Lie Groupoids and Algebroids

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1 Lie groupoids

Let us start this lecture by an example.

Example 1.1. Let M be smooth manifold and let G be a Lie group. Let us consider

$$\mathcal{G} := M \times G \times M. \tag{1}$$

We have two canonical maps $\mathsf{s} : \mathcal{G} \to M$ and $\mathsf{t} : \mathcal{G} \to M$ defined by

$$\mathsf{s}(m, g, m') := m', \ \ \mathsf{t}(m, g, m') := m. \tag{2}$$

Both are smooth surjective submersions. In particular, one can consider their fibered product

$$\begin{aligned} \mathcal{G} * \mathcal{G} &:= \{((m, g, m'), (n, h, n')) \in \mathcal{G} \times \mathcal{G} \mid \mathsf{s}(m, g, m') = \mathsf{t}(n, h, n')\} \\ &= \{((m, g, m'), (m', h, n')) \in \mathcal{G} \times \mathcal{G} \mid m, m', n' \in M \ g, h \in G\}. \end{aligned} \tag{3}$$

We can define a *partial multiplication map* $\mu : \mathcal{G} * \mathcal{G} \to \mathcal{G}$ by the formula

$$\mu((m, g, m'), (m', h, n')) \equiv (m, g, m') \cdot (m', h, n) := (m, gh, n'). \tag{4}$$

This map is obviously smooth. Finally, for each $m \in M$, we have the element $\mathbf{1}_m = (m, e, m)$, where $e \in G$ is the group unit. It follows that

$$\mathbf{1}_m \cdot (m, g, m') = (m, g, m'), \ \ (m, g, m') \cdot \mathbf{1}_{m'} = (m, g, m'), \tag{5}$$

and to every $(m, g, m') \in \mathcal{G}$, there is a unique element $(m, g, m')^{-1} := (m', g^{-1}, m)$, such that

$$(m, g, m') \cdot (m, g, m')^{-1} = \mathbf{1}_m, \ \ (m, g, m')^{-1} \circ (m, g, m) = \mathbf{1}_{m'}. \tag{6}$$

Note that the assignment $m \mapsto \mathbf{1}_m$ can be viewed as a smooth map $\mathbf{1} : M \to \mathcal{G}$.

This simple example gives the intuition to understand the following definition:

Definition 1.2. Let us consider the following data:

(i) a pair of smooth manifolds $\mathcal{G}$ and M;

(ii) a pair of smooth surjective submersions $\mathsf{s}, \mathsf{t} : \mathcal{G} \to \mathcal{M}$, where s is called the **source map** and t is called the **target map**;

(iii) a smooth map $\mu : \mathcal{G} * \mathcal{G} \to \mathcal{G}$, where $\mathcal{G} * \mathcal{G}$ is the fibered product over a pair of maps (s, t).

Then $(\mathcal{G}, M, \mathsf{s}, \mathsf{t}, \mu)$ is called a **Lie groupoid $\mathcal{G}$ over the base** M, if it satisfies the following axioms. We assume that each identity holds for all elements of $\mathcal{G}$, such that everything is well defined. One also uses the short-hand notation $g \cdot h := \mu(g, h)$.

(i) One has $\mathsf{s}(g \cdot h) = \mathsf{s}(h)$ and $\mathsf{t}(g \cdot h) = \mathsf{t}(g)$.

(ii) μ is associative, that is $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

(iii) For each $m \in M$, there exists an element $1_m \in \mathcal{G}$, such that $\mathsf{s}(1_m) = \mathsf{t}(1_m) = m$ and

$$g \cdot \mathbf{1}_{\mathsf{s}(g)} = g, \quad \mathbf{1}_{\mathsf{t}(h)} \cdot h = h. \tag{7}$$

Moreover, the formula $1(m) := 1_m$ has to define a smooth map $1 : M \to \mathcal{G}$.

(iv) For each $g \in \mathcal{G}$, there exists $g^{-1} \in \mathcal{G}$ with $\mathsf{s}(g^{-1}) = \mathsf{t}(g)$ and $\mathsf{t}(g^{-1}) = \mathsf{s}(g)$, such that

$$g \cdot g^{-1} = 1_{\mathsf{t}(g)}, \quad g^{-1} \cdot g = 1_{\mathsf{s}(g)}. \tag{8}$$

One usually writes $\mathcal{G} \rightrightarrows M$. The set $\mathcal{G}$ is called the **arrows** of $\mathcal{G} \rightrightarrows M$. μ is called the **partial multiplication**, 1 is the **object inclusion map**, g^{-1} is called the **inverse arrow of** g.

Exercise 1.3. Note that neither $1 : M \to \mathcal{G}$ and $\mathsf{i} : \mathcal{G} \to \mathcal{G}$ defined as $\mathsf{i}(g) := g^{-1}$ are not part of the data $(\mathcal{G}, \mathcal{M}, \mathsf{s}, \mathsf{t}, \mu)$. This is because if they exist, they are unique. Prove this.

Example 1.4. The structure in Example 1.1 is called a **trivial Lie groupoid**. It contains two special cases:

(a) If $G = \{e\}$, we have $\mathcal{G} \cong M \times M$ and $M \times M \rightrightarrows M$ is called the **product Lie groupoid**.

(b) If $M = \{*\}$, we have $\mathcal{G} \cong G$ and $G \rightrightarrows \{*\}$ is just a Lie group.

Remark 1.5. Lie groupoid can be viewed as a small category whose set of objects is M, its set of arrows if $\mathcal{G}$. Axioms $(i) - (iii)$ are just a reformulation of category axioms. The axiom (iv) is equivalent to it being a groupoid, that is a category with invertible arrows.

Definition 1.6. For each $m, n \in M$, we define the following subsets of $\mathcal{G}$:

(i) The **s-fiber over** m is a set $\mathcal{G}_m := \{g \in \mathcal{G} \mid \mathsf{s}(g) = m\}$;

(ii) The **t-fiber over** n is a set $\mathcal{G}^n := \{g \in \mathcal{G} \mid \mathsf{t}(g) = n\}$;

(iii) $\mathcal{G}_m^n := \mathcal{G}_m \cap \mathcal{G}^n$. In particular, $\mathcal{G}_m^m$ is called the **isotropy group at** m.

All of those subsets are actually closed embedded submanifolds of $\mathcal{G}$. For $\mathcal{G}_m^n$, the statement is non-trivial and it will be proved later.

They are some immediate consequences of the definition.

Proposition 1.7. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Then the following facts can be deduced:*

(i) The object inclusion map $1 : M \to \mathcal{G}$ is a closed embedding. In particular $\mathbf{1}(M) \subseteq \mathcal{G}$ is a closed embedded submanifold diffeomorphic to M.

(ii) The inversion map $\mathsf{i} : \mathcal{G} \to \mathcal{G}$ *is a diffeomorphism.*

Exercise 1.8. Prove the preceding proposition. Here are some hints

(i) Use the fact that $\mathsf{t} \circ \mathsf{i} = \mathbb{1}_M$ to argue that i is an injective immersion. Then prove that it is is a closed map. Every closed injective immersion is a closed embedding.

(ii) Define a smooth map $\theta : \mathcal{G} * \mathcal{G} \to \mathcal{G} \times_{\mathsf{t}} \mathcal{G}$ as $\theta(g, h) = (g, g \cdot h)$. Find its inverse to prove that θ is a bijection. Use a global rank theorem to prove that θ is a diffeomorphism.

Then observe that

$$\mathsf{i} = \pi_2 \circ \theta^{-1} \circ (\mathbb{1}_{\mathcal{G}}, \mathsf{1} \circ \mathsf{t}) \tag{9}$$

Finally, show that $\mathsf{i}^{-1} = \mathsf{i}$ to finish the proof.

Let us continue with more examples:

Example 1.9. Let $\theta : G \times M \to M$ be a left action of G on M. Let $\mathcal{G} := G \times M$.

(i) The source map is $\mathsf{s}(g, m) := m$. This is a smooth surjective submersion.

(ii) The target map is $\mathsf{t}(g, m) := g \cdot m$, that is $\mathsf{t} := \theta$. It is surjective as $\theta(e, m) = m$. It is a submersion as its restriction to a fiber $\{g\} \times M$ is a diffeomorphism.

(iii) Note that $((g, m), (h, n)) \in \mathcal{G} * \mathcal{G}$, if $m = h \cdot n$. We define the partial multiplication as

$$(g, m) \cdot (h, n) := (gh, n) \tag{10}$$

It is smooth as it just a restriction of a smooth map to the submanifold $\mathcal{G} * \mathcal{G}$. Then

$$\mathsf{s}((g, m) \cdot (h, n)) = \mathsf{s}(gh, n) = n = \mathsf{s}(h, n), \tag{11}$$

$$\mathsf{t}((g, m) \cdot (h, n)) = \mathsf{t}(gh, n) = (gh) \cdot n = g \cdot (h \cdot n) = g \cdot m = \mathsf{t}(g, m). \tag{12}$$

The associativity follows from the one of G.

(iv) The object inclusion map is $\mathbf{1}(m) = (e, m)$ and the inverse is $(g, m)^{-1} = (g^{-1}, g \cdot m)$.

One writes $\mathcal{G} = G \lessdot M$ and calls $G \lessdot M \rightrightarrows M$ the **action Lie groupoid** corresponding to the action θ.

Example 1.10. Let $\pi : P \to M$ be a principal G-bundle with the right action $R : P \times G \to P$.

There is an induced action on $P \times P$, namely set $(p, q) \cdot g := (p \cdot g, q \cdot g)$. This action is free an proper. Consequently, there is unique topology and smooth structure on the quotient

$$\mathcal{G} := \frac{P \times P}{G}, \tag{13}$$

making the canonical quotient map $\natural : P \times P \to \mathcal{G}$ into a surjective submersion. Let us write $[p, q] := \natural(p, q)$. By design, we thus have $[p \cdot g, q \cdot g] = [p, g]$.

(i) Let $\mathsf{s}([p, q]) := \pi(q)$ an $\mathsf{t}([p, q]) := \pi(p)$. They are obviously well-defined and surjective. Since $\mathsf{s} \circ \natural = \pi \circ \pi_2$ and $\mathsf{t} \circ \natural = \pi \circ \pi_1$, they are smooth surjective submersions.

(ii) Define a map $\delta : P \times_\pi P \to P$ to satisfy the formula $\delta(p \cdot g, p) := g$. Such a map is unique and it is smooth. Since for $([p, q], [p', q']) \in \mathcal{G} * \mathcal{G}$, one has $(q, p') \in P \times_\pi P$, one can define

$$[p, q] \cdot [p', q'] := [p, q' \cdot \delta(q, p')]. \tag{14}$$

Now, one has

$$\mathsf{s}([p, q' \cdot \delta(q, p')]) = \pi(q' \cdot \delta(q, p')) = \pi(a') = \mathsf{s}([p', q']), \tag{15}$$

$$\mathsf{t}([p, q' \cdot \delta(q, p')]) = \pi(p) = \mathsf{t}([p, q]). \tag{16}$$

Note that the associativity is non-trivial, it boils down to the fact that

$$\delta(p \cdot h, q) = \delta(p, q) \cdot h, \tag{17}$$

for all $(p, q) \in P \times_\pi P$ and $h \in G$. We leave the smoothness of μ as an excercise.

(iii) For each $m \in M$, fix and arbitrary $p \in \pi^{-1}(M)$. We declare $\mathtt{1}_m := [p, p]$. It is easy to see that this is well-defined. Observe that for each $[p', q'] \in \mathcal{G}^m$, one

$$\mathtt{1}_m \cdot [p', q'] = [p', p'] \cdot [p', q'] = [p', q' \cdot \delta(p', p')] = [p', q' \cdot e] = [p', q']. \tag{18}$$

The proof that for any $[p, q] \in \mathcal{G}_m$, onhe has $[p, q] \cdot \mathtt{1}_m = [p, q]$ is analogous.

(iv) The inverse is easily seen to be $[p, q]^{-1} = [q, p]$.

Lie algebroid $\frac{P \times P}{G} \rightrightarrows M$ is called the g**auge groupoid corresponding to** $\pi : P \to M$.

Exercise 1.11. (i) Prove that the action on $P \times P$ is proper.

Recall that the right action $\theta : N \times G \to N$ is proper, if the maps $(n, g) \mapsto (n, \theta(n, g))$ is a proper map. This is equivalent to the following statement: Suppose we are given sequences $\{n_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ in N and G, respectively. Then if both $\{n_k\}_{k=1}^\infty$ and $\{n_k \cdot g_k\}_{k=1}^\infty$ converge in N, some subsequence of $\{g_k\}_{k=1}^\infty$ must converge in G.

Finally, observe that if $\{p_k\}_{k=1}^\infty$ converges to $p \in P$, for large enough k, the terms of the sequence end up in $\pi^{-1}(U)$ for some open subset of U containing $\pi(p)$, such that $P \cong U \times G$. But since G acts along the fibers, the same is true for $\{p_k \cdot g_k\}_{k=1}^\infty$. Modify this argument for $P \times P$ and work locally to prove that the action is proper.

(ii) Prove that $\delta : P \times_\pi P \to G$ is smooth.

(iii) Prove that $\mu : \mathcal{G} * \mathcal{G} \to \mathcal{G}$ is smooth. Hint: prove that a smooth map $\natural \times \natural$ restricts to a surjective submersion

$$\natural \times \natural : P \times (P \times_\pi P) \times P \to \mathcal{G} * \mathcal{G} \tag{19}$$

The partial multiplication $\mu : \mathcal{G} * \mathcal{G}$ then fits into the commutative diagram

$$\begin{array}{ccc} P \times (P \times_\pi P) \times P & \xrightarrow{\hat{\mu}} & P \times P \\ \downarrow{\scriptstyle \natural \times \natural} & & \downarrow{\scriptstyle \natural} \\ \mathcal{G} * \mathcal{G} & \xrightarrow{\mu} & \mathcal{G} \end{array}, \tag{20}$$

where $\hat{\mu}(p, q, p', q') = (p, q' \cdot \delta(q, p'))$ is obviously smooth.

(iv) Prove that $1 : M \to \mathcal{G}$ is smooth. Hint: observe that 1 fits into the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & P \times P \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \natural} \\ M & \xrightarrow{1} & \mathcal{G} \end{array} \tag{21}$$

Example 1.12. Let M be an arbitrary manifold. Let $I = [0, 1]$ and let $\mathcal{G} := \Pi(M)$ be a set of homotopy classes of continuous curves $\gamma : I \to M$.

(i) One sets $\mathsf{s}([\gamma]) := \gamma(0)$ and $\mathsf{t}([\gamma]) = \gamma(1)$.

(ii) The partial multiplication is just the usual concatenation of curves, that is $[\gamma] \cdot [\gamma'] = [\gamma * \gamma']$, where $\gamma * \gamma'$ is defined as γ' on $[0, \frac{1}{2}]$ followed by γ on $[\frac{1}{2}, 1]$. It is a well-known fact that it has all the properties required of the partial multiplication.

(iii) One has $1_m := [e_m]$, where $e_m : I \to M$ is a constant curve valued at m. Again, it is a standard argument to see that 1_m works as a unit element at m.

(iv) Finally, one has $[\gamma]^{-1} = [\gamma^{-1}]$, where $\gamma^{-1}(t) := \gamma(1 - t)$.

$\Pi(M) \rightrightarrows M$ is called the **fundamental groupoid of** M.

One has to introduce a topology and a smooth structure on $\Pi(M)$. This can be done in a rather interesting indirect way. Suppose M is connected.

1) There exists a universal covering space $\pi : \widetilde{M} \to M$. Fix $m_0 \in M$. Then $\widetilde{M}$ is defined as a set of all homotopy classes of all continuous curves starting from m_0. For each $[\gamma] \in \widetilde{M}$, one has $\pi[\gamma] := \gamma(1)$. There is a topology and a smooth structure on $\widetilde{M}$ making π into a smooth covering.

2) The fundamental group $\pi_1(M, m_0)$ is at most countable, it thus forms a discrete Lie group. There is a canonical right action of $\pi_1(M, m_0)$ on $\widetilde{M}$, namely $[\gamma] \cdot [\omega] := [\gamma * \omega]$.

   One can show that this makes $\pi : \widetilde{M} \to M$ into a principal $\pi_1(M, m_0)$-bundle.

3) There is a canonical bijection $\Psi : \Pi(M) \to \frac{\widetilde{M} \times \widetilde{M}}{\pi_1(M, m_0)}$. Let $[\gamma] \in \Pi(M)$. Let $x := \gamma(0)$ and $y := \gamma(1)$. Since M is connected, there exist a curve $\gamma_0 : I \to M$ connecting m_0 and x. Then $\gamma * \gamma_0$ connects m_0 and y. One defines

$$\Psi([\gamma]) := [[\gamma * \gamma_0], [\gamma_0],]. \tag{22}$$

   With some work, one can show that Ψ is both injective and surjective, it intertwines both the source and target maps, and it is compatible with both partial multiplications.

   We can thus use Ψ to *declare* the topology and a smooth structure on $\Pi(M)$, making it into a Lie groupoid.

2 Morphisms, local triviality

Definition 2.1. Let $(\mathcal{G}, M, \mathsf{s}, \mathsf{t}, \mu)$ and $(\mathcal{H}, N, \mathsf{s}', \mathsf{t}', \mu')$ be a pair of Lie groupoids. A **morphism of Lie groupids** is a pair of smooth maps (F, f), where $F : \mathcal{G} \to \mathcal{H}$ and $f : M \to N$, such that

(i) $\mathsf{s}' \circ F = f \circ \mathsf{s}$ and $\mathsf{t}' \circ F = f \circ \mathsf{t}$;

(ii) $F(g) \cdot F(h) = F(g \cdot h)$ for all $(g, h) \in \mathcal{G} * \mathcal{G}$; Note that (i) ensures that $(F(g), F(h)) \in \mathcal{H} * \mathcal{H}$.

We say that $F : \mathcal{G} \to \mathcal{H}$ is a **LG morphism over** f. If $M = N$ and $f = \mathbb{1}_M$, we say that F is a **LG morphism over** M.

Exercise 2.2. Let $F : \mathcal{G} \to \mathcal{H}$ be a LG morphism over f.

(i) For each $m \in M$, one has $F(1_m) = 1_{f(m)}$.

(ii) One has $F(g^{-1}) = F(g)^{-1}$ for each $g \in \mathcal{G}$.

(iii) f is in fact completely determined by F, since it fits into the diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{F} & \mathcal{H} \\ {\scriptstyle \mathbf{1}}\uparrow & & \uparrow{\scriptstyle \mathbf{1}'} \\ M & \overset{f}{\dashrightarrow} & N \end{array} . \tag{23}$$

In fact, if F is smooth, f is automatically smooth.

(iv) Lie groupoids and their morphisms form a category.

Example 2.3. Let $\mathcal{G} \rightrightarrows M$ be any groupoid. Let $\chi : \mathcal{G} \to M \times M$ be defined as

$$\chi(g) := (\mathsf{t}(g), \mathsf{s}(g)). \tag{24}$$

Then χ is a morphism of $\mathcal{G}$ and the pair groupoid $M \times M$ called the **anchor of** $\mathcal{G}$.

Example 2.4. Let $q : E \to M$ be a vector bundle over M. For each $m, n \in M$, let

$$\Phi_m^n(E) := \mathrm{Iso}(E_m, E_n) \tag{25}$$

Let $\Phi(E) = \bigsqcup_{m,n} \Phi_m^n(E)$. For each $\varphi \in \Phi_m^n(E)$, define $\mathsf{s}(\varphi) = m$ and $\mathsf{t}(\varphi) = n$.

The partial multiplication is simply a composition of maps, $1_m = \mathbb{1}_{E_m}$ and $\mathsf{i}(\varphi) = \varphi^{-1}$.

One only has to define a topology and a smooth structure on $\Phi(E)$. This is done as follows. For all open subsets $U, V \subseteq M$, let $\Phi_U^V(E) := \mathsf{s}^{-1}(U) \cap \mathsf{t}^{-1}(V)$. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a local trivialization for E. Without the loss of generality, we may assume that we also have an atlas $\{(U_\alpha, \nu_\alpha)\}_{\alpha \in I}$ for M, that is $\nu_\alpha : U_\alpha \to \hat{U}_\alpha \subseteq \mathbb{R}^n$.

For each $\alpha, \beta \in I$, one produces a bijection

$$\Psi_{\alpha\beta} : \Phi_{U_\alpha}^{U_\beta}(E) \to \hat{U}_\beta \times \mathrm{GL}(k, \mathbb{R}) \times \hat{U}_\alpha \tag{26}$$

as follows. For any $\varphi \in \Phi_{U_\alpha}^{U_\beta}(E)$, let

$$\Psi_{\alpha\beta}(\varphi) := \Big(\nu_\beta(\mathsf{t}(\varphi)), \phi_{\beta,\mathsf{t}(\varphi)}^{-1} \circ \varphi \circ \phi_{\alpha,\mathsf{s}(\varphi)}, \nu_\alpha(\mathsf{s}(\varphi))\Big). \tag{27}$$

It is not difficult to find its inverse, since one finds.

$$\Phi_{\alpha\beta}^{-1}(y, \mathbf{A}, x) = \phi_{\beta,\nu_\beta^{-1}(y)} \circ \mathbf{A} \circ \phi_{\alpha,\nu_\alpha^{-1}(x)}^{-1} \in \mathrm{Iso}(E_{\nu_\beta^{-1}(y)}, E_{\nu_\alpha^{-1}(x)}). \tag{28}$$

$\{\Phi_{U_\alpha}^{U_\beta}(E)\}_{\alpha,\beta \in I}$ is an cover of $\Phi(E)$. Since $\mathrm{GL}(k, \mathbb{R}) \subseteq \mathbb{R}^{k \times k}$ is an open subset, we can now define a topology and a smooth structure on Φ by declaring $\{(\Phi_{U_\alpha}^{U_\beta}, \Psi_{\alpha\beta}\}_{\alpha,\beta \in I}$ into a smooth atlas.

It is then not difficult to prove that $\Phi(E) \rightrightarrows M$ forms a Lie groupoid called the **frame groupoid of** E.

Exercise 2.5. Finish some technical details in the above example.

(i) Calculate the transition maps of the above atlas and prove that they are smooth

(ii) Prove that s, t and μ all have the required properties. Hint: locally, everything boils down to the trivial Lie groupoid.

Example 2.6. The frame groupoid $\Phi(E)$ is in fact isomorphic to a gauge groupoid of the frame bundle $\pi : \mathrm{Fr}(E) \to M$.

Let us only construct a mapping from $\Phi(E)$ to the gauge groupoid. For any $\varphi \in \Phi(E)$, let us pick an arbitrary basis $q = (q_\mu)_{\mu=1}^k$ of $E_{\mathsf{s}(\varphi)}$. Then $p := (\varphi(q_\mu))_{\mu=1}^k$ is a basis of $E_{\mathsf{t}(\varphi)}$. Let

$$F(\varphi) := [p, q] \in \Phi. \tag{29}$$

It is easy to see that F is well-defined, since if $q' := q \cdot \mathbf{A}$ is any other basis, then the induced basis of $E_{\mathsf{t}(\varphi)}$ is $p' = p \cdot \mathbf{A}$, so $[p', q'] = [p, q]$. Since $\mathsf{s}'(F(\varphi)) = \mathsf{s}(\varphi)$ and $\mathsf{t}'(F(\varphi)) = \mathsf{t}(\varphi)$, the underlying map is $\mathbb{1}_M$. We proof of the smoothness of F is a straighforward exercise.

Definition 2.7. One says that a Lie groupoid $\mathcal{G} \rightrightarrows M$ is **transitive**, if the anchor map $\chi : \mathcal{G} \to M \times M$ is surjective.

It turns out that the set of arrows of any transitive Lie groupoid is necessarily bijective to the one of a trivial Lie groupoid.

Lemma 2.8. *For any $p \in M$, the set $\mathcal{G}$ is bijective to $M \times \mathcal{G}_p^p \times M$.*

Proof. Pick any reference point $p \in M$. It follows that the restriction $\mathsf{t}_p : \mathcal{G}_p \to M$ remains surjective. There is thus a map $\sigma : M \to \mathcal{G}_p$, such that $\mathsf{t}_p \circ \sigma = \mathbb{1}_M$.

Now, define $F : M \times \mathcal{G}_p^p \times M \to \mathcal{G}$ as

$$F(n, g, m) := \sigma(n) \cdot g \cdot \sigma(m)^{-1}. \tag{30}$$

This defines a bijection. Indeed, for any $g \in \mathcal{G}$, define

$$F^{-1}(g) := \big(\mathsf{t}(g), \sigma(\mathsf{t}(g))^{-1} \cdot g \cdot \sigma(\mathsf{s}(g)), \mathsf{s}(g)\big). \tag{31}$$

Note that the middle term is indeed in $\mathcal{G}_p^p$, since $\sigma(\mathsf{s}(g)) \in \mathcal{G}_p^{\mathsf{s}(g)}$ and $\sigma(\mathsf{t}(g))^{-1} \in \mathcal{G}_{\mathsf{t}(g)}^p$, so the multiplication is well-defined and the results in $\mathcal{G}_p^p$. The prove that inverse of F is easy. ■

Note that if σ is smooth, F is also smooth and defines a LG isomorphism of $\mathcal{G}$ and $M \times \mathcal{G}_p^p \times M$. However, in general case, such a global smooth σ may not exist. We cannot ensure this globally. However, there is a notion suitable for smooth setting.

Definition 2.9. One says that a Lie groupoid $\mathcal{G} \rightrightarrows M$ is **locally trivial**, if its anchor $\chi : \mathcal{G} \to M \times M$ is a smooth surjective submersion.

Example 2.10. Most of Lie groupoids we have already met are (at least in some cases) locally trivial.

(i) A trivial Lie groupoid is locally trivial.

(ii) An action Lie groupoid $G \ltimes M$ is locally trivial, iff $\theta : G \times M \to M$ is transitive. Indeed, one has $\chi(g, m) = (\theta(g, m), m)$. This map is surjective, iff θ is a transitive action. It is easy to see that this is always a submersion.

(iii) For any principal G-bundle $\pi : P \to M$, the corresponding gauge groupoid is locally trivial. This is because the anchor fits into the commutative diagram

$$\begin{array}{ccc} P \times P & & \\ \big\downarrow{\natural} & \searrow^{\pi\times\pi} & \\ \frac{P\times P}{G} & \xrightarrow{\chi} & M \times M \end{array} \quad . \tag{32}$$

Since $\pi \times \pi$ is a surjective submersion, so is χ. In particular, $\Pi(M)$ and $\Phi(E)$ are locally trivial.

Let us justify the name "locally trivial". Observe that for any open subset $U \subseteq M$, the subset $\mathcal{G}_U^U = \mathsf{s}^{-1}(U) \cap \mathsf{t}^{-1}(U)$ is open in $\mathcal{G}$. It follows that all structure maps of a Lie groupoid can be restricted and one obtain a **restricted Lie groupoid** $\mathcal{G}_U^U \rightrightarrows U$. First, let us make a following important observation.

Proposition 2.11. *Let $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid.*

Then for each $p \in M$, $\mathcal{G}_p^p$ is a closed embedded submanifold of $\mathcal{G}$ and with respect to the restriction of μ to $\mathcal{G}_p^p$, it forms a Lie group.

Proof. Observe that $\mathcal{G}_p^p = \chi^{-1}(p)$. Since χ is a surjective submersion, $\mathcal{G}_p^p$ is a closed embedded submanifold. Moreover, notice that $\mathcal{G}_p^p \times \mathcal{G}_p^p \subseteq \mathcal{G} * \mathcal{G}$ is also a closed embedded submanifold, so μ restricts to a smooth map $\mu_p : \mathcal{G}_p^p \times \mathcal{G}_p^p \to \mathcal{G}_p^p$. The fact that $(\mathcal{G}_p^p, \mu_p)$ forms a group is straightforward. ■

Proposition 2.12. *Let $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid.*

Then for every $p \in M$, there is an open neighborhood U of p, such that a restricted Lie groupoid $\mathcal{G}_U^U$ is isomorphic to the trivial Lie groupoid $U \times \mathcal{G}_p^p \times U$ over U.

Proof. Let $p \in M$ be an arbitrary but fixed point. Let us argue that $\mathsf{t}_p : \mathcal{G}_p \to M$ is also a surjective submersion. To do so, observe that $\sigma_p : M \to M \times M$ defined by $\sigma_p(n) = (n, p)$ is a closed embedding. Let $j_p : \mathcal{G}_p \to \mathcal{G}$ be the embedding. Then t_p fits into the pullback diagram

$$\begin{array}{ccc} \mathcal{G}_p & \xrightarrow{\mathsf{t}_p} & M \\ \big\downarrow{j_p} & & \big\downarrow{\sigma_p} \\ \mathcal{G} & \xrightarrow{\chi} & M \times M \end{array} \quad . \tag{33}$$

This means that $\mathcal{G}_p$ is precisely the inverse image submanifold of $\sigma_p(M)$ under χ. It is an easy exercise to check that χ being a surjective submersion implies that t_p is also.

Now, for any surjective submersion, there exists an open subset $U \subseteq M$ containing p and a smooth map $\sigma : U \to \mathcal{G}_p$, such that $\mathsf{t}_p \circ \sigma = \mathbb{1}_U$. One can now use the same tactic as in Lemma 2.8 to construct a Lie groupoid isomorphism $F : U \times \mathcal{G}_p^p \times U \to \mathcal{G}_U^U$. ■

Example 2.13. Let G be a Lie group with a Lie algebra $\mathfrak{g}$. We will now make T^*G into a Lie groupoid over $\mathfrak{g}^*$.

For any $\xi \in T_g^*G$, let us define $\mathsf{t}(\xi) := R_g^*(\xi)$ and $\mathsf{s}(\xi) := L_g^*(\xi)$.

Suppose $\xi \in T_g^*\mathcal{G}$ and $\eta \in T_h^*\mathcal{G}$ satisfy $\mathsf{s}(\xi) = \mathsf{t}(\eta)$, that is $L_g^*(\xi) = R_h^*(\eta)$. We propose that $\xi \bullet \eta \in T_{gh}^*\mathcal{G}$. We must ensure that

$$\mathsf{t}(\xi \bullet \eta) = \mathsf{t}(\xi), \;\; \mathsf{t}(\xi \bullet \eta) = \mathsf{s}(\eta). \tag{34}$$

This forces $R_k^*(\xi \bullet \eta) = R_g^*(\xi)$ and $L_k^*(\xi \bullet \eta) = L_h^*(\eta)$. This determines $\xi \bullet \eta$ uniquely as

$$\xi \bullet \eta = R_{h^{-1}}^*(\xi) = L_{g^{-1}}(\eta). \tag{35}$$

For each $\alpha \in \mathfrak{g}^*$, the unit element 1_α is α viewed as an element of $T_e^*\mathcal{G}$. Then

$$1_{\mathsf{t}(\eta)} \bullet \eta = L_{e^{-1}}(\eta) = \eta, \;\; \xi \bullet 1_{\mathsf{s}(\xi)} = R_{e^{-1}}^*(\xi) = \xi. \tag{36}$$

Finally, for any $\xi \in T_g^*\mathcal{G}$, one has $\xi^{-1} = I_{g^{-1}}^*(\xi)$. The check of all the required properties is left as an exercise.

Exercise 2.14. Work out the technical details.

Hint: observe that one has a diffeomorphism $F : T^*G \to G \times \mathfrak{g}^*$ taking $\xi \in T_g^*G$ to $F(\xi) = (g, L_{g^{-1}}^*(\xi))$. Show that F defines a Lie groupoid isomorphism with the action Lie groupoid $G \lessdot \mathfrak{g}^*$ with respect to the coadjoint action Ad_g^* of G on $\mathfrak{g}^*$.

3 Bisections

Recall that for a Lie group G, we have a class of diffeomorphisms forming a subgroup of $\mathrm{Diff}(G)$ isomorphic to G, namely left translations $\{L_g\}_{g \in G}$.

For a Lie groupoid $\mathcal{G} \rightrightarrows M$ and each $g \in \mathcal{G}$, the corresponding **left translation** $L_g(h) := g \cdot h$ is defined only for $h \in \mathcal{G}$ with $\mathsf{t}(h) = \mathsf{s}(g)$. Since $\mathsf{t}(g \cdot h) = \mathsf{t}(g)$, we see that L_g can be viewed as a smooth map

$$L_g : \mathcal{G}^{\mathsf{s}(g)} \to \mathcal{G}^{\mathsf{t}(g)}. \tag{37}$$

In fact, since $L_{g^{-1}}$ is its inverse, we see that it defines a diffeomorphism.

Now, since $\mathcal{G} = \bigsqcup_{m \in M} \mathcal{G}^m$, we may try to choose a collection $\{g_m\}_{m \in M}$ of elements of $\mathcal{G}$, each satisfying $\mathsf{s}(g_m) = m$, and define the resulting diffeomorphism by the whole family of left translation. Since $L_{g_m}(\mathcal{G}^m) = \mathcal{G}^{\mathsf{t}(g_m)}$ and we much reach every t-fiber of $\mathcal{G}$, we see that the map $m \mapsto \mathsf{t}(g_m)$ must be a bijection. Note that the resulting diffeomorphism preserves sources. This leads us to the following concept - we will consider diffeomorphisms of $\mathcal{G}$ which arise in this way:

Definition 3.1. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A **left translation of** $\mathcal{G}$ is a pair of diffeomorphisms $(\varphi, \varphi_\circ)$, where $\varphi : \mathcal{G} \to \mathcal{G}$ and $\varphi_\circ : M \to M$, such that

1. $\mathsf{t} \circ \varphi = \varphi_\circ \circ \mathsf{t}$, $\mathsf{s} \circ \varphi = \mathsf{s}$;
2. for each $m \in M$, the restriction $\varphi^m : \mathcal{G}^m \to \mathcal{G}^{\varphi_\circ(m)}$ is of the form L_{g_m} for some $g_m \in \mathcal{G}$.

Now, the assignment $m \mapsto g_m$ with $\mathsf{s}(g_m) = m$ can be viewed as a map $\sigma : M \to \mathcal{G}$ satisfying $\mathsf{s} \circ \sigma = \mathbb{1}_M$. As already noted, the composition $\mathsf{t} \circ \sigma : M \to M$ must be a bijection. This leads to the following definition.

Definition 3.2. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A **bisection of** $\mathcal{G}$ is a smooth map $\sigma : M \to \mathcal{G}$, such that $\mathsf{s} \circ \sigma = \mathbb{1}_M$ and $\mathsf{t} \circ \sigma : M \to M$ is a diffeomorphism. The set of bijections of $\mathcal{G}$ is denoted as $\mathscr{B}(\mathcal{G})$.

In fact, left translations and bisections are in one-to-one correspondence.

Proposition 3.3. *To each bisection σ of $\mathcal{G}$, there is an associated left translation $L_\sigma : \mathcal{G} \to G$. Every left translation of $\mathcal{G}$ is of this form for a unique bisection σ.*

Proof. Let $\sigma \in \mathscr{B}(G)$. We have to construct a pair of diffeomorphisms $\varphi : \mathcal{G} \to \mathcal{G}$ and $\varphi_\circ : M \to M$.

Obviously, let $\varphi_\circ := \mathsf{t} \circ \sigma$. This is a diffeomorphism by definition. For each $h \in \mathcal{G}$, define

$$\varphi(h) := \sigma(\mathsf{t}(h)) \cdot h. \tag{38}$$

This is well-defined and obviously smooth. The facts that $\mathsf{t} \circ \varphi = \varphi_\circ \circ \mathsf{t}$ and $\mathsf{s} \circ \varphi = \mathsf{s}$ are obvious. Its inverse is a smooth map

$$\varphi^{-1}(k) := \sigma(\varphi_\circ^{-1}(\mathsf{t}(k)))^{-1} \cdot k, \tag{39}$$

To check that this is well-defined, the target of $\sigma(\varphi_\circ^{-1}(\mathsf{t}(k)))$ must be $\mathsf{t}(k)$. But this is checked easily using the definition of $\varphi_\circ$. The proof that this is indeed an inverse to φ is straightforward and we leave it as an exercise. Finally, one has $\varphi^m = L_{\sigma(m)}$. Hence (φ, φ_0) is a left translation on $\mathcal{G}$, henceforth denoted as L_σ.

Conversely, let (φ, φ_0) be a left translation. Define $\sigma := \varphi \circ 1$. This is a smooth map. One has immediately obtains the required properties, since

$$\mathsf{s} \circ \sigma = \mathsf{s} \circ \varphi \circ 1 = \mathsf{s} \circ 1 = \mathbb{1}_M, \tag{40}$$

$$\mathsf{t} \circ \sigma = \mathsf{t} \circ \varphi \circ 1 = \varphi_0 \circ \mathsf{t} \circ 1 = \varphi_0 \circ \mathbb{1}_M = \varphi_0, \tag{41}$$

and one uses the fact that φ_0 is a diffeomorphism. Hence $\sigma \in \mathscr{B}(G)$. To see that $\varphi = L_\sigma$, note that necessarily $\varphi^m = L_{\sigma(m)}$ for each $m \in M$.

Indeed, we know that $\varphi^m = L_{g_m}$ for some $g_m \in \mathcal{G}$. But since $1_m \in \mathcal{G}^m$, one can write

$$g_m = L_{g_m}(1_m) = \varphi^m(1_m) \equiv (\varphi \circ 1)(m) = \sigma(m). \tag{42}$$

For an arbitrary $h \in \mathcal{G}$, one has $h \in \mathcal{G}^{\mathsf{t}(h)}$, so one can now write

$$\varphi(h) = \varphi^{\mathsf{t}(h)}(h) = L_{\sigma(\mathsf{t}(h))}(h) = \sigma(\mathsf{t}(h)) \cdot h \equiv L_\sigma(h). \tag{43}$$

Also note that σ was uniquely determined by φ. This finishes the proof. ■

Proposition 3.4. *Left translations form a subgroup of* $\mathrm{Diff}(\mathcal{G})$. *Consequently, there is a unique group structure induced on $\mathscr{B}(\mathcal{G})$.*

Proof. The first claim is easy to check.

Let $\sigma, \sigma' \in \mathscr{B}(\mathcal{G})$. Their product on $\mathscr{B}(\mathcal{G})$ is defined by requirement

$$L_{\sigma \star \sigma'} = L_\sigma \circ L_{\sigma'}. \tag{44}$$

By the above proof, one can write $\sigma \star \sigma' = (L_\sigma \circ L_{\sigma'}) \circ 1$. By plugging into the formulas for L_σ and $L_{\sigma'}$, one immediately obtains the expression

$$(\sigma \star \sigma')(m) = L_\sigma(L_{\sigma'}(1_m)) = L_\sigma(\sigma'(m)) = \sigma(\mathsf{t}(\sigma'(m))) \cdot \sigma'(m). \tag{45}$$

The group unit is obviously precisely the object inclusion map $\mathbf{1} : M \to \mathcal{G}$ and the group inverse is a bisection σ^{-1} defined by

$$\sigma^{-1}(m) := \sigma((\mathsf{t} \circ \sigma)^{-1}(m))^{-1}. \tag{46}$$

We can thus write $(L_\sigma)^{-1} = \sigma^{-1}$. This finishes the proof. ■

Example 3.5. Let us examine $\mathcal{B}(\mathcal{G})$ for the trivial Lie groupoid $\mathcal{G} = M \times G \times M$.

It is easy to see that the most general $\sigma \in \mathscr{B}(G)$ takes the form $\sigma(m) = (\varphi_\circ(m), \mathbf{g}(m), m)$, where $\varphi_\circ : M \to M$ is a diffeomorphism and $\mathbf{g} : M \to G$ is a smooth map. If $\sigma'(m) = (\varphi'_0(m), \mathbf{g}'(m), m)$, their product has the form

$$\begin{aligned}(\sigma \star \sigma')(m) &= \sigma(\mathsf{t}(\sigma'(m))) \cdot \sigma'(m) = \sigma(\varphi'_\circ(m)) \cdot \sigma'(m) \\ &= (\varphi_0(\varphi'_0(m)), \mathbf{g}(\varphi'_0(m)), \varphi'_0(m)) \cdot (\varphi'_0(m), \mathbf{g}'(m), m) \\ &= ((\varphi \circ \varphi')(m), \mathbf{g}(\varphi'_0(m)) \cdot \mathbf{g}(m), m).\end{aligned} \tag{47}$$

This shows that $\mathscr{B}(\mathcal{G}) \cong \mathrm{Diff}(M) \times C^\infty(M, G)$ with the multiplication given by $(\varphi, \mathbf{g}) \star (\varphi', \mathbf{g}') = (\varphi \circ \varphi', (\mathbf{g} \circ \varphi'_\circ) \cdot \mathbf{g})$. The group unit is $(\mathbb{1}_M, \mathbf{e})$, where $\mathbf{e}(m) := e$ for all $m \in M$.

Example 3.6. Let $\Phi(E)$ be the frame groupoid. Let us examine the group $\mathscr{B}(\Phi(E))$. We claim that it corresponds to the group of all vector bundle automorphisms of E.

First, suppose $\sigma : M \to \Phi(E)$ be a bisection. We shall define a corresponding vector bundle map $\overline{\sigma} : E \to E$ as follows. Recall that $q : E \to M$. For each $e \in E$, let

$$\overline{\sigma}(e) := [\sigma(q(e))](e) \tag{48}$$

Let $\varphi_\circ : M \to M$ be the diffeomorphism defined by σ, that is $\varphi_\circ = \mathsf{t} \circ \sigma$. Then

$$q(\overline{\sigma}(e)) = q([\sigma(q(e))](e)) = \mathsf{t}([\sigma(q(e))]) = \varphi_0(q(e)), \tag{49}$$

that is $\overline{\sigma}$ satisfies $q \circ \overline{\sigma} = \varphi_0 \circ q$. For each $m \in M$, its restriction to the fiber E_m is a linear isomorphism $\sigma(m) \in \mathrm{Iso}(E_m, E_{\varphi_\circ(m)})$. One only has to argue that it is smooth. We leave that as an exercise.

Conversely, $F : E \to E$ is a smooth vector bundle automorphism over $\varphi_\circ : M \to M$. For each $m \in M$, we will produce an element $\sigma(m) \in \Phi(E)$ satisfying $\mathsf{s}(\sigma(m)) = m$ and $\mathsf{t}(\sigma(m)) = \varphi_\circ(m)$. Obviously, set $\sigma(m) := F_m \in \mathrm{Iso}(E_m, E_{\varphi_0(m)})$. One only has to show that σ so defined is smooth. Again, this is given as an exercise.

It is easy to see that these assignments are inverse to each other and that the group product $\star$ on $\mathscr{B}(\Phi(E))$ corresponds to the composition of vector bundle automorphisms.

Exercise 3.7. Prove the smoothness claims of the previous example. Hint: use suitable local trivialization charts for E.

Example 3.8. Let $\mathcal{G} = G \lessdot M$ be the action algebroid corresponding to $\theta : G \times M \to M$. Every bisection is thus of the form $\sigma(m) = (\mathbf{g}(m), m)$, where $\mathbf{g} : M \to G$ is smooth.

By definition, $\mathsf{t} \circ \sigma : M \to M$ must be a difeomorphism. But this means that the map $m \mapsto \mathbf{g}(m) \cdot m$ must be a diffeomorphism of M. We see that

$$\mathscr{B}(\mathcal{G} \lessdot M) \cong \{\mathbf{g} \in \mathcal{C}^\infty(M, G) \mid m \mapsto \mathbf{g}(m) \cdot m \text{ is a diffeomorphism}\} \tag{50}$$

It is straightforward to see that the product on $\mathscr{B}(\mathcal{G} \lessdot M)$ can be identified with

$$(\mathbf{g} \star \mathbf{g}')(m) = \mathbf{g}(\mathbf{g}'(m) \cdot m) \cdot \mathbf{g}'(m). \tag{51}$$

Observe that for every $g \in G$, the constant map $\mathbf{g}(m) \equiv g$ has the required property, since $m \mapsto g \cdot m$ is indeed a diffeomorphism. In other words, G can be viewed as a subgroup of $\mathscr{B}(\mathcal{G} \lessdot M)$.

Example 3.9. Let $\pi : P \to M$ be a principal G-bundle. We claim (without proof) that the following diagram is actually a pullback:

$$\begin{array}{ccc} P \times P & \xrightarrow{\natural} & \frac{P\times P}{G} \\ \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle \mathsf{s}} \\ P & \xrightarrow{\pi} & M \end{array} \tag{52}$$

Now, suppose $\sigma : M \to \frac{P\times P}{G}$ is a bisection. In particular, one has $\mathsf{s} \circ \sigma = \mathbb{1}_M$ and $\varphi_\circ := \mathsf{t} \circ \sigma$ is a diffeomorphism.

Now the pair of maps $\sigma \circ \pi : P \to \frac{P\times P}{G}$ and $\mathbb{1}_P : P \to P$ fits into the pullback diagram

$$\begin{tikzcd} P \arrow[drr, bend left, "\sigma\circ\pi"] \arrow[dr, dashed, "\overline{\sigma}"] \arrow[ddr, bend right, "\mathbb{1}_P"'] & & \\ & P \times P \arrow[r, "\natural"] \arrow[d, "\pi_2"] & \frac{P\times P}{G} \arrow[d, "\mathsf{s}"] \\ & P \arrow[r, "\pi"] & M \end{tikzcd} . \tag{53}$$

There is thus a unique map $\overline{\sigma} : P \to P \times P$, such that $\pi_2 \circ \overline{\sigma} = \mathbb{1}_P$ and $\natural \circ \sigma = \sigma \circ \pi$. The first condition foces $\overline{\sigma}(p) = (\varphi(p), p)$ for some smooth map $\varphi : P \to P$. The second condition shows that necessarily

$$[\varphi(p), p] = \sigma(\pi(p)). \tag{54}$$

For the above equation to hold, one must have $[\varphi(p \cdot g), p \cdot g] = [\varphi(p), p]$ for every $p \in P$ and $g \in G$. But this is equivalent to $\varphi(p \cdot g) = \varphi(p) \cdot g$, that is $\varphi : P \to P$ is a G-equivariant map.

Finally, observe that

$$\varphi_\circ(\pi(p)) = \mathsf{t}(\sigma(\pi(p)) = \mathsf{t}([\varphi(p), p]) = \pi(\varphi(p)), \tag{55}$$

that is φ fits into the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\varphi_\circ} & M \end{array} , \tag{56}$$

that is $(\varphi, \varphi_\circ)$ is an endomorphism of the principal bundle $\pi : P \to G$. In fact, since $\varphi_\circ$ is a diffeomorphism, φ is automatically a diffeomorphism (prove that!) and thus $(\varphi, \varphi_\circ)$ is a **principal bundle automorphism** of $\pi : P \to M$.

Conversely, for any principal bundle automorphism $(\varphi, \varphi_\circ)$, we can use the formula (54) to define a map $\sigma : M \to \frac{P\times P}{G}$ which is well defined since φ is G-equivariant and smooth since π is a surjective submersion. It is easy to check that σ is a bisection, such that $\mathsf{t} \circ \sigma = \varphi_\circ$. It follows that $\mathscr{B}(\frac{P\times P}{G})$ is canonically isomorphic to the group of principal G-bundle automorphisms of $\pi : P \to M$.

We have the following related definitions.

Definition 3.10. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $\sigma \in \mathscr{B}(\mathcal{G})$.

(i) A **right translation by** σ is defined by $R_\sigma(g) := g \cdot \sigma((\mathsf{t} \circ \sigma)^{-1}(\mathsf{s}(g)))$, for all $g \in \mathcal{G}$.

(ii) A **conjugation by** σ is defined by $I_\sigma(g) := \sigma(\mathsf{t}(g)) \cdot g \cdot \sigma(\mathsf{s}(g))^{-1}$, for all $g \in \mathcal{G}$.

Both maps are diffeomorphism of $\mathcal{G}$.

There is another equivalent, more intrinsic definition of a bisection, symmetric with respect to the target and source maps.

Proposition 3.11. *Let $\sigma \in \mathscr{B}(\mathcal{G})$. Then its image $S := \sigma(M) \subseteq \mathcal{G}$ is a closed embedded submanifold, such that both $\mathsf{s}|_S : S \to M$ and $\mathsf{t}|_S : S \to M$ are diffeomorphisms.*

Conversely, any such submanifold $S \subseteq \mathcal{G}$ is obtained in this way from a unique bisection.

Proof. Since $\mathsf{s} \circ \sigma = \mathbb{1}_M$, it is easy to see that σ is an injective immersion. Similarly to Proposition 1.7, one proves that it is a closed map, hence a closed embedding. Hence $S := \sigma(M) \subseteq \mathcal{G}$ is a closed embedded submanifold and $\sigma : M \to S$ defines a diffeomorphism. Then $\mathsf{s}|_S : M \to S$ is its inverse and $\mathsf{t}|_S : S \to M$ is a diffeomorphism since $\mathsf{t} \circ \sigma : M \to M$ is a diffeomorphism.

Conversely, suppose $S \subseteq \mathcal{G}$ be a closed embedded submanifold having those properties. Let $\iota : S \to \mathcal{G}$ be the inclusion. Since $\mathsf{s}|_S : S \to M$ is a diffeomorphism, we can define $\sigma := \iota \circ (\mathsf{s}|_S)^{-1} : M \to \mathcal{G}$. But then $\mathsf{s} \circ \sigma = (\mathsf{s} \circ \iota) \circ (\mathsf{s}|_S)^{-1} = \mathsf{s}|_S \circ (\mathsf{s}|_S)^{-1} = \mathbb{1}_M$ and $\mathsf{t} \circ \sigma = (\mathsf{t} \circ \iota) \circ (\mathsf{s}|_S)^{-1} = \mathsf{t}|_S \circ (\mathsf{s}|_S)^{-1} : M \to M$ is a diffeomorphism. ■

There is no guarantee that the group $\mathscr{B}(\mathcal{G})$ contains anything else then its unit $\mathbf{1} : M \to \mathcal{G}$. Similarly to the space of sections of a fiber bundle, it is often convenient to consider local bisections only.

Definition 3.12. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Let $U \subseteq M$ be an open subset.

A **local bisection over** U is a smooth map $\sigma : U \to M$, such that $\mathsf{s} \circ \sigma = \mathbb{1}_U$ and $\varphi_\circ := \mathsf{t} \circ \sigma : U \to \varphi_\circ(U)$ is a diffeomorphism of U and an open subset $\varphi_\circ(U) \subseteq M$. A set of local bisections of $\mathcal{G}$ over U is denoted as $\mathscr{B}_U(\mathcal{G})$.

The **local left translation** induced by σ is a diffeomorphism $L_\sigma : \mathcal{G}^U \to \mathcal{G}^{\varphi_\circ(U)}$ defined by the same formula as for $U = M$.

It turns out that for every point of a Lie groupoid, there is always a local bisection

Proposition 3.13. *For each $g \in \mathcal{G}$, there exists $U \subseteq M$ and a local section $\sigma \in \mathscr{B}_U(\mathcal{G})$, such that $g = \sigma(\mathsf{s}(g))$.*

Proof. Let $g \in \mathcal{G}$ be a given point. Let $m := \mathsf{s}(g)$ and $n := \mathsf{t}(g)$. One can always find a linear subspace $I \subseteq T_g\mathcal{G}$, such that

$$\begin{aligned} T_g\mathcal{G} &= T_g(\mathcal{G}_m) \oplus I \equiv \ker(T_g\mathsf{s}) \oplus I, \\ T_g\mathcal{G} &= T_g(\mathcal{G}^n) \oplus I \equiv \ker(T_g\mathsf{t}) \oplus I. \end{aligned} \tag{57}$$

Since $\mathsf{s} : \mathcal{G} \to M$ is a surjective submersion, there is an open neighborhood U of m and $\sigma : U \to \mathcal{G}$, such that $\mathsf{s} \circ \sigma = \mathbb{1}_U$, and $\sigma(m) = g$ and $(T_m\sigma)(T_mM) = I$. But then $T_m(\mathsf{t} \circ \sigma) : T_mM \to T_nM$ is a linear isomorphism. This means that it is a local diffeomorphism and the claim follows. ■

Exercise 3.14. Prove the technical details. Hint: use local coordinates adapted to a submersion.

Corollary 3.15. *For each $m \in M$, the restriction $\mathsf{t}_m : \mathcal{G}_m \to M$ has a constant rank.*

Proof. Suppose $g, h \in \mathcal{G}_m$. We must argue that $T_g(\mathsf{t}_m)$ and $T_h(\mathsf{t}_m)$ have the same rank. Let $k := g \cdot h^{-1}$. By the above proposition, there exists $\sigma \in \mathscr{B}_U(\mathcal{G})$ with $\mathsf{t}(h) \in U$ and $\sigma(\mathsf{t}(h)) = k$. The corresponding left translation is a diffeomorphism $L_\sigma : \mathcal{G}^U \to \mathcal{G}^V$, where $V = (\mathsf{t} \circ \sigma)(U)$. By construction, one has $L_\sigma(h) = \sigma(\mathsf{t}(h)) \cdot h = k \cdot h = g$.

Since $\mathsf{s} \circ L_\sigma = \mathsf{s}$, it restricts to a diffeomorphism $L_\sigma : \mathcal{G}^U_m \to \mathcal{G}^V_m$, and $\mathsf{t} \circ L_\sigma = (\mathsf{t} \circ \sigma) \circ \mathsf{t}$ gives

$$\mathsf{t}_m \circ L_\sigma = (\mathsf{t} \circ \sigma) \circ t_m. \tag{58}$$

Evaluating the tangent maps thus gives $T_g(\mathsf{t}_m) \circ T_h(L_\sigma) = T_{\mathsf{t}(h)}(\mathsf{t} \circ \sigma) \circ T_h(\mathsf{t}_m)$. Since both $T_h(L_\sigma)$ and $T_{\mathsf{t}(h)}(\mathsf{t} \circ \sigma)$ are linear isomorphisms, this proves the claim. ■

Corollary 3.16. *For each $m, n \in M$, $\mathcal{G}^n_m \subseteq \mathcal{G}$ is a closed embedded submanifold. In particular, $\mathcal{G}^m_m \subseteq \mathcal{G}$ is a Lie group for any $m \in M$.*

Proof. For a given $m \in M$, $\mathsf{t}_m : \mathcal{G}_m \to M$ has a constant rank. Consequently, the level set $\mathcal{G}^n_m := \mathsf{t}_m^{-1}(n)$ is a closed embedded submanifold of $\mathcal{G}_m$, hence of $\mathcal{G}$, see Theorem 5.12 in [1]. If $\mathcal{G}^m_m$ is a closed embedded submanifold of $\mathcal{G}$, it follows that $\mathcal{G}^m_m \times \mathcal{G}^m_m$ is a closed embedded submanifold of $\mathcal{G} * \mathcal{G}$ and the restriction of μ gives the smooth multiplication on $\mathcal{G}^m_m$. ■

4 Actions

Lie groups often arise through their actions on manifolds. Similarly, Lie groupoids act on the fibered spaces over the base M. Imagine $f : N \to M$ is a fiber bundle. For each $m \in M$, we can "act" on the elements of the fiber $N_m := f^{-1}(m)$ by arrows $g \in \mathcal{G}$, which "start at" m, that is $\mathsf{s}(g) = m$. We obtain an element $g \triangleright n$ at the fiber over *the end of* g, that is $f(g \triangleright n) = \mathsf{t}(g)$. This leads to the following definition:

Definition 4.1. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $f : N \to M$ be a smooth map. Since $\mathsf{s} : \mathcal{G} \to M$ is a surjective submersion, the fibered product

$$\mathcal{G} * N := \{(g, n) \in \mathcal{G} \times N \mid \mathsf{s}(g) = f(n)\} \tag{59}$$

is a closed embedded submanifold. An **action of $\mathcal{G}$ on** $f : N \to M$ is a smooth map $\theta : \mathcal{G} * N \to N$ satisfying the following axiom. We write $g \triangleright n := \theta(g, n)$.

(i) $f(g \triangleright n) = \mathsf{t}(g)$.

(ii) $g \triangleright (h \triangleright n) = (g \cdot h) \triangleright n$ for all $(g, h) \in \mathcal{G} * \mathcal{G}$ and $n \in N$, such that $(h, n) \in \mathcal{G} * N$.

(iii) $\mathtt{1}_{f(n)} \triangleright n = n$ for all $n \in N$. Note that $(\mathtt{1}_{f(n)}, n) \in \mathcal{G} * N$ since $\mathsf{s}(\mathtt{1}_{f(n)}) = f(n)$.

For a given $n \in N$, we have $(g, n) \in \mathcal{G} * N$ whenever $\mathsf{s}(g) = f(n)$, that is $g \in \mathcal{G}_{f(n)}$. The **orbit of the point** $n \in N$ is then defined as a subset

$$\mathcal{G}[n] := \{g \triangleright n \mid g \in \mathcal{G}_{f(n)}\} \subseteq N. \tag{60}$$

Exercise 4.2. Prove the claim about $\mathcal{G} * N$. Hint: learn about the transversality.

Proposition 4.3. *Suppose $f : N \to M$ in the above definition is a surjective submersion.*

Then for every $m, m' \in M$, for every $g \in \mathcal{G}_m^{m'}$, the map $\theta_g : N_m \to N_{m'}$ is a diffeomorphism.

Proof. One needs the assumption to ensure that the fibers N_m are closed embedded submanifolds of N. The rest is clear. ∎

Definition 4.4. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $\theta_1 : \mathcal{G} * N_1 \to N_1$ and $\theta_2 : \mathcal{G} * N_2 \to N_2$ be actions of $\mathcal{G}$ over $f_1 : N \to M$ and $f_2 : N \to M$, respectively. A smooth map $\varphi : N_1 \to N_2$ is **$\mathcal{G}$-equivariant with respect to θ_1 and θ_2**, if $f_2 \circ \varphi = f_1$ and

$$\varphi(g \triangleright_1 n) = g \triangleright_2 \varphi(n), \tag{61}$$

for all $(g, n) \in \mathcal{G} * N_1$.

Example 4.5. Let $\pi_1 : M \times K \to M$ be the projection with an arbitrary manifold K. Observe that the fiber product of $\mathcal{G}$ and this space gives

$$\mathcal{G} * (M \times K) = \{(g, (\mathsf{s}(g), k)) \ (g, k) \in \mathcal{G} \times K\} \cong \mathcal{G} \times K. \tag{62}$$

The **trivial action** of $\mathcal{G}$ over π_1 is now defined as $\theta(g, (\mathsf{s}(g), k)) := (\mathsf{t}(g), k)$, for all $(g, k) \in \mathcal{G} \times F$. For any $g \in \mathcal{G}$, the induced map $\theta_g : F \to F$ is just the identity.

For each $(m, k) \in M \times K$, the corresponding orbit is

$$\mathcal{G}[(m, k)] = \{g \triangleright (m, k) \mid g \in \mathcal{G}_m = \{(\mathsf{t}(g), k) \mid g \in \mathcal{G}_m\} = \mathsf{t}(\mathcal{G}_m) \times \{k\}. \tag{63}$$

Before the formulation of the next example, we will need the following structure:

Proposition 4.6. *Let $\pi : P \to M$ be a principal G-bundle.*

Let $\theta : G \times N \to N$ be a Lie group action of G on a manifold N. Define a right action of G on $P \times N$ as $(p, n) \cdot g := (p \cdot g, g^{-1} \cdot n)$ for all $(p, n) \in P \times N$ and $g \in G$.

Then there is a unique fiber bundle structure $\varpi : \frac{P \times N}{G} \to N$ making the quotient map $\natural : P \times N \to \frac{P \times N}{G}$ into a surjective submersion. Moreover, the diagram

$$\begin{array}{ccc} P \times N & \xrightarrow{\ \natural\ } & \frac{P \times N}{G} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \varpi} \\ P & \xrightarrow{\ \pi\ } & M \end{array} \tag{64}$$

is a pullback in the category of smooth manifolds. $P \times_\theta N := \frac{P \times N}{G}$ is called the ***associated fibre bundle to P and the action θ.***

Proof. Let $\langle p, n \rangle \in P \times_\theta N$ denote the equivalence class of (p, n). To make (64) commutative, we must set

$$\varpi(\langle p, n \rangle) := \pi(p). \tag{65}$$

The topology and a smooth structure on the total space $P \times_\theta N$ is induced by the local trivialization. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a G-equivariant local trivialization for P. We have to produce bijections $\mu_\alpha : U_\alpha \times N \to \varpi^{-1}(U_\alpha)$. Let

$$\mu_\alpha(m, n) := \langle \phi_\alpha(m, e), n \rangle. \tag{66}$$

It is easy to see that $\varphi \circ \mu_\alpha = \pi_1$. Conversely, recall that for each $\alpha \in I$, there is a G-equivariant smooth map $\mathbf{g}_\alpha : \pi^{-1}(U_\alpha) \to G$ defined by $\mathbf{g}_\alpha := \pi_2 \circ \phi_\alpha^{-1}$. For each $\langle p, n \rangle \in \varpi^{-1}(U_\alpha)$, let

$$\mu_\alpha^{-1}\langle p, n \rangle := (\pi(p), \mathbf{g}_\alpha(p) \cdot n). \tag{67}$$

This well-defined, since $\mathbf{g}_\alpha(p \cdot g) = \mathbf{g}_\alpha(p) \cdot g$. It is easy to check that μ_α and μ_α^{-1} are inverse to each other. One only has to verify that the transition maps are smooth. Recall that the transition maps of P are given by smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$, such that $(\pi_2 \circ \phi_\alpha^{-1} \circ \phi_\beta)(m, g) = g_{\alpha\beta}(m) \cdot g$. It is then a straightforward calculation that

$$[h_{\alpha\beta}(m)](n) := (\pi_2 \circ \mu_\alpha^{-1} \circ \mu_\beta)(m, n) = g_{\alpha\beta}(m) \cdot n. \tag{68}$$

One thus has $h_{\alpha\beta}(m) = \theta_{g_{\alpha\beta}(m)} \in \mathrm{Diff}(N)$ and for each $n \in N$, the map $m \mapsto [h_{\alpha\beta}(m)](n)$ is smooth from $U_{\alpha\beta}$ to N. This gives us a topology and a smooth structure on $P \times_\theta N$ making $\varpi : P \times_\theta N \to M$ into a fiber bundle with a typical fiber N and a local trivialization $\{(U_\alpha, \mu_\alpha)\}_{\alpha \in I}$.

Let us argue that $\natural : P \times N \to P \times_\theta N$ becomes a smooth surjective submersion. This also fixes the topology and a smooth structure uniquely. Since $\natural(\pi^{-1}(U_\alpha) \subseteq \varpi^{-1}(U_\alpha)$, we can form the composition

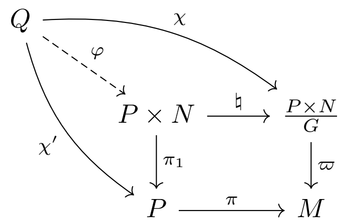
$$\mu_\alpha^{-1} \circ \natural \circ (\phi_\alpha \times \mathbb{1}_N) : (U_\alpha \times G) \times N \to U_\alpha \times N. \tag{69}$$

It suffices to prove that this is a smooth surjective submersion for each $\alpha \in I$. But one finds

$$[\mu_\alpha^{-1} \circ \natural \circ (\phi_\alpha \times \mathbb{1}_N)]((m, g), n) = (m, g \cdot n). \tag{70}$$

The claim follows from the fact that $\theta : G \times M \to M$ is a smooth surjective submersion.

It remains to prove that the diagram is in fact a pullback. Let Q be any manifold together with a pair of maps $\chi : Q \to P \times_\theta N$ and $\chi' : Q \to P$, satisfying $\varpi \circ \chi = \pi \circ \chi'$. We must construct a unique map $\varphi : Q \to P \times N$ fitting into the commutative diagram



(71)

The commutativity of the bottom-left triangle forces $\varphi(q) = (\chi'(q), \varphi'(q))$ for some smooth map $\varphi' : Q \to N$. The other triangle forces $\langle \chi'(q), \varphi'(q) \rangle = \chi(q)$ for all $q \in Q$. But an element representing a given class $\chi(q)$ is uniquely determined once we fix the element $\chi'(q)$. This shows that for each $q \in Q$, there is a unique element $\varphi'(q) \in N$ with this property, whence φ' exists and its uniquely determined.

We only have to prove the smoothness. Let us consider the open subset $V_\alpha := \chi^{-1}(\varpi^{-1}(U_\alpha))$. For each $q \in V_\alpha$, one can write $\chi(q) = \mu_\alpha(\pi(\chi'(q)), \hat{\chi}_\alpha(q))$, where $\hat{\chi}_\alpha : V_\alpha \to N$ is smooth. Similarly, one has $\chi'(V_\alpha) \subseteq \pi^{-1}(U_\alpha)$, so $\chi'(q) = \phi_\alpha(\pi(\chi'(q)), \hat{\mathbf{g}}_\alpha(q))$ for a smooth $\hat{\mathbf{g}}_\alpha : V_\alpha \to G$.

We claim that necessarily

$$\varphi(q) = (\phi_\alpha(\pi(\chi'(q)), \hat{\mathbf{g}}_\alpha(q)), \hat{\mathbf{g}}_\alpha^{-1}(q) \cdot \hat{\chi}_\alpha(q)), \tag{72}$$

for each $q \in V_\alpha$. By applying $\natural$ on both sides, one obtains

$$\natural(\varphi(q)) = \mu_\alpha\big(\pi(q), \hat{\mathbf{g}}_\alpha(q) \cdot (\hat{\mathbf{g}}_\alpha^{-1}(q) \cdot \hat{\chi}_\alpha(q))\big) = \mu_\alpha(\pi(q), \hat{\chi}_\alpha(q)) = \chi(q), \tag{73}$$

But the formula (72) shows that the restriction of φ to V_α is smooth. In fact, one has $\varphi'(q) = \hat{\mathbf{g}}_\alpha^{-1}(q) \cdot \hat{\chi}_\alpha(q)$ for all $q \in V_\alpha$. Since $\{V_\alpha\}_{\alpha \in I}$ covers Q, this proves the claim. ■

Exercise 4.7. (i) Show that sections of $P \times_\theta N$ are in one-to-one correspondence with G-equivariant smooth maps $\varphi : P \to N$.

(ii) Use the new knowledge to argue that (52) is a pullback.

Example 4.8. Let $\pi : P \to M$ be a principal bundle and let $\varphi : P \times_\theta N \to M$ be the fiber bundle associated to the left action θ. Write $\mathcal{G} := \frac{P \times P}{G}$ and $E := P \times_\theta N$.

Elements of $\mathcal{G} * E$ are pairs $([p,q], \langle r, n\rangle) \in \mathcal{G} \times E$ with $\mathsf{s}([p,q]) = \varphi(\langle r, n\rangle)$. But this means that $\pi(q) = \pi(r)$. One can thus arrange those so that $r = q$ and define

$$[p,q] \triangleright \langle q, n\rangle := \langle p, n\rangle. \tag{74}$$

To prove that this action is smooth, let $\natural : P \times P \to \mathcal{G}$ and $\natural' : P \times N \to E$ denote the quotient maps. Then observe that the corresponding map $\Theta : \mathcal{G} * E \to E$ fits into

$$\begin{array}{ccc} P \times (P \times_\rho P) \times N & \xrightarrow{\hat{\Theta}} & P \times N \\ \Big\downarrow{\scriptstyle \natural \times \natural'} & & \Big\downarrow{\scriptstyle \natural'} \\ \mathcal{G} * E & \xrightarrow{\Theta} & E \end{array}, \tag{75}$$

where $\hat{\Theta}(p,q,r,n) := (p, \delta(q,r)^{-1} \cdot n)$ is smooth. Checking the action axioms is easy, since

(i) $\varpi([p,q] \triangleright \langle q,n\rangle) = \pi(p) = \mathsf{t}([p,q])$,

(ii) $[p,q] \triangleright ([q,r] \triangleright \langle r,n\rangle) = [p,q] \triangleright \langle q,n\rangle = \langle p,n\rangle = [p,r] \triangleright \langle r,n\rangle = ([p,q] \cdot [q,r]) \triangleright \langle r,n\rangle$.

(iii) $1_{\varpi(\langle p,n\rangle)} \triangleright \langle p,n\rangle = 1_{\pi(p)} \triangleright \langle p,n\rangle = [p,p] \triangleright \langle p,n\rangle = \langle p,n\rangle$.

Now, to any Lie *groupoid* action, there is also an associated Lie groupoid.

Proposition 4.9. *Let $\theta : \mathcal{G} * N \to N$ be a Lie groupoid action of $\mathcal{G} \rightrightarrows M$ over $f : N \to M$.*

Then there is a canonical Lie groupoid with the base N on $\mathcal{G} * N$, defined as follows:

(i) The source map is $\mathsf{s}'(g,n) = n$; The target map is $\mathsf{t}'(g,n) := g \triangleright n$.

(ii) The multiplication is $(g, h \triangleright n) \cdot (h,n) := (g,n)$.

(iii) The object inclusion map is $\mathbf{1}'_n := (1_{f(n)}, n)$.

The resulting Lie groupoid is denoted as $\mathcal{G} \lessdot N$ and called the ***action Lie groupoid associated to the action θ of $\mathcal{G}$ on $f : N \to M$.***

Proof. Recall that fiber product $\mathcal{G} * N$ is a pullback, fitting into the commutative square

$$\begin{array}{ccc} \mathcal{G} * N & \xrightarrow{\pi_2'} & N \\ \Big\downarrow{\scriptstyle \pi_1'} & & \Big\downarrow{\scriptstyle f} \\ \mathcal{G} & \xrightarrow{\mathsf{s}} & M \end{array}, \tag{76}$$

where $\pi'_{1,2}$ are just the restrictions of the respective projections. Now, note that in fact $\mathsf{s}' \equiv \pi'_2$. It is a general fact that whenever s is a surjective submersion, then so is π'_2.

Next, note that $\mathsf{t}' = \theta$. To prove that θ is a surjective submersion, let us consider the groupoid inverse $\mathsf{i} : \mathcal{G} \to G$ and observe that we have a smooth map $(\mathsf{i} \circ \pi'_1, \theta) : \mathcal{G} * N \to \mathcal{G} * N$. Indeed, it maps each $(g, n) \in \mathcal{G} * N$ to $(g^{-1}, \theta(g, n))$, which satisfies $f(\theta(g, n)) = \mathsf{t}(g) = \mathsf{s}(g^{-1})$, so it is indeed in $\mathcal{G} * N$. This map is in fact a diffeomorphism, since it is its own inverse.

The action axioms imply the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{G} * N & \xrightarrow{(\mathsf{i}\circ\pi'_1,\theta)} & \mathcal{G} * N \\ & \searrow_{\pi'_2} & \downarrow \theta \\ & & N \end{array} \tag{77}$$

Since π'_2 is a surjective submersion and the horizontal map is a diffeomorphism, θ must be a surjective submersion. Finally, it follows that the multiplication $\mu' : (\mathcal{G} * N) * (\mathcal{G} * N) \to (\mathcal{G} * N)$ fits into the diagram

$$\begin{array}{ccc} (\mathcal{G} * N) * (\mathcal{G} * N) & \xrightarrow{\mu'} & \mathcal{G} * N \\ \downarrow & & \downarrow \\ (\mathcal{G} * N) \times (\mathcal{G} * N) & \xrightarrow{\pi'_1 \times \pi'_2} & \mathcal{G} \times N \end{array}, \tag{78}$$

where the vertical maps are embeddings of the respective fiber products. Finally, the object inclusion map $\mathbf{1}' : N \to \mathcal{G} * N$ fits into the commutative diagram

$$\begin{array}{ccc} & & \mathcal{G} * N \\ & \nearrow^{\mathbf{1}'} & \downarrow \\ N & \xrightarrow[(\mathbf{1}\circ\pi,\mathbb{1}_N)]{} & \mathcal{G} \times N \end{array} \tag{79}$$

That the described operations make $\mathcal{G} \lessdot N$ into a Lie groupoid is now easy to check. ■

Example 4.10. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Its multiplication $\mu : \mathcal{G} * \mathcal{G} \to \mathcal{G}$ can be viewed as a left action of $\mathcal{G}$ on $\mathsf{t} : \mathcal{G} \to M$.

The corresponding action Lie groupoid $\mathcal{G} \lessdot \mathcal{G}$ is given by $\mathsf{s}'(g, h) = h$, $\mathsf{s}'(g, h) = g \cdot h$ and $(g, h \cdot \ell) \cdot (h, \ell) = (g, \ell)$.

References

[1] J. Lee, *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics. Springer, 2012.