \cap \sigma(U)$ be a given point, and let $y = p_1(f_1)$. As $\varphi(f_1) \in \tau(W)$, it folows from (158) that $y \in W$. We will now argue that $\sigma(W)$ is an open neighborhood of f_1, such that $\sigma(W) \subseteq \varphi^{-1}(\tau(W)) \cap \sigma(U)$. It suffices to show that $\sigma(W) \subseteq \varphi^{-1}(\tau(W))$.

Let $f_1' \in \sigma(W)$. There is thus a unique $y' \in W$, such that $f_1' = \sigma(y')$. Moreover, by definition $\tau(y') \in (\varphi \circ \sigma)(U)$. There is thus a unique $x' \in U$, such that $\tau(y') = (\varphi \circ \sigma)(x')$. Applying p_2 it follows that in fact $y' = x'$. Whence $\varphi(f_1') = \varphi(\sigma(y')) = \tau(y')$. This shows that $f_1' \in \varphi^{-1}(\tau(W))$. Thus $\sigma(W) \subseteq \varphi^{-1}(\tau(W))$. We conclude that φ is continuous.

The implication (iii) $\Rightarrow$ (i) is trivial. ■

Now, we may examine the stalks of the sheaf $\Gamma(E, p)$. It turns out that they can be canonically identified (as sets) with the fibers of the fibration $p : E \to X$.

Proposition 8.5. *Let (E, p) be any stalk space and let $\Gamma(E, p)$ be the sheaf of its continuous sections. Let $E_x = p^{-1}(x)$ be the fiber over x. Then E_x has discrete topology.*

Moreover, for each $x \in X$, there exists a canonical isomorphism of E_x and the stalk $\Gamma(E, p)_x$.

Proof. We have to argue that every one-point set $\{e\} \subseteq E_x$ is open in the subspace topology. There exists some $U \in \mathbf{Op}_x(X)$ and $\sigma \in \Gamma_U(E)$, such that $e \in \sigma(U)$. In particular, we must have $e = \sigma(x)$. But every other point $e' \in E_x \cap \sigma(U)$ must be $e' = \sigma(x)$. Hence $\{e\} = E_x \cap \sigma(U)$. This proves that $\{e\}$ is open in the subspace topology.

Stalk at x is defined as a direct limit (hence a colimit) over the category $\mathbf{Op}_x(X)$. We may thus use the universality. For each $U \in \mathbf{Op}_x(X)$, one may consider a map $\mathrm{ev}_{U,x} : \Gamma_U(E) \to E_x$ which simply evaluates the section $\sigma \in \Gamma_U(E)$ at the point x. We claim that $\sigma(x) \equiv \mathrm{ev}_{U,x}(\sigma)$ is the germ of σ at x. First, note that if $V \subseteq U$ and $\rho^U_V : \Gamma_U(E) \to \Gamma_V(E)$ is the (sheaf) restriction, the diagram

$$\begin{array}{ccc} \Gamma_U(E) & \xrightarrow{\rho^U_V} & \Gamma_V(E) \\ & {\scriptstyle \mathrm{ev}_{U,x}} \searrow \quad \swarrow {\scriptstyle \mathrm{ev}_{V,x}} & \\ & E_x & \end{array} \tag{160}$$

commutes. We have to show that whenever we find a collection of maps $\{\tau_{U,x}\}_{U \in \mathbf{Op}_x(X)}$ and a set S, such that $\tau_{U,x} : \Gamma_U(E) \to S$ and we have the same commutative triangle for every $V \subseteq U$, that is $\tau_{U,x} = \tau_{V,x} \circ \rho^U_V$, there must exist a unique map $k : E_x \to S$, such that $\tau_{U,x} = k \circ e_{U,x}$ for all $U \in \mathbf{Op}_x(X)$.

To define k, let $e \in E_x$. As (E, p) is a stalk space, we have $e = \sigma(x) = \mathrm{ev}_{V,x}(\sigma)$ for some $V \in \mathbf{Op}_x(X)$ and $\sigma \in \Gamma_V(E)$. We must set $k(e) := k(\mathrm{ev}_{V,x}(\sigma)) = \tau_{V,x}(\sigma)$. We will now argue that this definition depends only on e, not on σ "extending it".

Hence, suppose that there is $W \in \mathbf{Op}_x(X)$ and $\tau \in \Gamma_W(E)$, such that $\tau(x) = \sigma(x) = e$. It follows from Lemma 8.4 that there is a non-empty open set $Z \subseteq V \cap W$, such that σ and τ coincide on this set. In terms of restriction maps, this means that $\rho^V_Z(\sigma) = \rho^W_Z(\tau)$. Thus, by assumption

$$\tau_{V,x}(\sigma) = \tau_{Z,x}(\rho^V_Z(\sigma)) = \tau_{Z,x}(\rho^W_Z(\tau)) = \tau_{W,x}(\tau). \tag{161}$$

This shows that k is well-defined. Clearly, it is a unique such map. ■

Now, we can finally examine the map $S : \mathbf{PSh}(X) \to \mathbf{StalkS}(X)$ in more detail.

Proposition 8.6. *The map $S : \boldsymbol{PSh}(X) \to \boldsymbol{StalkS}(X)$ which assigns to each presheaf $\mathcal{F}$ the stalk space $(S\mathcal{F}, p)$ defines a covariant functor.*

Proof. We only have to show that to any natural transformation $\varphi : \mathcal{F} \to \mathcal{G}$ of two presheaves over X, there exists a morphism $S(\varphi) : S\mathcal{F} \to S\mathcal{G}$ of the respective stalk spaces. Recall that to any map of presheaves and any $x \in X$, there is a unique map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ of the respective stalks, see Corollary 7.5. As each fiber $(\mathcal{SF})_x$ is precisely the stalk $\mathcal{F}_x$, we define $S(\varphi)$ "fiber-wise" using φ_x. Clearly, it then fits into the commutative diagram

$$\begin{array}{ccc} S\mathcal{F} & \xrightarrow{\;S(\varphi)\;} & S\mathcal{G} \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array} \quad . \tag{162}$$

According to Proposition 8.3 - (c), it suffices to prove that $S(\varphi)$ is an open map. The basis $\mathcal{B}$ of the topology on $S\mathcal{F}$ is formed by sets $\widetilde{s}(U)$, where $s \in \mathcal{F}(U)$ is a local section of $\mathcal{F}$ over $U \in \mathbf{Op}(X)$. We will now argue that $S(\varphi)(\widetilde{s}(U)) = \widetilde{\varphi_U(s)}(U)$. This is an open set. Every point $\widetilde{s}(U)$ can be uniquely written as s_x for a unique $x \in U$. Then $S(\varphi)(s_x) = \varphi_x(s_x) = (\varphi_U(s))_x$. This proves the inclusion $\subseteq$. The other one is also obvious - kind of from definition. This proves that $S(\varphi)$ maps every element of the topology basis to the open set in $S\mathcal{G}$, whence it is open.

It is easy to see that $S(\varphi \circ \psi) = S(\varphi) \circ S(\psi)$ and $S(1_{\mathcal{F}}) = 1_{S\mathcal{F}}$ and we conclude that S is indeed a functor from **PSh** to **StalkS**. ■

Now, note that we have a natural transformation $\eta : \mathcal{F} \to \Gamma S\mathcal{F}$, hence a presheaf morphism. It thus induces a morphism of the corresponding stalks. But note that according to Proposition 8.5, we can identify $\Gamma(S\mathcal{F})_x$ with the fiber $(S\mathcal{F})_x$ and thus with $\mathcal{F}_x$.

Proposition 8.7. *For each $x \in X$, the map of stalks $\eta_x : \mathcal{F}_x \to \widetilde{\mathcal{F}}_x \equiv \Gamma(S\mathcal{F})_x$ is the identity.*

Proof. The stalk map is defined as a unique map $\eta_x : \mathcal{F}_x \to \widetilde{\mathcal{F}}_x = F_x$ which for every $U \in \mathbf{Op}(X)$ and $s \in \mathcal{F}(U)$ satisfies the relation

$$\eta_x(s_x) = (\eta_U(s))_x = \mathrm{ev}_{U,x}(\eta_U(s)) = \mathrm{ev}_{U,x}(\widetilde{s}) = \widetilde{s}(x) = s_x. \tag{163}$$

This is, quite obviously, the identity. ■

We have already argued that there exists a canonical natural isomorphism $\eta_{\mathcal{F}} : \mathcal{F} \to \Gamma S\mathcal{F}$ for every sheaf $\mathcal{F}$. It can be easily seen that it is in fact natural in $\mathcal{F}$, whence defines a natural isomorphism η from the identity functor $1 : \mathbf{Sh}(X) \to \mathbf{Sh}(X)$ to the functor ΓS.

Proposition 8.8. *There exists a canonical natural isomorphism ϵ from the identity functor 1 on $\boldsymbol{StalkS}(X)$ to the functor $S\Gamma$.*

Proof. Let $(E,p) \in \mathbf{StalkS}(X)$. We have to define a stalk space map $\epsilon_E : (E,p) \to S(\Gamma(E,p))$. First, it must fit into the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\epsilon_E} & S(\Gamma(E,p)) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array} \quad , \tag{164}$$

where π is the canonical projection from the definition of the functor S. Let $e \in E_x$. Then $\epsilon_E(e)$ must be in the stalk $\Gamma(E,p)_x = E_x$ (see Proposition 8.5). The obvious choice is $\epsilon_E(e) = e$. According to Proposition 8.3, it suffices to check that ϵ_E is open. A topology basis $\mathcal{B}$ for E can be chosen to consist of sets $\sigma(U)$, where $U \in \mathbf{Op}(X)$ and $\sigma \in \Gamma_U(E)$.

But σ is a local section of the sheaf $\Gamma(E,p)$ and we may assign to it a section $\widetilde{\sigma} : U \to S(\Gamma(E,p))$ of the other stalk space. By Proposition 8.3, the set $\widetilde{\sigma}(U)$ is open in $S(\Gamma(E,p))$. We claim that $\epsilon_E(\sigma(U)) = \widetilde{\sigma}(U)$. Recall that the section $\widetilde{\sigma}$ assigns to each $x \in U$ the germ of the section σ at x, that is, in this case, $\widetilde{\sigma}(x) = \mathrm{ev}_{U,x}(\sigma) \equiv \sigma(x)$. We thus have

$$\epsilon_E(\sigma(x)) = \sigma(x) = \widetilde{\sigma}(x). \tag{165}$$

This proves that claim. Consequently, the map η_E is open and thus continuous. In fact, it has to be an isomorphism of the two stalk spaces. A local homeomorphism is a homeomorphism iff it is a bijection. But it clearly is a bijection. Finally, we must prove that it is natural in E, that is fitting into the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\epsilon_{E_1}} & S(\Gamma(E_1,p_1)) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle S\Gamma(\varphi)} \\ E_2 & \xrightarrow{\epsilon_{E_2}} & S(\Gamma(E_2,p_2)) \end{array} \tag{166}$$

for every morphism of stalk spaces $\varphi : (E_1,p_1) \to (E_2,p_2)$. Let $e_1 \in (E_1)_x$. Then $\epsilon_{E_1}(e_x) = e_1$. The map $S(\Gamma(\varphi))$ then simply applies the stalk map $\Gamma(\varphi)_x$ on e_1. This stalk map is determined uniquely by the diagram

$$\begin{array}{ccc} \Gamma_U(E_1) & \xrightarrow{\Gamma(\varphi)_U} & \Gamma_U(E_2) \\ \downarrow{\scriptstyle \mathrm{ev}_{U,x}} & & \downarrow{\scriptstyle \mathrm{ev}_{U,x}} \\ (E_1)_x & \overset{\Gamma(\varphi)_x}{\dashrightarrow} & (E_2)_x. \end{array} \tag{167}$$

Obviously, the only reasonable (and correct) choice is $\Gamma(\varphi)_x(e_1) = \varphi(e_1)$. The "clockwise" branch of the above diagram thus sends $e_1 \in (E_1)_x$ to $\varphi(e_2) \in S(\Gamma(E_2,p_2))_x = (E_2)_x$. But this is exactly what the "counter-clockwise" branch does. ■

Corollary 8.9. *The categories* $\boldsymbol{Sh}(X)$ *and* $\boldsymbol{StalkS}(X)$ *are equivalent.*

Proof. We have constructed functors $S : \mathbf{Sh}(X) \to \mathbf{StalkS}(X)$ and $\Gamma : \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$, such that $1 \cong S\Gamma$ and $1 \cong \Gamma S$, where $\cong$ denote the natural isomorphisms of the functors. This is a definition of the equivalence of categories. ■

Recall that for any presheaf $\mathcal{F}$, we have constructed a certain sheaf $\widetilde{\mathcal{F}}$ together with a natural transformation $\eta_{\mathcal{F}} : \mathcal{F} \to \widetilde{\mathcal{F}}$. But what if there is an another sheaf $\mathcal{G}$ together with a natural map $\tau : \mathcal{F} \to \mathcal{G}$? The claim is that it has to "factor through" $\widetilde{\mathcal{F}}$ anyway. This can be viewed as the **universality of the sheafification** process.

Proposition 8.10. *Let $\mathcal{F} \in \textbf{\textit{PSh}}(X)$ be an arbitrary presheaf. Suppose $\tau : \mathcal{F} \to \mathcal{G}$ is a natural transformation from $\mathcal{F}$ to a sheaf $\mathcal{G} \in \textbf{\textit{Sh}}(X)$. Then there is a unique natural transformation $\widetilde{\tau} : \widetilde{\mathcal{F}} \to \mathcal{G}$, such that the following diagram commutes:*

$$\begin{CD} \mathcal{F} @>{\eta_{\mathcal{F}}}>> \widetilde{\mathcal{F}} \\ @. {\searrow}^{\tau} @VV{\widetilde{\tau}}V \\ @. \mathcal{G} \end{CD} \tag{168}$$

Proof. First, let prove the uniqueness of $\widetilde{\tau}$. For every $x \in X$, we have the corresponding stalk maps, that is $\tau_x = \widetilde{\tau}_x \circ \eta_{\mathcal{F},x}$. We have already argued that $\eta_{\mathcal{F},x} : \mathcal{F}_x \to \widetilde{\mathcal{F}}_x \equiv \mathcal{F}_x$ is just an identity, whence $\widetilde{\tau}_x$ is uniquely determined. By Proposition 7.6, the map $\widetilde{\tau} : \widetilde{\mathcal{F}} \to \mathcal{G}$ of (pre)sheaves, where $\mathcal{G}$ is a sheaf, is uniquely determined (if it exists) by its stalk map. Hence it is unique.

To prove its existence, one may define $\widetilde{\tau}$ to complete the following diagram:

$$\begin{CD} \mathcal{F} @>{\eta_{\mathcal{F}}}>> \widetilde{\mathcal{F}} \\ @V{\tau}VV {\swarrow}_{\widetilde{\tau}} @VV{S\Gamma(\tau)}V \\ \mathcal{G} @>{\eta_{\mathcal{G}}}>> \widetilde{\mathcal{G}} \end{CD} \;. \tag{169}$$

This is possible as $\widetilde{\mathcal{G}}$ is a sheaf and by Corollary 7.15, $\eta_{\mathcal{G}}$ is a natural isomorphism. ∎

9 Stalk spaces of R-modules and commutative rings

So far we have considered only the presheaves valued in **Set** and all statements were discussed only on the set level. However, we would like to have something more suited for two important classes of sheaves, namely those valued R-**Mod** and **CRing**.

Definition 9.1. A **stalk space of R-modules** over X is a pair (E, p), where $p : E \to X$ is a surjective local homeomorphism, such that

(i) Every fiber $E_x = p^{-1}(x)$ is an R-module.

(ii) For every $\lambda \in R$, the map $e \mapsto \lambda \cdot e$ on E is continuous.

(iii) The additive inverse $-_E : E \to E$, defined fiberwise, is continuous.

(iv) Let $E \times_X E$ be a fibered product over X, that is

$$E \times_X E = \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\} \tag{170}$$

We can then define a map $+_E : E \times_X E \to E$, using the R-module structure of the fibers of E. This map must be continuous.

Lef $\mathcal{F} : X \to$ R-**Mod** be a presheaf. We will now prove that the stalk space $p : S\mathcal{F} \to X$, constructed in Section 7, is a stalk space of R-modules. This should not be too difficult as we already know things or two about the topology of $S\mathcal{F}$. Before doing so, let us prove a very useful general statement:

Proposition 9.2. *Let (E_1, p_1) and (E_2, p_2) be two stalk spaces over X. Then $E = E_1 \times_X E_2$ together with a canonical projection $\pi : E \to X$ and equipped with a canonical topology (subspace topology of the product topology) is a stalk space again.*

Proof. We have $\pi(e_1, e_2) := p_1(e_1) = p_2(e_2)$. It is surjective as for any $x \in X$, there are $e_1 \in p_1^{-1}(x)$ and $e_2 \in p_2^{-1}(x)$. Hence $\pi(e_1, e_2) = x$. It is continuous as to any $U \in \mathbf{Op}(X)$, one has $\pi^{-1}(U) = E \cap (p_1^{-1}(U) \times p_2^{-1}(U))$ which is an open subset of E. To prove that it is a stalk space, recall that we have a special topology bases for E_1 and E_2, respectively:

$$\mathcal{B}_1 = \{\sigma_1(U) \mid \sigma_1 \in \Gamma_U(E_1)\}, \tag{171}$$

$$\mathcal{B}_2 = \{\sigma_2(U) \mid \sigma_2 \in \Gamma_U(E_2)\}. \tag{172}$$

By definition, the basis $\mathcal{B}$ for the topology of E has the form

$$\mathcal{B} = \{E \cap (V_1 \times V_2) \mid V_1 \in \mathcal{B}_1,\ V_2 \in \mathcal{B}_2\}. \tag{173}$$

Now, let $(e_1, e_2) \in E$. Let $x = \pi(e_1, e_2)$. There are thus $U_1, U_2 \in \mathbf{Op}_x(X)$ and two local sections $\sigma_1 \in \Gamma_{U_1}(E_1)$ and $\sigma_2 \in \Gamma_{U_2}(E_2)$, such that $e_1 = \sigma_1(x)$ and $e_2 = \sigma_2(x)$, respectively. Let $W = E \cap (\sigma_1(U_1) \times \sigma_2(U_2))$. We claim that $\pi : W \to \pi(W)$ is a homeomorphism. Let us examine the set W a little bit. Let $(f_1, f_2) \in W$. We have $(f_1, f_2) = (\sigma_1(x_1), \sigma_2(x_2))$ for some $x_1 \in U_1$ and $x_2 \in U_2$. But then also $x_1 = p_1(f_1) = p_2(f_2) = x_2$. It follows that $W = (\sigma_1, \sigma_2)(U_1 \cap U_2)$. In particular, we find $\pi(W) = U_1 \cap U_2$, which is open in X. Moreover, the restriction of π to W is clearly bijective. To prove that π is a homeomorphism of W and $U_1 \cap U_2$, it suffices to show that it is open. This can be shown on the elements of the basis $\mathcal{B}_W$ of W. As $W \subseteq E$ is open, we may consider the basis

$$\mathcal{B}_W = \{V \in \mathcal{B} \mid V \subseteq W\} \tag{174}$$

Let $V \in \mathcal{B}_W$. There are thus two open subsets $Z_1, Z_2 \in \mathbf{Op}(X)$ and $\tau_1 \in \Gamma_{Z_1}(E_1)$ and $\tau_2 \in \Gamma_{Z_2}(E_2)$, such that $V = (\tau_1, \tau_2)(Z_1 \cap Z_2)$. In particular, $Z_1 \cap Z_2 \subseteq U_1 \cap U_2$. and $\pi(V) = Z_1 \cap Z_2 \subseteq U_1 \cap U_2 \equiv \pi(W)$ is open. Whence $\pi : W \to \pi(W)$ is an bijective continuous open map, hence a homeomorphism. ■

Now, we may prove the first of the two main statements of this section.

Proposition 9.3. *Let $\mathcal{F} : X \to$ R-**Mod** be a presheaf on X valued in the category of R-modules. Then the stalk space $p : S\mathcal{F} \to X$, with the topology described in the previous section, is a stalk space of R-modules.*

Proof. Write $E = S\mathcal{F}$. By definition $E_x = \mathcal{F}_x$, which is an R-module by construction. One only has to check the continuity properties (ii) - (iv). By Proposition 8.3, for maps from E to E, it suffices to check they are open to prove they are continuous.

Ad (ii): Let $\lambda \in R$. Let $\lambda_E : E \to E$ denote the corresponding fiberwise multiplication. Recall that the topology of E has the basis $\mathcal{B}$ consisting of the sets $\widetilde{s}(U)$, where $s \in \mathcal{F}(U)$ is a local section of $\mathcal{F}$ over U and $\widetilde{s}(x) = s_x$ for all $x \in U$. It suffices to prove that $\lambda_E(\widetilde{s}(U))$ is open. As $\mathcal{F}(U)$ is an R-module, we have also a section $\lambda \cdot s \in \mathcal{F}(U)$ and

$$\lambda_E(\widetilde{s}(U)) = \widetilde{(\lambda \cdot s)}(U), \tag{175}$$

which will prove the claim. Let $e \in \widetilde{s}(U)$. Thus $e = s_x$ for $x = p(e)$. The germ map $s \mapsto s_x$ is an R-module morphism, and thus

$$\lambda_E(e) = \lambda \cdot s_x = (\lambda \cdot s)_x = \widetilde{(\lambda \cdot s)}(x). \tag{176}$$

This proves the inclusion $\subseteq$. The other one is just reading the above equation backwards.

Ad (iii): This one is completely analogous, one shows that $-_E(\widetilde{s}(U)) = \widetilde{(-s)}(U)$.

Ad (iv): By Proposition 9.2, the space $\pi : E \times_X E \to X$ is again a stalk space. Now, the map $+_E : E \times_X E \to E$ becomes a map of stalks. To prove that it is continuous, it suffices to show that it is open. We may do it again on the topology basis $W = (\widetilde{s}, \widetilde{s}')(U \cap U')$, where $s \in \mathcal{F}(U)$ and $s' \in \mathcal{F}(U')$ for some $U, U' \in \mathbf{Op}(X)$. See the Proof of Proposition 9.2 to see that this is indeed a basis of topology for $E \times_M E$. Let $t \in \mathcal{F}(U \cap U')$ be the section defined as

$$t = \rho^U_{U \cap U'}(s) + \rho^{U'}_{U \cap U'}(s') \tag{177}$$

We will now argue that $+_E(W) = \widetilde{t}(U \cap U')$. Indeed, let $(e, e') \in W$. There is thus $x \in U \cap U'$, such that $(e, e') = (s_x, s'_x)$. Then

$$+_E\,(e, e') = s_x + s'_x = (\rho^U_{U \cap U'}(s) + \rho^{U'}_{U \cap U'}(s'))_x = \widetilde{t}(x), \tag{178}$$

where the second equality is the definition of the R-module structure of the stalk space $\mathcal{F}_x$. This proves the inclusion $\subseteq$. To prove the other one, simply read the equations backwards. ■

Now, for the converse. Suppose (E, p) is a stalk space of R-modules. We would like to show that the corresponding sheaf $\Gamma(E, p)$ is a sheaf of R-modules. First, let us note the following simple lemma:

Lemma 9.4. *Let (E, p) be a stalk space over X, and let $U \in \boldsymbol{Op}(X)$. Then $E_U := p^{-1}(U)$ together with a restriction of p is a stalk space over U, called the* ***restriction of** E **to** U.*

Proof. Clearly, $p : E_U \to U$ is a surjective continuous map. Only has to argue that it is a local homeomorphism. Let $e \in E_U$. There is thus an open subset $W \in \mathbf{Op}_e(E)$, such that $p : W \to p(W)$ is a homeomorphism. Then $p(W \cap E_U) = p(W) \cap U$. We have thus found $W \cap E_U \in \mathbf{Op}_e(E_U)$, such that $p(W \cap E_U) \in \mathbf{Op}_e(U)$. Clearly, the restriction of p to an open subset $W \cap E_U$ of W defines the homeomorphism of $W \cap E_U$ and its image $p(W) \cap U$. ■

Proposition 9.5. *Let (E, p) be a stalk space of R-modules. Then its sheaf $\Gamma(E, p)$ of sections is a sheaf of R-modules. The R-module structure is the unique one making the canonical bijection $E_x \to (\Gamma(E, p))_x$ into a R-module isomorphism.*

Proof. Let $x \in X$ and let $U \in \mathbf{Op}_x(X)$ be any its open neighborhood. Let $\sigma, \tau \in \Gamma_U(E)$. The canonical map $E_x \to (\Gamma(E, p))_x$ is a R-module isomorphism if the the ring structure on $\Gamma_U(E)$ makes the evaluation map $\mathrm{ev}_{U,x} : \Gamma_U(E) \to E_x$ into a R-module morphism. In other words, se must necessarily set

$$(\sigma + \tau)(x) = \mathrm{ev}_{U,x}(\sigma + \tau) := \sigma(x) + \tau(x), \tag{179}$$

and similarly for the additive inverse and R multiplication, that is

$$(-\sigma)(x)) := -\sigma(x), \;\; (\lambda \cdot \sigma)(x) := \lambda \cdot (\sigma(x)). \tag{180}$$

All the right-hand sides use the assumed R-module structure of the fiber E_x. One only has to check that point-wise defined operations on sections make them into a continous sections of E again. Now, define a map $\varphi : U \to E \times_X E$ as $\varphi(x) = (\sigma(x), \tau(x))$. Let $\pi : (E \times_X E)_U \to U$ define a restricted stalk space as given by Lemma 9.4. Then φ fits into the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & (E \times_X E)_U \\ \downarrow{\scriptstyle 1_U} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{1_U} & U \end{array} \;\; . \tag{181}$$

Indeed, one has $\pi(\varphi(x)) = \pi(\sigma(x), \tau(x)) = x$. We can view $1_U : U \to U$ as a stalk space over U. By Proposition 8.3, the map φ is then continuous if and only if it is open. But for any $V \in \mathbf{Op}(U)$, we have $\varphi(V) = (\sigma, \tau)(V)$. Such sets form a basis of $(E \times_X E)_U$ and, naturally, they are open.

Thus φ is continuous as a map from U to $(E \times_X E)_U$ and thus also as a map to $E \times_X E$. Now, finally, we may view the section $\sigma + \tau$ as a composition of two continuous maps, namely $\sigma + \tau = +_E \circ \varphi$. Hence $\sigma + \tau \in \Gamma_U(E)$.

The other two operations are much simpler, as we have $-\sigma = -_E \circ \sigma$ and $\lambda \cdot \sigma = \lambda_E \circ \sigma$, where λ_E is the fiber-wise left multiplication by $\lambda \in R$. This finishes the proof. ■

Naturally, there is also a version of stalk space maps suitable for the category R-**Mod**. It is not difficult to guess what the right definition might be.

Definition 9.6. Let (E_1, p_1) and (E_2, p_2) be two stalk spaces of R-modules. The map $\varphi : E_1 \to E_2$ is a **map of stalk spaces of R-modules**, if it is a map of stalk spaces and furthermore, the induced map $\varphi_{(x)} : (E_1)_x \to (E_2)_x$ is an R-module morphism.

One can now reprove all statements in the previous section, valid for the category of (pre)sheaves valued in R-**Mod** and the category of stalk spaces of R-modules with the above defined morphisms.

To conclude this section, we briefly mention the modification of the above definition to work for presheaves valued in the commutative rings category **CRing**.

Definition 9.7. A **stalk space of commutative rings** over X is a pair (E, p), where $p : E \to X$ is a surjective local homeomorphism, such that

(i) Every fiber $E_x = p^{-1}(x)$ is a commutative ring.

(ii) One can define a map $-_E : E \to E$ which to each $e \in E$ assigns its additive inverse (using the ring structure of the fiber). This map must be continuous.

(iii) Let $E \times_X E$ be a fibered product over X, that is

$$E \times_X E = \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\} \tag{182}$$

We can then define maps $+_E : E \times_X E \to E$ and $\cdot_E : E \times_X E \to E$, using the commutative ring structure of the fibers of E. These maps must be continuous.

It is easy to prove the analogues of propositions 9.3 and 9.5 suitable for the category of **CRing**.

To conclude this section, let us prove the following interesting observation. In general, there may not exist any section of a given stalk space $p : E \to X$. However, for stalk spaces of R-modules (or commutative rings), there is a particular element in every fiber E_x, namely the zero element which we denote as 0_x. One can thus define a **zero section**, which we denote as 0_E, defined by $0_E(x) = 0_x$. Is the map $0_E : X \to E$ continuous?

Proposition 9.8. *For each stalk space of R-modules (or commutative rings), there exists a canonical global **zero section** $0_E \in \Gamma_X(E)$.*

Proof. To prove that 0_E is continuous, it suffices to show that for each $x \in X$, there exists $U \in \mathbf{Op}_x(X)$, such that the restriction $0_E|_U : U \to E$ is continuous. Now, certainly, there is some $U \in \mathbf{Op}_x(X)$ together with a continous local section $\sigma : U \to E$.

Recall that E is a stalk space of R-modules and the fiber-wise multiplication map $\lambda_E : E \to E$ is continuous for every $\lambda \in R$. In particular, choose $\lambda = 0_R$, that is an additive zero of the commutative ring R. Then we can write $0_E|_U = \lambda_R \circ \sigma$. Indeed, for any $x \in U$, one has $(\lambda_R \circ \sigma)(x) = 0_R \cdot \sigma(x) = 0_x$, where we use the general R-module property $0_R \cdot E_x = 0_x$. But the composition $\lambda_R \circ \sigma$ is continuous and thus so is $0_E|_U$. The proof for **CRing** is analogous. ■

10 Morphisms of presheaves and sheaves

We will now consider presheaves and sheaves valued in the categories R-**Mod** or **CRing**. Suppose $\varphi \in \text{R-}\mathbf{Mod}(A, B)$ for two R-modules A and B. We can then define the following notions:

1. The **kernel** $\ker(\varphi)$ of φ defined as a set
$$\ker(\varphi) = \{a \in A \mid \varphi(a) = 0\}. \tag{183}$$
Since φ is R-linear and additive, it follows that $\ker(\varphi)$ is a R-submodule of A. The map φ is injective, if and only if $\ker(\varphi) = 0$.

2. The **image** $\operatorname{im}(\varphi)$ of φ is a subset of B given by
$$\operatorname{im}(\varphi) = \{b \in B \mid \text{there exists } a \in A \text{ such that } b = \varphi(a)\}. \tag{184}$$
Again, properties of φ imply that $\operatorname{im}(\varphi)$ is a R-submodule of B.

3. The **cokernel** $\operatorname{coker}(\varphi)$ of φ is the quotient R-module $B/\operatorname{im}(\varphi)$. The map φ is surjective, if and only if $\operatorname{coker}(\varphi) = 0$.

4. The **coimage** $\operatorname{coim}(\varphi)$ of φ is the quotient R-module $A/\ker(\varphi)$. There always exists a canonical map from $\operatorname{coim}(\varphi)$ to $\operatorname{im}(\varphi)$ which happens to be a R-module isomorphism. Indeed, there exists a map $\hat{\varphi}$ defined to fit into the commutative diagram
$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \operatorname{im}(\varphi) \\ \downarrow{\natural} & \overset{\hat{\varphi}}{\nearrow} & \\ A/\ker(\varphi) & & \end{array}, \tag{185}$$
where $\natural : A \to A/\ker(\varphi)$ is the canonical quotient map. It is easy to check that $\hat{\varphi}$ is a well-defined bijective R-module morphism, hence an isomorphism.

The idea is to generalize this notions to a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of two (pre)sheaves. On the level of presheaves, everything works as expected. However, for sheaves, the "image presheaf" of a sheaf is in general not a sheaf. In this section, we shall address these issues.

Proposition 10.1. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of two presheaves on X. Then the* ***kernel presheaf*** $\ker(\varphi)$ ***corresponding to*** φ *is for every $U \in \boldsymbol{Op}(X)$ defined by*
$$(\ker(\varphi))_U := \ker(\varphi_U). \tag{186}$$
The restriction morphisms are obtained by restricting those of $\mathcal{F}$. If $\mathcal{F}$ is a sheaf and $\mathcal{G}$ satisfies the monopresheaf axiom, $\ker(\varphi)$ *forms a sheaf.*

Proof. For every $V \subseteq U$, we have a commutative diagram
$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow{\rho^U_V} & & \downarrow{\tilde{\rho}^U_V} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}. \tag{187}$$

For every $s \in \ker(\varphi_U)$, we thus have $\varphi_V(\rho^U_V(s)) = \widetilde{\rho}^U_V(\varphi_U(s)) = \widetilde{\rho}^U_V(0) = 0$. This means that $\rho^U_V(\ker(\varphi_U)) \subseteq \ker(\varphi_V)$ and the restriction morphisms of $\mathcal{F}$ induce those of $\ker(\varphi)$. This makes $\ker(\varphi)$ into a presheaf.

Now, let us assume that $\mathcal{F}$ is a sheaf and $\mathcal{G}$ satisfies the monopresheaf axiom. Let $U \in \mathbf{Op}(X)$ and let $\mathcal{U} = \{U_i\}_{i \in I}$ be its open cover.

First, suppose we are given a collection $\{s_i\}_{i \in I}$, where $s_i \in \ker(\varphi_{U_i})$ and $\rho^{U_i}_{U_{ij}}(s_i) = \rho^{U_j}_{U_{ij}}(s_j)$ for all $i, j \in I$. As $\ker(\varphi_{U_i}) \subseteq \mathcal{F}(U_i)$, we may use the fact that $\mathcal{F}$ is a sheaf. There is thus a unique section $s \in \mathcal{F}(U)$, such that $s_i = \rho^U_{U_i}(s)$. We have to argue that $s \in \ker(\varphi_U)$. We have

$$\widetilde{\rho}^U_{U_i}(\varphi_U(s)) = \varphi_{U_i}(\rho^U_{U_i}(s)) = \varphi_{U_i}(s_i) = 0. \tag{188}$$

We thus have $\widetilde{\rho}^U_{U_i}(\varphi_U(s)) = \widetilde{\rho}^U_{U_i}(0) = 0$, and by the monopresheaf axiom for $\mathcal{G}$, we find $\varphi_U(s) = 0$. This proves the gluing axiom for $\ker(\varphi)$.

Now, let $s, t \in \ker(\varphi_U)$, such that $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$. This means that $s = t$ as elements of $\mathcal{F}(U)$, hence also as elements of $\ker(\varphi_U)$. This is a monopresheaf axiom for $\ker(\varphi)$. ■

In fact, in the category of R-modules, and thus also in the category of presheaves valued in R-modules, injective R-module morphisms are the monic morphisms (which are defined in every category).

Definition 10.2. Let $\varphi \in \text{R-}\mathbf{Mod}(A, B)$ for $A, B \in \text{R-}\mathbf{Mod}$. We say that φ is a **monic morphism**, if for any $K \in \text{R-}\mathbf{Mod}$ and any pair of R-module morphisms $\kappa, \kappa' \in \text{R-}\mathbf{Mod}(K, A)$, the equation $\varphi \circ \kappa = \varphi \circ \kappa'$ implies $\kappa = \kappa'$.

The same definition defines a **monic presheaf morphism** in the category $\mathbf{PSh}(X)$ of presheaves on X valued in the category R-**Mod**.

Before proving the main proposition about kernels, let us note that kernel of a R-module homomorphism (and thus also of a kernel presheaf of a presheaf morphism) has a certain universality property.

Proposition 10.3. *Suppose one has $K, A, B \in \text{R-}\boldsymbol{Mod}$ and a pair of morphisms $\kappa \in \text{R-}\boldsymbol{Mod}(K, A)$ and $\varphi \in \text{R-}\boldsymbol{Mod}(A, B)$, such that $\varphi \circ \kappa = 0$. Then there exists a unique R-module morphism $\hat{\kappa} : K \to \ker(\varphi)$ making the following diagram commutative:*

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & A \\ & \searrow^{\hat{\kappa}} & \uparrow I \\ & & \ker(\varphi) \end{array} \tag{189}$$

$I : \ker(\varphi) \to A$ is the inclusion. The same statement holds in the category of $\boldsymbol{PSh}(X)$ of presheaves on X valued in the category R-$\boldsymbol{Mod}$.

Proof. By definition $\text{im}(\kappa) \subseteq \ker(\varphi)$. Whence $\hat{\kappa}$ is just κ viewed as a map from K to $\ker(\varphi)$. ■

We can now formulate the usual relation of kernels and injectivity. If $\mathcal{F}$ is a sheaf, it can be slightly reformulated in terms of the corresponding stalk maps.

Proposition 10.4. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves. Then the following statements are equivalent:*

(i) $\ker(\varphi) = 0$ *(the trivial sheaf);*

(ii) *For every* $U \in \boldsymbol{Op}(X)$*, the map* φ_U *is injective;*

(iii) φ *is a monic morphism in the cateogory* $\boldsymbol{PSh}(X)$ *of presheaves valued in* R-$\boldsymbol{Mod}$*.*

(iv) *(only for* $\mathcal{F}$ *satisfying the monopresheaf axiom) The induced stalk map* $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ *is injective for all* $x \in X$*.*

Proof. The equivalence of (i) and (ii) is the property of the R-module morphism φ_U for each $U \in \mathbf{Op}(X)$.

Next, suppose (i) holds. We are to prove (iii). Let $\mathcal{H}$ be any presheaf together with a pair $\kappa, \kappa' : \mathcal{H} \to \mathcal{F}$ of presheaf morphisms, such that $\varphi \circ \kappa = \varphi \circ \kappa'$. We thus have $\varphi \circ (\kappa - \kappa') = 0$. By Proposition 10.3, there is a unique presheaf map $\mu : \mathcal{H} \to \ker(\rho)$, such that $\mathrm{I} \circ \mu = (\kappa - \kappa')$, where $\mathrm{I} : \ker(\varphi) \to \mathcal{F}$ is the inclusion. But $\ker(\rho) = 0$, whence $\mathrm{I} \circ \mu = 0$ and consequently, $\kappa' = \kappa$. Hence (i) implies (iii). Conversely, suppose (iii) holds. Consider $\mathcal{H} = \ker(\varphi)$ and let $\kappa = \mathrm{I}$ (the inclusion), $\kappa' = 0$ (the zero map). Then $\varphi \circ \kappa = \varphi \circ \kappa'$. As φ is a monic morphism, we have $\kappa = \kappa'$, that is $\mathrm{I} = 0$. But this can only happen if $\ker(\varphi) = 0$. Thus (i) holds.

Suppose (ii) holds. Let $s_x \in \mathcal{F}_x$ be a germ at x, such that $\varphi_x(s_x) = 0$, where $s \in \mathcal{F}(U)$ and $U \in \mathbf{Op}_x(X)$. This implies that $(\varphi_U(s))_x = 0$. There is thus an open subset $V \subseteq U$, such that $\widetilde{\rho}^U_V(\varphi_U(s))_x = 0$. This implies $\varphi_V(\rho^U_V(s)) = 0$. As φ_V is injective, one finds $\rho^U_V(s) = 0$. But this implies that $s_x = 0$, whence φ_x is injective. Thus (iv) holds.

Conversely, suppose φ_x is injective for all $x \in X$. Let $U \in \mathbf{Op}(X)$ be an open set and suppose $\varphi_U(s) = 0$ for some $s \in \mathcal{F}(U)$. This implies that $\varphi_x(s_x) = (\varphi_U(s))_x = 0$. Hence $s_x = 0$. There is thus some open neighborhood $V_x \in \mathbf{Op}_x(U)$, such that $\rho^U_{V_x}(s) = 0$. As $\mathcal{V} = \{V_x\}_{x \in U}$ forms an open cover, and we have $\rho^U_{V_x}(s) = \rho^U_{V_x}(0)$, the monopresheaf axiom for $\mathcal{F}$ implies $s = 0$. This concludes the proof of $(iv) \Rightarrow (ii)$. ∎

Definition 10.5. We say that $\varphi : \mathcal{F} \to \mathcal{G}$ is **injective**, if any of the above properties hold.

Note that we have just prove that for any presheaf $\mathcal{F}$, the assumption $\ker(\varphi) = 0$ implies $\ker(\varphi_x) = 0$ for every $x \in X$, whereas the inverse holds (in particular for) presheaves. As the trivial presheaf obviously has a trivial stalk, this suggests that $\ker(\varphi_x)$ may be somehow related to the stalk space of a presheaf $\ker(\varphi)$. This is discussed in the following proposition.

Proposition 10.6. *For any presheaf* $\mathcal{F}$ *and any presheaf map* $\varphi : \mathcal{F} \to \mathcal{G}$*, one can, for every* $x \in X$*, identify* $\ker(\varphi)_x = \ker(\varphi_x)$*.*

Proof. Let $\mathrm{I} : \ker(\rho) \to \mathcal{F}$ be the inclusion of the kernel presheaf into $\mathcal{F}$, that is $\mathrm{I}_U : \ker(\varphi_U) \to \mathcal{F}(U)$ is the inclusion of R-modules for every $U \in \mathbf{Op}(X)$. This is a presheaf map inducing a unique stalk map $\mathrm{I}_x : (\ker(\varphi))_x \to \mathcal{F}_x$ for every $x \in X$. We have shown that the implication $(i) \Rightarrow (iv)$ of Proposition 10.4 holds for every presheaf. Hence I_x is injective and we may identify $(\ker(\varphi))_x$ with $\mathrm{im}(\mathrm{I}_x)$. The statement of proposition is thus understood as the equality $\mathrm{im}(\mathrm{I}_x) = \ker(\varphi_x)$.

Let $t \in \mathrm{im}(\mathrm{I}_x)$. Equivalently, $t = s_x$ for some $s \in \ker(\varphi_U)$ and some $U \in \mathbf{Op}_x(X)$. Equivalently, $t = s_x$ for some $s \in \mathcal{F}(U)$ and some $U \in \mathbf{Op}(X)$, such that $\varphi_U(s) = 0$. Equivalently, t satisfies $\varphi_x(t) = 0$. Finally, this is equivalent to $t \in \ker(\varphi_x)$.

In fact, one may directly identify the germ maps $\rho'_{U,x} : \ker(\varphi_U) \to \ker(\varphi_x)$ - they are simple the restriction of $\rho_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x$ to the submodule $\ker(\varphi_U)$ (that is a composition with I_U). ∎

Now, recall that we have functors $S : \mathbf{PSh}(X) \to \mathbf{StalkS}(X)$ and $\Gamma : \mathbf{StalkS}(X) \to \mathbf{Sh}(X)$, where both stalk spaces and (pre)sheaves are assumed to work in the category R-**Mod** or **CRing**. The injectivity of a presheaf map $\varphi : \mathcal{F} \to \mathcal{G}$ thus somehow reflect on the map $S(\varphi) : S\mathcal{F} \to S\mathcal{G}$, and conversely, any stalk space map $\eta : (E, p) \to (E', p')$ should give some significant sheaf morphism $\Gamma(\eta) : \Gamma(E, p) \to \Gamma(E', p')$.

Proposition 10.7. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a presheaf map. Then if φ is injective, then $S(\varphi)$ is injective. The converse is true only for $\mathcal{F}$ satisfying the monopresheaf axiom. Let $\eta : (E, p) \to (E', p')$ be a morphism of stalk spaces. Then $\Gamma(\eta)$ is injective, if and only if $\eta : E \to E'$ is injective.*

Proof. A map $S(\varphi)$ of stalk spaces $S\mathcal{F}$ and $S\mathcal{G}$ is injective, if and only if its restriction to each fiber $(S\mathcal{F})_x = \mathcal{F}_x$ is injective. But this restriction is the stalk map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$. $S(\varphi)$ is thus injective if and only if for all $x \in X$, the map φ_x is injective. If φ is injective, then so is φ_x for any $x \in X$. This holds for any presheaf $\mathcal{F}$. The converse holds only for $\mathcal{F}$ satisfying the monopresheaf axiom. See (the proof of) Proposition 10.4.

Let $\eta : (E, p) \to (E', p')$ be a morphism of stalk spaces. For every $U \in \mathbf{Op}(X)$, the map $\Gamma(\eta)_U : \Gamma_U(E) \to \Gamma_U(E')$ composes a section $\sigma \in \Gamma_U(E)$ with η, that is $\Gamma(\eta)_U(\sigma) = \eta \circ \sigma$.

First, suppose that η is injective. Fix $U \in \mathbf{Op}(X)$. We have to show that $\Gamma(\eta)_U$ is injective. Suppose there is $\sigma \in \Gamma_U(E)$, such that $\eta \circ \sigma$ is a zero section in $\Gamma_U(E')$. This implies $\eta_{(x)}(\sigma(x)) = 0'_x$, where $0'_x$ is the zero element of the R-module E'_x and $\eta_{(x)} : E_x \to E'_x$ is an R-module morphism obtained from η by restriction. The injectivity of η implies the one of $\eta_{(x)}$, hence $\sigma(x) = 0$ for all $x \in U$. Thus $\sigma = 0$.

Conversely, suppose that $\Gamma(\eta)$ is injective. Let $e_1, e_2 \in E$ be two elements, such that $\eta(e_1) = \eta(e_2)$. As η is a stalk space map, we find $p(e_1) = p(e_2) \equiv x$. As E is a stalk space, there is $U \in \mathbf{Op}(X)$, and two sections $\sigma_1, \sigma_2 \in \Gamma_U(E)$, such that $e_1 = \sigma_1(x)$ and $e_2 = \sigma_2(x)$. The sections $\eta \circ \sigma_1$ and $\eta \circ \sigma_2$ in $\Gamma_U(E')$ thus coincide at x. Using Lemma 8.4, they must coincide on some open neighborhood $V \in \mathbf{Op}_x(U)$. But then $\Gamma(\eta)_V(\sigma_1|_V) = \Gamma(\eta)_V(\sigma_2|_V)$. By assumption, $\Gamma(\eta)_V$ is injective, whence $\sigma_1|_V = \sigma_2|_V$. In particular, we find that $e_1 = \sigma_1(x) = \sigma_2(x) = e_2$ and we conclude that η is injective. ∎

Remark 10.8. Note that the stalk space map $\varphi : (E, p) \to (E', p')$ is injective, if and only if it is a homeomorphism onto an open subset of E'.

Definition 10.9. Let $\mathcal{F}$ and $\mathcal{G}$ be a two (pre)sheaves on X. We say that $\mathcal{F}$ is a sub(pre)sheaf of $\mathcal{G}$, if for every $U \in \mathbf{Op}(X)$, $\mathcal{F}(U)$ is a R-submodule (or subring) of $\mathcal{G}(U)$, such that the collection $\{\mathrm{I}_U\}_{U \in \mathbf{Op}(X)}$ of inclusions $\mathrm{I}_U : \mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ forms a (pre)sheaf morphism.

Let (E_1, p_1) and (E_2, p_2) be two stalk spaces. We say that (E_1, p_1) is a **substalk space**, if $E_1 \in \mathbf{Op}(E_2)$, p_1 is a restriction of p_2 to E_1 and the fiber $(E_1)_x$ is an R-submodule (or subring) of $(E_2)_x$ for all $x \in X$.

We will now prove the useful criterion for comparing two subsheaves.

Proposition 10.10. *Suppose $\mathcal{F}, \mathcal{F}'$ are two subsheaves of a sheaf $\mathcal{G}$. Then $\mathcal{F} = \mathcal{F}'$ if and only if $\mathcal{F}_x = \mathcal{F}'_x$ for all $x \in X$, where we view $\mathcal{F}_x$ and $\mathcal{F}'_x$ as R-submodules of $\mathcal{G}_x$.*

Proof. First, note that if $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, there is the injective presheaf morphism $\mathrm{I} : \mathcal{F} \to \mathcal{G}$ induced for each $U \in \mathbf{Op}(X)$ by the inclusion $\mathrm{I}_U : \mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$. For each $x \in X$, there is thus $\mathrm{I}_x : \mathcal{F}_x \to \mathcal{G}_x$, which is injective by Proposition 10.4. We can (and will) thus identify $\mathcal{F}_x$ with its image $\mathrm{im}(\mathrm{I}_x) \subseteq \mathcal{G}_x$.

The only if part is trivial.

To show the if statement, we will now demonstrate that whenever $\mathcal{F}_x \subseteq \mathcal{F}'_x$ for all $x \in X$, then $\mathcal{F}$ is a subsheaf of $\mathcal{F}'$. Let ρ^U_V and ρ'^U_V denote the restriction maps of $\mathcal{F}$ and $\mathcal{F}'$, whereas λ^U_V the ones of $\mathcal{G}$. Suppose $s \in \mathcal{F}(U)$ is a local section of $\mathcal{F}$ over U. Its germ s_x is thus also an element of $\mathcal{F}'_x$. There is thus an open set $U_x \in \mathbf{Op}_x(U)$ together with a section $t^{(x)} \in \mathcal{F}'(U_x)$, such that $\mathrm{I}_{U_x}(\rho^U_{U_x}(s)) = \mathrm{I}'_{U_x}(t^{(x)})$. For any $x, y \in U$, let $U_{xy} = U_x \cap U_y$. Then

$$\begin{aligned}\mathrm{I}'_{U_{xy}}(\rho'^{U_x}_{U_{xy}}(t^{(x)})) &= \lambda^{U_x}_{U_{xy}}(\mathrm{I}'_x(t^{(x)})) = \lambda^{U_x}_{U_{xy}}(\mathrm{I}_{U_x}(\rho^U_{U_x}(s))) \\ &= \lambda^U_{U_{xy}}(\mathrm{I}_U(s)) = \cdots = \mathrm{I}'_{U_{xy}}(\rho'^{U_y}_{U_{xy}}(t^{(y)})).\end{aligned} \tag{190}$$

We have used the naturality of $\mathrm{I} : \mathcal{F} \to \mathcal{G}$ and $\mathrm{I}' : \mathcal{F}' \to \mathcal{G}$ several times. As $\mathrm{I}'_{U_{xy}}$ is injective, we obtain the relation

$$\rho'^{U_x}_{U_{xy}}(t^{(x)}) = \rho'^{U_y}_{U_{xy}}(t^{(y)}). \tag{191}$$

At this moment, we use the fact that $\mathcal{F}'$ is a sheaf. We have an open cover $\mathcal{U} = \{U_x\}_{x\in U}$ of U and a collections $\{t^{(x)}\}_{x\in U}$ of sections on this open cover, which agree on overlaps. There is thus a unique section $t \in \mathcal{F}'(U)$, such that $t^{(x)} = \rho'^U_{U_x}(t)$. Define the map $\varphi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ as $\varphi_U(s) = t$. It takes some work to see that this is, in fact, a canonical map.

Indeed, suppose we have found a different collection $\{u^{(x)}\}_{x\in U}$ of local sections, defined on a different open cover $\mathcal{U}' = \{V_x\}_{x\in U}$, and use them to define a section $v \in \mathcal{F}'(U)$. We have to argue why $t = v$. To this, we must use the fact that $\mathcal{F}'$ is a sheaf, so we can compare both sections locally. Let $\mathcal{W} = \{U_x \cap V_x\}_{x\in U}$. This is again an open cover. Write $W_x = U_x \cap V_x$. Then

$$\mathrm{I}'_{W_x}(\rho'^U_{W_x}(t)) = \mathrm{I}'_{W_x}\{\rho'^{U_x}_{W_x}(\rho'^U_{U_x}(t))\} = \lambda^{U_x}_{W_x}\{\mathrm{I}'_{U_x}(t^{(x)})\} = \lambda^U_{W_x}(\mathrm{I}_U(s)) = \cdots = \mathrm{I}'_{W_x}(\rho'^U_{W_x}(v)). \tag{192}$$

We have just use the naturality of I and I'. As I'_{W_x} is injective, we find that $\rho'^U_{W_x}(t) = \rho'^U_{W_x}(v)$ for all $x \in U$. The monopresheaf axiom for $\mathcal{F}'$ now implies $t = v$.

Now, in particular, if $\mathcal{U} = \{U_x\}_{x\in U}$ and $t^{(x)} \in \mathcal{F}'(U_x)$ satisfy the required condition, we may always pass to a collection $\mathcal{V} = \{V_x\}_{x\in U}$, where $V_x \subseteq U_x$ and $v^{(x)} = \rho'^{U_x}_{V_x}(t^{(x)})$. Indeed:

$$\mathrm{I}'_{V_x}(v^{(x)}) = \mathrm{I}'_{V_x}\{\rho'^{U_x}_{V_x}(t^{(x)})\} = \lambda^{U_x}_{V_x}\{\mathrm{I}'_{U_x}(t^{(x)})\} = \lambda^{U_x}_{V_x}\{\mathrm{I}_{U_x}(\rho^U_{U_x}(s))\} = \mathrm{I}_{V_x}(\rho^U_{V_x}(s)). \tag{193}$$

We can now prove that $\varphi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ is an R-module morphism. Let $s_1, s_2 \in \mathcal{F}(U)$. By the above discussion, we may find an open cover $\mathcal{U} = \{U_x\}_{x\in U}$ together with a collections $\{t_1^{(x)}\}_{x\in U}$ and $\{t_2^{(x)}\}_{x\in U}$ of local sections fitting into the appropriate relations for s_1 and s_2, respectively. The additivity of all maps implies that $t_1^{(x)} + t_2^{(x)}$ fits into the appropriate relation for $s_1 + s_2$. The uniqueness of the construction implies that $\varphi_U(s_1 + s_2)$ must be the global section of $\mathcal{F}'(U)$ which restricts onto $t_1^{(x)} + t_2^{(x)}$ on U_x. But this is clearly $t_1 + t_2 = \varphi_U(s_1) + \varphi_U(s_2)$. The R-linearity is an (easier) similar check.

Next, one must prove the naturality of φ_U, that is whenever $V \subseteq U$, we must verify that $\rho'^U_V(\varphi_U(s)) = \varphi_V(\rho^U_V(s))$ for every $s \in \mathcal{F}(U)$. Let $\mathcal{U} = \{U_x\}_{x\in U}$ and $t^{(x)} \in \mathcal{F}'(U_x)$ be the collection used to define $\varphi_U(s)$. For every $x \in V$, define $V_x = U_x \cap V$ and $v^{(x)} = \rho'^{U_x}_{V_x}(t^{(x)})$. Using the similar tricks as above, we find

$$\mathrm{I}'_{V_x}(v^{(x)}) = \mathrm{I}_{V_x}(\rho^V_{V_x}(\rho^U_V(s))), \tag{194}$$

for every $x \in V$. This implies that $\varphi_V(\rho^U_V(s)) = v$, where v is the unique section satisfying $\rho'^V_{V_x}(v) = v^{(x)}$. But it is an easy check that this equation is satisfied by $v = \rho'^U_V(\varphi_U(s))$.

One only has to check that $\varphi : \mathcal{F} \to \mathcal{F}'$ is the inclusion. This is equivalently stated as $\mathrm{I}' \circ \varphi = \mathrm{I}$. Using Proposition 7.6 and the fact that $\mathcal{G}$ is a sheaf, it suffices to compare their stalk maps, that is we have to prove $\mathrm{I}'_x \circ \varphi_x = \mathrm{I}_x$ for all $x \in X$. Let $s_x \in \mathcal{F}_x$ for some $s \in \mathcal{F}(U)$ and $U \in \mathbf{Op}_x(X)$. Then we can write

$$\begin{aligned} \mathrm{I}'_x(\varphi_x(s_x)) &= \mathrm{I}'_x((\varphi_U(s))_x) = (\mathrm{I}'_U(\varphi_U(s)))_x = (\lambda^U_{U_x}\{\mathrm{I}'_U(\varphi_U(s))\})_x \\ &= (\mathrm{I}'_{U_x}\{\rho'^U_{U_x}(\varphi_U(s))\})_x = (\mathrm{I}'_{U_x}\{t^{(x)}\})_x = (\mathrm{I}_{U_x}\{\rho^U_{U_x}(s)\})_x \\ &= (\lambda^U_{U_x}\{\mathrm{I}_U(s)\})_x = (\mathrm{I}_U(s))_x = \mathrm{I}_x(s_x). \end{aligned} \tag{195}$$

This concludes this lengthy and cumbersome proof of a quite useful property of sheaves. ■

Now, the kernel (pre)sheaf measures the injectivity of a given (pre)sheaf morphism. What about the surjectivity? For R-modules or commutative rings, surjectivity is measured by vanishing of the cokernels.

Proposition 10.11. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of two presheaves on X. Then we get the* ***presheaf cokernel*** $\mathrm{Pcoker}(\varphi)$ ***corresponding to*** φ *is for every $U \in \mathbf{Op}(X)$ defined by*

$$(\mathrm{Pcoker}(\varphi))_U := \mathrm{coker}(\varphi_U) \equiv \mathcal{G}(U)/\operatorname{im}(\varphi_U), \tag{196}$$

where its restriction morphisms $\bar{\rho}^U_V : \mathrm{coker}(\varphi_U) \to \mathrm{coker}(\varphi_V)$ are induced by those of $\mathcal{G}$. Moreover, the collection of canonical quotient maps $\mathrm{pcoker}_U : \mathcal{G}(U) \to \mathcal{G}(U)/\operatorname{im}(\varphi_U)$ defines a presheaf morphism $\mathrm{pcoker} : \mathcal{G} \to \mathrm{Pcoker}(\varphi)$.

Moreover, one has the following universal property. Suppose $\mathcal{H}$ is another presheaf on X, together with a presheaf morphism $\kappa : \mathcal{G} \to \mathcal{H}$, such that $\kappa \circ \varphi = 0$. Then there exists a unique presheaf morphism $\hat{\kappa} : \mathrm{Pcoker}(\varphi) \to \mathcal{H}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\kappa} & \mathcal{H} \\ \downarrow{\scriptstyle \mathrm{pcoker}} & \nearrow{\scriptstyle \hat{\kappa}} & \\ \mathrm{Pcoker}(\varphi) & & \end{array} \tag{197}$$

Proof. For any $U, V \in \mathbf{Op}(X)$ such that $V \subseteq U$, we have the usual commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \tilde{\rho}^U_V} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array} \tag{198}$$

We define the restriction maps $\bar{\rho}^U_V : \mathrm{coker}(\varphi_U) \to \mathrm{coker}(\varphi_V)$ to complete the commutative square:

$$\begin{array}{ccc} \mathcal{G}(U) & \xrightarrow{\mathrm{pcoker}_U} & \mathrm{coker}(\varphi_U) \\ \downarrow{\scriptstyle \tilde{\rho}^U_V} & & \downarrow{\scriptstyle \bar{\rho}^U_V} \\ \mathcal{G}(V) & \xrightarrow{\mathrm{pcoker}_V} & \mathrm{coker}(\varphi_V) \end{array} \tag{199}$$

We only have to argue that $\tilde{\rho}^U_V(\operatorname{im}(\varphi_U)) \subseteq \operatorname{im}(\varphi_V)$. But this follows immediately from the preceding diagram. It is easy to see that these restriction maps make $\mathrm{Pcoker}(\varphi)$ into a presheaf. Moreover, we immediately see that the collection $\{\mathrm{pcoker}_U\}_{U \in \mathbf{Op}(X)}$ defines a presheaf morphism pcoker. It remains to prove the universal property.

For each $U \in \mathbf{Op}(X)$, there is a unique R-module morphism $\hat{\kappa}_U : \mathrm{coker}(\varphi_U) \to \mathcal{H}$ satisfying the condition $\hat{\kappa}_U \circ \mathrm{pcoker}_U = \kappa_U$. This is because $\operatorname{im}(\varphi_U) \subseteq \ker(\kappa_U)$. Only has to argue that

$\{\hat{\kappa}_U\}_{U \in \mathbf{Op}(X)}$ forms a morphism of presheaves. Let $\lambda^U_V : \mathcal{H}(V) \to \mathcal{H}(U)$ be the restriction morphism of the presheaf $\mathcal{H}$. The R-module morphism pcoker_U is surjective, hence we only need to verify the commutativity of the diagram

$$\begin{array}{ccccc} \mathcal{G}(U) & \xrightarrow{\mathrm{pcoker}_U} & \mathrm{coker}(\varphi_U) & \xrightarrow{\hat{\kappa}_U} & \mathcal{H}(U) \\ & & \Big\downarrow{\scriptstyle \bar{\rho}^U_V} & & \Big\downarrow{\scriptstyle \lambda^U_V} \\ & & \mathrm{coker}(\varphi_V) & \xrightarrow{\hat{\kappa}_V} & \mathcal{H}(V) \end{array} \tag{200}$$

But here we use just definitions and the naturality of the collection $\{\kappa_U\}_U$ in U:

$$\lambda^U_V \circ \hat{\kappa}_U \circ \mathrm{pcoker}_U = \lambda^U_V \circ \kappa_U = \kappa_V \circ \widetilde{\rho}^U_V = \hat{\kappa}_V \circ \mathrm{pcoker}_V \circ \widetilde{\rho}^U_V = \hat{\kappa}_V \circ \bar{\rho}^U_V \circ \mathrm{pcoker}_U \,. \tag{201}$$

This concludes the proof. ■

The cokernel presheaf has similar properties to those of kernel presheaf. In particular, we know how does its stalk space look like.

Proposition 10.12. *For every $x \in X$, there is a canonical isomorphism of* $\mathrm{coker}(\varphi_x) = \mathcal{G}_x / \mathrm{im}(\varphi_x)$ *and the stalk space* $(P\,\mathrm{coker}(\varphi))_x$.

Proof. There is a canonical presheaf morphism $\mathrm{pcoker} : \mathcal{G} \to \mathrm{Pcoker}(\varphi)$. It thus induces a unique stalk map $\mathrm{pcoker}_x : \mathcal{G}_x \to \mathrm{Pcoker}(\varphi)_x$. We will now argue that this map is surjective and its kernel is precisely $\mathrm{im}(\varphi_x)$. Suppose $\mu \in \mathrm{Pcoker}(\varphi)_x$. There is thus $U \in \mathbf{Op}(X)$ and a section $\mu' \in \mathrm{coker}(\varphi_U)$, such that $\mu = \mu'_x$. On the other hand, there is a section $t \in \mathcal{G}(U)$, such that $\mu' = \mathrm{coker}_U(t)$. Thus $\mu = (\mathrm{coker}_U(t))_x = \mathrm{pcoker}_x(t_x)$.

Now, suppose $\nu \in \mathrm{im}(\varphi_x)$. There is thus $U \in \mathbf{Op}(X)$ and $s \in \mathcal{F}(U)$, such that $\nu = (\varphi_U(s))_x$. Hence $\mathrm{pcoker}_x(\nu) = (\mathrm{pcoker}_U(\varphi_U(s))_x = 0$. Thus $\ker(\varphi_x) \subseteq \ker(\mathrm{pcoker}_x)$. Conversely, let $\nu \in \ker(\mathrm{pcoker}_x)$. There is $U \in \mathbf{Op}_x(X)$ and $t \in \mathcal{G}(U)$, such that $\nu = t_x$. By assumption $\mathrm{pcoker}_x(\nu) = (\mathrm{pcoker}_U(t))_x = 0$. There is thus an open set $V \subseteq U$, such that $\bar{\rho}^U_V(\mathrm{pcoker}_U(t)) = 0$. Hence

$$0 = \bar{\rho}^U_V(\mathrm{pcoker}_U(t)) = \mathrm{pcoker}_V(\widetilde{\rho}^U_V(t)). \tag{202}$$

This implies that there is a section $s \in \mathcal{F}(V)$, such that $\widetilde{\rho}^U_V(t) = \varphi_V(s)$. Consequently, one finds

$$\nu = t_x = (\widetilde{\rho}^U_V(t))_x = (\varphi_V(s))_x = \varphi_x(s_x). \tag{203}$$

This proves that $\nu \in \mathrm{im}(\varphi_x)$ and this proves the inclusion $\ker(\mathrm{pcoker}_x) \subseteq \ker(\varphi_x)$. Finally, the canonical isomorphism $\mathrm{pcoker}'_x : \mathrm{coker}(\varphi_x) \to (\mathrm{Pcoker}(\varphi))_x$ completes the commutative diagram

$$\begin{array}{ccc} \mathcal{G}_x & \xrightarrow{\mathrm{pcoker}_x} & (\mathrm{Pcoker})_x \\ \Big\downarrow{\scriptstyle \natural_x} & \nearrow{\scriptstyle \mathrm{pcoker}'_x} & \\ \mathrm{coker}(\varphi_x) \equiv \mathcal{G}_x / \mathrm{im}(\varphi_x) & & \end{array} \,, \tag{204}$$

where $\natural_x : \mathcal{G}_x \to \mathrm{coker}(\varphi_x)$ is the quotient map. This finishes the proof. ■

Example 10.13. The presheaf cokernel is in general not a sheaf. Consider the topological space $X = \mathbb{C}$ and a sheaf $\mathcal{F} = C^\omega$ of holomorphic functions. Set $\mathcal{G} = \mathcal{F}$ and consider a presheaf map $\varphi := \frac{d}{dz}$ of complex differentiation. Then $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf morphism. Now, consider the open set $U = \mathbb{C} - \{0\}$ and a function $f(z) = \frac{1}{z}$. Its primitive function is not defined on entire U. In other words, the map φ_U is not surjective, and thus $\operatorname{coker}(\varphi_U) \neq 0$. Consequently, the presheaf $\operatorname{Pcoker}(\varphi)$ is non-trivial.

On the other hand, let $z \in \mathbb{C}$. Choose any $\mu \in \operatorname{coker}(\varphi_z)$. There is thus a holomorphic function $f \in C^\omega(U)$ for some $U \in \mathbf{Op}_z(\mathbb{C})$, such that $\mu = \natural_z(f_z)$, where $\natural_z : \mathcal{G}_z \to \operatorname{coker}(\varphi_z)$ is the quotient map. But there is a neighborhood $V \in \mathbf{Op}_z(U)$, where the function can be written as a power series and thus integrated to a holomorphic function on V. In other words, we have

$$\rho^U_V(f) = \varphi_V\Big(\int_V \rho^U_V(f)\Big). \tag{205}$$

But this implies that $f_z = \varphi_z\{(\int_V \rho^U_V(f))_z\} \in \operatorname{im}(\varphi_z)$. Consequently $\mu = 0$. Whence $\operatorname{coker}(\varphi_z) = 0$ for every $z \in \mathbb{C}$. By previous proposition, this shows that every stalk space $(\operatorname{Pcoker}(\varphi))_z$ is trivial.

We have already proved that the presheaf cokernel is non-trivial. This situation cannot happen for a sheaf. Indeed, if any sheaf $\mathcal{F}$ has a trivial stalk $\mathcal{F}_x$ for every $x \in X$, the trivial map $0 : \mathcal{F} \to \mathcal{F}$ and the identity map $1 : \mathcal{F} \to \mathcal{F}$ would then induce the same map of stalks (namely the trivial one). By Proposition 7.6, this would imply $1 = 0$, which in turn can happen only for $\mathcal{F} = 0$.

We will now try to fix this issue by considering the sheafification of the presheaf $\operatorname{Pcoker}(X)$. This has some issues on its own, which we shall discuss in the following propositions.

Definition 10.14. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of any two presheaves. Then the **sheaf cokernel** $\operatorname{Scoker}(\varphi)$ is defined as a sheafification of the presheaf cokernel $\operatorname{Pcoker}(\varphi)$.

Proposition 10.15. *There is a canonical presheaf morphism* $\operatorname{scoker} : \mathcal{G} \to \operatorname{Scoker}(\varphi)$. *The sheaf* $\operatorname{Scoker}(\varphi)$ *has the following universal property. Suppose* $\mathcal{H}$ *is any* ***sheaf*** *together with any presheaf morphism* $\kappa : \mathcal{G} \to \mathcal{H}$, *such that* $\kappa \circ \varphi = 0$. *Then there is a unique sheaf morphim* $\hat{\kappa} : \operatorname{Scoker}(\varphi) \to \mathcal{H}$ *filling the commutative diagram*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\kappa} & \mathcal{H} \\ \big\downarrow \text{scoker} & \nearrow_{\hat{\kappa}} & \\ \operatorname{Scoker}(\varphi) & & \end{array} . \tag{206}$$

Moreover, there is a canonical R-module isomorphism $\operatorname{scoker}'_x : \operatorname{coker}(\varphi_x) \to (\operatorname{Scoker}(\varphi))_x$.

Proof. Let $\eta : \operatorname{Pcoker}(\varphi) \to \operatorname{Scoker}(\varphi)$ be the canonical map from presheaf to its sheafification. Set $\operatorname{scoker} := \eta \circ \operatorname{pcoker}$. To prove the universal property, suppose $\kappa : \mathcal{G} \to \mathcal{H}$ be any presheaf morphism such that $\kappa \circ \varphi = 0$, where $\mathcal{H}$ is any sheaf. By the universal property for $\operatorname{Pcoker}(\varphi)$, there is a unique presheaf morphism $\widetilde{\kappa} : \operatorname{Pcoker}(\varphi) \to \mathcal{H}$, such that $\kappa = \widetilde{\kappa} \circ \operatorname{pcoker}$. As $\mathcal{H}$ is assumed to be a sheaf, we may employ another universality rule, namely there is a unique sheaf map $\hat{\kappa} : \operatorname{Scoker}(\varphi) \to \mathcal{H}$, such that $\hat{\kappa} \circ \eta = \widetilde{\kappa}$. We have found a sheaf map $\hat{\kappa} : \operatorname{Scoker}(\varphi) \to \mathcal{H}$, such that

$$\kappa = \widetilde{\kappa} \circ \operatorname{pcoker} = (\hat{\kappa} \circ \eta) \circ \operatorname{pcoker} = \hat{\kappa} \circ \operatorname{scoker}\,. \tag{207}$$

We only have to argue that this two-step construction gives a unique such map. Suppose we find some other map $\hat{\chi} : \operatorname{Scoker}(\varphi) \to \mathcal{H}$, such that $\kappa = \hat{\chi} \circ \operatorname{scoker}$. This implies

$$\kappa = (\hat{\kappa} \circ \eta) \circ \operatorname{pcoker} = (\hat{\chi} \circ \eta) \circ \operatorname{pcoker}\,. \tag{208}$$

The uniqueness statement in the universality property of $\operatorname{Pcoker}(\varphi)$ now implies $\hat{\kappa} \circ \eta = \hat{\chi} \circ \eta$. For each $x \in X$, we then obtain a relation of stalk maps, that is $\hat{\kappa}_x \circ \eta_x = \hat{\chi}_x \circ \eta_x$. It follows from Proposition 8.7 that this is equivalent to $\hat{\kappa}_x = \hat{\chi}_x$. Finally, as $\mathcal{H}$ is a sheaf, we may use Proposition 7.6 to see that $\hat{\kappa} = \hat{\chi}$. This proves the uniqueness.

For the final statement, we know from Proposition that there is a canonical isomorphism $\mathrm{pcoker}'_x : \mathrm{coker}(\varphi_x) \to (\mathrm{Pcoker}(\varphi))_x$. But there is also a canonical isomorphism $\eta_x : (\mathrm{Pcoker}(\varphi))_x \to (\mathrm{Scoker}(\varphi))_x$. A composition of these two maps clearly gives scoker'_x ■

Definition 10.16. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a presheaf morphism. We say that φ is an **epic morphism** if for any other presheaf $\mathcal{H}$ and any pair of presehaf morphisms $\kappa, \kappa' : \mathcal{G} \to \mathcal{H}$, the equation $\kappa \circ \varphi = \kappa' \circ \varphi$ implies $\kappa = \kappa'$.

Proposition 10.17. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a presheaf morphism. Then the following conditions are equivalent*

(i) $\mathrm{Pcoker}(\varphi) = 0$ *(trivial presheaf).*

(ii) For each $U \in \mathbf{Op}(X)$, the R-module morphism φ_U is surjective.

(iii) φ is an epic morphism.

Proof. The equivalence of (i) with (ii) is obvious.

Let us now prove the equivalence of (i) with (iii). Assume that $\mathrm{Pcoker}(\varphi) = 0$. Let $\kappa, \kappa' : \mathcal{G} \to \mathcal{H}$ be a pair of presheaf morphisms, such that $\kappa \circ \varphi = \kappa' \circ \varphi$. Thus $(\kappa - \kappa') \circ \varphi = 0$. By the universality of the presheaf cokernel, there is a unique presheaf map $\mu : \mathrm{Pcoker}(\varphi) \to \mathcal{G}$, such that $\kappa - \kappa' = \mu \circ \mathrm{pcoker}$. But as $\mathrm{Pcoker}(\varphi) = 0$, the right-hand side must be zero, thus $\kappa' = \kappa$.

Now, suppose that φ is epic. Choose $\mathcal{H} = \mathrm{Pcoker}(\varphi)$, $\kappa = \mathrm{pcoker}$ and $\kappa' = 0$. We have $0 = \kappa \circ \varphi = \kappa' \circ \varphi$. This implies $\kappa' = \kappa$, which is possible only if $\mathrm{Pcoker}(\varphi) = 0$. Hence the implication $(iii) \Rightarrow (i)$ holds and the proof is finished. ■

We may now modify this statement for the sheaf $\mathrm{Scoker}(\varphi)$.

Proposition 10.18. *Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of presheaves on X. Then the following conditions are equivalent:*

(i) $\mathrm{Scoker}(\varphi) = 0$.

(ii) For every $x \in X$, $\mathrm{coker}(\varphi_x) = 0$.

(iii) For every $x \in X$, φ_x is surjective.

(iv) For every $U \in \mathbf{Op}(X)$ and for every $t \in \mathcal{G}(U)$, there is some open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U, and a family of sections $\{s_i\}_{i \in I}$, where $s_i \in \mathcal{F}(U_i)$, such that $\varphi_{U_i}(s_i) = \widetilde{\rho}^U_{U_i}(t)$ for all $i \in I$.

(v) The morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is epic in the category of sheaves.

Any of these conditions follows from those in Proposition 10.17.

Proof. First, let us prove the equivalence of (i) with (ii).

As $\mathrm{Scoker}(\varphi)$ is a sheaf, it is trivial if and only if $(\mathrm{Scoker}(\varphi))_x = 0$ for all $x \in X$. As there is a natural isomorphism $\eta_x : (\mathrm{Pcoker}(\varphi))_x \cong (\mathrm{Scoker}(\varphi))_x$, this is equivalent to $(\mathrm{Pcoker}(\varphi))_x = 0$. By Proposition 10.12, this is equivalent to (ii).

The equivalence of (ii) and (iii) is the definition of R-module cokernel.

Now, suppose that (iii) holds. Let $U \in \mathbf{Op}(X)$ and $t \in \mathcal{G}(U)$. Now, for every $x \in U$, let us consider the germ $t_x \in \mathcal{G}_x$. As φ_x is surjective, there is a local section $s'^{(x)} \in \mathcal{F}(U_x)$ over some

$V_x \in \mathbf{Op}_x(X)$, such that $\varphi_x((s'^{(x)})_x) = t_x$. Unfolding the left-hand side, we have $(\varphi_{V_x}(s'^{(x)}))_x = t_x$. There is thus some $U_x \subseteq V_x \cap U$, such that $\widetilde{\rho}^{V_x}_{U_x}(\varphi_{V_x}(s'^{(x)})) = \widetilde{\rho}^U_{U_x}(t)$. Defining $s^{(x)} := \rho^{V_x}_{U_x}(s'^{(x)})$, we obtain an open cover $\mathcal{U} = \{U_x\}_{x \in U}$ of U together with a family of sections $\{s^{(x)}\}_{x \in U}$, where $s^{(x)} \in \mathcal{F}(U_x)$ and $\varphi_{U_x}(s^{(x)}) = \widetilde{\rho}^U_{U_x}(t)$ for all $x \in U$. Hence (iv) holds.

To prove $(iv) \Rightarrow (iii)$, let $x \in X$ be a arbitrary and fix $t_x \in \mathcal{G}_x$, where $t \in \mathcal{G}(U)$ for some $U \in \mathbf{Op}_x(X)$. There is $i \in I$ such that $x \in U_i$, and a section $s_i \in \mathcal{F}(U_i)$, such that $\varphi_{U_i}(s_i) = \widetilde{\rho}^U_{U_i}(t)$. But this immediately implies $\varphi_x((s_i)_x) = t_x$ and (iii) is true.

Now, let us prove the equivalence of (i) and (v). By epic in the category $\mathbf{Sh}(X)$, we mean that for any *sheaf* $\mathcal{H}$, and any pair of morphisms $\kappa, \kappa' : \mathcal{G} \to \mathcal{H}$, the equality $\kappa \circ \varphi = \kappa' \circ \varphi$ must imply $\kappa = \kappa'$. Then one uses the universality property for $\mathrm{Scoker}(\varphi)$ from Proposition 10.15 just like in the case of $\mathrm{Pcoker}(\varphi)$ in the proof of Proposition 10.17.

Finally, $\mathrm{Pcoker}(\varphi) = 0$ clearly implies $\mathrm{Scoker}(\varphi) = 0$ and any of the statements in Proposition 10.17 thus implies any of the equivalent statements of this one. ■

Example 10.19. The codition $\mathrm{Scoker}(\varphi) = 0$ is not enough to ensure the surjectivity of $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every $U \in \mathbf{Op}(X)$. We have already shown this in Example 10.13, as there $\ker(\varphi_x) = 0$. By Proposition 10.18, this is equivalent to $\mathrm{Scoker}(\varphi) = 0$. On the other hand, we have shown that $\mathrm{Pcoker}(\varphi) \neq 0$.

Remark 10.20. The concept of monic and epic morphisms is a universal definition in every category $\mathbf{C}$. In the category $\mathbf{PSh}(X)$ of presheaves of R-modules on X, this has an expected counterpart - maps of presheaves which for each $U \in \mathbf{Op}(X)$ define surjective R-module morphisms. However, for the (full sub)category $\mathbf{Sh}(X)$ of sheaves of R-modules on X, this leads to a weaker definition - not every morphism epic in a subcategory $\mathbf{Sh}(X)$ is thus epic in $\mathbf{PSh}(X)$.

Definition 10.21. We say that the presheaf map $\varphi : \mathcal{F} \to \mathcal{G}$ is **surjective**, if it is epic in the category $\mathbf{PSh}(X)$. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of sheaves, we say that it is **surjective** if it is epic in the category $\mathbf{Sh}(X)$.

Luckily, if we combine both injectivity and surjectivity, the criterions become easy again for sheaves.

Proposition 10.22. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a presheaf morphism, where $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X valued in the category* R-***Mod*** *or* ***CRing****. Then the following conditions are equivalent:*

*(i) φ is an isomorphism in the category **PSh**(X).*

*(ii) For every $U \in$ **Op**(X), the R-module morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is bijective.*

(iii) (Assuming both $\mathcal{F}$ and $\mathcal{G}$ are sheaves) φ_x is bijective for all $x \in X$.

*(iv) (Assuming both $\mathcal{F}$ and $\mathcal{G}$ are sheaves) φ is both monic and epic in the category **Sh**(X).*

Proof. Note that φ is an isomorphism in the category $\mathbf{PSh}(X)$, if there exists a presheaf morphism $\psi : \mathcal{G} \to \mathcal{F}$, such that $\psi \circ \varphi = 1_{\mathcal{F}}$ and $\varphi \circ \psi = 1_{\mathcal{G}}$. If (i) is true, for every $U \in \mathbf{Op}(X)$, we have $\psi_U \circ \varphi_U = 1_{\mathcal{F}(U)}$ and $\varphi_U \circ \psi_U = 1_{\mathcal{G}(U)}$. But this clearly implies (ii). Conversely, if for each $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is bijective, define $\psi_U := (\varphi_U)^{-1}$. It is a usual property of R-**Mod** and **CRing** that ψ_U is an R-module morphism (or ring morphism). One only has to show that the collection

$\{\psi_U\}_{U\in\mathbf{Op}(X)}$ defines a natural transformation $\psi : \mathcal{G} \to \mathcal{F}$, that is the diagram

$$\begin{array}{ccc} \mathcal{G}(U) & \xrightarrow{\psi_U} & \mathcal{F}(U) \\ \Big\downarrow{\scriptstyle \widetilde{\rho}^U_V} & & \Big\downarrow{\scriptstyle \rho^U_V} \\ \mathcal{G}(V) & \xrightarrow{\psi_V} & \mathcal{F}(V) \end{array} \tag{209}$$

commutes for all $U, V \in \mathbf{Op}(X)$ such that $V \subseteq U$. But for any $t \in \mathcal{G}(U)$, we can write

$$\rho^U_V(\psi_U(t)) = \psi_V \circ (\varphi_V \circ \rho^U_V)(\psi_U(t)) = \psi_V \circ (\widetilde{\rho}^U_V \circ \varphi_U)(\psi_U(t)) = \psi_V(\widetilde{\rho}^U_V(t)). \tag{210}$$

We have used the definitions and the naturality of φ. Hence (i) holds.

Next, the equivalence of (iii) and (iv) follows the equivalences $(iv) \Leftrightarrow (iii)$ of Proposition 10.4 and $(iii) \Leftrightarrow (v)$ of Proposition 10.18.

It only remains to prove the equivalence of (ii) and (iii) in the case when $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf morphism in $\mathbf{PSh}(X)$. The injectivity of φ_U for every $U \in \mathbf{Op}(X)$ is (in the case $\mathcal{F}$ is a sheaf) is equivalent to the injectivity of φ_x for every $x \in X$. This is $(ii) \Leftrightarrow (iv)$ of Proposition (10.4). The surjectivity of φ_U for every $U \in \mathbf{Op}(X)$ implies the surjectivity of φ_x for every $x \in X$. This is the statement (iii) of Proposition 10.18 which is implied by (ii) of Proposition 10.17.

It thus remains to prove that if φ_x is bijective for every $x \in X$, then φ_U is surjective for any $U \in \mathbf{Op}(X)$. Let $U \in \mathbf{Op}(X)$ and choose $t \in \mathcal{G}(U)$. From (iv) of Proposition 10.18 it follows that there is an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and a family of sections $\{s_i\}_{i\in I}$, such that $s_i \in \mathcal{F}(U_i)$ and $\varphi_{U_i}(s_i) = \widetilde{\rho}^U_{U_i}(t)$ for all $i \in I$. Now, apply $\widetilde{\rho}^{U_i}_{U_{ij}}$ on both sides of this equation, where $U_{ij} = U_i \cap U_j$. This gives the equality

$$\varphi_{U_{ij}}(\rho^{U_i}_{U_{ij}}(s_i)) = \widetilde{\rho}^{U_i}_{U_{ij}}(\varphi_{U_i}(s_i)) = \widetilde{\rho}^{U_i}_{U_{ij}}(\widetilde{\rho}^U_{U_i}(t)) = \widetilde{\rho}^U_{U_{ij}}(t) = \cdots = \varphi_{U_{ij}}(\rho^{U_j}_{U_{ij}}(s_j)). \tag{211}$$

Now, one has to use the assumed injectivity of $\varphi_{U_{ij}}$ to conclude that $\rho^{U_i}_{U_{ij}}(s_i) = \rho^{U_j}_{U_{ij}}(s_j)$. As $\mathcal{F}$ satisfies the gluing axiom, there is a unique section $s \in \mathcal{F}(U)$, such that $s_i = \rho^U_{U_i}(s)$. To finish the proof, we shall prove that $\varphi_U(s) = t$. As $\mathcal{G}$ satisfies the monopresheaf axiom, it suffices to compare the restrictions of both sections to U_i, for each $i \in I$. But

$$\widetilde{\rho}^U_{U_i}(\varphi_U(s)) = \varphi_{U_i}(\rho^U_{U_i}(s)) = \varphi_{U_i}(s_i) = \widetilde{\rho}^U_{U_i}(t). \tag{212}$$

This finishes the proof. ■

We will now argue why a sheaf cannot be isomorphic to a presheaf.

Proposition 10.23. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an isomorphism of presheaves over X. If $\mathcal{F}$ is a sheaf over X, then so is $\mathcal{G}$.*

Proof. Let $\psi : \mathcal{G} \to \mathcal{F}$ denote the natural inverse to φ. Let $U \in \mathbf{Op}(X)$ be any open set and let $\mathcal{U} = \{U_i\}_{i\in I}$ be any its open cover. Let us prove the gluing axiom for $\mathcal{G}$. Suppose we are given a colection $\{t_i\}_{i\in I}$, where $t_i \in \mathcal{G}(U_i)$, such that

$$\widetilde{\rho}^{U_i}_{U_{ij}}(t_i) = \widetilde{\rho}^{U_j}_{U_{ij}}(t_j), \tag{213}$$

for all $i, j \in I$, where one writes $U_{ij} = U_i \cap U_j$. Set $s_i := \psi_{U_i}(t_i)$. Then

$$\rho^{U_i}_{U_{ij}}(s_i) = \psi_{U_{ij}}(\widetilde{\rho}^{U_i}_{U_{ij}}(t_i) = \psi_{U_{ij}}(\widetilde{\rho}^{U_j}_{U_{ij}}(t_j)) = \rho^{U_j}_{U_{ij}}(s_j). \tag{214}$$

By the gluing axiom for $\mathcal{F}$, there is a unique section $s \in \mathcal{F}(U)$, such that $\rho^U_{U_i}(s) = s_i$. Set $t = \varphi_U(s)$. Then

$$\widetilde{\rho}^U_{U_i}(t) = \widetilde{\rho}^U_{U_i}(\varphi_U(s)) = \varphi_{U_i}(\rho^U_{U_i}(s)) = \varphi_{U_i}(s_i) = t_i. \tag{215}$$

This proves the gluing axiom for $\mathcal{G}$. Next, let $t, t' \in G(U)$, such that $\widetilde{\rho}^U_{U_i}(t) = \widetilde{\rho}^U_{U_i}(t')$ for all $i \in I$. We have to conlude that $t = t'$. Set $s = \psi_U(t)$ and $s' = \psi_U(t')$, respectively. Then

$$\rho^U_{U_i}(s) = \rho^U_{U_i}(\psi_U(t)) = \psi_{U_i}(\widetilde{\rho}^U_{U_i}(t)) = \psi_{U_i}(\widetilde{\rho}^U_{U_i}(t')) = \cdots = \rho^U_{U_i}(s'). \tag{216}$$

By monopresheaf axiom for $\mathcal{F}$, this proves that $s = s'$. As Ψ_U is a R-module isomorphism, we find that $t = t'$. Hence $\mathcal{G}$ satisfies the monopresheaf axiom. And the proof is finished. ■

Finally, to finish this section, let us briefly discuss the problem of kernels, images and quotients for stalk spaces. Are they also somehow related to the kernels, images and quotients of the corresponding section sheaf?

Suppose (E, p) and (E', p') are two stalk spaces of R-modules (or commutative rings) over X, and let $\varphi : E \to E'$ be the stalk space morphism. In particular, $\varphi_{(x)} : E_x \to E'_x$ is an R-module morphism (or ring morphism). It thus makes sense to define the subset $\ker(\varphi)$

$$\ker(\varphi) = \{e \in E \mid e \in \ker(\varphi_{(p(e))})\} = \bigsqcup_x \ker(\varphi_{(x)}). \tag{217}$$

It is natural to ask - is this a substalk space? If so, we may identify its sheaf of sections with certain subsheaf of $\Gamma(E, p)$. By definition, this amounts to proving that $\ker(\varphi)$ is an open subset of E. But this is surprisingly easy, as shows the following proposition.

Proposition 10.24. *The set* $\ker(\varphi) \subseteq E$ *is open. It thus forms a substalk space* $(\ker(\varphi), p)$ *of* (E, p). *Moreover, there is a canonical isomorphism* $\Gamma(\ker(\varphi), p') \cong \ker(\Gamma(\varphi))$, *where* $\Gamma(\varphi) : \Gamma(E, p) \to \Gamma(E', p')$ *is the sheaf morphism induced by* φ.

Proof. Consider the zero (global) section $0_{E'} \in \Gamma_X(E')$, defined as $0_{E'}(x) = 0'_x \in E'_x$. This is indeed a continous global section of E, see Proposition 9.8. The set $0_{E'}(X) \subseteq E'$ is open (in fact homeomorphic to X). The subset $\ker(\varphi)$ can be then defined as an inverse image of this open set, $\ker(\varphi) = \varphi^{-1}(0_{E'}(X))$. Hence it is open.

Next, let $i : \ker(\varphi) \to E$ denote the inclusion of the subspace $\ker(\varphi)$ to E. This is an injective stalk space map. By Proposition 10.7, the induced sheaf morphism $\Gamma(i) : \Gamma(\ker(\varphi), p) \to \Gamma(E, p)$ is also injective. We have to show that for each $U \in \mathbf{Op}(X)$, one has $\mathrm{im}(\Gamma(i)_U) = \ker(\Gamma(\varphi)_U)$.

Let $\sigma \in \Gamma_U(\ker(\varphi))$. This is equivalent to $\varphi_{(x)}(\sigma(x)) = 0'_x$ for all $x \in U$. In turn, this is equivalent to $(\varphi \circ \sigma)(x) = 0'_x$ for $x \in U$. This is is equivalent to $\Gamma(\varphi)_U(\sigma) = 0$. Finally, this is equivalent to $\sigma \in \ker(\Gamma(\varphi)_U)$. ■

One can quite naturally construct a quotient of a stalk space (E, p) by its substalk space (F, p). The result is a stalk space again. One can then construct an example for a special substalk space and identify its sheaf of sections. We will work in the category of R-**Mod**, so we assume that $F_x \subseteq E_x$ is an R-submodule. It thus makes sense to define a space $Q = \bigsqcup_{x \in X} E_x/F_x$. Define $q : Q \to X$ in an obvious way. We have also a canonical quotient map $\natural : E \to Q$, which is for each $x \in X$ defined by the quotient map $\natural_x : E_x \to E_x/F_x =: Q_x$.

We would like to discuss the topology on Q. We declare $V \subseteq Q$ open, if and only if $\natural^{-1}(V) \subseteq E$ is open. This is a usual quotient topology. We have to argue that it makes $q : Q \to X$ into the stalk space of R-modules. To do so, note that $\sigma \in \Gamma_U(E)$ is a local section of E over U, then

$\natural \circ \sigma : U \to Q$ is a local section of Q over U. It is continuous and $q \circ (\natural \circ \sigma) = p \circ \sigma = 1_U$. We will now argue that the basis of topology for Q can be taken to be

$$\mathcal{B}_Q = \{\natural(\sigma(U)) \mid \sigma \in \Gamma_U(E) \text{ for some } U \in \mathbf{Op}(X)\} \tag{218}$$

First, we will prove that $\natural : E \to Q$ is in fact an open map. This happens if and only if $\natural(\sigma(U))$ is open in Q for every $U \in \mathbf{Op}(X)$ and $\sigma \in \Gamma_U(E)$. This is equivalent to $W = \natural^{-1}(\natural(\sigma(U))$ being open in E.

Now, let $e \in W$ be any point, and let $x = p(e)$. By definition $\natural(e) \in \natural(\sigma(U))$ and it follows that $\natural_x(e) = \natural_x(\sigma(x))$. There is thus an element $f \in F_x$, such that $e = \sigma(x) + f$. Now, as F is a substalk space, there is some section $\tau \in \Gamma_Z(F)$, such that $\tau(x) = f$, for some $Z \in \mathbf{Op}_x(X)$. It can be viewed as a continuous section in $\Gamma_Z(E)$ which happens to take values in the submodule F_z for each $z \in Z$. Furthermore, we may assume $Z \subseteq U$ and define a section $\sigma' := \rho^U_Z(\sigma) + \tau$. Then $\sigma' \in \Gamma_Z(E)$. We claim that $\sigma'(Z) \in \mathbf{Op}_e(E)$ is contained in W. But we have

$$\natural(\sigma'(z)) = \natural_z(\sigma(z) + \tau(z)) = \natural_z(\sigma(z)) = \natural(\sigma(z)) \in \natural(\sigma(U)), \tag{219}$$

for all $z \in Z$. This proves the claim. Hence $\natural : E \to Q$ is open. Next, we can use the following general statement:

Proposition 10.25. *Suppose $\natural : X \to Y$ is a surjective map from the topological space X to the set Y. Suppose Y is equipped with a quotient topology, that is $V \subseteq Y$ is open if and only if $\natural^{-1}(V)$ is open in X. Moreover, assume that $\natural$ becomes an open map. Let $\mathcal{B}$ be a basis of topology of X.*

Then $\mathcal{B}_\natural = \{\natural(B) \mid B \in \mathcal{B}\}$ is a basis for the topology on Y.

Proof. As $\natural$ is open, we have $\mathcal{B}_\natural \subseteq \mathbf{Op}(Y)$. To show that it is a basis, every open set $V \subseteq Y$ has to be written as a union of some elements from $\mathcal{B}_\natural$. For each $y \in V$, let $x \in \natural^{-1}(y)$ be arbitrary. Such element exists as $\natural$ is surjective. As $\natural^{-1}(V)$ is open, there exist some element $B \in \mathcal{B}$, such that $x \in B \subseteq \natural^{-1}(V)$. But then $\natural(B) \subseteq V$. Stated differently, for every point $y \in V$ of a given open set V, there exists an element of $\mathcal{B}_\natural$ which contains y and forms a subset of V. In particular, V can be written union of some elements of $\mathcal{B}_\natural$. ■

This proposition shows that the family $\mathcal{B}_Q$ indeed forms a basis for Q. We have to argue that $q : Q \to X$ is a local homeomorphism. First, is it continuous? Let $U \in \mathbf{Op}(X)$. Then $q^{-1}(U) = \natural(p^{-1}(U))$. As $\natural : E \to Q$ is open and p is continous, we find that $q^{-1}(U)$ is open. Now, we claim that $\natural$ is a homeomorphism when restricted to any element of the baisis $\mathcal{B}_Q$. Indeed, for any $\sigma \in \Gamma_U(E)$, $\natural \circ \sigma : U \to \natural(\sigma(U))$ provides a continuous inverse to the restricted projection $q : \natural(\sigma(U)) \to U$. We conclude that (Q, q) forms a stalk space, such that $\natural : (E, p) \to (Q, q)$ becomes a surjective stalk space map.

One only has to discuss the R-module compatibility. Clearly each fiber $Q_x = E_x/F_x$ has a natural R-module structure. Two things are fairly simple. For example, let $\lambda \in R$ and let $\lambda_Q : Q \to Q$ be the fiber-wise multiplication map. By definition, it completes the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\lambda_E} & E \\ \downarrow \natural & & \downarrow \natural \\ Q & \xrightarrow{\lambda_Q} & Q \end{array} \tag{220}$$

For any $V \subseteq Q$ open, we have $\lambda_Q^{-1}(V) = \natural((\natural \circ \lambda_E)^{-1}(V))$. Indeed, suppose $\mu \in \lambda_Q^{-1}(V)$. By definition $\lambda_Q(\mu) \in V$. On the other hand, we have $\mu = \natural(e)$ for some $e \in E$. But then

$$\lambda_Q(\mu) = (\lambda_Q \circ \natural)(e) = (\natural \circ \lambda_E)(e) \tag{221}$$

Thus $e \in (\natural \circ \lambda_E)^{-1}(U)$ and as $\mu = \natural(e)$, we find that $\mu \in \natural((\natural \circ \lambda_E)^{-1}(U))$. Repeating the steps backwards proves the other inclusion. As $\natural \circ \lambda_E$ is continuous and $\natural$ is open, we see that $\natural_Q^{-1}(U)$ is open and thus λ_Q is continuous. The proof for the fiber-wise additive inverse $-_Q$ follows completely the same lines.

For the addition operation $+_Q$, note that the (continuous) map $\natural \times \natural : E \times E \to Q \times Q$ restricts to a map of the fibered products $\natural \times \natural : E \times_X E \to Q \times_X Q$. In fact, it defines the stalk map of the two stalk spaces (see Proposition 9.2). In particular, it is an open map. Finally, it is not difficult to see that $+_Q$ completes the commutative square

$$\begin{array}{ccc} E \times_X E & \xrightarrow{+_E} & E \\ \big\downarrow{\scriptstyle \natural\times\natural} & & \big\downarrow{\scriptstyle \natural} \\ Q \times_X Q & \xrightarrow{+_Q} & Q \end{array} . \tag{222}$$

As the left vertical map is open, we may use the same argument as for λ_Q to prove that $+_Q$ is continuous. We thus conclude this paragraph with a proposition.

Proposition 10.26. *Let $(F, p) \subseteq (E, p)$ be a substalk space of a stalk space of R-modules. Let (Q, q) be the fiber-wise quotient space. Then the usual quotient topology on Q makes it into a stalk space of R-modules.*

Now, let $\eta : (E, p) \to (E', p')$ be a stalk space map. Let $\operatorname{im}(\eta) \subseteq E'$ denote the image of this map. As every stalk space map is open, it follows that $(\operatorname{im}(\eta), p')$ is a substalk space of (E', p'). We can thus form its quotient $q' : E'/\operatorname{im}(\eta) \to X$. One can then examine its sheaf of sections $\Gamma(E'/\operatorname{im}(\eta), q')$. Can something be said about it?

Proposition 10.27. *There exists a canonical sheaf isomorphism $\psi : \operatorname{Scoker}(\Gamma(\eta)) \to \Gamma(E'/\operatorname{im}(\eta), q')$, where $\Gamma(\eta) : \Gamma(E, p) \to \Gamma(E', p')$ is the sheaf morphism induced by $\eta : E \to E'$.*

Proof. Let $\natural' : E' \to E'/\operatorname{im}(\eta)$ be the canonical quotient map. The induced sheaf morphism $\Gamma(\natural') : \Gamma(E') \to \Gamma(E'/\operatorname{im}(\eta))$ satisfies $\Gamma(\natural') \circ \Gamma(\eta) = 0$. By the universality property of $\operatorname{Scoker}(\Gamma(\eta))$, see Proposition 10.15, there is a unique sheaf morphism $\psi : \operatorname{Scoker}(\Gamma(\eta)) \to \Gamma(E'/\operatorname{im}(\varphi))$, such that $\Gamma(\natural') = \psi \circ \operatorname{scoker}$, where $\operatorname{scoker} : \Gamma(E') \to \operatorname{Scoker}(\Gamma(\eta))$ is the canonical sheaf map.

As both domain and codomain of ψ are sheaves, it suffices to prove, see Proposition 10.22, that for each $x \in X$, the induced stalk map $\psi_x : \operatorname{Scoker}(\Gamma(\eta))_x \to \Gamma(E'/\operatorname{im}(\varphi))_x$ is an R-module isomorphism. To this account, we must prove that the stalk map $\Gamma(\natural')_x : \Gamma(E')_x \to \Gamma(E'/\operatorname{im}(\varphi))_x$ is surjective and its kernel is precisely the kernel of $\operatorname{scoker}_x : \Gamma(E')_x \to \operatorname{Scoker}(\Gamma(\eta))_x$.

But this boils down to proving the surjectivity of $\natural'_x$ and the equality $\ker(\natural'_x) = \operatorname{im}(\eta_{(x)})$, for all $x \in X$, which is trivial. ∎

This proposition shows that although the stalk space $E'/\operatorname{im}(\eta)$ is well-defined, its sheaf of sections is not the naïve guess $\Gamma(E')/\Gamma(\operatorname{im}\Gamma(\eta))$ but rather its sheafification.

11 Exact sequences of presheaves

To define what an exact sequence is in a category, we need to have the notion of both kernels and images. With the category $\mathbf{Sh}(X)$ of sheaves (of R-modules or commutative rings), we immediately hit the same problem as with cokernels. For a sheaf map $\varphi : \mathcal{F} \to \mathcal{G}$, the naïve definition of the image $\text{im}(\varphi)$ forms a subpresheaf of $\mathcal{G}$, but usually not a sheaf. One can again reach for the help of a sheafification. However, the new sheaf is not a subsheaf of $\mathcal{G}$ anymore. To avoid this issue, one can resolve to the following trick:

Definition 11.1. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a presheaf morphism. We define **presheaf image** $\text{Pim}(\varphi)$ as the kernel of the canonical cokernel map $\text{pcoker}(\varphi) : \mathcal{G} \to \text{Pcoker}(\varphi)$, that is

$$\text{Pim}(\varphi) = \ker(\text{pcoker}(\varphi)) \tag{223}$$

The **sheaf image** $\text{Sim}(\varphi)$ is defined as the kernel of $\text{scoker}(\varphi) : \mathcal{G} \to \text{Scoker}(\varphi)$, that is

$$\text{Sim}(\varphi) = \ker(\text{scoker}(\varphi)) \tag{224}$$

It is easy to check that $\text{Pim}(\varphi)$ agrees with a naïve definition of the image presheaf. Let us argue that if $\mathcal{G}$ is a sheaf, there is always a canonical sheaf isomorphism from the sheafification of $\text{Pim}(\varphi)$ to $\text{Sim}(\varphi)$, which shows that two approaches to the definition of sheaf image are essentially equivalent.

Proposition 11.2. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a presheaf morphism, where $\mathcal{G}$ is a sheaf. Then there is a canonical sheaf isomorphism of the sheafification of* $\text{Pim}(\varphi)$ *and the sheaf* $\text{Sim}(\varphi)$*.*

Proof. Let $\text{J} : \text{Pim}(\varphi) \to \mathcal{G}$ denote the canonical inclusion of the subpresheaf $\text{Pim}(\varphi)$ into $\mathcal{G}$. Write

$$\mathcal{H} := \widetilde{\text{Pim}(\varphi)}. \tag{225}$$

By universality of a sheaffication, see Proposition 8.10, there is a unique sheaf map $\widetilde{\text{J}} : \mathcal{H} \to \mathcal{G}$, such that $\text{J} = \widetilde{\text{J}} \circ \eta$. Now, we claim that $\text{scoker}(\varphi) \circ \widetilde{\text{J}} = 0$. This is a comparison of two sheaf maps. It thus suffices to look at the induced stalk maps. This boils done to proving $\text{pcoker}_x(\varphi) \circ \text{J}_x = 0$ for all $x \in X$. But this is obvious. Finally, using the universality of the kernels, see Proposition 10.3, there is a unique map $\hat{\text{J}} : \mathcal{H} \to \ker(\text{scoker}(\varphi)) \equiv \text{Sim}(\varphi)$, such that $\hat{\text{J}} = \text{I} \circ \widetilde{\text{J}}$, where $\text{I} : \text{Sim}(\varphi) \to \mathcal{G}$ is the inclusion. It only remains to prove that $\hat{\text{J}}$ is a sheaf isomorphism. Both its domain and codomain are sheaves. By Proposition 10.22, it suffices to prove that the induced stalk map $\hat{\text{J}}_x$ is bijective for all $x \in X$. But this agains boils down to the obvious comparison $\text{im}(\varphi_x) = \ker(\text{coker}(\varphi_x))$. ■

Now, there come some useful but rather simple observations.

Proposition 11.3. *Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves.*

(i) If $\text{Pim}(\varphi)$ *happens to be a sheaf, then* $\text{Pim}(\varphi) = \text{Sim}(\varphi)$*.*

(ii) If φ is injective, then $\text{Pim}(\varphi) = \text{Sim}(\varphi)$*.*

Proof. To prove (i), note that if $\text{Pim}(\varphi)$ is a sheaf, we are comparing two subsheaves of the sheaf $\mathcal{G}$. By Proposition 10.10, it suffices to compare their stalk spaces. But

$$\text{Pim}(\varphi)_x = \ker(\text{pcoker}_x(\varphi)) = \ker(\text{scoker}_x(\varphi)) = \text{Sim}(\varphi)_x, \tag{226}$$

for all $x \in X$. This proves $\text{Pim}(\varphi) = \text{Sim}(\varphi)$. To prove (ii), note that if φ is injective, it can be viewed as a presheaf isomorphism $\varphi : \mathcal{F} \to \text{Pim}(\varphi)$. By Proposition 10.23, this implies that $\text{Pim}(\varphi)$ is a sheaf. The rest follows from (i). ■

Before proceeding further, let us again examine the concept of image on the level of stalk spaces. Suppose $\eta : (E, p) \to (E', p')$ is a map of stalk space (of R-modules or commutative rings). With no problems, we may define a substalk space $\mathrm{im}(\eta) \subseteq E'$. Is its sheaf of sections $\Gamma(\mathrm{im}(\eta), p)$ somehow related to the sheaf image of the induced map $\Gamma(\eta) : \Gamma(E, p) \to \Gamma(E', p')$? The answer is, as expected, yes.

Proposition 11.4. *There exists a canonical sheaf isomorphism* $\chi : \Gamma(\mathrm{im}(\eta), p') \to \mathrm{Sim}(\Gamma(\eta))$.

Proof. Let $j : \mathrm{im}(\eta) \to E'$ be the inclusion. We claim that the induced sheaf morphism $\Gamma(j) : \Gamma(\mathrm{im}(\eta), p') \to \Gamma(E', p')$ satisfies the equation

$$\mathrm{scoker}(\Gamma(\eta)) \circ \Gamma(j) = 0 \tag{227}$$

Recall that we have shown that there is a canonical sheaf isomorphism $\psi : \mathrm{Scoker}(\Gamma(\eta)) \to \Gamma(E'/\mathrm{im}(\eta), q')$ such that $\psi \circ \mathrm{scoker}(\Gamma(\eta)) = \Gamma(\natural')$, where $\natural' : E' \to E'/\mathrm{im}(\eta)$ is the quotient map. See the proof of Proposition 10.27. The above equation is thus equivalent to

$$\Gamma(\natural') \circ \Gamma(j) = 0, \tag{228}$$

which holds as $\natural' \circ j = 0$. Finally, the universality of the kernels, see Proposition 10.3, implies the existence of a unique sheaf homomorphism $\chi : \Gamma(\mathrm{im}(\eta), p') \to \ker(\mathrm{scoker}(\Gamma(\eta)) \equiv \mathrm{Sim}(\Gamma(\eta))$, such that $\mathrm{J}' \circ \chi = \Gamma(j)$, where $\mathrm{J}' : \mathrm{Sim}(\Gamma(\eta)) \to \Gamma(E', p')$ is the inclusion of a subpresheaf. Finally, one has to argue that χ is an isomorphism of sheaves. Again, by Proposition 10.22, it suffices to prove that the induced stalk map $\chi_x : \Gamma(\mathrm{im}(\eta), p')_x \to \mathrm{Sim}(\Gamma(\eta))_x$ is the isomorphism for all $x \in X$. Using the canonical identifications $\Gamma(\mathrm{im}(\eta), p')_x = \mathrm{im}(\eta)_x = \mathrm{im}(\eta_{(x)}$ and $\mathrm{Sim}(\Gamma(\eta))_x = \mathrm{im}(\Gamma(\eta)_x) = \mathrm{im}(\eta_{(x)})$, we find that χ_x is just the identity. ■

We have now make sense of kernels and images in both the category $\mathbf{PSh}(X)$ and $\mathbf{Sh}(X)$, targeted in R-**Mod** (or **CRing**). We can now define the concept of exact sequences in those categories.

Definition 11.5. Consider the sequence of (pre)sheaf morphisms of (pre)sheaves

$$\cdots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow \cdots \tag{229}$$

We say that it is **exact at $\mathcal{G}$ in the category of presheaves**, if $\mathrm{Pim}(\varphi) = \ker(\psi)$. We say that it is **exact at $\mathcal{G}$ in the category of sheaves**, if $\mathrm{Sim}(\varphi) = \ker(\psi)$.

Moreover, we say that any sequence of (pre)sheaf morphisms of (pre)sheaves is **exact in the category of (pre)sheaves**, if it is exact at each of its terms (in the category of (pre)sheaves).

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