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Doctoral Thesis

Strongly Singular Operators with Interactions Supported by Curves and Surfaces

Silně singulární operátory s interakcí nesenou křivkami a plochami

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Abstrakt

V této dizertaci zkoumáme spektrální vlastnosti Schrödingerových operátorů popisující δ'-interakci lokalizovanou na křivkách a plochách. Tyto operátory jsou sdruženy s následující kvadratickou formou

$$h_\beta^\Gamma(\psi) = (\nabla\psi, \nabla\psi)_{\mathbb{R}^n} + (\beta^{-1}(\psi|_{\Gamma+} - \psi|_{\Gamma-}), \psi|_{\Gamma+} - \psi|_{\Gamma-})_\Gamma$$

kde je síla vazby charakterizována funkcí β^{-1} a stopy funkce na hranici nosiče singulární interakce značíme $\psi|_{\Gamma\pm}$.
Odvodíme asymptotické chování diskrétního spektra pro případ silné vazby, tzn. $\beta \to 0_-$, pro uzavřené křivky a plochy. Rovněž pro tuto situaci spočteme esenciální spektrum. Uvažujeme C^4 křivky a plochy, kompaktní i nekonečné. Pro nekonečné křivky předpokládáme, že jsou asymptoticky přímé a obdobně pro nekonečné plochy předpokládáme, že jsou asymptoticky rovné. Pro tyto situace jsme schopni napsat první dva členy asymptotického rozvoje diskétních vlastních čísel, kde je druhý člen určen Schrödingerovým operátorem s potenciálem závislým na křivosti nosiče singulární interakce.
Dále dokážeme, že pro případ slabé vazby, tzn. $\beta \to -\infty$, je diskrétní spektrum prázdné pro každou kompaktní neuzavřenou varietu. Pro neuzavřené křivky odvodíme podmínku postačující na absenci diskrétního spektra v závislosti na vazbové konstantě β a křivosti křivky.

Abstract

The thesis is devoted to studies of spectral properties of the Schrödinger operators describing δ'-interaction supported on curves and surfaces. These operators are associated with the quadratic form

$$h_\beta^\Gamma(\psi) = (\nabla\psi, \nabla\psi)_{\mathbb{R}^n} + (\beta^{-1}(\psi|_{\Gamma+} - \psi|_{\Gamma-}), \psi|_{\Gamma+} - \psi|_{\Gamma-})_\Gamma$$

where the constant β^{-1} characterizes the coupling strength and $\psi|_{\Gamma\pm}$ are traces of the function at the boundaries of the interaction support.
We derive the asymptotic expression for the discrete spectrum in the strong coupling limit, i.e. $\beta \to 0_-$, for closed curves and surfaces as well as for the essential spectrum. We consider C^4 curves and surfaces, either compact or infinite. We assume that the infinite curves are asymptotically straight and the infinite surfaces are asymptotically planar. In this setting we derive the first two terms of the asymptotic expansion of the discrete spectrum where the second term is determined by Schrödinger type operator with effective potential expressed in the means of the interaction support curvatures.
We also prove that in the weak coupling limit, i.e. $\beta \to -\infty$, the discrete spectrum is empty for the case of compact non-closed manifolds. For the large class of non-closed curves we derive sufficient conditions for the absence of the discrete spectrum with respect to their coupling constant and the curvature of the curve.

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Contents

Introduction

In this work we study the Schrödinger operators describing δ'-interaction supported by various hypersurfaces. We are interested in the dependence of the spectrum on the geometry of the interaction support. The operators we are working with are associated with the quadratic forms

$$h_\beta^\Gamma(\psi) = (\nabla\psi, \nabla\psi)_{\mathbb{R}^n} + (\beta^{-1}(\psi|_{\Gamma+} - \psi|_{\Gamma-}), \psi|_{\Gamma+} - \psi|_{\Gamma-})_\Gamma \qquad (1)$$

where β^{-1} characterizes the coupling strength and $\psi|_{\Gamma\pm}$ are traces of the function at the boundaries of the interaction support.

In recent years the problem of quantum particles confined to curves, graphs, tubes, surfaces, layers, and other geometrically nontrivial objects, attracted a lot of attention. In physics they are used as models of various nanostructures and at the same time they present many interesting problems from a purely mathematical point of view. There are many ways how to treat these systems. Models using singular interactions give answers to these problems for several decades now. Historically possibly the most influential paper was [KP31], where the scattering of the electron on the crystal lattice was done using δ-interactions located at $\mathbb{Z}$ on a line. The great advantage of the models with singular interactions is that they are exactly solvable in the sense that their resolvents can be written explicitly and as a consequence we are able to derive their spectrum, eigenfunctions as well as to solve the corresponding scattering problem.

One possibility how to describe a quantum particle confined to a set of nontrivial geometry is to use models of quantum graphs. There is a wealth of literature studying such systems. We can mention few of them [BK13, H00, KS99] or the review [K08]. In this setting the configuration space is a metric graph. The edges are equipped with differential operator with boundary conditions at each vertex connecting wave functions. They arise as simplifications of various models in mathematics, physics and chemistry. We can mention the motion of a free-electron in polymer conjugated molecules, thin waveguides, photonic crystals and many others. This approach has several drawbacks. The first is the presence of ad hoc parameters in the boundary conditions. Another one is that the particle is strictly confined to graph edges and as a result the model neglects quantum tunneling effects which can be important in real situations when the edges are close to each other. Furthermore these systems neglect practically all geometric properties and take into account only the length of the edges.

These drawbacks can be treated using more realistic models of the so-called leaky quantum structures. The idea behind these models is to preserve the whole space for the motion of the particle and to keep the particle localized in the vicinity of the structure by an attractive singular potential. For an attractive δ-interaction it is described by a Hamiltonian which can be formally written as

$$-\Delta + \alpha\delta(\cdot - \Gamma)$$

with $\alpha < 0$ where Γ is the interaction support. An exhaustive review of these models was presented in [E08]. The main advantage of this approach is that there is no ambiguous parameters at the vertices and also that the tunneling between parts of the structure is possible.

One can try to use different singular interactions than the δ-interaction. If we consider manifolds of codimension one we can implement various singular interactions in the direction perpendicular to the manifold because in dimension one we have a larger variety of singular interactions. A prominent role among them is played by the δ'-interaction which is, in a sense, a counterpart to δ-interaction. At first glance the δ'-interaction seems to be less “natural”, however, it is not a mere mathematical construct because it can be estimated by a “triple layer” potential as it was shown in [CS98], which gives also interesting physical meaning to it. When δ'-interaction was used for the first time, it was for the symmetric case of concentric spheres in [AGS87, S88]. An elegant and rigorous definition for a general closed manifold was given in [BLL13] using operators associated with quadratic forms (1). Later based on this approach it was shown how to introduce δ'-interaction on a non-closed manifold in [JL16].

This work is divided into three main parts. The first one is devoted to rigorous definition of operators and their basic properties. The second one is the core of the thesis and contains a summary of published original results. We present them with a reasonable completeness, leaving out just some purely technical parts for which the reader is referred to the original papers included in the appendix. The last chapter contains additional unpublished results.

In the Chapter 1 we summarize rigorous definitions and basic spectral properties of the δ'-interaction in various settings. We are interested in the δ'-interaction supported by curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$. Locally δ'-interaction behaves as a one dimensional δ'-interaction in the orthogonal direction with respect to the interaction support. Due to this fact it is useful to study also the one dimensional case. An extensive study of finite number of δ'-interactions supported by points on a line was given in [AGHH05]. The operator describing N δ'-interactions on a line acts as

$$H_{B,Y} = -\Delta\,,$$
$$\mathcal{D}(H_{B,Y}) = \{\psi \in H^2(\mathbb{R} \setminus \{Y\} | \psi'(y_{i,-}) = \psi'(y_{i,+}) = \beta_i^{-1}(\psi(y_{i,+}) - \psi(y_{i,-}))\}$$

where the interactions are located at sites y_i with coupling constants β_i. At this point we note that a small negative β corresponds to the strong attractive coupling and a large negative β to the weak one. In several cases we also use a more general one dimensional setting than a straight line. For a general setting we can introduce δ'-coupling on a quantum graph as it was done in [E96].

For the description of δ'-interaction supported by a manifold we can use two basic equivalent approaches. One of them is making self-adjoint extensions by adding appropriate boundary conditions at the points with a singular interaction in the direction perpendicular to the interaction support. These conditions are locally equivalent to the conditions for the one dimensional case. the alternative way is to introduce our operator as an operator associated with correct semi-bounded, symmetric quadratic form (1). Both of these definitions are interchangeable if the support is sufficiently regular. We further split the case of δ'-interaction supported by hypersurface into two settings namely for closed and non-closed interaction supports. The situations for closed manifolds were studied in [BLL13]. The case of non-closed support is more complex because we have to treat the endpoints. In [JL16] it was solved by taking the quadratic form corresponding to the interaction supported by a closed manifold and restricting the domain to functions which are continuous at the point where we do not want the interaction. Alternative approach was derived in [MPS16] where it was solved via boundary condition of a self-adjoint operator.

The main results of the thesis, presented in Chapter 2, were published in the papers [EJ13, EJ14, J15] and [JL16]. The first three papers [EJ13, EJ14, J15] describe spectral behavior of the operator associated with the form (1) for a strong coupling limit, i.e. $\beta \to 0_-$, where the manifold Γ is a closed curve or a surface. A similar problem was treated in [EY02] where the authors derived the spectral asymptotics of strong δ-interaction supported by Jordan C^4 curve in $\mathbb{R}^2$. The spectral asymptotics of strong δ-interaction supported by a C^4 surface was treated in [EK03]. We assume that the manifold is at least C^4 with a neighborhood of finite thickness which do not cross itself. Furthermore for the case of infinite manifold we need to assume that the curves are asymptotically straight and that the surface is asymptotically planar. This asymptotic condition is added because it guarantees the existence of a discrete spectrum. For example the spectrum of strong δ-interaction supported by a periodic curve has a band structure without discrete eigenvalues as shown in [EY01]. The essential spectrum of our system is either

$$\sigma_{ess}(H_{\beta,\Gamma}) = \mathbb{R}^+$$

for the case of a compact manifold which is true for any coupling strength or

$$\sigma_{ess}(H_{\beta,\Gamma}) = [\epsilon(\beta), \infty) \ , \quad \epsilon(\beta) \to -\frac{4}{\beta^2}$$

for infinite manifolds in the asymptotic limit which holds for the strong coupling limit, i.e $\beta \to 0_-$. The limit infimum value of the essential spectrum $-\frac{4}{\beta^2}$ corresponds to the one dimensional ground state of one δ'-interaction on a line. We derive the first two terms of the asymptotic expansion of the discrete spectrum. The first term is manifold independent diverging for $\beta \to 0_-$ and the second term encodes the geometry of the manifold. The discrete eigenvalues admit an asymptotic expansion in the form

$$\lambda_i = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta \ln(|\beta|))$$

where we added the error term and μ_j are the eigenvalues with multiplicity taken into account of the following operators

$$S = -\partial_{ss} - \frac{1}{4}\gamma(s)^2 \quad \text{for curves}\,,$$
$$S = -\Delta_\Gamma - \frac{1}{4}(k_1 - k_2)^2 \quad \text{for surfaces}$$

where $\gamma(s)$ is the signed curvature of the curve, k_i are the principal curvatures of the surface and $-\Delta_\Gamma$ is the Laplace-Bertrami operator of the surface. We note that the technique used for the estimation of the eigenvalues could be in principle with few modification used also for the estimation of eigenvalues of δ'-interaction supported on non-closed manifolds. For the case of a compact curve we also derive the number of discrete eigenvalues

$$-\frac{2L}{\pi\beta} + \mathcal{O}(\ln|\beta|)\,.$$

This asymptotic estimate is dominated by a natural Weyl-type term.

In the paper [JL16] we studied the behavior of δ'-interaction supported on a non-closed curve in the weak coupling regime. It can be shown by a variational approach, in the same way as in [BEL13], that the δ'-interaction supported by a closed compact curve Γ has always at least one negative eigenvalue estimated from above by $\frac{L}{\beta S}$ where L is a length of the curve Γ and S is an area enclosed by that curve. However this is not true for non-closed curves. We show that for a certain class of curves there are no negative discrete eigenvalues for a sufficiently weak coupling, i.e. β negative and large enough. This behavior of an attractive potential was previously unknown in $\mathbb{R}^2$. It is a result of the fact that δ'-interaction is strongly singular and behaves differently from more regular potentials. Such behavior is well known in the weak coupling regime for Schrödinger operators with regular potentials in dimension $d \geq 3$, but not for $d = 1, 2$ [S76]. This behavior is exhibited by δ-interaction supported by either arbitrary compact hypersurfaces in $\mathbb{R}^3$ which was shown in [EF09] or for non-closed curves in $\mathbb{R}^3$ as proved in [EK08]. However attractive δ-interaction supported by any compact curve in $\mathbb{R}^2$ always has negative eigenvalues [ET04, KL16]. The essential spectrum for δ'-interaction supported on compact non-closed curve is

$$\sigma_{ess}(H_{\beta,\Gamma}) = \mathbb{R}^+\,.$$

For "monotone" curves (in the sense made precise below) parametrized as $\Gamma(r) = (r\sin(\phi(r)), r\cos(\phi(r)))$, $r \in (0, L)$, $L \in \mathbb{R}^+$ we can write the sufficient condition for the absence of the discrete spectrum in the following form

$$\beta(r) \leq -2\pi r\sqrt{1 + (r\phi'(r))^2}\,.$$

From this expression we see that bound states can arise from either bending the curve or from making the coupling stronger. This behavior arises from the fact that δ'-interaction on a loop of length d have a critical value for which the system have no negative eigenvalues, explicitly $-\beta < d$. One can check that for a sufficiently strong coupling our system has at least one negative discrete eigenvalue. For a straight line of length L we can estimate a critical value of β as

$$-\frac{2L}{\pi} < \beta < 0\,,$$

which is a sufficient condition for the existence of the bound state for constant β. The absence of the discrete spectrum in the weak coupling regime is present also for the δ'-interaction supported by hypersurfaces in higher dimensions.

The last chapter summarizes several unpublished results and work in progress. The chapter presents four new results. The first one treats a problem of δ'-interaction supported on a sharp angle. A similar problem, namely solving the spectrum for δ-interaction supported on sharp angle, was studied in [DR14]. We show that for a sufficiently small angle θ we can estimate the operator by a one-dimensional operator with Coulomb-like potential in the form $1/(\beta\theta r)$. The approach for this problem is based on results derived for two δ'-interactions on a loop.

The second result is an optimization of the lowest eigenvalue with respect to the position of N δ'-interactions on a loop. We show that for the case of even number of interactions, the optimal position with maximal lowest eigenvalue is for the totally symmetric case with the equal distance between the interactions. Studying these easier models can be very useful because the obtained results can be used quite often in treating more complicated problems in higher dimension.

The third result is a generalization of the result presented in [JL16]. The main theorem of [JL16] requires the interaction support to be a "monotone" curve, i.e. curve which "moves" away from one endpoint. This result was in the paper [JL16] generalized to a class of curves which are obtained by a linear fractional maps from monotone curves. We can further enlarge the class of curves for which the statement about the absence of negative eigenvalues for the weak coupling holds. We do not use linear fractional maps but any conformal maps from the unit disc to a different subset of $\mathbb{R}^2$. The only restriction is that our curve, before the transformation, has to connect the center of the unit disc to the boundary and be monotone.

The fourth and last presented result follows an idea due to Monique Dauge [D16]. It allows us to show that δ'-interaction supported by an arbitrary compact non-closed manifold in an arbitrary dimension $d \geq 2$ has no discrete eigenvalues in the weak coupling regime. A drawback of this approach is that it does not give an explicit lower bound for β.

1. Theoretical Background

In this section we summarize definitions of δ'-interactions in various settings as well as their basic properties. At the end of this chapter we give a short summary of some properties of conformal maps which are needed for the proofs of Theorems 2.4.4 and 3.3.1 in the chapters to follow.

1.1 δ'-Interaction in 1D

The δ'-interaction is by its nature a one-dimensional effect. In higher dimensions it is always introduced as an effect supported by a manifold of codimension 1. That is the reason why we study the 1D situation first. An extensive study of δ'-interactions on the line was given in [AGHH05]. The approach used for δ'-interaction on the line is based on self-adjoint extensions of an appropriate symmetric operator. A different approach based on operators associated with quadratic forms is employed later for the case of an interaction localized on closed and non-closed manifolds. However we can use both definitions in any setting.

1.1.1 One δ'-Interaction on Line

We start by introducing a closed, nonnegative operator, which acts as

$$H_y\psi(x) = -\Delta\psi(x)$$

with the domain $\mathcal{D}(H_y) = H_0^2(\mathbb{R} \setminus \{y\})$ where $y \in \mathbb{R}$. The adjoint of this operator can be easily shown to be

$$H_y^*\psi(x) = -\Delta\psi(x)$$

with the domain $\mathcal{D}(H_y^*) = H^2(\mathbb{R} \setminus \{y\})$ where $y \in \mathbb{R}$. A simple direct calculation shows that H_y has deficiency indices $(2, 2)$. This implies that H_y has a four-parameter family of self-adjoint extensions. It contains some subfamilies of particular importance. One consists of the well-known δ-interaction [AGHH05]. Another one, which is of importance for the present thesis, corresponds to δ'-interaction. The δ'-interaction is described by the following operator

$$-\Delta_{y,\beta}\psi(x) = -\Delta\psi(x) \tag{1.1}$$

with the domain $\mathcal{D}(-\Delta_{y,\beta}) = \{\psi \in H^2(\mathbb{R}\setminus\{y\}) | \psi'(y_-) = \psi'(y_+), \beta\psi'(y_+) = \psi(y_+) - \psi(y_-)\}$, where β is a fixed real number. A particular case is $\beta = 0$ which is understood as the absence of the interaction, i.e. $-\Delta_{y,0}$ coincides with the kinetic energy Hamiltonian on the line. The family of self-adjoint extensions contain also the case which can be identified with $\beta = \infty$ and which corresponds to the Neumann boundary condition at y decoupling the line into two halflines.

The resolvent of $-\Delta_{y,\beta}$ can be expressed by means of Krein's formula. Its explicit form is given by the following theorem [AGHH05]:

Theorem 1.1.1. *The resolvent of $-\Delta_{y,\beta}$ is given by*

$$(-\Delta_{y,\beta} - k^2)^{-1} = G_k - 2\beta k^2(2 - i\beta k)^{-1}(\overline{\tilde{G}_k(\cdot - y)}, \cdot)\tilde{G}_k(\cdot - y)$$

where $k^2 \in \rho(-\Delta_{y,\beta})$, $\Im k > 0$, $-\infty < \beta \leq \infty$, $y \in \mathbb{R}$. Furthermore, $G_k(x - x') = \frac{i}{2k}\exp(ik|x - x'|)$ is the resolvent kernel of the free Laplacian and

$$\tilde{G}_k(x - y) = \frac{i}{2k}\exp(ik(x - y)), \quad x > y\,,$$
$$\tilde{G}_k(x - y) = -\frac{i}{2k}\exp(ik(y - x)), \quad x > y\,, \quad \Im k > 0\,.$$

The spectral properties $-\Delta_{y,\beta}$ are summarized in the following theorem.

Theorem 1.1.2. *Let $-\infty < \beta \leq \infty$, $y \in \mathbb{R}$. Then the essential spectrum of $-\Delta_{y,\beta}$ is purely absolutely continuous and covers the nonnegative real axis*

$$\sigma_{ess}(-\Delta_{y,\beta}) = \sigma_{ac}(-\Delta_{y,\beta}) = \mathbb{R}_0^+, \quad \sigma_{sc}(-\Delta_{y,\beta}) = \emptyset\,.$$

If $-\infty < \beta < 0$, $-\Delta_{y,\beta}$ has precisely one negative, simple eigenvalue

$$\sigma_p(-\Delta_{y,\beta}) = \left\{-\frac{4}{\beta^2}\right\}$$

with the normalized eigenfunction

$$\psi(x) = \sqrt{-\frac{\beta}{8}} \exp\left(\frac{2}{\beta}(x-y)\right), \quad x > y\,,$$
$$\psi(x) = -\sqrt{-\frac{\beta}{8}} \exp\left(\frac{2}{\beta}(y-x)\right), \quad x < y\,.$$

For $\beta \geq 0$ or $\beta = \infty$, $-\Delta_{y,\beta}$ has no eigenvalues

$$\sigma_p(-\Delta_{y,\beta}) = \emptyset.$$

From the previous theorem we can conclude that $\beta > 0$ corresponds to the repulsive interaction and $\beta < 0$ to the attractive one. Furthermore, the strong coupling limit is $\beta \to 0_-$.

1.1.2 Finitely Many δ'-Interaction on Line

The concept from the previous section can be extended to finitely many δ'-interactions on the real line. Proofs and a more detailed discussion can be found in [AGHH05]. We start with the set of $N \in \mathbb{N}$ points $Y = \{y_1, \dots, y_N\} \subset \mathbb{R}^N$. We again introduce the closed, nonnegative minimal operator

$$H_Y\psi(x) = -\Delta\psi(x)$$

with the domain $\mathcal{D}(H_Y) = H_0^2(\mathbb{R} \setminus Y)$. The adjoint of this operator is

$$H_Y^*\psi(x) = -\Delta\psi(x)$$

with the domain $\mathcal{D}(H_Y^*) = H^2(\mathbb{R} \setminus Y)$. A direct calculation shows that this operator has deficiency indices $(2N, 2N)$. This means that there is a $4N^2$-parameter family of self-adjoint extensions. In particular, we are interested in families with local boundary conditions, i.e. the boundary conditions that couple the function values from left and right and the values of its derivatives always at one point. There is an N parameter family corresponding to N δ-interactions on the line and a different N parameter family corresponding to N δ'-interactions. The family corresponding to N δ'-interaction is the following

$$-\Delta_{Y,B}\psi(x) = -\Delta\psi(x) \tag{1.2}$$

with the domain $\mathcal{D}(-\Delta_{Y,B}) = \{\psi \in H^2(\mathbb{R}\backslash Y) | \psi'(y_{j-}) = \psi'(y_{j+}), \beta_j\psi'(y_{j+}) = \psi(y_{j+}) - \psi(y_{j-})\}$ where $B = \{\beta_1, \dots, \beta_N\}$ is the parameter family which has the meaning of coupling parameters. The particular case of $\beta_j = 0$ leads to the absence of the interaction at the point y_j. The case $\beta_j = \infty$ leads to Neumann boundary condition at y_j decoupling the line into two halflines with $j-1$ and $N-j$ δ'-interactions with Neumann boundary condition at the endpoint.

The following theorems describe some basic properties of these operators, namely their resolvents and spectra. The resolvent is obtained again by Krein's formula and has the following form [AGHH05]:

Theorem 1.1.3. *Let $\beta_j \neq 0$ for all $j \in \hat{N} = \{1, \ldots, N\}$. Then the resolvent of $-\Delta_{Y,B}$ can be written as*

$$(-\Delta_{Y,B} - k^2)^{-1} = G_k - \sum_{j,j' \in \hat{N}} [\Gamma_{Y,B}(k)]^{-1}_{jj'} (\overline{\tilde{G}_k(\cdot - y_{j'}), \cdot})\tilde{G}_k(\cdot - y_j)$$

where $k^2 \in \rho(-\Delta_{Y,B})$, $\Im k > 0$, $-\infty < \beta_j \leq \infty$, $\beta_j \neq 0$, $y_j \in Y$ for all $j \in \hat{N}$. $G_k(x - x') = \frac{i}{2k} \exp(ik|x - x'|)$ is the resolvent of free Laplacian on the line, the matrix $[\Gamma_{Y,B}(k)]_{jj'}$ can be written as

$$[\Gamma_{Y,B}(k)]_{jj'} = -(\beta_j k^2)^{-1} \delta_{jj'} + G_k(y_j - y_{j'})$$

and the function $\tilde{G}_k(x - y)$ is

$$\tilde{G}_k(x - y) = \frac{i}{2k} \exp(ik(x - y)) , \quad x > y ,$$
$$\tilde{G}_k(x - y) = -\frac{i}{2k} \exp(ik(y - x)) , \quad x < y , \quad \Im k > 0 .$$

The next theorem describes the spectrum of the operator $-\Delta_{Y,B}$. As in the case of single interaction case, the discrete spectrum is associated with the kernel of the matrix $[\Gamma_{Y,B}(k)]$:

Theorem 1.1.4. *Let $-\infty < \beta_j \leq \infty$, $y_j \in Y$ for $j \in \hat{N}$. If at most one parameter corresponds to separating boundary condition, i.e. $\beta_{j_0} = \infty$, then the point spectrum consists of at most N negative, simple eigenvalues. If at least two parameters lead to separating boundary conditions, then the point spectrum consists of at most N negative eigenvalues and infinitely many eigenvalues embedded in $[0, \infty)$ accumulating at ∞. In particular*

$$k^2 \in \sigma_p(-\Delta_{Y,B}) \cap (-\infty, 0) <=> \det[\Gamma_{Y,B}(k)] = 0 , \quad \Im k > 0 ,$$

and the multiplicity of the eigenvalue $k^2 < 0$ equals to multiplicity of the eigenvalue 0 of the matrix $[\Gamma_{Y,B}(k)]$. For an eigenvalue $E_0 = k_0^2$ of the operator $-\Delta_{Y,B}$ corresponding eigenfunctions are in the form

$$\psi_0(x) = \sum_{j=1}^{N} c_j \tilde{G}_{k_0}(x - y_j)\,, \quad \Im k_0 > 0$$

where $(c_1, \ldots, c_N)$ is an eigenvector of the matrix $[\Gamma_{Y,B}(k_0)]$ corresponding to the eigenvalue 0.
The remaining part of the spectrum is purely absolutely continuous and covers the nonnegative real axis

$$\sigma_{ess}(-\Delta_{Y,B}) = \sigma_{ac}(-\Delta_{Y,B}) = \mathbb{R}_0^+\,, \quad \sigma_{sc}(-\Delta_{Y,B}) = \emptyset\,.$$

1.1.3 δ'-coupling on Graph

It is possible to introduce δ'-interaction in a more general setting than connecting the endpoints of two intervals. To introduce δ'-coupling on a graph we need a metric graph equipped with a differential operator on its edges, typically a multiple of the Laplacian, and appropriate boundary conditions at the vertices. This setting is usually called "quantum graph". We work only with connected graphs because the spectrum of two disjoint graphs is union of the spectra for disjoint parts. We also limit ourselves for the sake of simplicity to graphs with a finite number of edges.

We consider a graph $\mathbb{G}$ constructed from p vertices and q edges where $p, q \in \mathbb{N}$. The lengths of the edges are represented by the vector $L = (l_i | i \in \hat{q})$ where $l_i \in \mathbb{R}^+ \cup \{\infty\}$ because we allow finite and semi-finite edges. The space $L^2(\mathbb{G})$ is composed of all square integrable functions on each edge, i.e. it is possible to write them as orthogonal direct sum of $L^2((0, l_i))$ spaces on edges

$$L^2(\mathbb{G}) = \bigoplus_{i=1}^{q} L^2((0, l_i))\,.$$

Our operator is constructed in an analogous way to the previous case from an appropriate symmetric operator by finding its self-adjoint extensions. We take the symmetric operator to act as

$$H_{\mathbb{G}} \psi_i = -\Delta \psi_i\,, \quad i \in \hat{q}\,.$$

with the domain $\Psi \in \{\psi_i \in H_0^2((0,l_i))|i \in \hat{q}\}$. The adjoint operator to $H_{\mathbb{G}}$ is

$$H_{\mathbb{G}}^* \psi_i = -\Delta \psi_i \,, \quad i \in \hat{q} \,.$$

with the domain $\Psi \in \{\psi_i \in H^2((0,l_i))|i \in \hat{q}\}$. This operator has deficiency indices $(2m+k, 2m+k)$, where m is the number of finite edges and k is the number of semi-infinite ones; in other words $2m+k$ is the number of endpoints of all edges. To make it self-adjoint we need to choose a subset in the domain of $H_{\mathbb{G}}^*$ by imposing appropriate boundary conditions. According to [GG91, H00, KS99] the general conditions can be written in the following way

$$A\tilde{\Psi} + B\tilde{\Psi}' = 0$$

where $\tilde{\Psi}$ is a vector composed of function boundary values at the endpoints of edges, $\tilde{\Psi}'$ is a vector composed of outgoing function derivatives at the endpoints of edges, $A, B \in \mathbb{C}^{2m+k,2m+k}$ and they fulfill

$$\begin{aligned} \operatorname{rank}(A|B) &= 2m+k \,, \\ AB^+ &\text{ is self-adjoint.} \end{aligned}$$

It can be easily shown that the choice of matrices A, B is not unique, e.g. for any regular matrix $C \in \mathbb{C}^{2m+k,2m+k}$ the matrix pairs (CA, CB) and (A, B) generate the same boundary conditions. There is a nice way how to get rid of this ambiguity. We can write matrices A, B as functions of one unitary matrix $U \in \mathbb{C}^{2m+k,2m+k}$ in the following way [KS00, H00]

$$\begin{aligned} A &= U - I \,, \\ B &= i(U + I) \,. \end{aligned} \tag{1.3}$$

This is a rather large class of possible boundary conditions even if we restrict ourselves to local boundary conditions at each vertex. By local boundary conditions we mean conditions, which couple only boundary values of functions from the endpoints of the edges located to the same vertex. For such a case we obtain the matrix U in the block diagonal form. We are able to write the conditions for each vertex in the form of (1.3) with a "smaller" unitary matrix $U \in \mathbb{C}^{l,l}$, where l is number of edges connected to the concrete vertex. We want to choose those generalizing δ'-interaction. The correct conditions for branched vertices are the following [E96]

$$\begin{aligned} \sum_{i \in \hat{l}} \psi_i'(0_+) &= 0 \,, \\ \psi_i(0_+) - \psi_j(0_+) + \frac{\beta}{2}(\psi_i'(0_+) - \psi_j'(+)) &= 0 \,, \quad i, j \in \hat{l} \end{aligned} \tag{1.4}$$

where the vertex in question has l edges. One can easily check that these conditions coincide with conditions of δ'-interaction on a line for the case $l = 2$. The operator describing δ'-interactions on the graph $\mathbb{G}$ is denoted by $-\Delta_{\mathbb{G},B}$. It acts as a free Laplacian on each edge

$$-\Delta_{\mathbb{G},B}\psi_i = -\psi_i''$$

with the domain $\Psi \in \{\psi_i \in H^2((0,l_i))|i \in q\}$, where Ψ satisfy conditions (1.4) for each vertex and $B = (\beta_1, \cdots, \beta_n)$ denotes coupling parameters at the vertices. The basic spectral properties of the operator $-\Delta_{\mathbb{G},B}$ are given in [BEH08].

Theorem 1.1.5. *Let $-\Delta_{\mathbb{G},B}$ be defined as above. If lengths of all edges are finite $l_i < \infty$ then $\sigma(-\Delta_{\mathbb{G},B})$ is purely discrete. If at least one edge is infinite $l_j = \infty$ then $\sigma_{ess}(-\Delta_{\mathbb{G},B}) = \mathbb{R}^+$.*

There is also a similar boundary condition corresponding to symmetrized version of δ'-interaction [E96]. This interaction has similar properties as δ'-interaction. A simple example is one δ'_s-interaction on the line, whose conditions are

$$\begin{aligned} \psi'(0_-) + \psi'(0_+) = 0\,, \\ \psi(0_+) + \psi(0_-) = \varsigma\psi'(0_+)\,, \quad \varsigma \in \mathbb{R}\,. \end{aligned}$$

This interaction has one eigenvalue $-\frac{4}{\varsigma^2}$ for $\varsigma < 0$ and has up to a sign the same reflection and transmission amplitudes for a plane wave as the δ'-interaction. The vertex conditions for δ'_s coupling for general vertex are

$$\begin{aligned} \psi_i'(0_+) = \psi_j'(0_+) = 0\,, \quad i, j \in \hat{l} \\ \sum_{i\in\hat{l}} \psi_i(0_+) = \varsigma\psi_j'((0_+)\,. \end{aligned}$$

where the vertex in question has l edges.

1.2 δ'-interaction Supported on Manifold of Codimension One

For the situations where the δ'-interaction is supported by non-trivial hypersurfaces in higher dimensions we use a different approach than for the case on the line. We define our operator via the first representation theorem, i.e. as an operator associated with semi-bounded, symmetric, densely-defined sesquilinear form. The case for closed manifolds was discussed to a large extent in [BLL13]. The situation for non-closed curves was treated in [JL16]. The procedure employed in [JL16] is applicable also for higher dimensions. We note that the approach via boundary conditions is possible as well [BLL13, MPS16].

1.2.1 δ'-interaction Supported by Closed Manifold

By a closed manifold we mean a manifold without a free boundary, i.e. either a closed manifold or an infinite one. Before we introduce the needed quadratic form we state several auxiliary results which were derived and proved in [BEL13].

Theorem 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then for any $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that*

$$\|\psi|_{\partial\Omega}\|^2_{\partial\Omega} \leq \epsilon\|\nabla\psi\|^2_\Omega + C(\epsilon)\|\psi\|^2_\Omega$$

holds for $\psi \in H^1(\Omega)$.

We assume that our manifold Γ separates $\mathbb{R}^n$ into two Lipschitz subsets $\Omega_+, \Omega_- \subset \mathbb{R}^n$ and $\Gamma = \partial\Omega_+ = \partial\Omega_-$. We can write $H^1(\mathbb{R}^n \setminus \Gamma) \ni \psi = \psi_+ \oplus \psi_- \in H^1(\Omega_+) \oplus H^1(\Omega_-)$. We introduce the notation $[\psi]_\Gamma = \psi|_{\Gamma+} - \psi|_{\Gamma-}$, where $\psi|_{\Gamma+}$ and $\psi|_{\Gamma-}$ denotes traces of the functions at the boundary of Γ. Now we can state the next auxiliary result.

Theorem 1.2.2. *Let $\Omega_\pm \subset \mathbb{R}^n$ be a Lipschitz domain and $\mathbb{R}^n \setminus \overline{\Omega_+} = \Omega_-$. Then for any $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that*

$$\|[\psi]_\Gamma\|^2_\Gamma \leq \epsilon\|\nabla\psi\|^2_{\mathbb{R}^n} + C(\epsilon)\|\psi\|^2_{\mathbb{R}^n}$$

holds for $\psi \in H^1(\mathbb{R}^n \setminus \Gamma)$.

We are ready to introduce the quadratic form which defines the operator describing a δ'-interaction on the manifold Γ

$$h^\Gamma_\beta(\psi) = \|\nabla\psi\|^2_{\mathbb{R}^n} + (\beta^{-1}[\psi]_\Gamma, [\psi]_\Gamma)_\Gamma \tag{1.5}$$

where $\beta^{-1}(x) \in L^\infty(\Gamma, \mathbb{R})$ with the domain $H^1(\mathbb{R}^n \setminus \Gamma)$. Using the previous theorem and employing properties of the form $\|\nabla\psi\|^2_{\mathbb{R}^n}$ we get that the form

h^Γ_β is semi-bounded, symmetric and densely-defined on $H^1(\mathbb{R}^n\setminus\Gamma)$. We define the operator $-\Delta_{\Gamma,\beta}$ corresponding to the quadratic form $h^\Gamma_\beta(\psi)$ via the first representation theorem. The operator $-\Delta_{\Gamma,\beta}$ acts as

$$\begin{aligned}-\Delta_{\Gamma,\beta}\psi(x) &= -\Delta\psi(x)\,,\\ \mathcal{D}(-\Delta_{\Gamma,\beta}) &= \{\psi \in H^{3/2}_\Delta(\mathbb{R}^n \setminus \Gamma)|\\ \partial_{\Gamma+}\psi|_{\Gamma+} = -\partial_{\Gamma-}\psi|_{\Gamma-}, \beta\partial_{\Gamma+}\psi|_{\Gamma+} &= \psi|_{\Gamma+} - \psi|_{\Gamma-}\}\end{aligned}$$

where $\partial_{\Gamma\pm}\psi|_{\Gamma\pm}$ are normal derivatives with respect to Γ and $H^{3/2}_\Delta(\mathbb{R}^n \setminus \Gamma) = \{\psi \in H^{3/2}(\mathbb{R}^n \setminus \Gamma)|\Delta\psi \in L^2(\mathbb{R}^n \setminus \Gamma)\}$. The authors in [BLL13] derived that if we restrict the coupling function to Sobolev space of order one of $L^\infty(\Gamma, \mathbb{R})$ functions, i.e. $\beta^{-1}(x) \in W^{1,\infty}(\Gamma)$ the domain of $-\Delta_{\Gamma,\beta}$ satisfies $\mathcal{D}(-\Delta_{\Gamma,\beta}) \subset H^2(\mathbb{R}^n \setminus \Gamma)$. The behavior of the spectrum was studied in [BLL13].

Theorem 1.2.3. *Let $-\Delta_{\Gamma,\beta}$ be defined as above and let the manifold Γ be compact and closed. Then $\sigma_{ess}(-\Delta_{\Gamma,\beta}) = \mathbb{R}^+$. Furthermore if $\beta < 0$ then $\sigma_d(-\Delta_{\Gamma,\beta}) \neq \emptyset$.*

1.2.2 δ'-interaction Supported by Non-closed Manifold

The approach which we use for δ'-interaction supported by a non-closed manifold was introduced in [JL16], where it was applied to δ'-interaction supported by a non-closed curve. An alternative definition of the operator was given in [MPS16] using boundary conditions. We start with the same setting as in the previous section. We take a connected but not necessarily bounded subset $\Lambda \subset \Gamma$ and introduce the linear space

$$F_\Lambda := \{\psi \in \mathcal{D}(\overline{\Omega_+}) \oplus \mathcal{D}(\overline{\Omega_-})|[\psi]_{\Gamma\setminus\Lambda} = 0\}$$

where $\mathcal{D}(\overline{\Omega}) := \{\psi|_\Omega|\psi \in \mathcal{D}(\mathbb{R}^n)\}$. It can be checked that F_Λ is a subspace of the Hilbert space $H^1(\Omega_+) \oplus H^1(\Omega_-)$. The closure of F_Λ in $H^1(\Omega_+) \oplus H^1(\Omega_-)$

$$H^1(\mathbb{R}^n \setminus \Lambda) := \overline{F_\Lambda}$$

is a Hilbert space. For the definition of the quadratic form we need the following estimate, which can be derived in the same way as in [JL16].

Theorem 1.2.4. *Let $\Gamma, \Lambda \subset \mathbb{R}^n$ and $\Omega_\pm \subset \mathbb{R}^n$ be as above. Then the following statements hold:*
i) $[\psi]_{\Gamma\setminus\Lambda} = 0\ \forall\psi \in H^1(\mathbb{R}^n \setminus \Lambda)$
ii) $\forall\epsilon > 0\ \exists C(\epsilon) > 0$

$$\|[\psi]_\Lambda\|_\Lambda^2 \leq \epsilon|\nabla\psi\|_{\mathbb{R}^n}^2 + C_\epsilon\|\psi\|_{\mathbb{R}^n}^2$$

holds for $\psi \in H^1(\mathbb{R}^n \setminus \Lambda)$.

Now we are ready to introduce the quadratic form

$$h_\beta^\Lambda(\psi) = \|\nabla\psi\|_{\mathbb{R}^n}^2 + (\beta^{-1}[\psi]_\Gamma, [\psi]_\Gamma)_\Lambda \tag{1.6}$$

where $\beta^{-1}(x) \in L^\infty(\Lambda, \mathbb{R})$ with the domain $H^1(\mathbb{R}^n \setminus \Lambda)$. The basic properties of this form were proved in [JL16].

Theorem 1.2.5. *Let $\Lambda \subset \mathbb{R}^n$, $\beta^{-1}(x) \in L^\infty(\Lambda, \mathbb{R})$ and F_Λ be as above. Then the quadratic form h_β^Λ is closed, densely defined, symmetric and lower-semibounded in Hilbert space $L^2(\mathbb{R}^n)$. Moreover, $F_\Lambda \subset \operatorname{dom} h_\beta^\Lambda$ is a core for this form.*

By the first representation theorem we define the operator $-\Delta_{\Lambda,\beta}$ to the quadratic form $h_\beta^\Gamma(\psi)$ which describes the δ'-interaction supported by a non-closed manifold.

The alternative approach presented in [MPS16] is based on the self-adjoint extension of a symmetric operator. We start by introducing the symmetric operator corresponding to a free Laplacian on $\mathbb{R}^n$ with omitted closed symmetric hypersurface. Next we introduce trace operators at the outer and inner boundary of the hypersurface. The domain of the self-adjoint operator is obtain by imposing proper boundary conditions. At the parts of the boundary where we want to have the δ'-interaction we introduce conditions corresponding to the δ'-interaction and at the other parts we impose conditions leading to sufficient smoothness of the functions.

In [MPS16] the authors used this approach for a δ'-interaction supported by an open subset Λ of a bounded closed manifold Γ which is the boundary of a subset $\Omega_- \subset \mathbb{R}^n$. The set Ω_- is assumed to be of the class $\mathcal{C}^{k,1}$, $k \geq 0$, i.e. local maps describing the manifold Γ are Lipschitz continuous together with their inverse up to the order k. For the definition of our operator we need to introduce two auxiliary operators. The operators on $\Omega_\pm$ act as

$$-\Delta^{max}_{\Omega_\pm} = -\Delta|_{\mathcal{D}(-\Delta^{max}_{\Omega_\pm})}\,,$$
$$\mathcal{D}(-\Delta^{max}_{\Omega_\pm}) := \{\psi_\pm \in L^2(\Omega_\pm) | -\Delta\psi_\pm \in L^2(\Omega_\pm)\}\,.$$

We denote one sided, zero-order, trace operators of co-normal derivative at the boundary of $\Omega_\pm$ by $\partial_{\Omega_\pm}\psi|_{\partial\Omega_\pm}$ and one sided, zero-order, trace operators at the boundary of $\Omega_\pm$ by $\psi|_{\partial\Omega_\pm} - \psi|_{\partial\Omega_\pm}$. Furthermore we need a set of functions $\mathcal{D}_{\beta,N,\Lambda} = \{\phi \in \{\psi \in H^{\frac{1}{2}}(\Gamma)|\text{supp}\psi \subseteq \overline{\Lambda}\}|((-\frac{1}{\beta} + \frac{1}{2}(\partial_{\Omega_+}\psi|_{\partial\Omega_+} + \partial_{\Omega_-}\psi|_{\partial\Omega_-})DL)\phi)|_\Lambda \in H^{\frac{1}{2}}(\Lambda)\}$ where we use shorthand $DL := DL_{\lambda_0}$ with $\lambda_0 > 0$. The double layer operator DL_z is defined as $DL_z : H^{-\frac{1}{2}}(\Gamma) \to L^2(\mathbb{R}^n)$ and is related to $-\Delta$ as $(DL_z\psi, \theta)_{L^2(\mathbb{R}^n)} := (\psi, \gamma_1(-A + \overline{z})^{-1}\theta)_{-\frac{1}{2},\frac{1}{2}}$ with $\theta \in L^2(\mathbb{R}^n)$. The linear operator defined as

$$-\Delta_{\Lambda,\beta} := (-\Delta^{max}_{\Omega_-} \oplus -\Delta^{max}_{\Omega_+})|_{\mathcal{D}(-\Delta_{\Lambda,\beta})}\,,$$
$$\mathcal{D}(-\Delta_{\Lambda,\beta}) = \Bigg\{\psi \in H^1(\mathbb{R}^n \setminus \overline{\Lambda}) \cap (\mathcal{D}(-\Delta^{max}_{\Omega_-}) \oplus \mathcal{D}(-\Delta^{max}_{\Omega_+}))\Bigg| \; (\psi|_{\partial\Omega_+} - \psi|_{\partial\Omega_-}) \in \mathcal{D}_{\beta,N,\Lambda}, \; \left(\beta\frac{1}{2}(\partial_{\Omega_+}\psi|_{\partial\Omega_+} + \partial_{\Omega_-}\psi|_{\partial\Omega_-}) - (\psi|_{\partial\Omega_+} - \psi|_{\partial\Omega_-})\right)\Bigg|_\Lambda = 0\Bigg\}$$

describes a δ'-interaction localized on non-closed curve Λ.

1.3 Conformal Mappings

For a later use we define conformal mappings and state several useful properties. For a complete review we refer the reader to any good text-book on complex variable e.g. [K99]. These transformations of the coordinates map a subset S_1 of the complex plane to a generally different subset S_2. The conformal mapping $M : S_1 \to S_2$ acts as

$$x_M = \Re(M(x+iy))\,, \quad y_M = \Im(M(x+iy))$$

where i is the complex unit and M is the analytic complex bijective function. The analyticity of the function M implies

$$\begin{aligned} \partial_x x_M &= \partial_y y_M\,, \\ \partial_x y_M &= -\partial_y x_M \end{aligned} \tag{1.7}$$

where we used shorthand $\partial_x = \frac{\partial}{\partial x}$. These equalities are in fact Cauchy-Riemann conditions. There is the famous Riemann mapping theorem which we use later on.

Theorem 1.3.1. *Let U be any simple connected open compact subset of $\mathbb{C}$. Then there exists such a conformal map $M(z) : U \to \mathbb{C}$ that $MU = D$ where D is the open unit disc centered around 0 in $\mathbb{C}$. Furthermore M is a homeomorphic continuous bijection between U and D.*

We also want to map the whole complex plane onto itself. We introduce the notation for the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The special class of conformal maps defined on $\hat{\mathbb{C}}$ is called linear fractional transformation. These maps $M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ act as

$$M(z) = \frac{az+b}{cy+d}\,, \quad ad - bc \neq 0\,.$$

The next theorem summarizes basic properties of the linear fractional transformations.

Theorem 1.3.2. *Any linear fractional transformation $M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is homeomorphism with respect to the standard topology on $\hat{\mathbb{C}}$ and its inverse M^{-1} is also a linear fractional transformation. The composition $M_1 \circ M_2$ of two linear fractional transformations M_1, M_2 is also a linear fractional transformation.*

Using equalities (1.7) we obtain the Jacobian J_M of the mapping M as

$$J_M = (\partial_x x_M)^2 + (\partial_y x_M)^2 = (\partial_x y_M)^2 + (\partial_y y_M)^2\,.$$

Furthermore equalities (1.7) yield the following relation

$$(\nabla x_M, \nabla y_M) = \partial_x x_M \partial_x y_M + \partial_y x_M \partial_y y_M = 0\,,$$

i.e. vectors $\nabla x_M, \nabla y_M$ are orthogonal to each other. Now we state the auxiliary lemma needed for the transformation of the quadratic forms corresponding to the free Laplacian.

Lemma 1.3.1. *Let M be an conformal map with the Jacobian J_M. Then for any $x \in \mathbb{R}^2$, $x \neq M^{-1}(\infty)$ and any function $u : \mathrm{dom}(M) \to \mathbb{C}$ differentiable at the point $M(x)$*

$$|(\nabla v)(x)|^2 = |(\nabla u)(M(x))|^2 J_M(x)$$

holds with $v = u \circ M$.

Proof. Using the expression for the Jacobian J_M, along with the orthogonality of the vectors $\nabla x_M, \nabla y_M$ and the chain rule for differentiation we obtain

$$\begin{aligned} |(\nabla v)(x)|^2 &= |(u_1' \circ M)\partial_x x_M + (u_2' \circ M)\partial_x y_M|^2 \\ &\quad + |(u_1' \circ M)\partial_y x_M + (u_2' \circ M)\partial_y y_M|^2 \\ &= \left(|u_1' \circ M|^2 + |u_2' \circ M|^2\right) J_M + 2\Re\left[(u_1' u_2') \circ (M) \cdot (\nabla x_M, \nabla y_M)\right] \\ &= \left(|u_1' \circ M|^2 + |u_2' \circ M|^2\right) J_M = |(\nabla u) \circ M|^2 J_M \end{aligned}$$

which completes the proof. □

2. Results

In this chapter we summarize the main results of the thesis. Most of them were published in [EJ13, EJ14, J15, JL16]. Each paper is discussed in one section. The first three papers describe spectral asymptotics of a strong δ'-interaction localized on closed curves and closed surfaces. The approach used in these works can be employed also for non-closed manifolds, but the approximating operators for the longitudinal part would be more complex. In [EJ13] we give the spectral asymptotics for the δ'-interaction localized on a closed compact curve as well as the asymptotics of the number of the negative eigenvalues. In [EJ14] we described spectral asymptotics for the δ'-interaction localized on either an infinite surface or a closed compact one. In [J15] we derived the spectral asymptotics for the δ'-interaction localized on the infinite curve. The last paper [JL16] discusses the absence of the discrete spectrum for the δ'-interaction localized on non-closed compact curves.

2.1 Spectral asymptotics of a strong δ' interaction on a planar loop

In this section we discuss the spectrum of a δ'-interaction localized on a closed compact curve in a plane. In the strong coupling regime $\beta \to 0_-$ we provide the asymptotic expansion of the eigenvalues as well as the asymptotic expression for the number of eigenvalues. The first two terms of the eigenvalues are a curve-independent term diverging as $\beta \to 0_-$ and the appropriate eigenvalues of a one-dimensional Schrödinger operator with effective potential depending on the geometry of the interaction support. The asymptotic expression for the number of eigenvalues is dominated by a natural Weyl-type term.

2.1.1 Formulation of Problem

We take a C^4 Jordan curve, i.e. closed and without self-intersections. We parametrize it by its arc length as

$$\Gamma : [0, L] \to \mathbb{R}^2\,, \quad s \to (\Gamma_1(s), \Gamma_2(s))$$

where $\Gamma_1(s), \Gamma_2(s) \in C^4(\mathbb{R})$. The operator we are interested in is associated with the quadratic form (1.5) which we rewrite in the following form

$$h^\Gamma_\beta(\psi) = (\nabla\psi, \nabla\psi)^2 + \beta^{-1}\int_0^L [\psi]^2_\Gamma ds\,, \quad \mathcal{D}(h^\Gamma_\beta(\psi)) = H^2(\mathbb{R}^2 \setminus \Gamma)$$

where we used the natural parametrization for the curve Γ in the integral. The operator associated with the form h^Γ_β is explicitly

$$\begin{aligned} -\Delta_{\Gamma,\beta}(\psi) &= -\Delta\psi\,, \\ \mathcal{D}(-\Delta_{\Gamma,\beta}) = \{\psi \in H^2(\mathbb{R}^2 \setminus \Gamma) | \beta\partial|_{\Gamma+}\psi(x)|_{\Gamma+} &= -\beta|_{\Gamma-}\psi(x)|_{\Gamma-} = [\psi(x)]_\Gamma\}\,. \end{aligned}$$

For the statement of our theorems we need to introduce an auxiliary operator

$$\begin{aligned} S_\Gamma &= -\frac{\partial^2}{\partial s^2} - \frac{\gamma(s)^2}{4}\,, \\ \mathcal{D}(S_\Gamma) = \{\psi \in H^2((0, L)) | \psi(0_+) &= \psi(L_-), \psi'(0_+) = \psi'(L_-)\} \end{aligned}$$

where γ is the signed curvature, i.e. $\gamma(s) = (\Gamma''_1\Gamma'_2 - \Gamma''_2\Gamma'_1)(s)$. We denote the eigenvalues of the operator S_Γ by μ_j with the multiplicity of eigenvalues taken into account. Now we are ready to state two main theorems of the paper [EJ13]. We state each theorem in a separate subsection, where we also include sketches of the proofs.

2.1.2 Asymptotics of Discrete Spectrum

Theorem 2.1.1. *Let Γ be a C^4 Jordan curve. For any $n \in \mathbb{N}$ there exists $\beta_n < 0$ such that*

$$|\sigma_{disc}(-\Delta_{\Gamma,\beta})| \geq n \quad \text{holds for} \quad \beta \in (\beta_n, 0)\,.$$

For any such β we denote by $\lambda_j(\beta)$ the j-th eigenvalue of $-\Delta_{\Gamma,\beta}$ counted with multiplicity taken into account. The asymptotic expansion for $\beta \to 0_-$ is

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta\ln(|\beta|))\,, \quad j \in \hat{n}\,.$$

The proof of this theorem is based on Dirichlet and Neumann bracketing. We estimate our operator from above and from below by suitable operators. To be able to do so, we need several auxiliary results. We start by introducing curvilinear coordinates s, u in the vicinity of the curve as

$$(x, y) = (\Gamma_1(s) + u\Gamma_2'(s), \Gamma_2(s) - u\Gamma_1'(s)) \,. \tag{2.1}$$

The regularity of the coordinates for small u is guaranteed by the fact that Γ is a C^4 Jordan curve; for the details we refer the reader to [EY02]. We choose a strip neighborhood $\Omega_a = \{x \in \mathbb{R}^2 | \mathrm{dist}(x, \Gamma) < a\}$. The neighborhood is chosen in such a way that the coordinates (2.1) are injective. We define two approximating operators to $-\Delta_{\Gamma,\beta}$ with imposed Neumann boundary condition and Dirichlet boundary condition at the boundary of Ω_a

$$-\Delta_{\Gamma,\beta}^{N,a} \leq -\Delta_{\Gamma,\beta} \leq -\Delta_{\Gamma,\beta}^{D,a} \,.$$

The operators $-\Delta_{\Gamma,\beta}^{D/N,a}$ have the same differential expression as $-\Delta_{\Gamma,\beta}$ but different domains $\mathcal{D}(-\Delta_{\Gamma,\beta}^{D,a}) = \{\psi \in \mathcal{D}(-\Delta_{\Gamma,\beta}) | \psi(s, a) = \psi(s, -a) = 0\}$ and $\mathcal{D}(-\Delta_{\Gamma,\beta}^{N,a}) = \{\psi \in H^2(\mathbb{R}^2 \setminus (\Gamma \cup \partial\Omega_a) | \beta\partial_u\psi(s, 0_+) = \beta\partial_u\psi(s, 0_-) = [\psi(s, 0)]_\Gamma, \partial_u\psi(s, a) = \partial_u\psi(s, -a) = 0\}$ where we used curvilinear coordinates. The operators $-\Delta_{\Gamma,\beta}^{D/N,a}$ can be written as direct sums of operators acting on parts of the plane separated by the boundary of Ω_a. For the study of the negative discrete spectrum we can neglect the parts acting on $\mathbb{R}^2 \setminus \overline{\Omega_a}$ due to the positivity of free Laplacian on these subsets. The parts corresponding to the strip neighborhood Ω_a are associated with the following quadratic forms

$$\begin{aligned}
h_{\beta,D}^{\Gamma,a}(\psi) &= \|\nabla\psi\|_{\Omega_a}^2 + \beta^{-1}\int_0^L [\psi]_\Gamma^2 ds \,, \\
\mathcal{D}(h_{\beta,D}^{\Gamma,a}(\psi)) &= \{\psi \in H^1(\Omega_a) | \psi(x)|_{\partial\Omega_a} = 0\} \,, \\
h_{\beta,N}^{\Gamma,a}(\psi) &= \|\nabla\psi\|_{\Omega_a}^2 + \beta^{-1}\int_0^L [\psi]_\Gamma^2 ds \,, \\
\mathcal{D}(h_{\beta,N}^{\Gamma,a}(\psi)) &= H^1(\Omega_a \setminus \Gamma) \,.
\end{aligned}$$

The next step is rewriting these forms by means of curvilinear coordinates (2.1) which was done in [EJ13].

Lemma 2.1.1. *Quadratic forms $h^{\Gamma,a}_{\beta,D}$ and $h^{\Gamma,a}_{\beta,N}$ are unitarily equivalent to quadratic forms $q^{\Gamma,a}_{\beta,D}$ and $q^{\Gamma,a}_{\beta,N}$*

$$
\begin{aligned}
q^{\Gamma,a}_{\beta,D}(\psi) &= \left\| \frac{\partial_s \psi}{g} \right\|^2_{\Omega_a} + \|\partial_u \psi\|^2_{\Omega_a} + (\psi, V\psi)_{\Omega_a} \\
&+\beta^{-1}\int_0^L [\psi]^2_\Gamma ds + \frac{1}{2}\int_0^L \gamma(s)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2) ds\,, \\
\mathcal{D}(q^{\Gamma,a}_{\beta,D}) &= \{\psi \in H^1((0,L)\times((-a,a)\setminus\{0\})) | \psi(0,u) = \psi(L,u), \\
&\qquad \psi(s,a) = \psi(s,-a) = 0\}\,, \\
q^{\Gamma,a}_{\beta,N}(\psi) &= q^{\Gamma,a}_{\beta,D}(\psi) - \int_0^L \frac{\gamma(s)}{2(1+a\gamma(s))}|\psi(s,a)|^2 ds \\
&+ \int_0^L \frac{\gamma(s)}{2(1+a\gamma(s))}|\psi(s,a)|^2 ds\,, \\
\mathcal{D}(q^{\Gamma,a}_{\beta,N}) &= \{\psi \in H^1((0,L)\times((-a,a)\setminus\{0\})) | \psi(0,u) = \psi(L,u)\}
\end{aligned}
$$

where we use shorthands $V(s,u) = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$ and $g(s,u) := 1 + u\gamma(s)$.

The quadratic forms $q^{\Gamma,a}_{\beta,D/N}$ are not simple enough to be handled easily so we replace the previous estimate by a cruder one. As an upper bound we introduce the operator associated with the quadratic form $q^{\Gamma,a}_{\beta,+}$

$$
\begin{aligned}
q^{\Gamma,a}_{\beta,+}(\psi) &= \frac{\|\partial_s \psi\|^2_{\Omega_a}}{(1-a\gamma_+)^2} + \|\partial_u \psi\|^2_{\Omega_a} + (\psi, V^+\psi)_{\Omega_a} \\
&+\beta^{-1}\int_0^L [\psi]^2_\Gamma ds + \frac{1}{2}\int_0^L \gamma(s)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2) ds\,, \\
&\qquad \mathcal{D}(q^{\Gamma,a}_{\beta,+}) = \mathcal{D}(q^{\Gamma,a}_{\beta,D})
\end{aligned}
$$

where we used shorthands $V^+(s) = \frac{a(\gamma'')_+}{2(1-a\gamma_+)^3} - \frac{\gamma^2}{4(1+a\gamma_+)^2}$, $f_+ = \max_x |f(x)|$ and we omitted the non-positive term $-\frac{5(u\gamma')^2}{4g^4}$. The operator corresponding to the form $q^{\Gamma,a}_{\beta,+}$ can be written as

$$
Q^{\Gamma,a}_{\beta,+} = U^{\Gamma,a}_+ \otimes I + \int^{\oplus}_{(0,L)} T^{\Gamma,a}_{\beta,+}(s) ds
$$

where the operator $T^{\Gamma,a}_{\beta,+}(s)$ is associated with the form

$$
[t^{\Gamma,a}_{\beta,+}(s)](\psi) = \|\partial_u \psi\|^2_{(-a,a)} + \beta^{-1}|\psi(0_+) - \psi(0_-)|^2 + \frac{1}{2}\gamma(s)(|\psi(0_+)|^2 - |\psi(0_-)|^2)\,.
$$

The operator $T^{\Gamma,a}_{\beta,+}(s)$ itself acts as

$$
\begin{aligned}
T^{\Gamma,a}_{\beta,+}(s)(\psi) &= -\psi''\,,\\
\mathcal{D}(T^{\Gamma,a}_{\beta,+}(s)) = \Big\{\psi \in H^2((-a,a)\setminus\{0\})\Big| &\psi(a) = \psi(-a) = 0,\\
\psi'(0_-) = \psi'(0_+) = \beta^{-1}(\psi(0_+) - \beta(0_-)) + \frac{\gamma(s)}{2}&(\psi(0_+) + \beta(0_-))\Big\}\,.
\end{aligned}
$$

We are interested only in the negative part of the spectrum, which is independent of s [EJ13].

Lemma 2.1.2. *Let $\frac{a}{\beta} > 2$. Then the operator $T^{\Gamma,a}_{\beta,+}(s)$ has exactly one negative eigenvalue $t_+ = -\kappa_+^2$ independent of s where*

$$
\kappa_+ = -\frac{2}{\beta} + \frac{4}{\beta}\exp(4a/\beta) + \mathcal{O}\left(\frac{\exp(8a/\beta)}{\beta}\right) \quad \text{as} \quad \beta \to 0_-\,.
$$

In a similar fashion we find a lower bound. We introduce the quadratic form $q^{\Gamma,a}_{\beta,-}$ defined as

$$
\begin{aligned}
q^{\Gamma,a}_{\beta,-}(\psi) &= \frac{\|\partial_s\psi\|^2_{\Omega_a}}{(1+a\gamma_+)^2} + \|\partial_u\psi\|^2_{\Omega_a} + (\psi, V^-\psi)_{\Omega_a}\\
+\beta^{-1}\int_0^L [\psi]^2_\Gamma ds &- \frac{1}{2}\int_0^L \gamma(s)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2)ds\\
&-\gamma_+\int_0^L (|\psi(s,a)|^2 + |\psi(s,-a)|^2)ds\,,\\
\mathcal{D}(q^{\Gamma,a}_{\beta,-}) &= \{\psi \in H^1((0,L)\times((-a,a)\setminus\{0\}))|\psi(0,u) = \psi(L,u)\}
\end{aligned}
$$

where $V^-(s) = -\frac{a(\gamma'')_+}{2(1-a\gamma_+)^3} - \frac{5(a(\gamma')_+)^2}{4(1-a\gamma_+)^4} - \frac{\gamma^2}{4(1-a\gamma_+)^2}$. The operator $Q^{\Gamma,a}_{\beta,-}$ associated with the quadratic form $q^{\Gamma,a}_{\beta,-}(\psi)$ can be written in the same fashion as

$$
Q^{\Gamma,a}_{\beta,-} = U^{\Gamma,a}_- \otimes I + \int^{\oplus}_{(0,L)} T^{\Gamma,a}_{\beta,-}(s)ds\,.
$$

Again we need to estimate the negative part of the spectrum of the operator $T^{\Gamma,a}_{\beta,-}(s)$ corresponding to the transversal part [EJ13].

Lemma 2.1.3. *Let $\frac{2}{\beta} > \gamma_+$. Then for $\beta \to 0_-$ the operator $T^{\Gamma,a}_{\beta,-}(s)$ has exactly one negative eigenvalue $t_- = -\kappa_-^2$ independent of s where*

$$\kappa_- = -\frac{2}{\beta} - \frac{4}{\beta}\frac{2+\beta\gamma_+}{2-\beta\gamma_+}\exp(4a/\beta) + \mathcal{O}\left(\frac{4}{\beta}\left(\frac{2+\beta\gamma_+}{2-\beta\gamma_+}\right)^2\exp(8a/\beta)\right).$$

The spectrum of the operators $U^{\Gamma,a}_{\pm}$ corresponding to the longitudinal variable s can be approximated by the spectrum of the operator S_Γ in the same way as in [EJ13].

Lemma 2.1.4. *There is a positive C independent of a and j such that*

$$|\mu_j^{\pm}(a) - \mu_j| \leq Caj^2$$

holds for $j \in \mathbb{N}$ and $0 < a < \frac{1}{2\gamma_+}$, where $\mu_j^{\pm}(a)$ are the eigenvalues of the operators $U^{\Gamma,a}_{\pm}$, respectively, with the multiplicity taken into account.

Now we have all we need to prove the first theorem. We define $a(\beta) = \frac{3}{4}|\beta|\ln(|\beta|)$. We denote the eigenvalues of the operators $T^{\Gamma,a(\beta)}_{\beta,\pm}(s)$ as $t^{\Gamma,a(\beta)}_{\beta,\pm,j}$ with the multiplicity taken into account. We already established that for β small enough exactly one eigenvalue $t^{\Gamma,a(\beta)}_{\beta,\pm,j}(s)$ is negative, i.e. the first one $t^{\Gamma,a(\beta)}_{\beta,\pm,1}(s) = t_\pm$ which is independent of s. The estimates we employed before can be written together as

$$Q^{\Gamma,a(\beta)}_{\beta,-} \oplus -\Delta^N_{\mathbb{R}^2\setminus\Omega_a} \leq -\Delta^{N,a(\beta)}_{\Gamma,\beta} \leq -\Delta_{\Gamma,\beta} \leq -\Delta^{D,a(\beta)}_{\Gamma,\beta} \leq Q^{\Gamma,a(\beta)}_{\beta,+} \oplus -\Delta^D_{\mathbb{R}^2\setminus\Omega_a} \tag{2.2}$$

where $\Delta^{D/N}_{\mathbb{R}^2\setminus\Omega_a}$ is Dirichlet or Neumann Laplacian on $\mathbb{R}^2 \setminus \Omega_a$, respectively. The eigenvalues of $Q^{\Gamma,a(\beta)}_{\beta,\pm}$ can be written as $t^{\Gamma,a(\beta)}_{\beta,\pm,j} + \mu_k^{\pm}(a(\beta))$ with $j,k \in \mathbb{N}$. We can write the following estimate

$$t^{\Gamma,a(\beta)}_{\beta,\pm,j} + \mu_k^{\pm}(a(\beta)) \geq \mu_1^{\pm}(a(\beta)) = \mu_1 + \mathcal{O}(\beta\ln(|\beta|)), \quad j > 1.$$

This estimate allows us to focus only on eigenvalues with $j = 1$. We denote

$$\omega_{\beta,\pm,k} = t^{\Gamma,a(\beta)}_{\beta,\pm,1} + \mu_k^{\pm}(a(\beta)).$$

Using the choice of $a(\beta)$ we obtain

$$\omega_{\beta,\pm,k} = -\frac{4}{\beta^2} + \mu_k + \mathcal{O}(\beta j^2\ln(|\beta|)) \quad \text{for} \quad \beta \to 0_-.$$

Combining the two previous expression we obtain that for each $n \in \mathbb{N}$ there exists $\beta(n) < 0$ such that

$$\omega_{\beta,+,n} \le 0\,, \quad \omega_{\beta,+,n} < t^{\Gamma,a(\beta)}_{\beta,+,j} + \mu^+_k(a(\beta))\,, \quad \omega_{\beta,-,n} < t^{\Gamma,a(\beta)}_{\beta,-,j} + \mu^-_k(a(\beta))$$

holds for $0 > \beta > \beta(n)$, $j > 1$ and $k \ge 1$. Thus the operators $Q^{\Gamma,a(\beta)}_{\beta,\pm}$ have the eigenvalues $\omega_{\beta,\pm,k}$ counting multiplicity for $k \le n$ for $\beta(n) < \beta < 0$. We denote the eigenvalues of $-\Delta^{D/N,a}_{\Gamma,\beta}$ as $\xi_{\beta,\pm,k}$ respectively. From the min-max principle and (2.2) we obtain

$$\omega_{\beta,-,k} \le \xi_{\beta,-,k}\,, \quad \xi_{\beta,+,k} \le \omega_{\beta,+,k}$$

for $k \in \hat{n}$ which also implies that $\xi_{\beta,+,n} < 0$. Using the min-max principle we can conclude that our operator $-\Delta_{\Gamma,\beta}$ has at least n eigenvalues in the interval $(-\infty, \xi_{\beta,+,n})$ and for all $k \in \hat{n}$

$$\xi_{\beta,-,k} \le \lambda_k \le \xi_{\beta,+,k}$$

holds which completes the proof.

2.1.3 Number of Eigenvalues with Respect to β

Theorem 2.1.2. *Let Γ be a C^4 Jordan curve. Then the counting function $\beta \mapsto \#|\sigma_{disc}(-\Delta_{\Gamma,\beta})|$ of the operator $-\Delta_{\Gamma,\beta}$ admits asymptotic expansion in the form*

$$\#|\sigma_{disc}(-\Delta_{\Gamma,\beta})| = -\frac{2L}{\pi\beta} + \mathcal{O}(|ln(|\beta|)|) \quad \text{as} \quad \beta \to 0_+$$

where L is the length of the curve Γ.

For the proof of this theorem we need two auxiliary results. First we introduce two notations. For a self-adjoint operator A with $\inf \sigma_{ess}(A) = 0$ we denote the number of negative eigenvalues as $N^-(A) := \#|\sigma_d(A) \cap \mathbb{R}^-|$. We also introduce $K^\pm_\beta = \{k \in \mathbb{N} | \omega_{\beta,+,k} < 0\}$. The following estimate was derived in [EJ13].

Lemma 2.1.5. *Let the operators $Q^{\Gamma,a(\beta)}_{\beta,\pm}$ be defined as above. Then for $\beta \to 0_-$ we have*

$$|K^\pm_\beta| = \frac{2L}{\pi\beta} + \mathcal{O}(|\ln(|\beta|)|)\,.$$

For the proof we need to estimate the second eigenvalue of the operator $T^{\Gamma,a}_{\beta,-}(s)$.

Lemma 2.1.6. *Let $T^{\Gamma,a}_{\beta,-}(s)$ defined as above, let $s \in [0, L)$ be fixed and let $0 < -\beta < 2a$. Then there are no eigenvalues of the operator $T^{\Gamma,a}_{\beta,-}(s)$ in the interval $\left[0, \min\left\{\frac{\gamma_+}{2a}, \left(\frac{\pi}{4a}\right)^2\right\}\right)$.*

From (2.2) we can estimate the number of negative eigenvalues in the following way

$$N^-(Q^{\Gamma,a(\beta)}_{\beta,-}) \le N^-(-\Delta^{N,a(\beta)}_{\Gamma,\beta}) \le N^-(-\Delta_{\Gamma,\beta}) \le N^-(-\Delta^{D,a(\beta)}_{\Gamma,\beta}) \le N^-(Q^{\Gamma,a(\beta)}_{\beta,+}).$$

Using the results of Lemmata 2.1.4 and 2.1.5 we can obtain that for small enough β we have $K^-_\beta = N^-(Q^{\Gamma,a(\beta)}_{\beta,-})$. This implies that

$$K^+_\beta \le |\sigma_d(-\Delta_{\Gamma,\beta})| = N^-(-\Delta_{\Gamma,\beta}) \le N^-(Q^{\Gamma,a(\beta)}_{\beta,-}) = K^-_\beta$$

which completes the proof.

2.2 Spectral asymptotics of a strong δ' interaction supported by a surface

In this section we describe the spectrum of δ'-interaction localized on a closed surface in the strong coupling regime. We study two settings - a closed compact surface and an infinite asymptotically planar surface. For both situations we derive the asymptotic expansions of the discrete spectrum. The second term of the discrete spectrum is determined by a Schrödinger type operator with an effective potential dependent on the curvatures of the interaction support. For an infinite surface we calculate the threshold of the essential spectrum.

2.2.1 Formulation of Problem

The operator, which we are interested in, is associated with the quadratic form (1.5), namely

$$h^\Gamma_\beta(\psi) = \|\nabla\psi\|^2_{\mathbb{R}^3} + \beta^{-1}\|[\psi]_\Gamma\|^2_\Gamma\,.$$

Due to the nature of the proofs we need to employ some additional conditions concerning the surface Γ. For the infinite one we assume the following:

(a1) Γ is C^4 smooth and allows a global normal parametrization with uniformly bounded elliptic tensor,

(a2) Γ can be enclosed in a symmetric layer neighborhood of a finite thickness with no self-intersections,

(a3) Γ is asymptotically planar, i.e. curvatures of the surface vanishes as the geodetic distance from one fixed point tends to the infinity and

(a4) Γ is not a plane.

By the last assumption we exclude the trivial case because it can be solved exactly by separation of variables. The case of finite surfaces is simpler than the case of infinite ones due to the compactness. For compact surfaces we require the absence of a free boundary:

(b) Γ is a C^4 smooth surface of finite genus.

For a closed finite surface we have no global parametrization. A finite surface Γ can be described by an atlas of maps representing normal parameterizations with a uniformly bounded elliptic tensor.
Now we recall various needed facts about the geometry of the surface and its neighborhoods. For a complete analysis we refer the reader to any good textbook on differential geometry, e.g. [F11]. We start with the infinite surface. We introduce normal coordinates on Γ by extending a local exponential map $\gamma : T_o\Gamma \to U_o$ with the origin $o \in \Gamma$ to the neighborhood U_0 of the point o. These normal coordinates are given by

$$s = (s_1, s_2) \to \exp_o \left(\sum_i s_i e_i(o) \right) \tag{2.3}$$

where we denoted the orthonormal basis of $T_o\Gamma$ as $(e_1(o), e_2(o))$. Using the assumption (a1) we can find a point $o \in \Gamma$ such that the map (2.3) can be extended to a global parametrization of Γ from $T_o\Gamma \equiv \mathbb{R}^2$ to Γ. In these coordinates the metric tensor of the surface Γ can be expressed as $g_{\mu\nu} = \gamma_{,\mu} \cdot \gamma_{,\nu}$. The inverse of the metric tensor is $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. The invariant surface element is denoted by $d\Gamma = \sqrt{g} ds_1 ds_2 = \det(g_{\mu\nu}) ds_1 ds_2$. The normal vector $n(s)$ can be calculated using the tangent vectors $\gamma_{,\mu}$ as $n(s) = \frac{\gamma_{,1} \times \gamma_{,2}}{|\gamma_{,1} \times \gamma_{,2}|}$.

We denote the Weingarten tensor as $h^\mu_\nu := -n_{,\nu} \cdot \gamma_{,\sigma} g^{\sigma\mu}$. The Gauss curvature K and mean curvature M are calculated as

$$K = \det(h^\nu_\mu) = k_1 k_2 \,,$$
$$M = \frac{1}{2}\mathrm{Tr}(h^\nu_\mu) = \frac{1}{2}(k_1 + k_2)\,.$$

The eigenvalues of the Weingarten tensor $k_{1,2}$ are called principal curvatures. By a direct calculation one can check the following identity $K - M^2 = -\frac{1}{4}(k_1 - k_2)^2$.

We need to parametrize the neighborhood of the surface Γ. Using the parametrization (2.3) we define the following mapping of a layer neighborhood Ω_d of the surface Γ, where $d > 0$ is the halfwidth of the layer

$$\mathcal{L} : D_d = \{(s,u) | s \in \mathbb{R}^2, u \in (-d,d)\} \to \gamma(s) + un(s)\,. \tag{2.4}$$

The assumption (a2) can be rephrased as

(a2) there is such $d_0 > 0$ that the mapping (2.4) is injective for any $0 < d < d_0$.

Using (a1) we can conclude that $\mathcal{L}$ is diffeomorphism. We can regard the layer Ω_d as a manifold with a boundary. In parametrization (2.4) the metric tensor is expressed as

$$G_{ij} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}$$

where $G_{\mu\nu} = (\delta^\sigma_\mu - uh^\sigma_\mu)(\delta^\rho_\sigma - uh^\rho_\sigma)g_{\rho\nu}$. We use the convention that Latin letters indices run through $1,2,3$ corresponding to coordinates (s_1, s_2, u) and Greek letters indices run through $1,2$ corresponding to the variables on the surface. The volume element in the layer is $d\Omega_d = \sqrt{G} ds_1 ds_2 du$ with $G = \det(G_{ij}) = g[(1-uk_1)(1-uk_2)]^2 = g(1 - 2Mu + Ku^2)^2$. It is useful to introduce the shorthand $\xi(s,u) = 1 - 2M(s)u + K(s)u^2$. Using the notion of curvatures we can express the assumption (a3) as

(a3) $K, M \to 0$ for $\sqrt{s_1^2 + s_2^2} \to \infty\,.$

There are several useful estimates which are the corollaries of the conditions $(a1) - (a3)$ [DEK01]. We have uniformly bounded principal curvatures k_1 and k_2. We denote $\rho = \frac{1}{\max\{\|k_1\|_\infty, \|k_2\|_\infty\}}$. The critical halfwidth introduced in (a2) is always smaller than $\rho > d_0$. For the values $d < \rho$ the following inequalities hold in the layer neighborhood Ω_d of Γ

$$\begin{aligned} C_-(d) \leq \xi(s,u) \leq C_+(d)\,, \\ C_-(d) g_{\mu\nu} \leq G_{\mu\nu} \leq C_+(d) g_{\mu\nu} \end{aligned} \tag{2.5}$$

where $C_\pm(d) = \left(1 \pm \frac{d}{\rho}\right)^2$. From the fact that the metric tensor is uniformly elliptic we also have

$$c_- \delta_{\mu\nu} \leq g_{\mu\nu} \leq c_+ \delta_{\mu\nu} \tag{2.6}$$

as a matrix inequality for some positive $c_\pm$.

For closed finite surfaces we need to replace the global parametrization by a finite atlas $\mathcal{A}$ of maps. In each map $\mathcal{M}_i$ we introduce normal coordinates in the same way as done for the infinite case operator. The latter of each part ran$(\mathcal{M}_i)$ is described by the map $\hat{\mathcal{M}}_i$ for given halfwidth of the layer $d > 0$ as

$$\hat{\mathcal{M}}_i : D_{i,d} = \{(s,u) | s \in \mathrm{dom}(\mathcal{M}_i), u \in (-d,d)\} \to \gamma_i(s) + un(s)\,.$$

Due to assumption (b) we have a critical value of $d_0 > 0$ such that every map $\hat{\mathcal{M}}_i : D_{i,d} \to \Omega_d$ from the atlas $\mathcal{A}$ is injective provided that $d < d_0$ and a diffeomorphism. Also $\hat{\mathcal{M}}_i(s_i,u_i) = \hat{\mathcal{M}}_j(s_j,u_j)$ implies $\mathcal{M}_i(s_i) = \mathcal{M}_j(s_j)$. The estimates of the metric tensor remain the same also for the case of finite surface Γ.

Similarly as for the case of the δ'-interaction supported by a closed curve we need a comparison operator. The comparison operator can be written as

$$S_\Gamma = -\Delta_\Gamma - \frac{1}{4}(k_1 - k_2)^2 = -\Delta_\Gamma + K - M^2$$

where Δ_Γ is the Laplace-Bertrami operator on the surface Γ and $k_{1,2}$ are the principal curvatures of the surface Γ. For the case of a compact surface Γ the spectrum of the operator S_Γ is purely discrete. For the case of a non-compact surface Γ with the potential vanishing at the infinity we have $\sigma_{ess}(S_\Gamma) = [0, \infty)$. Unless the infinite surface Γ is a plane, which is excluded by (a4), the discrete spectrum of S_Γ is non-empty. We denote the eigenvalues of the operator S_Γ by μ_j in ascending order with the multiplicity taken into account.

2.2.2 Behavior of Essential Spectrum

First we calculate the essential spectrum. The case of the compact surface Γ was treated in [BEL13]. For such a case the essential spectrum is $\sigma_{ess}(-\Delta_{\Gamma,\beta}) = [0,\infty)$. The excluded case of Γ being a plane can be calculated by separation of variables. For such a case we obtain $\sigma_{ess}(-\Delta_{\Gamma,\beta}) = \left[-\frac{4}{\beta^2},\infty\right)$. We show that under the assumption (a3) the essential spectrum remains the same at least in the considered limit.

Theorem 2.2.1. *Let an infinite surface Γ satisfy the assumptions (a1)-(a4). Then the essential spectrum satisfies $\sigma_{ess}(-\Delta_{\Gamma,\beta}) \subseteq [\epsilon(\beta),\infty)$, where $\epsilon(\beta) = -\frac{4}{\beta^2} + \mathcal{O}\left(\exp\left(\frac{c}{\beta}\right)\right)$ holds as $\beta \to 0_-$ for some constant $c > 0$.*

The proof of this theorem is based on the Neumann bracketing. We introduce the operator with added Neumann boundary condition at the boundary of the layer Ω_d. Such an operator can be written as a direct sum of two operators

$$-\Delta^N_{\mathbb{R}^3\setminus\Omega_d} \oplus H^{-,d}_{\beta,\Gamma} \leq -\Delta_{\Gamma,\beta}$$

where $-\Delta^N_{\mathbb{R}^3\setminus\Omega_d}$ is Neumann Laplacian on the set $\mathbb{R}^3 \setminus \Omega_d$ and the operator $H^{-,d}_{\beta,\Gamma}$ is associated with the following quadratic form

$$h^{-,d}_{\beta,\Gamma}(\psi) = \|\nabla\psi\|^2_{\Omega_d} + \beta^{-1}\|[\psi]_\Gamma\|^2_\Gamma$$

with the domain $\mathcal{D}(h^{-,d}_{\beta,\Gamma}) = H^1(\Omega_d \setminus \Gamma)$. Due to the fact, that the free Laplacian is positive, all the information about the negative spectrum is encoded in the operator $H^{-,d}_{\beta,\Gamma}$. The inclusion $\sigma_{ess}(-\Delta_{\Gamma,\beta}) \subseteq [\epsilon(\beta),\infty)$ is equivalent to

$$\inf \sigma_{ess}(-\Delta_{\Gamma,\beta}) \geq \epsilon(\beta)\,.$$

This inequality is satisfied if $\inf \sigma_{ess}(H^{-,d}_{\beta,\Gamma}) \geq \epsilon(\beta)$ for $d < d_0 < \rho$. Next we divide the surface Γ into two parts, namely $\Gamma^{int}_\tau = \{s \in \Gamma | |s| < \tau\}$ and $\Gamma^{ext}_\tau = \Gamma \setminus \overline{\Gamma^{int}_\tau}$. The corresponding layer neighborhoods of Ω_d are $D^{int}_{d,\tau} = \{(s,u) | s \in \Gamma^{int}_\tau, u \in (-d,d)\}$ and $D^{ext}_{d,\tau} = D_d \setminus \overline{D^{int}_{d,\tau}}$. The operators with the Neumann boundary conditions on the boundary of the sets $D^{int/ext}_{d,\tau}$ are denoted by $H^{-,d,int/ext}_{\beta,\Gamma,\tau}$. These operators are associated with the forms

$$(\psi, H^{-,d,int/ext}_{\beta,\Gamma,\tau}\psi)_{D^{int/ext}_{d,\tau}} = (\partial_i\psi, G^{ij}\partial_j\psi)_{D^{int/ext}_{d,\tau}} + \frac{1}{\beta}\|[\psi]_\Gamma\|^2_{\Gamma^{int/ext}_\tau}$$

with the domains $\mathcal{H}^1(D_{d,\tau}^{int/ext} \setminus \Gamma_\tau^{int/ext}, d\Omega)$, respectively. Neumann bracketing yields the following

$$H_{\beta,\Gamma}^{-,d} \geq H_{\beta,\Gamma,\tau}^{-,d,int} \oplus H_{\beta,\Gamma,\tau}^{-,d,ext} .$$

The set $\overline{D_d^{int}}$ is compact hence the spectrum of the operator $H_{\beta,\Gamma,\tau}^{-,d,int}$ is purely discrete. The min-max principle implies

$$\inf \sigma_{ess}(H_{\beta,\Gamma}^{-,d}) \geq \inf \sigma_{ess}(H_{\beta,\Gamma,\tau}^{-,d,ext}) .$$

Now it is sufficient to check that the expression $\inf \sigma_{ess}(H_{\beta,\Gamma,\tau}^{-,d,ext})$ can not be smaller then $\epsilon(\beta)$. By neglecting some positive terms we have the following

$$\begin{aligned}
(\psi, H_{\beta,\Gamma,\tau}^{-,d,ext}\psi)_{D_{d,\tau}^{ext}} &\geq \int_{D_{d,\tau}^{ext}} |\partial_3\psi|^2 d\Omega + \frac{1}{\beta}\int_{D_{d,\tau}^{ext}} |\psi(s_1,s_2,0_+) - \psi(s_1,s_2,0_-)|^2 d\Gamma \\
&\geq m_{\tau,d}^- \int_{D_{d,\tau}^{ext}} |\partial_3\psi|^2 d\Gamma du + \frac{1}{\beta}\int_{D_{d,\tau}^{ext}} |\psi(s_1,s_2,0_+) - \psi(s_1,s_2,0_-)|^2 d\Gamma \\
&\geq \frac{1}{\beta^2 m_{\tau,d}^+ m_{\tau,d}^-}\left[-4 - 16\exp\left(\frac{4d}{\beta}\right)\right] \|\psi\|_{D_{d,\tau}^{ext}}^2
\end{aligned}$$

where $m_{\tau,d}^+ = \sup_{D_{d,\tau}^{ext}} \xi$ and $m_{\tau,d}^- = \inf_{D_{d,\tau}^{ext}} \xi$. To prove the last inequality one has to use Lemma [EJ13, 3.3]. It implies, in particular,

$$\int_{-d}^{d} |\psi'(u)|^2 du + \frac{1}{\beta}|\psi(0_+) - \psi(0_-)|^2 \geq \left(-\frac{4}{\beta^2} - \frac{16}{\beta^2}\exp\left(\frac{4d}{\beta}\right)\right) \|\psi\|_{(-d,d)}^2$$

which holds for sufficiently small β satisfying $-\beta < 2d$. The fact that τ can be chosen arbitrarily large and $\lim_{\tau\to\infty} m_{\tau,d}^{\pm} = 1$ imply

$$\epsilon(\beta) \geq \left(-\frac{4}{\beta^2} - \frac{16}{\beta^2}\exp\left(\frac{4d}{\beta}\right)\right)$$

which completes the proof.

2.2.3 Asymptotics of Discrete Spectrum

In this part we present two theorems describing the spectrum of the operator $-\Delta_{\Gamma,\beta}$. The first theorem corresponds to an infinite surface and the second one to a compact surface. Both of them are proved in a similar fashion as it was done for the asymptotics of the spectrum in [EJ13, EK03].

Theorem 2.2.2. *Let the infinite surface* Γ *satisfy the assumptions (a1)-(a4). Then the operator* $-\Delta_{\Gamma,\beta}$ *has at least one isolated negative eigenvalue below the threshold of the essential spectrum for all negative* β *with* $|\beta|$ *small enough and the* j*-th eigenvalue behaves in the limit* $\beta \to 0_-$ *as*

$$\lambda_j = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta \ln |\beta|)\,.$$

The proof is based on Dirichlet and Neumann bracketing. We estimate our operator from above and below by operators with added either Dirichlet or Neumann boundary condition at the boundary of Ω_d. This can be written as

$$-\Delta^N_{\mathbb{R}^3\setminus\Omega_d} \oplus H^{-,d}_{\beta,\Gamma} \leq -\Delta_{\Gamma,\beta} \leq -\Delta^D_{\mathbb{R}^3\setminus\Omega_d} \oplus H^{+,d}_{\beta,\Gamma}$$

where $-\Delta^D_{\mathbb{R}^3\setminus\Omega_d}$ is Dirichlet Laplacian on the set $\mathbb{R}^3 \setminus \Omega_d$ and the operator $H^{+,d}_{\beta,\Gamma}$ is associated with the following quadratic form

$$h^{+,d}_{\beta,\Gamma}(\psi) = \|\nabla\psi\|^2_{\Omega_d} + \beta^{-1}\|[\psi]_\Gamma\|^2_\Gamma$$

with the domain $\mathcal{D}(h^{+,d}_{\beta,\Gamma}) = \{\psi \in H^1(\Omega_d \setminus \Gamma) | \psi(x)|_{\partial\Omega_d} = 0\}$. Due to the fact that the Dirichlet and the Neumann Laplacian are positive, all the information about the negative spectrum is encoded in the spectra of the operators $H^{\pm,d}_{\beta,\Gamma}$. The next step is to transform these operators into curvilinear coordinates. First we use unitary transformation in the form

$$U[\psi(x,y)] = \psi(s_1,s_2,u) : L^2(\Omega_d) \to L^2(D_d, d\Omega)\,.$$

The transformed operators $UH^{\pm,d}_{\beta,\Gamma}U^{-1}$ are associated with the quadratic forms

$$\tilde{h}^{\pm,d}_{\beta,\Gamma}(\psi) = h^{\pm,d}_{\beta,\Gamma}(U^{-1}\psi) = (\partial_i\psi, G^{ij}\partial_j\psi)_{D_d,d\Omega} + \beta^{-1}\|[\psi]_\Gamma\|^2_\Gamma$$

with the domains $\mathcal{D}(\tilde{h}^{-,d}_{\beta,\Gamma}) = H^1(D_d \setminus \Gamma, d\Omega)$ and $\mathcal{D}(\tilde{h}^{+,d}_{\beta,\Gamma}) = \{\psi \in H^1(D_d \setminus \Gamma, d\Omega) | \psi(x)|_{\partial\Omega_d} = 0\}$. Next we switch from the metric $d\Omega$ to $d\Gamma du$. This can be done by another unitary transformation [DEK01]

$$\tilde{U}\psi = \sqrt{\xi}\psi(s_1,s_2,u) : L^2(D_d, d\Omega) \to L^2((D_d, d\Gamma du))\,.$$

The transformed operators $F_{\beta,\Gamma}^{\pm,d} = \tilde{U}UH_{\beta,\Gamma}^{\pm,d}U^{-1}\tilde{U}^{-1}$ are associated with the quadratic forms

$$\zeta_{\beta,\Gamma}^{+,d}(\psi) = (\partial_\mu\psi, G^{\mu\nu}\partial_\nu\psi)_{D_d,d\Gamma du} + (\psi,(V_1+V_2)\psi)_{D_d,d\Gamma du} + \|\partial_3\psi\|_{D_d,d\Gamma du}$$
$$+\frac{1}{\beta}\|[\psi]_\Gamma\|_\Gamma^2 - \int_\Gamma M(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2)d\Gamma\,,$$
$$\zeta_{\beta,\Gamma}^{-,d}(\psi) = \zeta_{\beta,\Gamma}^{+,d}(\psi) + \int_\Gamma \varsigma(s,d)|\psi(s,d)|^2 d\Gamma - \int_\Gamma \varsigma(s,-d)|\psi(s,-d)|^2 d\Gamma$$

where $\varsigma(s,u) = \frac{M-Ku}{\xi}$ and curvature induced potentials are

$$V_1 = g^{-\frac{1}{2}}(g^{\frac{1}{2}}g^{\mu\nu}J_{,\mu})_{,\nu} + J_{,\mu}G^{\mu\nu}J_{,\nu}\,,$$
$$V_2 = \frac{K-M^2}{\xi^2}$$

with $J = \frac{\ln\xi}{2}$. The domains of the quadratic forms $\zeta_{\beta,\Gamma}^{\pm,d}$ are $\mathcal{D}(\zeta_{\beta,\Gamma}^{-,d}) = H^1(D_d\backslash\Gamma, d\Gamma du)$ and $\mathcal{D}(\zeta_{\beta,\Gamma}^{+,d}) = \{\psi \in H^1(D_d\backslash\Gamma, d\Gamma du) | \psi(x)|_{\partial D_d} = 0\}$. The operators $F_{\beta,\Gamma}^{\pm,d}$ are not yet suitable for our proof because they do not separate the transverse and surface variables, hence we make a slightly cruder estimates. We approximate the curvature induced potentials $V_{1,2}$ using inequalities (2.5) and (2.6) in the following way

$$\tilde{c}_- d \le V_1 \le \tilde{c}_+ d\,,$$
$$\frac{K-M^2}{C_-^2} \le V_2 \le \frac{K-M^2}{C_+^2}$$

where $\tilde{c}_\pm$ are appropriate constants, $d < d_0 < \rho$, and $C_\pm$ are the same constants as in (2.5). Now we are ready to introduce the cruder estimate operators $D_{\beta,\Gamma}^{\pm,d}$. Then the operators $F_{\beta,\Gamma}^{\pm,d}$ satisfy

$$\begin{aligned} U_{\beta,\Gamma}^{-,d}\otimes I + \int_\Gamma^\oplus T_{\beta,\Gamma}^{-,d}(s)d\Gamma = D_{\beta,\Gamma}^{-,d} \le F_{\beta,\Gamma}^{-,d} \le -\Delta_{\Gamma,\beta} \\ -\Delta_{\Gamma,\beta} \le F_{\beta,\Gamma}^{+,d} \le D_{\beta,\Gamma}^{+,d} = U_{\beta,\Gamma}^{+,d}\otimes I + \int_\Gamma^\oplus T_{\beta,\Gamma}^{+,d}(s)d\Gamma \end{aligned} \tag{2.7}$$

where the operators $T_{\beta,\Gamma}^{\pm,d}(s)$ correspond to the transversal part and the operators $U_{\beta,\Gamma}^{\pm,d}$ correspond to the motion on the surface Γ. The operators $U_{\beta,\Gamma}^{\pm,d}$ act as

$$U_{\beta,\Gamma}^{\pm,d}\psi = -C_\pm\Delta_\Gamma\psi + \frac{K-M^2}{C_\pm} + \tilde{c}_\pm d$$

with the domains $\mathcal{D}(U^{\pm,d}_{\beta,\Gamma}) = L^2(\mathbb{R}^2, d\Gamma)$. The operators $T^{\pm,d}_{\beta,\Gamma}(s)$ are explicitly

$$T^{\pm,d}_{\beta,\Gamma}(s)\psi = -\Delta\psi$$

with the domains

$$\mathcal{D}(T^{+,d}_{\beta,\Gamma}(s)) = \Big\{ f \in H^2((-d,d)\setminus\{0\}) \Big| f(d) = f(-d) = 0,$$
$$f'(0_-) = f'(0_+) = \frac{1}{\beta}(f(0_+) - f(0_-)) + M(f(0_+) + f(0_-)) \Big\}$$

and

$$\mathcal{D}(T^{-,d}_{\beta,\Gamma}(s)) = \Big\{ f \in H^2((-d,d)\setminus\{0\}) \Big| \mp \frac{\|M\|_\infty + d\|K\|_\infty}{C_-} f(\pm d) = f'(\pm d),$$
$$f'(0_-) = f'(0_+) = \frac{1}{\beta}(f(0_+) - f(0_-)) + M(f(0_+) + f(0_-)) \Big\}\,.$$

The discrete spectrum of the operators $U^{\pm,d}_{\beta,\Gamma}$ was solved in [EK03] with the following result.

Lemma 2.2.1. *The eigenvalues $\mu^\pm_j(d)$ of the operators $U^{\pm,d}_{\beta,\Gamma}$ satisfy*

$$\mu^\pm_j(d) = \mu_j + C^\pm_j d + \mathcal{O}(d^2) \quad \text{for} \quad d \to 0_+\,,$$

where μ_j is the j-th eigenvalue of the operator S_Γ and the constants $C^\pm_j$ are independent on d.

The spectrum of the operators $T^{\pm,d}_{\beta,\Gamma}(s)$ was discussed in [EJ13] and summarized in Lemmata 2.1.2 and 2.1.3. We rephrase them to obtain the symmetric estimate from above and below.

Lemma 2.2.2. *Let $\frac{d}{\beta} > 2$ and $\beta(\|M\|_\infty + d\|K\|_\infty) < 1$. Then the operators $T^{\pm,d}_{\beta,\Gamma}(s)$ have exactly one negative eigenvalue denoted by $t_\pm(d,\beta)$ respectively. For negative β with $|\beta|$ small enough these eigenvalues satisfy*

$$-\frac{4}{\beta^2} - \frac{16}{\beta^2}\exp\left(\frac{4d}{\beta}\right) \le t_-(d,\beta) \le -\frac{4}{\beta^2} \le t_+(d,\beta) \le -\frac{4}{\beta^2} + \frac{16}{\beta^2}\exp\left(\frac{4d}{\beta}\right)\,.$$

Using the previous two Lemmata we are able to write down the discrete spectrum of the operators $D^{\pm,d}_{\beta,\Gamma}$ as $t_\pm(d,\beta) + \mu^\pm_j(d)$. Choosing $d(\beta) = \beta \ln|\beta|$ we obtain the spectra of the operators $D^{\pm,d}_{\beta,\Gamma}$ explicitly as

$$t_\pm(d(\beta),\beta) + \mu^\pm_j(d(\beta)) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta\ln|\beta|)\,.$$

Using the min-max principle in combination with the inequalities (2.7) completes the proof of Theorem 2.2.2.

The next theorem describes the discrete spectrum for the situation of a compact surface.

Theorem 2.2.3. *Let the compact surface Γ satisfy the assumption (b). Then the operator $-\Delta_{\Gamma,\beta}$ has at least one isolated negative eigenvalue below the threshold of the essential spectrum for all $\beta < 0$ and the j-th eigenvalue behaves in the limit $\beta \to 0_-$ as*

$$\lambda_j = -\frac{4}{\beta^2} + \mu_j + (\beta \ln|\beta|)\,.$$

The existence of the negative eigenvalue for all negative β can be done by a variational argument as in [BEL13]. We take a test function ξ_Γ in the form of the characteristic function of the volume enclosed in Γ. This gives us an upper estimate of the ground state energy in the following form

$$\lambda_0 \leq \frac{h^\Gamma_\beta(\xi)}{\|\xi_\Gamma\|^2} = \frac{S}{\beta V} < 0$$

where S is the area of the surface Γ, V is the enclosed volume by the surface Γ and β is negative. The eigenvalue asymptotics is obtained in the same fashion as for the infinite surface with only minor changes so we omit the details.

2.3 Spectral asymptotics for δ' interaction supported by a infinite curve

In this section we describe the spectral asymptotics of the essential and discrete spectrum for δ' interaction supported by an infinite curve. We study the strong coupling regime, where $\beta \to 0_-$. We show that for the situation of asymptotically straight curves the essential spectrum remains at least asymptotically the same as for the case of the straight line. The first two terms of the asymptotic expansion of the discrete spectrum in the strong coupling limit are the following, the first one corresponds to δ'-interaction on the line and the other one corresponds to the longitudinal variable around the curve.

2.3.1 Formulation of Problem

The operator, which we are working with is associated with the quadratic form (1.5), i.e.

$$h^{\Gamma}_{\beta}(\psi) = \|\nabla\psi\|^2_{\mathbb{R}^3} + \beta^{-1}\|[\psi]_{\Gamma}\|^2_{\Gamma}\,.$$

The proof of the theorems requires to impose several additional conditions onto the curve Γ:

(c1) Γ is C^4 smooth,

(c2) Γ can be enclosed in a symmetric strip neighborhood of finite width with no self-intersections,

(c3) Γ is asymptotically straight, i.e. curvature of the curve vanishes as the distance from one fixed point tends to the infinity and

(c4) Γ is not a straight line.

The last assumption excludes the trivial case of a straight line because its spectral problem can be solved exactly by separation of variables. We parametrize the curve Γ by its arc length

$$\Gamma : \mathbb{R} \to \mathbb{R}^2 : s \to (\Gamma_1(s), \Gamma_2(s))\,.$$

We also need coordinates in the strip neighborhood $\Omega_d = \{x \in \mathbb{R}^2 | \mathrm{dist}(x, \Gamma) < d\}$. We introduce curvelinear coordinates around the curve Γ as

$$(x, y) = (\Gamma_1(s) + u\Gamma_2'(s), \Gamma_2(s) - u\Gamma_1'(s))\,.$$

Due to the conditions (c1) and (c2) there exists such a positive constant $\rho_0 > 0$ that the coordinates (s, u) are injective in the strip neighborhood of thickness $d < \rho_0$.

For the statement about the discrete spectrum we need an auxiliary operator S_Γ which acts as

$$S_\Gamma\psi = -\frac{d^2\psi}{ds^2} - \frac{\gamma(s)^2}{4}\psi$$

with the domain $\mathcal{D}(S_\Gamma) = H^2(\mathbb{R})$ where $\gamma(s) = (\Gamma_1''\Gamma_2' - \Gamma_1'\Gamma_2'')(s)$ is a signed curvature. We denote the eigenvalues of the operator S_Γ by μ_j with the multiplicity taken into the account. The main results of [J15] are presented with sketches of the proofs in the following subsections. For complete proofs we refer the reader to the mentioned paper.

2.3.2 Behavior of Essential Spectrum

We start with the behavior of the essential spectrum.

Theorem 2.3.1. *Let an infinite curve* Γ *satisfy conditions (c1)-(c3), then*

$$\sigma_{ess}(-\Delta_{\Gamma,\beta}) = [\epsilon(\beta), \infty)$$

where $\epsilon(\beta) \to -\frac{4}{\beta^2}$ *holds for* $\beta \to 0_-$.

The trivial case of a straight line can be solved by separation of variables. We obtain that $\sigma_{ess}(-\Delta_{\Gamma,\beta}) = \left[-\frac{4}{\beta^2}, \infty\right)$.
The proof for the non-trivial case is based on Neumann bracketing estimates of the operator $-\Delta_{\Gamma,\beta}$. The inclusion $\sigma_{ess}(-\Delta_{\Gamma,\beta}) = [\epsilon(\beta), \infty)$ can be rewritten as

$$\inf \sigma_{ess}(-\Delta_{\Gamma,\beta}) \geq \epsilon(\beta)\,.$$

We introduce the operator with added Neumann boundary condition at the boundary of Ω_d where $d \leq \rho$. The new operator satisfies

$$-\Delta_{\Gamma,\beta} \geq -\Delta^N_{\mathbb{R}^2\setminus\Omega_d} \oplus H^{N,d}_{\beta,\Gamma}$$

where $-\Delta^N_{\mathbb{R}^2\setminus\Omega_d}$ is the Neumann Laplacian on the set $\mathbb{R}^2\setminus\Omega_d$ and the operator $H^{N,d}_{\beta,\Gamma}$ is associated with the form

$$h^{N,d}_{\beta,\Gamma}(\psi) = \|\nabla\psi\|^2_{\Omega_d} + \beta^{-1}\|[\psi]_\Gamma\|^2_\Gamma$$

with the domain $\mathcal{D}(h^{N,d}_{\beta,\Gamma}) = H^1(\Omega_d\setminus\Gamma)$. The Neumann Laplacian is positive and as a result all the information about the negative spectrum is encoded in the operator $H^{N,d}_{\beta,\Gamma}$. Using the previous inequality it is sufficient to check $\inf \sigma_{ess}(H^{N,d}_{\beta,\Gamma}) \geq \epsilon(\beta)$. We rewrite the quadratic form $h^{N,d}_{\beta,\Gamma}$ to the unitarily equivalent form in the curvilinear coordinates which can be done in the same way as it was done in [EJ13]

Lemma 2.3.1. *The quadratic form* $h^{N,d}_{\beta,\Gamma}$ *is unitarily equivalent to the quadratic form*

$$q^{N,d}_{\beta,\Gamma}(\psi) = \left\|\frac{\partial_s\psi}{g}\right\|^2_{\Omega_d} + \|\partial_u\psi\|^2_{\Omega_d} + (\psi, V\psi)_{\Omega_d}$$
$$+\beta^{-1}\int_{\mathbb{R}} |\psi(s,0_+) - \psi(s,0_-)|^2 ds + \frac{1}{2}\int_{\mathbb{R}} \gamma(s)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2) ds$$
$$-\int_{\mathbb{R}} \frac{\gamma(s)}{2(1+d\gamma(s))}|\psi(s,d)|^2 ds + \int_{\mathbb{R}} \frac{\gamma(s)}{2(1-d\gamma(s))}|\psi(s,-d)|^2 ds$$

where the geometrically induced potential is

$$V(s,u) = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$$

and $g(s,u) = 1 + u\gamma(s)$ *with the domain* $\mathcal{D}(q_{\beta,\Gamma}^{-,d}) = H^1(\mathbb{R} \times ((-d,d) \setminus \{0\}))$.

Next we divide the strip neighborhood into two parts. We take the curve Γ and split it into two parts. We define them as $\Gamma_\tau^{int} = \{\Gamma(s) | S \leq \tau\}$ and $\Gamma_\tau^{ext} = \Gamma \setminus \overline{\Gamma_\tau^{int}}$. The corresponding strip neighborhoods are $\Omega_{d,\tau}^{int} = \{x(s,u) \in \Omega_d | s < \tau\}$ and $\Omega_{d,\tau}^{ext} = \Omega_d \setminus \overline{\Omega_{d,\tau}^{int}}$. We introduce the operators $Q_{\beta,\Gamma}^{-,d,int/ext}$ with added Neumann boundary condition at the boundary of $\Omega_{d,\tau}^{int}$ and $\Omega_{d,\tau}^{ext}$. These operators are associated with the following forms

$$\begin{aligned} q_{\beta,\Gamma,\tau}^{N,d,int/ext}(\psi) = \left\| \frac{\partial_s \psi}{g} \right\|^2_{\Omega_{d,\tau}^{int/ext}} + \|\partial_u \psi\|^2_{\Omega_{d,\tau}^{int/ext}} + (\psi, V\psi)_{\Omega_{d,\tau}^{int/ext}} \\ + \beta^{-1} \int_{\Gamma_\tau^{int/ext}} |\psi(s,0_+) - \psi(s,0_-)|^2 ds \\ + \frac{1}{2} \int_{\Gamma_\tau^{int/ext}} \gamma(s)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2) ds \\ - \int_{\Gamma_\tau^{int/ext}} \frac{\gamma(s)}{2(1+d\gamma(s))} |\psi(s,d)|^2 ds + \int_{\Gamma_\tau^{int/ext}} \frac{\gamma(s)}{2(1-d\gamma(s))} |\psi(s,-d)|^2 ds \end{aligned}$$

with the domains $\mathcal{D}(q_{\beta,\Gamma,\tau}^{N,d,int/ext}) = H^1(\Omega_{d,\tau}^{int/ext})$ respectively. Neumann bracketing implies

$$H_{\beta,\Gamma}^{-,d} \geq Q_{\beta,\Gamma,\tau}^{N,d,int} \oplus Q_{\beta,\Gamma,\tau}^{N,d,ext}.$$

The spectrum of the operator $Q_{\beta,\Gamma,\tau}^{N,d,int}$ is purely discrete. The min-max principle implies that

$$\inf \sigma_{ess}(H_{\beta,\Gamma}^{N,d}) \geq \inf \sigma_{ess}(Q_{\beta,\Gamma,\tau}^{N,d,ext}).$$

We denote the infimum of the potential as $V_{\tau,d} = \inf_{|s|>\tau, u\in(-d,d)} V(s,u)$. The assumption (c3) implies that

$$\lim_{\tau\to\infty} V_{\tau,d} = 0.$$

Now using the estimates derived in [EJ14] we can write

$$
\begin{aligned}
q_{\beta,\Gamma,\tau}^{N,d,ext}(\psi) &\leq \|\partial_u\psi\|^2_{\Omega^{ext}_{d,\tau}} + V_{\tau,d}\,\|\psi\|^2_{\Omega^{ext}_{d,\tau}} \\
+\beta^{-1}\int_{\Gamma^{ext}_\tau}|\psi(s,0_+)-\psi(s,0_-)|^2 ds &+ \frac{1}{2}\int_{\Gamma^{ext}_\tau}\gamma(s)(|\psi(s,0_+)|^2-|\psi(s,0_-)|^2)ds \\
-\int_{\Gamma^{ext}_\tau}\frac{\gamma(s)}{2(1+d\gamma(s))}|\psi(s,d)|^2 ds &+ \int_{\Gamma^{ext}_\tau}\frac{\gamma(s)}{2(1-d\gamma(s))}|\psi(s,-d)|^2 ds \\
&\leq \left(V_{\tau,d}-\frac{4}{\beta^2}-\frac{16}{\beta^2}\exp\left(\frac{4d}{\beta}\right)\right)\|\psi\|^2_{\Omega^{ext}_{d,\tau}}
\end{aligned}
$$

where we omitted some positive terms in the first inequality. The fact that we can choose τ arbitrarily large completes the proof.

2.3.3 Asymptotics of Discrete Spectrum

In this subsection we derive the behavior of the discrete spectrum for the strong coupling limit.

Theorem 2.3.2. *Let an infinite curve Γ satisfy assumptions (c1)-(c4), then the operator $-\Delta_{\Gamma,\beta}$ has at least one isolated eigenvalue below the threshold of the essential spectrum for all negative β with $|\beta|$ small enough. The j-th eigenvalue behaves in the strong coupling regime $\beta \to 0_-$ as*

$$
\lambda_j = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta\ln|\beta|)\,.
$$

The proof of this theorem is based on Dirichlet and Neumann bracketing. We estimate our operator from above and from below in the following way

$$
-\Delta^N_{\mathbb{R}^2\setminus\Omega_d} \oplus H^{N,d}_{\beta,\Gamma} \leq -\Delta_{\Gamma,\beta} \leq \oplus - \Delta^D_{\mathbb{R}^2\setminus\Omega_d} \oplus H^{D,d}_{\beta,\Gamma}
$$

where $-\Delta^D_{\mathbb{R}^2\setminus\Omega_d}$ is Dirichlet Laplacian on the set $\mathbb{R}^2\setminus\Omega_d$ and the operator $H^{D,d}_{\beta,\Gamma}$ is associated with the quadratic form

$$
h^{D,d}_{\beta,\Gamma}(\psi) = \|\nabla\psi\|^2_{\Omega_d} + \beta^{-1}\|[\psi]_\Gamma\|^2_\Gamma
$$

with the domain $\mathcal{D}(h^{D,d}_{\beta,\Gamma}) = \{\psi \in H^1(\Omega_d\setminus\Gamma) | \psi(x)|_{\partial\Omega_d} = 0\}$. All the information about the negative spectrum is encoded in the operators $H^{D/N,d}_{\beta,\Gamma}$ because Neumann and Dirichlet Laplacian are positive. The form $h^{D,d}_{\beta,\Gamma}$ can be rewritten in the curvilinear coordinates similarly as the form $h^{N,d}_{\beta,\Gamma}$.

Lemma 2.3.2. *The quadratic form $h^{D,d}_{\beta,\Gamma}$ is unitarily equivalent to the quadratic form*

$$q^{D,d}_{\beta,\Gamma}(\psi) = \left\| \frac{\partial_s \psi}{g} \right\|^2_{\Omega_d} + \|\partial_u \psi\|^2_{\Omega_d} + (\psi, V\psi)_{\Omega_d}$$
$$+\beta^{-1} \int_{\mathbb{R}} |\psi(s,0_+) - \psi(s,0_-)|^2 ds + \frac{1}{2} \int_{\mathbb{R}} \gamma(s)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2) ds$$

where $\Omega_d = \mathbb{R} \times \{(-d,0) \cup (0,d)\}$, the geometrically induced potential is

$$V(s,u) = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$$

and $g(s,u) = 1 + u\gamma(s)$ with the domain $\mathcal{D}(q^{-,d}_{\beta,\Gamma}) = H^1(\Omega_d \setminus \Gamma)$.

We also need cruder estimates. For this purpose we introduce the operators $Q^{\pm,d}_{\beta,\Gamma}$ which are associated with the forms $q^{\pm,d}_{\beta,\Gamma}$ and satisfy

$$Q^{-,d}_{\beta,\Gamma} \leq Q^{N,d}_{\beta,\Gamma} \leq -\Delta_{\Gamma,\beta} \leq Q^{D,d}_{\beta,\Gamma} \leq Q^{+,d}_{\beta,\Gamma}\,.$$

These forms can be written as follows

$$q^{+,d}_{\beta,\Gamma}(\psi) = \frac{\|\partial_s \psi\|^2_{\Omega_d}}{(1-d\gamma_+)^2} + \|\partial_u \psi\|^2_{\Omega_d} + (\psi, V^+\psi)_{\Omega_d}$$
$$+\beta^{-1} \int_0^L |\psi(s,0_+) - \psi(s,0_-)|^2 ds + \frac{1}{2} \int_0^L \gamma(d)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2) ds\,,$$
$$\mathcal{D}(q^{+,d}_{\beta,\Gamma}) = \mathcal{D}(q^{D,d}_{\beta,\Gamma})$$

where the estimating potential is $V^+(s) = \frac{d(\gamma'')_+}{2(1-d\gamma_+)^3} - \frac{\gamma^2}{4(1+d\gamma_+)^2}$ with $f_+ = \max_x |f(x)|$ and

$$q^{-,d}_{\beta,\Gamma}(\psi) = \frac{\|\partial_s \psi\|^2_{\Omega_d}}{(1+d\gamma_+)^2} + \|\partial_u \psi\|^2_{\Omega_d} + (\psi, V^-\psi)_{\Omega_d}$$
$$+\beta^{-1} \int_0^L |\psi(s,0_+) - \psi(s,0_-)|^2 ds - \frac{1}{2} \int_0^L \gamma(s)(|\psi(s,0_+)|^2 - |\psi(s,0_-)|^2) ds$$
$$-\gamma_+ \int_0^L (|\psi(s,d)|^2 + |\psi(s,-d)|^2) ds\,,$$
$$\mathcal{D}(q^{-,d}_{\beta,\Gamma}) = H^1(\Omega_d \setminus \Gamma)$$

where $V^-(s) = -\frac{d(\gamma'')_+}{2(1-d\gamma_+)^3} - \frac{5(d(\gamma')_+)^2}{4(1-d\gamma_+)^4} - \frac{\gamma^2}{4(1-d\gamma_+)^2}$. The operators $Q^{\pm,d}_{\beta,\Gamma}$ can be rewritten as the sums of the operators in the following way

$$
\begin{aligned}
Q^{+,d}_{\beta,\Gamma} &= U^{\Gamma,d}_+ \otimes I + \int_{\mathbb{R}}^{\oplus} T^{\Gamma,d}_{\beta,+}(s)ds\,, \\
Q^{-,d}_{\beta,\Gamma} &= U^{\Gamma,d}_- \otimes I + \int_{\mathbb{R}}^{\oplus} T^{\Gamma,d}_{\beta,-}(s)ds
\end{aligned}
\tag{2.8}
$$

where the operators corresponding to the transversal part are

$$
\begin{aligned}
T^{\Gamma,d}_{\beta,+}(s)\psi(u) &= -\psi''(u)\,, \\
\mathcal{D}(T^{\Gamma,d}_{\beta,+}(s)) = \Big\{\psi \in H^2((-d,d)\setminus\{0\})\Big|\psi(d) &= \psi(-d) = 0, \\
\psi'(0_-) = \psi'(0_+) = \beta^{-1}(\psi(0_+) - \beta(0_-)) + \frac{\gamma(s)}{2}&(\psi(0_+) + \beta(0_-))\Big\}\,, \\
T^{\Gamma,a}_{\beta,-}(s)\psi(u) &= -\psi''(u)\,, \\
\mathcal{D}(T^{\Gamma,d}_{\beta,-}(s)) = \Big\{\psi \in H^2((-d,d)\setminus\{0\})\Big| \mp\gamma_+\psi(\pm d) &= \psi'(\pm d), \\
\psi'(0_-) = \psi'(0_+) = \beta^{-1}(\psi(0_+) - \beta(0_-)) + \frac{\gamma(s)}{2}&(\psi(0_+) + \beta(0_-))\Big\}
\end{aligned}
$$

and the operators describing the longitudinal part are

$$
\begin{aligned}
U^{\Gamma,d}_+\psi(s) &= -\frac{\psi''(s)}{1-d\gamma_+} + V^+(s)\psi(s)\,, \\
\mathcal{D}(U^{\Gamma,d}_+) &= H^2(\mathbb{R})\,, \\
U^{\Gamma,d}_-\psi(s) &= -\frac{\psi''(s)}{1+d\gamma_+} + V^-(s)\psi(s)\,, \\
\mathcal{D}(U^{\Gamma,d}_-) &= H^2(\mathbb{R})\,.
\end{aligned}
$$

The spectrum of the operators $T^{\Gamma,d}_{\beta,\pm}(s)$ was already described in Lemma 2.2.2. The spectrum of the operators $U^{\Gamma,d}_\pm$ can be derived step by step as done in [EJ13].

Lemma 2.3.3. *Let $0 < d < \frac{1}{2\gamma_+}$ then there is a positive constant C independent of d and j such that*

$$|\mu_j^\pm(d) - \mu_j| \le Cdj^2$$

holds for all $j \in N$ where $\mu_j^\pm(d)$ are the eigenvalues of $U^{\Gamma,d}_\pm$, respectively, with the multiplicity taken into account.

Now we put $d(\beta) = \frac{3}{4}\beta \ln(|\beta|)$. Using the explicit form of the operators $Q_{\beta,\Gamma}^{\pm,d(\beta)}$ and previous Lemmata we obtain the discrete spectrum of the operators $Q_{\beta,\Gamma}^{\pm,d(\beta)}$ in the following form

$$\lambda_{q,\beta}^{\pm} = t_{\pm}(d(\beta),\beta) + \mu_j^{\pm}(d(\beta)) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta \ln(|\beta|))\,.$$

The proof is completed by using the inequality (2.8) and the min-max principle.

2.4 On absence of bound states for weakly attractive δ'-interactions supported on non-closed curves in $\mathbb{R}^2$

In this section we study δ'-interaction supported by either a bound curve with two free endpoints or an unbounded one with one free endpoint. We give the description of the essential spectrum. We show that for a weak coupling, i.e. $\beta \to -\infty$, there is no negative discrete eigenvalues below the threshold of the essential spectrum. We also derive explicit sufficient condition on β for the absence of the negative spectrum. The presented approach is applicable also for a manifolds in higher dimensions. For the details of the reasoning in this section including proofs we refer the reader to [JL16].

2.4.1 Formulation of Problem

We consider a piecewise-C^1 curve Γ defined by the following injective mapping γ

$$\gamma(s) : I \to \mathbb{R}^2\,, \quad \gamma(s) = (\gamma_1(s), \gamma_2(s))\,, \quad I = (0, L)$$

where $L \in (0,\infty]$ and γ_i are piecewise-C^1. Moreover if $|\gamma'(s)| = 1$ for almost all s, we say that γ is a natural parametrization of the curve Γ.

The operator $-\Delta_{\Gamma,\beta}$, which we study, is associated with the quadratic form (1.6), i.e.

$$h_\beta^\Gamma(\psi) = \|\nabla\psi\|_{\mathbb{R}^2}^2 + (\beta^{-1}[\psi]_\Gamma, [\psi]_\Gamma)_\Gamma\,.$$

where $\beta^{-1} \in L^\infty(\Gamma)$ and $[\psi]_\Gamma = \psi|_{\Gamma+} - \psi|_{\Gamma-}$ with $\psi|_{\Gamma+}$ and $\psi|_{\Gamma-}$ denoting traces of the functions at the boundary of Γ.

We also need to introduce the concept of monotone curves. We say that the curve is monotone if it can be parametrized in the following way

$$\gamma(r) = x_0 + (r\cos(\phi(r)), r\sin(\phi(r)))\,, \quad x_0 \in \mathbb{R}^2\,, \quad r \in (0,R)\,, \quad R \in (0,\infty]$$

where $\phi(r)$ is a piecewise-C^1 mapping $\phi(r) : (0,R) \to \mathbb{R}$. We can say that these curves go away from the starting point.

2.4.2 Essential Spectrum

We define the quasi-conical domain in the standard way [G66]. We say that a domain $\Omega \subset \mathbb{R}^m$ is quasi-conical if for any $n \in \mathbb{N}$ there exists $x_n \in \mathbb{R}^m$ such that the disc $D_n(x_n)$ of radius n centered around x_n is subset of Ω.

Theorem 2.4.1. *Let the curve Γ be a non-closed and piecewise-C^1 and let $\mathbb{R}^2 \setminus \Gamma$ be quasi-conical. Then the spectrum of the operator $-\Delta_{\Gamma,\beta}$ satisfies*

$$\sigma_{\text{ess}}(-\Delta_{\Gamma,\beta}) \supseteq [0,\infty)\,.$$

Theorem 2.4.2. *Let the non-closed curve Γ be bounded. Then the spectrum of the operator $-\Delta_{\Gamma,\beta}$ satisfies*

$$\sigma_{\text{ess}}(-\Delta_{\Gamma,\beta}) = [0,\infty)\,.$$

Proof of the first theorem can be done in the standard way using well chosen series of the test functions in the same way as in [JL16]. The second theorem is proved with the help of bracketing.

2.4.3 Positivity of Operator for Monotone Curves

We derive the sufficient condition under which the operator $-\Delta_{\Gamma,\beta}$ is positive. The complete proof is presented in [JL16].

Theorem 2.4.3. *Let a curve $\Gamma \subset \mathbb{R}^2$ be monotone piecewise-C^1. Then*

$$\sigma(-\Delta_{\Gamma,\beta}) \subseteq \mathbb{R}^+ \quad \text{for} \quad \beta(r) \le -2\pi r\sqrt{1 + (r\phi'(r))^2}\,, \quad r \in (0,R)\,. \tag{2.9}$$

Furthermore if $\mathbb{R}^2 \setminus \Gamma$ is quasi-conical, then $\sigma(-\Delta_{\Gamma,\beta}) = \mathbb{R}^+$.

For the proof of this theorem we need to introduce the operator $T_{d,\beta}$ describing one δ'-interaction with coupling constant β on a loop of length d. This operator acts as free Laplacian on the interval

$$T_{d,\beta} = -\Delta$$

with the domain $\mathcal{D}(T_{d,\beta}) = \{\psi \in H^2((0,d)) | \psi'(0_+) = \psi'(d_-), \beta\psi'(0_+) = \psi(0_+) - \psi(d_-)\}$. The quadratic form corresponding to the operator $T_{d,\beta}$ is

$$t_{d,\beta}(\psi) = (\nabla\psi, \nabla, \psi) + \frac{1}{\beta}|\psi(0_+) - \psi(d_-)|^2$$

with the domain $\mathcal{D}(t_{d,\beta}) = H^1((0,d))$. The next lemma is proved in [JL16] and it gives the condition for the positivity of $T_{d,\beta}$.

Lemma 2.4.1. *Let β satisfy $-1 \leq \frac{d}{\beta}$. Then the operator $T_{d,\beta}$ and the quadratic form $t_{d,\beta}$ are non-negative.*

With this result in mind we start rewriting and estimating the quadratic form h_β^Γ. We show that the form h_β^Γ is positive for any function from its core for $\beta \leq -2\pi r\sqrt{1 + (r\phi'(r))^2}$. The gradient can be expressed in polar coordinates as

$$|(\nabla u)(x,y)|^2 = |(\partial_r u)(r,\phi)|^2 + \frac{1}{r^2}|(\partial_\phi u)(r,\phi)|^2 \,.$$

The part of the form h_β^Γ corresponding to the gradient can be estimated in the following way

$$\begin{aligned} \|(\nabla u)(x,y)\|^2_{\mathbb{R}^2} &= \int_{\mathbb{R}^+}\int_{(0,2\pi)} |(\nabla u)(r,\phi)|^2 r\, dr\, d\phi \\ &\geq \int_{(0,R)} \frac{1}{r}\left(\int_{(0,2\pi)} |(\partial_\phi u)(r,\phi)|^2\, d\phi\right) dr \end{aligned}$$

where we omitted positive term $|(\partial_r u)(r,\phi)|^2$ and R is the same as in (2.9). Next using the estimate on β we rewrite the second term in the form h_β^Γ as

$$\begin{aligned} (\beta^{-1}[u]_\Gamma, [u]_\Gamma)_\Gamma = \int_\Gamma \beta^{-1}(s)|[u]_\Gamma|^2 ds = \\ \int_{(0,R)} \beta^{-1}(r)|u_+(r,\phi(r)) - u_-(r,\phi(r))|^2\sqrt{1 + (1 + r\phi'(r))^2} dr \\ \leq -\int_{(0,R)} \frac{1}{2\pi r}|u_+(r,\phi(r)) - u_-(r,\phi(r))|^2 dr \,. \end{aligned}$$

We introduce the shorthand

$$S(r) = \int_{(0,2\pi)} |(\partial_\phi u)(r,\phi)|^2 d\phi - \frac{1}{2\pi}|u_+(r,\phi(r)) - u_-(r,\phi(r))|^2$$

where $r \in (0,R)$. Due to the choice of u from the core of the form h_β^Γ we are able to identify the function u for fixed r with the piecewise-C^1 function ψ_r on the interval $(0,2\pi)$. Using this identity we obtain the relation $S(r) = t_{2\pi,-2\pi}(\psi_r)$. Using Lemma 2.4.1 we obtain that $S(r) \geq 0$ for all $r \in (0,R)$. Finally we arrive at

$$h_\beta^\Gamma(u) \geq \int_{(0,R)} \frac{S(r)}{r} dr \geq 0$$

which completes the proof.

A direct consequence of the previous theorem for a constant coupling parameter β is the following

Corollary 2.4.1. *Let a monotone bounded piecewise-C^1 curve $\Gamma \subset \mathbb{R}^2$ and $\beta^*(\Gamma) = \inf_{r\in(0,R)} -2\pi r\sqrt{1+(r\phi'(r))^2}$. Then*

$$\sigma(-\Delta_{\Gamma,\beta}) = \mathbb{R}^+ \quad \text{for} \quad \beta \leq \beta^*(\Gamma)\,.$$

2.4.4 Positivity of Operator-Generalization

Theorem 2.4.3 can be extended to curves, which can be obtain from monotone curves via linear fractional transformations introduced in Section 1.3.

Theorem 2.4.4. *Let $\Gamma \subset \mathbb{R}^2$ be a bounded piecewise-C^1 curve. Suppose that there exists a linear fractional transformation M such that $M(\infty), M^{-1}(\infty) \notin \Gamma$ and curve $M^{-1}(\Gamma)$ is monotone. Then*

$$\sigma(-\Delta_{\Gamma,\beta}) = [0,\infty)$$

for all $\beta \leq \beta^*(\Gamma)\sup_{z\in\Gamma}\sqrt{J_M(z)}$.

The proof is based on the transformation of the form h_β^Γ via a linear fractional transformation and careful analysis of the result. Next two Lemmata were proved in [JL16]. They describe transformation of the form corresponding to the operator $-\Delta_{\Gamma,\beta}$.

Lemma 2.4.2. *Let $\Lambda \subset \mathbb{R}^2$ be a bounded Lipschitz curve, let the set Ω satisfy $\Lambda \subset \Omega \subset \mathbb{R}^2$ and let M be a conformal mapping with $\mathcal{D}(M) = \Omega$. Furthermore let the conformal mapping M satisfy $M^{-1}(\infty) = \emptyset$ or $M^{-1}(\infty) \notin \Lambda$. Then for any $u \in H^1(\Omega \setminus \Lambda)$*

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla v|^2 dx$$

holds with $v = u \circ M$.

Lemma 2.4.3. *Let $\Lambda \subset \Omega \subset \mathbb{R}^2$ be a bounded Lipschitz curve parametrized by the mapping $\lambda : (0, L) \to \mathbb{R}^2$ and let M be a conformal mapping with $\mathcal{D}(M) = \Omega$. Furthermore let the conformal mapping M satisfy $M^{-1}(\infty) = \emptyset$ or $M^{-1}(\infty) \notin \Lambda$ and also $\beta^{-1} \in L^\infty(\Lambda, \mathbb{R})$. Then for any $u \in H^1(\Omega \setminus \Lambda)$*

$$(\beta^{-1}[u]_\Lambda, [u]_\Lambda)_\Lambda = (\tilde{\beta}^{-1}[v]_\Gamma, [v]_\Gamma)_\Gamma$$

holds with $v = u \circ M$, $\Gamma = M^{-1}(\Lambda)$, $\gamma = M^{-1} \circ \lambda$ and $\tilde{\beta}^{-1}(\gamma(s)) = \beta^{-1}(\lambda(s))\sqrt{J_M(\gamma(s))}$, $s \in (0, L)$.

Using Lemmata 2.4.2 and 2.4.3 we obtain

$$\begin{aligned} h^\Gamma_\beta(\psi) &= \|\nabla u\|^2_{\mathbb{R}^2} + (\beta^{-1}[u]_\Lambda, [u]_\Lambda)_\Lambda = \|\nabla v\|^2_{\mathbb{R}^2} + (\beta^{-1}[u]_\Gamma, [u]_\Gamma)_\Gamma \\ &= \|\nabla v\|^2_{D_R} + (\beta^{-1}[u]_\Gamma, [u]_\Gamma)_\Gamma + \|\nabla v\|^2_{\mathbb{R}^2 \setminus D_R} \geq \|\nabla v\|^2_{D_R} + (\beta^{-1}[u]_\Gamma, [u]_\Gamma)_\Gamma \end{aligned}$$

where D_R is a disc with the curve Γ starting at the center, ending at the boundary and $v = u \circ M$. Using the same procedure as in the last part of the proof for Theorem 2.4.3 gives us the desired result.

2.4.5 Examples

Let us present two examples given in [JL16]. The first one is an δ'-interaction supported by a line segment and the second one by a circle arc.

Corollary 2.4.2. *Let a curve Γ be a line segment of length L. Then the operator $-\Delta_{\Gamma,\beta}$ has no negative eigenvalues if $\beta \leq -\pi L$. Furthermore if $-\frac{2L}{\pi} \leq \beta \leq 0$ then $\sigma_d(-\Delta_{\Gamma,\beta}) \cap \mathbb{R}^- \neq \emptyset$.*

The absence of the negative spectrum for $\beta \leq -2\pi L$ is the direct result of Theorem 2.4.3. This estimate can be improved in the following way. We write the curve Γ as $\Gamma = \{(l, 0) | l \in (0, L)\}$. We take the operator $-\Delta_{\Gamma,\beta}$ and add

Neumann boundary condition at the points on the circles centered around the endpoints of the curve Γ with the diameter of $\frac{L}{2}$. Adding Neumann boundary condition splits the operator into 3 parts

$$-\Delta_{\Gamma,\beta}^{N,D_{\frac{L}{2}}((0,0))} \oplus -\Delta^N_{\mathbb{R}^2\setminus[D_{\frac{L}{2}}((0,0))\cup D_{\frac{L}{2}}((L,0))]} \oplus -\Delta_{\Gamma,\beta}^{N,D_{\frac{L}{2}}((L,0))} .$$

The Neumann Laplacian $-\Delta^N_{\mathbb{R}^2\setminus[D_{\frac{L}{2}}((0,0))\cup D_{\frac{L}{2}}((L,0))]}$ is positive by definition. It can be shown that the remaining two parts are positive in the same way as it was done in the proof of Theorem 2.4.3. Using the Neumann bracketing we obtain the desired result

$$-\Delta_{\Gamma,\beta}^{N,D_{\frac{L}{2}}((0,0))} \oplus -\Delta^N_{\mathbb{R}^2\setminus[(D_{\frac{L}{2}}((0,0))\cup D_{\frac{L}{2}}((L,0))]} \oplus -\Delta_{\Gamma,\beta}^{N,D_{\frac{L}{2}}((L,0))} \leq -\Delta_{\Gamma,\beta} .$$

The existence of the negative spectrum can be done with help of Dirichlet bracketing. We add Dirichlet boundary conditions at two lines $L_1 = \{(0,x)|x\in\mathbb{R}\}$ and $L_2 = \{(L,x)|x\in\mathbb{R}\}$. These lines split $\mathbb{R}^2$ into three sets

$$\Omega_1 = \mathbb{R}^- \times \mathbb{R}\,, \quad \Omega_2 = (0,L)\times\mathbb{R}\,, \quad \Omega_3 = (L,\infty)\times\mathbb{R}\,.$$

The operator with added Dirichlet boundary can be written as an orthogonal sum of three operators

$$-\Delta^D_{\Omega_1} \oplus -\Delta^{D,\Omega_2}_{\Gamma,\beta} \oplus -\Delta^D_{\Omega_3} .$$

Dirichlet Laplacians $-\Delta^D_{\Omega_1}$ and $-\Delta^D_{\Omega_3}$ are positive by definition and its spectra are $\sigma(-\Delta^D_{\Omega_1}) = \sigma(-\Delta^D_{\Omega_3}) = [0,\infty)$. We show that the operator $-\Delta^{D,\Omega_2}_{\Gamma,\beta}$ has at least one negative eigenvalue. We solve the spectral problem for the operator $-\Delta^{D,\Omega_2}_{\Gamma,\beta}$ by separation of variables. It consists of two one-dimensional operators. One is the Dirichlet Laplacian on the line segment of the length L and the second one is a particle on the line with added δ'-interaction at the point 0. In particular we obtain the lowest eigenvalue in the following form

$$\lambda_1(-\Delta^{D,\Omega_2}_{\Gamma,\beta}) = \frac{\pi^2}{L^2} - \frac{4}{\beta^2} .$$

The eigenvalue is negative for sufficiently small negative β, explicitly $-\frac{2L}{\pi} < \beta < 0$. Dirichlet bracketing completes the proof

$$-\Delta_{\Gamma,\beta} \leq -\Delta^D_{\Omega_1} \oplus -\Delta^{D,\Omega_2}_{\Gamma,\beta} \oplus -\Delta^D_{\Omega_3} .$$

The next example illustrates the use of the conformal maps for the situation of δ'-interaction supported by a circle arc.

Corollary 2.4.3. *Let a curve* Γ *be parametrized as*

$$\Gamma_\epsilon\{(R\cos\phi, R(1-\sin\phi))|\phi\in(\epsilon, 2\pi-\epsilon)\}\,, \quad R\in\mathbb{R}^+.$$

Then the operator $-\Delta_{\Gamma_\epsilon,\beta}$ *is positive if* $\beta \le -\frac{8\pi R}{\tan\frac{\epsilon}{2}}$.

The proof of this corollary is based on Theorem 2.4.4. We consider linear fractional transformation $M(z) = \frac{1}{z}$. One can check by direct calculation that the transformed coordinates x_M and y_M and the Jacobian of the transformation J_M are

$$x_M = \Re M(x+iy) = \frac{x}{x^2+y^2}\,,$$
$$y_M = \Im M(x+iy) = -\frac{y}{x^2+y^2}\,,$$
$$J_M(x,y) = \frac{1}{(x^2+y^2)^2}\,.$$

The inverse of the mapping M is $M^{-1} = \frac{1}{z}$. By mapping the curve Γ we obtain a line segment

$$\tilde{\Gamma} = M^{-1}(\Gamma) = \left\{\left(x, -\frac{1}{2R}\right)\middle|\, |x| < \frac{1}{2R\tan\frac{\epsilon}{2}}\right\}.$$

The coupling parameter is transformed as $\tilde{\beta}^{-1} = \beta^{-1}\sqrt{J_M(x)}$. Using Theorem 2.4.4 we obtain that the operator $-\Delta_{\Gamma_\epsilon,\beta}$ has no negative eigenvalues for

$$\frac{1}{\beta} \le -\frac{\tan\frac{\epsilon}{2}}{8R\pi}\,.$$

3. Unpublished Results

In this chapter we summarize various unpublished results. The first section describes a toy model of two δ'-interaction on the line and on the circle. The results from this toy model are used for the description of the spectral asymptotics for δ'-interaction supported on a sharp angle. We also derive the configuration with maximal ground state energy for $2n$ δ'-interactions on a loop. The next section generalize the result of [JL16] using conformal mapping from a unit disc to a different subset of $\mathbb{C}$. In the last section we prove that sufficiently weak δ'-interaction supported on non-closed compact manifold has no negative discrete eigenvalues.

3.1 Spectrum of δ'-interaction on Sharp Angle

We are interested in the behavior of the spectrum for the system where the δ'-interaction is supported by a line broken at a sharp angle. We study the relation between the angle and the discrete spectrum. A similar problem was studied for δ-interaction in [DR14], where the estimate on the number of eigenvalues and eigenvalues itself were given with respect to the angle. Cruder estimate on the number of eigenvalues for δ-interaction was also given in [EN03].

3.1.1 Warmup: Two δ' Interactions on Line

We start with the spectral asymptotics for the case of two merging δ'-interactions on the line and we use these results later on. We consider the operator

$$-\Delta_{\beta_1,\beta_2,d}\psi = -\Delta_{B,Y} \tag{3.1}$$

where $B = (\beta_1, \beta_2)$ and $Y = (-d, d)$. The operator (3.1) is associated with the quadratic form

$$q_{\beta_1,\beta_2,d}[\psi] = (\nabla\psi, \nabla\psi) + \beta_1^{-1}|\psi(-d_+) - \psi(-d_-)|^2 + \beta_2^{-1}|\psi(d_+) - \psi(d_-)|^2$$

with the domain $H^1(\mathbb{R}\backslash\{-d, d\})$. We are interested in the spectral properties for the limiting cases when d is either small or large. According to [AGHH05] the essential spectrum of the operator (3.1) is $\sigma_{ess}(-\Delta_{\beta_1,\beta_2,d}) = [0, \infty)$ and the discrete spectrum $\sigma_{disc}(-\Delta_{\beta_1,\beta_2,d})$ is composed from at most 2 negative eigenvalues depending on the values of parameters β_1, β_2. The solution of the eigenvalue equation

$$-\Delta_{\beta_1,\beta_2,d}\psi = -\kappa^2\psi\,,$$

where $\kappa > 0$, can be described by the secular equation in the form

$$\left|\begin{pmatrix} \frac{1}{\beta_1\kappa^2} + \frac{1}{2\kappa} & \frac{1}{2\kappa}\exp(-2\kappa d) \\ \frac{1}{2\kappa}\exp(-2\kappa d) & \frac{1}{\beta_2\kappa^2} + \frac{1}{2\kappa} \end{pmatrix}\right| = 0\,.$$

This is equivalent to the following equation

$$\kappa_{1,2} = \frac{-\frac{1}{\beta_1} - \frac{1}{\beta_2} \pm \sqrt{\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)^2 - \frac{4}{\beta_1\beta_2}(1 - \exp(-4\kappa d))}}{1 - \exp(-4\kappa d)}\,.$$

For $d \to \infty$ we can simplify the previous expression by taking the first term of the Taylor expansion with respect to $\exp(-4\kappa d)$ as

$$\kappa_{1,2} = -\frac{1}{\beta_1} - \frac{1}{\beta_2} \pm \sqrt{\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)^2 - \frac{4}{\beta_1\beta_2}} = \frac{1}{\beta_1} + \frac{1}{\beta_2} \pm \left|\frac{1}{\beta_1} + \frac{1}{\beta_2}\right|\,. \quad (3.2)$$

Depending on the sign of the parameters β_i we obtain either two negative eigenvalues, one negative eigenvalue or no negative eigenvalues due to the fact, that the eigenvalues correspond to $\kappa > 0$. These solutions can be written explicitly as

$$\kappa_{1,2} = -\frac{2}{\beta_{1,2}}\,.$$

This corresponds to the ground state energy of one δ' interaction on the line with coupling constant β_i. The situation for $d \to 0+$ is more complicated.

The right-hand side of equation (3.2) can be estimated by the first terms of the Taylor expansion for small d as

$$\kappa_{1,2} = \frac{A \pm \sqrt{A^2}}{4\kappa d} + \frac{A\sqrt{A^2} \pm (A^2+B)}{2\sqrt{A^2}}$$
$$+ \left(\frac{A}{3} \pm \frac{\sqrt{A^2}}{3} \mp \frac{\sqrt{A^2}B^2}{2A^4} \right) \kappa d \pm \frac{\sqrt{a^2}(A^2B^2+B^3)}{A^6} \kappa^2 d^2$$

where we used the following shorthands $A = -\frac{1}{\beta_1} - \frac{1}{\beta_2}$ and $B = -\frac{4}{\beta_1\beta_2}$. Without loss of generality we assume that $\beta_1 < \beta_2$. In general there are 4 possible situations which are summarized in the following table:

Situation	a)	b)	c)	d)
A	$A > 0$	$A > 0$	$A < 0$	$A < 0$
B	$B < 0$	$B > 0$	$B > 0$	$B < 0$
β_1	$\beta_1 < 0$	$\beta_1 < 0$	$\beta_1 < 0$	$\beta_1 > 0$
β_2	$\beta_2 < 0$	$\beta_2 > 0, \lvert\beta_1\rvert < \lvert\beta_2\rvert$	$\beta_2 > 0, \lvert\beta_1\rvert > \lvert\beta_2\rvert$	$\beta_2 > 0$
$\lvert\sigma_{disc}\rvert$	2	1	1	0

The eigenvalue asymptotics can be written for various situations as follows.

a) The spectrum for two attractive δ'-interactions consists of two discrete eigenvalues:

$$E_1 = -\frac{A}{2d} + \mathcal{O}(d^{-1/2}) = \frac{\beta_1+\beta_2}{2d\beta_1\beta_2} + \mathcal{O}(d^{-1/2}),$$
$$E_2 = -\frac{B^2}{4A^2} + \mathcal{O}(d) = -\frac{4}{(\beta_1+\beta_2)^2} + \mathcal{O}(d).$$

The first eigenvalue escapes to $-\infty$ as $d \to 0+$ and the second one converges to the eigenvalue of one attractive δ' interaction with interaction strength $\beta_1 + \beta_2$.

b) This situation corresponds to one attractive and one repulsive δ'-interaction where attractive interaction is stronger in the sense of quadratic forms $1/|\beta_1| > 1/|\beta_2|$. Such system has one eigenvalue

$$E = -\frac{A}{2d} + \mathcal{O}(d^{-1/2}) = \frac{\beta_1+\beta_2}{2d\beta_1\beta_2} + \mathcal{O}(d^{-1/2}).$$

The eigenvalue escapes to $-\infty$ as $d \to 0+$ and disappears.

c) This situation is again one attractive and one repulsive δ' interaction but now the repulsive one is stronger. This system has again one negative eigenvalue which can be written as

$$E = -\frac{4}{(\beta_1+\beta_2)^2} + \mathcal{O}(d)\,.$$

The eigenvalue $-\frac{4}{(\beta_1+\beta_2)^2}$ corresponds to eigenvalue of one attractive δ' interaction $\beta_1+\beta_2$.

d) For the case of two repulsive δ'-interactions there is no negative eigenvalues.

Next we show that by "merging" two δ'-interactions with coupling constants β_1 and β_2 we obtain one δ'-interaction with the coupling parameter equal to $\beta_1+\beta_2$. Furthermore we show that these operators converge in the norm resolvent sense. The operator $-\Delta_{0,\beta}$ describing one δ'-interaction localized at 0 was introduced by (1.1).

Theorem 3.1.1. *The operator $-\Delta_{\beta_1,\beta_2,d}$ converges to $-\Delta_{0,\beta_1+\beta_2}$ in the norm resolvent sense for $d\to 0_+$.*

Proof. The form of the resolvents for the operators $-\Delta_{\beta_1,\beta_2,d}$ and $-\Delta_{0,\beta_1+\beta_2}$ were given in Theorem 1.1.3. We need to show that the second term in the resolvents converge to each other. This can be checked by direct calculation. It is not hard to see that

$$\lim_{y_j\to 0}\tilde{G}_\kappa(x-y_j) = \tilde{G}_\kappa(x)\,.$$

From this follows that we need to calculate the following limit

$$\lim_{d\to 0}\sum_{i,j=1}^{2}[\Gamma_{\beta_1,\beta_2,d}(\kappa)]^{-1}_{i,j}$$

and show that it is equal to $\frac{2(\beta_1+\beta_2)\kappa^2}{2+\kappa(\beta_1+\beta_2)}$. Writing the inverse matrix explicitly we get

$$[\Gamma_{\beta_1,\beta_2,d}(\kappa)]^{-1}_{i,j} = \frac{1}{\det[\Gamma_{\beta_1,\beta_2,d}(\kappa)]}\begin{pmatrix} [\Gamma_{\beta_1,\beta_2,d}(\kappa)]_{2,2} & -[\Gamma_{\beta_1,\beta_2,d}(\kappa)]_{2,1} \\ -[\Gamma_{\beta_1,\beta_2,d}(\kappa)]_{1,2} & [\Gamma_{\beta_1,\beta_2,d}(\kappa)]_{1,1}\end{pmatrix} =$$

$$\frac{1}{\left(\frac{1}{\beta_1\kappa^2}+\frac{1}{2\kappa}\right)\left(\frac{1}{\beta_2\kappa^2}+\frac{1}{2\kappa}\right)-\frac{\exp(-4\kappa d)}{4\kappa^2}}\begin{pmatrix} \frac{1}{\beta_2\kappa^2}+\frac{1}{2\kappa} & -\frac{\exp(-2\kappa d)}{2\kappa} \\ -\frac{\exp(-2\kappa d)}{2\kappa} & \frac{1}{\beta_1\kappa^2}+\frac{1}{2\kappa}\end{pmatrix}$$

Taking the limit $d \to 0$ we obtain

$$\frac{1}{\left(\frac{1}{\beta_1\kappa^2}+\frac{1}{2\kappa}\right)\left(\frac{1}{\beta_2\kappa^2}+\frac{1}{2\kappa}\right)-\frac{1}{4\kappa^2}}\begin{pmatrix} \frac{1}{\beta_2\kappa^2}+\frac{1}{2\kappa} & -\frac{1}{2\kappa} \\ -\frac{1}{2\kappa} & \frac{1}{\beta_1\kappa^2}+\frac{1}{2\kappa} \end{pmatrix}$$
$$= \frac{1}{\frac{1}{\beta_1\kappa^2}\frac{1}{\beta_2\kappa^2}+\frac{1}{2\kappa}\left(\frac{1}{\beta_1\kappa^2}+\frac{1}{\beta_2\kappa^2}\right)}\begin{pmatrix} \frac{1}{\beta_2\kappa^2}+\frac{1}{2\kappa} & -\frac{1}{2\kappa} \\ -\frac{1}{2\kappa} & \frac{1}{\beta_1\kappa^2}+\frac{1}{2\kappa} \end{pmatrix}.$$

And finally we sum the entries in the matrix and we obtain the desired result. □

3.1.2 Merging of 2 δ'-interactions on Loop

The study of two δ'-interaction gives us an idea how the merging of two interactions work. As the next step we show that we encounter similar behavior for two δ'-interactions on a loop. With the idea of attractive interaction on an angle in mind we work only with two attractive δ'-interactions on a unit circle. We note that the scaling of coupling parameter β can be rephrased to the changing the length of the loop. We introduce the operator via a quadratic form in the following way

$$c_{\beta,\theta} = (\nabla\psi, \nabla\psi) + \beta|\psi(-\theta_+) - \psi(-\theta_-)|^2 + \beta|\psi(\theta_+) - \psi(\theta_-)|^2$$

with the domain $H^1((-\pi,\pi), \psi(-\pi) = \psi(\pi))$. The operator associated with this quadratic form can be written as

$$\begin{aligned} C_{\beta,\theta} &= -\Delta\,, \\ \mathcal{D}(C_{\beta,\theta}) = \{\psi \in H^2((-\pi,\pi)\setminus\{\theta,\theta\}) | \psi(-\pi) &= \psi(\pi), \psi'(-\pi) = \psi'(\pi), \\ \psi'(\theta_+) = \psi'(\theta_-) &= \beta^{-1}(\psi(\theta_+) - \psi(\theta_-)), \\ \psi'(-\theta_+) = \psi'(-\theta_-) &= \beta^{-1}(\psi(-\theta_+) - \psi(-\theta_-))\}\,. \end{aligned}$$

The next Lemma gives the estimate on the eigenvalues of the operator $C_{\beta,\theta}$.

Lemma 3.1.1. *Let $C_{\beta,\theta}$ be the operator describing two attractive δ'-interactions on the unit circle and let $-\pi < \beta < 0$. Then the negative eigenvalues can be in the limit $\theta \to 0_+$ written as*

$$\lambda_1 = \frac{1}{\theta\beta}\,,$$
$$\lambda_2 = -\frac{3(\pi+\beta)}{\theta^3 + (\pi-\theta)^3}\,.$$

Proof. The negative spectrum of the operator $C_{\beta,\theta}$ is determined by the following secular equations

$$-\kappa\beta = \coth(\kappa(\pi-\theta)) + \coth(\kappa\theta)\,, \tag{3.3}$$

$$-\kappa\beta = \tanh(\kappa(\pi-\theta)) + \tanh(\kappa\theta)\,. \tag{3.4}$$

These equation are obtainable by a straightforward computation from boundary conditions of δ'-interaction at either θ or $-\theta$. For the later purposes we need the limit cases for $\beta \to 0_-, -\infty$ and $\theta \to 0_+$. For both equations (3.3) and (3.4) the situation corresponding to $\beta \to 0_-$ leads in the limit to $\kappa = -\frac{2}{\beta}$ as long as $\kappa\theta \gg 1$ which is equivalent to $2\theta \gg -\beta$. For equation (3.3) the cases $\beta \to -\infty$ and $\theta \to 0$ can be treated simultaneously in the following way due to the fact that $\beta \to -\infty$ corresponds to the weak coupling and as a result we have $0 < \kappa \ll 1$. For $0 < \kappa\theta \ll 1$ we can estimate

$$-\beta = \frac{\coth(\kappa(\pi-\theta)) + \coth(\kappa\theta)}{\kappa} \approx \frac{1}{\theta\kappa^2}\,.$$

This can be rewritten as $\kappa^2 = -\frac{1}{\theta\beta}$. The condition for which the estimate $0 < \kappa\theta \ll 1$ holds is $\sqrt{-\frac{\theta}{\beta}} \ll 1$.

The equation (3.4) has no solution for the case $\beta \to -\infty$ due to the fact that

$$0 < \frac{\tanh(\kappa(\pi-\theta)) + \tanh(\kappa\theta)}{\kappa} < \pi\,.$$

Also from the previous inequality we can see that our system has a second negative eigenvalue only for $-\pi < \beta < 0$. For the case $\theta \to 0+$ we can rewrite the expression (3.4) as

$$-\beta\kappa = \tanh(\kappa(\pi-\theta)) + \tanh(\kappa\theta) \approx \kappa\pi - \frac{(\kappa\theta)^3}{3} - \frac{(\kappa(\pi-\theta))^3}{3}\,.$$

As a result we obtain $\kappa^2 = \frac{3(\pi+\beta)}{\theta^3+(\pi-\theta)^3}$. □

3.1.3 Spectral Asymptotics

We want to calculate the spectrum of the system where the δ'-interaction is supported by a broken line Γ with the angle $2\theta < \pi$ between the lines. The operator can be defined by its quadratic form (1.5)

$$h_{\beta,\Gamma}(\psi) = (\nabla\psi, \nabla\psi)^2_{\mathbb{R}^n} + (\beta^{-1}[\psi]_\Gamma, [\psi]_\Gamma)_\Gamma$$

with the domain $H^1(\mathbb{R}^2 \setminus \Gamma)$.

Theorem 3.1.2. *Let Γ be a broken line with the angle $2\theta < \pi$. Then the operator $-\Delta_{\Gamma,\beta}$ can be estimated from below by the operator $-\Delta + V(r)$ with the domain $H^2(\mathbb{R}^+)$ where*

$$\begin{aligned} V(r) &= \frac{1}{4r^2} + \frac{1}{r\beta\theta} \quad \text{for} \quad r \to 0_+\,, \\ V(r) &= \frac{1}{4r^2} - \frac{4}{\beta^2} \quad \text{for} \quad r \to \infty\,. \end{aligned}$$

Proof. Rewriting the quadratic form $h_{\beta,\Gamma}$ in polar coordinates we obtain the following

$$\begin{aligned} h_{\beta,\Gamma}(\psi) = &\int_{\mathbb{R}^+\times(-\pi,\pi)} r\left(|\partial_r\psi|^2 + \frac{|\partial_\varphi\psi|^2}{r^2}\right) dr\, d\varphi \\ &+\beta^{-1}\sum_{\phi=-\theta,\theta}\int_{\mathbb{R}^+} |\psi(r,\phi_+) - \psi(r,\phi_-)|^2\, dr \end{aligned}$$

with the domain $H^1(\mathbb{R}\times(-\pi,\pi), \psi(r,-\pi) = \psi(r,\pi), r\, dr\, d\varphi)$. We transform the quadratic form $h_{\beta,\Gamma}$ using the unitary transformation $\tilde{\psi} = \frac{\psi}{\sqrt{r}}$ as

$$\begin{aligned} \tilde{h}_{\beta,\Gamma}(\psi) = &\int_{\mathbb{R}^+\times(-\pi,\pi)} |\partial_r\psi|^2 + \frac{|\psi|^2}{4r^2} + \frac{|\partial_\varphi\psi|^2}{r^2}\, dr\, d\varphi \\ &+\beta^{-1}\sum_{\phi=-\theta,\theta}\int_{\mathbb{R}^+} \frac{1}{r}|\psi(r,\phi_+) - \psi(r,\phi_-)|^2\, dr \end{aligned}$$

with the domain $H^1(\mathbb{R}\times(-\pi,\pi), \psi(r,-\pi) = \psi(r,\pi), dr\, d\varphi)$. This is equivalent to the following form

$$\begin{aligned} \tilde{h}_{\beta,\Gamma}(\psi) &= \int_{\mathbb{R}^+\times(-\pi,\pi)} |\partial_r\psi|^2 + \frac{|\psi|^2}{4r^2}\, dr\, d\varphi \\ &+ \int_{\mathbb{R}^+}\frac{1}{r^2}\left(\int_{-\pi}^{\pi} |\partial_\varphi\psi|^2\, d\varphi + \beta^{-1}\sum_{\phi=-\theta,\theta}\int_{\mathbb{R}^+}\frac{1}{r}|\psi(r,\phi_+) - \psi(r,\phi_-)|^2\right) dr \\ &= \int_{\mathbb{R}^+\times(-\pi,\pi)} |\partial_r\psi|^2 + \frac{|\psi|^2}{4r^2}\, dr\, d\varphi + \int_{\mathbb{R}^+}\frac{1}{r^2}c_{\frac{\beta}{r},\theta}(\psi)\, dr \end{aligned}$$

where $c_{\frac{\beta}{r},\theta}$ is a quadratic form describing 2 δ'-interactions localized on a unit circle. We can estimate this quadratic form from below as

$$\tilde{h}_{\beta,\Gamma}[\psi] \geq \int_{\mathbb{R}^+\times(-\pi,\pi)} |\partial_r\psi|^2 + \frac{|\psi|^2}{4r^2}\, dr\, d\varphi + \int_{\mathbb{R}^+} \frac{1}{r^2} \inf_r c_{\frac{\beta}{r},\theta}(\psi)\, dr\,.$$

From the previous section we have that $\inf_r c_{\frac{\beta}{r},\theta}(\psi)$ behaves for small r as $\frac{r}{\beta\theta}$ and for the large r as $-\frac{4r^2}{\beta^2}$. Using this we can estimate the spectrum using an operator on the halfline with the potential

$$V(r) = \frac{1}{4r^2} + \frac{1}{r\beta\theta} \quad \text{for} \quad r \to 0_+\,,$$
$$V(r) = \frac{1}{4r^2} - \frac{4}{\beta^2} \quad \text{for} \quad r \to \infty\,.$$

□

We note that the operator $-\Delta + \frac{1}{4r^2}$ on the halfline is critical. Furthermore it can be checked by direct calculation that the essential spectrum for the setting of δ'-interaction supported by a broken line is $-\Delta_{\Gamma,\beta} = \left[-\frac{4}{\beta}, \infty\right)$. The operator by which we estimate acts as a free Laplacian with an added Coulomb interaction. We can estimate the spectrum by the number of eigenvalues of such operator which are smaller than $-4/\beta^2$.

3.2 Spectrum of δ'-interactions Supported by Star Graph

We consider an operator describing δ'-interaction supported by a star graph. We are interested in the setting with maximal infimum of the spectrum. For this purpose we study a toy model of finite number of δ'-interactions on a loop of the length d. We used this model later on for a special setting with strength of δ'-interaction decaying as $1/r$.

3.2.1 Even Number of δ'-interactions on Loop

We define an operator describing m δ'-interactions on the loop. We take a metric graph composed of m edges connected by appropriate boundary conditions. At each edge our operator acts as a second derivative

$$C_{\mathbf{d},\beta}\psi_k(x) = \psi_k''(x) \quad \text{for all} \quad x \in (0, d_k), k \in m$$

where $\mathbf{d} = \{d_j\}_{j=1}^m$ are the lengths of the edges and ψ_k is the function on the k-th edge. The domain of our operator is

$$\mathcal{D}(C_{\mathbf{d},\beta}) = \{\psi \in \bigoplus_{j=1}^{m} H^2((0,d_j)) | \psi_1'(0_+) = \psi_m'(d_{m,-}) = \frac{1}{\beta}\psi_0(0_+) - \psi_m(d_{m,-}),$$
$$\psi_j'(0_+) = \psi_{j-1}'(d_{j-1,-}) = \frac{1}{\beta}\psi_j(0_+) - \psi_{j-1}(d_{j-1,-}) \text{ for } (j-1) \in \widehat{m-1}\}\,.$$

The boundary conditions can be rewritten in the matrix form as

$$A\Gamma\psi + B\Gamma'\psi = 0\,,$$

$$\Gamma\psi = \{\psi_1(0_+), \psi_1(d_{1,-}), \ldots, \psi_m(0_+), \psi_m(d_{m,-})\}^T\,,$$
$$\Gamma'\psi = \{\psi_1'(0_+), -\psi_1'(d_{1,-}), \ldots, \psi_m'(0_+), -\psi_m'(d_{m,-})\}^T$$

where A, B are appropriate $2m \times 2m$ matrices. The matrices A, B can be expressed as

$$A = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & a & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & a & 0 \\ -1 & 0 & \ldots & 0 & 1 \end{pmatrix},$$
$$B = -\beta I$$

where $a = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and I is the identity matrix. According to [BL10] we have $\Gamma'\psi_\kappa = M(\kappa)\Gamma\psi_\kappa$ where ψ_κ is the eigenfunction of $C_{\mathbf{d},\beta}$ corresponding to the eigenvalue $-\kappa^2$ and

$$M(\kappa) = \begin{pmatrix} m_1(\kappa) & & 0 \\ & \ddots & \\ 0 & & m_m(\kappa) \end{pmatrix},$$

$$m_i(\kappa) = \frac{\kappa}{\sinh(\kappa d_i)} \begin{pmatrix} -\cosh(\kappa d_i) & 1 \\ 1 & -\cosh(\kappa d_i) \end{pmatrix}.$$

Furthermore we have the following

$$-\kappa^2 \in \sigma_p(C_{\mathbf{d},\beta}) \Leftrightarrow 0 \in \sigma(A + M(\kappa)B)\,.$$

We denote $X(\mathbf{d},\kappa) = A + M(\kappa)B$. The ground state energy of the operator $C_{\mathbf{d},\beta}$ corresponds to the highest eigenvalue of the matrix $C_{\mathbf{d},\beta}$. In the following lemma we treat the situation for even number of edges. We note that the situation with odd number of edges is more complicated because the lowest eigenvalue of the matrix $X(\mathbf{d},\kappa)$ has no symmetry. This lack of symmetry makes the proof used here inapplicable.

Lemma 3.2.1. *Let $C_{\mathbf{d},\beta}$ be an operator defined above with $2n$ edges and $\sum_{i=1}^{2n} d_i = L$. The maximal value of the ground state energy of this operator is obtained for the situation $d_i = d = L/2n$.*

Proof. First we show that the eigenvector $C_0 = (-1,-1,1,1,...,-1,-1,1,1)^T$ corresponds to the highest eigenvalue of the matrix $X(\mathbf{d},\kappa)$ for the symmetric case, i.e. all the edges with the same length $d = d_i = d_j$. It is easy to check by direct calculation that it is an eigenvector. We know the following

$$C_0^T X(\mathbf{d},\kappa) C_0 = 4n + 2\sum_{i=1}^{n} \kappa\beta \coth(\kappa d_i) + 2\sum_{i=1}^{n} \kappa\beta \sinh^{-1}(\kappa d_i)\,.$$

We are interested in the value κ for which $C_0^T X(\mathbf{d},\kappa) C_0 = 0$, i.e. $2 + \kappa\beta\coth(\kappa d) - \kappa\beta\sinh^{-1}(\kappa d) = 0$. Now we calculate $C^T X(\mathbf{d},\kappa) C$ for a general vector C and show that it is always nonpositive which implies that the vector C_0 corresponds to the highest eigenvalue of the matrix $X(\mathbf{d},\kappa)$. Direct calculation shows

$$\begin{aligned} C^T X(\mathbf{d},\kappa) C = |c_1 - c_{2n}|^2 + \sum_{i=1}^{n-1} |c_{2i} - c_{2i+1}|^2 \\ + \sum_{i=1}^{n} [-\kappa\beta\coth(\kappa d)(|c_{2i}^2| + |c_{2i-1}|^2) \\ -\kappa\beta\sinh^{-1}(\kappa d)(|c_{2i} + c_{2i-1}|^2 - |c_{2i}|^2 - |c_{2i-1}|^2)]\,. \end{aligned}$$

We substitute $-2 + \frac{\kappa\beta}{\sinh(\kappa d)} = -\kappa\beta\coth(\kappa d)$ into the previous expression and we obtain

$$C^T X(\mathbf{d},\kappa)C = |c_1 - c_{2n}|^2 + \sum_{i=1}^{n-1} |c_{2i} - c_{2i+1}|^2$$

$$-2\sum_{i=1}^{2n} |c_i|^2 + \sum_{i=1}^{n}[-\kappa\beta \sinh^{-1}(\kappa d)(|c_{2i} + c_{2i-1}|^2 - 2|c_{2i}|^2 - 2|c_{2i-1}|^2)]\,.$$

One can check that $|a+b|^2 \leq 2|a|^2 + 2|b|^2$ for arbitrary a, b and as a direct result we obtain $C^T X(\mathbf{d},\kappa)C \leq 0$.
Now we will show that $C_0^T X(\mathbf{d},\kappa)C_0 \leq C_0^T X(\mathbf{d}',\kappa)C_0$ for any positive vectors $\mathbf{d}'$ and that the equality holds only for $\mathbf{d} = \mathbf{d}'$. We take $C_0^T X(\mathbf{d},\kappa)C_0$ as a function of d_i and we find the minimum with the help of Lagrange multipliers with the condition $\sum_{i=1}^{n} d_i = L$. We obtain the following equations for the d_i

$$\partial_{d_i}\left(C_0^T X(\mathbf{d},\kappa)C_0 - \lambda\left(\sum_{i=1}^{2n} d_i - L\right)\right) = 0, \quad \sum_{i=1}^{2n} d_i - L = 0\,.$$

These equations are equivalent to

$$\lambda = -2\kappa^2\beta\frac{1-\cosh(\kappa d_i)}{\sinh^2(\kappa d_i)}, \quad \sum_{i=1}^{2n} d_i - L = 0\,.$$

The functions $\frac{1-\cosh(\kappa d_i)}{\sinh^2(\kappa d_i)}$ are monotonous with respect to d_i which along with the latter implies that the minimum is obtained for $d_i = \frac{L}{2n}$. □

3.2.2 Optimal Geometry for Maximum of Spectrum

For the special situation of a star graph with even number of edges with decaying δ'-interaction we are able to show that the maximum of the spectrum is obtained for the symmetric case.

Theorem 3.2.1. *Let Γ be a star graph with $2n$ edges and let the $\beta(r) = -\frac{C}{r}$ with $C \in \mathbb{R}^+$ and r being the distance from the center of the star graph Γ. Let θ_i denote the angle between the edge i-th and $i+1$-th of the star graph Γ. Then the maximum of the discrete spectrum $\max_{\theta_i}\left\{\inf\left[\sigma_d\left(-\Delta_{\Gamma,\beta(r)}\right)\right]\right\}$ is obtained for $\theta_i = \frac{\pi}{n}$ for all $i \in \widehat{2n}$.*

Proof. We start by rewriting the form (1.5) in polar coordinates in the same way as in the previous section

$$\tilde{h}_\beta^\Gamma(\psi) = \int_{\mathbb{R}^+\times(-\pi,\pi)} |\partial_r\psi|^2 + \frac{|\psi|^2}{4r^2} + \frac{|\partial_\varphi\psi|^2}{r^2}\, dr\, d\varphi$$
$$+ \sum_{\phi=\theta_i} \int_{\mathbb{R}^+} \frac{1}{\beta(r)r} |\psi(r,\phi_{i,+}) - \psi(r,\phi_{i,-})|^2\, dr$$

with the domain $H^1(\mathbb{R}\times(-\pi,\pi), \psi(r,-\pi) = \psi(r,\pi), dr\, d\varphi)$. This is equivalent to the following form

$$\tilde{h}_{\beta,\Gamma}(\psi) = \int_{\mathbb{R}^+\times(-\pi,\pi)} |\partial_r\psi|^2 + \frac{|\psi|^2}{4r^2}\, dr\, d\varphi$$
$$+ \int_{\mathbb{R}^+} \frac{1}{r^2}\left(\int_{-\pi}^{\pi} |\partial_\varphi\psi|^2\, d\varphi + \sum_{\phi=\theta_i}\int_{\mathbb{R}^+} \frac{1}{C}|\psi(r,\phi_{i,+}) - \psi(r,\phi_{i,-})|^2\right) dr$$
$$= \int_{\mathbb{R}^+\times(-\pi,\pi)} |\partial_r\psi|^2 + \frac{|\psi|^2}{4r^2}\, dr\, d\varphi + \int_{\mathbb{R}^+} \frac{1}{r^2} c_{\theta,\beta}(\psi)\, dr$$

where we used shorthand $c_{\theta,\beta}$ with $\theta = \{\theta_i | i \in \widehat{2n}\}$ for the quadratic form corresponding to the operator $C_{\theta,\beta}$. Using Lemma 3.2.1 we obtain that the effective potential in the form $\tilde{h}_{\beta,\Gamma}$ has a maximum for $\theta_i = \frac{\pi}{n}$ for all $i \in \widehat{2n}$ which completes the proof. □

3.3 Conformal Maps of Unit Disc

In this section we present one possible generalization of the result presented in [JL16]. We are able to generalize Theorem 2.4.4 using conformal maps from the unit disc to a general subset of $\mathbb{C}$. In this way we are able to cover a much larger set of curves because limiting oneself to linear fractional transformations is quite restrictive. Using Riemann mapping theorem we are able to map the unit disc to any subset of $\mathbb{C}$. The only question which remains to solve is how precisely the interior of the disc is mapped.

We consider curves Γ satisfying the following conditions:

(cm1) Γ is a compact piecewise C^1 smooth curve,

(cm2) Γ is obtainable by a conformal mapping M from a curve $\tilde{\Gamma}$. The conformal map M transforms the unit disc D to a different subset of $\mathbb{C}$. Furthermore the preimage $\tilde{\Gamma}$ of the curve Γ is a monotone curve parametrized as

$$\tilde{\Gamma}(r) = (r\cos\phi(r), r\sin\phi(r)) \quad \text{for} \quad r \in (0,1)$$

with the endpoints at the boundary of the unit disc D and the center of the disc D.

Theorem 3.3.1. *Let the curve $\Gamma \subset \mathbb{R}^2$ satisfy (cm1) and (cm2). Then*

$$\sigma(-\Delta_{\Gamma,\beta}) = [0,\infty)$$

for all $\beta(z) \leq -2\pi r\sqrt{1+(r\phi'(r))^2}\sqrt{J_M(z)}$ where r is related to $\tilde{\Gamma}$.

Proof. We start with the operator defined on the unit disc D as

$$\begin{aligned} -\Delta_{\tilde{\Gamma},\tilde{\beta},D} &= -\Delta\,, \\ \mathcal{D}(-\Delta_{\tilde{\Gamma},\tilde{\beta},D}) &= \{\psi \in \mathbb{H}^2(D \setminus \tilde{\Gamma})| \\ \partial_{\Gamma+}\psi|_{\Gamma+} = -\partial_{\Gamma-}\psi|_{\Gamma-}, \tilde{\beta}\partial_{\Gamma+}\psi|_{\Gamma+} &= \psi|_{\Gamma+} - \psi|_{\Gamma-}\}\,. \end{aligned}$$

Using the same technique as in the proof of Theorem 2.4.3 we obtain that there is no negative eigenvalues of the operator $-\Delta_{\tilde{\Gamma},\tilde{\beta},D}$ as long as $\tilde{\beta}(r) \leq -2\pi r\sqrt{1+(r\phi'(r))^2}$. Now using the conformal mapping M we rewrite the the quadratic form $h_{\tilde{\Gamma},\tilde{\beta},D}$ associated with the operator $-\Delta_{\tilde{\Gamma},\tilde{\beta},D}$ as

$$\begin{aligned} h_{\tilde{\Gamma},\tilde{\beta},D}(\psi) &= (\nabla u, \nabla u)^2_D + (\tilde{\beta}^{-1}[u]_{\tilde{\Gamma}}, [u]_{\tilde{\Gamma}})_{\tilde{\Gamma}}\,, \\ h_{\Gamma,\beta,M(D)}(\psi) &= (\nabla v, \nabla v)^2_{M(D)} + (\beta^{-1}[u]_\Gamma, [u]_\Gamma)_\Gamma \end{aligned}$$

where $\tilde{\beta}^{-1} = \beta^{-1}\sqrt{J_M(z)}$. We denote the operator associated with the quadratic form $h_{\Gamma,\beta,M(D)}$ as $-\Delta_{\Gamma,\beta,M(D)}$. We introduce a new operator defined as a direct sum of the operator $-\Delta_{\Gamma,\beta,M(D)}$ and Neumann Laplacian $-\Delta^N_{\mathbb{R}^2\setminus M(D)}$ on a set $\mathbb{R}^2 \setminus M(D)$. Both of these operators are positive and its sum corresponds to the operator $-\Delta_{\Gamma,\beta}$ with added Neumann boundary condition at the boundary of $M(D)$. Using Neumann bracketing

$$0 \leq -\Delta_{\Gamma,\beta,M(D)} \oplus -\Delta^N_{\mathbb{R}^2\setminus M(D)} \leq -\Delta_{\Gamma,\beta}$$

we complete the proof. □

3.4 Absence of Negative Eigenvalues for Non-closed Hypersurface

In this section we prove that for a sufficiently small coupling δ'-interaction supported by a non-closed compact manifold has no negative discrete eigenvalues. The result is similar to one presented in Theorem 3.3.1 and 2.4.4, where it was proved for a certain class of curves with the estimate on the coupling strength. The advantage of the following theorem is, that it works for any non-closed compact manifold; the price we pay is that we do not obtain a quantitative estimate. The idea of the proof was suggested by Monique Dauge [D16].

We consider the operator $-\Delta_{\Lambda,\beta}$ associated with the quadratic form (1.6), i.e.

$$h_\beta^\Lambda(\psi) = (\nabla\psi, \nabla\psi)^2_{\mathbb{R}^n} + \big(\beta^{-1}[\psi]_\Lambda, [\psi]_\Lambda\big)_\Lambda$$

where $\beta^{-1}(x) \in L^\infty(\Lambda, \mathbb{R})$ with the domain $H^1(\mathbb{R}^n \setminus \Lambda)$.

Theorem 3.4.1. *Let Λ be a non-closed compact Lipschitz manifold of the codimension 1. Then there exists β_0 such that the operator $-\Delta_{\Lambda,\beta}$ is positive as long as $\beta < \beta_0$. Furthermore the spectrum is $\sigma(-\Delta_{\Lambda,\beta}) = \sigma_{ess}(-\Delta_{\Lambda,\beta}) = \mathbb{R}^+$.*

Proof. The proof of the fact $\sigma_{ess}(-\Delta_{\Lambda,\beta}) \subset \mathbb{R}^+$ can be done in the same way as in [JL16]. We show that the operator $-\Delta_{\Lambda,\beta}$ is non-negative. First we take a subset $B \subset \mathbb{R}^n$ such that $\Lambda \subset B$ and $B \setminus \Lambda$ is connected. We introduce a new operator with the added Neumann boundary condition. This operator can be written as a direct sum of Neumann Laplacian $-\Delta^N_{\mathbb{R}^n\setminus B}$ on a set $\mathbb{R}^n \setminus B$ and the operator $-\Delta^B_{\Lambda,\beta}$ associated with the following form

$$h_\beta^{\Lambda,B}(\psi) = (\nabla\psi, \nabla\psi)^2_B + \big(\beta^{-1}[\psi]_\Lambda, [\psi]_\Lambda\big)_\Lambda$$

with the domain $H^1(B \setminus \Lambda)$. Using Neumann bracketing we obtain, in the sense of ordering forms,

$$-\Delta^N_{\mathbb{R}^n\setminus B} \oplus -\Delta^B_{\Lambda,\beta} \leq -\Delta_{\Lambda,\beta}\,.$$

The Neumann Laplacian is positive and as a result it is sufficient to check that the operator $-\Delta^B_{\Lambda,\beta}$ is positive. We show that the form $h_\beta^{\Lambda,B}$ is positive for sufficiently large negative β. We use the inequality

$$0 \leq \|[\psi]_\Lambda\|_\Lambda^2 \leq \|\psi|_{\Lambda+}\|_\Lambda^2 + \|\psi|_{\Lambda-}\|_\Lambda^2 \leq \tilde{C}(\|\nabla\psi\|_B^2 + \|\psi\|_B^2)\,, \tag{3.5}$$

which is a direct result of [GM08, Lemma 2.5]. We introduce the notation $\overline{\psi} = \int_{B\setminus\Lambda} \psi(x)dx$ for the average of the function ψ. Direct calculation shows that

$$[\psi - \overline{\psi}]_\Lambda = [\psi]_\Lambda\,,$$
$$\nabla(\psi - \overline{\psi}) = \nabla\psi\,.$$

Now we estimate the norm $\|\psi - \overline{\psi}\|_B^2$ in the following way

$$\lambda_2^N \|\psi - \overline{\psi}\|_B^2 \leq \|\nabla(\psi - \overline{\psi})\|_B^2$$

where λ_j^N denotes j-th eigenvalue including the multiplicity of Neumann Laplacian on set $B\setminus\Lambda$. The previous inequality holds due to the fact that the constant function is the eigenfunction corresponding to the lowest eigenvalue of the Neumann Laplacian and $\psi - \overline{\psi}$ is an orthogonal projection of the function ψ with respect to the ground state of the Neumann Laplacian. Now we rewrite the inequality (3.5) for the function $\psi - \overline{\psi}$ as follows

$$\|[\psi]_\Lambda\|_\Lambda^2 = \|[\psi - \overline{\psi}]_\Lambda\|_\Lambda^2 \leq \tilde{C}(\|\nabla(\psi - \overline{\psi})\|_B^2 + \|\psi - \overline{\psi}\|_B^2)\,,$$
$$\leq \tilde{C}\left(\|\nabla(\psi - \overline{\psi})\|_B^2 + \frac{\|\nabla(\psi - \overline{\psi})\|_B^2}{\lambda_2^N}\right) \leq \tilde{C}\left(1 + \frac{1}{\lambda_2^N}\right)\|\nabla\psi\|_B^2\,.$$

These inequalities imply that

$$\|[\psi]_\Lambda\|_\Lambda^2 \leq -\beta\|\nabla\psi\|_B^2$$

holds for $-\beta \geq \tilde{C}\left(1 + \frac{1}{\lambda_2^N}\right)$. Rewriting the latter we obtain

$$0 \leq \|\nabla\psi\|_B^2 + \frac{1}{\beta}\|[\psi]_\Lambda\|_\Lambda^2$$

which completes the proof. □

Summary

In this thesis we studied spectral properties of Schrödinger operators describing δ'-interactions supported by curves and surfaces. The results can be divided into two main groups. One is concerned with the strong coupling limit asymptotics of the spectrum and the second one with the absence of the discrete spectrum for δ'-interaction supported by a non-closed hypersurfaces. The first terms in the asymptotic expansion for the discrete eigenvalues in a strong coupling limit can be written as

$$\lambda_j = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta \ln |\beta|)$$

where β^{-1} corresponds to the interaction strength and μ_j depends on the geometry of the interaction support. The essential spectrum is either positive real line axis

$$\sigma_{ess}(H_{\Gamma,\beta}) = [0, \infty)$$

for the case of a compact manifold Γ and for an infinite manifold Γ with a sufficiently fast decaying interaction. For the case of a constant coupling strength and an infinite manifold which is asymptotically flat the essential spectrum behaves as

$$\sigma_{ess}(H_{\Gamma,\beta}) = [\epsilon, \infty) \, .$$

where $\epsilon \to -\frac{4}{\beta^2}$. The most interesting property of the spectrum studied and proved in this work is the absence of the negative spectrum for sufficiently weak δ'-interaction supported by a non-closed manifold. Such behavior was previously unknown for attractive potential in $\mathbb{R}^2$. We can rephrase the result in a way that there exists a critical value β^* such that for any $\beta \leq \beta^*$ the discrete spectrum is empty. If we consider a curve we can show that bending the curve results in the effective decrease of the value of β^*.

There are several open questions to be addressed. Among them we can list the optimization of the support shape with respect to the discrete spectrum or the challenging question of determining the precise value of β^* for a general non-closed manifold.

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Appendix-Publications Connected with Thesis

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(3) M. Jex: Spectral asymptotics for a δ' interaction supported by an infinite curve, *Mathematical Results in Quantum Mechanics: Proceedings of the QMath12 Conference*, World Scientific Publishing Co., 2015, pp. 259-265.

(4) M. Jex, V. Lotoreichik: On absence of bound states for weakly attractive δ'-interactions supported on non-closed curves in $\mathbb{R}^2$, *J. Phys. A: Math. Gen.* **57** (2016), 022101 (20pp).

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Spectral asymptotics of a strong δ' interaction on a planar loop

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Abstract
We consider a generalized Schrödinger operator in $L^2(\mathbb{R}^2)$ with an attractive strongly singular interaction of δ' type characterized by the coupling parameter $\beta > 0$ and supported by a C^4 smooth closed curve Γ of length L without self-intersections. It is shown that in the strong-coupling limit, $\beta \to 0_+$, the number of eigenvalues behaves as $\frac{2L}{\pi\beta} + \mathcal{O}(|\ln\beta|)$, and furthermore, that the asymptotic behavior of the jth eigenvalue in the same limit is $-\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta|\ln\beta|)$, where μ_j is the jth eigenvalue of the Schrödinger operator on $L^2(0, L)$ with periodic boundary conditions and the potential $-\frac{1}{4}\gamma^2$, where γ is the signed curvature of Γ.

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1. Introduction

Schrödinger operators with singular interactions supported by manifolds of a lower dimension have been studied for several decades beginning with the early works [Ku78, BT92]. In recent years they have attracted attention as a model of a quantum particle confined to sets of nontrivial geometry and a possible alternative to usual quantum graphs [BK13], having two advantages over the latter. The first is that they lack the abundance of free parameters associated with the vertex coupling. The second, perhaps physically more important, is that the confinement is not strict and certain tunneling between parts of the graph is allowed. One usually speaks about 'leaky' quantum graphs and describes them using Hamiltonians which can be formally written as $-\Delta - \alpha\delta(\cdot - \Gamma),\ \alpha > 0,$ where Γ is the support of the attractive singular interaction. A discussion of such operators and a survey of their properties can be found in [Ex08].

One can think of the singular interaction as a δ potential in the direction perpendicular to Γ, at least at the points where the manifold supporting the interaction is smooth. If the codimension of Γ is one, however, there are other singular interactions which can be considered, a prime example being the one coming from the one-dimensional δ' interaction [AGH05], that is, operators which can be formally written as

$$H = -\Delta - \beta^{-1}\delta'(\cdot - \Gamma).$$

The formal expression has to be taken with a substantial pinch of salt, of course, because in contrast to the δ interaction, which can be approximated by naturally scaled regular potentials, the problem of approximating δ' is considerably more complicated[4]—see [Še86, CS98, ENZ01] and also [CAZ03+, GH10]. What is important for our present purpose, however, is that irrespective of the meaning of such an interaction, there is a mathematically sound way of defining the above operator through boundary conditions, and moreover, one can also specify it using the associated quadratic form [BLL13].

Apart from the definition, one is naturally interested in spectral properties of such operators, in particular, in relation to the geometry of Γ. In the case of the δ-type singular interaction, we know, for instance, that Γ in the form of a broken or bent line gives rise to a nontrivial discrete spectrum [EI01] and a similar result can also be proven for the δ' interaction [BEL13]. In this paper we want to demonstrate another manifestation of the relation between eigenvalues of H and the shape of Γ. It is inspired by [EY02] in which it was shown how the eigenvalues coming from a δ interaction supported by a $\mathcal{C}^4$ Jordan curve Γ behave in the strong-coupling regime, $\alpha \to \infty$, namely, that after a renormalization consisting of subtracting the Γ-independent divergent term they are in the leading order given by the respective eigenvalue of a one-dimensional Schrödinger operator with a potential determined by the curvature of Γ.

Here we are going to show that in the δ' case, where the strong-coupling limit is $\beta \to 0_+$, we have an analogous result, namely that the asymptotic expansion of the eigenvalues starts from a Γ-independent divergent term followed by the appropriate eigenvalues of a one-dimensional Schrödinger operator, the same as in the δ case. We will also be able to derive an asymptotic expression for the number of eigenvalues dominated by a natural Weyl-type term. In the next section we state the problem properly and formulate the indicated results; the next two sections are devoted to the proofs. The technique is similar to that of [EY02], however, the argument is slightly more complicated because the present form of the associated quadratic form does not allow one to estimate the operator in question using operators with separated variables. In conclusion, we shall comment briefly on possible extensions of the results.

2. Formulation of the problem and main results

We consider a closed curve Γ without self-intersections identified with the graph of

$$\Gamma : [0, L] \to \mathbb{R}^2, \quad s \mapsto (\Gamma_1(s), \Gamma_2(s)),$$

with the component functions $\Gamma_1, \Gamma_2 \in C^4(\mathbb{R})$. We assume conventionally that the curve is parameterized by its arc length, in other words, $\Gamma_1'^2 + \Gamma_2'^2 = 1$. The operator we are interested in acts as the Laplacian outside the interaction support,

$$(H_\beta\psi)(x) = -(\Delta\psi)(x)$$

for $x \in \mathbb{R}^2 \setminus \Gamma$, and its domain is $\mathcal{D}(H_\beta) = \{\psi \in H^2(\mathbb{R}^2 \setminus \Gamma) \mid \partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) = \psi'(x)|_\Gamma,\ -\beta\psi'(x)|_\Gamma = \psi(x)|_{\partial_+\Gamma} - \psi(x)|_{\partial_-\Gamma}\}$, where n_Γ is normal to Γ, which for definiteness

[4] It is important to bear in mind that δ' is *not* approximated by squeezed potentials of zero mean [CAZ03+, GH10] which illustrates that the name, invented originally by Grossmann, Høegh-Krohn, and Mebhkhout, is unfortunate and can lead a reader unfamiliar with the concept to false conclusions. Some authors proposed alternative terms, see e.g. [CS99], but the name stuck and we shall use it, keeping in mind that δ' is *not* a distributional potential.

is supposed to be the outer one, and $\psi(x)|_{\partial_\pm\Gamma}$ are the traces of the function ψ in the regions separated by the curve. The quadratic form associated with this operator is well known [BLL13, proposition 3.15]; it is bounded from below for any $\beta > 0$. In order to write it, we employ the locally orthogonal curvilinear coordinates (s, u) in the vicinity of the curve introduced in relation (3.1) below. With an abuse of notation we write the value of a function $\psi \in H^1(\mathbb{R}^2\setminus\Gamma)$ as $\psi(s, u)$; then we have

$$h_\beta[\psi] = \|\nabla\psi\|^2 - \beta^{-1}\int_\Gamma |\psi(s, 0_+) - \psi(s, 0_-)|^2\,\mathrm{d}s.$$

To state our main theorem we introduce the following operator:

$$S = -\frac{\partial^2}{\partial s^2} - \frac{1}{4}\gamma(s)^2, \tag{2.1}$$

where γ denotes the signed curvature of the loop, $\gamma(s) := (\Gamma_1''\Gamma_2' - \Gamma_1'\Gamma_2'')(s)$. The domain of this operator is $\mathcal{D}(S) = \{\psi \in H^2(0, L) \mid \psi(0) = \psi(L),\ \psi'(0) = \psi'(L)\}$. We denote by μ_j the jth eigenvalue of S with the multiplicity taken into account.

Theorem 2.1. *One has $\sigma_{\mathrm{ess}}(H_\beta) = [0,\infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that*

$$\#\sigma_{\mathrm{disc}}(H_\beta) \geqslant n \quad \text{holds for} \quad \beta \in (0, \beta_n).$$

For any such β we denote by $\lambda_j(\beta)$ the jth eigenvalue of H_β, again counted with its multiplicity. Then the asymptotic expansions

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta|\ln\beta|), \qquad j = 1, \ldots, n,$$

are valid in the limit $\beta \to 0_+$. The error term here depends on j and the eigenvalues of the two operators are numbered in ascending order.

Theorem 2.2. *The counting function $\beta \mapsto \#\sigma_{\mathrm{disc}}(H_\beta)$ admits the asymptotic expansion*

$$\#\sigma_{\mathrm{disc}}(H_\beta) = \frac{2L}{\pi\beta} + \mathcal{O}(|\ln\beta|) \quad \text{as} \quad \beta \to 0_+.$$

3. Proof of theorem 2.1

The essential spectrum of H_β is found in [BLL13, theorem 3.16]. To prove the claim about the discrete one we first need a few auxiliary results. To begin with, we introduce locally orthogonal curvilinear coordinates s and u which allow us to write points (x_1, x_2) in the vicinity of the curve as

$$(x_1, x_2) = (\Gamma_1(s) - u\Gamma_2'(s), \Gamma_2(s) + u\Gamma_1'(s)). \tag{3.1}$$

Since Γ is supposed to be a C^4 smooth closed Jordan curve, it is not difficult to establish that the map (3.1) is injective for all u small enough; for a detailed proof see [EY02].

We choose a strip neighborhood $\Omega_a := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < a\}$ of Γ with the half-width a small enough to ensure the injectivity of (3.1) on Ω_a, and use bracketing to get a two-sided estimate of the operator H_β by imposing the Dirichlet and Neumann conditions at the boundary of Ω_a, i.e.

$$H_N(\beta) \leqslant H_\beta \leqslant H_D(\beta), \tag{3.2}$$

where both the estimating operators correspond to the same differential expression and $\mathcal{D}(H_N(\beta)) = \{\psi \in \mathcal{D}(H_\beta) \mid \partial_{u_+}\psi(s, a) = \partial_{u_-}\psi(s, -a) = 0\}$, while the other is $\mathcal{D}(H_D(\beta)) = \{\psi \in \mathcal{D}(H_\beta) \mid \psi(s, a) = \psi(s, -a) = 0\}$. The operators $H_D(\beta)$ and $H_N(\beta)$ are obviously direct sums of operators corresponding to the parts of the plane separated by the boundary conditions, and since their parts referring to $\mathbb{R}^2 \setminus \overline{\Omega}_a$ are positive, we can neglect them when considering the discrete spectrum. The parts of $H_N(\beta)$ and $H_D(\beta)$ referring to the strip Ω_a are associated with the following quadratic forms,

$$h_{N,\beta}[f] = \|\nabla f\|^2 - \beta^{-1}\int_\Gamma |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s,$$
$$h_{D,\beta}[f] = \|\nabla f\|^2 - \beta^{-1}\int_\Gamma |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s,$$

respectively, the former being defined on $H^1(\Omega_a \setminus \Gamma)$, the latter on $\tilde{H}_0^1(\Omega_a \setminus \Gamma)$ understood as a set of functions which are locally H^1 and vanish at the boundary of Ω_a. Our first task is to rewrite these forms in terms of the curvilinear coordinates s and u.

Lemma 3.1. *Quadratic forms $h_{N,\beta}$, $h_{D,\beta}$ are unitarily equivalent to quadratic forms $q_{N,\beta}$ and $q_{D,\beta}$ which can be written as*

$$q_D[f] = \left\|\frac{\partial_s f}{g}\right\|^2 + \|\partial_u f\|^2 + (f, Vf) - \beta^{-1}\int_0^L |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^L \gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,\mathrm{d}s$$
$$q_N[f] = q_D[f] - \int_0^L \frac{\gamma(s)}{2(1+a\gamma(s))}|f(s,a)|^2\,\mathrm{d}s + \int_0^L \frac{\gamma(s)}{2(1-a\gamma(s))}|f(s,-a)|^2\,\mathrm{d}s$$

defined on $\tilde{H}_0^1((0,L)\times((-a,0)\cup(0,a)))$ and $H^1((0,L)\times((-a,0)\cup(0,a)))$, respectively, with periodic boundary conditions in the variable s. The geometrically induced potential in these formulæ is given by $V = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$ with $g(s,u) := 1 + u\gamma(s)$, and we employ here the conventional shorthands, $\partial_s = \frac{\partial}{\partial s}$ etc.

Proof. We express ∂_s and ∂_u as linear combinations of ∂_{x_1} and ∂_{x_2} with the coefficients $\partial_s x_1 = \Gamma_1' - u\Gamma_2''$, $\partial_s x_2 = \Gamma_2' + u\Gamma_1''$, $\partial_u x_1 = \Gamma_2'$, and $\partial_u x_2 = -\Gamma_1'$. Working out the inverse coordinate transformation we get

$$\partial_{x_1} = g^{-1}(-\Gamma_1'\partial_s - (\Gamma_2' + u\Gamma_1'')\partial_u), \qquad \partial_{x_2} = g^{-1}(-\Gamma_2'\partial_s + (\Gamma_1' - u\Gamma_2'')\partial_u),$$

where $g = (\Gamma_1' - u\Gamma_2'')\Gamma_1' + (\Gamma_2' + u\Gamma_1'')\Gamma_2' = 1 + u\gamma$ because $\Gamma_1'^2 + \Gamma_2'^2 = 1$ holds by assumption. The last relation gives $\Gamma_1'\Gamma_1'' + \Gamma_2'\Gamma_2'' = 0$ which in turn implies $\gamma^2 = \Gamma_1''^2 + \Gamma_2''^2$. Using these identities we can check by a direct computation [EŠ89] that

$$q_{j,\beta}[Uf] = h_{j,\beta}[f]$$

where $(Uf)(s,u) := \sqrt{1+u\gamma(s)}\, f(x_1(s,u), x_2(s,u))$ holds for $j = D, N$ and all functions $f \in \mathcal{D}(h_{j,\beta})$, which proves the claim. □

The forms $q_{N,\beta}$ and $q_{D,\beta}$ are still not easy to handle and we are going to replace the estimate (3.2) by a cruder one in terms of the following forms associated with operators. As for the upper bound, we introduce the quadratic form $q_{a,\beta}^+$ acting as

$$q_{a,\beta}^+[f] = \|\partial_u f\|^2 + (1 - a\|\gamma\|_\infty)^{-2}\|\partial_s f\|^2 + (f, V^{(+)}f)$$
$$-\beta^{-1}\int_0^L |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s + \frac{1}{2}\int_0^L \gamma(s)(|f(s,0_+)|^2 - |f(s,0_-)|^2)\,\mathrm{d}s,$$

where $V^{(+)} := \frac{a\gamma_+''}{2(1-a\|\gamma\|_\infty)^3} - \frac{\gamma^2}{4(1+a\|\gamma\|_\infty)^2}$ with $\gamma_+'' := (\gamma'')_+$ and the positive (negative) part given by the standard convention, $f_\pm := \frac{1}{2}(|f| \pm f)$; we have neglected here the non-positive term $-\frac{5}{4}(u\gamma')^2 g^{-4}$. In contrast to the argument used in the δ interaction case [EY02] the operator $Q^+_{a,\beta}$ associated with this form does not have separated variables, however, one can write it as $Q^+_{a,\beta} = U^+_a \otimes I + \int^{\oplus}_{[0,L)} T^+_{a,\beta}(s)\,\mathrm{d}s$ and we are going to show that the spectrum of the second part associated with the form

$$t^+_{a,\beta}(s)[f] := \|f'\|^2 - \frac{1}{\beta}|f(0_+) - f(0_-)|^2 + \frac{1}{2}\gamma(s)(|f(s,0_+)|^2 - |f(s,0_-)|^2)$$

is independent of s. The operator itself acts as $T^+_{a,\beta}(s)f = -f''$ with the domain

$$\mathcal{D}(T^+_{a,\beta}(s)) = \{f \in H^2((-a,a)\setminus\{0\}) \mid f(a) = f(-a) = 0,$$
$$f'(0_-) = f'(0_+) = -\beta^{-1}(f(0_+) - f(0_-)) + \tfrac{1}{2}\gamma(s)(f(0_+) + f(0_-))\}.$$

Lemma 3.2. *The operator $T^+_{a,\beta}(s)$ has exactly one negative eigenvalue $t_+ = -\kappa_+^2$ provided $\frac{a}{\beta} > 2$ which is independent of s and such that*

$$\kappa_+ = \frac{2}{\beta} - \frac{4}{\beta}\,\mathrm{e}^{-4a/\beta} + \mathcal{O}(\beta^{-1}\,\mathrm{e}^{-8a/\beta}) \quad \textit{holds as} \quad \beta \to 0.$$

Proof. An eigenfunction corresponding to the eigenvalue $-\kappa^2$ and obeying the conditions $f(\pm a) = 0$ and $f'(0_-) = f'(0_+)$ is, up to a multiplicative constant, equal to $\sinh(\kappa(x \mp a))$ for $\pm x \in (0,a)$. The function is odd, hence $f(0_-) = -f(0_+)$ and the s-dependent term does not influence the eigenvalue; the spectral condition is easily seen to be

$$\kappa = \frac{2}{\beta}\tanh(\kappa a). \tag{3.3}$$

We are interested in the asymptotic behavior of the solution as $\beta \to 0_+$. Let us rewrite the condition as $\beta = \frac{2}{\kappa}\tanh(\kappa a)$; since the right-hand side is monotonous as a function of $\kappa > 0$ it is clear that there is at most one eigenvalue and that this happens if $\beta < 2a$. Furthermore, the right-hand side is less that $\frac{2}{\kappa}$ which means that $\kappa < 2\beta^{-1}$ and the inequality turns to an equality as $\beta \to 0$ and $\kappa \to \infty$. Next we employ the Taylor expansion

$$\frac{2}{\beta}\tanh(\kappa a) = \frac{2}{\beta}(1 - 2\,\mathrm{e}^{-2\kappa a} + 2\,\mathrm{e}^{-4\kappa a} + \mathcal{O}(\mathrm{e}^{-6\kappa a})),$$

and since $\kappa \to \frac{2}{\beta}$ as $\beta \to 0$, relation (3.3) yields the sought result. □

Next we estimate in a similar fashion the operator with the Neumann boundary condition which we need to get a lower bound. To this aim we employ the quadratic form $q^-_{a,\beta}$, defined as

$$\begin{aligned} q^-_{a,\beta}[f] = \|\partial_u f\|^2 + (1 + a\|\gamma\|_\infty)^{-2}\|\partial_s f\|^2 + (f, V^{(-)}f) \\ -\beta^{-1}\int_0^L |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s - \frac{1}{2}\int_0^L \gamma(s)(|f(s,0_+)|^2 - |f(s,0_-)|^2)\,\mathrm{d}s \\ -\|\gamma\|_\infty\int_0^L |f(s,a)|^2\,\mathrm{d}s - \|\gamma\|_\infty\int_0^L |f(s,-a)|^2\,\mathrm{d}s, \end{aligned}$$

where $V^{(-)} = -\frac{a\gamma_+''}{2(1-a\|\gamma\|_\infty)^3} - \frac{5(a\gamma_+')^2}{4(1-a\|\gamma\|_\infty)^4} - \frac{\gamma^2}{4(1-a\|\gamma\|_\infty)^2}$. As before, the operator associated with the quadratic form can be written as $Q^-_{a,\beta} = U^-_a \otimes I + \int^{\oplus}_{[0,L)} T^-_{a,\beta}(s)\,\mathrm{d}s$, where the operator $T^-_{a,\beta}(s)$ referring to the transverse variable acts for any $s \in [0,L)$ as $T^-_{a,\beta}(s)f = -f''$ with the domain

$$\mathcal{D}(T^-_{a,\beta}(s)) = \{f \in H^2((-a,a)\setminus\{0\}) \mid \mp\|\gamma\|_\infty f(\pm a) = f'(\pm a)\,,$$
$$f'(0_-) = f'(0_+) = -\beta^{-1}(f(0_+) - f(0_-)) + \tfrac{1}{2}\gamma(s)(f(0_+) + f(0_-))\}. \tag{3.4}$$

We are going to estimate the spectrum of $T^-_{a,\beta}(s)$ and check its independence of s.

Lemma 3.3. *The operator $T^-_{a,\beta}(s)$ has exactly one negative eigenvalue $t_- = -\kappa_-^2$ for any β small enough; the latter is independent of s and for $\beta \to 0$ we have*

$$\kappa_- = \frac{2}{\beta} + \frac{4}{\beta}\frac{2-\beta\|\gamma\|_\infty}{2+\beta\|\gamma\|_\infty}\,\mathrm{e}^{-4a/\beta} + \mathcal{O}\left(-\frac{4}{\beta}\left(\frac{2-\beta\|\gamma\|_\infty}{2+\beta\|\gamma\|_\infty}\right)^2 \mathrm{e}^{-8a/\beta}\right).$$

Proof. The function satisfying $f''(x) = \kappa^2 f(x)$ for $x \neq 0$ together with the boundary conditions $\mp\|\gamma\|_\infty f(\pm a) = f'(\pm a)$, which has its derivative continuous at $x = 0$, is of the form

$$f(x) = \begin{cases} A\,\mathrm{e}^{\kappa x} + B\,\mathrm{e}^{-\kappa x} & \text{if} \quad x \in (-a, 0) \\ C\,\mathrm{e}^{\kappa x} + D\,\mathrm{e}^{-\kappa x} & \text{if} \quad x \in (0, a) \end{cases}$$

The constant A is arbitrary, while for the others the requirements imply $B = AZ\,\mathrm{e}^{-2\kappa a}$ with $Z := \frac{\kappa-\|\gamma\|_\infty}{\kappa+\|\gamma\|_\infty}$ and $D = -A,\ C = -B$. The remaining property from (3.4) leads to

$$\kappa(A-B) = \frac{1}{\beta}(A+B-C-D) + \frac{1}{2}\gamma(s)(A+B+C+D),$$

and since the last term vanishes we can rewrite the spectral condition as

$$\kappa = \frac{2}{\beta}\frac{1+Z\,\mathrm{e}^{-2\kappa a}}{1-Z\,\mathrm{e}^{-2\kappa a}}.$$

As before, we are interested in the regime $\beta \to 0_+$. Note that as long as $Z > 0$ we have $\kappa > 2\beta^{-1}$, hence κ is large and $\xi = Z\,\mathrm{e}^{-2\kappa a}$ is small and the expansion

$$\kappa = \frac{2}{\beta}\frac{1+\xi}{1-\xi} = \frac{2}{\beta}(1+\xi)(1+\xi+\xi^2+\mathcal{O}(\xi^3))$$

yields the stated behavior of κ as $\beta \to 0_+$. Since we are interested in the strong-coupling situation, we may assume $Z > 0$ without loss of generality. This assumption is satisfied for $2\beta^{-1} > \|\gamma\|_\infty$, and the uniqueness of the eigenvalue is a consequence of the above spectral condition and the monotonicity of the function $\kappa \mapsto \frac{1}{\kappa}\frac{1+Z\,\mathrm{e}^{-2\kappa a}}{1-Z\,\mathrm{e}^{-2\kappa a}}$ which can be checked by a direct computation. □

Next we estimate the eigenvalues of the operators $U_a^\pm$, referring to the longitudinal part in the expressions for $Q^\pm_{a,\beta}$ in a similar way to [EY02]; they correspond to the second and third terms in the definition of the quadratic forms $q^\pm_{a,\beta}$.

Lemma 3.4. *There is a positive C independent of a and j such that*

$$|\mu_j^\pm(a) - \mu_j| \leqslant Caj^2$$

holds for $j \in \mathbb{N}$ and $0 < a < \frac{1}{2\|\gamma\|_\infty}$, where $\mu_j^\pm(a)$ are the eigenvalues of $U_a^\pm$, respectively, with the multiplicity taken into account.

Proof. We employ the operator $S_0 = -\partial_s^2$ with the periodic boundary conditions, i.e. the domain $\mathcal{D}(S_0) = \{f \in L^2((0,L)) \mid f(0) = f(L),\ f'(0) = f'(L)\}$; its eigenvalues, counting multiplicity, are $4[\frac{j}{2}]^2\frac{\pi^2}{L^2}$, $j = 1, 2, \dots$, where $[\cdot]$ as usual denotes the entire part. Its difference from our comparison operator (2.1) on $L^2(0, L)$ is easily estimated,

$$\|S - S_0\| \leqslant \tfrac{1}{4}\|\gamma\|_\infty^2,$$

and consequently, by the min–max principle we have

$$\left|\mu_j - 4\left[\frac{j}{2}\right]^2 \frac{\pi^2}{l^2}\right| \leqslant \frac{1}{4}\|\gamma\|_\infty^2 \tag{3.5}$$

for $j \in \mathbb{N}$. Next we can use another simple estimate,

$$U_a^+ - \frac{1}{(1-a\|\gamma\|_\infty)^2} S = \frac{a\gamma_+''}{2(1-a\|\gamma\|_\infty)^3} - \frac{\gamma^2}{4(1+a\|\gamma\|_\infty)^2} + \frac{\gamma^2}{4(1-a\|\gamma\|_\infty)^2},$$

and since the last two terms combine to $a\|\gamma\|_\infty\gamma^2(1-a^2\|\gamma\|_\infty^2)^{-2}$, we infer that

$$\left|\mu_j^+ - \frac{\mu_j}{(1-a\|\gamma\|_\infty)^2}\right| \leqslant c_0 a \tag{3.6}$$

holds for some $c_0 > 0$ and any $j \in \mathbb{N}$. Combining now (3.5) and (3.6) we get

$$\begin{aligned}|\mu_j^+ - \mu_j| &\leqslant \left|\mu_j^+ - \frac{\mu_j}{(1-a\|\gamma\|_\infty)^2}\right| + |\mu_j| \cdot \left|\frac{1-(1-a\|\gamma\|_\infty)^2}{(1-a\|\gamma\|_\infty)^2}\right| \\ &\leqslant c_0 a + c_1 a|\mu_j| \leqslant Caj^2\end{aligned}$$

with suitable constants. The second inequality is checked in a similar way: we use

$$\begin{aligned}U_a^- - \frac{1}{(1+a\|\gamma\|_\infty)^2} S = &-\frac{a\gamma_+''}{2(1-a\|\gamma\|_\infty)^3} - \frac{5a^2(\gamma_+')^2}{4(1-a\|\gamma\|_\infty)^4} \\ &-\frac{a\|\gamma\|_\infty}{(1-a\|\gamma\|_\infty)^2(1+a\|\gamma\|_\infty)^2}\gamma^2,\end{aligned}$$

which implies

$$\left\|U_a^- - \frac{1}{(1+a\|\gamma\|_\infty)^2} S\right\| \leqslant \tilde{c}_0 a + \tilde{c}_1 a^2 \leqslant c_2 a,$$

where in the second inequality we employed the fact that a is bounded. With help of the min–max principle we then get

$$\left|\mu_j^- - \frac{\mu_j}{(1+a\|\gamma\|_\infty)^2}\right| \leqslant c_2 a,$$

hence finally we arrive at the inequality

$$|\mu_j^- - \mu_j| \leqslant c_2 a + |\mu_j| \left|\frac{1-(1+a\|\gamma\|_\infty)^2}{(1+a\|\gamma\|_\infty)^2}\right| \leqslant c_2 a + c_3 a|\mu_j| \leqslant Caj^2$$

valid for a suitable C which completes the proof. □

Now we are ready to prove our first main result.

We define $a(\beta) = -\frac{3}{4}\beta \ln\beta$ and denote the eigenvalues of the operators $T_{a(\beta),\beta}^{\pm}$ as $t_{\pm,\beta}^j$, respectively, taking their multiplicities into account. From Lemmata 3.2 and 3.3 we know that $t_{\pm,\beta}^1 = t_\pm$ for small enough β, while $t_{\pm,\beta}^j \geqslant 0$ holds for $j > 1$. Collecting the estimates worked out above we have

$$\begin{aligned}Q_{a(\beta),\beta}^- = U_{a(\beta)}^- \otimes I + \int_{(0,L)}^{\oplus} T_{a(\beta),\beta}^-(s)\, \mathrm{d}s \leqslant H_N(\beta) \leqslant H_\beta \\ \leqslant H_D(\beta) \leqslant U_{a(\beta)}^+ \otimes I + \int_{(0,L)}^{\oplus} T_{a(\beta),\beta}^+(s)\, \mathrm{d}s = Q_{a(\beta),\beta}^+\end{aligned} \tag{3.7}$$

and the eigenvalues of the operators $Q_{a(\beta),\beta}^{\pm}$ between which we squeeze our singular Schrödinger operator H_β are naturally $t_{\pm,\beta}^k + \mu_j^{\pm}(a(\beta))$ with $k, j \in \mathbb{N}$. Those with $k \geqslant 2$ and $j \in \mathbb{N}$ are uniformly bounded from below in view of the inequality

$$t_{\pm,\beta}^k + \mu_j^{\pm}(a(\beta)) \geqslant \mu_1^{\pm}(a(\beta)) = \mu_1 + \mathcal{O}(-\beta\ln\beta), \tag{3.8}$$

hence we can focus on $k = 1$ only. For $j \in \mathbb{N}$ we denote

$$\omega^j_{\pm,\beta} = t^1_{\pm,\beta} + \mu^\pm_j(a(\beta)).$$

With our choice of $a(\beta)$ we have $e^{-4\kappa a} = \beta^3$, so from the above lemmata we get $\kappa_\pm = \frac{2}{\beta} + \mathcal{O}(\beta)$ and $\mu^\pm_j(a(\beta))$ differ from μ_j by $\mathcal{O}(-\beta j^2 \ln \beta)$; putting these estimates together we can conclude that

$$\omega^j_{\pm,\beta} = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(-\beta \ln \beta) \quad \text{as} \quad \beta \to 0_+ \tag{3.9}$$

with the error term in general dependent on j. Combining (3.8) and (3.9) we can conclude that to any $n \in \mathbb{N}$ there is a $\beta(n) > 0$ such that

$$\omega^n_{+,\beta} \leqslant 0, \quad \omega^n_{+,\beta} < t^k_{+,\beta} + \mu^+_j(a(\beta)) \quad \text{and} \quad \omega^n_{-,\beta} < t^k_{-,\beta} + \mu^-_j(a(\beta))$$

holds for $\beta \leqslant \beta(n)$, $k \geqslant 2$, and $j \geqslant 1$. Hence the jth eigenvalue of $Q^\pm_{a(\beta),\beta}$, counting multiplicity, is $\omega^j_{\pm,\beta}$ for all $j \leqslant n$ and $\beta \leqslant \beta(n)$. Furthermore, for $\beta \leqslant \beta(n)$ we denote $\xi^j_+(\beta)$ and $\xi^j_-(\beta)$ as the jth eigenvalue of $H_D(\beta)$ and $H_N(\beta)$, respectively; then from (3.7) and the min–max principle we obtain

$$\omega^n_{-,\beta} \leqslant \xi^j_-(\beta), \quad \xi^j_+(\beta) \leqslant \omega^n_{+,\beta}$$

for $j = 1, 2, \ldots, n$, which in particular implies $\xi^n_+(\beta) < 0$. Using the min–max principle once again we conclude that H_β has at least n eigenvalues in the interval $(-\infty, \xi^n_+(\beta))$ and for any $1 \leqslant j \leqslant n$ we have $\xi^j_-(\beta) \leqslant \lambda_j \leqslant \xi^j_+(\beta)$ which completes the proof.

4. Proof of theorem 2.2

For a self-adjoint operator A with $\inf \sigma_{\text{ess}}(A) = 0$ we put $N^-(A) := \#\{\sigma_d(A) \cap (-\infty, 0)\}$. In view of (3.7) the eigenvalue number of H_β can be estimated as

$$N^-(Q^-_{a,\beta}) \leqslant N^-(H_N(\beta)) \leqslant \#\sigma_d(H_\beta) \leqslant N^-(H_D(\beta)) \leqslant N^-(Q^+_{a,\beta}). \tag{4.1}$$

In order to use this estimate we define

$$K^\pm_\beta = \{j \in \mathbb{N} |\, \omega^j_{\pm,\beta} < 0\}$$

and derive the following asymptotic expansions of these quantities.

Lemma 4.1. *In the strong-coupling limit, $\beta \to 0_+$, we have*

$$\#K^\pm_\beta = \frac{2L}{\pi\beta} + \mathcal{O}(|\ln \beta|). \tag{4.2}$$

Proof. We choose K such that $\beta^{-1} > K > 0$ and $(\beta^{-1} - K)^2 < \beta^{-2} - 4\beta - 16^{-1}\|\gamma\|^2_\infty$; this can be obviously done for all sufficiently small β. With the preceding proof in mind we can write

$$K^+_\beta = \left\{j \in \mathbb{N} |\, t^1_{+,\beta} + \mu^+_j(a(\beta)) < 0\right\}.$$

Lemma 3.2 allows us to make the following estimate,

$$K^+_\beta \supset \left\{ j \in \mathbb{N} \middle|\, \mu_j + Ca(\beta) j^2 < \frac{4}{\beta^2} - \frac{16}{\beta^2} e^{-4a(\beta)/\beta} = \frac{4}{\beta^2} - 16\beta \right\};$$

using further (3.5) and the indicated choice of K we infer that

$$
\begin{aligned}
K_\beta^+ &\supset \left\{ j \in \mathbb{N} |\, 4\left[\frac{j}{2}\right]^2 \frac{\pi^2}{L^2} + Ca(\beta)j^2 < \frac{4}{\beta^2} - 16\beta - \frac{1}{4}\|\gamma\|_\infty^2 \right\} \\
&\supset \left\{ j \in \mathbb{N} \Big| \, j^2 \frac{\pi^2}{L^2} - \frac{3}{4} C\beta \ln\beta \, j^2 < 4\left(\frac{1}{\beta} - K\right)^2 \right\} \\
&\supset \left\{ j \in \mathbb{N} |\, j < 2\left(\frac{1}{\beta} - K\right)\left(\frac{\pi^2}{L^2} - \frac{3}{4} C\beta \ln\beta\right)^{-1/2} \right\}.
\end{aligned}
$$

We employ the Taylor expansion $(M+x)^{-1/2} = M^{-1/2} - \frac{1}{2} x M^{-3/2} + \mathcal{O}(x^2)$; since we are interested in the asymptotics $\beta \to 0_+$, we rewrite the right-hand side of the last inequality as

$$
2\left(\frac{1}{\beta} - K\right)\left(\frac{\pi^2}{L^2} - \frac{3}{4} C \ln\beta\right)^{-\frac{1}{2}} \simeq 2\left(\frac{1}{\beta} - K\right)\left[\frac{L}{\pi} + \frac{3}{8} C\beta \ln\beta \left(\frac{L}{\pi}\right)^3\right],
$$

which allows us to infer that

$$
\# K_\beta^+ \geqslant \frac{2L}{\pi\beta} + \mathcal{O}(|\ln\beta|) \tag{4.3}
$$

holds as $\beta \to 0_+$. In a similar way we estimate $\# K_\beta^-$. First we choose a number K' satisfying $0 < K' < \left(4\beta + \frac{\|\gamma\|_\infty^2}{16}\right)^{1/2}$ and note that $\frac{1}{\beta^2} + 4\beta + \frac{\|\gamma\|_\infty^2}{16} < (\frac{1}{\beta} + K')^2$. Then we have

$$
\begin{aligned}
K_\beta^- &= \left\{ j \in \mathbb{N} \big| \, t_{-,\beta}^1 + \mu_j^-(a(\beta)) < 0 \right\} \\
&\subset \left\{ j \in \mathbb{N} \Big| \, \mu_j - Ca(\beta)j^2 < \frac{4}{\beta^2} + \frac{16}{\beta^2} \frac{2 - \beta\|\gamma\|_\infty}{2 + \beta\|\gamma\|_\infty} \mathrm{e}^{-4a(\beta)/\beta} \right\} \\
&\subset \left\{ j \in \mathbb{N} \Big| \, \mu_j + \frac{3}{4} C\beta \ln\beta \, j^2 < \frac{4}{\beta^2} + 16\beta \frac{2 - \beta\|\gamma\|_\infty}{2 + \beta\|\gamma\|_\infty} \right\}.
\end{aligned}
$$

With the help of the fact that $2(j-1) \geqslant j$ for $j > 1$ we further have

$$
\begin{aligned}
K_\beta^- &\subset \{1\} \cup \left\{ j \geqslant 2 |\, \left(\frac{(j-1)\pi}{L}\right)^2 + \frac{3}{4} C\beta \ln\beta (j-1)^2 < \frac{4}{\beta^2} + 16\beta + \frac{\|\gamma\|_\infty^2}{4} \right\} \\
&\subset \{1\} \cup \left\{ j \geqslant 2 |\, (j-1)^2 < \left(\frac{4}{\beta^2} + 16\beta + \frac{\|\gamma\|_\infty^2}{4}\right)\left(\left(\frac{\pi}{L}\right)^2 + \frac{3}{4} C\beta \ln\beta\right)^{-1} \right\} \\
&\subset \{1\} \cup \left\{ j \geqslant 2 |\, j < 1 + 2\left(\frac{1}{\beta} + K'\right)\left(\left(\frac{\pi}{L}\right)^2 + \frac{3}{4} C\beta \ln\beta\right)^{-1/2} \right\}.
\end{aligned}
$$

Now we can estimate the expression on the right-hand side of the last inequality in the asymptotic regime $\beta \to 0_+$ as

$$
2\left(\frac{1}{\beta} + K'\right)\left(\left(\frac{\pi}{L}\right)^2 + \frac{3}{4} C\beta \ln\beta\right)^{-\frac{1}{2}} \simeq \frac{2L}{\pi\beta} + \mathcal{O}(|\ln\beta|).
$$

In combination with the above inclusions this leads to

$$
\# K_\beta^- \leqslant \frac{2l}{\pi\beta} + \mathcal{O}(|\ln\beta|) \tag{4.4}
$$

as $\beta \to 0_+$. Finally, we know that $t_{+,\beta}^1 < t_{-,\beta}^1$ which implies $K_\beta^+ \subset K_\beta^-$, and this together with (4.3) and (4.4) concludes the proof. □

We also need to estimate the second eigenvalue of the operators $T_{a(\beta),\beta}^-(s)$.

Lemma 4.2. *$T^-_{a,\beta}(s)$ with a fixed $s \in [0, L)$ has no eigenvalues in $\big[0, \min\big\{\frac{\|\gamma\|_\infty}{2a}, \big(\frac{\pi}{4a}\big)^2\big\}\big)$ provided $0 < \beta < 2a$.*

Proof. Let us check first that zero is not an eigenvalue. The corresponding eigenfunction should have to be linear and the conditions $\mp\|\gamma\|_\infty f(\pm a) = f'(\pm a)$ and $f'(0_-) = f'(0_+)$ would require $f(x) = \pm A(\mp\|\gamma\|_\infty x + 1 + \|\gamma\|_\infty a)$ for $\pm x \in (0, a)$, and as in lemma 3.3 the spectral condition would read $-\|\gamma\|_\infty = \frac{2}{\beta}(1 + \|\gamma\|_\infty a)$ which cannot be true because the right-hand side is positive. Furthermore, the spectral condition for an eigenvalue $k^2 > 0$ is found again as in lemma 3.3; after a simple calculation we find that it reads

$$\frac{1}{2}\beta = \frac{1}{k}\frac{\|\gamma\|_\infty \tan ka + k}{\|\gamma\|_\infty - k\tan ka}.$$

The right-hand side can be estimated by $\frac{1+\|\gamma\|_\infty a}{\|\gamma\|_\infty - k^2 a}$ provided that $ka < \frac{\pi}{2}$ and at the same time $\|\gamma\|_\infty - k\tan ka > 0$; by finding the value for which this expression equals $\frac{1}{2}\beta$ we would obviously get a lower bound to k. Rewriting the condition as

$$-ka^2 = (1 + \|\gamma\|_\infty a)\frac{2}{\beta} - \|\gamma\|_\infty$$

we see that the left-hand side is negative while the right-hand side is positive under our assumption, hence one has to determine the restriction coming from the condition $\|\gamma\|_\infty - k\tan ka > 0$. In particular, for $ka < \frac{1}{4}\pi$ this is true provided $\|\gamma\|_\infty - 2k^2 a > 0$, which means that the spectral problem has no solution if k^2 is smaller than either $\frac{\|\gamma\|_\infty}{2a}$ or $\big(\frac{\pi}{4a}\big)^2$, which concludes the argument. □

Now we are ready to prove our second main result.
We begin by showing that the relation

$$N^-(Q^-_{a(\beta),\beta}) = \#K^-_\beta \tag{4.5}$$

holds for any sufficiently small $\beta > 0$. We know that all the eigenvalues of $Q^-_{a(\beta),\beta}$ can be written as $\{t^j_{-,\beta} + \mu^-_k(a(\beta))\}_{j,k\in\mathbb{N}}$ with the multiplicity taken into account. From the previous lemma we have $t^2_{-,\beta} > \min\big\{\frac{\|\gamma\|_\infty}{2a}, \big(\frac{\pi}{4a}\big)^2\big\}$ which together with $|\mu^-_j(a) - \mu_j| \leqslant Caj^2$ implies the existence of a β_0 such that

$$t^j_{-,\beta} + \mu^-_k(a(\beta)) > 0$$

holds for $j > 1,\ k \geqslant 1$, and $\beta \in (0, \beta_0)$. This implies

$$\begin{aligned} N^-(Q^-_{a(\beta),\beta}) &= \#\{(k, j) \in \mathbb{N}^2 |\, t^k_{-,\beta} + \mu^-_j(a(\beta)) < 0\} \\ &= \#\{j \in \mathbb{N} |\, t^1_{-,\beta} + \mu^-_j(a(\beta)) < 0\} =: K^-_\beta, \end{aligned}$$

i.e. the relation (4.5); combining it with (4.1) we obtain

$$\#K^+_\beta \leqslant \#\sigma_d(H_\beta) \leqslant N^-(Q^-_{a,\beta}) = \#K^-_\beta,$$

which by virtue of lemma 4.1 concludes the proof.

5. Concluding remarks

We have seen that, despite very different eigenfunctions, the δ' 'leaky loops' behave in the strong-coupling regime similarly to their δ counterparts: the number of negative eigenvalues is given in the leading order by a Weyl-type term, and the eigenvalues themselves are after a natural renormalization determined by the one-dimensional Schrödinger operator with the known curvature-induced potential.

The question is whether and how the current results can be extended. The bracketing technique we used would work for infinite smooth curves Γ without ends, provided that suitable regularity assumptions were imposed. If, on the other hand, the curve is finite or semi-infinite the situation becomes more complicated because one has to impose appropriate boundary conditions at the endpoints of the interval on which the comparison operator (2.1) is defined. One can modify the present argument to get an estimate on the number of eigenvalues because there those boundary conditions play no role, the counting functions in the Dirichlet and Neumann case differing by an $\mathcal{O}(1)$ term. For an eigenvalue position estimate, on the other hand, this is not sufficient and one conjectures that the *Dirichlet* comparison operator has to be used. For a two-dimensional open arc Γ supporting a δ interaction this conjecture has recently been proved [EP12]; the argument is more complicated because one cannot use operators with separated variables. We believe that the same method could work in the δ' case too—however, the question is not simple and we postpone discussing it to another paper.

On the other hand, finding the asymptotics in the case when Γ is not smooth, or even has branching points, represents a much harder problem and the answer is not known even in the δ case, although some inspiration can be found in the squeezing limits of Dirichlet tubes; see, e.g., [CE07].

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Spectral asymptotics of a strong δ' interaction supported by a surface

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ABSTRACT

We derive asymptotic expansion for the spectrum of Hamiltonians with a strong attractive δ' interaction supported by a smooth surface in $\mathbb{R}^3$, either infinite and asymptotically planar, or compact and closed. Its second term is found to be determined by a Schrödinger type operator with an effective potential expressed in terms of the interaction support curvatures.

1. Introduction

Quantum mechanics of particles confined to curves, graphs, tubes, surfaces, layers, and other geometrically nontrivial objects is a rich and inspirative subject. On one hand it is useful physically, in particular, to describe various nanostructures, and at the same time it offers numerous interesting mathematical problems. Models of "leaky" structures [1] in which the confinement is realized by an attractive potential have the advantage that they take quantum tunneling into account. The potential is often taken singular, of the δ type, because it is easier to handle [2].

Very recently also more singular couplings of the δ' type attracted attention. The corresponding Hamiltonians can be formally written as

$$H_\beta = -\Delta - \beta^{-1}\delta'(\cdot - \Gamma), \tag{1}$$

where Γ is a smooth surface supporting the interaction. Some prefer to write the interaction term as $\beta^{-1}(\delta'(\cdot - \Gamma), \cdot)\delta'(\cdot - \Gamma)$ to stress that the interaction is invariant with respect to mirror reflection. What is important, however, is that either of the expressions is purely formal. A proper definition which employs the standard δ' concept [3] will be given below, here we only note that we write β^{-1} to underline that a *strong* δ' interaction corresponds to *small* values of the parameter β. We also note that investigation of such δ' interactions is not just a mathematical exercise. Due to a seminal idea of Cheon and Shigehara [4] made rigorous in [5,6] they can be approximated by a scaled "tripple-layer" potential combination. The possibility of forming such systems with barriers which become more opaque as the energy increases is no doubt physically attractive.

The subject of this letter is the strong coupling asymptotics of bound states of operators (1) with an attractive δ' interaction supported by a finite or infinite surface in $\mathbb{R}^3$. The analogous problem for δ interaction supported by infinite surface was solved in [7]. As in this case, we are going to show that the asymptotics is determined by the geometry of Γ. As a byproduct, we will demonstrate the existence of bound states for sufficiently small β for non-planar infinite surfaces which are asymptotically planar, in a way alternative to the argument proposed recently in [8].

2. The Hamiltonian

The first thing to do is to define properly the operator (1). It acts, of course, as Laplacian outside of the surface Γ

$$(H_\beta\psi)(x) = -(\Delta\psi)(x)$$

for $x \in \mathbb{R}^3 \setminus \Gamma$ and the interaction will be expressed through suitable boundary conditions on the surface which, in accord with [3], would include continuity of the normal derivative together with a jump of the function value. Specifically, the domain of the operator will be

$$\mathcal{D}(H_\beta) = \Big\{\psi \in H^2(\mathbb{R}^3 \setminus \Gamma) \;\Big|\; \partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) =: \psi'(x)\big|_\Gamma, \\ -\beta\psi'(x)\big|_\Gamma = \psi(x)\big|_{\partial_+\Gamma} - \psi(x)\big|_{\partial_-\Gamma}\Big\},$$

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where n_Γ is the normal to Γ and $\psi(x)|_{\partial_\pm\Gamma}$ are the appropriate traces of the function ψ; all these quantities exist in view of the Sobolev embedding theorem. Being interested in the attractive δ' interactions, we choose the above form of boundary conditions with $\beta > 0$. Another way to define the operator H_β is by the means of the associated quadratic form as discussed in [2]. Its domain is $H^1(\mathbb{R}^3 \setminus \Gamma)$ and the form value for a function $\psi \in H^1(\mathbb{R}^3 \setminus \Gamma)$ is given by

$$h_\beta[\psi] = \|\nabla\psi\|^2 - \beta^{-1}\big\|\psi(x)\big|_{\partial_+\Gamma} - \psi(x)\big|_{\partial_-\Gamma}\big\|^2_{L^2(\Gamma)}. \tag{2}$$

As indicated we are interested in the spectrum of H_β in the strong-coupling regime, $\beta \to 0_+$, for two kinds of surfaces Γ. The first is an infinite surface of which we assume that:

(a1) Γ is C^4 smooth and allows a global normal parametrization with uniformly bounded elliptic tensor,
(a2) Γ has no "near self-intersections", i.e. there exists its symmetric layer neighborhood of a finite thickness which does not intersect with itself,
(a3) Γ is asymptotically planar in the sense that its curvatures vanish as the geodetic distance from a fixed point tends to infinity,

and finally

(a4) trivial case is excluded, Γ is not a plane.

In fact, the assumption (a1) can be weakened in a way similar to [9], however, for the sake of simplicity we stick to the existence of a global normal parametrization. The second class to consider are finite surfaces. The compactness makes the assumptions simpler in this case, on the other hand, we have to require additionally absence of a boundary:

(b) Γ is a closed C^4 smooth surface of a finite genus.

In this case no global parametrization exists, of course, but the geometry of Γ can be described by an atlas of maps representing normal parameterizations with a uniformly bounded elliptic tensor.

3. Geometric preliminaries

Let us collect now some needed facts about the geometry of the surface and its neighborhoods; for a more complete information we refer, e.g., to [10]. We consider infinite surfaces first and we introduce normal coordinates on Γ starting from a local exponential map $\gamma : T_o\Gamma \to U_o$ with the origin $o \in \Gamma$ to the neighborhood U_o of the point o; the coordinates s are given by

$$s = (s_1, s_2) \to \exp_o\left(\sum_i s_i e_i(o)\right) \tag{3}$$

where $\{e_1(o), e_2(o)\}$ is an orthonormal basis of $T_o\Gamma$. By assumption (a1) one can find a point $o \in \Gamma$ such that the map (3) can be extended to a global diffeomorphism from $T_o\Gamma \simeq \mathbb{R}^2$ to Γ.

Using these coordinates, we express components of the surface metric tensor $g_{\mu\nu}$ as $g_{\mu\nu} = \gamma_{,\mu}\cdot\gamma_{,\nu}$ and denote $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. The invariant surface element is denoted as $\mathrm{d}\Gamma = g^{\frac{1}{2}}\mathrm{d}^2 s$ where $g := \det g_{\mu\nu}$. The unit normal $n(s)$ is defined as the cross product of the linearly independent tangent vectors $\gamma_{,\mu}$, i.e. $n(s) = \frac{\gamma_{,1}\times\gamma_{,2}}{|\gamma_{,1}\times\gamma_{,2}|}$. The Gauss curvature K and mean curvature M can be calculated by means of the Weingarten tensor $h^\nu_\mu := -n_{,\mu}\cdot\gamma_{,\sigma}g^{\sigma\nu}$,

$$K = \det h^\nu_\mu = k_1 k_2, \qquad M = \frac{1}{2}\operatorname{Tr} h^\nu_\mu = \frac{1}{2}(k_1 + k_2).$$

We recall that the eigenvalues of h^ν_μ are the principal curvatures $k_{1,2}$ and that the identity $K - M^2 = -\frac{1}{4}(k_1 - k_2)^2$ holds.

We also need neighborhoods of the surface Γ. A layer Ω_d of halfwidth $d > 0$ will be defined as the image of $D_d := \{(s,u) : s \in \mathbb{R}^2, u \in (-d,d)\}$ by the map

$$\mathcal{L}: \quad D_d \ni q \equiv (s,u) \to \gamma(s) + un(s) \tag{4}$$

This definition provides at the same time a parametrization of Ω_d, and the assumption (a2) can be rephrased as

(a2) there is a $d_0 > 0$ such that the map (4) is injective for any $d < d_0$.

Moreover, in view of (a1) such an $\mathcal{L}$ is a diffeomorphism, which will be crucial for the considerations to follow. The layer Ω_d can be regarded as a manifold with a boundary and characterized by the metric tensor which can be expressed in the parametrization (4) as

$$G_{ij} = \begin{pmatrix} (G_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix},$$

where $G_{\mu\nu} = (\delta^\sigma_\mu - uh^\sigma_\mu)(\delta^\sigma_\rho - uh^\sigma_\rho)g_{\rho\nu}$. We use here the convention in which the Latin indices run through $1, 2, 3$, numbering the coordinates (s_1, s_2, u) in Ω_d, and the Greek ones through $1, 2$. The volume element of the manifold Ω_d can be written in the form $\mathrm{d}\Omega_d := \sqrt{G}\,\mathrm{d}^2 s\,\mathrm{d}u$ with

$$G := \det G_{ij} = g\big[(1 - uk_1)(1 - uk_2)\big]^2 = g\big(1 - 2Mu + Ku^2\big)^2;$$

with the future purpose in mind we introduce a shorthand for the last factor, $\xi(s,u) := 1 - 2M(s)u + K(s)u^2$. The curvatures also allow us to express more explicitly the next assumption:

(a3) $K, M \to 0$ as $|s| := \sqrt{s_1^2 + s_2^2} \to \infty$.

Recall next a few useful estimates made possible by the assumption (a3), cf. [11]. In combination with (a1) and (a2) it implies that the principal curvatures k_1 and k_2 are uniformly bounded. We set

$$\rho := \big(\max\{\|k_1\|_\infty, \|k_2\|_\infty\}\big)^{-1};$$

note that $\rho > d_0$ holds for the critical halfwidth of assumption (a2). It can be checked easily that for a given $d < \rho$ the following inequalities are satisfied in the layer neighborhood Ω_d of Γ,

$$C_-(d) \le \xi \le C_+(d), \tag{5}$$

where $C_\pm := (1 \pm d\rho^{-1})^2$, and this in turn implies

$$C_-(d)g_{\mu\nu} \le G_{\mu\nu} \le C_+(d)g_{\mu\nu}. \tag{6}$$

Since the metric tensor $g_{\mu\nu}$ uniformly elliptic by assumption, we also have

$$c_-\delta_{\mu\nu} \le g_{\mu\nu} \le c_+(d)\delta_{\mu\nu} \tag{7}$$

as a matrix inequality for some positive constants $c_\pm$.

Let us briefly describe modifications needed if we pass to closed surfaces. As we have indicated a global parametrization is replaced now by a finite atlas $\mathcal{A}$ of maps; in each part $\mathcal{M}_i$ we introduce normal coordinates and define layer neighborhoods by the maps $\hat{\mathcal{M}}_i$ on $D_{i,d} := \{(s,u) : s \in \operatorname{dom}\mathcal{M}_i, u \in (-d,d)\}$ with a given $d > 0$,

$$\hat{\mathcal{M}}_i: \quad \mathrm{D}_{i,d} \ni q \equiv (s,u) \to \gamma_i(s) + un(s) \tag{8}$$

In view of assumption (b) there is a critical $d_0 > 0$ such that every map $\hat{\mathcal{M}}_i : D_{i,d} \to \Omega_d$ from $\mathcal{A}$ is injective provided $d < d_0$ and a diffeomorphism. Furthermore, $\hat{\mathcal{M}}_i(s_i,u_i)=\hat{\mathcal{M}}_j(s_j,u_j)$ implies $\mathcal{M}_i(s_i)=\mathcal{M}_j(s_j)$. The above estimates of the metric tensor remains valid also for compact Γ.

4. The results

As in the case of a δ interaction supported by a surface, the asymptotics is determined by the geometry of Γ. To state the results, we introduce the following comparison operator,

$$S=-\Delta_\Gamma-\frac{1}{4}(k_1-k_2)^2=-\Delta_\Gamma+K-M^2, \tag{9}$$

where Δ_Γ is the Laplace–Bertrami operator on the surface Γ and $k_{1,2}$ are the principal curvatures of Γ. The spectrum of S is purely discrete if Γ is compact. In the noncompact case the potential vanishes at infinity and has negative values unless Γ is a plane which is, however, excluded by assumption (a4). Consequently, $\sigma_{\mathrm{ess}}(S)=[0,\infty)$ and the discrete spectrum is nonempty. We denote the eigenvalues of S, arranged in the ascending order with the multiplicity taken into account, as μ_j.

First we inspect the essential spectrum in the strong-coupling regime:

Theorem 1. *Let an infinite surface Γ satisfy assumptions (a1)–(a4), then $\sigma_{\mathrm{ess}}(H_\beta)\subseteq[\epsilon(\beta),\infty)$, where $\epsilon(\beta)=-\frac{4}{\beta^2}+\mathcal{O}(\mathrm{e}^{-c/\beta})$ holds as $\beta\to 0_+$ for some constant $c>0$.*

We note that in case of a compact Γ we have $\sigma_{\mathrm{ess}}(H_\beta)=[0,\infty)$; a proof can be found in [8]. The next two theorems describe the asymptotics of the negative point spectrum of H_β.

Theorem 2. *Let an infinite surface Γ satisfy assumptions (a1)–(a4), then H_β has at least one isolated eigenvalue below the threshold of the essential spectrum for all sufficiently small $\beta>0$, and the j-th eigenvalue behaves in the limit $\beta\to 0_+$ as*

$$\lambda_j=-\frac{4}{\beta^2}+\mu_j+\mathcal{O}(-\beta\ln\beta).$$

Theorem 3. *Let a compact surface Γ satisfy assumption (b), then H_β has at least one isolated eigenvalue below the threshold of the essential spectrum for all $\beta>0$, and the j-th eigenvalue behaves in the limit $\beta\to 0_+$ as*

$$\lambda_j=-\frac{4}{\beta^2}+\mu_j+\mathcal{O}(-\beta\ln\beta).$$

5. Bracketing estimates

The basic idea is analogous to the one used in [7], namely to estimate the operator H_β from above and below, in a tight enough manner, by suitable operators for which we are able to calculate the spectrum directly. The starting point for such estimates is the bracketing trick, that is, imposing additional Dirichlet/Neumann conditions at the boundary of the neighborhood Ω_d of the surface Γ. We introduce quadratic forms h_β^+ and h_β^-, both of them given by the formula

$$\|\nabla\psi\|^2_{L^2(\Omega_d)}-\beta^{-1}\int_\Gamma\left|\psi(s,0_+)-\psi(s,0_-)\right|^2\mathrm{d}\Gamma$$

with the domains $\mathcal{D}(h_\beta^+)=\tilde{H}_0^1(\Omega_d\setminus\Gamma)$ and $\mathcal{D}(h_\beta^-)=H^1(\Omega_d\setminus\Gamma)$, respectively, the former being understood as a set of functions which are locally H^1 and vanish at the boundary of Ω_a. We denote the self-adjoined operators associated with these forms as $H_\beta^\pm$. By the standard bracketing argument we get

$$-\Delta^N_{\mathbb{R}^3\setminus\Omega_d}\oplus H_\beta^-\le H_\beta\le-\Delta^D_{\mathbb{R}^3\setminus\Omega_d}\oplus H_\beta^+, \tag{10}$$

where $-\Delta^{D,N}_{\mathbb{R}^3\setminus\Omega_d}$ is the Dirichlet–Laplacian and Neumann–Laplacian respectively on the set $\mathbb{R}^3\setminus\Omega_d$. The operators $-\Delta^{D,N}_{\mathbb{R}^3\setminus\Omega_d}$ are positive, hence all the information about the negative spectrum is encoded in the operators $H_\beta^\pm$.

The next step is to transform the operators $H_\beta^\pm$ into the curvilinear coordinates (s,u). This is done by means of the unitary transformation

$$U\psi=\psi\circ\mathcal{L}:L^2(\Omega_d)\to L^2(D_d,\mathrm{d}\Omega).$$

By $(\cdot,\cdot)_G$ we denote the scalar product in $L^2(D_d,\mathrm{d}\Omega)$. The operators $UH_\beta^\pm U^{-1}$ acting on this space are associated with the forms

$$h_\beta^\pm(U^{-1}\psi)=\left(\partial_i\psi,G^{ij}\partial_j\psi\right)_G-\beta^{-1}\int_\Gamma\left|\psi(s,0_+)-\psi(s,0_-)\right|^2\mathrm{d}\Gamma$$

having the domains $\tilde{H}_0^1(D_d\setminus\Gamma,\mathrm{d}\Omega)$ and $\tilde{H}^1(D_d\setminus\Gamma,\mathrm{d}\Omega)$, respectively. Next we employ another unitary transformation, inspired by [11], with the aim to get rid of the transverse coordinate dependence, i.e. switch from the metric $\mathrm{d}\Omega$ to $\mathrm{d}\Gamma\,\mathrm{d}u$ by

$$\tilde{U}\psi=\xi^{\frac{1}{2}}\psi:L^2(D_d,\mathrm{d}\Omega)\to L^2(D_d,\mathrm{d}\Gamma\mathrm{d}u).$$

Similarly as before, we denote the scalar product in $L^2(D_d,\mathrm{d}\Gamma\mathrm{d}u)$ as $(\cdot,\cdot)_g$ and consider the operators

$$F_\beta^\pm:=\tilde{U}UH_\beta^\pm U^{-1}\tilde{U}^{-1}$$

which act in $L^2(D_d,\mathrm{d}\Gamma\mathrm{d}u)$. The quadratic forms $\zeta_\beta^\pm$ associated with $F_\beta^\pm$ can be calculated as $h_\beta^\pm(\tilde{U}^{-1}U^{-1}\psi)$ with the result

$$\begin{aligned}\zeta_\beta^+[\psi]&=\left(\partial_\mu\psi,G^{\mu\nu}\partial_\nu\psi\right)_g+\left(\psi,(V_1+V_2)\psi\right)_g+\|\partial_3\psi\|_g\\&\quad-\beta^{-1}\int_\Gamma\left|\psi(s,0_+)-\psi(s,0_-)\right|^2\mathrm{d}\Gamma\\&\quad-\int_\Gamma M\left(\left|\psi(s,0_+)\right|^2-\left|\psi(s,0_-)\right|^2\right)\mathrm{d}\Gamma\\\zeta_\beta^-[\psi]&=\zeta_\beta^+[\psi]+\int_\Gamma\varsigma(s,d)\left|\psi(s,d)\right|^2\mathrm{d}\Gamma\\&\quad-\int_\Gamma\varsigma(s,-d)\left|\psi(s,-d)\right|^2\mathrm{d}\Gamma,\end{aligned}$$

where $\varsigma=\frac{M-Ku}{\xi}$, the two curvature-induced potentials are

$$V_1=g^{-\frac{1}{2}}\left(g^{\frac{1}{2}}G^{\mu\nu}J_{,\mu}\right)_{,\nu}+J_{,\mu}G^{\mu\nu}J_{,\nu},\qquad V_2=\frac{K-M^2}{\xi^2}$$

with $J=\frac{\ln\xi}{2}$. The corresponding form domains are $\tilde{H}_0^1(D_d\setminus\Gamma,\mathrm{d}\Gamma\mathrm{d}u)$ and $\tilde{H}^1(D_d\setminus\Gamma,\mathrm{d}\Gamma\mathrm{d}u)$, respectively.

6. Proof of Theorem 1

In the excluded case when Γ is a plane, the spectrum is easily found by separation of variables. Since a δ' interaction in one dimension has a single eigenvalue equal to $-\frac{4}{\beta^2}$, cf. [3, Sec. I.4], we get $\sigma(H_\beta)=\sigma_{\mathrm{ess}}(H_\beta)=[-\frac{4}{\beta^2},\infty)$. We want to show that under the assumption (a3) the essential spectrum does not change, at least asymptotically. We employ an estimate which follows from Lemma 4 that we will prove below, namely

$$\int_{-d}^{d}\left|\frac{\mathrm{d}f}{\mathrm{d}u}\right|^2\mathrm{d}u-\beta^{-1}\left|f(0_+)-f(0_-)\right|^2 \geq\left(-\frac{4}{\beta^2}-\frac{16}{\beta^2}\exp\left(-\frac{4d}{\beta}\right)\right)\|f\|_{L^2(-d,d)}. \tag{11}$$

As we shall see the inequality holds for sufficiently small β and $\frac{d}{\beta}>2$. The inclusion $\sigma_{\mathrm{ess}}(H_\beta)\subseteq[\epsilon(\beta),\infty)$ is equivalent to

$$\inf\sigma_{\mathrm{ess}}(H_\beta)\geq\epsilon(\beta)$$

which will be satisfied if $\inf\sigma_{\mathrm{ess}}(H_\beta^-)\geq\epsilon(\beta)$ for H_β^- acting in $L^2(\Omega_d)$ for $d<g_0<\rho$. This is obvious from inequalities (10) and the fact that the operator $-\Delta^N_{\mathbb{R}^3\setminus\Omega_d}$ is positive and cannot thus contribute to the negative part of the spectrum. In the next step we divide the surface Γ into two parts, namely $\Gamma_\tau^{\mathrm{int}}:=\{s\in\Gamma\,|\,r(s)<\tau\}$ and $\Gamma_\tau^{\mathrm{ext}}:=\Gamma\setminus\overline{\Gamma_\tau^{\mathrm{int}}}$. The layer neighborhoods corresponding to $\Gamma_\tau^{\mathrm{int}}$ and $\Gamma_\tau^{\mathrm{ext}}$ are $D_\tau^{\mathrm{int}}=\{(s,u)\in D_d\,|\,s\in\Gamma_\tau^{\mathrm{int}}\}$ and $D_\tau^{\mathrm{ext}}=D_d\setminus\overline{D_\tau^{\mathrm{int}}}$. We introduce the Neumann operators on respective neighborhoods, $H_{\beta,\tau}^{-,z}$ for $z=\mathrm{int},\mathrm{ext}$ associated with the forms

$$\left(\partial_i\psi,G^{ij}\partial_j\psi\right)_G-\beta^{-1}\int_{\Gamma_\tau^z}\left|\psi(s,0_+)-\psi(s,0_-)\right|^2\mathrm{d}\Gamma$$

defined on $\tilde{H}^1(D_\tau^z\setminus\Gamma,\mathrm{d}\Omega)$. Using once more Neumann bracketing we get $H_\beta^-\geq H_{\beta,\tau}^{-,\mathrm{int}}\oplus H_{\beta,\tau}^{-,\mathrm{ext}}$. The inner part is compact, hence the spectrum of $H_{\beta,\tau}^{-,\mathrm{int}}$ is purely discrete. Consequently, the min–max principle implies

$$\inf\sigma_{\mathrm{ess}}\left(H_\beta^-\right)\geq\inf\sigma_{\mathrm{ess}}\left(H_{\beta,\tau}^{-,\mathrm{ext}}\right),$$

and it is sufficient to check that the right-hand side cannot be smaller than $\epsilon(\beta)$. The quantities $m_\tau^+:=\sup_{\Gamma_\tau^{\mathrm{ext}}}\xi$ and $m_\tau^-:=\inf_{\Gamma_\tau^{\mathrm{ext}}}\xi$ tend to one as $\tau\to\infty$ in view of assumption (a3). We have the following estimate,

$$\begin{aligned}\left(\psi,H_{\beta,\tau}^{-,\mathrm{ext}}\psi\right)_G&\geq\int_{D_\tau^{\mathrm{ext}}}\left|\partial_3\psi(q)\right|^2\mathrm{d}\Omega\\&\quad-\beta^{-1}\int_{\Gamma_\tau^{\mathrm{ext}}}\left|\psi(s,0_+)-\psi(s,0_-)\right|^2\mathrm{d}\Gamma\\&\geq m_\tau^-\int_{D_\tau^{\mathrm{ext}}}\left|\partial_3\psi(q)\right|^2\mathrm{d}\Gamma\,\mathrm{d}u\\&\quad-\beta^{-1}\int_{\Gamma_\tau^{\mathrm{ext}}}\left|\psi(s,0_+)-\psi(s,0_-)\right|^2\mathrm{d}\Gamma\\&\geq\frac{1}{\beta^2m_\tau^+m_\tau^-}\left[-4-16\exp\left(-\frac{4d}{\beta}\right)\right]\\&\quad\times\int_{D_\tau^{\mathrm{ext}}}\left|\psi(q)\right|^2\mathrm{d}\Omega,\end{aligned}$$

and since τ is arbitrary, we obtain $\epsilon(\beta)\geq-\frac{4}{\beta^2}-\frac{16}{\beta^2}\exp(-\frac{4d}{\beta})$.

7. Proof of Theorem 2

To prove the second theorem, we will need several auxiliary results. The operators $F_\beta^\pm$ are still not suitable to work with and so we replace them with a slightly cruder bounds. First we estimate the values of the potentials V_1 and V_2. With the help of inequalities (5)–(7) we are able to check that

$$dv^-\leq V_1\leq dv^+$$

holds for suitable numbers $v^\pm$ and $d<d_0<\rho$. On the other hand, V_2 can be estimated as

$$C_-^{-2}\left(K-M^2\right)\leq V_2\leq C_+^{-2}\left(K-M^2\right),$$

where $C_\pm$ are the same as in (5). This allows us to replace (10) with the estimates using operators $D_\beta^\pm$,

$$\begin{aligned}D_{d,\beta}^-&:=U_d^-\otimes I+\int_\Gamma^\oplus T_{d,\beta}^-(s)\,\mathrm{d}\Gamma\leq F_\beta^-\leq H_\beta\\H_\beta&\leq F_\beta^+\leq U_d^+\otimes I+\int_\Gamma^\oplus T_{d,\beta}^+(s)\,\mathrm{d}\Gamma=:D_{d,\beta}^+\end{aligned}\tag{12}$$

where

$$U_d^\pm=-C_\pm\Delta_\Gamma+C_\pm^{-2}\left(K-M^2\right)+v^\pm d$$

with the domain $\mathcal{D}(U_d^\pm)=L^2(\mathbb{R}^2,\mathrm{d}\Gamma)$ and the transverse part acts as

$$T_{d,\beta}^\pm(s)\psi=-\Delta\psi$$

with the domains

$$\begin{aligned}\mathcal{D}\left(T_{a,\beta}^+(s)\right)=\Big\{&f\in H^2\left((-a,a)\setminus\{0\}\right)\,\Big|\,f(a)=f(-a)=0,\\&f'(0_-)=f'(0_+)=-\beta^{-1}\left(f(0_+)-f(0_-)\right)\\&+M\left(f(0_+)+f(0_-)\right)\Big\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}\left(T_{a,\beta}^-(s)\right)=\Bigg\{&f\in H^2\left((-a,a)\setminus\{0\}\right)\,\Big|\,\mp\frac{\|M\|_\infty+d\|K\|_\infty}{C_-}f(\pm a)\\&=f'(\pm a),\\&f'(0_-)=f'(0_+)=-\beta^{-1}\left(f(0_+)-f(0_-)\right)\\&+M\left(f(0_+)+f(0_-)\right)\Bigg\},\end{aligned}$$

respectively. The negative spectrum is described by the following result the proof of which can be found in [12].

Lemma 4. *Each of the operators $T_{d,\beta}^\pm(s)$ has exactly one negative eigenvalue $t_\pm(d,\beta)$, respectively, which is independent of s provided that $\frac{d}{\beta}>2$ and $\beta(\|M\|_\infty+d\|K\|_\infty)<1$. For all $\beta>0$ sufficiently small these eigenvalues satisfy the inequalities*

$$\begin{aligned}-\frac{4}{\beta^2}-\frac{16}{\beta^2}\exp\left(-\frac{4d}{\beta}\right)&\leq t_-(d,\beta)\leq-\frac{4}{\beta^2}\leq t_+(d,\beta)\\&\leq-\frac{4}{\beta^2}+\frac{16}{\beta^2}\exp\left(-\frac{4d}{\beta}\right).\end{aligned}$$

On the other hand, the spectrum of the operators $U_d^\pm$ has the asymptotic expansion governed by the operator S which we can adopt from [7]:

Lemma 5. *The eigenvalues of $U_d^\pm$ satisfy the relations*

$$\mu_j^\pm(d)=\mu_j+C_j^\pm d+\mathcal{O}\left(d^2\right)\quad\text{for } d\to0,$$

where μ_j is the j-th eigenvalue of the operator S and the constants $C_j^\pm$ are independent on d.

With these prerequisites we are ready to prove the second theorem. We put $d(\beta) = -\beta \ln \beta$. Using the fact that each of the operators $T^{\pm}_{d,\beta}(s)$ has exactly one negative eigenvalue $t_{\pm}(d(\beta), \beta)$ together with the explicit form of $D^{\pm}_{d,\beta}$ we can write their spectra as $t_{\pm}(d(\beta), \beta) + \mu^{\pm}_j(d(\beta))$, where $\mu^{\pm}_j$ are the eigenvalues of the operators $U^{\pm}_d$. Using now Lemmata 4 and 5 we are able to rewrite this as

$$t_{\pm}\big(d(\beta), \beta\big) + \mu^{\pm}_j\big(d(\beta)\big) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}\big(\beta|\ln\beta|\big),$$

hence the min–max principle in combination with inequalities (12) conclude the argument.

8. Proof of Theorem 3

The existence of isolated eigenvalues can be checked variationally as in [8]. For a test function ξ one chooses characteristic function of the volume enclosed by the surface Γ; this yields an estimate of the ground state energy from above,

$$\lambda_0 \le \frac{h_\beta(\xi)}{\|\xi\|^2} = \beta^{-1}\frac{S}{V} \tag{13}$$

where h_β is quadratic (2), S is the area of the surface Γ and V is the volume enclosed by Γ. The proof of the asymptotic expansion proceed with minimum modifications as for the infinite surface, hence we omit the details.

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Spectral asymptotics for δ' interaction supported by a infinite curve

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We consider a generalized Schrödinger operator in $L^2(\mathbb{R}^2)$ describing an attractive δ' interaction in a strong coupling limit. δ' interaction is characterized by a coupling parameter β and it is supported by a C^4-smooth infinite asymptotically straight curve Γ without self-intersections. It is shown that in the strong coupling limit, $\beta \to 0_+$, the eigenvalues for a non-straight curve behave as $-\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta|\ln\beta|)$, where μ_j is the j-th eigenvalue of the Schrödinger operator on $L^2(\mathbb{R})$ with the potential $-\frac{1}{4}\gamma^2$ where γ is the signed curvature of Γ.

Keywords: δ' interaction, quantum graphs, spectral theory

1. Introduction

The quantum mechanics describing the particle confined to various manifolds is studied quite extensively. It is very useful for describing various nanostructures in physics but it also offers a large variety of interesting problems from the purely mathematical point of view. Systems where the confinement is realized by a singular attractive potential, so called 'leaky' quantum graphs [1], have the advantage that they take quantum tunneling effects into account in contrast to quantum graphs [2]. The confining potential is often taken to be of the δ type. One can think also about more singular types of potentials namely δ' type based on the concept of δ' interaction in one dimension [3].

We are interested in the spectrum of the operator which can be formally written as

$$H = -\Delta - \beta^{-1}\delta'(\cdot - \Gamma)$$

where δ' interaction is supported by an infinite curve Γ in $\mathbb{R}^2$. We are interested in the strong coupling regime which corresponds to small values of the parameter β. We derive spectral asymptotics of discrete and essential spectra. As a byproduct we obtain that for a non-straight curve the bound state arises for sufficiently small β in an alternative way to one presented in [4].

2. Formulation of the Problem and Results

We consider a curve Γ parameterized by its arc length

$$\Gamma : \mathbb{R} \to \mathbb{R}^2 , \quad s \mapsto (\Gamma_1(s), \Gamma_2(s)) ,$$

where $\Gamma_1(s), \Gamma_2(s) \in C^4(\mathbb{R})$ are component functions. We denote signed curvature as $\gamma(s) := (\Gamma_1''\Gamma_2' - \Gamma_1'\Gamma_2'')(s)$. We introduce several conditions for the curve Γ as:
(Γ1) Γ is C^4 smooth curve,
(Γ2) Γ has no "near self-intersections", i.e. there exists its strip neighborhood of a finite thickness which does not intersect with itself,
(Γ3) Γ is asymptotically straight in the sense that $\lim_{|s|\to\infty} \gamma(s) = 0$ and
(Γ4) Γ is not a straight line.

The operator, we are interested in, acts as a free Laplacian outside of the interaction support

$$(H_\beta \psi)(x) = -(\Delta\psi)(x)$$

for $x \in \mathbb{R}^2 \setminus \Gamma$ with the domain which can be written as $\mathcal{D}(H_\beta) = \{\psi \in H^2(\mathbb{R}^2 \setminus \Gamma) \mid \partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) = \psi'(x)|_\Gamma,\ -\beta\psi'(x)|_\Gamma = \psi(x)|_{\partial_+\Gamma} - \psi(x)|_{\partial_-\Gamma}\}$. The vector n_Γ denotes the normal to Γ and $\psi(x)|_{\partial_\pm\Gamma}$ are the appropriate traces of the function ψ. For the purpose of the proofs we introduce curvelinear coordinates (s, u) along the curve in the same way as done in [6], i.e.

$$(x, y) = \left(\Gamma_1(s) + u\Gamma_2'(s), \Gamma_2(s) - u\Gamma_1'(s)\right) . \tag{1}$$

As a result of the conditions (Γ1) and (Γ2) it can be shown that the map (1) is injective for all u small enough. We denote d as a maximum for which the map (1) is injective. A strip neighborhood around Γ of thickness $a < d$ is denoted by $\Omega_a := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < a\}$.

The quadratic form associated with the operator H_β was derived in [5] and it can be written as

$$h_\beta[\psi] = \|\nabla\psi\|^2 - \beta^{-1} \int_{\mathbb{R}} |\psi(s, 0_+) - \psi(s, 0_-)|^2 \,\mathrm{d}s .$$

where we used the curvelinear coordinates in the strip neighborhood of the curve Γ for the functions $\psi \in C(\mathbb{R}^2) \cap H^1(\mathbb{R}^2 \setminus \Gamma)$ as $\psi(s, u)$. We also need to introduce the operator defined on the line as

$$S = -\frac{\partial^2}{\partial s^2} - \frac{1}{4}\gamma(s)^2 , \tag{2}$$

with the domain $\mathcal{D}(S) = H^2(\mathbb{R})$. The eigenvalues of the operator S are denoted by μ_j with the multiplicity taken into account. Now we are ready to write down the main results of our paper.

Theorem 2.1. *Let an infinite curve Γ satisfy conditions (Γ1)–(Γ3), then $\sigma_{\mathrm{ess}}(H_\beta) \subseteq [\epsilon(\beta), \infty)$, where $\epsilon(\beta) \to -\frac{4}{\beta^2}$ holds as $\beta \to 0_+$.*

Theorem 2.2. *Let an infinite curve* Γ *satisfy assumptions* $(\Gamma 1)$–$(\Gamma 4)$*, then* H_β *has at least one isolated eigenvalue below the threshold of the essential spectrum for all sufficiently small* $\beta > 0$*, and the* j*-th eigenvalue behaves in the strong coupling limit* $\beta \to 0_+$ *as*

$$\lambda_j = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(-\beta \ln(\beta))\,.$$

3. Bracketing estimates

For the proofs of both theorems we will need estimates of our operator H_β via Dirichlet and Neumann bracketing as done in [7]. We introduce the operators with added either Dirichlet or Neummann boundary conditions at the boundary of the strip neighborhood Ω_a of Γ. We introduce quadratic forms h_β^+ and h_β^- on the strip neighborhood of Γ which can be written as

$$h_\beta^\pm[\psi] = \|\nabla\psi\|^2 - \beta^{-1}\int_{\mathbb{R}} |\psi(s,0_+) - \psi(s,0_-)|^2\,\mathrm{d}s$$

with the domains $\mathcal{D}(h_\beta^+) = \tilde{H}_0^1(\Omega_a \setminus \Gamma)$ and $\mathcal{D}(h_\beta^-) = \tilde{H}^1(\Omega_a \setminus \Gamma)$. The operators associated with the quadratic forms $h_\beta^\pm$ are denoted by $H_\beta^\pm$, respectively. With the help of Dirichlet-Neumann bracketing we are able to write the following inequality

$$-\Delta^N_{\mathbb{R}^2\setminus\Omega_a} \oplus H_\beta^- \le H_\beta \le -\Delta^D_{\mathbb{R}^2\setminus\Omega_a} \oplus H_\beta^+\,, \qquad (3)$$

where $-\Delta^{N,D}_{\mathbb{R}^2\setminus\Omega_a}$ denotes either Neumann or Dirichlet Laplacian on $\mathbb{R}^2 \setminus \Omega_a$ respectively. Neumann Laplacian and Dirichlet Laplacian are positive and as a result all the information about the negative spectrum, which we are interested in, is encoded in the operators $H_\beta^\pm$.
Now we rewrite the quadratic forms $h_\beta^\pm$ in the curvelinear coordinates (1). We obtain expression which are analogical to those obtained in [6], i.e.

Lemma 3.1. *Quadratic forms* h_β^+*,* h_β^- *are unitarily equivalent to quadratic forms* q_β^+ *and* q_β^- *which can be written as*

$$q^+[f] = \|\frac{\partial_s f}{g}\|^2 + \|\partial_u f\|^2 + (f, Vf) - \beta^{-1}\int_{\mathbb{R}} |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s$$
$$+\frac{1}{2}\int_{\mathbb{R}} \gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,\mathrm{d}s$$

$$q^-[g] = q_D[g] - \int_{\mathbb{R}} \frac{\gamma(s)}{2(1+a\gamma(s))}|f(s,a)|^2\,\mathrm{d}s + \int_{\mathbb{R}} \frac{\gamma(s)}{2(1-a\gamma(s))}|f(s,-a)|^2\,\mathrm{d}s$$

defined on $\tilde{H}_0^1(\mathbb{R}\times((-a,0)\cup(0,a)))$ *and* $\tilde{H}^1(\mathbb{R}\times((-a,0)\cup(0,a)))$*, respectively. The geometrically induced potential in these formulæ is given by*

$$V(s,u) = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$$

with $g(s,u) := 1 + u\gamma(s)$*.*

The proof of this lemma can be done step by step as done in [6] so we omit the details.

We will also need cruder estimates by quadratic forms $b_\beta^\pm[f]$ which satisfy $b_\beta^-[f] \le q^-[f] \le h_\beta[f] \le q^+[f] \le b_\beta^+[f]$. The quadratic form $b_\beta^+[f]$ can be written as

$$b_\beta^+[f] = \|\partial_u f\|^2 + (1-a\gamma_+)^{-2}\|\partial_s f\|^2 + (f, V^{(+)}f)$$
$$-\beta^{-1}\int_{\mathbb{R}} |f(s,0_+) - f(s,0_-)|^2 \mathrm{d}s + \frac{1}{2}\int_{\mathbb{R}} \gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,\mathrm{d}s$$

where $V^{(+)} := \frac{a(\gamma'')_+}{2(1-a\gamma_+)^3} - \frac{\gamma^2}{4(1+a\gamma_+)^2}$ and $f_+ := \max_{s\in\mathbb{R}}|f|$ denotes maximum of $|f|$. The quadratic form $b_\beta^-[f]$ can be written as

$$b_\beta^-[f] = \|\partial_u f\|^2 + (1+a\gamma_+)^{-2}\|\partial_s f\|^2 + (f, V^{(-)}f)$$
$$-\beta^{-1}\int_{\mathbb{R}} |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s - \frac{1}{2}\int_{\mathbb{R}} \gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,\mathrm{d}s$$
$$-\gamma_+\int_{\mathbb{R}} |f(s,a)|^2\,\mathrm{d}s - \gamma_+\int_{\mathbb{R}} |f(s,-a)|^2\,\mathrm{d}s$$

where $V^{(-)} = -\frac{a(\gamma'')_+}{2(1-a\gamma_+)^3} - \frac{5(a(\gamma')_+)^2}{4(1-a\gamma_+)^4} - \frac{\gamma^2}{4(1-a\gamma_+)^2}$. The operators $B_\beta^\pm$ associated with $b_\beta^\pm[f]$ can be written as $B_\beta^\pm = U_a^\pm \otimes I + \int_{\mathbb{R}}^{\oplus} T_{a,\beta}^\pm(s)\,\mathrm{d}s$ where $U_a^\pm$ corresponds to the longitudinal variable s and $T_{a,\beta}^\pm(s)$ corresponds to the transversal variable u. The operators $T_{a,\beta}^\pm(s)$ act as $T_{a,\beta}^\pm(s)f = -f''$ with the domains

$$\mathcal{D}(T_{a,\beta}^+(s)) = \big\{f \in H^2((-a,a)\setminus\{0\}) \mid f(a) = f(-a) = 0\,,$$
$$f'(0_-) = f'(0_+) = -\beta^{-1}(f(0_+) - f(0_-)) + \frac{1}{2}\gamma(s)(f(0_+) + f(0_-))\big\}$$
$$\mathcal{D}(T_{a,\beta}^-(s)) = \big\{f \in H^2((-a,a)\setminus\{0\}) \mid \mp\gamma_+ f(\pm a) = f'(\pm a)\,,$$
$$f'(0_-) = f'(0_+) = -\beta^{-1}(f(0_+) - f(0_-)) + \frac{1}{2}\gamma(s)(f(0_+) + f(0_-))\big\}\,.$$

The operators $U_a^\pm$ act as $U_a^\pm f = -(1\mp a\gamma_+)^{-2}f'' + V^{(\pm)}f$ with the domain $\mathcal{D}(U_a^\pm) = H^2(\mathbb{R})$. The operators $T_{a,\beta}^\pm(s)$ depend on the variable s, however, their negative spectrum is independent of s. Now we state two lemmata estimating the eigenvalues of operators $T_{a,\beta}^\pm(s)$ and $U_a^\pm$. Their proofs can be found in [6] so we omit the details.

Lemma 3.2. *Each of the operators $T_{a,\beta}^\pm(s)$ has exactly one negative eigenvalue $t_\pm(a,\beta)$, respectively, which is independent of s provided that $\frac{a}{\beta} > 2$ and $\frac{2}{\beta} > \gamma_+$. For all $\beta > 0$ sufficiently small these eigenvalues satisfy the inequalities*

$$-\frac{4}{\beta^2} - \frac{16}{\beta^2}\exp\left(-\frac{4a}{\beta}\right) \le t_-(d,\beta) \le -\frac{4}{\beta^2} \le t_+(d,\beta) \le -\frac{4}{\beta^2} + \frac{16}{\beta^2}\exp\left(-\frac{4a}{\beta}\right)\,.$$

Lemma 3.3. *There is a positive C independent of a and j such that*

$$|\mu_j^\pm(a) - \mu_j| \le Caj^2$$

holds for $j \in \mathbb{N}$ *and* $0 < a < \frac{1}{2\gamma_+}$, *where* $\mu_j^{\pm}(a)$ *are the eigenvalues of* $U_a^{\pm}$, *respectively, with the multiplicity taken into account.*

Now we are ready to prove our main theorems.

4. Proof of Theorem 2.1

First we prove the trivial case for the straight line. By separation of variables the spectrum is $\sigma(H_\beta) = \sigma_{\text{ess}}(H_\beta) = \big[-\frac{4}{\beta^2}, \infty\big)$.

The case for non-straight curve is done similarly as for the singular interaction supported by nonplanar surfaces in [8, 9]. The inclusion $\sigma_{\text{ess}}(H_\beta) \subseteq [\epsilon(\beta), \infty)$ can be rewritten as

$$\inf \sigma_{\text{ess}}(H_\beta) \geq \epsilon(\beta)\,.$$

The inequality $H_\beta \geq H_\beta^- \oplus -\Delta^N_{\mathbb{R}^2\setminus\Omega_d}$ implies that it is sufficient to check $\inf \sigma_{\text{ess}}(H_\beta^-) \geq \epsilon(\beta)$ in $L^2(\Omega_a)$ for $a < d$ because the operator $-\Delta^N_{\mathbb{R}^3\setminus\Omega_d}$ is positive. Next we divide the curve Γ into two parts. First part is defined as $\Gamma_\tau^{\text{int}} := \{\Gamma(s)|s<\tau\}$ and the second one $\Gamma_\tau^{\text{ext}} := \Gamma\setminus\overline{\Gamma_\tau^{\text{int}}}$. The corresponding strip neighborhoods are defined as $\Omega_a^{\text{int}} := \{x(s,u)\in\Omega_a|s<\tau\}$ and $\Omega_a^{\text{ext}} := \{x(s,u)\in\Omega_a|s>\tau\}$. We introduce Neumann decoupled operators on $\Omega_a^{\text{int,ext}}$ as

$$H_{\beta,\tau}^{-,\text{int}} \oplus H_{\beta,\tau}^{-,\text{ext}}\,.$$

The operators $H_{\beta,\tau}^{-,\omega}$, $\omega = \text{int}, \text{ext}$ are associated with quadratic forms $h_{\beta,\tau}^{-,\omega}$ which can be written as

$$\begin{aligned} h_{\beta,\tau}^{-,\omega} = \|\frac{\partial_s f}{g}\|^2 + \|\partial_u f\|^2 + (f, Vf) - \beta^{-1}\int_{\Gamma_\tau^\omega} |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s \\ + \frac{1}{2}\int_{\Gamma_\tau^\omega} \gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,\mathrm{d}s \\ - \int_{\Gamma_\tau^\omega} \frac{\gamma(s)}{2(1+a\gamma(s))}|f(s,a)|^2\,\mathrm{d}s + \int_{\Gamma_\tau^\omega} \frac{\gamma(s)}{2(1-a\gamma(s))}|f(s,-a)|^2\,\mathrm{d}s \end{aligned}$$

with the domains $\tilde{H}^1(\Omega_a^\omega)$. Neumann bracketing implies that $H_{\beta,\tau}^- \geq H_{\beta,\tau}^{-,\text{int}} \oplus H_{\beta,\tau}^{-,\text{ext}}$. The spectrum of the operator $H_{\beta,\tau}^{-,\text{int}}$ is purely discrete [10] and as a result min-max principle implies that

$$\inf \sigma_{\text{ess}}(H_{\beta,\tau}^-) \geq \inf \sigma_{\text{ess}}(H_{\beta,\tau}^{-,\text{ext}})\,.$$

We denote the following expression $V_\tau := \inf_{|s|>\tau, u\in(-a,a)} V(s,u)$. The assumption $(\Gamma 2)$ gives us that

$$\lim_{\tau\to\infty} V_\tau = 0$$

With the help of Lemma 3.2 we can write the following estimates

$$
\begin{aligned}
h_{\beta,\tau}^{-,\mathrm{ext}}[f] \geq \|\partial_u f\|^2 + V_\tau \|f\|^2 - \beta^{-1} \int_{\Gamma_\tau^{\mathrm{ext}}} |f(s,0_+) - f(s,0_-)|^2 \, \mathrm{d}s \\
+ \frac{1}{2} \int_{\Gamma_\tau^{\mathrm{ext}}} \gamma(s) \big(|f(s,0_+)|^2 - |f(s,0_-)|^2 \big) \, \mathrm{d}s \\
- \int_{\Gamma_\tau^{\mathrm{ext}}} \frac{\gamma(s)}{2(1+a\gamma(s))} |f(s,a)|^2 \, \mathrm{d}s + \int_{\Gamma_\tau^{\mathrm{ext}}} \frac{\gamma(s)}{2(1-a\gamma(s))} |f(s,-a)|^2 \, \mathrm{d}s \\
\geq \left(V_\tau - \frac{4}{\beta^2} - \frac{16}{\beta^2} \exp\left(-\frac{4a}{\beta} \right) \right) \|f\|^2
\end{aligned}
$$

Because we can choose τ arbitrarily large we obtain the the desired result.

5. Proof of Theorem 2.2

For the proof of the second theorem we use the inequalities (3) and Lemmata 3.2 and 3.3. First we put $a(\beta) = -\frac{3}{4}\beta \ln \beta$. Now with the explicit form of $B_\beta^\pm$ in mind and the fact that $T_{a,\beta}^\pm(s)$ have exactly one negative eigenvalue we have that the spectra of $B_\beta^\pm$ can be written as $t_\pm(d(\beta),\beta) + \mu_j^\pm(d(\beta))$. Using Lemmata 3.2 and 3.3 we obtain

$$
t_\pm(a(\beta),\beta) + \mu_j^\pm(a(\beta)) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta |\ln \beta|) \, .
$$

The min-max principle along with the inequality (3) completes the proof.

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On absence of bound states for weakly attractive δ'-interactions supported on non-closed curves in $\mathbb{R}^2$

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Let $\Lambda \subset \mathbb{R}^2$ be a non-closed piecewise-C^1 curve, which is either bounded with two free endpoints or unbounded with one free endpoint. Let $u_\pm|_\Lambda \in L^2(\Lambda)$ be the traces of a function u in the Sobolev space $H^1(\mathbb{R}^2 \setminus \Lambda)$ onto two faces of Λ. We prove that for a wide class of shapes of Λ the Schrödinger operator $\mathsf{H}_\omega^\Lambda$ with δ'-interaction supported on Λ of strength $\omega \in L^\infty(\Lambda;\mathbb{R})$ associated with the quadratic form $H^1(\mathbb{R}^2 \setminus \Lambda) \ni u \mapsto \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \int_\Lambda \omega |u_+|_\Lambda - u_-|_\Lambda|^2 \mathrm{d}s$ has no negative spectrum provided that ω is pointwise majorized by a strictly positive function explicitly expressed in terms of Λ. If, additionally, the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical, we show that $\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty)$. For a bounded curve Λ in our class and non-varying interaction strength $\omega \in \mathbb{R}$, we derive existence of a constant $\omega_* > 0$ such that $\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty)$ for all $\omega \in (-\infty, \omega_*]$; informally speaking, bound states are absent in the weak coupling regime. © *2016 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4939749]

I. INTRODUCTION

In this paper, we study the self-adjoint operator corresponding to the formal differential expression

$$-\Delta - \omega\delta'(\cdot - \Lambda), \qquad \text{on } \mathbb{R}^2,$$

with the δ'-interaction supported on a non-closed piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$, which is either bounded with two free endpoints or unbounded with one free endpoint, here $\omega \in L^\infty(\Lambda;\mathbb{R})$ is called the strength of the interaction. More precisely, for any function u in the Sobolev space $H^1(\mathbb{R}^2 \setminus \Lambda)$ its traces $u_\pm|_\Lambda$ onto two faces of Λ turn out to be well-defined as functions in $L^2(\Lambda)$, and employing the shorthand notation $[u]_\Lambda := u_+|_\Lambda - u_-|_\Lambda$ we introduce the following symmetric sesquilinear form:

$$\begin{aligned} \mathfrak{a}_\omega^\Lambda[u,v] &:= (\nabla u, \nabla v)_{L^2(\mathbb{R}^2;\mathbb{C}^2)} - (\omega[u]_\Lambda, [v]_\Lambda)_{L^2(\Lambda)}, \\ \operatorname{dom} \mathfrak{a}_\omega^\Lambda &:= H^1(\mathbb{R}^2 \setminus \Lambda), \end{aligned} \tag{1.1}$$

which is closed, densely defined, and semibounded in the Hilbert space $L^2(\mathbb{R}^2)$; see Proposition 3.1. Let $\mathsf{H}_\omega^\Lambda$ be defined as the unique self-adjoint operator representing the form $\mathfrak{a}_\omega^\Lambda$ in the usual manner. We regard $\mathsf{H}_\omega^\Lambda$ as the *Schrödinger operator with δ'-interaction of strength ω supported on Λ*.

The aim of this paper is to demonstrate a peculiar spectral property of $\mathsf{H}_\omega^\Lambda$. Namely, we show absence of negative spectrum for $\mathsf{H}_\omega^\Lambda$ under not too restrictive assumptions on the shape of Λ and assuming that the strength ω is pointwise majorized by a strictly positive function explicitly expressed in terms of the shape of Λ. The important point to note here is that the discovered phenomenon is non-emergent for δ'-interactions supported on loops in $\mathbb{R}^2$, cf., [Ref. 3, Theorem 4.4].

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The basic geometric ingredient in our paper is the concept of *monotone curves*. A non-closed piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ is monotone if it can be parametrized via the piecewise-C^1 mapping $\varphi\colon (0,R) \to \mathbb{R}$, $R \in (0,+\infty]$, as

$$\Lambda = \{\mathsf{x}_0 + (r\cos\varphi(r), r\sin\varphi(r)) \in \mathbb{R}^2 \colon r \in (0,R)\}, \tag{1.2}$$

here, $\mathsf{x}_0 \in \mathbb{R}^2$ is fixed. For example, a circular arc subtending an angle $\theta \le \pi$ is monotone, whereas a circular arc subtending an angle $\theta > \pi$ is not.

In the next theorem, which is the first main result of our paper, we provide a condition on ω ensuring absence of negative spectrum for the operator $\mathsf{H}_\omega^\Lambda$ with Λ being monotone. The statement of Theorem A below is contained in Theorem 4.2, in Subsection IV B.

Theorem A. *Let a monotone piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ be parametrized as in (1.2) via $\varphi\colon (0,R) \to \mathbb{R}$, $R \in (0,+\infty]$. Then the spectrum of $\mathsf{H}_\omega^\Lambda$ satisfies*

$$\sigma(\mathsf{H}_\omega^\Lambda) \subseteq [0,+\infty) \qquad \text{if} \quad \omega(r) \le \frac{1}{2\pi r\sqrt{1+(r\varphi'(r))^2}}, \quad \text{for} \quad r \in (0,R).$$

If ω is majorized as above and, additionally, the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical, then $\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty)$.

Roughly speaking, a domain $\Omega \subset \mathbb{R}^2$ is quasi-conical if it contains a disc of arbitrary large radius; see Subsection III B for details. In Proposition 4.7, we demonstrate that, in general, the operator $\mathsf{H}_\omega^\Lambda$ may have negative spectrum if the δ'-interaction is "sufficiently strong."

Operators $\mathsf{H}_\omega^\Lambda$ with non-varying strengths $\omega \in \mathbb{R}$ are of special interest. One can derive from Theorem A that for a bounded monotone Λ one can find a constant $\omega_* > 0$ such that

$$\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty), \qquad \text{for all } \omega \in (-\infty,\omega_*]. \tag{1.3}$$

In other words, there are no bound states in the weak coupling regime. Computation of the largest constant $\omega_* > 0$ such that (1.3) still holds presents a more delicate problem, which will be considered elsewhere.

In the formulation of the second main result of the paper we use the notion of a *linear fractional transformation (LFT)*. The complex plane $\mathbb{C}$ can be extended up to the *Riemann sphere* $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with a suitable topology and for $a,b,c,d \in \mathbb{C}$ such that $ad - bc \ne 0$ one defines the LFT as

$$M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \qquad M(z) := \frac{az+b}{cz+d},$$

with the exception of the points $z = \infty$ and $z = -d/c$ if $c \ne 0$, which have to be treated separately; see Subsection II C. The next theorem generalizes Theorem A to the case of curves, which are images of monotone curves under LFTs; the statement of this theorem is contained in Theorem 4.12, in Subsection IV C. Here, we confine ourselves to non-varying interaction strengths only.

Theorem B. *Let $\Lambda \subset \mathbb{R}^2$ be a bounded piecewise-C^1 curve. Suppose that there exists a LFT $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $M(\infty), M^{-1}(\infty) \notin \Lambda$ and that $M^{-1}(\Lambda)$ is monotone. Then there exists a constant $\omega_* > 0$ such that*

$$\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty), \qquad \text{for all } \omega \in (-\infty,\omega_*].$$

In the main body of the paper also an explicit formula for ω_* in the above theorem is provided. Using Theorem B we can treat, e.g., any circular arc, since it can be mapped via a suitable LFT to a subinterval of the straight line in $\mathbb{R}^2$; see Example 4.13. One may even conjecture that for any bounded Λ there exists an $\omega_* > 0$ such that $\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty)$, for all $\omega \in (-\infty,\omega_*]$.

Our proofs rely on the min-max principle applied to the form $\mathfrak{a}_\omega^\Lambda$ in (1.1) on a suitable core. A further important ingredient in our analysis is careful investigation of a model one-dimensional problem with a point δ'-interaction on the loop.

The results of this paper contribute to a prominent topic in spectral theory: existence/non-existence of weakly coupled bound states for Schrödinger-type operators. Absence of bound states in the weak coupling regime holds for Schrödinger operators with regular potentials in space dimensions $d \geq 3$ (but not for $d = 1, 2!$); see Ref. 38. Also such an effect occurs for δ-interactions supported on arbitrary compact hypersurfaces in $\mathbb{R}^3$ (see Ref. 13) and for δ-interactions on compact non-closed curves in $\mathbb{R}^3$ (see Ref. 17). However, for δ-interactions in $\mathbb{R}^2$ supported on arbitrary compact curves such an effect is non-existent.[19,26]

Schrödinger operators with δ'-interactions supported on hypersurfaces are attractive from physical point of view, because they exhibit rather unusual scattering properties, cf., [Ref. 1, Chap. I.4]. These operators are also physically relevant in photonic crystals theory.[20] As a mathematical abstraction they were perhaps first studied in Refs. 2 and 37, where interactions were supported on spheres. A rigorous definition of such operators with interactions supported on general hypersurfaces has been posed in [Ref. 12, Section 7.2] as an open question. Such Hamiltonians with interactions supported on closed hypersurfaces without free boundaries have been rigorously defined in Ref. 5 using two approaches: via the theory of self-adjoint extensions of symmetric operators and by means of form methods. Spectral properties of them were investigated in several subsequent works.[3,4,14–16,24,32] In the recent preprint[33] Schrödinger operators with δ'-interactions supported on non-closed curves and surfaces are defined via the theory of self-adjoint extensions and their scattering properties are discussed.

Let us briefly outline the structure of the paper. Section II presents some preliminaries: Sobolev spaces, geometry of curves, linear fractional transformations, and a model one-dimensional spectral problem. Section III provides a rigorous definition of the operator $\mathsf{H}_\omega^\Lambda$ and a characterisation of its essential spectrum. Section IV contains proofs of our main results, formulated in Theorems A and B, as well as some related results and examples. In Section V, final remarks are given and two open questions are posed. A couple of standard proofs of identities related to LFTs are outsourced to the Appendix.

A. Notations

By $D_R(\mathsf{x}) := \{\mathsf{x} \in \mathbb{R}^2 \colon |\mathsf{x} - \mathsf{x}_0| < R\}$, we denote the open disc of the radius $R > 0$ with the center $\mathsf{x}_0 \in \mathbb{R}^2$. If such a disc is centered at the origin, we use the shorthand notation $D_R := D_R(0)$. By definition, we set $D_\infty := \mathbb{R}^2$. For a self-adjoint operator T, we denote by $\sigma_{\rm ess}(\mathsf{T})$, $\sigma_{\rm d}(\mathsf{T})$, and $\sigma(\mathsf{T})$ its essential, discrete, and full spectra, respectively. For an open set $\Omega \subset \mathbb{R}^2$, the space of smooth compactly supported functions and the first order Sobolev space are denoted by $\mathcal{D}(\Omega)$ and by $H^1(\Omega)$, respectively.

II. PRELIMINARIES

This section contains some preliminary material that will be used in the main part of this paper. In Subsection II A, we provide basic facts on the Sobolev space H^1, in particular, we define the Sobolev space $H^1(\mathbb{R}^2 \setminus \Lambda)$ for a non-closed Lipschitz curve Λ. In Subsection II B, we introduce the notions of a piecewise-C^1 curve and of a monotone curve. The concept of the linear fractional transformation is discussed in Subsection II C. A model spectral problem for one-dimensional Schrödinger operator with one-center δ'-interaction on a loop is considered in Subsection II D and a sufficient condition for absence of negative eigenvalues in this spectral problem is established.

A. Sobolev spaces

Let $\Omega \subset \mathbb{R}^2$ be a simply connected *Lipschitz domain* from the class described in [Ref. 39, Chap. VI]. This class of Lipschitz domains includes (as a subclass) Lipschitz domains with compact boundaries as in [Ref. 35, Chap. 3], *hypographs* of uniformly Lipschitz functions, and some other domains with non-compact boundaries. In what follows the Hilbert spaces $L^2(\Omega)$, $L^2(\Omega; \mathbb{C}^2)$, $L^2(\partial\Omega)$, and $H^1(\Omega)$ are defined as usual; see, e.g., [Ref. 35, Chap. 3] and Ref. 34. For the sake of brevity, we denote the scalar products in both $L^2(\Omega)$ and $L^2(\Omega; \mathbb{C}^2)$ by $(\cdot,\cdot)_\Omega$ without any danger of confusion. The scalar product in $L^2(\partial\Omega)$ is abbreviated by $(\cdot,\cdot)_{\partial\Omega}$. The space of functions on $\overline{\Omega}$ smooths up to the boundary $\partial\Omega$ is defined as

$$\mathcal{D}(\overline{\Omega}) := \{u|_{\Omega} : u \in \mathcal{D}(\mathbb{R}^2)\}.$$

By [Ref. 35, Theorem 3.29], see also Refs. 34 and 39, the space $\mathcal{D}(\overline{\Omega})$ is dense in both $L^2(\Omega)$ and $H^1(\Omega)$. The natural restriction mapping $\mathcal{D}(\overline{\Omega}) \ni u \mapsto u|_{\partial\Omega} \in L^2(\partial\Omega)$ can be extended by continuity up to the whole space $H^1(\Omega)$; see, e.g., [Ref. 35, Theorem 3.37] and Ref. 34. The corresponding extension by continuity $H^1(\Omega) \ni u \mapsto u|_{\partial\Omega} \in L^2(\partial\Omega)$ is called *the trace mapping*. The statement of the first lemma in this subsection appears in several monographs and papers in various forms; see, e.g., [Ref. 3, Lemma 2.6] and [Ref. 21, Lemma 2.5] for two different proofs of this statement.

Lemma 2.1. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain. Then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\|u|_{\partial\Omega}\|^2_{\partial\Omega} \le \varepsilon \|\nabla u\|^2_{\Omega} + C(\varepsilon)\|u\|^2_{\Omega}$$

holds for all $u \in H^1(\Omega)$.

The following hypothesis will be used throughout the paper.

Hypothesis 2.1. Let $\Omega_+ \subset \mathbb{R}^2$ be a simply connected Lipschitz domain from the above class, whose complement $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega}_+$ is a Lipschitz domain from the same class. Set $\Sigma := \partial\Omega_+ = \partial\Omega_-$ and suppose that $\Lambda \subset \Sigma$ is a connected subarc of Σ, which is not necessarily bounded if Σ is unbounded.

Obviously, the orthogonal sum $H^1(\Omega_+) \oplus H^1(\Omega_-)$ is a Hilbert space with respect to the scalar product

$$(u_+ \oplus u_-, v_+ \oplus v_-)_1 := (u_+, v_+)_{H^1(\Omega_+)} + (u_-, v_-)_{H^1(\Omega_-)}, \quad u_\pm, v_\pm \in H^1(\Omega_\pm).$$

The norm associated to this scalar product is denoted by $\|\cdot\|_1$. Let us define the jump of the trace as

$$[u]_\Sigma := u_+|_\Sigma - u_-|_\Sigma, \qquad u = u_+ \oplus u_- \in H^1(\Omega_+) \oplus H^1(\Omega_-).$$

The Hilbert space $L^2(\Sigma)$ can be decomposed into the orthogonal sum

$$L^2(\Sigma) = L^2(\Lambda) \oplus L^2(\Sigma \setminus \Lambda).$$

The scalar products in $L^2(\Lambda)$ and $L^2(\Sigma \setminus \Lambda)$ will further be denoted by $(\cdot,\cdot)_\Lambda$ and $(\cdot,\cdot)_{\Sigma\setminus\Lambda}$. Clearly enough, the restrictions of $u_\pm|_\Sigma$ for a $u_\pm \in H^1(\Omega_\pm)$ to the arcs $\Sigma \setminus \Lambda$ and Λ satisfy $u_\pm|_{\Sigma\setminus\Lambda} \in L^2(\Sigma \setminus \Lambda)$ and $u_\pm|_\Lambda \in L^2(\Lambda)$. Let us also introduce the notations

$$[u]_\bullet := u_+|_\bullet - u_-|_\bullet, \quad \bullet = \Lambda, \Sigma \setminus \Lambda, \quad u = u_+ \oplus u_- \in H^1(\Omega_+) \oplus H^1(\Omega_-).$$

The linear space

$$F_\Lambda := \{u \in \mathcal{D}(\overline{\Omega_+}) \oplus \mathcal{D}(\overline{\Omega_-}) : [u]_{\Sigma\setminus\Lambda} = 0\} \tag{2.1}$$

is a subspace of the Hilbert space $H^1(\Omega_+) \oplus H^1(\Omega_-)$, and its closure in $H^1(\Omega_+) \oplus H^1(\Omega_-)$

$$H^1(\mathbb{R}^2 \setminus \Lambda) := \overline{F_\Lambda}^{\|\cdot\|_1} \tag{2.2}$$

is itself a Hilbert space with respect to the same scalar product $(\cdot,\cdot)_1$.

Remark 2.2. The above construction of the space $H^1(\mathbb{R}^2 \setminus \Lambda)$ can easily be translated to the higher space dimensions, in which case Λ will be a hypersurface with free boundary (open hypersurface).

Remark 2.3. The space $H^1(\mathbb{R}^2 \setminus \Lambda)$ can also be defined in an alternative way. The set $\mathbb{R}^2 \setminus \Lambda$ is an open subset of $\mathbb{R}^2$. Hence, one can define for any $u \in L^2(\mathbb{R}^2)$ its weak partial derivatives $\partial_1 u$ and $\partial_2 u$ by means of the test functions in $\mathcal{D}(\mathbb{R}^2 \setminus \Lambda)$; see, e.g., [Ref. 35, Chap. 3]. Then the space $H^1(\mathbb{R}^2 \setminus \Lambda)$ is given by

$$H^1(\mathbb{R}^2 \setminus \Lambda) = \{u \in L^2(\mathbb{R}^2) : \partial_1 u, \partial_2 u \in L^2(\mathbb{R}^2)\}.$$

We are not aiming to provide here an argumentation that this new definition gives rise to the same space as in (2.2). It is only important here that the equivalence of these definitions automatically implies that the space $H^1(\mathbb{R}^2 \setminus \Lambda)$ is independent of the continuation of the arc Λ up to Σ. Another way of verifying the independence of the space $H^1(\mathbb{R}^2 \setminus \Lambda)$ from a continuation of Λ can be found in Ref. 9.

Next proposition collects some useful properties of the traces of functions in $H^1(\mathbb{R}^2 \setminus \Lambda)$ onto $\Sigma \setminus \Lambda$ and onto Λ.

Proposition 2.4. Let the curves $\Sigma, \Lambda \subset \mathbb{R}^2$, and the domains $\Omega_\pm \subset \mathbb{R}^2$ be as in Hypothesis 2.1. Let the Hilbert space $(H^1(\mathbb{R}^2 \setminus \Lambda), (\cdot,\cdot)_1)$ be as in (2.2). Then the following statements hold.

(i) $[u]_{\Sigma\setminus\Lambda} = 0$, *for all* $u \in H^1(\mathbb{R}^2 \setminus \Lambda)$.
(ii) *For any* $\varepsilon > 0$ *there exists a constant* $C(\varepsilon) > 0$ *such that*

$$\|[u]_\Lambda\|^2_\Lambda \leq \varepsilon \|\nabla u\|^2_{\mathbb{R}^2} + C(\varepsilon)\|u\|^2_{\mathbb{R}^2},$$

for all $u \in H^1(\mathbb{R}^2 \setminus \Lambda)$.

Proof. (i) It can be easily checked that the continuity of the trace mappings

$$H^1(\Omega_\pm) \ni u_\pm \mapsto u_\pm|_\Sigma \in L^2(\Sigma)$$

implies that the mapping

$$H^1(\Omega_+) \oplus H^1(\Omega_-) \ni u \mapsto [u]_{\Sigma\setminus\Lambda} \in L^2(\Sigma \setminus \Lambda)$$

is well-defined and continuous. Note that for any $u \in H^1(\mathbb{R}^2 \setminus \Lambda)$ there exists an approximating sequence $(u_n)_n \subset F_\Lambda$ (cf., (2.2)) such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$. Hence, we obtain

$$[u]_{\Sigma\setminus\Lambda} = \lim_{n\to\infty} [u_n]_{\Sigma\setminus\Lambda} = 0.$$

(ii) By Lemma 2.1 for any $\varepsilon > 0$ there exist constants $C_\pm(\varepsilon) > 0$ such that

$$\|u_\pm|_\Sigma\|^2_\Sigma \leq (\varepsilon/2)\|\nabla u_\pm\|^2_{\Omega_\pm} + C_\pm(\varepsilon)\|u_\pm\|^2_{\Omega_\pm}, \tag{2.3}$$

for all $u \in H^1(\Omega_+) \oplus H^1(\Omega_-)$. Set then $C(\varepsilon) := \max\{2C_+(\varepsilon), 2C_-(\varepsilon)\}$. Using the result of item (i) and bounds (2.3) we obtain that for any $\varepsilon > 0$ and any $u = u_+ \oplus u_- \in H^1(\mathbb{R}^2 \setminus \Lambda) \subset H^1(\Omega_+) \oplus H^1(\Omega_-)$ holds

$$\begin{aligned}\|[u]_\Lambda\|^2_\Lambda = \|[u]_\Sigma\|^2_\Sigma &\leq 2\|u_+|_\Sigma\|^2_\Sigma + 2\|u_-|_\Sigma\|^2_\Sigma \\ &\leq \varepsilon\|\nabla u_+\|^2_{\Omega_+} + \varepsilon\|\nabla u_-\|^2_{\Omega_-} + 2C_+(\varepsilon)\|u_+\|^2_{\Omega_+} + 2C_-(\varepsilon)\|u_-\|^2_{\Omega_-} \\ &\leq \varepsilon\|\nabla u\|^2_{\mathbb{R}^2} + C(\varepsilon)\|u\|^2_{\mathbb{R}^2}.\end{aligned}$$

□

Remark 2.5. For $\omega_1, \omega_2 \in L^\infty(\Lambda; \mathbb{R})$ by writing $\omega_1 \leq \omega_2$ we will always implicitly mean that $\omega_2 - \omega_1 \geq 0$ almost everywhere.

B. On curves in $\mathbb{R}^2$

We begin this subsection by defining the notion of a piecewise-C^1 curve. It should be emphasized that, especially for unbounded curves, definition of a piecewise-C^1 curve is non-unique in the mathematical literature.

Definition 2.6. A non-closed curve $\Lambda \subset \mathbb{R}^2$ satisfying Hypothesis 2.1 is called piecewise-C^1 *if it can be parametrized via a piecewise-C^1 mapping*

$$\lambda : I \to \mathbb{R}^2, \qquad \lambda(s) := (\lambda_1(s), \lambda_2(s)), \quad I := (0, L),\ L \in (0, +\infty], \tag{2.4}$$

such that $\lambda(I) = \Lambda$ and λ is injective. If, moreover, $|\lambda'(s)| = 1$ for almost all $s \in I$, then such a parametrization is called natural *and L is then called the length of Λ.*

We require in the above definition, that the curve Λ satisfies Hypothesis 2.1, to avoid increasing oscillations at infinity for unbounded curves.

Further, we proceed to define a (non-standard) concept of a *monotone curve*. The authors have not succeeded to find a common name for this concept in the literature on geometry.

Definition 2.7. A piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ is called monotone *if it can be parametrized via a piecewise-C^1 mapping $\varphi\colon (0,R) \to \mathbb{R}$ with $R \in (0,+\infty]$ such that*

$$\Lambda = \{\mathsf{x}_0 + (r\cos\varphi(r), r\sin\varphi(r)) \in \mathbb{R}^2 \colon r \in (0,R)\},$$

with some fixed $\mathsf{x}_0 \in \mathbb{R}^2$.

Informally speaking, a curve Λ is monotone if the distance (measured in $\mathbb{R}^2$) from one of its endpoints is always increasing when travelling along Λ from this endpoint towards another endpoint or towards infinity.

Remark 2.8. For a curve $\Lambda \subset \mathbb{R}^2$ as in Definition 2.7 any function $\omega \in L^\infty(\Lambda)$ can be viewed as a function of the argument $r \in (0,R)$.

C. Linear fractional transformations

For later purposes we introduce LFTs and state several useful properties of them. To work with LFT it is more convenient to deal with the *extended complex plane (Riemann sphere)* $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ rather than the usual complex plane. The complex plane itself as a subset of $\widehat{\mathbb{C}}$ can be naturally identified with the Euclidean plane $\mathbb{R}^2$ and occasionally we will use this identification.

For the purpose of convenience the extended complex plane $\widehat{\mathbb{C}}$ is endowed with a suitable topology: a sequence $(z_n)_n \in \widehat{\mathbb{C}}$ converges to $z \in \widehat{\mathbb{C}}$ if one of the following conditions holds:

(i) $z = \infty$ and there exists $N \in \mathbb{N}$ such that $z_n = \infty$ for all $n \geq N$;
(ii) $z = \infty$ and any infinite subsequence $(z_{n_k})_k \subset \mathbb{C}$ of $(z_n)_n$ satisfies $\lim_{k\to\infty} |z_{n_k}| = \infty$;
(iii) $z \in \mathbb{C}$, there exists $N \in \mathbb{N}$ such that $z_n \neq \infty$ for all $n \geq N$, and $\lim_{n\to\infty} z_n = z$ in the sense of convergence in $\mathbb{C}$.

This definition of topology can also be easily reformulated in terms of open sets. The above topology on $\widehat{\mathbb{C}}$ is equivalent to the topology of $\mathbb{S}^2$ (unit sphere in $\mathbb{R}^3$). A natural homeomorphism between $\widehat{\mathbb{C}}$ and $\mathbb{S}^2$ is called *stereographic projection*; see, e.g., [Ref. 29, Section 6.2.3].

For $a,b,c,d \in \mathbb{C}$ such that $ad - bc \neq 0$ the mapping $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a LFT if one of the two conditions holds:

(i) $c = 0$, $d \neq 0$, $M(\infty) := \infty$, and $M(z) := (a/d)z + (b/d)$ for $z \in \mathbb{C}$.
(ii) $c \neq 0$, $M(\infty) := a/c$, $M(-d/c) := \infty$, and $M(z) := \frac{az+b}{cz+d}$ for $z \in \mathbb{C}$, $z \neq -d/c$.

The following statement can be found in [Ref. 29, Section 6.2.3].

Proposition 2.9. Any LFT $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism with respect to the above topology on $\widehat{\mathbb{C}}$ and its inverse M^{-1} is also a LFT. The composition $M_1 \circ M_2$ of two LFTs M_1, M_2 is a LFT as well.

It is convenient to introduce $M_1(x,y) := \operatorname{Re} M(x + \mathsf{i}y)$ and $M_2(x,y) := \operatorname{Im} M(x + \mathsf{i}y)$. Then *Cauchy-Riemann equations*

$$\partial_x M_1 = \partial_y M_2, \qquad \partial_x M_2 = -\partial_y M_1, \tag{2.5}$$

hold pointwise in $\mathbb{R}^2$ except the point $M^{-1}(\infty)$. In view of these equations the Jacobian J_M of the mapping M can be computed (again except the point $M^{-1}(\infty)$) by the formulae

$$\mathsf{J}_M = (\partial_x M_1)^2 + (\partial_y M_1)^2 = (\partial_x M_2)^2 + (\partial_y M_2)^2, \tag{2.6}$$

also the following relation turns out to be useful:

$$\langle \nabla M_1, \nabla M_2 \rangle = \partial_x M_1 \partial_x M_2 + \partial_y M_1 \partial_y M_2 = 0, \tag{2.7}$$

i.e., the vectors ∇M_1 and ∇M_2 are orthogonal to each other.

Next, auxiliary lemma is of purely technical nature and is proven for convenience of the reader.

Lemma 2.10. Let M be a LFT with the Jacobian J_M. Then for any $x \in \mathbb{R}^2$, $x \neq M^{-1}(\infty)$, and any function $u\colon \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{C}$ differentiable at the point $M(x)$

$$|(\nabla v)(x)|^2 = |(\nabla u)(M(x))|^2 \mathsf{J}_M(x)$$

holds with $v = u \circ M$.

Proof. Using relations (2.6), (2.7), and the chain rule for differentiation we obtain

$$\begin{aligned}|\nabla v|^2 &= \big|(u'_x \circ M)\partial_x M_1 + (u'_y \circ M)\partial_x M_2\big|^2 + \big|(u'_x \circ M)\partial_y M_1 + (u'_y \circ M)\partial_y M_2\big|^2 \\ &= \Big(|u'_x \circ M|^2 + |u'_y \circ M|^2\Big)\mathsf{J}_M + 2\mathrm{Re}\left[\big((u'_x u'_y) \circ M\big) \cdot \big\langle \nabla M_1, \nabla M_2\big\rangle\right] \\ &= \Big(|u'_x \circ M|^2 + |u'_y \circ M|^2\Big)\mathsf{J}_M = |(\nabla u) \circ M|^2 \mathsf{J}_M.\end{aligned}$$

The claim is thus shown. □

D. Point δ'-interaction on a loop

In this subsection, we introduce an auxiliary self-adjoint Schrödinger operator $\mathsf{T}_{d,\omega}$ acting in the Hilbert space $(L^2(I), (\cdot,\cdot)_I)$ with $I := (0,d)$ and corresponding to a point δ'-interaction on the one-dimensional loop of length $d > 0$. Employing the following shorthand notation:

$$[\psi]_{\partial I} := \psi(d-) - \psi(0+), \qquad \psi \in H^2(I),$$

we define

$$\mathsf{T}_{d,\omega}\psi := -\psi'', \quad \mathrm{dom}\,\mathsf{T}_{d,\omega} := \big\{\psi \in H^2(I) \colon \psi'(0+) = \psi'(d-) = \omega[\psi]_{\partial I}\big\}, \tag{2.8}$$

where $\omega \in \mathbb{R}$; see Refs. 1, 7, 10, 18, 23, and 27 for the investigations of more general operators of this type. Note that $\omega = 0$ corresponds to Neumann boundary conditions at the endpoints. Next proposition states a spectral property of $\mathsf{T}_{d,\omega}$, which is useful for our purposes.

Proposition 2.11. The self-adjoint operator $\mathsf{T}_{d,\omega}$ in the Hilbert space $L^2(I)$, defined in (2.8), is non-negative if $d\omega \leq 1$.

Proof. We prove this proposition via construction of an explicit condition for the negative spectrum of $\mathsf{T}_{d,\omega}$ and its analysis. Obviously, the spectrum of $\mathsf{T}_{d,\omega}$ is discrete (due to the compact embedding of $H^2(I)$ into $L^2(I)$). An eigenfunction of $\mathsf{T}_{d,\omega}$, which corresponds to a negative eigenvalue $\lambda = -\kappa^2 < 0$ ($\kappa > 0$) is characterized by the following two conditions:

$$-\psi''(x) = -\kappa^2\psi(x), \tag{2.9a}$$

$$\psi'(0+) = \psi'(d-) = \omega[\psi]_{\partial I}. \tag{2.9b}$$

Condition (2.9a) is satisfied by a function, which can be represented in the form

$$\psi(x) = A\exp(\kappa x) + B\exp(-\kappa x), \qquad x \in (0,d),$$

with some constants $A, B \in \mathbb{C}$. Simple computations yield

$$\begin{aligned}\psi(0+) &= A + B, & \psi(d-) &= A\exp(\kappa d) + B\exp(-\kappa d),\\ \psi'(0+) &= \kappa A - \kappa B, & \psi'(d-) &= \kappa A\exp(\kappa d) - \kappa B\exp(-\kappa d).\end{aligned}$$

The above identities and condition (2.9b) together imply

$$A = \frac{1 - \exp(-\kappa d)}{1 - \exp(\kappa d)}B, \tag{2.10a}$$

$$\kappa A - \kappa B = \omega\Big(A\big(\exp(\kappa d) - 1\big) + B\big(\exp(-\kappa d) - 1\big)\Big). \tag{2.10b}$$

Substituting formula (2.10a) into (2.10b), we arrive at

$$\kappa B\left(\frac{1-\exp(-\kappa d)}{1-\exp(\kappa d)}-1\right)=\omega\Big(-B(1-\exp(-\kappa d))+B(\exp(-\kappa d)-1)\Big)$$

that is equivalent to

$$\exp(-\kappa d)-\exp(\kappa d)=\frac{2\omega}{\kappa}\Big(1-\exp(-\kappa d)\Big)\Big(1-\exp(\kappa d)\Big).$$

Making several steps further in the computations, we obtain

$$\begin{aligned}1&=\frac{2\omega}{\kappa}\frac{(1-\exp(-\kappa d))(1-\exp(\kappa d))}{\exp(\kappa d)(\exp(-2\kappa d)-1)}=\frac{2\omega}{\kappa}\frac{\exp(\kappa d)-1}{\exp(\kappa d)(\exp(-\kappa d)+1)}\\&=\frac{2\omega}{\kappa}\frac{1-\exp(-\kappa d)}{1+\exp(-\kappa d)}.\end{aligned}$$

Define then the following function:

$$\Theta_\omega(\kappa):=\frac{2\omega}{\kappa}\frac{1-\exp(-\kappa d)}{1+\exp(-\kappa d)},\qquad \kappa>0.$$

Hence, the point $\lambda=-\kappa^2$ is a negative eigenvalue of $\mathsf{T}_{d,\omega}$ if and only if $\Theta_\omega(\kappa)=1$. Let us consider the following auxiliary function:

$$f(x):=\frac{1-\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}},\qquad x\geq 0,$$

which is clearly continuously differentiable, and whose derivative is given by

$$f'(x)=\frac{2}{(\mathrm{e}^{x/2}+\mathrm{e}^{-x/2})^2},\qquad x\geq 0.$$

Hence, using the standard inequality $a+1/a>2$, $a\in(0,+\infty)$, $a\neq 1$, we get $f'(x)<1/2$ for all $x>0$. Applying the mean value theorem to f, we obtain

$$f(x)=f(0)+f'(\xi)(x-0)=f'(\xi)x<\frac{x}{2},$$

here, $\xi\in(0,x)$. Finally, note that

$$0\leq\Theta_\omega(\kappa)=\frac{2\omega}{\kappa}f(\kappa d)<d\omega.$$

Thus, for $d\omega\leq 1$ the equation $\Theta_\omega(\kappa)=1$ has no positive roots and the claim follows. □

According to, e.g., Ref. 28, the operator $\mathsf{T}_{d,\omega}$ represents the sesquilinear form

$$\mathfrak{a}_{d,\omega}[\psi,\varphi]:=(\psi',\varphi')_I-\omega[\psi]_{\partial I}[\overline{\varphi}]_{\partial I},\qquad \mathrm{dom}\,\mathfrak{a}_{d,\omega}:=H^1(I),\tag{2.11}$$

and we can derive the following simple corollary of Proposition 2.11.

Corollary 2.12. Let the sesquilinear form $\mathfrak{a}_{d,\omega}$ be as in (2.11). If $d\omega\leq 1$, then $\mathfrak{a}_{d,\omega}[\psi]\geq 0$ for all $\psi\in H^1(I)$.

Remark 2.13. Consider the non-negative symmetric operator

$$S\psi:=-\psi'',\qquad \mathrm{dom}\,S:=H^2_0(I),$$

in $L^2(I)$. The operator S is known to have deficiency indices (2,2). One may consider self-adjoint extensions of S in $L^2(I)$. The self-adjoint operator $\mathsf{T}_{d,\omega}$ with $d\omega=1$ turns to be the *Krein-von Neumann extension* of S (the "smallest" non-negative self-adjoint extension of S), i.e., for any other non-negative self-adjoint extension T of S

$$(T+a)^{-1}\leq(\mathsf{T}_{d,\omega}+a)^{-1}$$

holds for all $a>0$; see, e.g., [Ref. 36, Corollary 10.13, Theorem 14.25, Example 14.14].

III. DEFINITION OF THE OPERATOR AND ITS ESSENTIAL SPECTRUM

In this section, we rigorously define using form methods Schrödinger operators with δ'-interactions supported on non-closed curves as in Hypothesis 2.1 and characterise their essential spectra. In the latter characterisation the notion of a quasi-conical domain plays an essential role.

A. Definition of the operator via its sesquilinear form

Schrödinger operators with δ'-interactions supported on closed hypersurfaces were defined and investigated in Refs. 2, 3, 5, and 14–16. The goal of this subsection is to define rigorously Schrödinger operator with δ'-interactions supported on a non-closed curve Λ satisfying Hypothesis 2.1. In the case of a bounded $C^{2,1}$-smooth curve Λ our definition of the operator agrees with the one in the recent preprint,[33] where this Hamiltonian is defined using the theory of self-adjoint extensions of symmetric operators.

Let $\omega \in L^\infty(\Lambda;\mathbb{R})$ and denote by $\|\omega\|_\infty$ its sup-norm. Recall the definition of the sesquilinear form $\mathfrak{a}^\Lambda_\omega$ in (1.1)

$$\mathfrak{a}^\Lambda_\omega[u,v] := (\nabla u, \nabla v)_{\mathbb{R}^2} - (\omega[u]_\Lambda, [v]_\Lambda)_\Lambda, \qquad \operatorname{dom}\mathfrak{a}^\Lambda_\omega := H^1(\mathbb{R}^2 \setminus \Lambda). \tag{3.1}$$

If $\omega \equiv 0$, we occasionally write $\mathfrak{a}^\Lambda_{\mathrm{N}}$ instead of $\mathfrak{a}^\Lambda_\omega$.

Proposition 3.1. Let $\Lambda \subset \mathbb{R}^2$ be as in Hypothesis 2.1, let $\omega \in L^\infty(\Lambda;\mathbb{R})$, and let the linear space F_Λ be as in (2.1). Then the sesquilinear form $\mathfrak{a}^\Lambda_\omega$ in (3.1) is closed, densely defined, symmetric, and lower-semibounded in the Hilbert space $L^2(\mathbb{R}^2)$. Moreover, $F_\Lambda \subset \operatorname{dom}\mathfrak{a}^\Lambda_\omega$ is a core for this form.

Proof. Since $\mathfrak{a}^\Lambda_\omega[u,u] \in \mathbb{R}$ for all $u \in \operatorname{dom}\mathfrak{a}^\Lambda_\omega$, the form $\mathfrak{a}^\Lambda_\omega$ is, clearly, symmetric. It is straightforward to see the chain of inclusions $\mathcal{D}(\mathbb{R}^2) \subset F_\Lambda \subset \operatorname{dom}\mathfrak{a}^\Lambda_\omega$. Density of $\operatorname{dom}\mathfrak{a}^\Lambda_\omega$ in $L^2(\mathbb{R}^2)$ follows thus from the density of $\mathcal{D}(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$; for the latter see, e.g., [Ref. 35, Corollary 3.5].

The norm induced in the conventional way by the form $\mathfrak{a}^\Lambda_{\mathrm{N}}$ on its domain $H^1(\mathbb{R}^2 \setminus \Lambda)$ is easily seen to be equivalent to the norm $\|\cdot\|_1$ introduced in Subsection II A. Hence, the form $\mathfrak{a}^\Lambda_{\mathrm{N}}$ is closed and the space F_Λ, being dense in $H^1(\mathbb{R}^2 \setminus \Lambda)$, is a core for it, cf., [Ref. 36, Dfn. 10.2]. Let us then introduce an auxiliary form

$$\mathfrak{a}'[u,v] := (\omega[u]_\Lambda, [v]_\Lambda)_\Lambda, \qquad \operatorname{dom}\mathfrak{a}' := H^1(\mathbb{R}^2 \setminus \Lambda).$$

Using Proposition 2.4 (ii) we get for all $\varepsilon > 0$ the following bound:

$$|\mathfrak{a}'[u,u]| \le \varepsilon\,\|\omega\|_\infty \mathfrak{a}^\Lambda_{\mathrm{N}}[u,u] + C(\varepsilon)\|\omega\|_\infty \|u\|^2_{\mathbb{R}^2},$$

with some $C(\varepsilon) > 0$. Choosing $\varepsilon < \frac{1}{\|\omega\|_\infty}$ in the above bound, we obtain that $\mathfrak{a}'$ is relatively bounded with respect to $\mathfrak{a}^\Lambda_{\mathrm{N}}$ with form bound <1. Hence, by [Ref. 36, Theorem 10.21] *(KLMN theorem)* the form $\mathfrak{a}^\Lambda_\omega = \mathfrak{a}^\Lambda_{\mathrm{N}} + \mathfrak{a}'$ is closed and the space F_Λ is a core for it. □

Definition 3.2. The self-adjoint operator $\mathsf{H}^\Lambda_\omega$ in $L^2(\mathbb{R}^2)$ corresponding to the form $\mathfrak{a}^\Lambda_\omega$ via the first representation theorem (see, e.g., *Ref. 25, Chap. VI, Theorem 2.1]) is called Schrödinger operator with δ'-interaction of strength ω supported on Λ.*

If ω is a non-negative function, then we occasionally say that the respective δ'-interaction is *attractive*.

B. Essential spectrum

In this subsection, we characterise the essential spectrum of the operator $\mathsf{H}^\Lambda_\omega$. To this aim, we require the following auxiliary lemma.

Lemma 3.3. Let the self-adjoint operator $\mathsf{H}^\Lambda_\omega$ be as in Definition 3.2. Then for any $u \in \mathcal{D}(\mathbb{R}^2 \setminus \Lambda)$ holds

$$u \in \operatorname{dom}\mathsf{H}^\Lambda_\omega \quad \textit{and} \quad \mathsf{H}^\Lambda_\omega u = -\Delta u. \tag{3.2}$$

Proof. Let Σ and $\Omega_\pm$ be as in Hypothesis 2.1. Let $u \in \mathcal{D}(\mathbb{R}^2 \setminus \Lambda) \subset F_\Lambda \subset \operatorname{dom} \mathfrak{a}_\omega^\Lambda$ and $v \in \operatorname{dom} \mathfrak{a}_\omega^\Lambda$. Define $u_\pm := u \upharpoonright \Omega_\pm$ and $v_\pm := v \upharpoonright \Omega_\pm$. With these notations in hands we get

$$\mathfrak{a}_\omega^\Lambda[u,v] = (\nabla u_+, \nabla v_+)_{\Omega_+} + (\nabla u_-, \nabla v_-)_{\Omega_-},$$

where the boundary term in (3.1) vanished due to the choice of u. Applying the first Green identity (see, e.g. [Ref. 35, Lemma 4.1] and also [Ref. 3, Sec. 2]) to the above formula, we get

$$\begin{aligned} \mathfrak{a}_\omega^\Lambda[u,v] = (-\Delta u_+, v_+)_{\Omega_+} + (-\Delta u_-, v_-)_{\Omega_-} \\ + \big(\partial_{\nu_+} u_+|_{\Sigma\setminus\Lambda} + \partial_{\nu_-} u_-|_{\Sigma\setminus\Lambda}, v|_{\Sigma\setminus\Lambda}\big)_{\Sigma\setminus\Lambda} \\ + \big(\partial_{\nu_+} u_+|_\Lambda, v_+|_\Lambda\big)_\Lambda + \big(\partial_{\nu_-} u_-|_\Lambda, v_-|_\Lambda\big)_\Lambda = (-\Delta u, v)_{\mathbb{R}^2}, \end{aligned}$$

where we employed that $\partial_{\nu_\pm} u_\pm|_\Lambda = 0$, that $\partial_{\nu_+} u_+|_{\Sigma\setminus\Lambda} + \partial_{\nu_-} u_-|_{\Sigma\setminus\Lambda} = 0$, and that $[v]_{\Sigma\setminus\Lambda} = 0$; for the latter, cf., Proposition 2.4 (i). Finally, the first representation theorem yields (3.2). □

Next, we define the notion of the quasi-conical domain; see Ref. 22 and also [Ref. 11, Def. X.6.1].

Definition 3.4. A domain $\Omega \subset \mathbb{R}^2$ is called quasi-conical *if for any $n \in \mathbb{N}$ there exists $x_n \in \mathbb{R}^2$ such that $D_n(x_n) \subset \Omega$. Recall that here $D_n(x_n)$ is the disc of radius n with the center x_n.*

Using this notion, we prove that positive semi-axis lies inside the spectrum of $\mathsf{H}_\omega^\Lambda$ if the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical. The technique of this proof is rather standard.

Proposition 3.5. Let the curve $\Lambda \subset \mathbb{R}^2$ as in Hypotheses 2.1 be such that the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical. Then the spectrum of the self-adjoint operator $\mathsf{H}_\omega^\Lambda$ in Definition 3.2 satisfies

$$\sigma(\mathsf{H}_\omega^\Lambda) \supseteq [0, +\infty). \tag{3.3}$$

Proof. First, for any $k \in \mathbb{R}^2$, define the sequence

$$u_n(x) := v_n(x) \mathrm{e}^{\mathrm{i}k\cdot x}, \qquad n \in \mathbb{N},$$

where $v_n(x) := n^{-1} v(n^{-1}x)$, $n \in \mathbb{N}$, and v is a non-trivial function in $\mathcal{D}(\mathbb{R}^2)$ with $\operatorname{supp} v \subset D_1$ and such that $\|v\|_{\mathbb{R}^2} = 1$. The prefactor in the definition of v_n is chosen in such a way that also each v_n satisfies $\|v_n\|_{\mathbb{R}^2} = 1$. In fact, we have (by direct computations)

$$\|v_n\|_{\mathbb{R}^2} = 1, \qquad \|\nabla v_n\|_{\mathbb{R}^2} = \frac{\|\nabla v\|_{\mathbb{R}^2}}{n}, \qquad \|\Delta v_n\|_{\mathbb{R}^2} = \frac{\|\Delta v\|_{\mathbb{R}^2}}{n^2}. \tag{3.4}$$

Second, we set

$$w_n(x) := u_n(x - x_n), \qquad n \in \mathbb{N},$$

with x_n corresponding to the quasi-conical domain $\mathbb{R}^2 \setminus \Lambda$ according to Definition 3.4. Hence, we get

$$\operatorname{supp} w_n \subset D_n(x_n) \subset \mathbb{R}^2 \setminus \Lambda,$$

and therefore $w_n \in \mathcal{D}(\mathbb{R}^2 \setminus \Lambda)$ for all $n \in \mathbb{N}$. It is clear in view of Lemma 3.3 that each w_n belongs to $\operatorname{dom} \mathsf{H}_\omega^\Lambda \supset \mathcal{D}(\mathbb{R}^2 \setminus \Lambda)$.

A direct computation yields

$$|-\Delta w_n - |k|^2 w_n|^2 \le 2|\Delta v_n|^2 + 4|k\cdot\nabla v_n|^2 \le 2|\Delta v_n|^2 + 4|k|^2 \cdot |\nabla v_n|^2.$$

Using (3.4) and Lemma 3.3, we therefore have

$$\|\mathsf{H}_\omega^\Lambda w_n - |k|^2 w_n\|_{\mathbb{R}^2}^2 = \|-\Delta w_n - |k|^2 w_n\|_{\mathbb{R}^2}^2 \le 2\|\Delta v_n\|_{\mathbb{R}^2}^2 + 4|k|^2 \|\nabla v_n\|_{\mathbb{R}^2}^2 \to 0, \qquad n \to \infty.$$

Since the choice of $k \in \mathbb{R}^2$ was arbitrary, we conclude applying Weyl's criterion (see [Ref. 41, Sec. 7.4] and also [Ref. 30, Theorem 4]) that $[0, +\infty) \subseteq \sigma(\mathsf{H}_\omega^\Lambda)$. □

We emphasize that not for every non-closed curve $\Lambda \subset \mathbb{R}^2$ the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical; e.g., for the *Archimedean spiral*, defined in polar coordinates (r, φ) by the equation $r(\varphi) := a + b\varphi$, $\varphi \in \mathbb{R}_+$, $a, b > 0$, the domain $\mathbb{R}^2 \setminus \Lambda$ is not of this type.

In the case of bounded curves, we show that the essential spectrum of $\mathsf{H}_\omega^\Lambda$ coincides with the set $[0,+\infty)$.

Proposition 3.6. Let the bounded curve $\Lambda \subset \mathbb{R}^2$ be as in Hypothesis 2.1 and let the self-adjoint operator $\mathsf{H}_\omega^\Lambda$ be as in Definition 3.2. Then its essential spectrum is characterised as

$$\sigma_{\rm ess}(\mathsf{H}_\omega^\Lambda) = [0,+\infty).$$

Proof. Let the curve $\Sigma \subset \mathbb{R}^2$ and the domains $\Omega_\pm \subset \mathbb{R}^2$ be as in Hypothesis 2.1, in particular, $\Lambda \subset \Sigma$. Let us also set $c := \|\omega\|_\infty$. Consider the sesquilinear form

$$\begin{aligned} \mathfrak{a}_c^\Sigma[u,v] &:= (\nabla u, \nabla v)_{\mathbb{R}^2} - c([u]_\Sigma, [v]_\Sigma)_\Sigma, \\ \mathrm{dom}\,\mathfrak{a}_c^\Sigma &:= H^1(\Omega_+) \oplus H^1(\Omega_-). \end{aligned} \tag{3.5}$$

According to [Ref. 3, Prop. 3.1] the form $\mathfrak{a}_c^\Sigma$ is closed, densely defined, symmetric, and lower-semibounded in $L^2(\mathbb{R}^2)$. The self-adjoint operator H_c^Σ in $L^2(\mathbb{R}^2)$ representing the form $\mathfrak{a}_c^\Sigma$, satisfies

$$\sigma_{\rm ess}(\mathsf{H}_c^\Sigma) = [0,+\infty), \tag{3.6}$$

see [Ref. 3, Theorem 4.2] and also [Ref. 5, Theorem 3.16]. The sesquilinear forms $\mathfrak{a}_\omega^\Lambda$ and $\mathfrak{a}_c^\Sigma$ in (3.1) and (3.5), respectively, naturally satisfy the ordering

$$\mathfrak{a}_c^\Sigma \prec \mathfrak{a}_\omega^\Lambda$$

in the sense of [Ref. 25, Section VI.2.5], see also [Ref. 6, Section 10.2.3]. Indeed, first, $\mathrm{dom}\,\mathfrak{a}_\omega^\Lambda \subset \mathrm{dom}\,\mathfrak{a}_c^\Sigma$ and second, for any $u \in \mathrm{dom}\,\mathfrak{a}_\omega^\Lambda$ the inequality $\mathfrak{a}_c^\Sigma[u,u] \le \mathfrak{a}_\omega^\Lambda[u,u]$ holds due to the choice of the constant $c \ge 0$. Hence, using (3.6) and [Ref. 6, Section 10.2, Theorem 4] we arrive at

$$0 = \inf \sigma_{\rm ess}(\mathsf{H}_c^\Sigma) \le \inf \sigma_{\rm ess}(\mathsf{H}_\omega^\Lambda).$$

Therefore, we end up with the following inclusion:

$$\sigma_{\rm ess}(\mathsf{H}_\omega^\Lambda) \subseteq [0,+\infty). \tag{3.7}$$

Moreover, for simple geometric reasons for any bounded curve Λ the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical and hence by Proposition 3.5 the opposite inclusion

$$\sigma_{\rm ess}(\mathsf{H}_\omega^\Lambda) \supseteq [0,+\infty) \tag{3.8}$$

holds as well. The claim then follows from these two inclusions ((3.7) and (3.8)). □

IV. NON-NEGATIVITY OF $\mathsf{H}_\omega^\Lambda$

This section plays the central role in the present paper. We obtain various sufficient conditions for the operator $\mathsf{H}_\omega^\Lambda$ to be non-negative. Under additional assumptions, we also show that positive spectrum of $\mathsf{H}_\omega^\Lambda$ comprises the whole positive real axis and thus the operator $\mathsf{H}_\omega^\Lambda$ has no bound states. In the proofs, we use the min-max principle for self-adjoint operators, a reduction to the one-dimensional model problem discussed in Subsection II D, and some insights from geometry and complex analysis.

A. An auxiliary lemma

In this subsection, we prove a lemma, based on which we show non-negativity of the operators $\mathsf{H}_\omega^\Lambda$ under certain assumptions on ω. For the formulation of this lemma, we require the following hypothesis, the assumptions of which are grouped in three logical blocks labelled by capital latin letters.

Hypothesis 4.1. **(A)** Let a monotone piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ be parametrized via the mapping $\varphi\colon (0,R) \to \mathbb{R}$, $R \in (0,+\infty]$, as in Definition 2.7 with $\mathsf{x}_0 = 0$.

(B) Suppose that piecewise-C^1 domains $G_\pm \subset D_R$ satisfy the following conditions:

$$G_+ \cap G_- = \varnothing, \quad \overline{D_R} = \overline{G_+ \cup G_-}, \quad \text{and} \quad \Lambda \subset \overline{G_+} \cap \overline{G_-}.$$

Set $\Sigma \coloneqq \overline{G_+} \cap \overline{G_-}$. In particular, the inclusion $\Lambda \subset \Sigma$ holds.

(C) Let the function $\omega \in L^\infty(\Lambda; \mathbb{R})$ as a function of the distance r from the origin satisfy

$$\omega(r) \le \frac{1}{2\pi r\sqrt{1 + (r\varphi'(r))^2}}, \qquad \text{for} \quad r \in (0, R). \tag{4.1}$$

We further deal with the space $H^1(G_+) \oplus H^1(G_-) \subset L^2(D_R)$. Let us introduce also the following notations:

$$[u]_\bullet \coloneqq u_+|_\bullet - u_-|_\bullet, \quad \bullet = \Lambda \text{ or } \Sigma \setminus \Lambda, \quad u = u_+ \oplus u_- \in H^1(G_+) \oplus H^1(G_-).$$

Clearly, one can define polar coordinates (r, φ) on D_R, which are connected with the usual Cartesian coordinates via standard relations $x = r\cos\varphi$ and $y = r\sin\varphi$. The points $(r, \varphi + 2\pi k)$ with $k \in \mathbb{Z}$ are identified with each other. The disc D_R in the polar coordinate system is given by $D_R = \{(r,\varphi) : r \in [0, R), \varphi \in [0, 2\pi)\}$.

For the substantial simplification of further computations we make use of the following shorthand notation:

$$\mathfrak{f}^\Lambda_{D_R,\omega}[u] \coloneqq \|\nabla u\|^2_{D_R} - (\omega[u]_\Lambda, [u]_\Lambda)_\Lambda, \qquad u \in \mathcal{D}(\overline{G_+}) \oplus \mathcal{D}(\overline{G_-}), \tag{4.2}$$

where all the objects are as in Hypothesis 4.1. Now, we formulate and prove the following lemma.

Lemma 4.1. Assume that Hypothesis 4.1 holds. Then $\mathfrak{f}^\Lambda_{D_R,\omega}[u] \ge 0$ *for all* $u \in \mathcal{D}(\overline{G_+}) \oplus \mathcal{D}(\overline{G_-})$ *such that* $[u]_{\Sigma\setminus\Lambda} = 0$.

Proof. Let $u \in \mathcal{D}(\overline{G_+}) \oplus \mathcal{D}(\overline{G_-})$ be such that $[u]_{\Sigma\setminus\Lambda} = 0$. The proof of $\mathfrak{f}^\Lambda_{D_R,\omega}[u] \ge 0$ is then split in three steps.

Step 1. For any $(x, y) \in D_R \setminus \Sigma$ the value $|(\nabla u)(x, y)|^2$ can be expressed in polar coordinates (r, φ) as

$$|(\nabla u)(x, y)|^2 = |(\partial_r u)(r, \varphi)|^2 + \frac{1}{r^2}|(\partial_\varphi u)(r, \varphi)|^2.$$

Using the above expression for the gradient, we obtain the following estimate:

$$\|\nabla u\|^2_{D_R} = \int_0^{2\pi}\int_0^R |(\nabla u)(r, \varphi)|^2 r \mathsf{d}r\mathsf{d}\varphi \le \int_0^R \frac{1}{r}\left(\int_0^{2\pi} |(\partial_\varphi u)(r, \varphi)|^2 \mathsf{d}\varphi\right)\mathsf{d}r, \tag{4.3}$$

in which we have thrown away a positive term in the second step. Interchanging of the integrals in the above computation can be justified by Fubini's theorem (see, e.g., [Ref. 40, Chap. 2, Theorem 3.1]).

Step 2. Using the mapping $\varphi : (0, R) \to \mathbb{R}$ as in Hypothesis 4.1 (A) we define the following auxiliary function:

$$\mathfrak{j}(r) \coloneqq \sqrt{1 + (r\varphi'(r))^2}, \qquad r \in (0, R).$$

The curvilinear integral along Λ in (4.2) can be rewritten in terms of the mapping $\varphi : (0, R) \to \mathbb{R}$ and the function $\mathfrak{j}$ in the conventional way and then further estimated with the help of assumption (4.1)

$$\begin{aligned}(\omega[u]_\Lambda, [u]_\Lambda)_\Lambda &= \int_0^R \omega(r)\mathfrak{j}(r)|u_+(r, \varphi(r)) - u_-(r, \varphi(r))|^2 \mathsf{d}r \\ &\le \int_0^R \frac{1}{2\pi r}\left|u_+(r, \varphi(r)) - u_-(r, \varphi(r))\right|^2 \mathsf{d}r.\end{aligned} \tag{4.4}$$

Step 3. Define the following function:

$$S(r) \coloneqq \int_0^{2\pi} |(\partial_\varphi u)(r, \varphi)|^2 \mathsf{d}\varphi - \frac{1}{2\pi}\left|u_+(r, \varphi(r)) - u_-(r, \varphi(r))\right|^2, \tag{4.5}$$

where $r \in (0,R)$. Thanks to the choice of u, for all $r \in (0,R)$ the function $[0,2\pi) \ni \varphi \mapsto u(r,\varphi)$ can naturally be identified with the piecewise-C^1 function ψ_r on the interval $I = (0,2\pi)$, which by [Ref. 36, App. E] belongs to $H^1(I)$. Moreover, the relation $S(r) = \mathfrak{a}_{d,\omega}[\psi_r]$ holds with the form $\mathfrak{a}_{d,\omega}$ as in (2.11), where $d = 2\pi$ and $\omega = 1/2\pi$. In particular, $d\omega = 2\pi/2\pi = 1$ and by Corollary 2.12 we obtain

$$S(r) \geq 0, \qquad \text{for all} \quad r \in (0,R).$$

Finally, using (4.3), (4.4) and non-negativity of $S(r)$ we arrive at

$$\mathfrak{f}^{\Lambda}_{D_R,\omega}[u] \geq \int_0^R \frac{S(r)}{r}\mathrm{d}r \geq 0.$$

□

B. Non-negativity of $\mathsf{H}^{\Lambda}_{\omega}$ for monotone Λ

In this subsection, we obtain various explicit sufficient conditions on ω ensuring non-negativity of $\mathsf{H}^{\Lambda}_{\omega}$ assuming that Λ is monotone. General results are illustrated with two examples: an Archimedean spiral and a subinterval of the straight line in $\mathbb{R}^2$.

Theorem 4.2. *Let a monotone piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ be parametrized via $\varphi\colon (0,R) \to \mathbb{R}$, $R \in (0,+\infty]$, as in Definition 2.7. Let the self-adjoint operator $\mathsf{H}^{\Lambda}_{\omega}$ be as in Definition 3.2 with $\omega \in L^\infty(\Lambda;\mathbb{R})$. Then*

$$\sigma(\mathsf{H}^{\Lambda}_{\omega}) \subseteq [0,+\infty) \qquad \text{if} \quad \omega(r) \leq \frac{1}{2\pi r\sqrt{1+(r\varphi'(r))^2}}, \quad \text{for} \quad r \in (0,R).$$

If ω is majorized as above, and additionally, the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical, then $\sigma(\mathsf{H}^{\Lambda}_{\omega}) = [0,+\infty)$.

Proof. Let Σ and $\Omega_\pm$ be as in Hypothesis 2.1. Without loss of generality, we assume that $\mathsf{x}_0 = 0$ in Definition 2.7.

Let us define the complement $\Omega_c := \mathbb{R}^2 \setminus \overline{D_R}$ of the disc D_R, the curve $\Gamma := \Sigma \cap D_R$, and the domains $G_\pm := \Omega_\pm \cap D_R$. It is straightforward to see that the tuple $\{D_R, G_+, G_-, \Lambda, \omega\}$ satisfies Hypothesis 4.1.

Let $u \in F_\Lambda$ and define $u_R := u \upharpoonright D_R$, $u_c := u \upharpoonright \Omega_c$. Then it holds that

$$u_R \in \mathcal{D}(\overline{G_+}) \oplus \mathcal{D}(\overline{G_-}) \qquad \text{and} \qquad [u_R]_{\Gamma\setminus\Lambda} = 0.$$

Hence, using Lemma 4.1 we get

$$\mathfrak{a}^{\Lambda}_{\omega}[u,u] = \mathfrak{f}^{\Lambda}_{D_R,\omega}[u_R] + \|\nabla u_c\|^2_{\Omega_c} \geq \mathfrak{f}^{\Lambda}_{D_R,\omega}[u_R] \geq 0.$$

Since F_Λ is a core for the form $\mathfrak{a}^{\Lambda}_{\omega}$, we get by [Ref. 8, Theorem 4.5.3] that the self-adjoint operator $\mathsf{H}^{\Lambda}_{\omega}$ is non-negative. If, additionally, the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical, Proposition 3.6 implies that

$$\sigma(\mathsf{H}^{\Lambda}_{\omega}) = [0,+\infty).$$

□

Example 4.3. Let the piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ be defined as

$$\Lambda := \{(r\cos(r), r\sin(r)) \in \mathbb{R}^2 \colon r \in \mathbb{R}_+\}.$$

Obviously, this curve is monotone in the sense of Definition 2.7 with $\mathsf{x}_0 = 0$ and $\varphi(r) := r$, $r \in (0,+\infty)$. The curve Λ is a special case of an Archimedean spiral. Theorem 4.2 yields that

$$\sigma(\mathsf{H}^{\Lambda}_{\omega}) \subseteq [0,+\infty) \qquad \text{if} \quad \omega(r) \leq \frac{1}{2\pi r\sqrt{1+r^2}}, \quad \text{for} \quad r > 0.$$

The case of a non-varying interaction strength ω is of special interest. In the rest of this subsection, we assume for the sake of demonstrativeness that $\omega \in \mathbb{R}$ is a constant. Define also the following characteristic of a bounded monotone piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ (parametrized as in Definition 2.7):

$$\omega_*(\Lambda) := \inf_{r \in (0,R)} \frac{1}{2\pi r \sqrt{1 + (r\varphi'(r))^2}}. \tag{4.6}$$

It is not difficult to see that $0 < \omega_*(\Lambda) < +\infty$.

The following corollary is a direct consequence of Theorem 4.2, Proposition 3.6, and simple geometric argumentation.

Corollary 4.4. Let $\Lambda \subset \mathbb{R}^2$ be a bounded monotone piecewise-C^1 curve and let the self-adjoint operator $\mathsf{H}_\omega^\Lambda$ be as in Definition 3.2 with non-varying strength $\omega \in \mathbb{R}$. Then

$$\sigma(\mathsf{H}_\omega^\Lambda) = [0, +\infty), \qquad \textit{for all } \omega \in (-\infty, \omega_*],$$

with $\omega_* = \omega_*(\Lambda) > 0$ defined in (4.6).

To illustrate this corollary we provide an example.

Example 4.5. Consider the interval of length $L > 0$ in the plane:

$$\Lambda := \{(x,0) \in \mathbb{R}^2 : 0 < x < L\}. \tag{4.7}$$

Clearly, the interval Λ is monotone in the sense of Definition 2.7 with $\mathsf{x}_0 = 0$ and $\varphi(r) = 0$, $r \in (0, L)$. Then we get from Corollary 4.4, using formula (4.6), that

$$\sigma(\mathsf{H}_\omega^\Lambda) = [0, +\infty) \qquad \textit{for all } \omega \in \big(-\infty, \tfrac{1}{2\pi L}\big].$$

Remark 4.6. Let Λ be as in (4.7). It is worth noting that the result of the above example can be improved in the following way. Define the points $\mathsf{x}_0 = (0,0)$, $\mathsf{x}_1 = (0, L)$, the intervals

$$\Lambda_0 := \{(x,0) \in \mathbb{R}^2 : 0 < x < L/2\}, \qquad \Lambda_1 := \{(x,0) \in \mathbb{R}^2 : L/2 < x < L\},$$

the discs $D_{L/2}(\mathsf{x}_0)$, $D_{L/2}(\mathsf{x}_1)$, and the complement

$$\Omega_c := \mathbb{R}^2 \setminus \big(\overline{D_{L/2}(\mathsf{x}_0) \cup D_{L/2}(\mathsf{x}_1)}\big)$$

of the closure of their union. Let $u \in F_\Lambda$ and define $u_k := u \upharpoonright D_{L/2}(\mathsf{x}_k)$, $k = 0, 1$, $u_c := u \upharpoonright \Omega_c$. Assuming that $\omega \in (-\infty, \frac{1}{\pi L}]$, we get by Lemma 4.1 that

$$\mathfrak{a}_\omega^\Lambda[u,u] = \mathfrak{f}^{\Lambda_0}_{D_{L/2}(\mathsf{x}_0),\omega}[u_0] + \mathfrak{f}^{\Lambda_1}_{D_{L/2}(\mathsf{x}_1),\omega}[u_1] + \|\nabla u_c\|^2_{\Omega_c} \geq 0.$$

Thus, the operator $\mathsf{H}_\omega^\Lambda$ is non-negative and by Proposition 3.6 we get $\sigma(\mathsf{H}_\omega^\Lambda) = [0, +\infty)$.

One may expect that for a sufficiently large coupling constant $\omega > 0$ or for a sufficiently long curve Λ negative spectrum of the self-adjoint operator $\mathsf{H}_\omega^\Lambda$ is non-empty. In the next proposition, we confirm this expectation via an example.

Proposition 4.7. Let $\Lambda \subset \mathbb{R}^2$ be as in (4.7) and the self-adjoint operator $\mathsf{H}_\omega^\Lambda$ be as in Definition 3.2 with non-varying strength $\omega \in \mathbb{R}$. Then

$$\sigma_d(\mathsf{H}_\omega^\Lambda) \cap (-\infty, 0) \neq \varnothing, \qquad \textit{for all } \omega \in \big(\tfrac{\pi}{2L}, +\infty\big).$$

Proof. Let us split the plane $\mathbb{R}^2$ into three domains

$$\Omega_1 := (-\infty, 0) \times \mathbb{R}, \qquad \Omega_2 := (0, L) \times \mathbb{R}, \qquad \Omega_3 := (L, +\infty) \times \mathbb{R},$$

via straight lines

$$\Pi_1 := \{0\} \times \mathbb{R}, \qquad \Pi_2 := \{L\} \times \mathbb{R},$$

as indicated in Figure 1.

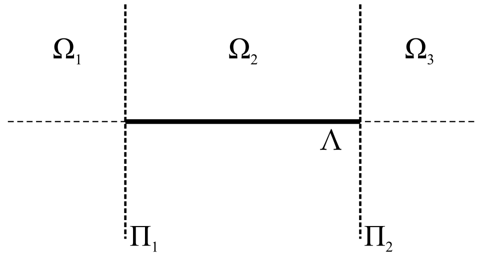

FIG. 1. Splitting of $\mathbb{R}^2$ into three domains $\{\Omega_k\}_{k=1}^3$.

Consider the sesquilinear form

$$\mathfrak{a}_{\omega,\mathrm{D}}^{\Lambda}[u,v] := \mathfrak{a}_{\omega}^{\Lambda}[u,v], \qquad \mathrm{dom}\,\mathfrak{a}_{\omega,\mathrm{D}}^{\Lambda} := \{u \in \mathrm{dom}\,\mathfrak{a}_{\omega}^{\Lambda} : u|_{\Pi_k} = 0,\ k = 1,2\}.$$

It is not difficult to check that the sesquilinear form $\mathfrak{a}_{\omega,\mathrm{D}}^{\Lambda}$ is closed, symmetric, densely defined, and semibounded in $L^2(\mathbb{R}^2)$. This form induces via the first representation theorem the self-adjoint operator $\mathsf{H}_{\omega,\mathrm{D}}^{\Lambda}$ in $L^2(\mathbb{R}^2)$, which can be represented as the orthogonal sum $\mathsf{H}_1 \oplus \mathsf{H}_2 \oplus \mathsf{H}_3$ with respect to the decomposition $L^2(\mathbb{R}^2) = \oplus_{k=1}^3 L^2(\Omega_k)$. Note that H_1 and H_3 are both non-negative and their spectra are given by the set $[0,+\infty)$. The spectrum of H_2 can be computed via separation of variables in the strip Ω_2. In particular, the ground state of H_2 corresponds to the eigenvalue

$$\lambda_1(\mathsf{H}_2) = \frac{\pi^2}{L^2} - 4\omega^2,$$

where we used that the one-dimensional Schrödinger operator on the full-line with one-center point δ'-interaction of strength $\omega > 0$ has the lowest eigenvalue $-4\omega^2$, cf., [Ref. 1, Chap. I.4], where not ω, but $\beta = 1/\omega$ is called the strength of δ'-interaction.

If the assumption in the formulation of the proposition holds, then $\lambda_1(\mathsf{H}_2) < 0$ and the operator $\mathsf{H}_{\omega,\mathrm{D}}^{\Lambda}$ has at least one negative eigenvalue.

It remains to note that by Proposition 3.6 we have $\sigma_{\mathrm{ess}}(\mathsf{H}_{\omega}^{\Lambda}) = [0,+\infty)$ and that the form ordering

$$\mathfrak{a}_{\omega}^{\Lambda} \prec \mathfrak{a}_{\omega,\mathrm{D}}^{\Lambda}$$

can easily be verified, which yields by [Ref. 6, Section 10.2, Theorem 4] that the operator $\mathsf{H}_{\omega}^{\Lambda}$ has at least one negative eigenvalue. □

C. Absence of bound states for $\mathsf{H}_{\omega}^{\Lambda}$ and LFTs

In this subsection, we show absence of bound states in the weak coupling regime for a class of bounded non-monotone piecewise-C^1 curves, which are (with minor restrictions) images of bounded monotone curves under LFTs. Since the identical transform $M(z) = z$ is a LFT, this class is certainly larger than the class of bounded monotone curves. As an example, we treat δ'-interaction supported on a circular arc subtending an angle $\theta > \pi$.

First, we provide for convenience of the reader two standard claims on change of variables under LFT. The proofs of them are outsourced to the Appendix.

Lemma 4.8. Let $\Lambda \subset \mathbb{R}^2$ be a bounded curve as in Hypothesis 2.1, let the space F_Λ be as in (2.1), and let $M: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a LFT such that $M(\infty), M^{-1}(\infty) \notin \Lambda$. Then for any $u \in F_\Lambda$

$$\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x = \int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}x \qquad (4.8)$$

holds with $v := u \circ M$.

Remark 4.9. The function v itself in the formulation of the above lemma is continuous and piecewise smooth, but it is not necessarily compactly supported or square-integrable.

Lemma 4.10. Let $\Lambda \subset \mathbb{R}^2$ be a bounded piecewise-C^1 curve, parametrized via the mapping $\lambda\colon I \to \mathbb{R}^2$, $I := (0, L)$, as in Definition 2.6, let the space F_Λ be as in (2.1) and let $\omega \in L^\infty(\Lambda;\mathbb{R})$. For a LFT $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with J_M as in (2.6) and such that $M(\infty), M^{-1}(\infty) \notin \Lambda$ define $\Gamma := M^{-1}(\Lambda)$, $\gamma := M^{-1} \circ \lambda$ and

$$\widetilde{\omega}(\gamma(s)) := \omega(\lambda(s))\sqrt{\mathsf{J}_M(\gamma(s))}, \qquad s \in I. \tag{4.9}$$

Then the relation

$$\big(\omega[u]_\Lambda, [u]_\Lambda\big)_\Lambda = \big(\widetilde{\omega}[v]_\Gamma, [v]_\Gamma\big)_\Gamma$$

holds for any $u \in F_\Lambda$ and $v := u \circ M$.

Remark 4.11. Note that the function v in the formulation of the above lemma does not belong to F_Γ in general. However, $v_\pm := v \upharpoonright M^{-1}(\Omega_\pm)$ with $\Omega_\pm$ as in Hypothesis 2.1 are well-defined and continuous up to Γ. Hence, the restrictions $v_\pm|_\Gamma$ are meaningful and $[v]_\Gamma := v_+|_\Gamma - v_-|_\Gamma$ is well-defined.

Now, we can formulate the key result of this subsection, whose proof with all the above preparations is rather short.

Theorem 4.12. *Let $\Lambda \subset \mathbb{R}^2$ be a bounded piecewise-C^1 curve and let the self-adjoint operator $\mathsf{H}^\Lambda_\omega$ in $L^2(\mathbb{R}^2)$ be as in Definition 3.2 with non-varying strength $\omega \in \mathbb{R}$. Suppose that there exists a LFT $M\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

(a) $M(\infty), M^{-1}(\infty) \notin \Lambda$,
(b) $\Gamma := M^{-1}(\Lambda)$ *is monotone.*

Let the constant $\omega_*(\Gamma) > 0$ be associated to Γ via (4.6). Then it holds that

$$\sigma(\mathsf{H}^\Lambda_\omega) = [0, +\infty) \qquad \text{for all } \omega \in (-\infty, \omega_*],$$

where $\omega_* := \frac{\omega_*(\Gamma)}{\sup_{\mathsf{z}\in\Gamma}\sqrt{\mathsf{J}_M(\mathsf{z})}}$.

Proof. Step 1. Suppose that the curve Λ is parametrized via the mapping $\lambda\colon (0, L) \to \mathbb{R}^2$ (as in Definition 2.6). Define the mapping $\gamma := M^{-1} \circ \lambda$. Due to assumption (a) the curve Γ is bounded and the mapping γ parametrizes it. Without loss of generality suppose that the curve Γ is monotone in the sense of Definition 2.7 with $\mathsf{x}_0 = \gamma(0) = 0$ and with $\varphi\colon (0, R) \to \mathbb{R}$, $R = |\gamma(L)|$. Consider the complement $\Omega_{\rm c} := \mathbb{R}^2 \setminus \overline{D_R}$ of the disc D_R. Let the curve Σ and the domains $\Omega_\pm$ be associated to Λ as in Hypothesis 2.1.

Define auxiliary domains $G_\pm := M^{-1}(\Omega_\pm) \cap D_R$. Thus, the splitting

$$D_R = G_+ \,\dot\cup\, M^{-1}(\Sigma) \,\dot\cup\, G_-$$

holds. Let $\widetilde{\omega}$ be defined via the formula (4.9) in Lemma 4.10. Hence, we obtain

$$\widetilde{\omega} \le \omega \sup_{z\in\Gamma} \sqrt{\mathsf{J}_M(z)} \le \omega_* \sup_{z\in\Gamma} \sqrt{\mathsf{J}_M(z)} = \omega_*(\Gamma).$$

Summarizing, the tuple $\{D_R, G_+, G_-, \Gamma, \widetilde{\omega}\}$ fulfils Hypothesis 4.1.

Step 2. Let $u \in F_\Lambda$ with F_Λ as in (2.1) and define the composition $v := u \circ M$. Set $v_R := v \upharpoonright D_R$ and $v_{\rm c} := v \upharpoonright \Omega_{\rm c}$. Using Lemmas 4.8 and 4.10 we obtain

$$\begin{aligned} \mathfrak{a}^\Lambda_\omega[u,u] &= \|\nabla u\|^2_{\mathbb{R}^2} - (\omega[u]_\Lambda, [u]_\Lambda)_\Lambda = \|\nabla v\|^2_{\mathbb{R}^2} - (\widetilde{\omega}[v]_\Gamma, [v]_\Gamma)_\Gamma \\ &= \|\nabla v_R\|^2_{D_R} - (\widetilde{\omega}[v_R]_\Gamma, [v_R]_\Gamma)_\Gamma + \|\nabla v_{\rm c}\|^2_{\Omega_{\rm c}} \ge \mathfrak{f}^\Gamma_{D_R,\widetilde{\omega}}[v_R, v_R] \ge 0, \end{aligned} \tag{4.10}$$

where we applied Lemma 4.1 in the last step. Hence, the operator $\mathsf{H}^\Lambda_\omega$ is non-negative.

Step 3. Since the curve Λ is bounded, Proposition 3.6 applies, and we arrive at $\sigma_{\rm ess}(\mathsf{H}^\Lambda_\omega) = [0, +\infty)$. The results of Step 2 and Step 3 imply the claim. ◻

To conclude this subsection, we show that a model of sufficiently weak δ'-interaction of non-varying strength supported on a circular arc subtending the angle $2\pi - 2\varepsilon$ ($\varepsilon \in (0,\pi)$) has no bound states in the weak coupling regime. We emphasize that circular arcs subtending angles $\theta > \pi$ are non-monotone and the results of the previous subsection do not apply to them.

Example 4.13. The circular arc (see Figure 2) can be efficiently parametrized as follows:

$$\Lambda := \big\{(R\sin\varphi, R(1-\cos\varphi)) \in \mathbb{R}^2 \colon \varphi \in (\varepsilon, 2\pi-\varepsilon)\big\}, \tag{4.11}$$

where $\varepsilon \in (0,\pi)$ and $R > 0$ is the radius of the underlying circle. Consider the LFT $M(z) := 1/z$. One easily sees that

$$M_1(x,y) = \operatorname{Re} M(x+\mathrm{i}y) = \frac{x}{x^2+y^2},$$

and according to (2.6) the Jacobian J_M of this LFT is given by the formula

$$\begin{aligned}\mathsf{J}_M(x,y) &= \big((\partial_x M_1)^2 + (\partial_y M_1)^2\big)(x,y)\\ &= \frac{(x^2-y^2)^2}{(x^2+y^2)^4} + \frac{4x^2y^2}{(x^2+y^2)^4} = \frac{1}{(x^2+y^2)^2}.\end{aligned} \tag{4.12}$$

Next observe that $M(\infty) = M^{-1}(\infty) = 0 \notin \Lambda$. Moreover, this LFT is inverse to itself and under the LFT $M^{-1}(z) = 1/z$ the arc $\Lambda \subset \mathbb{R}^2$ is mapped onto the interval

$$\Gamma := M^{-1}(\Lambda) = \big\{\big(x, -\tfrac{1}{2R}\big) \in \mathbb{R}^2 \colon |x| < \cot(\varepsilon/2)/(2R)\big\},$$

which is obviously monotone in the sense of Definition 2.7. Compute further $\omega_*(\Gamma)$ defined in (4.6)

$$\omega_*(\Gamma) = \inf_{r\in(0,|\Gamma|)} \frac{1}{2\pi r} = \frac{1}{2\pi|\Gamma|} = \frac{R}{2\pi\cot(\varepsilon/2)},$$

here $|\Gamma|$ in the length of Γ. Moreover, we obtain from (4.12) that

$$\sup_{\mathsf{z}\in\Gamma} \sqrt{\mathsf{J}_M(\mathsf{z})} = 4R^2.$$

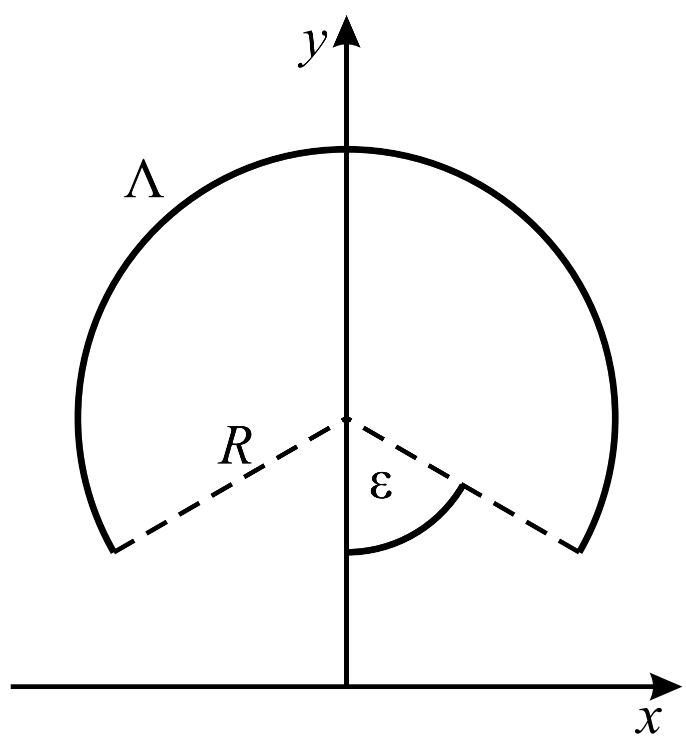

FIG. 2. The circular arc of radius $R > 0$ subtending the angle $2\pi - 2\varepsilon$ with $\varepsilon \in (0,\pi)$.

Hence, Theorem 4.12 implies that

$$\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty), \qquad \text{for all } \omega \in \big(-\infty, (8\pi R)^{-1}\tan(\varepsilon/2)\big].$$

V. REMARKS AND OPEN QUESTIONS

In the present paper, we have analysed from various perspectives a new effect of absence of the negative spectrum for Hamiltonians with δ'-interaction supported on non-closed curves in $\mathbb{R}^2$. Quite a few questions remain open and we wish to formulate two of them.

Comparing Example 4.5 and Proposition 4.7 one may pose the following question.

Open Question A. Let the constant $L > 0$ be fixed and the interval Λ be as in (4.7). The problem is to find the critical strength $\omega_{\rm cr}(L) > 0$ such that the operator $\mathsf{H}_\omega^\Lambda$ is non-negative if and only if $\omega \in \mathbb{R}$ satisfies $\omega \le \omega_{\rm cr}(L)$.

The same question as above can be asked for other shapes of Λ, but the authors do not expect that an exact formula for the critical strength can be found.

On one hand, our method of the proof does not allow to cover curves of generic shape. On the other hand, despite many attempts, we have not found out any example of a bounded non-closed curve, for which bound states in the weak coupling regime do exist. A general open question can be posed.

Open Question B. Is it true that for any bounded sufficiently smooth non-closed curve $\Lambda \subset \mathbb{R}^2$ there exists a constant $\omega_* > 0$ such that $\sigma(\mathsf{H}_\omega^\Lambda) = [0,+\infty)$ for all $\omega \in (-\infty, \omega_*]$?

It is worth noting that the program carried out in Subsection IV C for linear fractional transformations can be generalized by means of Neumann bracketing to arbitrary conformal maps. This could be a possible way to answer Question B.

Finally, we mention that several assumptions play only technical role and can be removed with additional efforts. Namely, assuming that Λ is a subarc of the boundary of a Lipschitz domain is technical as well assuming that the curve Λ is piecewise-C^1 in some of the formulations instead of just being Lipschitz.

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APPENDIX: PROOFS OF LEMMAS 4.8 AND 4.10

Proof of Lemma 4.8. Consider the following two open sets:

$$E := \mathbb{R}^2 \setminus (\{M(\infty)\} \cup M(\Lambda)) \quad \text{and} \quad F := \mathbb{R}^2 \setminus \Lambda.$$

By the formula in Lemma 2.10 and using that $\mathbb{R}^2 \setminus E$ is a null set, we get

$$\int_{\mathbb{R}^2} |\nabla v|^2 \mathsf{d}x = \int_E |\nabla v|^2 \mathsf{d}x = \int_E |(\nabla u)\circ M|^2 \mathsf{J}_M \mathsf{d}x.$$

According to Proposition 2.9, we have that $M^{-1}\colon E \to F$ is a bijection which is additionally everywhere differentiable in E, cf., (2.5). Hence, we can apply the substitution rule for Lebesgue integrals (e.g., [Ref. 31, Theorem 8.21, Corollary 8.22]) and get

$$\int_{\mathbb{R}^2} |\nabla v|^2 \mathsf{d}x = \int_F |(\nabla u)\circ M \circ M^{-1}|^2 \mathsf{J}_M (\mathsf{J}_M)^{-1} \mathsf{d}x$$
$$= \int_F |\nabla u|^2 \mathsf{d}x = \int_{\mathbb{R}^2} |\nabla u|^2 \mathsf{d}x,$$

where in the last step we employed that $\mathbb{R}^2 \setminus F$ is a null set. □

Proof of Lemma 4.10. Observe first that by definition of the curvilinear integral we have

$$(\widetilde{\omega}[v]_\Gamma, [v]_\Gamma)_\Gamma = \int_0^L \widetilde{\omega}(\gamma(s))|v_+(\gamma(s)) - v_-(\gamma(s))|^2 |\gamma'(s)| \mathsf{d}s. \tag{A1}$$

Using elementary composition rules, we also note

$$\begin{aligned} v_+(\gamma(s)) - v_-(\gamma(s)) &= (u_+ \circ M \circ M^{-1} \circ \lambda)(s) - (u_- \circ M \circ M^{-1} \circ \lambda)(s) \\ &= u_+(\lambda(s)) - u_-(\lambda(s)), \end{aligned} \tag{A2}$$

where $u_\pm = u \upharpoonright \Omega_\pm$ and $v_\pm = v \upharpoonright M^{-1}(\Omega_\pm)$. Observe also that $\lambda = M \circ \gamma$. Using (2.6) and (2.7), we obtain

$$\begin{aligned} |\lambda'(s)|^2 &= \big((\nabla M_1 \circ \gamma)(s) \cdot \gamma'(s)\big)^2 + \big((\nabla M_2 \circ \gamma)(s) \cdot \gamma'(s)\big)^2 \\ &= |(\nabla M_1 \circ \gamma)(s)|^2 \cdot |\gamma'(s)|^2 \cos^2\alpha + |(\nabla M_2 \circ \gamma)(s)|^2 \cdot |\gamma'(s)|^2 \sin^2\alpha \\ &= \mathsf{J}_M(\gamma(s)) \cdot |\gamma'(s)|^2, \end{aligned}$$

where α is the angle between ∇M_1 and γ'. Thanks to (A1) and using (A2), we arrive at

$$\begin{aligned} (\widetilde{\omega}[v]_\Gamma, [v]_\Gamma)_\Gamma &= \int_0^L \widetilde{\omega}(\gamma(s))|v_+(\gamma(s)) - v_-(\lambda(s))|^2\, |\gamma'(s)| \mathsf{d}s \\ &= \int_0^L \frac{\widetilde{\omega}(\gamma(s))}{\sqrt{\mathsf{J}_M(\gamma(s))}}\, |u_+(\lambda(s)) - u_-(\lambda(s))|^2\, |\lambda'(s)| \mathsf{d}s. \end{aligned}$$

Finally, employing (4.9) we end up with the desired relation

$$(\widetilde{\omega}[v]_\Gamma, [v]_\Gamma)_\Gamma = (\omega[u]_\Lambda, [u]_\Lambda)_\Lambda.$$

□

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