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DOCTORAL THESIS

GRADED CONTRACTIONS OF $\mathrm{sl}(3,\mathbb{C})$

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Contents

Introduction

During the centuries, physical theories have been developed with the growing knowledge of the world. Older theories were extended or replaced by the new ones. It usually turns out that older theories are valid only under certain conditions which are connected with observation abilities in the time of their origin. For example classical mechanics, developed in 17th century, works well only for macroscopic objects and velocities which are low in comparison with the velocity of light. In such case a new theory should coincide with the old one in predictions for phenomena where these conditions are satisfied. This is the so called correspondence principle.

Classical mechanics was replaced in the 20th century by two new theories: relativistic mechanics (for velocities close to the velocity of light) and quantum mechanics (for microscopic objects). The correspondence principle provides the connection between these newer theories and classical mechanics. If the velocity of light goes to infinity, the relativistic mechanics transforms into the classical one and its underlying symmetry group – Poincaré group — singularly transforms into Galilei group. Assuming the Planck constant tending to zero one gets another limiting process from the quantum mechanics to the classical one, which corresponds to the singular transition of Heisenberg algebra into abelian one.

The mathematical formulation of the correspondence principle for relativistic mechanics was given by Inönü and Wigner [43]. So called Inönü–Wigner contractions (IW–contractions) similar to the examples given above were introduced and studied. The mathematical concept of the limiting process (contraction) between Lie algebras was already known from the work of Segal [69], where it was formulated in terms of limiting process of bases. The general definition of contraction in terms of the family of non–singular linear operators was given by Saletan [67]. All these contractions represent continuous contractions.

The currently used definition of the continuous contraction [55, 75] is the following one. Let $\mathcal{L}$ be a finite–dimensional Lie algebra with Lie bracket $[.,.]$ and $U : (0,1] \longrightarrow \mathrm{GL}(\mathcal{L})$ be a continuous map which maps real numbers ε into regular linear operators U_ε on the vector space $\mathcal{L}$. If there exists a limit $[x,y]_0 := \lim\limits_{\varepsilon\to 0+}[x,y]_\varepsilon = \lim\limits_{\varepsilon\to 0+} U_\varepsilon^{-1}[U_\varepsilon x, U_\varepsilon y]$ for any $x,y \in \mathcal{L}$, then $[.,.]_0$ is a well–defined Lie bracket and the vector space $\mathcal{L}$ together with $[.,.]_0$

is called one–parametric continuous contraction of the Lie algebra $\mathcal{L}$. The older Saletan's definition required the existence of the limit $U_0 := \lim_{\varepsilon \to 0+} U_\varepsilon$ while the simple IW–contractions used only diagonal U_ε with eigenvalues ε and 1 only. So called generalized IW–contractions [22, 38] consider that the eigenvalues of the diagonal U_ε are integer powers of the contraction parameter ε, allowing positive as well as negative exponents.

The topological approach to the continuous contraction known as degenerations [8, 10] of Lie algebras represents another generalization of this problem. The degeneration of a given n–dimensional Lie algebra $\mathcal{L}$ over the field K is any element from the Zariski closure of the orbit of $\mathcal{L}$ under the action of the group $\mathrm{GL}(n, K)$ on the variety of all n–dimensional Lie algebras over K. The relations of contractions and deformations of Lie algebras were also studied for example in [25, 51, 75].

The useful necessary criteria for the existence of the continuous contraction between two algebras were introduced in works [8, 10, 55]. These criteria enabled the systematical study of continuous contractions among four–dimensional Lie algebras [8, 55]. Let us note that the contractions of three–dimensional Lie algebras were already obtained in [73].

In this work we concentrate on the algebraical approach to contractions – so called graded contractions. Graded contractions were originally introduced in [17] as a generalization of Inönü–Wigner contractions with the aim of the systematical study of IW–contractions. Unfortunately, it appeared that the results of graded contractions do not cover all possible IW–contractions [73, 77] and thus, graded contractions do not represent a generalization of IW–contractions. However, they can still be considered as a possible connection of different Lie algebras.

In contrast to continuous contractions, graded contractions represent a purely algebraical standpoint. Instead of a limit of a continuous family of isomorphic algebras which fulfil Jacobi identities, the commutation relations among the graded subspaces are multiplied by contraction parameters which have to satisfy the system of quadratic equations (contraction system) resulting from the Jacobi identities. The complexity of this system depends on the number of grading subspaces and the structure of grading.

There are two types of graded contractions: continuous graded contractions which correspond to IW–contractions and discrete graded contractions which possess no equivalent in continuous contractions. These discrete contractions yield, among other things, special continuous parametric families of non–isomorphic Lie algebras as graded contractions of a single Lie algebra. Such continuous parametric families could be interesting for the deformation theory of Lie algebras. However, the physical interpretation of discrete graded contractions has not been found yet. It is even claimed that it cannot exist [77].

The concept of graded contractions is not restricted only to Lie algebras. It can be used for any graded algebras. The only difference is the origin of the contraction system which is derived from the inner composition law of the given algebra. Many examples of graded contractions were studied in literature. These are for example, the graded contractions of inhomogeneous algebras [14, 61], central extensions [16], affine algebras [19, 42], Jordan algebras [45], Virasoro algebras [48]. Physically motivated examples related to the kinematical and conformal group of space–time were presented in [18, 72]. The attempt to obtain all graded contractions of $\mathbb{Z}_2^N$-graded $\mathrm{so}(N+1)$ for arbitrary natural number N was made in [39], however, only continuous graded contractions were studied herein.

The general solution of the graded contractions — considering only so called generic case — was achieved in [76, 77]. Since this solution depends solely on the grading group (the structure of the Lie algebra does not matter at all), it is obtained simultaneously for all Lie algebras which allow the given grading. However, this approach is in a certain sense too general. For a concrete Lie algebra the number of results obtained in this way is significantly smaller than in our approach.

The concept of graded contractions was also extended to the contractions of representations which are compatible with a given grading [54]. Since the complete representation theory exists only for simple Lie algebras, it can be helpful for the construction of the representations of solvable Lie algebras which are obtained via graded contractions of some simple Lie algebra. The examples of graded contractions of representations can be found in [50, 60, 71].

The contractions of simple Lie algebras have been often studied because of their many interesting physical applications. Simple Lie algebras can be contracted into solvable ones but not conversely. Therefore, it might be expected that the contractions of simple Lie algebras will lead to wider structures than the contractions of other types of Lie algebras. Hence, the systematical study of the contractions of simple Lie algebras is desirable.

The graded contractions of the lowest dimensional simple Lie algebra $\mathrm{sl}(2,\mathbb{C})$ are already known [17] for both its fine gradings. Let us recall that the graded contractions of the Cartan graded $\mathrm{sl}(2,\mathbb{C})$ are $\mathcal{A}_{2,1} \oplus \mathcal{A}_1, \mathcal{A}_{3,1}, \mathcal{A}_{3,3}, \mathcal{A}_{3,4}, \mathcal{A}_{3,5}^{(a)}$, while from the Pauli graded $\mathrm{sl}(2,\mathbb{C})$ only $\mathcal{A}_{3,1}, \mathcal{A}_{3,4}$ were obtained. Let us also note that there is only one three–dimensional Lie algebra $\mathcal{A}_{3,2}$ which is not a graded contraction of $\mathrm{sl}(2,\mathbb{C})$.

The goal of this work is to continue in a systematical study of graded contractions with the examination of the graded contractions for all four fine gradings of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. This task has already been solved for one of these gradings (Cartan grading) in [1, 13]. A summary about the Pauli graded contractions of $\mathrm{sl}(3,\mathbb{C})$ was also published [37].

All contractions of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$ form a subset of all 8–dimensional complex Lie algebras. Since these are still not completely classified, the classification of the graded contraction of $\mathrm{sl}(3,\mathbb{C})$ is a difficult task. Let us recall that there exists a complete classification of all simple Lie algebras, while the solvable Lie algebras were classified only for the dimensions not greater than six. For the nilpotent Lie algebras the classification is done up to dimension 8. Our work thus partially contributes to the classification problem of Lie algebras of the dimensions higher than 6. We have developed an algorithm which has enabled us to classify all presented graded contractions of $\mathrm{sl}(3,\mathbb{C})$, including the ranges of parameters for the parametric families.

Let us note that a computer program for the computation of graded contractions was written [5] in 1993, but it is inaccessible to us. As it deals with two–term contraction equations only, it is not suitable for our purpose. We have used our own program written in MAPLE 8 which enables us to use the symmetries during the computation of graded contractions and also serves for the classification of the resulting Lie algebras.

This work is divided into three parts. The first part formed by chapters 1.–4. provides the theoretical background and describes the algorithms which are used in the second part. In the second part(chapters 5.–8.) the complete investigation of all graded contractions for all fine gradings of $\mathrm{sl}(3,\mathbb{C})$ is performed. The obtained classification results are postponed to the third part which consists of three Appendices. These contain all contraction matrices and the tables of resulting Lie algebras and of the invariant functions for one–parametric families of Lie algebras.

In chapter 1 we introduce the notation and recall the basic facts from the theory of Lie algebras. Chapter 2 is devoted to gradings of Lie algebras. The basic definitions are recalled. The symmetries of the gradings are introduced and described for so called group gradings. The overview of all group gradings of $\mathrm{sl}(3,\mathbb{C})$ is also given.

In chapter 3 the definition of the graded contraction is introduced and its basic properties described. Let us note that in contrast to the known literature our definition of the graded contraction is explicitly formulated and is slightly different from the previously used concept, where the structure of algebras was ignored and group gradings were supposed. The definition of the continuous and discrete graded contractions is recalled and the criteria for distinguishing among them are presented. The role of the symmetry group of the grading in the construction and solution of the system of contraction equations is also described. The equivalence of contraction matrices, leading to the isomorphism among the corresponding graded contractions, is defined. Finally, the algorithm which we used for the computing of contraction matrices for $\mathrm{sl}(3,\mathbb{C})$ is described.

Chapter 4 deals with the identification problem of Lie algebras. The description of algorithms used for the direct decomposition, Levi decomposition and the computation of nilradical are presented. The set of numerical invariants containing the dimensions of generalized derivations and the number of formal invariants is introduced. With the exception of some parametric continua of Lie algebras, this set together with invariant functions form a sufficient tool for distinguishing among contracted Lie algebras of $\mathrm{sl}(3,\mathbb{C})$. Finally, the algorithm of the identification and its possible extensions are given.

In Chapters 5,6,7 all graded contractions of the Pauli, Gell-Mann and Cartan graded $\mathrm{sl}(3,\mathbb{C})$ are obtained. Starting from the description of the grading and its symmetry group we always give the construction of the system of contraction equations based on the knowledge of the orbits in the set of relevant grading indices under the symmetry group of the grading. The solution of the contraction system and the higher–order identities are found. The classification of the results is also described. The isomorphic graded contractions are specified and the structure of the resulting decomposable Lie algebras is given. The non–isomorphic graded contractions are tabulated in Appendix B. Several examples illustrating the manipulation with solutions and the identification of Lie algebras are added. Chapter 7 also contains the comparison of our results for the Cartan graded contractions of $\mathrm{sl}(3,\mathbb{C})$ with previous results obtained in [1, 13].

Chapter 8 contains the graded contractions of the Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$ which are non–solvable Lie algebras. Moreover, the comparison of results from all four grading is presented. Finally, different approaches to graded contractions are investigated and their possible outcomes are compared.

Chapter 1

Lie algebras

In this chapter we recall some basic definitions and properties of linear and Lie algebras. The proofs of stated propositions can be found in [28, 44]. All vector spaces and algebras will be considered over the field of complex numbers $\mathbb{C}$, which is algebraically closed of characteristic zero. Moreover, in the second part of this chapter only finite–dimensional vector spaces and algebras will be considered.

1.1 Linear Algebras

By a complex linear **algebra** $\mathcal{A}$ we will understand a vector space $\mathcal{A}$ over $\mathbb{C}$ on which bilinear map (multiplication) $\mu : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is given. The bilinearity of μ means that for all $x, y, z \in \mathcal{A}$ and for all $\alpha \in \mathbb{C}$ it holds

$$\mu(\alpha x + y, z) = \alpha\mu(x, z) + \mu(y, z), \quad \mu(z, \alpha x + y) = \alpha\mu(z, x) + \mu(z, y). \tag{1.1}$$

According to the properties of multiplication μ on $\mathcal{A}$ we recognize following types of algebras:

- **commutative (abelian) algebra** if μ is commutative.
- **associative algebra** if multiplication μ (usually denoted by $\cdot$) is associative, i.e. for all $x, y, z \in \mathcal{A}$ holds
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z. \tag{1.2}$$
- **Lie algebra** if multiplication μ (usually denoted by $[\,,]$ and called **Lie bracket**) fulfills following two conditions

  (1) $[x, x] = 0, \quad \forall x \in \mathcal{A}$ (anti–commutativity)

  (2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in \mathcal{A}$ (Jacobi identity).

The first condition can be equivalently rewritten as

(1') $[x, y] = -[y, x], \quad \forall x, y \in \mathcal{A}$.

- **Jordan algebra** if multiplication μ (denoted by $\circ$) for all $x, y \in \mathcal{A}$ satisfies

  (1) $x \circ y = y \circ x$

  (2) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$.

If an algebra $\mathcal{A}$ is of finite dimension $n \in \mathbb{N}$ and $\mathcal{E} = (e_1, \ldots, e_n)$ is a basis of $\mathcal{A}$, then the numbers $c^k_{i,j} \in \mathbb{C}$ defined by equations

$$\mu(e_i, e_j) = \sum_{k=1}^{n} c^k_{ij} e_k \tag{1.3}$$

are called **structural constants** of the algebra $\mathcal{A}$ with respect to the basis $\mathcal{E}$.

For vector subspaces B, C of an algebra $\mathcal{A}$ we will denote by $\mu(B, C)$ the vector subspace of $\mathcal{A}$ spanned by all products $\mu(b, c)$, where $b \in B, c \in C$, i.e.

$$\mu(B, C) = \operatorname{span}_{\mathbb{C}} \{\mu(b, c) \,|\, b \in B,\, c \in C\}. \tag{1.4}$$

A vector subspace B of $\mathcal{A}$ is called a **subalgebra** of $\mathcal{A}$ if B is closed with respect to multiplication on $\mathcal{A}$, i.e. $\mu(B, B) \subset B$. A subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called an **ideal** of $\mathcal{A}$ if $\mu(\mathcal{A}, \mathcal{B}) \subset \mathcal{B}$ and $\mu(\mathcal{B}, \mathcal{A}) \subset \mathcal{B}$.

Let $\mathcal{B}$ be an ideal of $\mathcal{A}$, then the factor vector space

$$\mathcal{A}/\mathcal{B} = \{[x] = x + \mathcal{B} \,|\, x \in \mathcal{A}\} \tag{1.5}$$

together with the well-defined multiplication $\mu([x], [y]) = [\mu(x, y)]$, $\forall x, y \in \mathcal{A}$ is called the **factor algebra** of $\mathcal{A}$ according to the ideal $\mathcal{B}$.

The set

$$C(\mathcal{A}) = \{x \in \mathcal{A} \,|\, \mu(x, y) = \mu(y, x),\ \forall y \in \mathcal{A}\} \tag{1.6}$$

forms a vector subspace of $\mathcal{A}$ and is called the **center** of algebra $\mathcal{A}$.

Let $\mathcal{A}, \widetilde{\mathcal{A}}$ be algebras with multiplications $\mu, \widetilde{\mu}$, then a linear mapping $h : \mathcal{A} \longrightarrow \widetilde{\mathcal{A}}$ is called a **homomorphism** of algebras $\mathcal{A}, \widetilde{\mathcal{A}}$, if for all $x, y \in \mathcal{A}$ it holds

$$h(\mu(x, y)) = \widetilde{\mu}(h(x), h(y)). \tag{1.7}$$

The **kernel** of h

$$\ker h = \{x \in \mathcal{A} \,|\, h(x) = 0\} \tag{1.8}$$

forms an ideal of $\mathcal{A}$. If the homomorphism h is onto $\widetilde{\mathcal{A}}$ and $\ker h = 0$, then h is called an **isomorphism** and algebras $\mathcal{A}, \widetilde{\mathcal{A}}$ are said to be **isomorphic** and denoted by $\mathcal{A} \cong \widetilde{\mathcal{A}}$. An isomorphism from $\mathcal{A}$ onto $\mathcal{A}$ is called an **automorphism** of algebra $\mathcal{A}$. The set of all automorphisms of $\mathcal{A}$ forms a group $\mathrm{Aut}(\mathcal{A})$ called **automorphism group** of algebra $\mathcal{A}$.

The isomorphism $\cong$ is an equivalence relation on the set of all algebras Ω and the cosets of this equivalence $[\mathcal{A}] = \left\{\widetilde{\mathcal{A}} \in \Omega \,|\, \widetilde{\mathcal{A}} \cong \mathcal{A}\right\}$ are called **isomorphism classes**. Let M be a non–empty set and $\Theta \subset \Omega$ a set of algebras such that $\mathcal{A} \in \Theta \Rightarrow [\mathcal{A}] \subset \Theta$. Any map $\xi : \Theta \longrightarrow M$ which for all $\mathcal{A}, \widetilde{\mathcal{A}} \in \Theta$ fulfils $\mathcal{A} \cong \widetilde{\mathcal{A}} \Rightarrow \xi(\mathcal{A}) = \xi(\widetilde{\mathcal{A}})$ is called an **invariant characteristic** of Θ or shortly an **invariant**.

Let $\mathcal{A}_1, \mathcal{A}_2$ be algebras with multiplications μ_1, μ_2. The direct sum of their vector spaces

$$\mathcal{A} = \{(x_1, x_2) \,|\, x_1 \in \mathcal{A}_1,\ x_2 \in \mathcal{A}_2\} \tag{1.9}$$

forms, together with a multiplication μ given by

$$\mu((x_1, x_2), (y_1, y_2)) = (\mu_1(x_1, y_1), \mu_2(x_2, y_2)), \tag{1.10}$$

a new algebra $\mathcal{A}$ called a **direct sum of algebras** $\mathcal{A}_1, \mathcal{A}_2$ and denoted $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Algebras $\mathcal{A}_1, \mathcal{A}_2$ can be identified with ideals in $\mathcal{A}$

$$\mathcal{A}_1 \cong \{(x_1, 0) \,|\, x_1 \in \mathcal{A}_1\}, \qquad \mathcal{A}_2 \cong \{(0, x_2) \,|\, x_2 \in \mathcal{A}_2\}, \tag{1.11}$$

therefore, we will say that $\mathcal{A}_1, \mathcal{A}_2$ are ideals in algebra $\mathcal{A}$. The direct sum of two associative, Lie or Jordan algebras is also an associative, Lie or Jordan algebra, respectively.

Let $\mathcal{A}$ be an associative algebra. If we define a new multiplication on $\mathcal{A}$ (so called **commutator**) by formula

$$[x, y] = x \cdot y - y \cdot x, \quad \forall x, y \in \mathcal{A}, \tag{1.12}$$

then the vector space $\mathcal{A}$ together with this commutator $[\,,\,]$ forms a Lie algebra $\mathcal{A}_{\mathcal{L}}$ called the Lie algebra of an associative algebra $\mathcal{A}$. If a new product on $\mathcal{A}$ is defined by

$$x \circ y = \frac{1}{2}(x \cdot y + y \cdot x), \quad \forall x, y \in \mathcal{A}, \tag{1.13}$$

then vector space $\mathcal{A}$ becomes a Jordan algebra denoted by $\mathcal{A}_{\mathcal{J}}$.

Let V be a vector space over $\mathbb{C}$. We denote by $\mathrm{End}(V)$ a vector space of all linear operators on V. Considering $\mathrm{End}(V)$ together with composition of linear operators we get associative algebra $\mathrm{End}(V)$. The Lie algebra of this associative algebra will be denoted by

$\mathrm{gl}(V) = \mathrm{End}(V)_{\mathcal{L}}$ and the Jordan algebra by $\mathrm{jor}(V) = \mathrm{End}(V)_{\mathcal{J}}$. If the dimension of V is $n \in \mathbb{N}$, we can choose a basis $\mathcal{E} = (e_1, \ldots, e_n)$ and for $A \in \mathrm{gl}(V)$ we can write

$$Ae_j = \sum_{i=1}^{n} A_{ij} e_i, \quad A_{ij} \in \mathbb{C} \tag{1.14}$$

and uniquely assign to A a $n \times n$ complex matrix $A^{\mathcal{E}} = (A_{ij})$. In this way we get an isomorphism between $\mathrm{gl}(V)$ and the Lie algebra of all $n \times n$ complex matrices $\mathrm{gl}(n, \mathbb{C})$. If $\mathcal{A}$ is an algebra, then we will use also notation $\mathrm{End}(\mathcal{A}), \mathrm{gl}(\mathcal{A})$ and $\mathrm{jor}(\mathcal{A})$ for an associative, Lie and Jordan algebras of all linear operators on $\mathcal{A}$.

Let $h : \mathcal{A} \longrightarrow \widetilde{\mathcal{A}}$ be an isomorphism of algebras $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, then map $g : \mathrm{End}(\mathcal{A}) \longrightarrow \mathrm{End}(\widetilde{\mathcal{A}})$, defined for all $X \in \mathrm{End}(\mathcal{A})$ by formula

$$g(X) = hXh^{-1} \tag{1.15}$$

is an isomorphism of associative algebras $\mathrm{End}(\mathcal{A})$ and $\mathrm{End}(\widetilde{\mathcal{A}})$, and consequently of Lie algebras $\mathrm{gl}(\mathcal{A}), \mathrm{gl}(\widetilde{\mathcal{A}})$ and Jordan algebras $\mathrm{jor}(\mathcal{A}), \mathrm{jor}(\widetilde{\mathcal{A}})$, thus

$$\mathcal{A} \cong \widetilde{\mathcal{A}} \implies \mathrm{End}(\mathcal{A}) \cong \mathrm{End}(\widetilde{\mathcal{A}}) \implies \begin{array}{l} \mathrm{gl}(\mathcal{A}) \cong \mathrm{gl}(\widetilde{\mathcal{A}}) \\ \mathrm{jor}(\mathcal{A}) \cong \mathrm{jor}(\widetilde{\mathcal{A}}). \end{array} \tag{1.16}$$

A linear operator $D \in \mathrm{End}(\mathcal{A})$ is called a **derivation** of an algebra $\mathcal{A}$ if for all $x, y \in \mathcal{A}$ it holds

$$D\mu(x, y) = \mu(Dx, y) + \mu(x, Dy). \tag{1.17}$$

The set of all derivations of $\mathcal{A}$ forms a Lie subalgebra of $\mathrm{gl}(\mathcal{A})$ and will be denoted by $\mathrm{der}(\mathcal{A})$. It can be easily proved, using (1.15), that

$$\mathcal{A} \cong \widetilde{\mathcal{A}} \Longrightarrow \mathrm{der}(\mathcal{A}) \cong \mathrm{der}(\widetilde{\mathcal{A}}). \tag{1.18}$$

1.2 Lie algebras

From now on we will consider only finite–dimensional vector spaces and Lie algebras. Let $\mathcal{L}$ be a Lie algebra of a finite dimension $n \in \mathbb{N}$. The multiplication (Lie bracket) on $\mathcal{L}$ is denoted by $[\,,]$ and fulfils for all $x, y, z \in \mathcal{L}$ following conditions

$$[x, y] = -[y, x] \tag{1.19}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{1.20}$$

If $\mathcal{X} = (x_1, \ldots, x_n)$ is a basis of $\mathcal{L}$, then, due to bilinearity of Lie bracket $[\,,]$ and Jacobi identity, it is sufficient if for all $i, j, k = 1, \ldots, n$ it holds

$$[x_i, x_j] = -[x_j, x_i] \tag{1.21}$$

$$[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] = 0, \tag{1.22}$$

or equivalently in terms of structural constants we have

$$c_{ij}^k = -c_{ji}^k, \quad \forall i, j, k = 1, \ldots, n \tag{1.23}$$

$$\sum_{l=1}^{n} (c_{il}^m c_{jk}^l + c_{jl}^m c_{ki}^l + c_{kl}^m c_{ij}^l) = 0, \quad \forall i, j, k = 1, \ldots, n. \tag{1.24}$$

Let V be a vector space of a finite dimension, a homomorphism $\rho : \mathcal{L} \longrightarrow \mathrm{gl}(V)$ of Lie algebras $\mathcal{L}$ and $\mathrm{gl}(V)$ is called a **representation** of a Lie algebra $\mathcal{L}$ on the vector space V. Considering $\mathcal{L}$ as a representation space and mapping $\mathrm{ad}_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathrm{gl}(\mathcal{L})$ defined for all $x \in \mathcal{L}$ by

$$(\mathrm{ad}_{\mathcal{L}}\, x)y = [x, y], \quad \forall y \in \mathcal{L} \tag{1.25}$$

we get so called **adjoint representation** of $\mathcal{L}$. It follows from Jacobi identity, that $\mathrm{ad}_{\mathcal{L}}\, x$ is for any $x \in \mathcal{L}$ a derivation of the Lie algebra $\mathcal{L}$. Such derivations are called **inner**. The set of all inner derivations $\mathrm{ad}(\mathcal{L}) = \{\mathrm{ad}_{\mathcal{L}}\, x \,|\, x \in \mathcal{L}\}$ forms an ideal in $\mathrm{der}(\mathcal{L})$. Let ρ be a representation of $\mathcal{L}$ on V, a mapping $f : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ defined by formula

$$f(x, y) = \mathrm{Tr}(\rho(x)\rho(y)), \quad \forall x, y \in \mathcal{L} \tag{1.26}$$

is an invariant, symmetric, bilinear form called a **trace form** of representation ρ. Invariance of f means that for all $x, y, z \in \mathcal{L}$ it holds

$$f([x, y], z) = f(x, [y, z]). \tag{1.27}$$

The trace form of the adjoint representation of $\mathcal{L}$

$$K_{\mathcal{L}}(x, y) = \mathrm{Tr}(\mathrm{ad}_{\mathcal{L}}(x)\, \mathrm{ad}_{\mathcal{L}}(y)), \quad \forall x, y \in \mathcal{L} \tag{1.28}$$

is called the **Killing form** of $\mathcal{L}$.

The kernel of the adjoint representation of $\mathcal{L}$ forms an ideal in $\mathcal{L}$ and is equal to the center of $\mathcal{L}$, i.e.

$$C(\mathcal{L}) = \ker(\mathrm{ad}_{\mathcal{L}}) = \{x \in \mathcal{L} \,|\, [x, y] = 0, \ \forall y \in \mathcal{L}\}\,. \tag{1.29}$$

In addition to the center, there are more significant ideals in any Lie algebra $\mathcal{L}$, such as so called **derived** or **commutator algebra** $D(\mathcal{L})$ of $\mathcal{L}$ defined by

$$D(\mathcal{L}) = [\mathcal{L}, \mathcal{L}], \tag{1.30}$$

or following series of ideals in $\mathcal{L}$:

- **derived series** $D^0(\mathcal{L}) \supset D^1(\mathcal{L}) \supset \ldots \supset D^k(\mathcal{L}) \supset \ldots$ defined by

$$D^0(\mathcal{L}) = \mathcal{L}, \qquad D^{k+1}(\mathcal{L}) = [D^k(\mathcal{L}), D^k(\mathcal{L})], \qquad k \in \mathbb{N}_0 \tag{1.31}$$

- **lower central series** $\mathcal{L}^1 \supset \mathcal{L}^2 \supset \ldots \supset \mathcal{L}^k \supset \ldots$ defined by

$$\mathcal{L}^1 = \mathcal{L}, \qquad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \qquad k \in \mathbb{N} \tag{1.32}$$

- **upper central series** $C^1(\mathcal{L}) \subset C^2(\mathcal{L}) \subset \ldots \subset C^k(\mathcal{L}) \subset \ldots$ defined by

$$C^1(\mathcal{L}) = C(\mathcal{L}), \qquad C^{k+1}(\mathcal{L})/C^k(\mathcal{L}) = C(\mathcal{L}/C^k(\mathcal{L})), \qquad k \in \mathbb{N}. \tag{1.33}$$

Note that for any $j, k \in \mathbb{N}$ it holds

$$D^k(\mathcal{L}) \subset \mathcal{L}^{k+1}, \qquad [\mathcal{L}^j, \mathcal{L}^k] \subset \mathcal{L}^{j+k}, \qquad [\mathcal{L}, C^{k+1}(\mathcal{L})] \subset C^k(\mathcal{L}). \tag{1.34}$$

According to occurrence of ideals and the behavior of the series of ideals we define following types of Lie algebras. We say that a Lie algebra $\mathcal{L}$ is

a) **abelian**, if $D(\mathcal{L}) = 0$,

b) **nilpotent**, if there exists $k \in \mathbb{N}$ such that $\mathcal{L}^k = 0$,

c) **solvable**, if there exists $k \in \mathbb{N}$ such that $D^k(\mathcal{L}) = 0$,

d) **semisimple**, if $\mathcal{L}$ has no nonzero solvable ideal,

e) **simple**, if $\mathcal{L}$ has no ideals (except 0 and $\mathcal{L}$) and $D(\mathcal{L}) \neq 0$.

A Lie algebra $\mathcal{L}$ is abelian if and only if $C(\mathcal{L}) = \mathcal{L}$. Such algebras have trivial multiplication (identically equal to 0) and differ only in their dimensions.

Any nilpotent Lie algebra is according to (1.34) solvable. If $\mathcal{L}$ is nilpotent and $\mathcal{L} \neq 0$, then $C(\mathcal{L}) \neq 0$. Since it holds for every $k \in \mathbb{N}$

$$\mathcal{L}^{k+1} = 0 \Longleftrightarrow C^k(\mathcal{L}) = \mathcal{L} \tag{1.35}$$

we have that $\mathcal{L}$ is nilpotent if and only if there exists $k \in \mathbb{N}$ such that $C^k(\mathcal{L}) = \mathcal{L}$. Recall that a linear operator $A \in \mathrm{End}(V)$ is called nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.

Theorem 1.1 (Engel). *A Lie algebra $\mathcal{L}$ is nilpotent if and only if $\mathrm{ad}_{\mathcal{L}}\, x$ is nilpotent for every $x \in \mathcal{L}$.*

It follows from definition that every subalgebra of a nilpotent or solvable Lie algebra is also nilpotent or solvable, respectively. Let $h : \mathcal{L} \longrightarrow \widetilde{\mathcal{L}}$ be a homomorphism of Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, then for all $k \in \mathbb{N}$ it holds

$$h(D^k(\mathcal{L})) = D^k(h(\mathcal{L})), \quad h(\mathcal{L}^k) = (h(\mathcal{L}))^k. \tag{1.36}$$

Moreover, if h is an isomorphism, then

$$h(C^k(\mathcal{L})) = C^k(h(\mathcal{L})). \tag{1.37}$$

Therefore, every homomorphic image of a nilpotent or solvable Lie algebra is nilpotent or solvable as well.

Theorem 1.2. *A Lie algebra $\mathcal{L}$ is solvable*

1. *if there exists a solvable ideal $\mathcal{A}$ in $\mathcal{L}$ such that $\mathcal{L}/\mathcal{A}$ is solvable.*

2. *if and only if $D(\mathcal{L})$ is nilpotent.*

Theorem 1.3 (Lie)**.** *Let $\mathcal{L}$ be a solvable Lie subalgebra of* $\mathrm{gl}(V)$*. Then there exists a basis in V such that all matrices of linear operators from $\mathcal{L}$ are upper triangular in this basis.*

If $\mathcal{A}, \mathcal{B}$ are two ideals in $\mathcal{L}$, then $\mathcal{A}+\mathcal{B}$, $\mathcal{A}\cap\mathcal{B}$ and $[\mathcal{A}, \mathcal{B}]$ are also ideals in $\mathcal{L}$. Moreover, it holds that the sum of any two nilpotent or solvable ideals in $\mathcal{L}$ is also a nilpotent or solvable ideal, respectively. The sum $N(\mathcal{L})$ of all nilpotent ideals in $\mathcal{L}$ is a maximal nilpotent ideal in $\mathcal{L}$ called a **nilradical**. The sum $R(\mathcal{L})$ of all solvable ideals in $\mathcal{L}$ is a maximal solvable ideal in $\mathcal{L}$ called a **radical**.

Theorem 1.4. *The radical $R(\mathcal{L})$ of a Lie algebra $\mathcal{L}$ is the orthogonal complement $D(\mathcal{L})^{\perp}$ of $D(\mathcal{L})$ with respect to the Killing form $K_{\mathcal{L}}$, i.e.*

$$R(\mathcal{L}) = \{x \in \mathcal{L} \mid \mathrm{Tr}(\mathrm{ad}_{\mathcal{L}}(x)\, \mathrm{ad}_{\mathcal{L}}(y)) = 0,\ \forall y \in D(\mathcal{L})\}\,. \tag{1.38}$$

Using the radical we can rewrite the definition of a semisimple Lie algebra equivalently as $\mathcal{L}$ is semisimple if $R(\mathcal{L}) = 0$. Moreover, $\mathcal{L}$ is semisimple if and only if $\mathcal{L}$ contains no abelian ideal $\neq 0$. The following theorem gives conditions for a Lie algebra to be solvable or semisimple in the terms of its Killing form.

Theorem 1.5 (Cartan's criterion)**.** *Let $\mathcal{L}$ be a Lie algebra, $K_{\mathcal{L}}$ the Killing form of $\mathcal{L}$, then*

1. *$\mathcal{L}$ is solvable if and only if $K_{\mathcal{L}}(x, y) = 0$ for all $x, y \in D(\mathcal{L})$*

2. $\mathcal{L}$ is semisimple if and only if $K_{\mathcal{L}}$ is non–degenerate, i.e.

$$\mathcal{L}^{\perp} = \{x \in \mathcal{L} \,|\, K_{\mathcal{L}}(x, y) = 0, \ \forall y \in \mathcal{L}\} = 0. \tag{1.39}$$

The corollary of this theorem says that $\mathcal{L}$ is solvable if and only if $K_{\mathcal{L}}(x, x) = 0$ for all $x \in D(\mathcal{L})$.

Properties of semisimple Lie algebras are summarized in the following theorem.

Theorem 1.6. *Let $\mathcal{L}$ be a semisimple Lie algebra.*

1. *$\mathcal{L}$ is a direct sum of simple ideals, uniquely up to their ordering of factors.*
2. *All ideals and factor algebras of $\mathcal{L}$ are semisimple.*
3. *$\mathcal{L}$ is perfect, i.e. $D(\mathcal{L}) = \mathcal{L}$.*
4. *$\mathcal{L}$ is complete, i.e. $\mathrm{ad}(\mathcal{L}) = \mathrm{der}(\mathcal{L})$ and $C(\mathcal{L}) = 0$.*

Any simple Lie algebra is according to the definition semisimple. All simple Lie algebras are already classified. There are five so called exceptional simple Lie algebras and four infinite series of so called classical simple Lie algebras. We will be mainly interested in one of these series called **special linear Lie algebras** $\mathrm{sl}(n, \mathbb{C}), n \geq 2$. The Lie algebra $\mathrm{sl}(n, \mathbb{C})$ is a subalgebra of $\mathrm{gl}(n, \mathbb{C})$ formed by all $n \times n$ complex matrices with zero trace i.e.

$$\mathrm{sl}(n, \mathbb{C}) = \{A \in \mathrm{gl}(n, \mathbb{C}) \,|\, \mathrm{Tr}(A) = 0\}. \tag{1.40}$$

Let $\mathcal{L}_1, \mathcal{L}_2$ be two Lie algebras with Lie brackets $[\,,]_1, [\,,]_2$ and $d : \mathcal{L}_2 \longrightarrow \mathrm{der}(\mathcal{L}_1)$ a homomorphism of Lie algebras. The direct sum of the vector spaces $\mathcal{L}_1, \mathcal{L}_2$

$$\mathcal{L} = \{(x_1, x_2) \,|\, x_1 \in \mathcal{L}_1, \ x_2 \in \mathcal{L}_2\} \tag{1.41}$$

together with a Lie bracket $[\,,]_d$ defined by

$$[(x_1, x_2), (y_1, y_2)]_d = ([x_1, y_1]_1 + d(x_2)y_1 - d(y_2)x_1, [x_2, y_2]_2) \tag{1.42}$$

forms a new Lie algebra $\mathcal{L}$ called **semidirect sum** of Lie algebras $\mathcal{L}_1, \mathcal{L}_2$. For this semidirect sum we will use the notation $\mathcal{L} = \mathcal{L}_1 \triangleleft \mathcal{L}_2$, where $\mathcal{L}_1$ identified with $\{(x_1, 0) \,|\, x_1 \in \mathcal{L}_1\}$ forms an ideal in $\mathcal{L}$ and $\mathcal{L}_2$ identified with $\{(0, x_2) \,|\, x_2 \in \mathcal{L}_2\}$ is a subalgebra in $\mathcal{L}$. If, in particular, $d \equiv 0$, we get definition of direct sum of Lie algebras and $\mathcal{L}_1, \mathcal{L}_2$ become ideals in $\mathcal{L}$.

For any Lie algebra $\mathcal{L}$ it holds that $\mathcal{L}/R(\mathcal{L})$ is semisimple and

$$[\mathcal{L}, R(\mathcal{L})] \subset N(\mathcal{L}). \tag{1.43}$$

Theorem 1.7 (Levi)**.** *Let $\mathcal{L}$ be a Lie algebra with radical $R(\mathcal{L})$. Then there exists a semisimple subalgebra $\mathcal{S}$ of $\mathcal{L}$ called Levi factor, such that $\mathcal{L} = R(\mathcal{L}) \triangleleft \mathcal{S}$.*

It follows from this theorem that $[\mathcal{L}, R(\mathcal{L})] = D(\mathcal{L}) \cap R(\mathcal{L})$ and together with (1.43) we have that the radical of $D(\mathcal{L})$ is nilpotent

$$R(D(\mathcal{L})) = D(\mathcal{L}) \cap R(\mathcal{L}) = [\mathcal{L}, R(\mathcal{L})] \subset N(\mathcal{L}). \tag{1.44}$$

Theorem 1.8 (Malcev-Harish-Chandra)**.** *Let $\mathcal{L} = \mathcal{R} \triangleleft \mathcal{S}$, where $\mathcal{R}$ is a solvable ideal and $\mathcal{S}$ is a semisimple subalgebra and let $\widetilde{\mathcal{S}}$ be a semisimple subalgebra of $\mathcal{L}$. Then there exists an automorphism $a \in \mathrm{Aut}(\mathcal{L})$ such that $a(\widetilde{\mathcal{S}}) \subset \mathcal{S}$.*

Corollary 1.9. *Any semisimple subalgebra of a Lie algebra $\mathcal{L}$ can be embedded in a Levi factor of $\mathcal{L}$.*

Corollary 1.10. *Let $\mathcal{S}_1, \mathcal{S}_2$ be semisimple subalgebras of a Lie algebra $\mathcal{L}$ such that $\mathcal{L} = R(\mathcal{L}) \triangleleft \mathcal{S}_1 = R(\mathcal{L}) \triangleleft \mathcal{S}_2$. Then there exist an automorphism $a \in \mathrm{Aut}(\mathcal{L})$ such that $a(\mathcal{S}_1) = \mathcal{S}_2$.*

We will call a Lie algebra $\mathcal{L}$ **decomposable** if there exists nonzero ideals $\mathcal{L}_1, \mathcal{L}_2$ in $\mathcal{L}$ such that $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$. In the opposite case we will call $\mathcal{L}$ **indecomposable**. We will call a Lie algebra $\mathcal{L}$ **Levi decomposable** if $\mathcal{L}$ has a nontrivial Levi decomposition i.e. $\mathcal{L} = R(\mathcal{L}) \triangleleft \mathcal{S}$, where $\mathcal{S}$ is semisimple and $[R(\mathcal{L}), \mathcal{S}] \neq 0$.

Theorem 1.11 (Ado)**.** *Let $\mathcal{L}$ be a Lie algebra, $N(\mathcal{L})$ its nilradical. Then there exists a faithful (one–to–one) representation $\rho : \mathcal{L} \longrightarrow \mathrm{gl}(V)$ such that $\rho(N(\mathcal{L}))$ is composed of nilpotent linear operators.*

Chapter 2

Gradings of Lie algebras

Graded contractions of Lie algebras are contractions which preserve chosen grading of a given Lie algebra. Therefore, before approaching graded contractions, we devote this chapter to the description of Lie gradings, i.e. gradings of Lie algebra. We start with basic definitions concerning Lie gradings. Then we define a symmetry group of Lie grading, which will play a crucial role during the search for graded contractions, and describe relations between gradings and automorphisms of Lie algebras. We focus on the special case of Lie gradings – so called group gradings, and we describe their construction. There were classified fine group gradings for classical simple Lie algebras in terms of MAD–groups in [33]. We describe relation of symmetry group and MAD–group for fine group gradings. In the end we present the overview of all group gradings of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$ taken from [63].

2.1 Basic definitions

Let $\mathcal{L}$ be a finite–dimensional Lie algebra over $\mathbb{C}$. A decomposition

$$\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i \tag{2.1}$$

of the vector space $\mathcal{L}$ into a direct sum of its vector subspaces $L_i \neq 0,\ i \in I$ is called a **grading** of Lie algebra $\mathcal{L}$, if for any pair of indices $i, j \in I$ there exists $k \in I$ such that

$$[L_i, L_j] \subset L_k. \tag{2.2}$$

Vector subspaces L_i are called **grading subspaces**. The number of grading subspaces is equal to the cardinality $|I|$ of the index set I. A pair of indices $i, j \in I$ is called

- **relevant** if $[L_i, L_j] \neq 0$,
- **irrelevant** if $[L_i, L_j] = 0$.

Relation (2.2) allows us to define binary operation $\diamond$ on the index set I. For any relevant pair $i, j \in I$ we define

$$i \diamond j = k \in I \iff [L_i, L_j] \subset L_k. \tag{2.3}$$

For irrelevant pairs we can choose $k \in I$ arbitrarily. It follows from anti–commutativity of Lie bracket, that $\diamond$ is commutative operation.

It is sometimes possible to decompose grading subspaces of the grading Γ such that new vector subspaces form also grading of $\mathcal{L}$. A grading $\widetilde{\Gamma} : \mathcal{L} = \bigoplus_{j \in J} \widetilde{L}_j$ is called a **refinement** of a grading $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$, if for any $j \in J$ there exists $i \in I$ such that $\widetilde{L}_j \subset L_i$. Moreover, if $|J| > |I|$, then $\widetilde{\Gamma}$ is called **proper refinement** of Γ. A grading Γ which has no proper refinement is called **fine** grading. If all grading subspaces of Γ are one–dimensional, then Γ is obviously fine and we call it **finest** grading. In the opposite way, if it is possible by composition of some grading subspaces of Γ to get another grading $\bar{\Gamma}$, we call it **coarsening** of grading Γ. Thus, $\bar{\Gamma}$ is coarsening of Γ if and only if Γ is refinement of $\bar{\Gamma}$. Refinements and coarsenings allow us to define partial ordering on the set of all gradings of $\mathcal{L}$, where the maximal elements are fine gradings and the minimal element is a trivial grading (with only one grading subspace $\mathcal{L}$).

If $g \in \mathrm{Aut}(\mathcal{L})$ is an automorphism of $\mathcal{L}$, then for all $i, j \in I$ there exists $k \in I$ such that

$$[g(L_i), g(L_j)] = g[L_i, L_j] \subset g(L_k) \tag{2.4}$$

and the decomposition

$$\Gamma^g : \mathcal{L} = \bigoplus_{i \in I} g(L_i) \tag{2.5}$$

is also grading of $\mathcal{L}$. Gradings Γ and Γ^g have the same structure (number of grading subspaces, their dimensions and relations between them). Such gradings are called equivalent. More precisely, two gradings $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$ and $\widetilde{\Gamma} : \mathcal{L} = \bigoplus_{j \in J} \widetilde{L}_j$ are called **equivalent** $\Gamma \cong \widetilde{\Gamma}$, if $|I| = |J|$ and there exists $g \in \mathrm{Aut}(\mathcal{L})$ such that

$$(\forall i \in I)(\exists j \in J)(g(L_i) = \widetilde{L}_j). \tag{2.6}$$

Note that a grading of a given Lie algebra $\mathcal{L}$ depends only on its grading subspaces i.e. two gradings $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$ and $\widetilde{\Gamma} : \mathcal{L} = \bigoplus_{j \in J} \widetilde{L}_j$ are equal $\Gamma = \widetilde{\Gamma}$ if there exists bijection $f : I \longrightarrow J$ such that for all $i \in I$ it holds $L_i = \widetilde{L}_{f(i)}$.

2.2 Symmetries of Lie gradings

Every automorphism $g \in \mathrm{Aut}(\mathcal{L})$ maps a grading $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$ of Lie algebra $\mathcal{L}$ onto an equivalent grading $\Gamma^g \cong \Gamma$. Now we will be interested only in such automorphisms $g \in \mathrm{Aut}(\mathcal{L})$ which preserve grading Γ, i.e. for which $\Gamma^g = \Gamma$ holds.

Let us denote the symmetry group of the set I by S_I i.e. S_I is the set of all bijections from I onto I. An automorphism $g \in \mathrm{Aut}(\mathcal{L})$ is called a **symmetry** of grading $\Gamma : \mathcal{L} = \bigoplus_{i\in I} L_i$ if there exists such a permutation $\pi_g \in S_I$ that

$$g(L_i) = L_{\pi_g(i)}, \ \forall i \in I. \tag{2.7}$$

We show that the set of all symmetries of Γ

$$\mathrm{Aut}(\Gamma) = \left\{ g \in \mathrm{Aut}(\mathcal{L}) \,|\, \exists \pi_g \in S_I, \ g(L_i) = L_{\pi_g(i)} \right\} \tag{2.8}$$

forms a subgroup of $\mathrm{Aut}(\mathcal{L})$ called a **symmetry group** of grading Γ. $\mathrm{Aut}(\Gamma)$ contains identity and for $g_1, g_2 \in \mathrm{Aut}(\Gamma)$ there exists $\pi_1, \pi_2 \in S_I$ such that $g_1(L_i) = L_{\pi_1(i)}, \ g_2(L_i) = L_{\pi_2(i)}$ and for all $i \in I$ we have

$$(g_1 g_2^{-1})(L_i) = g_1(g_2^{-1}(L_{\pi_2(\pi_2^{-1}(i))})) = g_1(L_{\pi_2^{-1}(i)}) = L_{\pi_1(\pi_2^{-1}(i))} = L_{(\pi_1\pi_2^{-1})(i)}. \tag{2.9}$$

Thus, $g_1 g_2^{-1} \in \mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ is a subgroup of $\mathrm{Aut}(\mathcal{L})$.

It follows from (2.9) that the mapping $\Delta_\Gamma : \mathrm{Aut}(\Gamma) \longrightarrow S_I$ defined by

$$\Delta_\Gamma(g) = \pi_g \tag{2.10}$$

is a homomorphism of groups $\mathrm{Aut}(\Gamma)$ and S_I, so called **permutation representation** of the group $\mathrm{Aut}(\Gamma)$ on the set I. The kernel of this representation Δ_Γ

$$\mathrm{Stab}(\Gamma) = \ker(\Delta_\Gamma) = \{ g \in \mathrm{Aut}(\Gamma) \,|\, g(L_i) = L_i, \ \forall i \in I \} \tag{2.11}$$

is the **stabilizer** of Γ in $\mathrm{Aut}(\Gamma)$. Therefore, the stabilizer of Γ is a normal subgroup of $\mathrm{Aut}(\Gamma)$ and according to the isomorphism theorem for groups, we have

$$\mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma) \cong \Delta_\Gamma(\mathrm{Aut}(\Gamma)). \tag{2.12}$$

This group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ will play crucial role in the concept of graded contractions as the symmetry group of system of contraction equations.

Let us denote the set of all diagonal automorphisms of the grading Γ by

$$\mathrm{Diag}(\Gamma) = \{ g \in \mathrm{Aut}(\Gamma) \,|\, \forall i \in I, \exists \lambda_i \in \mathbb{C}\backslash\{0\}, \ g|_{L_i} = \lambda_i \,\mathrm{Id}_{L_i} \}. \tag{2.13}$$

This set forms an abelian subgroup in $\mathrm{Aut}(\Gamma)$. Moreover, it lies in the center of $\mathrm{Stab}(\Gamma)$ and thus, it is a normal subgroup in $\mathrm{Stab}(\Gamma)$. We show that $\mathrm{Diag}(\Gamma)$ is also a normal subgroup in $\mathrm{Aut}(\Gamma)$. It is sufficient to prove that for all $g \in \mathrm{Aut}(\Gamma)$ and for all $h \in \mathrm{Diag}(\Gamma)$ it holds $g^{-1}hg \in \mathrm{Diag}(\Gamma)$. Let us take an arbitrary $x \in L_i$, then

$$(g^{-1}hg)(x) = g^{-1}(h(g(x)) = g^{-1}(\lambda_{\pi(i)} g(x)) = \lambda_{\pi(i)} g^{-1}(g(x)) = \lambda_{\pi(i)} x \tag{2.14}$$

where $\pi = \Delta_\Gamma(g)$. Thus, $(g^{-1}hg)|_{L_i} = \lambda_{\pi(i)} \,\mathrm{Id}_{L_i}$ and $g^{-1}hg \in \mathrm{Diag}(\Gamma)$.

2.3 Group gradings of Lie algebras

In many cases of gradings, the index set I with the commutative operation $\diamond$ has a group structure or at least can be embedded into an abelian group. Such gradings are called group gradings and we will devote this section to their description. More precisely: a grading $\Gamma : \mathcal{L} = \bigoplus_{i\in I} L_i$ is called **group grading** or **semigroup grading** if there exist an abelian group or semigroup G, respectively, and an injective mapping $f : I \longrightarrow G$ such that for any relevant pair of indices $i, j \in I$ it holds

$$f(i \diamond j) = f(i) + f(j), \tag{2.15}$$

where $+$ denotes binary operation in G. The group (semigroup) G is called **grading group (semigroup)**. Further on we will assume, without loss of generality, that I is a subset of G if not, one can take $f(I)$ as an index set instead of I.

In [59] it was asserted that any grading is a semigroup grading, and further that any grading of simple Lie algebra is a group grading. The first assertion was disproved in [24] by an explicit example of grading constructed on 16–dimensional nilpotent Lie algebra. However, it is still possible that the second assertion is true, for simple Lie algebras no counter examples are known. Let us note that graded contractions do not depend on the existence of group structure of the index set I.

A refinement or coarsening of a group grading is called **group refinement** or **group coarsening**, respectively, if it is a group grading. A **fine group grading** is a group grading which has no proper group refinement. Group refinements and coarsenings define partial ordering on the set of all group gradings of $\mathcal{L}$.

Great advantage of group gradings is their easy construction. Let $g \in \mathrm{Aut}(\mathcal{L})$ be a diagonalizable automorphism of $\mathcal{L}$. For any eigenvalue λ in the spectrum $\sigma(g) \subset \mathbb{C}\backslash\{0\}$ of g, we denote by L_λ the eigenspace of g in $\mathcal{L}$ corresponding to λ, i.e.

$$L_\lambda = \ker(g - \lambda\,\mathrm{Id}_{\mathcal{L}}). \tag{2.16}$$

For any two eigenvalues $\lambda, \mu \in \sigma(g)$ and corresponding eigenvectors $x \in L_\lambda,\ y \in L_\mu$ we have

$$g[x, y] = [g(x), g(y)] = [\lambda x, \mu y] = \lambda\mu[x, y]. \tag{2.17}$$

Thus, $[x, y]$ is either zero or $\lambda\mu$ is eigenvalue of g. Taking all possible $x \in L_\lambda,\ y \in L_\mu$ we get either $[L_\lambda, L_\mu] = 0$ or $[L_\lambda, L_\mu] \subset L_{\lambda\mu}$. Eigenspaces corresponding to different eigenvalues are disjoint and, since g is diagonalizable, their sum forms whole $\mathcal{L}$. Therefore,

$$\mathcal{L} = \bigoplus_{\lambda\in\sigma(g)} L_\lambda \tag{2.18}$$

is a grading of Lie algebra $\mathcal{L}$. For any pair of relevant indices $\lambda, \mu \in \sigma(g)$ we have

$$\lambda \diamond \mu = \lambda\mu \tag{2.19}$$

and the spectrum $\sigma(g)$ of automorphism g is a subset in multiplicative group $\mathbb{C}\backslash\{0\}$. Hence this grading is a group grading and we will denote it $\mathrm{Gr}(g)$.

Let us take another diagonalizable automorphism $h \in \mathrm{Aut}(\mathcal{L})$ which commutes with g. Then for any $\lambda \in \sigma(g)$ and all $x \in L_\lambda$ we have

$$g(h(x)) = h(g(x)) = h(\lambda x) = \lambda h(x). \tag{2.20}$$

Thus, $h(L_\lambda) = L_\lambda$ and it can be split up into eigenspaces of h. These eigenspaces, indexed by pairs from $\sigma(g) \times \sigma(h)$, are common eigenspaces of g and h and form a group refinement $\mathrm{Gr}(g, h)$ of grading $\mathrm{Gr}(g)$. Similarly, any set $S \subset \mathrm{Aut}(\mathcal{L})$ of diagonalizable mutually commuting automorphisms determines some group grading $\mathrm{Gr}(S)$.

On the other hand, we have for any grading Γ the group of mutually commuting diagonal automorphisms $\mathrm{Diag}(\Gamma)$ and it holds:

1. Γ is a refinement of $\mathrm{Gr}(\mathrm{Diag}(\Gamma))$
2. For any set $S \subset \mathrm{Aut}(\mathcal{L})$ of mutually commuting diagonalizable automorphisms is

$$S \subset \mathrm{Diag}(\mathrm{Gr}(S)). \tag{2.21}$$

The following theorem is proved in [70]. It says that any group grading Γ is generated by certain set of mutually commuting diagonalizable automorphisms for example by $\mathrm{Diag}(\Gamma)$.

Theorem 2.1. *Let $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$ be a group grading of a finite–dimensional complex Lie algebra $\mathcal{L}$. Then*

1. *there exists a set of automorphisms $S \subset \mathrm{Aut}(\mathcal{L})$ such that $\mathrm{Gr}(S) = \Gamma$*
2. $\mathrm{Gr}(\mathrm{Diag}(\Gamma)) = \Gamma$.

Having group grading $\Gamma = \mathrm{Gr}(S)$ we can obtain its group refinement by enlarging the set S of mutually commuting diagonalizable automorphism in $\mathrm{Aut}(\mathcal{L})$. Once the set S cannot be enlarged, it becomes an abelian group $\mathcal{G}$ so called **MAD–group** (maximal abelian group of diagonalizable automorphisms). The following theorem proved in [70] states that a MAD–group generates a fine group grading and

$$\mathcal{G} = \mathrm{Diag}(\mathrm{Gr}(\mathcal{G})). \tag{2.22}$$

Theorem 2.2. *Let Γ be a group grading of a finite–dimensional complex Lie algebra $\mathcal{L}$. Then Γ is a fine group grading if and only if the set* $\mathrm{Diag}(\Gamma)$ *is a MAD–group in* $\mathrm{Aut}(\mathcal{L})$.

This theorem provides one-to-one correspondence between fine group gradings and MAD–groups in $\mathrm{Aut}(\mathcal{L})$. And together with another theorem from [70] the classification of all non–equivalent group gradings is converted into classification of non–conjugated MAD–groups in $\mathrm{Aut}(\mathcal{L})$.

Theorem 2.3. *Let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathrm{Aut}(\mathcal{L})$ be MAD–groups on finite–dimensional complex Lie algebra $\mathcal{L}$. Let $\Gamma_1 = \mathrm{Gr}(\mathcal{G}_1)$ and $\Gamma_2 = \mathrm{Gr}(\mathcal{G}_2)$ be the fine group grading generated by $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Then gradings Γ_1 and Γ_2 are equivalent if and only if the MAD–groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are conjugated i.e. if there exists $h \in \mathrm{Aut}(\mathcal{L})$ such that $h\mathcal{G}_1 h^{-1} = \mathcal{G}_2$.*

Non-conjugated MAD–groups were found and classified in [31, 32, 33] for all simple classical complex Lie algebras except D_4.

2.4 Symmetries of group gradings

Another advantage of group gradings lies in the construction of their symmetry group. Let us suppose a group grading $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$ generated by a set of mutually commuting diagonalizable automorphisms $S \subset \mathrm{Aut}(\mathcal{L})$ i.e. $\Gamma = \mathrm{Gr}(S)$. We show that the **normalizer** of S

$$\mathcal{N}(S) = \{g \in \mathrm{Aut}(\mathcal{L}) \,|\, gS = Sg\} \tag{2.23}$$

is a subgroup of the symmetry group $\mathrm{Aut}(\Gamma)$. Since normalizer $\mathcal{N}(S)$ is a subgroup in $\mathrm{Aut}(\mathcal{L})$, it is sufficient to prove that $\mathcal{N}(S) \subset \mathrm{Aut}(\Gamma)$. Let us take $g \in \mathcal{N}(S)$, then for any $h \in S$ there exists $\widetilde{h} \in S$ such that $hg = g\widetilde{h}$. Let $\widetilde{\lambda}_i$ be an eigenvalue of $\widetilde{h}$ such that $\widetilde{h}|_{L_i} = \widetilde{\lambda}_i \,\mathrm{Id}_{L_i}$ then for all $x \in L_i$ we have

$$h(gx) = g(\widetilde{h}x) = g(\widetilde{\lambda}_i x) = \widetilde{\lambda}_i(gx) \tag{2.24}$$

and gx is common eigenvector of all $h \in S$. Thus, there exists $j \in S$ such that $gx \in L_j$ for all $x \in L_i$, i.e. $gL_i \subset L_j$. Since g is an automorphism of $\mathcal{L} = \bigoplus_{i \in I} L_i$ and for all $i \in I$ there exists $j \in I$ such that $gL_i \subset L_j$, we have that $gL_i = L_j$. Thus, g is a symmetry of the grading $\Gamma = \mathrm{Gr}(S)$ and

$$\mathcal{N}(S) \subset \mathrm{Aut}(\mathrm{Gr}(S)). \tag{2.25}$$

Similarly, for the **centralizer** of S in $\mathrm{Aut}(\mathcal{L})$

$$\mathcal{C}(S) = \{g \in \mathrm{Aut}(\mathcal{L}) \,|\, gh = hg, \ \forall h \in S\} \tag{2.26}$$

it holds

$$\mathcal{C}(S) \subset \mathrm{Stab}(\mathrm{Gr}(S)) \tag{2.27}$$

and therefore, $\mathcal{C}(S)$ is a subgroup of $\mathrm{Stab}(\mathrm{Gr}(S))$.

We focus now on fine group gradings only. Let $\mathcal{G}$ be a MAD–group in $\mathrm{Aut}(\mathcal{L})$ and $\Gamma = Gr(\mathcal{G})$, then $\mathcal{G} = \mathrm{Diag}(\Gamma)$. Since $\mathrm{Diag}(\Gamma)$ is a normal subgroup in $\mathrm{Aut}(\Gamma)$, we have $\mathrm{Aut}(\Gamma) \subset \mathcal{N}(\mathcal{G})$ and together with (2.25) we get

$$\mathcal{N}(\mathcal{G}) = \mathrm{Aut}(\Gamma). \tag{2.28}$$

Thus, the symmetry group of the fine group grading Γ is the normalizer of its generating MAD-group. Moreover, $\mathrm{Diag}(\Gamma)$ lies in the center of $\mathrm{Stab}(\Gamma)$ and therefore, $\mathrm{Stab}(\Gamma) \subset \mathcal{C}(\mathcal{G})$, which with respect to (2.27) gives

$$\mathcal{C}(\mathcal{G}) = \mathrm{Stab}(\Gamma). \tag{2.29}$$

Using these results we can rewrite the formula (2.12) into the following form

$$\Delta_\Gamma(\mathrm{Aut}(\Gamma)) \cong \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma) = \mathcal{N}(\mathcal{G})/\mathcal{C}(\mathcal{G}). \tag{2.30}$$

Note that for finest grading it holds $\mathrm{Diag}(\Gamma) = \mathrm{Stab}(\Gamma)$.

2.5 Group gradings of Lie algebra $\mathrm{sl}(3,\mathbb{C})$

From all Lie algebras we are mainly interested in the simple Lie algebra $\mathrm{sl}(3,\mathbb{C})$. This Lie algebra $\mathrm{sl}(3,\mathbb{C})$ has four non–equivalent fine group gradings [59] called Gell–Mann (Orthogonal), Cartan (Toroidal), Pauli and Γ_4. All these fine group gradings can be obtained using MAD–groups method developed in [33]. Gell-Mann and Cartan grading are formed by one two–dimensional subspace and six one–dimensional ones. Pauli and Γ_4 grading are finest. We leave detailed description of these gradings and their symmetries [34, 35, 36] to the following chapters.

Since $\mathrm{sl}(3,\mathbb{C})$ is a simple Lie algebra, the other group gradings can be obtained – according to the following theorem from [23] – as coarsenings of the fine ones.

Theorem 2.4. *Let Γ be a group grading of a finite–dimensional simple complex Lie algebra $\mathcal{L}$. Then there exists a finitely generated abelian group G and an index set $J \subset G$ such that:*

1. *the grading subspaces of Γ are indexed by elements from the set J, i.e. $\Gamma : \mathcal{L} = \bigoplus_{j\in J} L_j$; and*

2. *for any coarsening* Γ' *of the grading* Γ *there exists a group homomorphism* f *such that the grading subspaces of* Γ' *are indexed by* $J' \subset f(G)$, *i.e.* $\Gamma : \mathcal{L} = \bigoplus_{i \in J'} L'_i$ *and* $L'_i = \bigoplus_{f(k)=i} L_k$ *for any* $i \in J'$.

The group $f(G)$ *is unique up to isomorphism. The group* G *is called the* **universal group** *of the group grading* Γ.

Complete classification of group gradings of $\mathrm{sl}(3,\mathbb{C})$ was given in [63]. Here we present only oriented graph describing hierarchy of all 17 group grading of $\mathrm{sl}(3,\mathbb{C})$. The graph of the figure 2.1 shows successive refinements of group gradings leading from the trivial one (whole $\mathrm{sl}(3,\mathbb{C})$) to the four fine group gradings. Nodes of the graph stand for inequivalent gradings, links with arrows indicating refinements. The graph exhibits 8 levels corresponding to the number of grading subspaces: the numbers increase downwards, from 1 to 8, starting from the level of $\mathrm{sl}(3,\mathbb{C})$ itself. Let us mention that for the Lie algebra $\mathrm{sl}(3,\mathbb{C})$ no other gradings than the above listed group gradings are known.

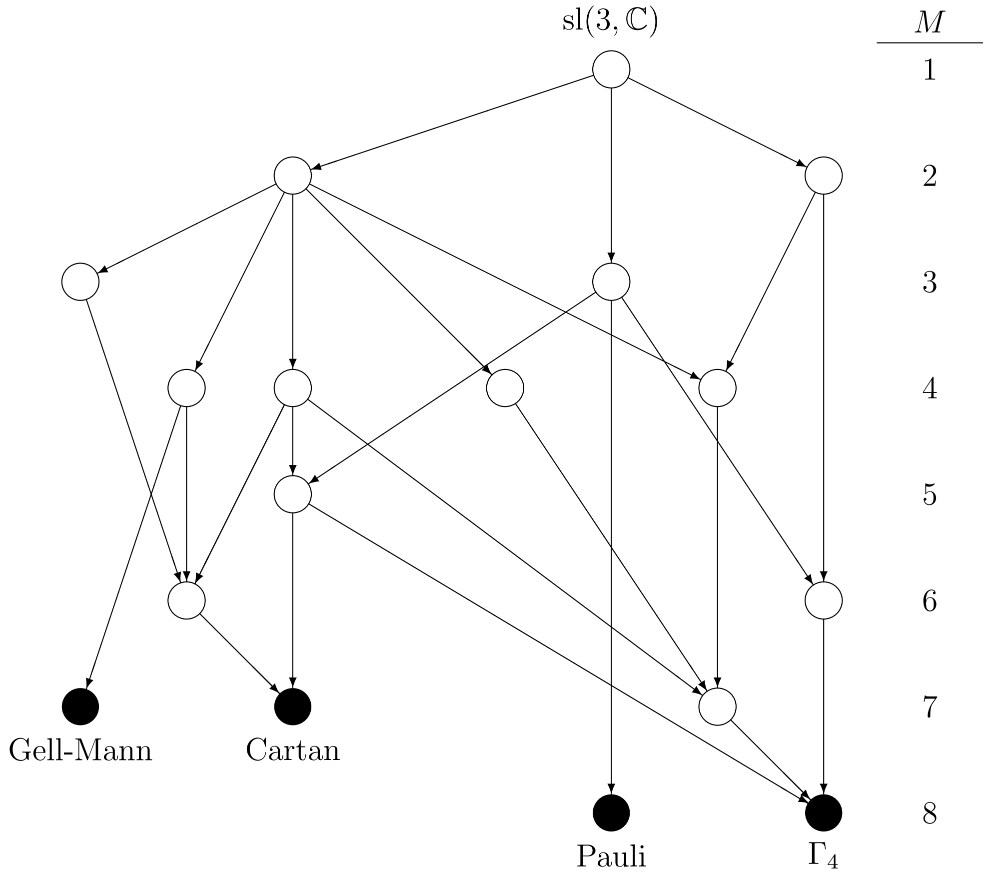

Figure 2.1: The hierarchy of 17 group gradings of sl$(3,\mathbb{C})$ [63]; The gradings are distributed into 8 levels according to the number M of their grading subspaces. Nodes of the graph stand for non–equivalent gradings, links (arrows) indicate refinements. Black circles denote fine gradings.

Chapter 3

Graded contractions

Graded contraction is a process of transition from a given Lie algebra with a chosen grading to another Lie algebra with the same grading by change of its Lie bracket. Graded contractions are also the Lie algebras resulting in this process.

The main goal of this work is to describe the concept of graded contractions and apply it to the group gradings of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. In this chapter we will concentrate on the description of graded contractions. Graded contractions of Lie algebras were originally introduced in [17] as an algebraic method for computing classical continuous contractions and then studied e.g. in [13, 54, 39, 58, 74, 76, 77].

We start with the definition of Γ-graded contraction, contraction matrices $\mathsf{C}_\Gamma(\mathcal{L})$ and the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$. Then we investigate some relations of graded contractions which correspond to different gradings. We introduce the normalization matrix as a solution of $\mathsf{S}_\Gamma(\mathcal{L})$ and define strong equivalence (normalization) for contraction matrices. Two types of contraction matrices — discrete and continuous — are distinguished. We also define the action of permutation representation $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ of the symmetry group of grading Γ on the set of all contraction matrices $\mathsf{C}_\Gamma(\mathcal{L})$ and show how it can simplify notation of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$. Using this action and strong equivalence, we define a new equivalence of contraction matrices and we describe an algorithm which produces all solutions of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ up to this equivalence.

3.1 Basic definitions

Let $\mathcal{L}$ be a complex Lie algebra of finite dimension with a Lie bracket $[\,,]$. Let $\Gamma : \mathcal{L} = \bigoplus_{i\in I} L_i$ be a grading of $\mathcal{L}$ with $|I| = m \in \mathbb{N}$ grading subspaces. Then any complex Lie algebra $\mathcal{L}^\varepsilon$ with a Lie bracket $[\,,]_\varepsilon$ which satisfies two conditions

1. the underlying vector space of $\mathcal{L}^\varepsilon$ is the underlying vector space of $\mathcal{L}$, i.e. $\mathcal{L}^\varepsilon = \bigoplus_{i\in I} L_i$

2. for all $i, j \in I$ there exits $\varepsilon_{ij} \in \mathbb{C}$ such that for all $x \in L_i$ and $y \in L_j$ it holds

$$[x, y]_\varepsilon = \varepsilon_{ij}[x, y] \tag{3.1}$$

is called Γ**-graded contraction** of Lie algebra $\mathcal{L}$. Complex numbers ε_{ij} are called **contraction parameters**.

It follows from the first condition that both the Lie algebra and its contraction have the same dimensions. The second condition guarantees that for any $i, j \in I$ there exists $k \in I$ such that

$$[L_i, L_j]_\varepsilon = \varepsilon_{ij}[L_i, L_j] \subset \varepsilon_{ij} L_k \tag{3.2}$$

and thus, $\mathcal{L}^\varepsilon$ is also Γ-graded, i.e. $\Gamma : \mathcal{L}^\varepsilon = \bigoplus_{i \in I} L_i$ is a grading of $\mathcal{L}^\varepsilon$.

If $i, j \in I$ is an irrelevant pair of indices for grading Γ of $\mathcal{L}$, i.e. $[L_i, L_j] = 0$, then (3.1) implies that it will also be irrelevant pair for grading Γ of $\mathcal{L}^\varepsilon$ and thus, ε_{ij} can be chosen arbitrarily. Therefore, we will call contraction parameter ε_{ij}

- **relevant** if $[L_i, L_j] \neq 0$
- **irrelevant** if $[L_i, L_j] = 0$.

We will consider all irrelevant contraction parameters to be zeros.

During the search of all Γ-graded contractions of Lie algebra $\mathcal{L}$ we will follow the procedure introduced in [17]. We start with Γ-graded Lie algebra $\mathcal{L} = \bigoplus_{i \in I} L_i$ and define a new bilinear map $[\,,]_\varepsilon$ on its underlying vector space according to (3.1)

$$[x, y]_\varepsilon = \varepsilon_{ij}[x, y], \quad \forall x \in L_i, \ \forall y \in L_j, \ \forall i, j \in I, \tag{3.3}$$

where ε_{ij} are arbitrary complex numbers. Since $[\,,]_\varepsilon$ is bilinear, it is determined by this condition unambiguously on whole $\mathcal{L}$. In order to get a Lie algebra, bilinear map $[\,,]_\varepsilon$ has to be anti–commutative and must fulfil the Jacobi identity.

Anti–commutativity of bracket $[\,,]_\varepsilon$ is, owing to bilinearity of this bracket, equivalent to the following condition. For all $i, j \in I$ and for all $x \in L_i, y \in L_j$ it holds

$$\varepsilon_{ij}[x, y] = [x, y]_\varepsilon = -[y, x]_\varepsilon = -\varepsilon_{ji}[y, x] \tag{3.4}$$

and therefore, for all relevant pairs of indices $i, j \in I$ we have

$$\varepsilon_{ij} = \varepsilon_{ji}. \tag{3.5}$$

For irrelevant pairs $i, j \in I$ we postulate $\varepsilon_{ij} = \varepsilon_{ji} = 0$. It is often convenient to order the index set I and view the set of all contraction parameters as a symmetric $m \times m$ matrix

$\varepsilon = (\varepsilon_{ij})$ — so called **contraction matrix** or in short just **contraction**. The set of all contractions of Γ-graded Lie algebra $\mathcal{L}$ will be denoted by $\mathsf{C}_\Gamma(\mathcal{L})$. We order the set I by fixing one bijection $\mathcal{O} : I \longrightarrow \{1, 2, \dots, m\}$ and we will write simply $i < j$ for $i, j \in I$ instead of $\mathcal{O}(i) < \mathcal{O}(j)$.

The Jacobi identity for our new bilinear map $[\,,]_\varepsilon$, i.e.

$$[x, [y, z]_\varepsilon]_\varepsilon + [y, [z, x]_\varepsilon]_\varepsilon + [z, [x, y]_\varepsilon]_\varepsilon = 0, \quad \forall x, y, z \in \mathcal{L}, \tag{3.6}$$

is a trilinear condition and therefore, it is equivalent to the following condition. For all $i, j, k \in I$ it holds

$$[x, [y, z]_\varepsilon]_\varepsilon + [y, [z, x]_\varepsilon]_\varepsilon + [z, [x, y]_\varepsilon]_\varepsilon = 0, \quad \forall x \in L_i, \forall y \in L_j, \forall z \in L_k. \tag{3.7}$$

Using (3.3) we can rewrite this condition into the following system of equations for variables ε_{ij} called **contraction system** $\mathsf{S}_\Gamma(\mathcal{L})$: for all $i, j, k \in I$

$$\begin{aligned} e(i\,j\,k): \quad & \varepsilon_{jk}\varepsilon_{i,j\diamond k}[x, [y, z]] + \varepsilon_{ki}\varepsilon_{j,k\diamond i}[y, [z, x]] + \varepsilon_{ij}\varepsilon_{k,i\diamond j}[z, [x, y]] = 0 \\ & \forall x \in L_i, \forall y \in L_j, \forall z \in L_k. \end{aligned} \tag{3.8}$$

These equations $e(i\,j\,k)$ are called **contraction equations** and do not depend on the order of indices i, j, k. In other words, triplets (i, j, k) and (j, i, k) lead to the same contraction equations $e(i\,j\,k) = e(j\,i\,k)$, taking in to account that ε is symmetric. Thus, we can equivalently demand the validity of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ only for all indices $i \leq j \leq k$. However, it will be more convenient to handle these triplets as unordered triplets $(i\,j\,k)$ and demand the validity of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ only for all unordered triplets of indices.

Let us consider the index set I and the set I^n of all n-tuples with entries in I. Let S_n denote the symmetric group of the set $\{1, 2, \dots, n\}$. We define equivalence relation on I^n as follows: two n-tuples $(x_1, \dots, x_n), (y_1, \dots, y_n) \in I^n$ are equivalent if and only if there exists $\sigma \in S_n$ such that

$$x_i = y_{\sigma(i)}, \quad \forall i = 1, 2, \dots, n. \tag{3.9}$$

Cosets $(x_1 \dots x_n) = \big\{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \,|\, \sigma \in S_n\big\}$ defined by this equivalence are called **unordered n-tuples** and the set of all unordered n-tuples with entries in I is denoted I^n_u.

Any contraction equation $e(i\,j\,k)$ is in fact the system of equations (3.8) generated by all $x \in L_i$, $y \in L_j$, $z \in L_k$. Owing to linearity of Lie bracket $[\,,]$, this system is equivalent (has the same solutions) to the system generated by basis vectors of grading subspaces L_i, L_j, L_k only. If Γ is finest grading, then each contraction equation $e(i\,j\,k)$ can be represented by one equation.

Using the Jacobi identity for $[\,,]$

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0,\quad \forall x,y,z\in\mathcal{L}, \tag{3.10}$$

one can rewrite contraction equation $e(i\,j\,k)$ as follows

$$e(i\,j\,k):\;(\varepsilon_{jk}\varepsilon_{i,j\diamond k}-\varepsilon_{ij}\varepsilon_{k,i\diamond j})[x,[y,z]]+(\varepsilon_{ki}\varepsilon_{j,k\diamond i}-\varepsilon_{ij}\varepsilon_{k,i\diamond j})[y,[z,x]]=0, \tag{3.11}$$

where $x\in L_i, y\in L_j, z\in L_k$. In special case, when there exist $x'\in L_i, y'\in L_j, z'\in L_k$ such that $[x',[y',z']]$ and $[y',[z',x']]$ are linearly independent, contraction equation (3.11) is equivalent to the following two–term equations:

$$e(i\,j\,k):\;\varepsilon_{jk}\varepsilon_{i,j\diamond k}=\varepsilon_{ij}\varepsilon_{k,i\diamond j}=\varepsilon_{ki}\varepsilon_{j,k\diamond i}. \tag{3.12}$$

However, this condition is not always fulfilled and three–term equations can arise in system of contraction equations. Let us note that in some works [39, 74, 76] only two–term equations are considered in the system of contraction equations.

Notice that irrelevant contraction parameters do not appear in the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ and the set of all solutions of $\mathsf{S}_\Gamma(\mathcal{L})$ gives the set of all contraction matrices $\mathsf{C}_\Gamma(\mathcal{L})$. Thus, we can summarize the results into following proposition.

Proposition 3.1. *Any contraction matrix $\varepsilon=(\varepsilon_{ij})\in\mathsf{C}_\Gamma(\mathcal{L})$ is symmetric and its relevant contraction parameters satisfy the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$*

$$\begin{gathered}\forall(i\,j\,k)\in I_u^3,\quad e(i\,j\,k):\;\varepsilon_{jk}\varepsilon_{i,j\diamond k}[x,[y,z]]+\varepsilon_{ki}\varepsilon_{j,k\diamond i}[y,[z,x]]+\varepsilon_{ij}\varepsilon_{k,i\diamond j}[z,[x,y]]=0\\ \forall x\in L_i,\forall y\in L_j,\forall z\in L_k.\end{gathered} \tag{3.13}$$

Irrelevant contraction parameters can be considered equal to zeros.

3.2 Contractions of different gradings

We have defined Γ-graded contractions of Lie algebra $\mathcal{L}$. All these Γ-graded contraction of all possible gradings Γ of $\mathcal{L}$ are together called **graded contractions** of $\mathcal{L}$. Since we are interested in all graded contractions, it will be useful to know how graded contractions of Lie algebra $\mathcal{L}$ which correspond to different gradings Γ and $\widetilde{\Gamma}$ are related to each other. In general case there is no known connection between such graded contractions. However, if there is a relation between gradings Γ and $\widetilde{\Gamma}$, then there is also a relation between the corresponding graded contractions. We will consider two types of relations, namely equivalence and refinement.

Let us take an automorphism $g \in \mathrm{Aut}(\mathcal{L})$ and suppose two equivalent gradings

$$\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i \qquad \text{and} \qquad \Gamma^g : \mathcal{L} = \bigoplus_{i \in I} g(L_i). \tag{3.14}$$

For any Γ-graded contraction $\mathcal{L}^\varepsilon$ of Lie algebra $\mathcal{L}$ we can construct a new Lie algebra $\mathcal{L}^\varepsilon_g$, where the Lie bracket $[\,,]^g_\varepsilon$ is defined by formula

$$[x, y]^g_\varepsilon = g[g^{-1}x, g^{-1}y]_\varepsilon, \qquad \forall x, y \in \mathcal{L}, \tag{3.15}$$

i.e. $g : \mathcal{L}^\varepsilon \longrightarrow \mathcal{L}^\varepsilon_g$ is an isomorphism of Lie algebras. Then for all $x \in g(L_i)$ and all $y \in g(L_j)$ we have

$$[x, y]^g_\varepsilon = g[g^{-1}x, g^{-1}y]_\varepsilon = \varepsilon_{ij} g[g^{-1}x, g^{-1}y] = \varepsilon_{ij}[x, y]. \tag{3.16}$$

Thus, $\mathcal{L}^\varepsilon_g$ is a Γ^g-graded contraction of $\mathcal{L}$ and any Γ-graded contraction is isomorphic to some of Γ^g-graded contractions. Moreover, $\mathsf{C}_\Gamma(\mathcal{L}) \subset \mathsf{C}_{\Gamma^g}(\mathcal{L})$ and considering Γ^g and mapping g^{-1} one can prove the reversed inclusion. Hence $\mathsf{C}_\Gamma(\mathcal{L}) = \mathsf{C}_{\Gamma^g}(\mathcal{L})$ for any $g \in \mathrm{Aut}(\mathcal{L})$. Since for any two equivalent gradings there exists an automorphism $g \in \mathrm{Aut}(\mathcal{L})$, we conclude that any two equivalent gradings lead to the same set of graded contractions (up to isomorphism). Therefore, we will consider only inequivalent gradings of Lie algebra $\mathcal{L}$.

Let us suppose now that grading

$$\widetilde{\Gamma} : \mathcal{L} = \bigoplus_{i \in I} \bigoplus_{j \in J_i} \widetilde{L}_j \tag{3.17}$$

is a refinement of the grading $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$, i.e. $L_i = \bigoplus_{j \in J_i} \widetilde{L}_j$ for all $i \in I$. Let $\mathcal{L}^\varepsilon$ be a Γ-graded contraction of $\mathcal{L}$. Then any Γ-graded contraction $\mathcal{L}^\varepsilon$ of $\mathcal{L}$ with Lie bracket

$$[x, y]_\varepsilon = \varepsilon_{ij}[x, y], \qquad \forall x \in L_i = \bigoplus_{k \in J_i} \widetilde{L}_k, \quad \forall y \in L_j = \bigoplus_{l \in J_j} \widetilde{L}_l, \tag{3.18}$$

is also a $\widetilde{\Gamma}$-graded contraction of $\mathcal{L}$. And any $\widetilde{\Gamma}$-graded contraction with contraction matrix $\widetilde{\varepsilon} \in \mathsf{C}_{\widetilde{\Gamma}}(\mathcal{L})$ is Γ-graded contraction if and only if for all $i, j \in I$ there exists $\varepsilon_{ij} \in \mathbb{C}$ such that

$$\widetilde{\varepsilon}_{kl} = \varepsilon_{ij}, \quad \forall k \in J_i, \forall l \in J_j. \tag{3.19}$$

Thus, considering the refinement $\widetilde{\Gamma}$ we get all Γ-graded contractions as contractions with contraction matrices which are formed by blocks of the same elements. Hence all graded contractions of given Lie algebra $\mathcal{L}$ can be obtained from fine gradings of $\mathcal{L}$ as mentioned above.

We can conclude that during the search of all graded contractions of a given Lie algebra it is sufficient to consider all inequivalent gradings which have no proper refinement. Therefore, in next chapters we will investigate only graded contractions of the Lie algebra $\mathrm{sl}(3, \mathbb{C})$ which correspond to its inequivalent fine gradings.

3.3 Successive graded contractions

A natural question which can arise is, what happens if graded contractions of graded contraction are investigated. Let us suppose that $\mathcal{L}^\varepsilon$ given by $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ is a Γ-graded contraction of Lie algebra $\mathcal{L} = \bigoplus_{i\in I} L_i$. Let $\mathcal{L}^{\varepsilon,\gamma}$ given by $\gamma \in \mathsf{C}_\Gamma(\mathcal{L}^\varepsilon)$ be a Γ-graded contraction of $\mathcal{L}^\varepsilon$. Then according to the definition of graded contractions we have for Lie brackets

$$[x,y]_{\varepsilon,\gamma} = \gamma_{ij}[x,y]_\varepsilon = \gamma_{ij}\varepsilon_{ij}[x,y], \qquad \forall x \in L_i, \forall y \in L_j,\ \forall i,j \in I. \tag{3.20}$$

Thus, $\mathcal{L}^{\varepsilon,\gamma}$ is also Γ-graded contraction of Lie algebra $\mathcal{L}$ and the corresponding contraction matrix is $\kappa = (\varepsilon_{ij}\gamma_{ij}) \in \mathsf{C}_\Gamma(\mathcal{L})$. This leads us to the definition of so called **elementwise matrix multiplication** $\bullet$ (also known as Hadamard product)

$$\kappa = \varepsilon \bullet \gamma \iff \kappa_{ij} = \varepsilon_{ij}\gamma_{ij}, \quad \forall i,j \in I. \tag{3.21}$$

Note that this multiplication is obviously commutative and associative.

Lemma 3.2. *Any Γ-graded contraction $\mathcal{L}^{\varepsilon,\gamma}$ of a Γ-graded contraction $\mathcal{L}^\varepsilon$ of Lie algebra $\mathcal{L}$ is a Γ-graded contraction $\mathcal{L}^{\varepsilon\bullet\gamma}$ of $\mathcal{L}$ i.e. $\varepsilon \bullet \gamma \in \mathsf{C}_\Gamma(\mathcal{L})$ for all $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ and all $\gamma \in \mathsf{C}_\Gamma(\mathcal{L}^\varepsilon)$.*

It remains to determine which of Γ-graded contractions of Lie algebra $\mathcal{L}$ are simultaneously Γ-graded contractions of $\mathcal{L}^\varepsilon$. $\mathcal{L}^\kappa$ is a Γ-graded contraction of $\mathcal{L}^\varepsilon$ if there exist $\gamma_{ij} \in \mathbb{C}$ such that for all $i,j \in I$ it holds

$$\kappa_{ij}[x,y] = [x,y]_\kappa = \gamma_{ij}[x,y]_\varepsilon = \gamma_{ij}\varepsilon_{ij}[x,y], \quad \forall x \in L_i, \forall y \in L_j. \tag{3.22}$$

If $i,j \in I$ is an irrelevant pair of indices for grading Γ of $\mathcal{L}$ then $\kappa_{ij} = \varepsilon_{ij} = \gamma_{ij} = 0$. If i,j is a relevant pair of indices then we have $\kappa_{ij} = \gamma_{ij}\varepsilon_{ij}$. If $\varepsilon_{ij} = 0$ then $\gamma_{ij} = 0$ (is irrelevant) and $\kappa_{ij} = 0$. On the other hand if $\varepsilon_{ij} \neq 0$ we can put $\gamma_{ij} = \kappa_{ij}\varepsilon_{ij}^{-1}$. Thus, the only condition on contraction matrix $\kappa \in \mathsf{C}_\Gamma(\mathcal{L})$ is $\kappa_{ij} = 0$ if $\varepsilon_{ij} = 0$. Therefore, we define the **support** of contraction matrix $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ as the set $S(\varepsilon)$ of all unordered pairs of grading indices $(i\,j)$ for which $\varepsilon_{ij} \equiv \varepsilon_{ji} \neq 0$

$$S(\varepsilon) = \left\{(i\,j) \in I_u^2 \,\middle|\, \varepsilon_{ij} \neq 0\right\} \tag{3.23}$$

and we conclude this section with following proposition.

Proposition 3.3. *All Γ-graded contractions of $\mathcal{L}^\varepsilon$ are all Γ-graded contractions $\mathcal{L}^\kappa$ of $\mathcal{L}$ for which contraction matrix $\kappa \in \mathsf{C}_\Gamma(\mathcal{L})$ fulfils $S(\kappa) \subset S(\varepsilon)$.*

Note that if the contraction system admits only two–term equations (3.12), then the set $\mathsf{C}_\Gamma(\mathcal{L})$ is closed with respect to elementwise matrix multiplication $\bullet$. Moreover, for any contraction matrix $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ so called **projection** $\widehat{\varepsilon}$ with property $\widehat{\varepsilon} \bullet \widehat{\varepsilon} = \widehat{\varepsilon}$ defined by

$$\widehat{\varepsilon}_{ij} = \langle \begin{array}{l} 0 \iff \varepsilon_{ij} = 0 \\ 1 \iff \varepsilon_{ij} \neq 0 \end{array} \qquad \forall i, j \in I, \tag{3.24}$$

is also a contraction matrix in $\mathsf{C}_\Gamma(\mathcal{L})$ and each subset of $\mathsf{C}_\Gamma(\mathcal{L})$ formed by all contraction matrices with the same support is an abelian group with respect to multiplication $\bullet$.

3.4 Trivial contractions

Observing the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ in the form (3.11) one can immediately write some of its solutions. These solutions are contraction matrices $\varepsilon = (c)$ which have all relevant contraction parameters equal to the same complex number $c \in \mathbb{C}$. However, these contractions lead, as we will see further on, only to two types of Lie algebras. If $c = 0$, then we get abelian Lie algebra and if $c \neq 0$, we get Lie algebra isomorphic with $\mathcal{L}$. Therefore, these solutions of $\mathsf{S}_\Gamma(\mathcal{L})$ are called **trivial solutions**. Similarly, we will say that graded contraction $\mathcal{L}^\varepsilon$ of Lie algebra $\mathcal{L}$ is **trivial** if $\mathcal{L}^\varepsilon \cong \mathcal{L}$ or $\mathcal{L}^\varepsilon$ is abelian.

An important example of trivial Γ-graded contraction is given by a regular linear map $h : \mathcal{L} \longrightarrow \mathcal{L}$ for which grading subspaces are eigen–subspaces. Let us assume that h is defined on grading subspaces $L_i, i \in I$, by relations

$$hx = a_i x, \qquad \forall x \in L_i, \tag{3.25}$$

where $0 \neq a_i \in \mathbb{C}$ are arbitrary nonzero complex numbers. We define a new Lie bracket $[\,,]_\alpha$ on the vector space $\mathcal{L}$ as follows

$$[x, y]_\alpha = h^{-1}[hx, hy], \quad \forall x, y \in \mathcal{L}, \tag{3.26}$$

and obtain a new Lie algebra $\mathcal{L}^\alpha$ which is isomorphic to $\mathcal{L}$ via h. This corresponds to change (renormalization) of basis of grading subspaces $L_i \to a_i L_i$. For all relevant pairs of indices $i, j \in I$ we have

$$[x, y]_\alpha = h^{-1}[hx, hy] = \frac{a_i a_j}{a_{i\diamond j}}[x, y], \quad \forall x \in L_i, \forall\, y \in L_j, \tag{3.27}$$

and therefore, $\mathcal{L}^\alpha$ is Γ-graded contraction of $\mathcal{L}$ with the contraction matrix $\alpha = (\alpha_{ij}) \in \mathsf{C}_\Gamma(\mathcal{L})$ where relevant contraction parameters are

$$\alpha_{ij} = \frac{a_i a_j}{a_{i\diamond j}}, \qquad \forall i, j \in I. \tag{3.28}$$

This matrix α is called **normalization matrix**. The importance of normalization matrix lies in the fact that it will enable us to classify solutions of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$.

Let us consider a Γ-grading $\mathcal{L}^\varepsilon$ of the Lie algebra $\mathcal{L}$. We use regular map h given by (3.25) and define a new Lie bracket $[\,,]_\mu$ on $\mathcal{L}$ as follows

$$[x,y]_\mu = h^{-1}[hx,hy]_\varepsilon, \qquad \forall x,y \in \mathcal{L}. \tag{3.29}$$

We get Lie algebra $\mathcal{L}^\mu$ isomorphic with $\mathcal{L}^\varepsilon$. Then similarly as in (3.27) we have for all $i,j \in I$

$$[x,y]_\mu = h^{-1}[hx,hy]_\varepsilon = \frac{a_i a_j}{a_{i\diamond j}}[x,y]_\varepsilon = \frac{a_i a_j}{a_{i\diamond j}}\varepsilon_{ij}[x,y], \quad \forall x \in L_i, \forall y \in L_j. \tag{3.30}$$

Therefore, $\mathcal{L}^\mu$ is also Γ-grading of the Lie algebra $\mathcal{L}$ and the corresponding contraction matrix is $\mu = \alpha \bullet \varepsilon$. Taking the inverse of the map h we get that $\varepsilon = \widetilde{\alpha} \bullet \mu$, where $\widetilde{\alpha}$ is a normalization matrix with relevant parameters of the form

$$\widetilde{\alpha}_{ij} = \frac{a_{i\diamond j}}{a_i a_j}. \tag{3.31}$$

Proposition 3.4. *Let $\mathcal{L}^\varepsilon$ be a Γ-graded contraction of Lie algebra $\mathcal{L} = \bigoplus_{i\in I} L_i$. Then $\mathcal{L}^\mu$, where $\mu = \alpha \bullet \varepsilon$, is for any normalization matrix $\alpha = (\frac{a_i a_j}{a_{i\diamond j}})$, $a_i \neq 0, \forall i \in I$ a Γ-graded contraction of $\mathcal{L}$ and the Lie algebras $\mathcal{L}^\mu$ and $\mathcal{L}^\varepsilon$ are isomorphic.*

Thus, any two solutions ε and μ of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ for which there exists a normalization matrix α such that $\mu = \alpha \bullet \varepsilon$ lead to isomorphic Γ-graded contractions. Moreover, the condition of existence of normalization matrix defines an equivalence on the set $\mathsf{C}_\Gamma(\mathcal{L})$ of all solutions of contraction system $\mathsf{S}_\Gamma(\mathcal{L})$. Two contractions $\varepsilon, \mu \in \mathsf{C}_\Gamma(\mathcal{L})$ are called **strongly equivalent** $\varepsilon \approx \mu$ if there exists a normalization matrix α such that $\varepsilon = \alpha \bullet \mu$, i.e. if there exist nonzero complex numbers $a_i \in \mathbb{C}, \forall i \in I$ such that

$$\varepsilon_{ij} = \frac{a_i a_j}{a_{i\diamond j}}\mu_{ij}, \qquad \forall i,j \in I. \tag{3.32}$$

Since we will be interested in non-isomorphic results only, we will normalize every solution — replace it by strongly equivalent solution which consists of as many 0's and 1's as possible. For example, any trivial nonzero solution is normalized to the solution which has all relevant contraction parameters equal to 1.

3.5 Continuous and discrete graded contractions

All Γ-graded contractions of the given Lie algebra $\mathcal{L}$ can be divided into two types. A solution $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ is called **continuous** if there exists a continuous set of solutions $\varepsilon(t) \in \mathsf{C}_\Gamma(\mathcal{L})$,

$0 < t \leq 1$ such that for all relevant contraction parameters one has

$$\varepsilon_{ij}(1) = 1, \quad \varepsilon_{ij}(t) \neq 0, \quad \varepsilon_{ij} = \lim_{t \to 0} \varepsilon_{ij}(t). \tag{3.33}$$

A solution which is not continuous is called **discrete**.

Continuous graded contractions are related to the classical continuous contractions [43, 55, 67, 69, 75]. For example, if for given continuous graded contraction $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$, the corresponding set of solutions has the form

$$\varepsilon_{ij} = t^{n_i + n_j - n_{i \diamond j}} \qquad n_i, n_j, n_{i\diamond j} \in \mathbb{Z}, \quad i, j \in I, \tag{3.34}$$

then ε is so called **generalized Inönü–Wigner contraction** [74]. However, discrete graded contractions have no equivalent in classical continuous contraction [77].

An efficient tool for distinguishing between continuous and discrete graded contractions was developed in [74]. So called higher–order identities allow us to identify discrete graded contractions. Let us consider an equation where on both sides stand products of r relevant contraction parameters. If this equation is satisfied for all contractions $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ for which all relevant contraction parameters do not vanish, it will be also satisfied for any limit of these solutions, i.e. for any continuous solution. Thus, we call any equation "$P_{\mathsf{i}} = P_{\mathsf{j}}$" of the type

$$P_{\mathsf{i}}(\varepsilon) := \varepsilon_{i_1}\varepsilon_{i_2}\ldots\varepsilon_{i_r} = \varepsilon_{j_1}\varepsilon_{j_2}\ldots\varepsilon_{j_r} =: P_{\mathsf{j}}(\varepsilon), \tag{3.35}$$

where $r \in \mathbb{N}$ and $\mathsf{i} = \{i_1, i_2, \ldots, i_r\}, \mathsf{j} = \{j_1, j_2, \ldots, j_r\}$ are disjoint sets of relevant pairs of grading indices, **higher–order identity of order** r, if it holds for all contraction matrices ε without zeros on all relevant positions, but is violated by some contraction matrix with zero on some relevant position. There are two possible types of violation of a given higher–order identity "$P_{\mathsf{i}} = P_{\mathsf{j}}$". We say that contraction $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ **violates** higher–order identities "$P_{\mathsf{i}} = P_{\mathsf{j}}$"

- **strongly** if $0 \neq P_{\mathsf{i}}(\varepsilon) \neq P_{\mathsf{j}}(\varepsilon) \neq 0$,
- **weakly** if $0 = P_{\mathsf{i}}(\varepsilon) \neq P_{\mathsf{j}}(\varepsilon)$ or $P_{\mathsf{i}}(\varepsilon) \neq P_{\mathsf{j}}(\varepsilon) = 0$.

Proposition 3.5. *Let a solution $\varepsilon \in \mathsf{C}_\Gamma(L)$ of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ be a continuous graded contraction. Then ε satisfies all higher–order identities.*

Higher–order identities can be constructed from the two term equations of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ by multiplying these equations and reducing all terms which are common for both sides of resulting equation. Another possibility is to construct higher–order identities from the knowledge of the explicit form of solutions without zeros on relevant positions.

Let us mention that in all cases of graded contractions of $\mathrm{sl}(3, \mathbb{C})$ which will be presented in next chapters it holds:

- All contraction matrices without zeros on relevant positions are normalization matrices.
- All continuous graded contractions are generalized Inönü–Wigner contractions (3.34).
- Any discrete solution of $\mathsf{S}_\Gamma(\mathcal{L})$ violates at least one of presented higher-order identities.

3.6 Action of the symmetry group of the grading

In previous chapter we have found that there is the symmetry group $\mathrm{Aut}(\Gamma)$ (2.8) which leaves the grading $\Gamma : \mathcal{L} = \bigoplus_{i\in I} L_i$ of the Lie algebra $\mathcal{L}$ unchanged. Now we will show that the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ is also preserved under certain action of this symmetry group or rather its permutation representation $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ (2.12).

First we recall some basic definitions concerning group actions. For a group G and a set $X \neq \emptyset$ a mapping $\psi : G \times X \longrightarrow X$ is called **left** or **right action** of the group G on the set X if for all $x \in X$ it holds

1. $\forall g, h \in G, \quad \psi(gh, x) = \psi(g, \psi(h, x))$ or $\psi(gh, x) = \psi(h, \psi(g, x))$, respectively, and
2. $\psi(e, x) = x$, where $e \in G$ is the unit element.

If ψ is an action of G on X then the relation $a \equiv_\psi b \Leftrightarrow \exists g \in G,\ \psi(g, a) = b$ is an equivalence on the set X and the equivalence class corresponding to element $a \in X$

$$[a]_\psi = \{b \in X \,|\, b \equiv_\psi a\} = \{b \in X \,|\, \exists g \in G,\ b = \psi(g, a)\} \tag{3.36}$$

is called an **orbit** of $a \in X$.

Any contraction equation $e(i\,j\,k)$ (3.8) is determined by unordered triplet $(i\,j\,k) \in I_u^3$ of grading indices. Variables of these equations, i.e. relevant contraction parameters $\varepsilon_{ij} = \varepsilon_{ji}$, are determined by unordered pairs $(i\,j) \in I_u^2$. Therefore, we denote the **set of all relevant unordered pairs of grading indices** $i, j \in I$ as

$$\mathcal{I} = \big\{(i\,j) \in I_u^2 \,|\, [L_i, L_j] \neq 0\big\} \tag{3.37}$$

and the **set of relevant contraction parameters** as

$$\mathcal{E} = \{\varepsilon_k \,|\, k \in \mathcal{I}\}\,. \tag{3.38}$$

Since we will work mainly with unordered n-tuples of grading indices, we define a left action of S_I (the symmetry group of the index set I) on the set I_u^n of all unordered n-tuples of grading indices. Any permutation $\pi \in S_I$ acts on unordered n-tuple $(x_1\, x_2\, \ldots\, x_n) \in I_u^n$ as follows

$$\pi(x_1\, x_2\, \ldots\, x_n) = (\pi(x_1)\, \pi(x_2)\, \ldots\, \pi(x_1)). \tag{3.39}$$

We will be interested in restrictions of this action to subgroup $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ of S_I and subsets of I_u^n. Let us consider any permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ and corresponding automorphism $g \in \mathrm{Aut}(\Gamma)$ i.e. $\pi = \Delta_\Gamma(g)$. Let $(i\, j) \in \mathcal{I}$ be an unordered relevant pair of grading indices, then

$$[L_{\pi(i)}, L_{\pi(j)}] = [gL_i, gL_j] = g[L_i, L_j] \neq 0. \tag{3.40}$$

Thus, $(\pi(i)\, \pi(j))$ is also an unordered relevant pair of grading indices and (3.39) is well defined action of the group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ on the set $\mathcal{I}$. Moreover,

$$L_{\pi(i)\diamond\pi(j)} \supset [L_{\pi(i)}, L_{\pi(j)}] = [gL_i, gL_j] = g[L_i, L_j] \subset g(L_{i\diamond j}) = L_{\pi(i\diamond j)}. \tag{3.41}$$

And therefore, it holds for all relevant pairs of grading indices $i, j \in I$

$$\pi(i \diamond j) = \pi(i) \diamond \pi(j). \tag{3.42}$$

Clearly, if $(i\, j)$ is irrelevant pair, then also $(\pi(i)\, \pi(j))$ is an irrelevant pair for any $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$.

For any permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ and any contraction matrix $\varepsilon = (\varepsilon_{ij})$ we define an **action of π on the contraction matrix** $\varepsilon \mapsto \varepsilon^\pi$ by formula

$$(\varepsilon^\pi)_{ij} = \varepsilon_{\pi(i)\pi(j)}, \qquad \forall i, j \in I. \tag{3.43}$$

We observe that the action on variables $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ is, due to (3.40), an action of the group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ on the set of all relevant contraction parameters $\mathcal{E}$. Since all irrelevant contraction parameters are equal to zero, the action (3.43) leaves irrelevant positions $I_u^2 \backslash \mathcal{I}$ of matrix ε unchanged. We prove now that (3.43) is well defined action of the group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ on the set of all contraction matrices $\mathsf{C}_\Gamma(\mathcal{L})$.

Consider any permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ and automorphism $g \in \mathrm{Aut}(\Gamma)$ such that $g(L_i) = L_{\pi(i)}$ for all $i \in I$. Then for any contraction matrix $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ equation

$$[x, y]_{\varepsilon^\pi} = \varepsilon_{\pi(i)\pi(j)}[x, y] = \varepsilon_{\pi(i)\pi(j)} g^{-1}[gx, gy] = g^{-1}[gx, gy]_\varepsilon \tag{3.44}$$

holds for all $x \in L_i, y \in L_j$ and $i, j \in I$. Thus, g is an isomorphism between Lie algebras $\mathcal{L}^{\varepsilon^\pi}$ and $\mathcal{L}^\varepsilon$. And it follows from the first equality of (3.44) that $\varepsilon^\pi \in \mathsf{C}_\Gamma(\mathcal{L})$. Thus, we have proved the following proposition.

Proposition 3.6. *Let $\mathcal{L}^\varepsilon$ be a Γ-graded contraction of a Γ-graded Lie algebra $\mathcal{L} = \bigoplus_{i\in I} L_i$ and $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$. Then $\mathcal{L}^{\varepsilon^\pi}$ is also a Γ-graded contraction of $\mathcal{L}$ and the Lie algebras $\mathcal{L}^\varepsilon$ and $\mathcal{L}^{\varepsilon^\pi}$ are isomorphic, $\mathcal{L}^{\varepsilon^\pi} \cong \mathcal{L}^\varepsilon$.*

Let us focus on the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$. We have shown that for any permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ the substitution $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ preserves the set $\mathsf{C}_\Gamma(\mathcal{L})$ of solutions of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$. Now we verify that it preserves also the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$. We write any contraction equation $e(i\,j\,k) \in \mathsf{S}_\Gamma(\mathcal{L})$ in the form

$$e(i\,j\,k):\ \varepsilon_{jk}\varepsilon_{i,j\diamond k}[x,[y,z]] + \text{cyclically} = 0, \quad \forall x \in L_i, \forall y \in L_j, \forall z \in L_k. \tag{3.45}$$

For any permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ and contraction equation $e(i\,j\,k) \in \mathsf{S}_\Gamma(\mathcal{L})$ we define the action

$$e(i\,j\,k) \mapsto e(\pi(i)\ \pi(j)\ \pi(k)). \tag{3.46}$$

Taking $g \in \mathrm{Aut}(\Gamma)$ which corresponds to π, i.e. $g(L_i) = L_{\pi(i)}, \forall i \in I$, we can write contraction equation $e(\pi(i)\ \pi(j)\ \pi(k))$ in the form

$$e(\pi(i)\ \pi(j)\ \pi(k)):\ \varepsilon_{\pi(j)\pi(k)}\varepsilon_{\pi(i),\pi(j)\diamond\pi(k)}[gx,[gy,gz]] + \text{cyclically} = 0, \tag{3.47}$$

for all $x \in L_i, y \in L_j, z \in L_k$. Since g is an automorphism of $\mathcal{L}$ and for all relevant pairs of grading indices (3.42) holds (terms with irrelevant grading indices do not appear in contraction equations) we have

$$g(\varepsilon_{\pi(j)\pi(k)}\varepsilon_{\pi(i),\pi(j\diamond k)}[x,[y,z]] + \text{cyclically}) = 0, \quad \forall x \in L_i, \forall y \in L_j, \forall z \in L_k. \tag{3.48}$$

And this equation is equivalent to

$$\varepsilon_{\pi(j)\pi(k)}\varepsilon_{\pi(i),\pi(j\diamond k)}[x,[y,z]] + \text{cyclically} = 0, \quad \forall x \in L_i, \forall y \in L_j, \forall z \in L_k, \tag{3.49}$$

which is exactly the equation $e(i\,j\,k)$ where the substitution $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ is effected. Thus, we have verified the invariance of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ (up to equivalence of solutions) under the substitution of contraction parameters $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$. Moreover, this leads us to the new method of construction of the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$. Namely, having one equation, we can generate a whole orbit of equations merely by substituting $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ till all permutations π from $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ are exhausted. Since any equation is determined by unordered triplet of grading indices, these orbits of equations correspond to the orbits in I_u^3 with respect to the action of the group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$. Thus, **construction of the system of contraction equations** consists in following three steps

- Determine orbits in I_u^3 with respect to the action (3.39) of the group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$.

- Choose the representatives of these orbits and generate contraction equations (3.8).
- From each of these equations generate the orbit by substitutions $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$, where π runs through whole group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$.

3.7 Equivalence of contraction matrices

We have seen, in Proposition 3.4, that strongly equivalent contraction matrices lead to isomorphic Lie algebras as well as any two permutations $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ of contraction matrix, according to Proposition 3.6. This leads us to the following definition of equivalence of contraction matrices. Two contraction matrices $\varepsilon_1, \varepsilon_2 \in \mathsf{C}_\Gamma(\mathcal{L})$ are called **equivalent**, $\varepsilon_1 \sim \varepsilon_2$, if there exits a normalization matrix α and permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ such that

$$\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi. \tag{3.50}$$

In order to verify that the relation $\sim$ is well defined equivalence, we firstly notice that the "inversion" $\widetilde{\alpha}$ and every permutation α^π, $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ of any normalization matrix α

$$\alpha_{ij} = \frac{a_i a_j}{a_{i\diamond j}}, \quad \widetilde{\alpha}_{ij} = \frac{a_{i\diamond j}}{a_i a_j}, \quad \alpha^\pi_{ij} = \frac{a_{\pi(i)}a_{\pi(j)}}{a_{\pi(i)\diamond\pi(j)}}, \quad \forall (i\, j) \in I,\ 0 \neq a_i \in \mathbb{C},\ \forall i \in I \tag{3.51}$$

are also normalization matrices. Taking identical permutation $\pi = \mathrm{Id}$ and normalization matrix α generated by numbers $a_i = 1, \forall i \in I$ we get reflexivity of $\sim$, i.e. that $\varepsilon \sim \varepsilon$ for all $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$. If we multiply the relation (3.50) by normalization matrix $\widetilde{\alpha}$, we get

$$\widetilde{\alpha} \bullet \varepsilon_1 = \widetilde{\alpha} \bullet \alpha \bullet \varepsilon_2^\pi = \varepsilon_2^\pi. \tag{3.52}$$

Considering inverse permutation π^{-1} we have

$$\varepsilon_2 = (\varepsilon_2^\pi)^{\pi^{-1}} = (\widetilde{\alpha} \bullet \varepsilon_1)^{\pi^{-1}} = (\widetilde{\alpha})^{\pi^{-1}} \bullet \varepsilon_1^{\pi^{-1}}. \tag{3.53}$$

Thus, the relation $\sim$ is also symmetric. Suppose now that $\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi$ and $\varepsilon_2 = \beta \bullet \varepsilon_3^\sigma$ then

$$\varepsilon_1 = \alpha \bullet \varepsilon_2 = \alpha \bullet (\beta \bullet \varepsilon_3^\sigma)^\pi = \alpha \bullet \beta^\pi \bullet (\varepsilon_3^\sigma)^\pi = (\alpha \bullet \beta^\pi) \bullet \varepsilon_3^{(\pi\sigma)}. \tag{3.54}$$

Since $(\alpha \bullet \beta^\pi)$ is normalization matrix and $\pi\sigma \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$, we have verified also transitivity of $\sim$. Thus, relation $\sim$ defined by (3.50) is an equivalence relation and combining propositions 3.4 and 3.6 we get the following result.

Proposition 3.7. *Let $\varepsilon_1, \varepsilon_2 \in \mathsf{C}_\Gamma(\mathcal{L})$ be equivalent contraction matrices, i.e. $\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi$ for some normalization matrix α and some permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$, then Lie algebras $\mathcal{L}^{\varepsilon_1}$ and $\mathcal{L}^{\varepsilon_2}$ are isomorphic, i.e.*

$$\varepsilon_1 \sim \varepsilon_2 \Longrightarrow \mathcal{L}^{\varepsilon_1} \cong \mathcal{L}^{\varepsilon_2}. \tag{3.55}$$

Let us emphasize that the implication (3.55) can not be reversed. There exist Γ-graded contractions which are isomorphic in spite of the fact that their corresponding contraction matrices are not equivalent. Note that in some works [74, 76] the equivalence of contraction matrices is defined in the same manner as our strong equivalence (3.32).

We discuss now some properties of equivalent and strongly equivalent contractions. Any two strongly equivalent contractions $\varepsilon_1 \approx \varepsilon_2$, $(\varepsilon_1 = \alpha \bullet \varepsilon_2)$ are also equivalent and have the same support $S(\varepsilon_1) = S(\varepsilon_2)$. We show that they also violate the same higher–order identities in the same way. If "$P_{\mathsf{i}} = P_{\mathsf{j}}$" is higher order identity then

$$P_{\mathsf{i}}(\varepsilon_1) = P_{\mathsf{i}}(\alpha \bullet \varepsilon_2) = P_{\mathsf{i}}(\alpha)P_{\mathsf{i}}(\varepsilon_2) = P_{\mathsf{j}}(\alpha)P_{\mathsf{j}}(\varepsilon_2) = P_{\mathsf{j}}(\alpha \bullet \varepsilon_2) = P_{\mathsf{j}}(\varepsilon_1)$$

and since α fulfills all higher–order identities we have

$$P_{\mathsf{i}}(\varepsilon_1) = P_{\mathsf{j}}(\varepsilon_1) \iff P_{\mathsf{i}}(\varepsilon_2) = P_{\mathsf{j}}(\varepsilon_2).$$

Moreover, if ε_1 violates "$P_{\mathsf{i}} = P_{\mathsf{j}}$" strongly, then

$$\frac{P_{\mathsf{i}}(\varepsilon_1)}{P_{\mathsf{j}}(\varepsilon_1)} = \frac{P_{\mathsf{i}}(\alpha)P_{\mathsf{i}}(\varepsilon_2)}{P_{\mathsf{j}}(\alpha)P_{\mathsf{j}}(\varepsilon_2)} = \frac{P_{\mathsf{i}}(\varepsilon_2)}{P_{\mathsf{j}}(\varepsilon_2)} \neq 0 \tag{3.56}$$

and the complex number $\frac{P_{\mathsf{i}}(\varepsilon_1)}{P_{\mathsf{j}}(\varepsilon_1)}$ is the same for all contractions strongly equivalent with ε_1.

Let us now consider two equivalent contractions $\varepsilon_1, \varepsilon_2 \in \mathsf{C}_\Gamma(\mathcal{L})$, $(\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi)$. These contractions have generally different supports $S(\varepsilon_1) \neq S(\varepsilon_2)$ with the same number of elements $|S(\varepsilon_1)| = |S(\varepsilon_2)|$. Therefore, we define the **number of zeros** in contraction matrix ε as the number of relevant contraction parameters which are equal to zero $\nu(\varepsilon) = |\mathcal{E}| - |S(\varepsilon)|$. This number of zeros is the same for all equivalent contractions. Moreover, we show that equivalent contractions $\varepsilon_1 \sim \varepsilon_2$ violate the same number of higher–order identities in the same way. Any higher–order identity "$P_{\mathsf{i}} = P_{\mathsf{j}}$" is determined by sets $\mathsf{i} = \{i_1, \dots, i_r\}, \mathsf{j} = \{j_1, \dots, j_r\}$ of unordered relevant pairs of indices. If we define an action of permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\mathcal{L}))$ on the set i as $\pi(\mathsf{i}) = \{\pi(i_1), \dots, \pi(i_r)\}$, we can write for both sides of "$P_{\mathsf{i}} = P_{\mathsf{j}}$"

$$P_{\mathsf{i}}(\varepsilon_2^\pi) = (\varepsilon_2^\pi)_{i_1} \dots (\varepsilon_2^\pi)_{i_r} = (\varepsilon_2)_{\pi^{-1}(i_1)} \dots (\varepsilon_2)_{\pi^{-1}(i_r)} = P_{\pi^{-1}(\mathsf{i})}(\varepsilon_2) \tag{3.57}$$

and since ε_2^π and ε_1 are strongly equivalent, we have

$$P_{\mathsf{i}}(\varepsilon_1) = P_{\mathsf{j}}(\varepsilon_1) \iff P_{\mathsf{i}}(\varepsilon_2^\pi) = P_{\mathsf{j}}(\varepsilon_2^\pi) \iff P_{\pi^{-1}(\mathsf{i})}(\varepsilon_2) = P_{\pi^{-1}(\mathsf{j})}(\varepsilon_2). \tag{3.58}$$

It follows from the relation (3.58) that having one higher–order identity "$P_{\mathsf{i}} = P_{\mathsf{j}}$" we can generate whole orbit of higher–order identities "$P_{\pi(\mathsf{i})} = P_{\pi(\mathsf{j})}$" where $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$.

We have shown that two equivalent solutions violate higher–order identities and thus fulfil sufficient condition of discreteness at the same time. Let us now consider that $\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi$

and ε_2 is continuous, i.e. there exists continuous family of contraction matrices $\varepsilon(t) \in \mathsf{C}_\Gamma(\mathcal{L})$, $0 < t \leq 1$ without zeros on relevant positions such that $\varepsilon_2 = \lim_{t\to 0} \varepsilon(t)$. Then, clearly, $\alpha \bullet \varepsilon^\pi(t) \in \mathsf{C}_\Gamma(\mathcal{L})$, $0 < t \leq 1$ is also an continuous family of contraction matrices without zeros and $\varepsilon_1 = \alpha \bullet \varepsilon_2^\pi = \lim_{t\to 0} \alpha \bullet \varepsilon^\pi(t)$, thus ε_1 is also continuous contraction.

Proposition 3.8. *Any two equivalent solutions have the same number of zeros, violate the same number of higher–order identities in the same way and are simultaneously discrete or continuous. Moreover, strongly equivalent solutions have the same support and violate the same higher–order identities in the same way.*

3.8 Solving the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$

In this section we present the algorithm based on the following theorem from [40] for solving the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$.

Theorem 3.9. *Let $\mathsf{C}_\Gamma(\mathcal{L})$ be the set of all contraction matrices, $\mathcal{I}$ the set of relevant unordered pairs of grading indices, $\mathsf{S}_\Gamma(\mathcal{L})$ the contraction system for Γ-graded Lie algebra $\mathcal{L} = \bigoplus_{i\in I} L_i$. For any subset $\mathcal{Q} \subset \mathsf{C}_\Gamma(\mathcal{L})$ and $\mathcal{P} = \{k_1, k_2, \ldots, k_m\} \subset \mathcal{I}$ we denote*

$$\begin{aligned}\mathcal{R}_0 &= \{\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L}) \,|\, (\forall \varepsilon' \in \mathcal{Q})(\varepsilon \nsim \varepsilon')\}\,,\\ \mathcal{R}^0 &= \{\varepsilon \in \mathcal{R}_0 \,|\, (\forall k \in \mathcal{P})(\varepsilon_k \neq 0)\}\,.\end{aligned}$$

Then a solution $\varepsilon \in \mathcal{R}_0$ is not equivalent to any solution in $\mathcal{R}^0$ if and only if the following system of equations holds:

$$\begin{aligned}\varepsilon_{\pi_1(k_1)}\varepsilon_{\pi_1(k_2)} \ldots \varepsilon_{\pi_1(k_m)} &= 0\\ &\vdots\\ \varepsilon_{\pi_n(k_1)}\varepsilon_{\pi_n(k_2)} \ldots \varepsilon_{\pi_n(k_m)} &= 0;\end{aligned} \tag{3.59}$$

here the set of permutations $\{\pi_1, \pi_2, \ldots, \pi_n\}$ exhaust all elements of the symmetry group $\mathrm{Aut}(\Gamma)$ of the grading Γ.

Proof. Consider $\varepsilon \in \mathcal{R}_0$ and suppose that there exists $\varepsilon' \in \mathcal{R}^0$ such that $\varepsilon' \sim \varepsilon$, i.e. there exist normalization matrix α and permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ such that $\varepsilon' = \alpha \bullet \varepsilon^\pi \in \mathcal{R}^0$. Thus, $(\alpha \bullet \varepsilon^\pi)_k \neq 0$ for all $k \in \mathcal{P}$ and, since $\alpha_k \neq 0$ for all relevant k, this is equivalent to $(\varepsilon^\pi)_k = \varepsilon_{\pi(k)} \neq 0$ for all $k \in \mathcal{P}$. Hence $\varepsilon \in \mathcal{R}_0$ is equivalent to some solution in $\mathcal{R}^0$ if and only if there exists $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ such that for all $k \in \mathcal{P}$ it holds $\varepsilon_{\pi(k)} \neq 0$. Therefore, $\varepsilon \in \mathcal{R}_0$ is not equivalent to any solution in $\mathcal{R}^0$ if and only if for any permutation $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ there exists $k \in \mathcal{P}$ such that $\varepsilon_{\pi(k)} = 0$, i.e. if and only if equations (3.59) are satisfied. □

The system of equations (3.59) is called **non–equivalence system** for the sets $\mathcal{Q} \subset \mathsf{C}_\Gamma(\mathcal{L})$ and $\mathcal{P} \subset \mathcal{I}$ and is, as well as the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$, invariant under the substitution of contraction parameters $\varepsilon_{ij} \mapsto \frac{\alpha_i \alpha_i}{\alpha_{i \diamond j}} \varepsilon_{\pi(i)\pi(j)}$, where $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ and $\alpha_i \neq 0$. For any system S of equations we denote $\mathcal{R}(S)$ the set of all its solutions. Repeated use of the Theorem 3.9 leads to the following algorithm for evaluation of solutions:

0. We start with $\mathcal{Q}_0 = \emptyset$ and the set of assumptions $\mathcal{P}^0 \subset \mathcal{I}$. Then $\mathcal{R}_0 = \mathsf{C}_\Gamma(\mathcal{L})$ and we compute all solutions which have nonzeros on assumed positions $\mathcal{P}^0$
$$\mathcal{R}^0 = \left\{ \varepsilon \in \mathsf{C}_\Gamma(\mathcal{L}) \,|\, (\forall k \in \mathcal{P}^0)(\varepsilon_k \neq 0) \right\}$$
and write the non–equivalence system $\mathcal{S}^0$ (3.59) for the sets $\mathcal{Q}_0 = \emptyset$ and $\mathcal{P}^0$.

1. We set $\mathcal{Q}_1 = \mathcal{R}^0$ and take assumptions $\mathcal{P}^1 \subset \mathcal{I}$, $\mathcal{P}^1 \neq \mathcal{P}^0$. Then
$$\mathcal{R}_1 = \left\{ \varepsilon \in \mathsf{C}_\Gamma(\mathcal{L}) \,|\, (\forall \varepsilon' \in \mathcal{R}^0)(\varepsilon \not\sim \varepsilon') \right\} = \mathcal{R}(\mathsf{S}_\Gamma(\mathcal{L}) \cup \mathcal{S}^0)$$
is the set of all solutions of the system $\mathsf{S}_\Gamma(\mathcal{L}) \cup \mathcal{S}^0$. Furthermore, we explicitly evaluate
$$\mathcal{R}^1 = \left\{ \varepsilon \in \mathcal{R}(\mathsf{S}_\Gamma(\mathcal{L}) \cup \mathcal{S}^0) \,|\, (\forall k \in \mathcal{P}^1)(\varepsilon_k \neq 0) \right\}$$
and write the non–equivalence system $\mathcal{S}^1$ (3.59) for the sets $\mathcal{Q}_1 = \mathcal{R}^0$ and $\mathcal{P}^1$.

2. We set $\mathcal{Q}_2 = \mathcal{R}^0 \cup \mathcal{R}^1$ and take assumptions $\mathcal{P}^2 \subset \mathcal{I}$. Then $\mathcal{R}_2 = \mathcal{R}(\mathsf{S}_\Gamma(\mathcal{L}) \cup \mathcal{S}^0 \cup \mathcal{S}^1)$ and we evaluate the set $\mathcal{R}^2 = \{\varepsilon \in \mathcal{R}_2 \,|\, (\forall k \in \mathcal{P}^2)(\varepsilon_k \neq 0)\}$ and write non–equivalence system $\mathcal{S}^2$. We continue till we have evaluated the whole $\mathsf{C}_\Gamma(\mathcal{L})$, i.e. till for some $n \in \mathbb{N}$ the set $\mathcal{R}_n$ is empty or consists only of trivial solution $\varepsilon = (0)$. Then all solutions of $\mathsf{S}_\Gamma(\mathcal{L})$ (up to equivalence) are collected in sets $\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^2, \ldots, \mathcal{R}^{n-1}$ and $\mathcal{R}_n$.

The efficiency of this algorithm depends on the choice of the assumption sets $\mathcal{P}^0, \mathcal{P}^1, \ldots, \subset \mathcal{I}$. For example taking $\mathcal{P}^0 = \{k\} \subset \mathcal{I}$ we get $\mathcal{S}^0 = \left\{ \varepsilon_{\pi(k)} = 0 \,|\, \pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma)) \right\}$, i.e. solutions inequivalent with those in $\mathcal{R}^0$ must have zeros on the whole orbit of relevant contraction parameters represented by ε_k. Clearly, if $\mathcal{I}$ forms only one orbit with respect to the action of $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$, then there is only one such solution — the trivial solution $\varepsilon = (0)$. Therefore, it is more convenient to take the sets of assumptions consisting of at least two elements.

3.9 Graded contraction summary

We summarize the concept of graded contractions into the algorithm which was used for their computation. Let us suppose that we have grading $\Gamma : \mathcal{L} = \bigoplus_{i \in I} L_i$ of a complex Lie algebra $\mathcal{L}$ of finite dimension $n \in \mathbb{N}$. For computing it is more convenient to replace the set I of grading indices by the set of natural numbers $\{1, \ldots, m\}$.

1. We take the permutation representation $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ of the symmetry group $\mathrm{Aut}(\Gamma)$ of the grading Γ and construct orbits of its action on the set of unordered triplets of grading indices I_u^3.

2. From each orbit we choose one representative - unordered triplet $(i\,j\,k)$ and compute corresponding contraction equation $e(i\,j\,k)$, i.e. a set of equations (3.8) generated by all basis elements of grading subspaces L_i, L_j and L_k. Then we use group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ and generate from these contraction equations the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ (3.13).

3. Using Theorem 3.9 we solve the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ with results stored in sets $\mathcal{R}^0, \mathcal{R}^1, \ldots$. Each set $\mathcal{R}^i$ consists of all solutions of the contraction system and some non–equivalence system under certain assumptions.

4. Since equivalent contraction matrices have the same number of vanishing elements (elements which are equal to zero), we discuss when elements of contraction matrix $\varepsilon \in \mathcal{R}^i$ vanish and divide sets $\mathcal{R}^i$ into the subsets $\mathcal{R}^i_j$ according to numbers $j \in \mathbb{N}_0$ of zeros in contraction matrices ε.

5. In order to get non–equivalent contraction matrices, we collect in sets $\mathcal{R}^i_j(k)$ all solutions from $\mathcal{R}^i_j$ with the same supports. These sets were formed by only one parametric matrix in all cases which we have computed. If there are more than one solution in some set $\mathcal{R}^i_j(k)$ then only solutions are considered which are not strongly equivalent.

6. Each set $\mathcal{R}^i_j(k)$ can be represented by projection $\hat{\varepsilon}(k)$ of its arbitrary element (3.24). It follows from the fact that $\mathcal{R}^i$ include all solutions of certain system with the symmetry group $\Delta_\Gamma(\mathrm{Aut}(\Gamma))$ under certain assumptions that if there exists $\pi \in \Delta_\Gamma(\mathrm{Aut}(\Gamma))$ such that $(\hat{\varepsilon}(k_1))^\pi = \hat{\varepsilon}(k_2)$, then all solutions in $\mathcal{R}^i_j(k_1)$ are equivalent to solutions in $\mathcal{R}^i_j(k_2)$. Thus, the set $\mathcal{R}^i_j(k_1)$ can be omitted.

7. Using normalization matrix α (3.28) we normalize every solution ε, i.e. we replace it by a strongly equivalent solution $\alpha \bullet \varepsilon$ which has as many as possible contraction parameters equal to 1.

8. We construct higher–order identities and decide which solutions are discrete and which are continuous. Continuous solutions are also found as generalized Inönü–Wigner contraction.

Chapter 4

Identification of Lie algebras

Let us assume that we have solved the contraction system $\mathsf{S}_\Gamma(\mathcal{L})$ and found all non–equivalent contraction matrices $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ for Γ-graded contractions of the Lie algebra $\mathcal{L}$. These matrices are divided into sets according to the number of zeros and marked as continuous or discrete. Each contraction matrix $\varepsilon \in \mathsf{C}_\Gamma(\mathcal{L})$ corresponds to a new Lie algebra $\mathcal{L}^\varepsilon$. Some contraction matrices depend on one or more complex parameters and therefore lead to whole continuously parameterized families of Lie algebras.

Thus we have now, as a result, all Lie algebras (up to isomorphism) which are Γ-graded contractions of Lie algebra $\mathcal{L}$. However, among these Lie algebras $\mathcal{L}^\varepsilon$ there can still be some isomorphic ones. In order to purify resulting Γ-graded contractions from isomorphic cases and in order to list them properly, we have to identify each Lie algebra $\mathcal{L}^\varepsilon$.

Identification of a given Lie algebra usually means the determination of its isomorphism class. Since a complete classification of Lie algebras in dimension 8 is not known, it can be a difficult task (including the completion of this classification). However, it will be sufficient for us to classify Γ-graded contractions of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Therefore, by identification of $\mathcal{L}^\varepsilon$ we mean finding all resulting algebras which are isomorphic with $\mathcal{L}^\varepsilon$ and finding invariant characteristics which distinguish $\mathcal{L}^\varepsilon$ from the rest of non–isomorphic results.

Unfortunately, existing methods [4, 66] do not allow us to recognize all resulted graded contractions of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Therefore, we enlarge these methods with computation of some invariants. We start with investigation of the structure of the chosen Γ-graded contraction $\mathcal{L}^\varepsilon$, describe algorithms from [66] for finding the decomposition, the Levi decomposition and the nilradical of the Lie algebra, which we have used. Then we introduce Casimir operators [2] as the invariants of Lie algebras. We present numerical invariants such as dimensions of vector spaces formed by certain operators on Lie algebras [41]. Finally we describe the problem of seeking isomorphism and whole identification procedure.

Algorithms for Lie algebras [66] are based on their structural constants. In the whole chapter we will consider a Γ-graded complex Lie algebra $\mathcal{L} = \bigoplus_{i=1}^{m} L_i$ with dimension $n \in \mathbb{N}$. Let $n_i \in \mathbb{N}$ denote dimension of grading subspace L_i and $\mathcal{E} = (e_1, \dots, e_n)$ be basis of $\mathcal{L}$ such that vectors $e_1, \dots, e_{n_1}$ form basis of L_1, vectors $e_{n_1+1}, \dots, e_{n_1+n_2}$ form basis of L_2 and so on. For any contraction matrix $\varepsilon \in \mathsf{S}_\Gamma(\mathcal{L})$ we construct a new $n \times n$ matrix ε^* such that every element ε_{ij} in matrix ε is replaced by the $n_i \times n_j$ matrix with all entries equal to ε_{ij}. Let c_{ij}^k be the structural constants of $\mathcal{L}$ with respect to the basis $\mathcal{E}$, then the structural constants of Γ-graded contraction $\mathcal{L}^\varepsilon$ with respect to $\mathcal{E}$ are given by the commutators

$$[e_i, e_j]_\varepsilon = \sum_{k=1}^{n} \varepsilon_{ij}^* c_{ij}^k e_k, \qquad \forall i, j \in \{1, \dots, n\}. \tag{4.1}$$

In the following sections we will describe particular steps of our identification procedure.

4.1 Central decomposition

First step in identification procedure lies in the special case of the direct decomposition — so called **central decomposition** of Lie algebras, where one of resulting ideals is abelian. Suppose that $\mathcal{L}$ has such decomposition, i.e.

$$\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2, \quad [\mathcal{L}_1, \mathcal{L}_1] = 0, \quad [\mathcal{L}_1, \mathcal{L}_2] = 0, \quad [\mathcal{L}_2, \mathcal{L}_2] \subset \mathcal{L}_2, \tag{4.2}$$

then clearly $\mathcal{L}_1 \subset C(\mathcal{L})$ and $D(\mathcal{L}) = D(\mathcal{L}_2) \subset \mathcal{L}_2$. Thus, the central decomposition can exist only if $C(\mathcal{L})$ is not subset of $D(\mathcal{L})$, i.e.

$$C(\mathcal{L}) \cap D(\mathcal{L}) \neq C(\mathcal{L}) \tag{4.3}$$

Let us consider now that condition (4.3) holds for $\mathcal{L}$, then there exists a nonzero complementary subalgebra $\mathcal{A}$ in the center of $\mathcal{L}$ such that

$$C(\mathcal{L}) = \mathcal{A} \oplus (C(\mathcal{L}) \cap D(\mathcal{L})), \qquad \mathcal{A} \cap D(\mathcal{L}) = 0. \tag{4.4}$$

This abelian algebra $\mathcal{A}$ is called **maximal central component** of $\mathcal{L}$ and if we find its complementary vector space $\mathcal{L}'$ in $\mathcal{L}$ such that $D(\mathcal{L}) \subset \mathcal{L}'$, then we get the central decomposition

$$\mathcal{L} = \mathcal{A} \oplus \mathcal{L}'. \tag{4.5}$$

Since ideals $\mathcal{A}$ and $\mathcal{L}'$ in $\mathcal{L}$ are complements, they are not unique. However, they are unique up to a central automorphism of $\mathcal{L}$, see [66].

During the decomposition we determine the complementary algebra $\mathcal{A}$ and its basis. Then we merge this basis with the basis of $D(\mathcal{L})$ and complete them to the basis of $\mathcal{L}$. Having this new basis, the decomposition is obvious and we replace the original Lie algebra $\mathcal{L}$ by its non–abelian part denoted by $\mathcal{L}'$. Further on we will investigate only non–abelian parts of Lie algebras.

4.2 Direct decomposition

In this step we show that the existence of the direct decomposition of a given n–dimensional complex Lie algebra $\mathcal{L}$ is equivalent to the existence of certain a $n \times n$ complex matrix.

Let $R = \mathbb{C}^{n,n}$ denote the ring of $n \times n$ complex matrices. Any nonzero element $E \in R$ satisfying $E^2 = E$ is called an **idempotent** in the ring R. The unit element $\mathbf{1}_n$ is called **trivial idempotent**. A complex number λ is an eigenvalue of an idempotent $E \in R$ if there exists nonzero vector $x \in \mathbb{C}^n$ such that $Ex = \lambda x$. It follows from the equation

$$\lambda x = Ex = E^2 x = E(\lambda x) = \lambda^2 x \tag{4.6}$$

and from the Jordan normal form of matrix E that the spectrum of any nontrivial idempotent E is $\sigma(E) = \{0, 1\}$.

Let us consider now that there is a nontrivial idempotent E in the **centralizer of the adjoint representation** of $\mathcal{L}$ in the ring R

$$C_R(\mathrm{ad}(\mathcal{L})) = \{X \in R \,|\, y \in \mathcal{L}, [X, \mathrm{ad}_{\mathcal{L}}(y)] = X \,\mathrm{ad}_{\mathcal{L}}(y) - \mathrm{ad}_{\mathcal{L}}(y) X = 0\} \,. \tag{4.7}$$

Then we have for all $x, y \in \mathcal{L}$

$$[Ex, Ey] = \mathrm{ad}_{\mathcal{L}}(Ex)Ey = E\,\mathrm{ad}_{\mathcal{L}}(Ex)y = -E\,\mathrm{ad}_{\mathcal{L}}(y)Ex = -E^2\,\mathrm{ad}_{\mathcal{L}}(y)x = E[x, y] \tag{4.8}$$

and thus, this idempotent E is an endomorphism of the Lie algebra $\mathcal{L}$. Let $\mathcal{L}_0$ and $\mathcal{L}_1$ denote the eigen–subspaces of idempotent E corresponding to eigenvalues 0 and 1. For any two eigenvectors $x \in \mathcal{L}_\mu, y \in \mathcal{L}_\nu, \mu, \nu \in \{0, 1\}$ we have

$$E[x, y] = [Ex, Ey] = [\mu x, \nu y] = \mu\nu[x, y]. \tag{4.9}$$

Hence $[x, y] \in \mathcal{L}_{\mu\nu}$ is either zero or eigenvector corresponding to eigenvalue $\mu\nu$. For $\mu = \nu$ we have $[\mathcal{L}_\mu, \mathcal{L}_\mu] \subset \mathcal{L}_\mu$ and $\mathcal{L}_\mu$ is a subalgebra of $\mathcal{L}$. If $\mu \neq \nu$, then $E[x, y] = 0$ and $[x, y] \in \mathcal{L}_0$. We show that in this case is $[x, y] = 0$. Suppose that $\mu = 0$ and $\nu = 1$, then $Ex = 0$, $Ey = y$ and

$$[x, y] = [x, Ey] = \mathrm{ad}_{\mathcal{L}}(x)Ey = E\,\mathrm{ad}_{\mathcal{L}}(x)y = E[x, y] = 0. \tag{4.10}$$

Therefore, $[\mathcal{L}_0, \mathcal{L}_1] = 0$ and the Lie algebra $\mathcal{L}$ is direct sum of its ideals $\mathcal{L}_0, \mathcal{L}_1$.

On the other hand, if $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, then there exists a basis $(x_1, \ldots, x_{n_0}, y_1, \ldots, y_{n_1})$ of $\mathcal{L}$ such that $(x_1, \ldots, x_{n_0})$ is a basis of $\mathcal{L}_0$ and $(y_1, \ldots, y_{n_1})$ basis of $\mathcal{L}_1$. In this basis any inner derivation is a block diagonal matrix of the form

$$\mathrm{ad}_{\mathcal{L}_0}(x) \oplus \mathrm{ad}_{\mathcal{L}_1}(y) = \begin{pmatrix} \mathrm{ad}_{\mathcal{L}_0}(x) & 0 \\ 0 & \mathrm{ad}_{\mathcal{L}_1}(y) \end{pmatrix}, \quad x \in \mathcal{L}_0, y \in \mathcal{L}_1 \tag{4.11}$$

and therefore, matrices

$$E_0 = \mathbf{0}_{n_0} \oplus \mathbf{1}_{n_1} = \begin{pmatrix} \mathbf{0}_{n_0} & 0 \\ 0 & \mathbf{1}_{n_1} \end{pmatrix}, \qquad E_1 = \mathbf{1}_{n_0} \oplus \mathbf{0}_{n_1} = \begin{pmatrix} \mathbf{1}_{n_0} & 0 \\ 0 & \mathbf{0}_{n_1} \end{pmatrix} \tag{4.12}$$

are idempotents in $C_R(\mathrm{ad}(\mathcal{L}))$. Thus, we have proved the following theorem.

Theorem 4.1. *Lie algebra $\mathcal{L}$ is decomposable into the direct sum of its ideals if and only if there exists a nontrivial idempotent E in the centralizer $C_R(\mathrm{ad}(\mathcal{L}))$ of the adjoint representation of $\mathcal{L}$. In such case the decomposition has the form*

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \tag{4.13}$$

where $\mathcal{L}_0, \mathcal{L}_1$ are eigen–subspaces of the idempotent E corresponding to the eigenvalues $0, 1$.

Now we describe the algorithm which we have used for finding idempotents and decompositions. This algorithm is based on an algorithm from [66], where also the theoretical background for these algorithms can be found.

1. Starting with the n-dimensional complex Lie algebra $\mathcal{L}$ for which $C(\mathcal{L}) \subset D(\mathcal{L})$, choose a basis of $\mathcal{L}$ and determine $n \times n$ complex matrices of the adjoint representation $\mathrm{ad}(\mathcal{L})$ of $\mathcal{L}$ with respect to this basis.

2. Compute the centralizer $A = C_R(\mathrm{ad}(\mathcal{L}))$ of $\mathrm{ad}(\mathcal{L})$ in R, i.e. an associative algebra formed by all complex $n \times n$ matrices which commute with all elements in $\mathrm{ad}(\mathcal{L})$, and its **Jacobson radical** $J(A)$ (maximal nilpotent ideal, $J(A)^k = 0$ for some $k \in \mathbb{N}$) according to the formula

$$J(A) = \{x \in A \,|\, \mathrm{Tr}(xy) = 0, \ \forall y \in A\}\,. \tag{4.14}$$

3. It is proved in [66] that $\mathcal{L}$ is indecomposable if and only if $\dim(A) - \dim(J(A)) = 1$. In other cases it holds $\dim(A) - \dim(J(A)) > 1$ and $\mathcal{L}$ is decomposable.

4. Choose a basis $(x_1, , \ldots, x_\nu, b_1, \ldots, b_\mu)$ of A such that $(x_1, , \ldots, x_\nu)$ is a basis of $J(A)$, $b_1 = \mathbf{1}_n$ and $\mathrm{Tr}(b_i) = 0, \ 2 \le i \le \mu$. Find a basis element $b_r, 2 \le r \le \mu$ which has reducible minimal polynomial $m_r(t) \in \mathbb{C}[t]$ (monic polynomial with lowest degree in t for which $m_r(b_r) = 0$). Factorize m_r into two mutually prime nonconstant polynomials f_1, f_2. Using Euclidean division algorithm find polynomials P_1, P_2 satisfying

$$P_1 f_1 + P_2 f_2 = 1. \tag{4.15}$$

5. Complex matrix $E = P_1(b_r)f_1(b_r)$ is an idempotent in A and its eigen–subspaces $\mathcal{L}_0, \mathcal{L}_1$ form components of the direct decomposition of $\mathcal{L}$.

We use this algorithm and decompose any decomposable Lie algebra into the direct sum of indecomposable components. From now on we deal only with indecomposable Lie algebras.

4.3 Dimensions of the series

For any Lie algebra $\mathcal{L}$ we compute the derived series $D^k(\mathcal{L})$, the lower central series $\mathcal{L}^k$ and the upper central series $C^k(\mathcal{L})$ defined by relations (1.31), (1.32) and (1.33). Since we are dealing with complex algebras of finite dimension, these series of ideals have only a finite number of different elements. In any series there is an element which is different from the previous one but is the same as all following elements; we will call such element the **last element** of the series and ignore all following elements. The last element in the upper central series is the so called **hypercenter**.

It follows from (1.36) and (1.37) that the structure of any ideal in the mentioned series is an invariant characteristic of Lie algebra. We will use dimensions of these ideals and define the following invariant characteristics of Lie algebra $\mathcal{L}$:

$$DS(\mathcal{L}) = (\dim(D^0(\mathcal{L})), \ldots, \dim(D^k(\mathcal{L}))) \tag{4.16}$$

$$CS(\mathcal{L}) = (\dim(\mathcal{L}^1), \ldots, \dim(\mathcal{L}^k)) \tag{4.17}$$

$$US(\mathcal{L}) = (\dim(C^1(\mathcal{L})), \ldots, \dim(C^k(\mathcal{L}))). \tag{4.18}$$

Since we are dealing with 8-dimensional algebras we will not separate numbers in round bracket by commas. The first number in $DS(\mathcal{L})$ and $CS(\mathcal{L})$ is the dimension of $\mathcal{L}$. If the last number in $DS(\mathcal{L})$ is zero, then $\mathcal{L}$ is solvable and the number of nonzeros in $DS(\mathcal{L})$ is called the **solvability rank** of $\mathcal{L}$. If the last number in $CS(\mathcal{L})$ is zero, then $\mathcal{L}$ is nilpotent and the number of nonzeros in $CS(\mathcal{L})$ is called the **nilpotency rank** of $\mathcal{L}$. First number in $US(\mathcal{L})$ is the dimension of the center of $\mathcal{L}$ and the last is the dimension of hypercenter.

We divide all Lie algebras into classes according to the invariants $DS(\mathcal{L}), CS(\mathcal{L})$ and $US(\mathcal{L})$ and decide which algebras are nilpotent or solvable.

4.4 Levi decomposition

Indecomposable Lie algebras which are not solvable are either simple or admit nontrivial Levi decomposition. In order to obtain this decomposition we have used an algorithm from [66]. Before we approach the description of this algorithm, we recall that according to the

Levi theorem 1.7, any Lie algebra $\mathcal{L}$ can be written as a semidirect sum $\mathcal{L} = R(\mathcal{L}) \triangleleft \mathcal{S}$ of its radical $R(\mathcal{L})$ and semisimple subalgebra $\mathcal{S}$. The radical is unique and basis independent. Semisimple subalgebra $\mathcal{S}$ is isomorphic to the factor algebra $\mathcal{L}/R(\mathcal{L})$ and is unique up to an automorphism of $\mathcal{L}$. We obtain the Levi decomposition of $\mathcal{L}$ as follows.

1. Find the radical $R(\mathcal{L})$ of the Lie algebra $\mathcal{L}$, according to Theorem 1.4 it is

$$R(\mathcal{L}) = \{x \in \mathcal{L} \mid \operatorname{Tr}(\operatorname{ad}_{\mathcal{L}}(x) \operatorname{ad}_{\mathcal{L}}(y)) = 0, \forall y \in D(\mathcal{L})\}\,. \tag{4.19}$$

If $\mathcal{L} = R(\mathcal{L})$, then $\mathcal{L}$ is solvable and $\mathcal{S} = 0$ (excluded in our case). If $R(\mathcal{L}) = 0$, then $\mathcal{L} = \mathcal{S}$ is semisimple (in our case simple). In all other cases $\mathcal{L}$ has nontrivial Levi decomposition.

2. If the radical $R(\mathcal{L})$ is abelian, then choose a basis $(r_1, \ldots, r_\rho, a_{\rho+1}, \ldots, a_n)$ for $\mathcal{L}$ such that $(r_1, \ldots, r_\rho)$ is a basis for $R(\mathcal{L})$. Let commutation relations in this basis be

$$[a_i, a_k] = \sum_{l=\rho+1}^{n} c_{ik}^l a_l + \sum_{p=1}^{\rho} f_{ik}^p r_p, \qquad [r_p, r_q] = 0, \qquad [a_i, r_q] = \sum_{p=1}^{\rho} h_{iq}^p r_p,$$
$$\rho + 1 \le i, k, l \le n, \qquad 1 \le p, q \le \rho. \tag{4.20}$$

Replace the basis elements a_i by $s_i = a_i + \sum_{p=1}^{\rho} x_{ip} r_p$ such that $[s_i, s_k] = \sum_{l=\rho+1}^{n} c_{ik}^l s_l$, i.e. coefficients $x_{ip} \in \mathbb{C}$ must satisfy the following system of inhomogeneous linear equations

$$\sum_{l=\rho+1}^{n} c_{ik}^l x_{lq} - \sum_{p=1}^{\rho} h_{ip}^q x_{kp} + \sum_{p=1}^{\rho} h_{kp}^q x_{ip} = f_{ik}^q, \quad 1 \le q \le \rho, \quad \rho + 1 \le i, k \le n. \tag{4.21}$$

The existence of solution of this system follows from the existence of the Levi decomposition of $\mathcal{L}$. Now $(s_{\rho+1}, \ldots, s_n)$ forms a basis of semisimple subalgebra $\mathcal{S}$. If $R(\mathcal{L})$ is not abelian, then continue.

3. Semisimple subalgebra $\mathcal{S}$ is perfect and therefore, $\mathcal{S}$ lies in any ideal in the derived series of $\mathcal{L}$. Find the last element of the derived series, i.e. $D^k(\mathcal{L})$, $k \in \mathbb{N}$ such that $D^{k-1}(\mathcal{L}) \neq D^k(\mathcal{L}) = D^{k+1}(\mathcal{L})$. The Levi decomposition of Lie algebra $\mathcal{L}$ can be obtained now from the Levi decomposition of the perfect Lie algebra $D^k(\mathcal{L}) = R(D^k(\mathcal{L})) \triangleleft \mathcal{S}$ by extending basis of $R(D^k(\mathcal{L}))$ to the basis of $R(\mathcal{L})$.

4. From now on we assume that $\mathcal{L}$ is perfect. Since the radical $R(\mathcal{L})$ is not abelian, $D(R(\mathcal{L}))$ is nonzero ideal in $\mathcal{L}$. Choose a basis $(r_1, \ldots, r_{\rho'}, r_{\rho'+1}, \ldots, r_\rho, a_{\rho+1}, \ldots, a_n)$

for $\mathcal{L}$ such that $(r_1, \ldots, r_{\rho'})$ is a basis for $D(R(\mathcal{L}))$, $(r_{\rho'+1}, \ldots, r_\rho)$ basis for a complement of $D(R(\mathcal{L}))$ in $R(\mathcal{L})$ and $(a_{\rho+1}, \ldots, a_n)$ for a complement of $R(\mathcal{L})$ in $\mathcal{L}$. Construct the factor algebra $\overline{\mathcal{L}} = \mathcal{L}/D(R(\mathcal{L}))$. It has dimension $\dim \overline{\mathcal{L}} = n - \rho' < n$ and the commutation relations for $\overline{\mathcal{L}}$ are obtained by setting $r_1 = \ldots = r_{\rho'} = 0$ in the commutation relations for $\mathcal{L}$. $\overline{\mathcal{L}}$ is a perfect Lie algebra with abelian radical $R(\mathcal{L})/D(R(\mathcal{L}))$. According to step 2 obtain its Levi decomposition $\overline{\mathcal{L}} = R(\overline{\mathcal{L}}) \triangleleft \overline{\mathcal{S}}$. From residue classes in $\overline{\mathcal{S}}$ construct elements of $\mathcal{L}$ and obtain a proper subalgebra of $\mathcal{L}$, namely

$$\mathcal{L}_1 = D(R(\mathcal{L})) + \mathcal{S}_1, \tag{4.22}$$

satisfying $\dim \mathcal{L}_1 = n - \rho + \rho' < n$. If $D(R(\mathcal{L}_1))$ is abelian, obtain a Levi decomposition of $\mathcal{L}_1$; if not, continue until, after a finite number of steps, an algebra $\mathcal{L}_k$ with an abelian radical is acquired. For this algebra we obtain $\mathcal{L}_k = R(\mathcal{L}_k) \triangleleft \mathcal{S}$, where $\mathcal{S}$ is the desired semisimple subalgebra which appears in the Levi decomposition of $\mathcal{L} = R(\mathcal{L}) \triangleleft \mathcal{S}$.

Semisimple subalgebra $\mathcal{S}$ can be further decomposed, using algorithm from section 4.2, into direct sum of simple Lie algebras. Structure of radical $R(\mathcal{L})$ and semisimple subalgebra $\mathcal{S}$ are invariant for the given Lie algebra $\mathcal{L}$. Thus, we divide all classes with non–solvable Lie algebras into new classes according to the types of radicals and semisimple subalgebras.

4.5 Nilradical

Nilradical, as a maximal nilpotent ideal in a Lie algebra, represents another invariant characteristic of the given Lie algebra $\mathcal{L}$. However, for nilpotent and semisimple Lie algebras the nilradical is trivial. We used the following algorithm taken from [66] and computed the nilradical for all solvable non–nilpotent Lie algebras and for all algebras which have nontrivial Levi decomposition.

1. Determine the radical $R(\mathcal{L})$. Nilradical is solvable ideal and thus, $N(\mathcal{L}) \subset R(\mathcal{L})$ and it follows from (1.43) that $N(\mathcal{L}) = N(R(\mathcal{L}))$. From now on we replace $\mathcal{L}$ by $R(\mathcal{L})$ and assume that $\mathcal{L}$ is solvable.

2. Calculate the ideals $D(\mathcal{L})$ and $D^2(\mathcal{L})$, these are nilpotent and thus belong into $N(\mathcal{L})$ and
$$N(\mathcal{L})/D^2(\mathcal{L}) = N(\mathcal{L}/D^2(\mathcal{L})). \tag{4.23}$$
From now on we consider the algebra $\mathcal{L}/D^2(\mathcal{L})$ instead of $\mathcal{L}$, thus we assume that $\mathcal{L}$ is solvable and its derived algebra $D(\mathcal{L})$ is abelian.

3. Calculate the hypercenter of $\mathcal{L}$, i.e. the last element $C^k(\mathcal{L})$ in the upper central sequence of $\mathcal{L}$. This is also nilpotent ideal contained in $N(\mathcal{L})$ and

$$N(\mathcal{L})/C^k(\mathcal{L}) = N(\mathcal{L}/C^k(\mathcal{L})). \tag{4.24}$$

We have thus reduced the problem of finding nilradical of an arbitrary complex finite–dimensional Lie algebra to that of finding $N(\mathcal{L})$ for a solvable Lie algebra $\mathcal{L}$ with abelian derived algebra and $C(\mathcal{L}) = 0$.

4. Introduce a basis $(u_1, \ldots, u_m)$ for the derived algebra $D(\mathcal{L})$ and extend it to a basis $(u_1, \ldots, u_m, x_1, \ldots, x_{n-m})$ of $\mathcal{L}$.

5. Choose a basis element of $D(\mathcal{L})$, say u_1, and an element of the complement, say x_j. Define $u_{0j} = u_1,\ u_{1j} = [x_j, u_{0j}], \ldots, u_{ij} = [x_j, u_{i-1,j}]$. Thus generate a chain of elements in $D(\mathcal{L})$,

$$S_i = \{u_{0j}, u_{1j}, \ldots, u_{ij}\} \tag{4.25}$$

such that the set is linearly dependent but the set S_{i-1} is linearly independent. Thus, there exists a set of numbers $a_{kj} \in \mathbb{C}$, not all vanishing, such that

$$u_{ij} + a_{1j}u_{i-1,j} + \cdots + a_{ij}u_{0j} = 0. \tag{4.26}$$

Using coefficients in (4.26), form the polynomial

$$f_j(q) = q^i + a_{1j}q^{i-1} + \cdots + a_{i-1,j}q + a_{ij}. \tag{4.27}$$

According to the properties of this polynomial continue with one of the following steps.

6. If $a_{ij} = 0$, form the ideal $B_1 = [x_j, D(\mathcal{L})]$ and determine the nilradical of $\mathcal{L}/B_1$. Construct the algebra M defined by $M/B_1 = N(\mathcal{L}/B_1)$ and find its nilradical $N(M)$; then $N(\mathcal{L}) = N(M)$.

7. If $a_{ij} \neq 0$ and $f_j(q)$ is not square free, find a nonconstant polynomial $g_j(q)$ which divides $f_j(q)$ and is square free. Form the ideal

$$B_2 = g_j(\mathrm{ad}_{\mathcal{L}}(x_j))D(\mathcal{L}), \tag{4.28}$$

where $n \times m$ matrix $D(\mathcal{L})$, whose columns represent basis elements of the derived algebra, is multiplied by $n \times n$ matrix $g_j(\mathrm{ad}_{\mathcal{L}}(x_j))$ and the result is matrix B_2, whose columns spanned over $\mathbb{C}$ form ideal B_2. Proceed further as with the ideal B_1, i.e. put $M/B_2 = N(\mathcal{L}/B_2),\ B_2 \subset M$ and $N(\mathcal{L}) = N(M)$.

8. If $a_{ij} \neq 0$ and $f_j(q)$ is square free and if $j < n - m$, then replace j by $j + 1$ and return to step 5. If $j = n - m$, then determine the centralizer in $\mathcal{L}$ of the element $u_1 = u_{0j}$ used in step 5:

$$M = C_{\mathcal{L}}(u_1) = \{y \in \mathcal{L} \,|\, [y, u_1] = 0\}\,. \tag{4.29}$$

We have $\dim M \geq \dim D(\mathcal{L})$ (since $D(\mathcal{L})$ is abelian). If $\dim M = \dim D(\mathcal{L})$, the $N(\mathcal{L}) = D(\mathcal{L})$. If $\dim M > \dim D(\mathcal{L})$, then $N(\mathcal{L}) = N(M)$, where $N(M)$ is found returning to step 2, but using M in place of $\mathcal{L}$.

4.6 Casimir operators

Casimir operators of a Lie algebra $\mathcal{L}$ are well known invariants used in representation theory for the labelling of irreducible representations of $\mathcal{L}$. As elements of the universal enveloping algebra of $\mathcal{L}$, they are polynomials in generators (basis elements) of $\mathcal{L}$ which commute with all elements in $\mathcal{L}$. The form of the Casimir operators depends on the choice of basis in $\mathcal{L}$. However, there are some characteristics which are basis independent, such as degrees of these polynomials or number of functionally independent polynomials, and thus represent invariant characteristics for $\mathcal{L}$.

More precisely, let us consider the tensor algebra $T(\mathcal{L})$ of the vector space $\mathcal{L}$, i.e.

$$T(\mathcal{L}) = \bigoplus_{k=0}^{\infty} T^k(\mathcal{L}), \tag{4.30}$$

where $T^0(\mathcal{L}) = \mathbb{C}$, $T^1(\mathcal{L}) = \mathcal{L}$ and $T^k(\mathcal{L}) = \mathcal{L} \otimes \mathcal{L} \otimes \ldots \otimes \mathcal{L}$ (k times) is k-th tensor power of the vector space $\mathcal{L}$. Note that the bilinear multiplication $\otimes$ in algebra $T(\mathcal{L})$ is associative. Let I be the two–sided ideal in $T(\mathcal{L})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$, where $x, y \in \mathcal{L}$, i.e.

$$I = \mathrm{span}_{\mathbb{C}} \{a \otimes (x \otimes y - y \otimes x - [x, y]) \otimes b \,|\, x, y \in \mathcal{L},\ a, b \in T(\mathcal{L})\}\,, \tag{4.31}$$

then the associative algebra $U(\mathcal{L}) = T(\mathcal{L})/I$ is called the **universal enveloping algebra** of $\mathcal{L}$ [20]. The product of u and v in $U(\mathcal{L})$ is denoted simply by uv. Vector subspaces of $U(\mathcal{L})$

$$U_m(\mathcal{L}) = \mathrm{span}_{\mathbb{C}} \{1, x_1 x_2 \ldots x_p \in U(\mathcal{L}) \,|\, x_1, x_2, \ldots, x_p \in \mathcal{L},\ p \leq m\} \tag{4.32}$$

form the increasing sequence $U_0(\mathcal{L}) \equiv \mathbb{C} \subset U_1(\mathcal{L}) \equiv \mathbb{C} \oplus \mathcal{L} \subset U_2(\mathcal{L}) \subset \ldots \subset U_m(\mathcal{L}) \subset \ldots$ of subspaces in $U(\mathcal{L})$ called **canonical filtration** of $U(\mathcal{L})$. For nonzero $u \in U(\mathcal{L})$ we call the number

$$\mathrm{dg}(u) = \min \{m \in \mathbb{N}_0 \,|\, u \in U_m(\mathcal{L})\} \tag{4.33}$$

degree or **filtration** of u. The center of $U(\mathcal{L})$

$$Z(\mathcal{L}) = \{u \in U(\mathcal{L}) \,|\, vu = uv, \ \forall v \in U(\mathcal{L})\} \tag{4.34}$$

is an abelian subalgebra in $U(\mathcal{L})$. Considering the composition π of canonical mappings $\mathcal{L} \longrightarrow T(\mathcal{L}) \longrightarrow U(\mathcal{L})$ we can identify the Lie algebra $\mathcal{L}$ with the subalgebra $\pi(\mathcal{L})$ in Lie algebra $U(\mathcal{L})_{\mathcal{L}}$ of the associative algebra $U(\mathcal{L})$. An element $u \in U(\mathcal{L})$ is called a **Casimir operator** of the Lie algebra $\mathcal{L}$ if

$$[x, u] = xu - ux = 0, \qquad \forall x \in \mathcal{L}. \tag{4.35}$$

Thus, the set of all Casimir operators of the Lie algebra $\mathcal{L}$ is the center of its universal enveloping algebra $U(\mathcal{L})$.

There are several methods [6, 57, 62] for computing Casimir operators of Lie algebras. The classical method [57] suggests to represent elements of basis $(e_1, \dots, e_n)$ in $\mathcal{L}$ by the vector fields

$$e_i \longrightarrow \widehat{x}_i = \sum_{j,k=1}^{n} c_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{4.36}$$

which have the same commutation rules and act on the space of continuously differentiable functions $F(x_1, \dots, x_n)$ of n variables. A function F is called a **formal invariant** of $\mathcal{L}$ if it solves the following linear system of first-order partial differential equations

$$\widehat{x}_i F(x_1, \dots, x_n) = \sum_{j,k=1}^{n} c_{ij}^k x_k \frac{\partial F}{\partial x_j}(x_1, \dots, x_n) = 0, \qquad \forall i = 1, \dots, n. \tag{4.37}$$

Let us note that any continuously differentiable function of formal invariants is also a formal invariant. According to the classical theorem in [20], the number of functionally independent solutions of system (4.37) is

$$\tau(\mathcal{L}) = \dim(\mathcal{L}) - \sup_{x_1, \dots, x_n} \operatorname{rank} M_{\mathcal{L}} \tag{4.38}$$

where $M_{\mathcal{L}}$ is skew–symmetric matrix with entries $(M_{\mathcal{L}})_{ij} = \sum_{k=1}^{n} c_{ij}^k x_k$ and $x_1, \dots, x_n$ are independent complex numbers. A maximal set of functionally independent formal invariants is called **fundamental set of invariants** and the cardinality of this set $\tau(\mathcal{L})$ is invariant characteristic for the Lie algebra $\mathcal{L}$.

Formal invariants which are polynomials in variables $x_1, \dots, x_n$ are in one to one correspondence with Casimir operators of the Lie algebra $\mathcal{L}$. This correspondence is provided by symmetrization [2, 21]: any term $x_{k_1} \dots x_{k_p}$ of polynomial $F(x_1, \dots, x_n)$ in commuting variables x_i is replaced by symmetric term in non–commuting basis elements $e_1, \ldots, e_n \in \mathcal{L}$ as follows

$$x_{k_1} \ldots x_{k_p} \mapsto \frac{1}{p!} \sum_{\sigma \in S_p} e_{k_{\sigma(1)}} \ldots e_{k_{\sigma(p)}}. \tag{4.39}$$

Formal invariants which are not polynomials (usually rational or exponential functions) correspond via symmetrization (4.39) to so called **generalized Casimir operators**. It is known [2] that for semisimple and nilpotent Lie algebras the fundamental set of invariants can be constructed of polynomials.

We compute for each Lie algebra $\mathcal{L}$ the number of independent formal invariants $\tau(\mathcal{L})$. Furthermore, we find and list all independent Casimir operators for each of the resulting nilpotent Lie algebras. Since these are all polynomials, we use the Poincaré–Birkhoff–Witt theorem [21] which says: if $(e_1, e_2, \ldots, e_n)$ is a basis for $\mathcal{L}$, then the set of all elements of the form

$$e_1^{k_1} e_2^{k_2} \ldots e_n^{k_n}, \qquad k_1, k_2 \ldots, k_n \in \mathbb{N}_0 \tag{4.40}$$

forms a basis for $U(\mathcal{L})$. From this basis we generate a general element $u^{(k)} \in U(\mathcal{L})$ of order $k \in \mathbb{N}$ and require

$$[e_i, u^{(k)}] = e_i u^{(k)} - u^{(k)} e_i = 0, \qquad \forall i = 1, \ldots, n. \tag{4.41}$$

This leads to a system of linear homogeneous equations and its solutions correspond to all Casimir operators of $\mathcal{L}$ up to order k. If the number $\tau(\mathcal{L})$ of independent Casimir operators is not obtained among these solutions, we have to increase the order k of general element.

Since the Lie algebra $\mathrm{sl}(3, \mathbb{C})$ has two Casimir operators — one of order two and one of order three, it was sufficient to consider $k = 3$. Let us note that computed Casimir operators are not necessarily symmetric, since they are written in the ordered basis (4.40). The increasing sequence of orders of all independent Casimir operators (up to certain order) is also an invariant characteristic of the Lie algebra $\mathcal{L}$.

The Casimir operators of contracted Lie algebras were also studied (mainly for continuous contraction) in [1, 11, 78].

4.7 Generalized derivations

Since for any two isomorphic Lie algebras are their algebras of derivations (1.18) also isomorphic, the structure of the algebra of derivations of $\mathcal{L}$ is an invariant characteristic for the Lie algebra $\mathcal{L}$. Thus, any invariant of $\mathrm{der}(\mathcal{L})$ is also an invariant of $\mathcal{L}$. However, the search for invariants of $\mathrm{der}(\mathcal{L})$ is usually more difficult because of higher dimension. Therefore, of all invariants of $\mathrm{der}(\mathcal{L})$ we will use mainly the dimension of $\mathrm{der}(\mathcal{L})$. Let us note that there is also possibility to search for the algebra of derivations of $\mathrm{der}(\mathcal{L})$ and so on and construct sequence $\mathrm{der}^k(\mathcal{L}) = \mathrm{der}(\mathrm{der}^{k-1}(\mathcal{L})), k \in \mathbb{N}$, so called **tower of derivation algebras** of $\mathcal{L}$ [64].

In order to get more invariants, similar to $\dim(\mathrm{der}(\mathcal{L}))$, we have generalized the concept of derivations in [III, VI] as follows. Let α, β, γ be arbitrary complex numbers; then a linear operator $A \in \mathrm{End}(\mathcal{L})$ is called $(\alpha, \beta, \gamma)-$**derivation** of $\mathcal{L}$ if for all $x, y \in \mathcal{L}$ it holds

$$\alpha A[x, y] = \beta[Ax, y] + \gamma[x, Ay]. \tag{4.42}$$

Let us note that several non–equivalent generalizations of derivations have been studied in [7, 30, 49]. For given $\alpha, \beta, \gamma \in \mathbb{C}$, the set of all $(\alpha, \beta, \gamma)-$derivations of $\mathcal{L}$ forms a vector subspace of $\mathrm{End}(\mathcal{L})$ which we denote by $\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L})$.

Let $h : \mathcal{L} \longrightarrow \widetilde{\mathcal{L}}$ be an isomorphism of Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ and A an $(\alpha, \beta, \gamma)-$derivation of $\mathcal{L}$. Then for all $x, y \in \widetilde{\mathcal{L}}$ we have

$$\alpha A[h^{-1}x, h^{-1}y]_{\mathcal{L}} = \beta[Ah^{-1}x, h^{-1}y]_{\mathcal{L}} + \gamma[h^{-1}x, Ah^{-1}y]_{\mathcal{L}}. \tag{4.43}$$

Applying the isomorphism h to this equality we get

$$\alpha hAh^{-1}[x, y]_{\widetilde{\mathcal{L}}} = \beta[hAh^{-1}x, y]_{\widetilde{\mathcal{L}}} + \gamma[x, hAh^{-1}y]_{\widetilde{\mathcal{L}}}, \tag{4.44}$$

i.e. hAh^{-1} is an $(\alpha, \beta, \gamma)-$derivation of $\widetilde{\mathcal{L}}$. Since the mapping $g : A \longrightarrow hAh^{-1}$ is an isomorphism of associative algebras $\mathrm{End}(\mathcal{L})$ and $\mathrm{End}(\widetilde{\mathcal{L}})$, we conclude that

$$g(\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L})) = \mathrm{der}_{(\alpha,\beta,\gamma)}(\widetilde{\mathcal{L}}). \tag{4.45}$$

Thus, the dimensions of the vector spaces $\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L})$ are invariants for $\mathcal{L}$.

It follows directly from the definition of spaces $\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L})$ that for any $\delta \in \mathbb{C} \setminus \{0\}$ it holds

$$\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L}) = \mathrm{der}_{(\alpha\delta,\beta\delta,\gamma\delta)}(\mathcal{L}) = \mathrm{der}_{(\alpha,\gamma,\beta)}(\mathcal{L}). \tag{4.46}$$

Moreover, the anti–commutativity of the Lie bracket leads to

$$\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L}) = \mathrm{der}_{(0,\beta-\gamma,\gamma-\beta)}(\mathcal{L}) \cap \mathrm{der}_{(2\alpha,\beta+\gamma,\beta+\gamma)}(\mathcal{L}). \tag{4.47}$$

Using (4.46) and (4.47) we have obtained a complete classification of all vector spaces $\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L})$ [VI]. Some of these vector spaces form subalgebras of $\mathrm{End}(\mathcal{L})$, $\mathrm{gl}(\mathcal{L})$ or $\mathrm{jor}(\mathcal{L})$. We have shown in [VI] that there are only the following types of spaces $\mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L})$:

- associative algebra $\mathrm{der}_{(0,0,0)}(\mathcal{L}) = \mathrm{End}(\mathcal{L})$ of all linear operators on the vector space $\mathcal{L}$.

- associative algebra $\mathrm{der}_{(1,0,0)}(\mathcal{L}) = \{A \in \mathrm{End}(\mathcal{L}) \,|\, A(D(\mathcal{L})) = 0\}$ of all linear operators on $\mathcal{L}$ which map derived algebra $D(\mathcal{L})$ to the zero vector; thus its dimension is

$$\dim(\mathrm{der}_{(1,0,0)}(\mathcal{L})) = \mathrm{codim}(D(\mathcal{L}))\dim(\mathcal{L}). \tag{4.48}$$

- associative algebra $\mathrm{der}_{(0,1,0)}(\mathcal{L}) = \{A \in \mathrm{End}(\mathcal{L}) \,|\, A(\mathcal{L}) \subset C(\mathcal{L})\}$ of all linear operators on $\mathcal{L}$ which map whole $\mathcal{L}$ into its center $C(\mathcal{L})$; thus its dimension is

$$\dim(\mathrm{der}_{(0,1,0)}(\mathcal{L})) = \dim(\mathcal{L})\dim(C(\mathcal{L})). \tag{4.49}$$

- associative algebra $\mathrm{der}_{(1,1,0)}(\mathcal{L}) = C_{\mathrm{End}(\mathcal{L})}(\mathrm{ad}(\mathcal{L}))$, i.e. the centralizer of the adjoint representation of $\mathcal{L}$ in $\mathrm{End}(\mathcal{L})$.
- Jordan algebra $\mathrm{der}_{(0,1,-1)}(\mathcal{L}) = \{A \in \mathrm{End}(\mathcal{L}) \,|\, [Ax, y] = [x, Ay],\ \forall x, y \in \mathcal{L}\}$.
- Jordan algebra $\mathrm{der}_{(1,1,-1)}(\mathcal{L}) = \mathrm{der}_{(0,1,-1)}(\mathcal{L}) \cap \mathrm{der}_{(1,0,0)}(\mathcal{L})$.
- Lie algebra $\mathrm{der}_{(0,1,1)}(\mathcal{L}) = \{A \in \mathrm{End}(\mathcal{L}) \,|\, [Ax, y] = -[x, Ay],\ \forall x, y \in \mathcal{L}\}$.
- Lie algebra $\mathrm{der}_{(1,1,1)}(\mathcal{L}) = \mathrm{der}(\mathcal{L})$ of the derivations of $\mathcal{L}$.
- one–parametric sets of vector spaces $\mathrm{der}_{(\delta,1,0)}(\mathcal{L}), \mathrm{der}_{(\delta,1,1)}(\mathcal{L})$, where $\delta \in \mathbb{C} \setminus \{0, 1\}$.

Thus, we have eight algebras and two parametric sets of vector spaces, all uniquely determined by the Lie algebra $\mathcal{L}$. It follows from (4.45) and the fact that mapping g is an isomorphism of associative algebras that the structures of these algebras form also invariants of $\mathcal{L}$. Since the structure of associative algebras $\mathrm{End}(\mathcal{L}), \mathrm{der}_{(1,0,0)}(\mathcal{L})$ and $\mathrm{der}_{(0,1,0)}(\mathcal{L})$ depends only on the dimension of $\mathcal{L}$, its derived algebra $D(\mathcal{L})$ and its center $C(\mathcal{L})$, invariants of these algebras will not distinguish between Lie algebras in the same class (formed in section 4.3). Therefore, these associative algebras are, from our point of view, useless.

Considering all possible intersections of vector spaces of generalized derivations we get only two new vector spaces:

- associative algebra $\mathrm{der}_{(1,0,0)}(\mathcal{L}) \cap \mathrm{der}_{(0,1,0)}(\mathcal{L})$ with dimension $\mathrm{codim}(D(\mathcal{L}))\dim(C(\mathcal{L}))$.
- Lie algebra $\mathrm{der}_{(1,1,1)}(\mathcal{L}) \cap \mathrm{der}_{(0,1,1)}(\mathcal{L})$.

Structure of both these intersections forms also invariants for $\mathcal{L}$. However, the associative algebra $\mathrm{der}_{(1,0,0)}(\mathcal{L}) \cap \mathrm{der}_{(0,1,0)}(\mathcal{L})$ is for any indecomposable Lie algebra $\mathcal{L}$ uniquely determined by dimensions of $\mathcal{L}, D(\mathcal{L})$ and $C(\mathcal{L})$ and therefore is also useless for our purpose.

We adopt the following notation for the list of invariants of $\mathcal{L}$ given by dimensions of the algebras of generalized derivations:

$$\begin{aligned}\dim_{(\alpha,\beta,\gamma)}(\mathcal{L}) = \ & [\dim(\mathrm{der}(\mathcal{L})), \dim(\mathrm{der}_{(0,1,1)}(\mathcal{L})), \dim(\mathrm{der}_{(1,1,0)}(\mathcal{L})), \\ & \dim(\mathrm{der}(\mathcal{L}) \cap \mathrm{der}_{(0,1,1)}(\mathcal{L})), \dim(\mathrm{der}_{(1,1,-1)}(\mathcal{L})), \dim(\mathrm{der}_{(0,1,-1)}(\mathcal{L}))].\end{aligned} \tag{4.50}$$

Furthermore, we use remaining one–parametric sets of vector spaces $\mathrm{der}_{(\alpha,1,1)}(\mathcal{L}), \mathrm{der}_{(\alpha,1,0)}(\mathcal{L})$ for the definition of the invariant functions of $\mathcal{L}$. Functions $\psi_{\mathcal{L}}, \psi^0_{\mathcal{L}} : \mathbb{C} \longrightarrow \mathbb{N}_0$ defined by relations

$$\psi_{\mathcal{L}}(\alpha) = \dim(\mathrm{der}_{(\alpha,1,1)}(\mathcal{L})) \tag{4.51}$$

$$\psi^0_{\mathcal{L}}(\alpha) = \dim(\mathrm{der}_{(\alpha,1,0)}(\mathcal{L})) \tag{4.52}$$

are called **invariant functions** corresponding to $(\alpha, \beta, \gamma)-$derivations of Lie algebra $\mathcal{L}$. It follows from the relations between vector spaces of generalized derivations that for all $\alpha \in \mathbb{C}$ it holds

$$\mathrm{codim}(D(\mathcal{L}))\dim(C(\mathcal{L})) \leq \ \psi^0_{\mathcal{L}}(\alpha) \ \leq \dim(\mathrm{der}_{(0,1,-1)}(\mathcal{L})), \tag{4.53}$$

$$\mathrm{codim}(D(\mathcal{L}))\dim(C(\mathcal{L})) \leq \ \psi_{\mathcal{L}}(\alpha) \ \leq (\dim(\mathcal{L}))^2. \tag{4.54}$$

Invariant functions turned out to be very useful especially for recognizing Lie algebras whose commutation relations depend on parameters. We have used them mainly for one–parametric families of Lie algebras. During the identification of graded contractions, invariant function $\psi_{\mathcal{L}}$ appeared to be more effective than function $\psi^0_{\mathcal{L}}$. Let us note that the invariant function $\psi_{\mathcal{L}}(\alpha)$ alone provides a complete set of invariants sufficient for classification of all three–dimensional complex Lie algebras [41]. Let us note that the same generalization of derivations is possible for any commutative or anti–commutative algebra.

The great advantage of these invariants is their easy computation. If c^k_{ij} are structural constants of the Lie algebra $\mathcal{L}$ in basis $(e_1, \ldots, e_n)$, then the matrix $A = (A_{ij}) \in \mathbb{C}^{n,n}$ of any $(\alpha, \beta, \gamma)-$derivation of $\mathcal{L}$ in this basis is given by the system of homogeneous linear equations:

$$\sum_{m=1}^{n} (\alpha c^m_{ij} A_{km} - \beta c^k_{mj} A_{mi} - \gamma c^k_{im} A_{mj}) = 0, \qquad i, j, k = 1, 2, \ldots n. \tag{4.55}$$

Thus, the computation of our invariants is reduced to the computation of rank of $n^2 \times n^3$ complex matrix.

4.8 Twisted cocycles

Invariant characteristics of Lie algebras can also be extracted from Chevalley cohomology of Lie algebras. For example, the dimensions of cohomology spaces with respect to adjoint and trivial representation were computed (up to dimension 7) and used as invariants for 7–dimensional nilpotent Lie algebras in [10]. It is also known that derivations of a Lie algebra $\mathcal{L}$ are cocycles of the dimension one corresponding to the adjoint representation of $\mathcal{L}$. The question whether there exists some equivalent for generalized derivations in cohomology theory was positively answered in [VII, 41].

Let us first recall the basic terms concerning cohomology of Lie algebras. Let V be a vector space over $\mathbb{C}$ and ρ representation of $\mathcal{L}$ on the vector space V. A $q-$linear map $c : \underbrace{\mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L}}_{q-times} \longrightarrow V$ is called a $V-$**cochain** of dimension q of $\mathcal{L}$, if c is alternating, i.e. $c(x_1, \ldots, x_q) = 0$ if $x_i = x_j$ for some $i \neq j$ or equivalently (over $\mathbb{C}$) if for all pairs of indices i, j, $(1 \leq i < j \leq q)$, it holds

$$c(x_1, \ldots, \underset{i}{x_i}, \ldots, \underset{j}{x_j}, \ldots, x_q) + c(x_1, \ldots, \underset{i}{x_j}, \ldots, \underset{j}{x_i}, \ldots, x_q) = 0. \tag{4.56}$$

We denote the vector space of all V-cochains of the dimension q for $q \in \mathbb{N}$ by $C^q(\mathcal{L}, V)$ and $C^0(\mathcal{L}, V) = V$. We define a map $d : C^q(\mathcal{L}, V) \longrightarrow C^{q+1}(\mathcal{L}, V)$ for $q = 0, 1, 2, \ldots$ by

$$\begin{aligned} dc(x) &= \rho(x)c, \quad c \in C^0(\mathcal{L}, V) \\ dc(x_1, \ldots, x_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} \rho(x_i) c(x_1, \ldots, \widehat{x_i}, \ldots, x_{q+1}) + \\ &+ \sum_{\substack{i,j=1 \\ i<j}}^{q+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{q+1}) \end{aligned} \tag{4.57}$$

where the symbol $\widehat{x_i}$ means that the term x_i is omitted. It is proved, for example in [28], that for this map d it holds $dd = 0$.

$V-$cochain $z \in C^q(\mathcal{L}, V)$ is called a **cocycle** of the dimension q corresponding to the representation ρ if $dz = 0$. The set of all cocycles of the dimension q corresponding to ρ is denoted by $Z^q(\mathcal{L}, \rho)$. An element $w \in C^q(\mathcal{L}, V)$ is called **coboundary** if there exists $c \in C^{q-1}(\mathcal{L}, V)$ such that $dc = w$. The set of all coboundaries $B^q(\mathcal{L}, \rho) = dC^{q-1}(\mathcal{L}, V)$ of the dimension q and the set $Z^q(\mathcal{L}, \rho)$ are vector subspaces of $C^q(\mathcal{L}, V)$. Since $dd = 0$, we have $B^q(\mathcal{L}, \rho) \subset Z^q(\mathcal{L}, \rho)$. The factor vector space $Z^q(\mathcal{L}, \rho)/B^q(\mathcal{L}, \rho) = H^q(\mathcal{L}, \rho)$ is then called a **cohomology space** of the dimension q of $\mathcal{L}$ with respect to the representation ρ. It follows directly from the definition that cocycles and coboundaries of the dimension one corresponding to the adjoint representation of $\mathcal{L}$ are

$$Z^1(\mathcal{L}, \mathrm{ad}_{\mathcal{L}}) = \mathrm{der}(\mathcal{L}), \qquad B^1(\mathcal{L}, \mathrm{ad}_{\mathcal{L}}) = \mathrm{ad}(\mathcal{L}). \tag{4.58}$$

The concept of cocycles was generalized [VII, 41] in the following way. Let ρ be a representation of the complex Lie algebra $\mathcal{L}$ on the vector space V and $\kappa = (\kappa_{ij})$ a $(q+1)\times(q+1)$ complex symmetric matrix. We call any V–cochain $c \in C^q(\mathcal{L}, V),\ q \in \mathbb{N}$ a κ–**twisted cocycle** or shortly κ–**cocycle** of the dimension q corresponding to the representation ρ if it satisfies

$$\begin{aligned} 0 = &\sum_{i=1}^{q+1}(-1)^{i+1}\kappa_{ii}\rho(x_i)c(x_1,\ldots,\widehat{x_i},\ldots,x_{q+1})+ \\ &+\sum_{\substack{i,j=1\\ i<j}}^{q+1}(-1)^{i+j}\kappa_{ij}c([x_i,x_j],x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_{q+1}) \end{aligned} \tag{4.59}$$

for all $x_1\ldots, x_{q+1} \in \mathcal{L}$. The set of all κ–cocycles of the dimension q, denoted by $Z^q(\mathcal{L}, \rho, \kappa)$, forms a vector subspace of $C^q(\mathcal{L}, V)$. Considering one–dimensional κ-cocycles corresponding to the adjoint representation of $\mathcal{L}$ we get (α, β, γ)–derivations of $\mathcal{L}$:

$$Z^1\left(\mathcal{L}, \mathrm{ad}_{\mathcal{L}}, \left(\begin{smallmatrix}\beta & \alpha\\ \alpha & \gamma\end{smallmatrix}\right)\right) = \mathrm{der}_{(\alpha,\beta,\gamma)}(\mathcal{L}). \tag{4.60}$$

Let us note that there is no suitable generalization of mapping d (4.57) compatible with (4.59) and thus no twisted coboundaries and twisted cohomology.

Since we are interested in comparable invariant characteristics, we have to consider only those representations which are given uniquely by the examined Lie algebra, i.e. adjoint or trivial representation. It was proved in [41] that if $h : \mathcal{L} \longrightarrow \widetilde{\mathcal{L}}$ is an isomorphism of Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, then the mapping $f : C^q(\mathcal{L}, \mathcal{L}) \longrightarrow C^q(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}})$ defined for all $c \in C^q(\mathcal{L}, \mathcal{L})$ by relation

$$(fc)(\widetilde{x}_1,\ldots,\widetilde{x}_q) = hc(h^{-1}\widetilde{x}_1,\ldots,h^{-1}\widetilde{x}_q), \qquad \forall \widetilde{x}_1,\ldots,\widetilde{x}_q \in \widetilde{\mathcal{L}} \tag{4.61}$$

is an isomorphism of vector spaces $C^q(\mathcal{L}, \mathcal{L}), C^q(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}})$. Moreover, for any complex symmetric $(q + 1) \times (q + 1)$ matrix κ it holds

$$f(Z^q(\mathcal{L}, \mathrm{ad}_{\mathcal{L}}, \kappa)) = Z^q(\widetilde{\mathcal{L}}, \mathrm{ad}_{\widetilde{\mathcal{L}}}, \kappa). \tag{4.62}$$

Thus, the dimensions of the vector spaces $Z^q(\mathcal{L}, \mathrm{ad}_{\mathcal{L}}, \kappa)$ are invariants of the Lie algebra $\mathcal{L}$.

In [41] vector spaces of two–dimensional κ–cocycles corresponding to $\mathrm{ad}_{\mathcal{L}}$ were explored. These vector spaces denoted by

$$\mathrm{coc}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)}(\mathcal{L}) = Z^2\left(\mathcal{L}, \mathrm{ad}_{\mathcal{L}}, \begin{pmatrix}\beta_1 & \alpha_2 & \alpha_3\\ \alpha_2 & \beta_3 & \alpha_1\\ \alpha_3 & \alpha_1 & \beta_2\end{pmatrix}\right) \tag{4.63}$$

consist of all V–cochains $B \in C^2(\mathcal{L}, \mathcal{L})$ which for all $x, y, z \in \mathcal{L}$ satisfy

$$\begin{aligned} 0 = &\ \alpha_1 B(x, [y, z]) + \alpha_2 B(z, [x, y]) + \alpha_3 B(y, [z, x]) \\ &+ \beta_1[x, B(y, z)] + \beta_2[z, B(x, y)] + \beta_3[y, B(z, x)]. \end{aligned} \tag{4.64}$$

Classification of spaces $\mathrm{coc}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)}(\mathcal{L})$, done in [41], leads to 16 possible spaces which depend on two, three or four parameters. Besides this classification we choose, rather empirically following our calculations, two one–parametric sets of vector spaces $\mathrm{coc}_{(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)}$ and define new invariant functions. We call functions $\varphi_{\mathcal{L}}, \varphi^0_{\mathcal{L}} : \mathbb{C} \longrightarrow \mathbb{N}_0$ defined by formulas

$$\varphi_{\mathcal{L}}(\alpha) = \dim(\mathrm{coc}_{(1,1,1,\alpha,\alpha,\alpha)}(\mathcal{L})) \qquad (4.65)$$

$$\varphi^0_{\mathcal{L}}(\alpha) = \dim(\mathrm{coc}_{(0,1,1,\alpha,1,1)}(\mathcal{L})) \qquad (4.66)$$

the **invariant functions** corresponding to two–dimensional twisted cocycles of the adjoint representation of a Lie algebra $\mathcal{L}$. Computation of these invariant functions consists in solution of a system of linear homogeneous equations, i.e. in the determination of the rank of a matrix.

Let us note that the invariant function $\varphi_{\mathcal{L}}$ together with the invariant function $\psi_{\mathcal{L}}$ defined in the previous section classify all four–dimensional complex Lie algebras [VII, 41].

4.9 Isomorphism

There are only two possible ways how to prove that two given algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ of the same dimension n are isomorphic. The first is the explicit calculation of their isomorphism. The second way (possible only for low–dimensional Lie algebras) is based on the existing classification. If there exists in the desired dimension classification of Lie algebras given by invariant characteristics, it is sufficient to compute and compare these invariant characteristics. In any case if two Lie algebras differ in some invariant characteristic, then they are not isomorphic. Since there is no complete classification in the dimension 8, we have to follow the first way. We try to find explicitly isomorphisms for all algebras which are in the same class.

Let us consider that c^k_{ij} and $\tilde{c}^k_{ij}$ are structural constants of Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ with respect to bases $\mathcal{E} = (e_1, \ldots, e_n)$ and $\widetilde{\mathcal{E}} = (\widetilde{e}_1, \ldots, \widetilde{e}_n)$. Then the regular linear mapping $A : \mathcal{L} \longrightarrow \widetilde{\mathcal{L}}$ defined by relations

$$Ae_i = \sum_{j=1}^{n} A_{ji}\widetilde{e}_j, \qquad A_{ji} \in \mathbb{C}, \qquad (4.67)$$

is isomorphism if and only if the following system of $n^2(n-1)/2$ quadratic equations is satisfied

$$\sum_{r=1}^{n} c^r_{ij} A_{kr} = \sum_{\mu,\nu=1}^{n} \tilde{c}^k_{\mu\nu} A_{\mu i} A_{\nu j}, \qquad i = 1, \ldots, n-1, \quad j = i, \ldots, n, \quad k \in \hat{n}. \qquad (4.68)$$

Using computers (system MAPLE 8) it is usually more convenient to solve the system (4.68) first and then test its solutions on regularity, i.e. if $\det(A_{ij}) \neq 0$. Computation can be simplified by choice of bases $\mathcal{E}, \widetilde{\mathcal{E}}$. For example, a basis which follows lower or upper central series, immediately fix some zero elements in the matrix (A_{ij}). However, since the system (4.68) is not a system of linear equations, there is no guarantee that the computer will find all its solutions. Thus, only the existence of an isomorphism can be proved in this way. Sometimes, it is possible to prove the nonexistence of a regular solution of the system (4.68) by hand.

Let us note that there exists also an algorithm for solving systems of nonlinear equations such as (4.68). This algorithm based on so called Gröbner basis was successfully used in [15] for finding isomorphisms of four–dimensional Lie algebras. However, it seems to be not suitable for us since, according to [46], the program already fails to terminate in a reasonable time in the dimension 6 in some cases and we are working mainly in the dimension 8.

4.10 Identification procedure

We summarize the whole identification procedure, which we have used for classifying graded contractions of the Lie algebra sl$(3, \mathbb{C})$. The individual steps of this procedure, described in detail in previous sections, are applied in the following order:

- **Central decomposition** – we separate a maximal central component whenever it is possible and continue with identification of non–abelian parts of Lie algebras only.

- **Direct decomposition** – we decompose all Lie algebras into a sum of indecomposable ones and continue with identification of these indecomposable Lie algebras.

- **Numerical invariants** – we compute derived, lower and upper central series, algebras of generalized derivation and number of formal invariants and divide all algebras into classes according to their invariants
$$\mathrm{inv}(\mathcal{L}) = DS(\mathcal{L}), CS(\mathcal{L}), US(\mathcal{L}), \dim_{(\alpha,\beta,\gamma)}(\mathcal{L}), \tau(\mathcal{L}). \tag{4.69}$$

- **Levi decomposition** – we determine radical and Levi decomposition of any non–solvable Lie algebra and refine classes of these non–solvable Lie algebras according to the types of radicals and semisimple parts.

- **Isomorphisms** – we test all Lie algebras in the same class upon isomorphisms. We keep only one representative of the isomorphic Lie algebras.

- **Nilradical** – we determine nilradical for any solvable Lie algebra and refine classes of solvable non–nilpotent Lie algebras according to the types of nilradicals.

- **Casimir operators** – we compute Casimir operators for all nilpotent Lie algebras and, whenever necessary, use their orders as invariants.

- **Invariant functions** – we compute invariant functions ψ, φ, φ^0 for all one–parametric families of Lie algebras and for Lie algebras which have not been distinguished from each other yet.

We took the following convention for parametric families of Lie algebras. A parametric family is considered and counted as one Lie algebra (parametric Lie algebra) whenever it has the same derived, lower and upper central series for all values of its parameters. The values of the parameters for which numerical invariants or Casimir operators will differ from general ones will be marked and corresponding Lie algebras will be considered only as special cases of this parametric Lie algebra.

During the identification we change the basis of the given Lie algebra such that first vectors of the new basis form gradually the basis of the center, the nilradical and the radical.

There are also following possibilities how to distinguish two non–isomorphic Lie algebras, which were not used in the main identification procedure:

- Compare the structure of ideals in their derived, lower and upper central series.

- Compare the structure of algebras of generalized derivations.

- Build the tower of generalized derivations, i.e. generalized derivations of algebras of generalized derivations and so on.

- Choose other possible invariant functions which are related with twisted cocycles of the dimension two or more.

Let us note that there are more invariants which we have found in literature and did not use for classification of our results. These invariant characteristics of the Lie algebra $\mathcal{L}$ are:

(1) The maximal dimension of abelian subalgebra and the maximal dimension of abelian ideal. These are used in [55].

(2) Megaideals, ideals and characteristic subalgebras were determined for low–dimensional Lie algebras in [65]. A vector subspace $V \subset \mathcal{L}$ which is invariant under any automorphism of $\mathcal{L}$, i.e. $g(V) \subset V,\ \forall g \in \mathrm{Aut}(\mathcal{L})$, is called a **megaideal** of the Lie algebra $\mathcal{L}$. A vector subspace $V \subset \mathcal{L}$ which is invariant under any derivation of $\mathcal{L}$, i.e. $D(V) \subset V,\ \forall D \in \mathrm{der}(\mathcal{L})$, is called a **characteristic subalgebra** of $\mathcal{L}$. All megaideals and characteristic subalgebras of $\mathcal{L}$ are ideals of $\mathcal{L}$.

(3) The structure of all possible subalgebras of the Lie algebra $\mathcal{L}$. Algorithm for finding all subalgebras, which are inequivalent with respect to the group of inner automorphisms was described in [56].

(4) The structure of factor algebras such as $\mathcal{L}/C(\mathcal{L})$ as well as existence of some special elements, for example $x \in \mathcal{L}$ such that $\dim([x, \mathcal{L}]) = 1$, is considered in [68].

(5) The dimension of null space of the Killing form, i.e.

$$\nu(\mathcal{L}) = \dim \{x \in \mathcal{L} \,|\, K_{\mathcal{L}}(x, y) = \mathrm{Tr}(\mathrm{ad}_{\mathcal{L}}(x)\, \mathrm{ad}_{\mathcal{L}}(y)) = 0, \ \forall y \in \mathcal{L}\} .$$

(6) The dimension of the space of all invariant symmetric bilinear forms, i.e. all bilinear mapping $\omega : \mathcal{L} \times \mathcal{L} \longrightarrow \mathbb{C}$ satisfying

$$\omega(x, y) = \omega(y, x), \qquad \omega([x, y], z) = \omega(x, [y, z]), \qquad \forall x, y, z \in \mathcal{L}. \tag{4.70}$$

(7) The dimensions of all cohomology spaces with respect to adjoint and trivial representation were used in [10, 52, 53].

(8) The dimension and structure of Lie algebra of prederivations [9]. Any $P \in \mathrm{gl}(\mathcal{L})$ is called **prederivation** of $\mathcal{L}$ if for all $x, y, z \in \mathcal{L}$ it holds

$$P([x, [y, z]]) = [P(x), [y, z]] + [x, [P(y), z]] + [x, [y, [P(z)]]. \tag{4.71}$$

The set of all prederivations of $\mathcal{L}$ forms a subalgebra $\mathrm{Pder}(\mathcal{L})$ of the Lie algebra $\mathrm{gl}(\mathcal{L})$ and contains $\mathrm{der}(\mathcal{L})$.

(9) The characteristic sequence for nilpotent Lie algebras [29]. Let $\mathcal{L}$ be a nilpotent complex finite–dimensional Lie algebra. For any $x \in \mathcal{L}$ we denote by $c(x) = (n_1, n_2, \ldots, n_p)$ the ordered sequence $n_1 \geq n_2 \geq \ldots \geq n_p$ of dimensions of the Jordan blocks of the nilpotent operator $\mathrm{ad}_{\mathcal{L}}(x)$. Since x is an eigenvector of $\mathrm{ad}_{\mathcal{L}}(x)$, we have always $n_p = 1$. Let $x, y \in \mathcal{L}$ and $c(x) = (n_1, n_2, \ldots, n_p = 1)$, $c(y) = (m_1, m_2, \ldots, m_q = 1)$ be corresponding sequences, we say that $c(x) \geq c(y)$ if there exists i such that $n_1 = m_1, n_2 = m_2, \ldots, n_{i-1} = m_{i-1}$ and $n_i > m_i$. This defines the lexicographical ordering in the set of all sequences. The ordered sequence

$$c(\mathcal{L}) = \sup \{c(x) \,|\, x \in \mathcal{L} \setminus D(\mathcal{L})\} \tag{4.72}$$

is called a **characteristic sequence** of the nilpotent Lie algebra $\mathcal{L}$ and a vector $x \in \mathcal{L}$ for which $c(x) = c(\mathcal{L})$ is called a **characteristic vector** of $\mathcal{L}$.

(10) The minimal numbers used for two–step nilpotent ($\mathcal{L}^3 = [[\mathcal{L}, \mathcal{L}], \mathcal{L}]] = 0$) Lie algebras in [27]. Chose a basis $(x_1 + D(\mathcal{L}), \dots, x_s + D(\mathcal{L}))$ for $\mathcal{L}/D(\mathcal{L})$ such that

$$\mathrm{mn}(\mathcal{L}) = (\dim(\mathrm{ad}_{\mathcal{L}}(x_1)(\mathcal{L})), \dots, \dim(\mathrm{ad}_{\mathcal{L}}(x_s)(\mathcal{L}))) \tag{4.73}$$

is minimal s–tuple in lexicographic order. Then $\mathrm{mn}(\mathcal{L})$ is invariant of $\mathcal{L}$.

(11) Ratios of the eigenvalues of $\mathrm{ad}_{\mathcal{L}}(x)$ for generic element x in $\mathcal{L}$ are used as invariants in classification of 4–dimensional complex Lie algebras [3]. Moreover, the following invariants for non–nilpotent Lie algebras are defined there. If the numbers

$$\chi_1(\mathcal{L}) = \frac{p_{222}(x)}{p_{111}^2(x)}, \qquad \chi_2(\mathcal{L}) = \frac{p_{333}(x)}{p_{111}^3(x)}, \qquad \chi_3(\mathcal{L}) = \frac{p_{333}^2(x)}{p_{222}^3(x)}, \tag{4.74}$$

where

$$p_{111}(x) = -\operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x)), \qquad p_{222}(x) = \frac{1}{2}[(\operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x)))^2 - \operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x))^2], \tag{4.75}$$

$$p_{333}(x) = \frac{1}{6}[(\operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x)))^3 - 3\operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x))\operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x))^2 + 2\operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x))^3], \tag{4.76}$$

are defined and independent of $x \in \mathcal{L}$, then they are invariants of $\mathcal{L}$.

(12) The other trace–based invariants are used for non–nilpotent Lie algebras in [8]. Let $i, j \in \mathbb{N}$, if the number

$$c_{ij}(\mathcal{L}) = \frac{\operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(x))^i \operatorname{Tr}(\mathrm{ad}_{\mathcal{L}}(y))^j}{\operatorname{Tr}((\mathrm{ad}_{\mathcal{L}}(x))^i(\mathrm{ad}_{\mathcal{L}}(y))^j)} \tag{4.77}$$

is defined and independent of elements $x, y \in \mathcal{L}$, then we call it (i, j)–**invariant** of Lie algebra $\mathcal{L}$.

(13) The rank of the Lie algebra $\mathcal{L}$ is defined as the dimension of its Cartan subalgebra. The **Cartan subalgebra** is a nilpotent subalgebra $\mathcal{H}$ of $\mathcal{L}$ which is its own normalizer, i.e. $\mathcal{H} = \{x \in \mathcal{L} \,|\, [x, \mathcal{H}] \subset \mathcal{H}\}$. The set of all roots of $\mathcal{H}$ in $\mathcal{L}$, i.e. all linear maps $\alpha : \mathcal{H} \to \mathbb{C}$ for which

$$L(\alpha) = \big\{y \in \mathcal{L} \,|\, \forall x \in \mathcal{H}, \exists k \in \mathbb{N}, (\mathrm{ad}_{\mathcal{L}}(x) - \alpha(x))^k(y) = 0\big\} \neq \{0\}, \tag{4.78}$$

forms a root system. The role of root systems of Lie algebras is well known from the classification of simple Lie algebras [28].

(14) The rank of the nilpotent Lie algebra $\mathcal{L}$ is defined as the dimension of the **maximal torus** T on $\mathcal{L}$, i.e. the maximal (in the sense of inclusion) abelian subalgebra T in $\mathrm{der}(\mathcal{L})$ consisting of semisimple (eigenvectors form a basis of $\mathcal{L}$) endomorphisms. For

any linear map $\alpha \in T^*$ (T^* is the dual space of the vector space T) we define a vector subspace of $\mathcal{L}$

$$L_\alpha = \{x \in \mathcal{L} \,|\, t(x) = \alpha(t)x, \ \forall t \in T\}\,. \tag{4.79}$$

The set

$$W(T) = \{(\alpha, \dim(L_\alpha)) \,|\, \alpha \in T^*, \ \dim(L_\alpha) > 0\} \tag{4.80}$$

is called the **weight system** associated to $\mathcal{L}$. Two weight systems $W(T)$ and $W'(T')$ are equivalent if $\dim(T) = \dim(T')$ and the linear representation of T in $\mathcal{L}$ is equivalent to that of T' on $\mathcal{L}'$. Equivalence classes of weight systems were used as invariants in [12].

The invariants mentioned above were not used in our identification procedure for various reasons. First of all, we were looking for a uniform way of describing contracted Lie algebras. For example, we have not found a suitable algorithm for computing (1) and (2). These can be obtained from (3), however, the process of searching (3) is too laborious. The comparison of the factor algebras (4) is a lower–dimensional task, however, it works well only in some cases and, similarly as the searching for special elements, it requires an individual treatment of the investigated algebras. Numerical invariants (5),(6) and (8) are easily computable but have relatively small discerning ability and do not significantly extend our set of invariants $\mathrm{inv}(\mathcal{L})$. Invariants (9) and (10) are not convenient for us, since their computation is based on finding the extremal values over elements of a given Lie algebra and thus present vast difficulties on a computer. Invariants (11) and (12) could be very helpful especially for parametric families of Lie algebras, but unfortunately, they are not defined for nilpotent Lie algebras. The computation of (13) and (14) and subsequently their juxtaposition is too laborious and not suitable for our approach. Finally, cohomologies (7) are helpful, but they are not able to classify parametric families of Lie algebras. Therefore, we have preferred rather twisted cocycles over cohomologies.

Chapter 5

Pauli graded contractions of $\mathrm{sl}(3,\mathbb{C})$

The content of this chapter was already published in [II].

5.1 Pauli grading of $\mathrm{sl}(3,\mathbb{C})$

A finest grading called Pauli grading was found [58] for any classical simple Lie algebra in the series $A_{n-1} = \mathrm{sl}(n,\mathbb{C})$, $n \geq 2$. The name Pauli grading is deduced from generalized $n \times n$ Pauli matrices which form the bases for grading subspaces of the Pauli grading of $\mathrm{sl}(n,\mathbb{C})$ in its defining n–dimensional representation. Let us define $n \times n$ complex matrices

$$Q_n = \mathrm{diag}(1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}), \qquad P_n = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{5.1}$$

where $\omega_n = \exp(2\pi i/n)$. These matrices fulfil

$$P_n^n = Q_n^n = \mathbf{1}_n, \qquad P_n Q_n = \omega_n Q_n P_n, \tag{5.2}$$

and their products — generalized Pauli matrices — form so called **Pauli's group**

$$\Pi_n = \left\{ \omega_n^k Q_n^i P_n^j \,|\, i,j,k = 0,1,\ldots,n-1 \right\}. \tag{5.3}$$

In our case $n = 3$ the Pauli grading is one of the four fine group gradings of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. The Pauli grading decomposes $\mathrm{sl}(3,\mathbb{C})$ into eight 1–dimensional subspaces. In the defining 3–dimensional representation, the basis vectors (generators) are 3×3 generalized

Pauli matrices:

$$\begin{aligned}\Gamma_P : \mathrm{sl}(3,\mathbb{C}) &= L_{01} \oplus L_{02} \oplus L_{10} \oplus L_{20} \oplus L_{11} \oplus L_{22} \oplus L_{12} \oplus L_{21} \\ &= \mathbb{C}Q \oplus \mathbb{C}Q^2 \oplus \mathbb{C}P \oplus \mathbb{C}P^2 \oplus \mathbb{C}PQ \oplus \mathbb{C}P^2Q^2 \oplus \mathbb{C}PQ^2 \oplus \mathbb{C}P^2Q \\ &= \mathbb{C}\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \oplus \\ &\oplus \mathbb{C}\begin{pmatrix}0&\omega&0\\0&0&\omega^2\\1&0&0\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0&0&\omega\\1&0&0\\0&\omega^2&0\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0&\omega^2&0\\0&0&\omega\\1&0&0\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0&0&\omega^2\\1&0&0\\0&\omega&0\end{pmatrix}\end{aligned} \tag{5.4}$$

where $\omega = \exp(2\pi i/3)$ and $Q = Q_3, P = P_3$. Putting $L_{rs} := \mathrm{span}_{\mathbb{C}}\{X_{rs}\} \equiv \mathbb{C}X_{rs} = \mathbb{C}Q^rP^s$, and using (5.2) we have the commutation relations

$$[X_{rs}, X_{r's'}] = (\omega^{sr'} - \omega^{rs'})X_{r+r',s+s' \pmod 3}\,. \tag{5.5}$$

The index set I for the Pauli grading consists of 8 ordered couples (r,s), where $r,s = 0,1,2$ with the exception of $(0,0)$ and the commutative operation $\diamond$ is a componentwise addition (mod 3). Thus, the grading group is the additive abelian group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

The symmetry group $\Delta_{\Gamma_P}(\mathrm{Aut}(\Gamma_P))$ of the Pauli grading was described in detail in [34, 35, 36]. It was shown there that $\Delta_{\Gamma_P}(\mathrm{Aut}(\Gamma_P))$ is isomorphic to a finite matrix group

$$H_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}_3,\ ad - bc \neq 0 \pmod 3 \right\}. \tag{5.6}$$

This group has 48 elements and contains the subgroup $SL(2,\mathbb{Z}_3)$ of order 24 formed by all matrices with determinant equal to 1. If we denote $\pi_A \in \Delta_{\Gamma_P}(\mathrm{Aut}(\Gamma_P))$ the permutation of index set $I \subset \mathbb{Z}_3 \times \mathbb{Z}_3$ corresponding to the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H_3$, then the action of π_A on the indices $(i,j) \in I$ is given by

$$\pi_A(i,j) = (i,j)A = ((ia + cj) \mod 3, (ib + jd) \mod 3). \tag{5.7}$$

For computer processing it is more convenient to choose some ordering $\mathcal{O}$ of index set I and replace I by the set $\{1, 2 \ldots, 8\}$. Our ordering $\mathcal{O}$ corresponds to the order of grading subspaces in (5.4), i.e.

$$\mathcal{O} \quad \begin{matrix} (0,1) & (0,2) & (1,0) & (2,0) & (1,1) & (2,2) & (1,2) & (2,1) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \tag{5.8}$$

We will also the use notation for basis vectors $X_{rs} = e_{\mathcal{O}(r,s)}$. The commutation relations of $\mathrm{sl}(3,\mathbb{C})$ in basis $(e_1, \ldots, e_n)$ are written in Table 5.1.

Table 5.1: Commutation relations of Pauli graded sl$(3,\mathbb{C})$

$\mathcal{L}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	0	$(\omega-1)e_5$	$(\omega^2-1)e_8$	$(\omega-1)e_7$	$(\omega^2-1)e_4$	$(\omega-1)e_3$	$(\omega^2-1)e_6$
e_2		0	$(\omega^2-1)e_7$	$(\omega-1)e_6$	$(\omega^2-1)e_3$	$(\omega-1)e_8$	$(\omega^2-1)e_5$	$(\omega-1)e_4$
e_3			0	0	$(1-\omega)e_8$	$(1-\omega^2)e_2$	$(1-\omega^2)e_6$	$(1-\omega)e_1$
e_4				0	$(1-\omega^2)e_1$	$(1-\omega)e_7$	$(1-\omega)e_2$	$(1-\omega^2)e_5$
e_5					0	0	$(\omega-\omega^2)e_4$	$(\omega^2-\omega)e_2$
e_6						0	$(\omega^2-\omega)e_1$	$(\omega-\omega^2)e_3$
e_7							0	0
e_8								0

5.2 Contraction system for the Pauli grading

In this section we will describe the construction of the system of contraction equations S_P for the Pauli grading of sl$(3,\mathbb{C})$ using the symmetry group of this grading. We start with overview of the orbits of the action of the symmetry group $H_3 \cong \Delta_{\Gamma_P}(\mathrm{Aut}(\Gamma_P))$ on the sets I, I_u^2, I_u^3.

The index set $I = \mathbb{Z}_3 \times \mathbb{Z}_3 \setminus \{(0,0)\}$ forms a whole one orbit. It corresponds to the fact that the symmetry group $\mathrm{Aut}(\Gamma_P)$ transforms an arbitrary grading subspace to any of the other grading subspaces. There are three orbits of unordered pairs of grading indices in I_u^2. Two of them are formed by irrelevant pairs: 8–point orbit represented by $((0,1)(0,1))$ and 4–point orbit represented by $((0,1)(0,2))$. The remaining 24–point orbit, represented by point $((0,1)(1,0))$, is equal to the set of all relevant pairs of unordered grading indices $\mathcal{I}$, i.e.

$$\mathcal{I} = \{((0,1)A(1,0)A) \mid A \in H_3\}\,. \tag{5.9}$$

These relevant pairs of indices correspond to 24 relevant contraction parameters. Irrelevant parameters are equal to zero and thus, the explicit form of the contraction matrix ε with respect to chosen ordering $\mathcal{O}$ is:

$$\varepsilon = \begin{pmatrix} 0 & 0 & \varepsilon_{(01)(10)} & \varepsilon_{(01)(20)} & \varepsilon_{(01)(11)} & \varepsilon_{(01)(22)} & \varepsilon_{(01)(12)} & \varepsilon_{(01)(21)} \\ 0 & 0 & \varepsilon_{(02)(10)} & \varepsilon_{(02)(20)} & \varepsilon_{(02)(11)} & \varepsilon_{(02)(22)} & \varepsilon_{(02)(12)} & \varepsilon_{(02)(21)} \\ \varepsilon_{(01)(10)} & \varepsilon_{(02)(10)} & 0 & 0 & \varepsilon_{(10)(11)} & \varepsilon_{(10)(22)} & \varepsilon_{(10)(12)} & \varepsilon_{(10)(21)} \\ \varepsilon_{(01)(20)} & \varepsilon_{(02)(20)} & 0 & 0 & \varepsilon_{(20)(11)} & \varepsilon_{(20)(22)} & \varepsilon_{(20)(12)} & \varepsilon_{(20)(21)} \\ \varepsilon_{(01)(11)} & \varepsilon_{(02)(11)} & \varepsilon_{(10)(11)} & \varepsilon_{(20)(11)} & 0 & 0 & \varepsilon_{(11)(12)} & \varepsilon_{(11)(21)} \\ \varepsilon_{(01)(22)} & \varepsilon_{(02)(22)} & \varepsilon_{(10)(22)} & \varepsilon_{(20)(22)} & 0 & 0 & \varepsilon_{(22)(12)} & \varepsilon_{(22)(21)} \\ \varepsilon_{(01)(12)} & \varepsilon_{(02)(12)} & \varepsilon_{(10)(12)} & \varepsilon_{(20)(12)} & \varepsilon_{(11)(12)} & \varepsilon_{(22)(12)} & 0 & 0 \\ \varepsilon_{(01)(21)} & \varepsilon_{(02)(21)} & \varepsilon_{(10)(21)} & \varepsilon_{(20)(21)} & \varepsilon_{(11)(21)} & \varepsilon_{(22)(21)} & 0 & 0 \end{pmatrix}. \tag{5.10}$$

Let us note that the orbits of the symmetry group H_3 on the sets I and I_u^2 coincide with the orbits of its subgroup $SL(2,\mathbb{Z}_3)$.

Contraction equations are labelled by unordered triplets of grading indices I_u^3. The equation $e((i,j)(k,l)(m,n)) \in \mathsf{S}_P$ is given by

$$\varepsilon_{(i,j)(k+m,l+n)}\varepsilon_{(k,l)(m,n)}[X_{ij}[X_{kl},X_{mn}]] + \text{ cyclically } = 0, \tag{5.11}$$

where the word "cyclically" means that the two remaining terms are obtained from the first one by the substitutions: $(ij) \to (mn)$, $(kl) \to (ij)$, $(mn) \to (kl)$ and $(ij) \to (kl)$, $(kl) \to (mn)$, $(mn) \to (ij)$, respectively. Using (5.5) we have

$$[\varepsilon_{(i,j)(k+m,l+n)}\varepsilon_{(k,l)(m,n)}(\omega^{lm}-\omega^{kn})(\omega^{j(k+m)}-\omega^{i(l+n)}) + \text{ cyclically}\,]X_{i+k+m,j+l+n} = 0. \tag{5.12}$$

The equation (5.12) is identically fulfilled for any unordered triplet $((i,j)(k,l)(m,n))$ which has at least two identical indices, for example $(i,j)=(k,l)$. Thus, the number of contraction equations is given by the combination number $\binom{8}{3} = 56$. Moreover, it follows from (5.12) that the equations for which it simultaneously holds $i+k+m=0$ and $j+l+n=0$ are also fulfilled identically (where operation $+$ is considered in $\mathbb{Z}_3$, i.e. $+ \mod 3$). This situation arises in eight cases. Hence, the contraction system consists of 48 equations.

The set I_u^3 of unordered triplets of grading indices is decomposed into six orbits with respect to the action of the symmetry group H_3. There are three 8–point orbits represented by points $((0,1)(0,1)(0,1)), ((0,1)(0,1)(0,2)), ((0,1)(1,0)(2,2))$ and one 48–point orbit represented by $((0,1)(0,1)(1,0))$, which lead to identically fulfilled equations. Remaining two 24–point orbits represented by points $((0,1)(1,0)(0,2))$ and $((0,1)(1,0)(1,1))$ lead to two sets of contraction equations. Let us note that the orbits of H_3 and its subgroup $SL(2,\mathbb{Z}_3)$ on the set I_u^3 coincide except the 48–point orbit which splits into two 24–point orbits of $SL(2,\mathbb{Z}_3)$ represented by points $((0,1)(0,1)(1,0))$ and $((0,1)(0,1)(2,0))$.

The contraction system S_P is now generated by the action of $SL(2,\mathbb{Z}_3)$ from two equations

$e((0,1)(1,0)(0,2))$ and $e((0,1)(1,0)(1,1))$. These equations

$$[\varepsilon_{(01)(12)}\varepsilon_{(10)(02)}(1-\omega^2)(\omega-1)+\varepsilon_{(02)(11)}\varepsilon_{(01)(10)}(\omega-1)(\omega^2-1)+0\,]X_{10}=0, \tag{5.13}$$

$$[\varepsilon_{(01)(21)}\varepsilon_{(10)(11)}(1-\omega)(\omega^2-1)+\varepsilon_{(10)(12)}\varepsilon_{(11)(01)}(1-\omega)(1-\omega^2)+0\,]X_{22}=0, \tag{5.14}$$

can be simplified and the whole system of 48 contraction equations can be written simply as

$$\mathsf{S}_P^a:\ \varepsilon_{(01)(12)A}\varepsilon_{(02)(10)A}-\varepsilon_{(01)(10)A}\varepsilon_{(02)(11)A}=0,\quad \forall A\in SL(2,\mathbb{Z}_3), \tag{5.15}$$

$$\mathsf{S}_P^b:\ \varepsilon_{(01)(21)A}\varepsilon_{(10)(11)A}-\varepsilon_{(01)(11)A}\varepsilon_{(10)(12)A}=0,\quad \forall A\in SL(2,\mathbb{Z}_3), \tag{5.16}$$

where we have used the abbreviation $\varepsilon_{(ij)(kl)A}=\varepsilon_{(ij)A(kl)A}$.

It turns out that the system of contraction equations S_P contains linearly dependent equations which can be eliminated. It follows from the fact that quadruples of grading indices [(01)(12)][(02)(10)] and [(01)(10)][(02)(11)] lie in the same $SL(2,\mathbb{Z}_3)$–orbit in the set $\mathcal{I}_u^2$ (the pairs of indices in the bracket [ ] and the pairs of these brackets are unordered). Remaining quadruples [(01)(21)][(10)(11)] and [(01)(11)][(10)(12)] lie in different orbits in $\mathcal{I}_u^2$ and thus cannot be transformed to each other. Therefore, the equation of the subsystem S_P^a generated by the unit matrix can be written in the form

$$\varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X}-\varepsilon_{(01)(10)}\varepsilon_{(02)(11)}=0,\quad X=\begin{pmatrix}1&2\\0&1\end{pmatrix}. \tag{5.17}$$

Adding this equation and the equation of S_P^a generated by matrix $A=X$

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2}-\varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X}=0, \tag{5.18}$$

we get

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2}-\varepsilon_{(01)(10)}\varepsilon_{(02)(11)}=0. \tag{5.19}$$

Since $X^3=1$ holds, the equation (5.19) is the equation of S_P^a generated by matrix $A=X^2$. Hence, we conclude that the left cosets of $SL(2,\mathbb{Z}_3)$ with respect to the cyclic subgroup $\{1,X,X^2\}$ generate the triplets of dependent equations. By Lagrange's theorem, the number of these cosets is $24/3=8$. In this way we obtained 8 equations (one to each coset) which can be eliminated from the system S_P^a. Therefore, the system of contraction equations S_P for the Pauli grading of $\mathrm{sl}(3,\mathbb{C})$ consists of 40 linearly independent equations.

5.3 Finding the solution of S_P

We solve the system of contraction equations S_P using the Theorem 3.9. First of all we choose the appropriate sequence of assumptions. The system S_P can be solved explicitly under the assumption that two of its variables do not vanish. Since all relevant contraction parameters lie in the same orbit, we can choose the index $k \in \mathcal{I}$ of the first variable arbitrarily. In order to find the appropriate second variable, we define the equivalence relation $\equiv^k$ on the set $\mathcal{I}^k := \mathcal{I} \setminus \{k\}$ as follows

$$i, j \in \mathcal{I}^k, \qquad i \equiv^k j \;\Leftrightarrow\; (\exists \pi \in \Delta_{\Gamma_P}(\mathrm{Aut}(\Gamma_P)))(\pi\,(i\ k) = (j\ k)), \tag{5.20}$$

where $(i\ j) \in \mathcal{I}_u^2$ denotes an unordered pair $i, j \in \mathcal{I}$ and $\pi\,(i\ k) := (\pi(i)\ \pi(k))$. We choose the index $k = (01)(10)$ and decompose $\mathcal{I}^k$ into nine equivalence classes $\mathcal{I}_1^k, \ldots, \mathcal{I}_9^k$ of the equivalence $\equiv^k$ listed in Table 5.2.

Table 5.2: The equivalence classes of $\equiv^{(01)(10)}$

$\mathcal{I}_1^k$	(11)(12), (11)(21), (22)(12), (22)(21)
$\mathcal{I}_2^k$	(01)(11), (10)(11), (01)(12), (10)(21)
$\mathcal{I}_3^k$	(02)(22), (20)(22), (02)(21), (20)(12)
$\mathcal{I}_4^k$	(01)(20), (02)(10)
$\mathcal{I}_5^k$	(01)(22), (10)(22)
$\mathcal{I}_6^k$	(01)(21), (10)(12)
$\mathcal{I}_7^k$	(02)(11), (20)(11)
$\mathcal{I}_8^k$	(02)(12), (20)(21)
$\mathcal{I}_9^k$	(02)(20)

The solutions of S_P are found in five consecutive steps. In each of the following steps, $k = (01)(10)$ is fixed, and it is assumed that the corresponding $\varepsilon_k \neq 0$. Let in $\mathcal{R}^m$ further $\varepsilon_i \neq 0$ be assumed. Then, in the next step, in evaluating $\mathcal{R}^{m+1}$, one finds the following: the non–equivalence system $\mathcal{S}^m$ and previous assumption $\varepsilon_k \neq 0$ imply zeros on all positions j, $j \equiv^k i$. It is advantageous to take pairs of indices from the three largest classes in Table 5.2, namely, $(22)(21) \in \mathcal{I}_1^k$, $(10)(11) \in \mathcal{I}_2^k$, $(02)(22) \in \mathcal{I}_3^k$, in order to evaluate the sets $\mathcal{R}^0$, $\mathcal{R}^1$ and $\mathcal{R}^2$, $\mathcal{R}^3$ respectively.

Now, we list the five steps and then make more detailed explanation.

1. $\mathcal{R}^0 = \left\{\varepsilon \in \mathcal{R}(\mathsf{S}_P) \mid \varepsilon_{(01)(10)} \neq 0,\ \varepsilon_{(22)(21)} \neq 0\right\}$

   $\mathcal{S}^0 :\ \varepsilon_{(01)(10)A}\varepsilon_{(22)(21)A} = 0 \quad \forall A \in H_3$

2. $\mathcal{R}^1 = \left\{\varepsilon \in \mathcal{R}(\mathsf{S}_P \cup \mathcal{S}^0) \mid \varepsilon_{(01)(10)} \neq 0,\ \varepsilon_{(10)(11)} \neq 0,\ \varepsilon_{(01)(22)} \neq 0\right\}$

   $\mathcal{S}^1 :\ \varepsilon_{(01)(10)A}\varepsilon_{(10)(11)A}\varepsilon_{(01)(22)A} = 0 \quad \forall A \in H_3$

3. $\mathcal{R}^2 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_P \cup \mathcal{S}^0 \cup \mathcal{S}^1) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(10)(11)} \neq 0\}$

$\mathcal{S}^2 :\ \varepsilon_{(01)(10)A}\varepsilon_{(10)(11)A} = 0 \quad \forall A \in H_3$

4. $\mathcal{R}^3 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_P \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2) \mid \varepsilon_{(01)(10)} \neq 0,\, \varepsilon_{(02)(22)} \neq 0\}$

$\mathcal{S}^3 :\ \varepsilon_{(01)(10)A}\varepsilon_{(02)(22)A} = 0 \quad \forall A \in H_3$

5. $\mathcal{R}^4 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_P \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3) \mid \varepsilon_{(01)(10)} \neq 0\}$

$\mathcal{S}^4 :\ \varepsilon_{(01)(10)A} = 0 \quad \forall A \in H_3$

Step 1. In the rest of this subsection the parameters $a, b, c, \dots$ are arbitrary complex numbers. Explicit solution under the assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(22)(21)} \neq 0$ can be written as four parametric matrices. These matrices in $\mathcal{R}^0$ can be equivalently replaced by renormalized matrices $\mathcal{R}^0_{nor} = \{\varepsilon^0_1, \varepsilon^0_2, \varepsilon^0_3, \varepsilon^0_4\}$ where

$$\varepsilon^0_1 = \begin{pmatrix} 0&0&1&0&0&1&a&b\\ 0&0&0&0&0&0&c&d\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1\\ a&c&0&0&0&0&0&0\\ b&d&0&1&0&1&0&0 \end{pmatrix}, \qquad \varepsilon^0_2 = \begin{pmatrix} 0&0&1&d&ad&1&1&a\\ 0&0&bad&bad&bad&bd&b&ab\\ 1&bad&0&0&dcba&c&cba&cba\\ d&bad&0&0&dcba&d&c&1\\ ad&bad&dcba&dcba&0&0&cba&ac\\ 1&bd&c&d&0&0&cb&1\\ 1&b&cba&c&cba&cb&0&0\\ a&ab&cba&1&ac&1&0&0 \end{pmatrix},$$

$$\varepsilon^0_3 = \begin{pmatrix} 0&0&1&0&0&1&0&0\\ 0&0&bd&0&0&ad&a&b\\ 1&bd&0&0&0&c&0&cb\\ 0&0&0&0&0&d&0&1\\ 0&0&0&0&0&0&0&0\\ 1&ad&c&d&0&0&ac&1\\ 0&a&0&0&0&ac&0&0\\ 0&b&cb&1&0&1&0&0 \end{pmatrix}, \quad \varepsilon^0_4 = \begin{pmatrix} 0&0&1&0&ad&1&0&a\\ 0&0&bd&0&0&0&0&b\\ 1&bd&0&0&0&c&0&cb\\ 0&0&0&0&0&d&0&1\\ ad&0&0&0&0&0&0&ac\\ 1&0&c&d&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ a&b&cb&1&ac&1&0&0 \end{pmatrix}.$$

Step 2. Note that the system of 48 equations $\mathcal{S}^0$ together with $\varepsilon_{(01)(10)} \neq 0$ enforces zeros on all positions from $\mathcal{I}^k_1$. Moreover, the assumption $\varepsilon_{(10)(11)} \neq 0$ and $\mathcal{S}^0$ enforces further 4 zeros. Then the assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(10)(11)} \neq 0$, $\varepsilon_{(01)(22)} \neq 0$ gives us a single solution:

$$\mathcal{R}^1_{nor} = \{\varepsilon^1\}, \qquad \text{where} \quad \varepsilon^1 = \begin{pmatrix} 0&0&1&a&b&1&0&0\\ 0&0&c&0&0&d&0&0\\ 1&c&0&0&1&e&0&0\\ a&0&0&0&0&f&0&0\\ b&0&1&0&0&0&0&0\\ 1&d&e&f&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \tag{5.21}$$

Step 3. Further solutions with assumption $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(10)(11)} \neq 0$, inequivalent to those in $\mathcal{R}^1$ and $\mathcal{R}^0$, are listed below:

$$\mathcal{R}^2_{nor} = \{\varepsilon^2_1, \varepsilon^2_2, \varepsilon^2_3, \varepsilon^2_4, \varepsilon^2_5, \varepsilon^2_6, \varepsilon^2_7, \varepsilon^2_8\}$$

$$\varepsilon^2_1 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & b & 0 & c \\ a & 0 & 0 & 0 & ac & d & e & ac \\ 0 & 0 & 1 & ac & 0 & 0 & 0 & 0 \\ 0 & 0 & b & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & c & ac & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^2_2 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & b & 1 & d \\ a & 0 & 0 & 0 & ad & 0 & c & ad \\ 0 & 0 & 1 & ad & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & d & ad & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon^2_3 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & b & 0 & c \\ a & 0 & 0 & 0 & ac & d & 0 & ac \\ 0 & 0 & 1 & ac & 0 & 0 & 0 & 0 \\ 0 & 0 & b & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & ac & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^2_4 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & b & 1 & c \\ a & 0 & 0 & 0 & ac & 0 & 0 & ac \\ 0 & 0 & 1 & ac & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & ac & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon^2_5 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & bd & b & c \\ a & 1 & 0 & 0 & ac & d & bd & ac \\ 0 & 0 & 1 & ac & 0 & 0 & 0 & 0 \\ 0 & 0 & bd & d & 0 & 0 & 0 & 0 \\ 0 & 0 & b & bd & 0 & 0 & 0 & 0 \\ 0 & 0 & c & ac & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^2_6 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & b \\ a & 1 & 0 & 0 & ab & c & d & ab \\ 0 & 0 & 1 & ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & b & ab & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon^2_7 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & b \\ a & 1 & 0 & 0 & ab & 0 & c & ab \\ 0 & 0 & 1 & ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & b & ab & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^2_8 = \begin{pmatrix} 0 & 0 & 1 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & e & 0 & 0 \\ 1 & c & 0 & 0 & 1 & 0 & 0 & 0 \\ a & d & 0 & 0 & 0 & f & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4. Now we can of course ignore the equations $\mathcal{S}^1$ because they are satisfied identically due to the system $\mathcal{S}^2$. We list the next set

$$\mathcal{R}^3_{nor} = \{\varepsilon^3_1, \varepsilon^3_2, \varepsilon^3_3, \varepsilon^3_4\}$$

$$\varepsilon^3_1 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^3_2 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 1 & 0 & 0 \\ 1 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon^3_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^3_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 5. The systems $\mathcal{S}^0$, $\mathcal{S}^2$, $\mathcal{S}^3$ together with $\varepsilon_{(01)(10)} \neq 0$ give us 12 zeros and further 20 non–trivial conditions. Adding two zeros following from S_P we obtain 3 solutions:

$$\mathcal{R}^4_{nor} = \{\varepsilon^4_1, \varepsilon^4_2, \varepsilon^4_3\}$$

$$\varepsilon^4_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & b & 0 & 0 & c & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & d & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & e & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^4_2 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & 0 & 0 & 0 & 0 \\ 1 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ a & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon^4_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since all pairs of relevant indices lie in one orbit (the whole set $\mathcal{I}$), the system $\mathcal{S}^4 : \varepsilon_k = 0, \forall k \in \mathcal{I}$ enforces zeros on all 24 positions. This precisely means that now only the trivial zero solution is inequivalent to solutions in $\mathcal{R}^0$, $\mathcal{R}^1$, $\mathcal{R}^2$, $\mathcal{R}^3$, $\mathcal{R}^4$, i.e. we have evaluated the whole $\mathcal{R}(\mathsf{S}_P)$ up to equivalence.

5.4 Solutions

Once we have solved the system S_P, we have to list the complete set of its inequivalent normalized solutions. Therefore, we take now each solution matrix and discuss all possible combinations of zero or nonzero parameters. Then we divide the sets of solutions $\mathcal{R}^k_{nor}$ in new sets according to the number of zeros $\nu(\varepsilon)$ in contraction matrices and eliminate equivalent solutions from these sets like in following examples. In this way we get 188 non–equivalent contraction matrices. The results are given in the Appendix A.1.

Example 5.1. For instance, take the matrix ε^0_2 in the set $\mathcal{R}^0_{nor}$ and let all its parameters be non–vanishing. This is clearly the only solution of S_P without zeros, i.e. $\nu(\varepsilon^0_2) = 0$. Our question is whether or not it is possible to normalize it to the trivial contraction matrix ε_0 which has all relevant epsilons equal to unity. Then the resulting graded contractions would be isomorphic to the algebra sl$(3, \mathbb{C})$ for arbitrary nonzero values of parameters in ε^0_2. We have verified that the system of 24 equations corresponding to the matrix equality $\varepsilon^0_2 \bullet\ \alpha = \varepsilon_0$ has a general solution in $\mathbb{C}\backslash\{0\}$. The matrix ε^0_2 with nonzero parameters is then equivalent to the trivial solution ε_0 and the corresponding graded contraction is isomorphic to sl$(3, \mathbb{C})$.

Example 5.2. The set $\mathcal{R}^1_{nor}$ contains six solutions with 16 zeros. These solutions are given by contraction matrix ε^1 (5.21), where only one of the parameters a, b, c, d, e, f is equal to zero and the rest is considered nonzero. The projections of these solutions are

$$\hat{\varepsilon}^1_a = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (\hat{\varepsilon}^1_f)^{\left(\begin{smallmatrix} 1 & 0 \\ 2 & 2 \end{smallmatrix}\right)}, \quad \hat{\varepsilon}^1_c = \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (\hat{\varepsilon}^1_f)^{\left(\begin{smallmatrix} 0 & 1 \\ 2 & 2 \end{smallmatrix}\right)}, \quad \hat{\varepsilon}^1_e = \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\hat{\varepsilon}^1_b = \begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (\hat{\varepsilon}^1_f)^{\left(\begin{smallmatrix} 2 & 2 \\ 1 & 0 \end{smallmatrix}\right)}, \quad \hat{\varepsilon}^1_d = \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = (\hat{\varepsilon}^1_f)^{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}, \quad \hat{\varepsilon}^1_f = \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where zeros are shown as dots. Except $\hat{\varepsilon}^1_e$, they are all generated by action of H_3 on the solution $\hat{\varepsilon}^1_f$. Thus, for example, solution ε^1_a and solution $(\varepsilon^1_f)^{\left(\begin{smallmatrix} 1 & 0 \\ 2 & 2 \end{smallmatrix}\right)}$ (equivalent to the solution ε^1_f) have the same supports. Hence, they are both solutions with 16 zeros of the system $\mathsf{S}_P \cup \mathcal{S}^0$ under the same assumptions $\varepsilon_{(01)(10)} \neq 0$, $\varepsilon_{(10)(11)} \neq 0$, $\varepsilon_{(01)(22)} \neq 0$. Since all such solutions are $\varepsilon^1_a, \ldots, \varepsilon^1_f$ and since $\varepsilon^1_a, \ldots, \varepsilon^1_f$ have different supports, the solution $(\varepsilon^1_f)^{\left(\begin{smallmatrix} 1 & 0 \\ 2 & 2 \end{smallmatrix}\right)}$ has to be strongly equivalent (since we have used the set $\mathcal{R}^1_{nor}$ which was already normalized, in the case of set $\mathcal{R}^1$ there is an equality instead of strong equivalence) to the solution ε^1_a. Therefore, the only non–equivalent solutions with 16 zeros in the set $\mathcal{R}^1_{nor}$ are ε^1_e and ε^1_f.

These five–parametric solutions can be further normalized. For example, the contraction matrix ε_f^1 is strongly equivalent to one–parametric contraction matrix marked as $\varepsilon^{16,2}$

$$\varepsilon_f^1 = \begin{pmatrix} \cdot & \cdot & 1 & a & b & 1 & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & d & \cdot & \cdot \\ 1 & c & \cdot & \cdot & 1 & e & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & d & e & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \approx \quad \varepsilon^{16,2} = \begin{pmatrix} \cdot & \cdot & q & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ q & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \alpha \bullet \varepsilon_f^1, \quad \alpha_{ij} = \frac{a_i a_j}{a_{i+j}}, \; i, j \in I. \quad (5.22)$$

In order to see this, it is sufficient to verify the following system of equations:

$$\begin{gathered} \frac{a_{01}a_{10}}{a_{11}} = q, \; a\frac{a_{01}a_{20}}{a_{21}} = 1, \; b\frac{a_{01}a_{11}}{a_{12}} = 1, \; \frac{a_{01}a_{22}}{a_{20}} = 1, \\ c\frac{a_{02}a_{10}}{a_{12}} = 1, \; d\frac{a_{02}a_{22}}{a_{21}} = 1, \; \frac{a_{10}a_{11}}{a_{21}} = 1, \; e\frac{a_{10}a_{22}}{a_{02}} = 1. \end{gathered} \quad (5.23)$$

This system has a solution in $\mathbb{C} \setminus \{0\}$ for $q = \frac{bd}{ac}$.

5.5 Higher–order identities

In order to determine which solutions are discrete, we construct higher–order identities. As an example of a third order identity for the Pauli grading one can give the following equation:

$$\varepsilon_{(01)(10)}\varepsilon_{(01)(11)}\varepsilon_{(02)(22)} = \varepsilon_{(01)(20)}\varepsilon_{(01)(22)}\varepsilon_{(02)(10)}. \quad (5.24)$$

It can either be deduced directly from the system S_P, or we can use the fact (following from example 5.1) that all solutions with all nonzero relevant elements can be written in the form of the normalization matrix α. Hence the identity

$$\frac{a_{(01)}a_{(10)}}{a_{(11)}} \frac{a_{(01)}a_{(11)}}{a_{(12)}} \frac{a_{(02)}a_{(22)}}{a_{(21)}} = \frac{a_{(01)}a_{(20)}}{a_{(21)}} \frac{a_{(01)}a_{(22)}}{a_{(20)}} \frac{a_{(02)}a_{(10)}}{a_{(12)}} \quad (5.25)$$

is evidently satisfied. Putting the contraction matrix $\varepsilon^{16,2}$ (5.22) into the identity (5.24) we get $q = 1$. Thus, the identity (5.24) is violated by any contraction matrix $\varepsilon^{16,2}$ with $q \neq 1$ and such matrix corresponds to discrete contraction.

Applying the symmetry group H_3 to (5.24), we can write the 24-point orbit of higher-order identities in the form

$$\varepsilon_{(01)(10)A}\varepsilon_{(01)(12)A}\varepsilon_{(02)(21)A} = \varepsilon_{(01)(22)A}\varepsilon_{(01)(21)A}\varepsilon_{(02)(12)A} \qquad \forall A \in H_3. \quad (5.26)$$

Note that the action is effective only for 24 elements of the subgroup $SL(2, \mathbb{Z}_3)$.

We have found a set of second and third order identities. In all we have 104 such identities, 24 of them being of second order. Table 5.3 lists their representative points and the number of the resulting identities under the action of $SL(2, \mathbb{Z}_3)$. For each solution of the system S_P we were able to decide that one of two exclusive alternatives holds:

- either we found that a solution violates some of 104 identities listed in Table 1 and therefore, it is *discrete*
- or we explicitly found a continuous path of the form (3.34), hence the solution is *continuous*

Thus, it was not necessary to investigate the completeness of our set of higher-order identities. Among all 188 solutions there were 81 continuous and 99 discrete ones. Remaining 8 contraction matrices were continuous only for isolated values of its parameters, otherwise they were discrete.

Table 5.3: Orbits of 2nd and 3rd order identities for the Pauli grading

Order	Representative equation	Number of equations
2	$\varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = \varepsilon_{(01)(20)}\varepsilon_{(02)(21)}$	*24*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(11)}\varepsilon_{(01)(12)} = \varepsilon_{(01)(20)}\varepsilon_{(01)(22)}\varepsilon_{(01)(21)}$	*8*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(12)}\varepsilon_{(02)(21)} = \varepsilon_{(01)(22)}\varepsilon_{(01)(21)}\varepsilon_{(02)(12)}$	*24*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(11)}\varepsilon_{(02)(21)} = \varepsilon_{(01)(22)}\varepsilon_{(01)(21)}\varepsilon_{(02)(10)}$	*24*
3	$\varepsilon_{(01)(10)}\varepsilon_{(01)(11)}\varepsilon_{(02)(22)} = \varepsilon_{(01)(20)}\varepsilon_{(01)(22)}\varepsilon_{(02)(10)}$	*24*

5.6 Identification of contracted Lie algebras

The set of 188 inequivalent solutions of the system of contraction equations for the Pauli graded Lie algebra $\mathrm{sl}(3,\mathbb{C})$ was divided into 13 groups according to the numbers ν of zeros among 24 relevant entries in the contraction matrices ε. These solutions are denoted $\varepsilon^{\nu,i}$, where the second index i is numbering solutions with the same ν. Correspondingly, the contracted Lie algebra corresponding to solution $\varepsilon^{\nu,i}$ is denoted $\mathcal{P}_{\nu,i}$. The following table gives the number of solutions ε corresponding to each ν:

Number of zeros ν	0	9	12	15	16	17	18	19	20	21	22	23	24
Number of solutions	1	1	2	7	7	17	36	45	42	21	7	1	1

Among the 188 solutions there are two trivial solutions. One trivial solution $\varepsilon^{24,1}$, with 24 zeros, corresponds to the 8–dimensional abelian Lie algebra while the other trivial solution $\varepsilon^{0,1}$, without zeros, corresponds to the initial Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Among the remaining

186 nontrivial solutions, 11 solutions depend on one nonzero complex parameter a and two depend on two nonzero complex parameters a, b. The corresponding parametric families of Lie algebras — the parametric Lie algebras — are denoted by $\mathcal{P}_{\nu,i}(a)$, $\mathcal{P}_{\nu,i}(a,b)$. Each of these parametric Lie algebras will be counted as one algebra.

We started our identification procedure with 186/13 (it means 186 algebras, 13 among them parametric) non–abelian Lie algebras. The results of the direct decomposition were following:

- 70/1 algebras allowed the central decomposition. Further, only non–abelian parts $\mathcal{P}'_{\nu,i}$ (with dimensions lower than 8) of these algebras $\mathcal{P}_{\nu,i}$ were investigated.
- 12 algebras were decomposable into the direct sum of two non–abelian indecomposable ideals.

Thus, we have to identify 198/13 indecomposable Lie algebras. These algebras are now divided according to their dimensions as follows:

Dimension	3	4	5	6	7	8
Number of algebras	11	14	9	12	42/1	110/12

The computation of derived, lower and upper central series reveal that there are 24/4 solvable and 174/9 nilpotent Lie algebras. These algebras are divided according to their numerical invariants inv($\mathcal{P}$) (4.69) into 18 classes of solvable (non–nilpotent) and 118 classes of nilpotent Lie algebras. Since all investigated algebras are solvable, we can skip the Levi decomposition which is trivial in all cases.

We have found 4 isomorphic algebras, which can be omitted, in classes of solvable Lie algebras and 52 isomorphic algebras in classes of nilpotent Lie algebras. Moreover, with respect to our convention in 4.10, we extend the definition of following two nilpotent and two solvable parametric Lie algebras for zero value of parameters:

$$\begin{aligned} &\mathcal{P}_{15,5}(a,0) := \mathcal{P}_{16,2}(\tfrac{1}{a}), && \mathcal{P}_{12,2}(0,b) := \mathcal{P}_{15,7}(\tfrac{1}{b^2}), \\ &\mathcal{P}_{18,25}(0) := \mathcal{P}_{19,26}, && \mathcal{P}_{15,6}(0) := \mathcal{P}_{16,6}. \end{aligned} \tag{5.27}$$

The nilpotent Lie algebras $\mathcal{P}_{18,25}(a)$ and $\mathcal{P}_{19,26}$ are not in the same inv($\mathcal{P}$)–class. Thus, after the extension we lose one nilpotent class. Therefore, we have now $24-6=18$ solvable Lie algebras in 18 classes and $174-54=120$ nilpotent Lie algebras in 117 classes. Since all classes of solvable Lie algebras consist of one algebra only, the solvable Lie algebras are now identified up to ranges of parameters for the parametric Lie algebras. For these solvable non–nilpotent Lie algebras nilradicals are computed and listed in the resulting tables.

There are still two classes of nilpotent Lie algebras which contain at least two algebras. The first class consists of algebras $\mathcal{P}_{19,31}$ and $\mathcal{P}_{20,11}$. The second one consists of algebras $\mathcal{P}_{16,3}(a), \mathcal{P}_{17,9}$ and $\mathcal{P}_{17,12}$. The computation of Casimir operators shows that the Lie algebra $\mathcal{P}_{20,11}$ has four independent Casimir operators of orders $1, 1, 2, 2$ while the Lie algebra $\mathcal{P}_{19,31}$ has four independent Casimir operators of orders $1, 1, 2, 3$. Therefore, these algebras are not isomorphic. Since all three algebras in the second class have the same Casimir operators, we have to use invariant function to distinguish them. Comparing the following tables with invariant function $\psi(\alpha)$

$\mathcal{P}_{16,3}(a), a \neq 0$

α	0	
$\psi(\alpha)$	19	16

$\mathcal{P}_{17,9}$

α	0	-2	
$\psi(\alpha)$	19	17	16

$\mathcal{P}_{17,12}$

α	0	$-\frac{1}{2}$	
$\psi(\alpha)$	19	17	16

we conclude that the Lie algebras $\mathcal{P}_{16,3}(a), \mathcal{P}_{17,9}, \mathcal{P}_{17,12}$ are not isomorphic. Blank spaces in the tables denote general value of complex variable α, different from all previous listed values, e.g. it holds $\psi_{\mathcal{P}_{16,3(a)}}(\alpha) = 16$ for $\alpha \neq 0$ while $\psi_{\mathcal{P}_{17,9}}(\alpha) = 16$ for $\alpha \neq 0, -2$.

Now, we have identified all of the 198/13 indecomposable algebras up to ranges of parameters for parametric algebras. We get 18/3 solvable non–nilpotent Lie algebras and 120/6 nilpotent ones. These indecomposable Lie algebras are listed together with their invariant characteristics in Appendix B.1. All resulting decomposable Lie algebras can be written as their direct sums. There are 8 mutually non–isomorphic decomposable Lie algebras among the graded contractions of Pauli graded $\mathrm{sl}(3, \mathbb{C})$:

solvable and discrete:

$\mathcal{P}_{18,32} \cong \mathcal{P}'_{21,9} \oplus \mathcal{P}'_{21,9}$,

$\mathcal{P}_{19,36} \cong \mathcal{P}'_{21,9} \oplus \mathcal{P}'_{22,1}$,

$\mathcal{P}_{20,20} \cong \mathcal{P}'_{21,9} \oplus \mathcal{P}'_{23,1} \oplus \mathcal{A}_1$,

nilpotent and discrete:

$\mathcal{P}_{20,10} \cong \mathcal{P}'_{21,2} \oplus \mathcal{P}'_{23,1}$,

$\mathcal{P}_{20,21} \cong \mathcal{P}'_{22,1} \oplus \mathcal{P}'_{22,1}$,

$\mathcal{P}_{21,4} \cong \mathcal{P}'_{22,1} \oplus \mathcal{P}'_{23,1} \oplus \mathcal{A}_1$,

nilpotent and continuous:

$\mathcal{P}_{21,15} \cong \mathcal{P}'_{22,3} \oplus \mathcal{P}'_{23,1}$,

$\mathcal{P}_{22,2} \cong \mathcal{P}'_{23,1} \oplus \mathcal{P}'_{23,1} \oplus 2\mathcal{A}_1$,

where $\mathcal{A}_1$ stands for one–dimensional abelian Lie algebra. Remaining decomposable algebras are nilpotent and isomorphic to those listed above:

$$\mathcal{P}_{20,22} \cong \mathcal{P}_{20,21}, \qquad \mathcal{P}_{21,10} \cong \mathcal{P}_{21,6} \cong \mathcal{P}_{21,4}, \qquad \mathcal{P}_{22,6} \cong \mathcal{P}_{22,2}.$$

For the purpose of completeness, we also list the remaining isomorphisms among indecomposable nilpotent Lie algebras:

$$\begin{array}{lll}
\mathcal{P}'_{22,5} \cong \mathcal{P}'_{22,4} \cong \mathcal{P}'_{22,3}, & \mathcal{P}'_{21,8} \cong \mathcal{P}'_{21,5} \cong \mathcal{P}'_{21,3}, & \mathcal{P}'_{21,18} \cong \mathcal{P}'_{21,13} \cong \mathcal{P}'_{21,11}, \\
\mathcal{P}'_{21,17} \cong \mathcal{P}'_{21,14} \cong \mathcal{P}'_{21,12}, & \mathcal{P}'_{20,29} \cong \mathcal{P}'_{20,26} \cong \mathcal{P}'_{20,19}, & \mathcal{P}'_{20,28} \cong \mathcal{P}'_{20,23} \cong \mathcal{P}'_{20,14}, \\
\mathcal{P}'_{20,24} \cong \mathcal{P}'_{20,15} \cong \mathcal{P}'_{20,13}, & \mathcal{P}_{20,37} \cong \mathcal{P}_{20,32}, & \mathcal{P}_{20,34} \cong \mathcal{P}_{20,31}, \\
\mathcal{P}_{19,39} \cong \mathcal{P}_{19,33} \cong \mathcal{P}_{19,31}, & \mathcal{P}_{20,27} \cong \mathcal{P}_{20,18} \cong \mathcal{P}_{20,17}, & \mathcal{P}_{20,36} \cong \mathcal{P}_{20,33}, \\
\mathcal{P}_{19,40} \cong \mathcal{P}_{19,38} \cong \mathcal{P}_{19,25}, & \mathcal{P}_{20,30} \cong \mathcal{P}_{20,25} \cong \mathcal{P}_{20,12}, & \mathcal{P}_{18,31} \cong \mathcal{P}_{18,26}, \\
\mathcal{P}_{19,35} \cong \mathcal{P}_{19,34} \cong \mathcal{P}_{19,28}, & \mathcal{P}_{18,33}(a) \cong \mathcal{P}_{18,30}(a) \cong \mathcal{P}_{18,25}(a), & \mathcal{P}_{19,37} \cong \mathcal{P}_{19,30} \cong \mathcal{P}_{19,26}.
\end{array}$$

Let us note that isomorphic graded contractions were always of the same type, i.e. all discrete or all continuous.

The overview with number of results is summarized in Table 5.4. The number of contracted Lie algebras are divided there according to the dimension of their non–abelian parts and their types.

Table 5.4: The number of nontrivial graded contractions of the Pauli graded $\mathrm{sl}(3, \mathbb{C})$

Dimension of non–abelian part	Solvable		Nilpotent		Total
	Indec.	Dec.	Indec.	Dec.	
3			1		1
4	1		1		2
5	1		4		5
6	1		9	1	11
7	4	1	28	1	34
8	11	2	77	3	93
					146

Including two trivial contractions we have obtained 148 non–isomorphic contracted Lie algebras as graded contractions of the Pauli graded Lie algebra $\mathrm{sl}(3, \mathbb{C})$. Among them there are 7 one–parametric and 2 two–parametric families of Lie algebras. We used the invariant functions to determine the ranges of parameters for one–parametric families as in the following example. These functions are given in Appendix C.1. Let us note that, due to higher dimensions of corresponding matrices and number of their parameters, a proper determination of invariant functions for two–parametric families could not be obtained.

Example 5.3. For the one–parametric Lie algebra $\mathcal{P}_{17,13}(a)$, $a \neq 0$ with nonzero commutation relations:

$$\begin{aligned} &[e_3, e_6] = e_1,\ [e_3, e_7] = e_2,\ [e_4, e_8] = e_1,\ [e_5, e_8] = e_2, \\ &[e_6, e_7] = e_3,\ [e_6, e_8] = -ae_4,\ [e_7, e_8] = e_5, \end{aligned} \tag{5.28}$$

we have found an isomorphism:

$$\mathcal{P}_{17,13}(a) \cong \mathcal{P}_{17,13}\left(\frac{1}{a}\right), \tag{5.29}$$

which leads to the following restriction for the parameter a:

$$a \in \mathbb{C}_{10} := \{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \cup \{z \in \mathbb{C} \,|\, |z| = 1, \mathrm{Im}(z) \geq 0\}\,. \tag{5.30}$$

Numerical invariants are different only for value $a = -1$:

$$\begin{aligned} \mathrm{inv}(\mathcal{P}_{17,13}(a)) &= (850)(8520)(258)\ [19,19,8,9,7,18]\ 4 \quad \text{for } a = -1, \\ \mathrm{inv}(\mathcal{P}_{17,13}(a)) &= (850)(8520)(258)\ [17,19,8,9,7,18]\ 4 \quad \text{for } a \neq 0,-1. \end{aligned} \tag{5.31}$$

In order to distinguish among algebras given by different values of the parameter a in the set C_{10}, we compute invariant functions $\psi^0(\alpha), \psi(\alpha)$ and $\varphi^0(\alpha)$. Functions $\psi^0(\alpha), \varphi^0(\alpha)$

$a \neq 0$

α	0	1	
$\psi^0(\alpha)$	16	8	7

$a \neq 0$

α	
$\varphi^0(\alpha)$	15

are in this case useless. However, the invariant function $\psi(\alpha)$

$a = -1$

α	1	0	-1	
$\psi(\alpha)$	19	19	17	16

$a = 1$

α	-1	0	1	
$\psi(\alpha)$	19	19	17	16

$a \neq 0, \pm 1$

α	0	$-a$	$-\frac{1}{a}$	-1	1	
$\psi(\alpha)$	19	17	17	17	17	16

is different for different $a, b \in \mathbb{C}_{10}$ and, therefore, there is no isomorphism among the Lie algebras $\mathcal{P}_{17,13}(a)$ with $a \in \mathbb{C}_{10}$. In such case, we use a notation $a \in \mathbb{C}_{10}^{*}$.

Chapter 6

Gell–Mann graded contractions of $\mathrm{sl}(3,\mathbb{C})$

In this chapter we find and identify all graded contractions of the Gell–Mann graded Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Since the Gell–Mann grading is not finest we can expect less solutions than in the case of the Pauli grading.

6.1 Gell–Mann grading of $\mathrm{sl}(3,\mathbb{C})$

The Gell–Mann grading of the Lie algebra $\mathrm{sl}(n,\mathbb{C})$ (also known as the orthogonal grading) was first described in [47]. This fine group grading decomposes $\mathrm{sl}(n,\mathbb{C})$ into $n(n-1)$ one–dimensional grading subspaces and one $(n-1)$–dimensional abelian subalgebra. In the defining n–dimensional representation of $\mathrm{sl}(n,\mathbb{C})$, the bases of these grading subspaces are formed by generalized Gell–Mann matrices:

$$M_{jk} = -i(E_{jk} - E_{kj}), \qquad N_{jk} = E_{jk} + E_{kj}, \qquad 1 \le j < k \le n, \tag{6.1}$$

where E_{jk} is the $n \times n$ matrix whose element on position j,k is equal to one and other elements are equal to zero. The basis of abelian subalgebra is formed by diagonal traceless matrices

$$D_j = E_{jj} - E_{j+1,j+1}, \qquad j = 1,2,\ldots,n-1. \tag{6.2}$$

The Gell–Mann grading of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$ has the form:

$$\begin{aligned}\Gamma_G : \mathrm{sl}(3,\mathbb{C}) &= L_{001} \oplus L_{111} \oplus L_{101} \oplus L_{011} \oplus L_{110} \oplus L_{010} \oplus L_{100}, \\ &= \mathrm{span}_{\mathbb{C}}\{e_1, e_2\} \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8\end{aligned} \tag{6.3}$$

where the basis $(e_1,\ldots,e_8)$ is in the defining 3–dimensional representation formed by Gell-Mann matrices [26]. Since the computation of the graded contractions does not depend on a concrete choice of bases for grading subspaces, we took the following basis:

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_8 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.4}$$

The commutation relations of $\mathrm{sl}(3,\mathbb{C})$ corresponding to this basis are in Table 6.1.

Table 6.1: Commutation relations of the Gell–Mann graded $\mathrm{sl}(3,\mathbb{C})$

$\mathcal{L}$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	0	$-2e_6$	$-e_8$	e_7	$-2e_3$	e_5	$-e_4$
e_2	0	0	e_6	$-e_8$	$-2e_7$	e_3	$-2e_5$	$-e_4$
e_3	$2e_6$	$-e_6$	0	$-e_7$	$-e_8$	$2e_1$	$-e_4$	$-e_5$
e_4	e_8	e_8	e_7	0	$-e_6$	$-e_5$	e_3	$2(e_1+e_2)$
e_5	$-e_7$	$2e_7$	e_8	e_6	0	e_4	$2e_2$	e_3
e_6	$2e_3$	$-e_3$	$-2e_1$	e_5	$-e_4$	0	$-e_8$	e_7
e_7	$-e_5$	$2e_5$	e_4	$-e_3$	$-2e_2$	e_8	0	$-e_6$
e_8	e_4	e_4	e_5	$-2(e_1+e_2)$	$-e_3$	$-e_7$	e_6	0

The index set I for the Gell–Mann grading of $\mathrm{sl}(3,\mathbb{C})$ consists of 7 ordered triplets (i,j,k), where $i,j,k = 0,1$ with the exception of $(0,0,0)$ and the commutative operation $\diamond$ is the componentwise addition $\pmod 2$. Thus, the grading group is the additive abelian group $\mathbb{Z}_2^3$ and $I = \mathbb{Z}_2^3 \setminus (0,0,0)$.

The symmetry group $\Delta_{\Gamma_G}(\mathrm{Aut}(\Gamma_G))$ of the Gell–Mann grading was described in detail in [34, 36]. This symmetry group $\Delta_{\Gamma_G}(\mathrm{Aut}(\Gamma_G))$ is isomorphic to the stability subgroup SL_G of the point $(0,0,1)$ in the finite matrix group $SL(3,\mathbb{Z}_2)$, i.e.

$$SL_G = \left\{ \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{Z}_2,\ ad-bc=1 \pmod 2 \right\}. \tag{6.5}$$

The number of elements in SL_G (24) can be easily counted [I] as the order of the group $|SL(2,\mathbb{Z}_2)| = 6$ multiplied by 4. The action of SL_G on the index set I is given as follows. If

$\pi_A \in \Delta_{\Gamma_P}(\mathrm{Aut}(\Gamma_P))$ is the permutation of the index set $I \subset \mathbb{Z}_2^3$ corresponding to the matrix $A = \left(\begin{smallmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{smallmatrix}\right) \in SL_G$, then for $(i,j,k) \in I$ it holds

$$\pi_A(i,j,k) = (i,j,k)A = ((ia+cj) \mod 2, (ib+jd) \mod 2, (ie+jf+k) \mod 2). \tag{6.6}$$

Let us note that the ordering $\mathcal{O}$ of the index set I is given by the order of grading subspaces in (6.3) and $\mathcal{O}(I) = \{1,2,\ldots,7\}$.

6.2 Contraction system for the Gell-Mann grading

Using the symmetry group SL_G, we construct the system of contraction equations S_G for the Gell-Mann grading of sl$(3,\mathbb{C})$. At first, we describe the orbits of the action of $SL_G \cong \Delta_{\Gamma_G}(\mathrm{Aut}(\Gamma_G))$ on the sets I, I_u^2, I_u^3.

The index set $I = \mathbb{Z}_2^3 \setminus \{(0,0,0)\}$ consists of two orbits. One orbit is formed by a single point $(0,0,1)$ and the second — six–point orbit — is represented by point $(0,1,0)$. It corresponds to the fact that the symmetry group $\mathrm{Aut}(\Gamma_G)$ leaves the first grading subspace invariant while the rest of the grading subspaces are permuted.

The set of unordered pairs of grading indices I_u^2 splits under the action of SL_G into five orbits. Two of them are formed by irrelevant pairs of grading indices: 6–point orbit represented by $((0,1,0)(0,1,0))$ and 1–point orbit represented by $((0,0,1)(0,0,1))$. Remaining three orbits are formed by relevant pairs $\mathcal{I}$ of grading indices:

- 12–point orbit represented by $((0,1,0)(1,0,0))$,
- 6–point orbit represented by $((0,1,0)(0,0,1))$, corresponding relevant contraction parameters will be marked by superscript $\circ$,
- 3–point orbit represented by $((0,1,0)(0,1,1))$, corresponding relevant contraction parameters will be marked by superscript $\bullet$.

The irrelevant parameters form the diagonal of the contraction matrix ε and thus its explicit form with respect to the chosen ordering $\mathcal{O}$ is:

$$\varepsilon = \begin{pmatrix}
0 & \varepsilon^\circ_{(001)(111)} & \varepsilon^\circ_{(001)(101)} & \varepsilon^\circ_{(001)(011)} & \varepsilon^\circ_{(001)(110)} & \varepsilon^\circ_{(001)(010)} & \varepsilon^\circ_{(001)(100)} \\
\varepsilon^\circ_{(001)(111)} & 0 & \varepsilon_{(111)(101)} & \varepsilon_{(111)(011)} & \varepsilon^\bullet_{(111)(110)} & \varepsilon_{(111)(010)} & \varepsilon_{(111)(100)} \\
\varepsilon^\circ_{(001)(101)} & \varepsilon_{(111)(101)} & 0 & \varepsilon_{(101)(011)} & \varepsilon_{(101)(110)} & \varepsilon_{(101)(010)} & \varepsilon^\bullet_{(101)(100)} \\
\varepsilon^\circ_{(001)(011)} & \varepsilon_{(111)(011)} & \varepsilon_{(101)(011)} & 0 & \varepsilon_{(011)(110)} & \varepsilon^\bullet_{(011)(010)} & \varepsilon_{(011)(100)} \\
\varepsilon^\circ_{(001)(110)} & \varepsilon^\bullet_{(111)(110)} & \varepsilon_{(101)(110)} & \varepsilon_{(011)(110)} & 0 & \varepsilon_{(110)(010)} & \varepsilon_{(110)(100)} \\
\varepsilon^\circ_{(001)(010)} & \varepsilon_{(111)(010)} & \varepsilon_{(101)(010)} & \varepsilon^\bullet_{(011)(010)} & \varepsilon_{(110)(010)} & 0 & \varepsilon_{(010)(100)} \\
\varepsilon^\circ_{(001)(100)} & \varepsilon_{(111)(100)} & \varepsilon^\bullet_{(101)(100)} & \varepsilon_{(011)(100)} & \varepsilon_{(110)(100)} & \varepsilon_{(010)(100)} & 0
\end{pmatrix}. \tag{6.7}$$

Contraction equations are labelled by unordered triplets of grading indices I_u^3. The equation $e((i,j,k)(l,m,n)(p,q,r)) \in \mathsf{S}_G$ is given by

$$\begin{gathered}\varepsilon_{(i,j,k)(l+p,m+q,n+r)}\varepsilon_{(l,m,n)(p,q,r)}[x_{ijk}[x_{lmn},x_{pqr}]] + \text{ cyclically } = 0,\\ \forall x_{ijk} \in L_{(i,j,k)},\ \forall x_{lmn} \in L_{(l,m,n)},\ \forall x_{pqr} \in L_{(p,q,r)},\end{gathered} \tag{6.8}$$

where the word "cyclically" means that the two remaining terms are obtained from the first one by permutation of its grading indices. Since $[x_{ijk}[x_{lmn},x_{pqr}]] \in L_{i+l+p,j+m+q,k+n+r}$ and $L_{000} = \{0\}$, the equation is fulfilled for any unordered triplet $((i,j,k)(l,m,n)(p,q,r))$ for which $i+l+p=0$, $j+m+q=0$ and $k+n+r=0$, where the operation $+$ is considered in $\mathbb{Z}_2$, i.e. $+ \mod 2$. Moreover, if there are two grading indices equal in the unordered triplet $((i,j,k)(l,m,n)(p,q,r))$, then the corresponding equation has only two terms. And if these indices correspond to the grading subspace which forms an abelian subalgebra of $\mathrm{sl}(3,\mathbb{C})$, then the equation (6.8) is also identically fulfilled.

The set I_u^3 of unordered triplets of grading indices is decomposed into 11 orbits with respect to the action of the symmetry group SL_G. The representatives and the number of points in these orbits are summarized in Table 6.2. Since all grading subspaces form abelian subalgebras of $\mathrm{sl}(3,\mathbb{C})$, only the orbits of the last column lead to nontrivial contraction equations.

Table 6.2: The representatives and the number of points in orbits of action SL_G on I_u^3

Trivial equations				Nontrivial equations	
(0,1,0)(0,1,0)(1,0,0)	24	(0,1,0)(0,0,1)(0,0,1)	6	(0,1,0)(1,0,0)(0,1,1)	12
(0,1,0)(0,1,0)(0,1,0)	6	(0,1,0)(1,0,0)(1,1,0)	4	(0,1,0)(1,0,0)(0,0,1)	12
(0,1,0)(0,1,0)(0,1,1)	6	(0,1,0)(0,0,1)(0,1,1)	3	(0,1,0)(1,0,0)(1,1,1)	4
(0,1,0)(0,1,0)(0,0,1)	6	(0,0,1)(0,0,1)(0,0,1)	1		

We write the contraction equation corresponding to the point $((0,1,0)(1,0,0)(1,1,1))$. Acting by SL_G on this contraction equation we generate 4 equations which form the first part S_G^1 of the system S_G. Using the commutation relations (Table 6.1) we get

$$\begin{aligned}\varepsilon^{\bullet}_{(011)(010)}\varepsilon_{(111)(100)}[e_7,[e_8,e_3]] + \varepsilon^{\bullet}_{(101)(100)}\varepsilon_{(111)(010)}[e_8,[e_3,e_7]] +&\\ + \varepsilon^{\bullet}_{(111)(110)}\varepsilon_{(010)(100)}[e_3,[e_7,e_8]] = 0,&\end{aligned}$$

$$\varepsilon^{\bullet}_{(011)(010)}\varepsilon_{(111)(100)}(-2e_2)+\varepsilon^{\bullet}_{(101)(100)}\varepsilon_{(111)(010)}(2e_1+2e_2)+\varepsilon^{\bullet}_{(111)(110)}\varepsilon_{(010)(100)}(-2e_1)=0.$$

Since e_1 and e_2 are linearly independent vectors, we obtain the following two–term equations

$$\underbrace{\varepsilon^{\bullet}_{(011)(010)}\varepsilon_{(111)(100)}}_{a}=\underbrace{\varepsilon^{\bullet}_{(101)(100)}\varepsilon_{(111)(010)}}_{b}=\underbrace{\varepsilon^{\bullet}_{(111)(110)}\varepsilon_{(010)(100)}}_{c}. \tag{6.9}$$

This comprises two independent equalities $a=b$, $b=c$ and one dependent $a=c$. Considering the action of SL_G on $\mathcal{I}^2_u$, one can see that the indices of the terms a,b,c lie in the same 12–point orbit. Moreover, the matrix $X=\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)$ transforms the equation $b=c$ into the equation $a=c$ and the matrix $Y=\left(\begin{smallmatrix}0&1&0\\1&1&1\\0&0&1\end{smallmatrix}\right)$ transforms $b=c$ into $a=b$. Thus, the whole S^1_G is generated from the equation $b=c$ by action of SL_G. Since the stability subgroup $H=\{1,X,Y,Y^2,XY,XY^2\}$ of the point $((0,1,0)(1,0,0)(1,1,1))$ provides all six permutations of the terms a,b,c, each of four left cosets of SL_G — with respect to the group H — generates six linearly dependent equations. Among these six equations only two are linearly independent, therefore, we get 8 linearly independent equations.

The representative point $((0,1,0)(1,0,0)(0,0,1))$ contains the index of the 2–dimensional grading subspace L_{001}. Therefore, it leads to two equations

$$\varepsilon_{(101)(010)}\varepsilon^{\circ}_{(001)(100)}[e_7,[e_8,e_i]]+\varepsilon_{(011)(100)}\varepsilon^{\circ}_{(001)(010)}[e_8,[e_i,e_7]]+\varepsilon^{\circ}_{(001)(110)}\varepsilon_{(010)(100)}[e_i,[e_7,e_8]]=0,$$

where $i=1,2$. Using the commutation relations (Table 6.1) we have

$$(-\varepsilon_{(101)(010)}\varepsilon^{\circ}_{(001)(100)}-\varepsilon_{(011)(100)}\varepsilon^{\circ}_{(001)(010)}+2\varepsilon^{\circ}_{(001)(110)}\varepsilon_{(010)(100)})e_3=0,$$
$$(-\varepsilon_{(101)(010)}\varepsilon^{\circ}_{(001)(100)}+2\varepsilon_{(011)(100)}\varepsilon^{\circ}_{(001)(010)}-\varepsilon^{\circ}_{(001)(110)}\varepsilon_{(010)(100)})e_3=0.$$

By summing and subtracting these equations we obtain new two–term equations

$$\underbrace{\varepsilon^{\circ}_{(001)(100)}\varepsilon_{(101)(010)}}_{a}=\underbrace{\varepsilon^{\circ}_{(001)(010)}\varepsilon_{(011)(100)}}_{b}=\underbrace{\varepsilon^{\circ}_{(001)(110)}\varepsilon_{(010)(100)}}_{c}. \tag{6.10}$$

Considering the action of SL_G, the matrix $X=\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)$ transforms the term a into the term b while the term c is unchanged. In fact, the index of the term c, i.e. $[(001)(110)][(010)(100)]$, lies in 12–point orbit in $\mathcal{I}^2_u$ while the indices of a,b belong to the one 24–point orbit. The stability subgroup of $((0,1,0)(1,0,0)(0,0,1))$ is a cyclic group $\{1,X\}$ and, therefore, the whole system S^2_G consists of 24 linearly independent equations generated from $b=c$ by action of SL_G.

The last representative point $((0,1,0)(1,0,0)(0,1,1))$ leads to three–term equation

$$\varepsilon_{(111)(010)}\varepsilon_{(011)(100)}[e_7,[e_8,e_5]] + \varepsilon^{\circ}_{(001)(100)}\varepsilon^{\bullet}_{(011)(010)}[e_8,[e_5,e_7]] + \\ + \varepsilon_{(011)(110)}\varepsilon_{(010)(100)}[e_5,[e_7,e_8]] = 0,$$

$$(2\underbrace{\varepsilon^{\circ}_{(001)(100)}\varepsilon^{\bullet}_{(011)(010)}}_{a} - \underbrace{\varepsilon_{(111)(010)}\varepsilon_{(011)(100)}}_{b} - \underbrace{\varepsilon_{(011)(110)}\varepsilon_{(010)(100)}}_{c})e_4 = 0. \tag{6.11}$$

The indices of b, c belong to the same 24–point orbit, while the index of a belongs to 12–point orbit in $\mathcal{I}^2_u$. The matrix $Z = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in SL_G$ transforms b into c while a is preserved. Since the stability subgroup of $((0,1,0)(1,0,0)(0,1,1))$ is $\{1, Z\}$, we get 12 linearly independent three–term equations by the action of SL_G on (6.11).

Thus, the system S_G consists of the following three subsystems:

$$\mathsf{S}^1_G : \varepsilon^{\bullet}_{(101)(100)A}\varepsilon_{(111)(010)A} = \varepsilon^{\bullet}_{(111)(110)A}\varepsilon_{(010)(100)A}, \quad \forall A \in SL_G, \tag{6.12}$$

$$\mathsf{S}^2_G : \varepsilon^{\circ}_{(001)(010)A}\varepsilon_{(011)(100)A} = \varepsilon^{\circ}_{(001)(110)A}\varepsilon_{(010)(100)A}, \quad \forall A \in SL_G, \tag{6.13}$$

$$\mathsf{S}^3_G : 2\varepsilon^{\circ}_{(001)(100)A}\varepsilon^{\bullet}_{(011)(010)A} = \varepsilon_{(111)(010)A}\varepsilon_{(011)(100)A} + \varepsilon_{(011)(110)A}\varepsilon_{(010)(100)A}, \forall A \in SL_G, \tag{6.14}$$

where we have used the abbreviation $\varepsilon_{(ij)(kl)A} = \varepsilon_{(ij)A(kl)A}$.

Observing the form of terms in these subsystems and considering that the contraction parameters $\varepsilon, \varepsilon^{\circ}, \varepsilon^{\bullet}$ belong into the different orbits, it is obvious that equations from the different subsystems are linearly independent. Thus, the contraction system S_G for the Gell–Mann graded Lie algebra sl$(3,\mathbb{C})$ consists of 44 linearly independent equations. 32 of these equations are two–term equations and remaining 12 are three–term equations.

6.3 Finding solution of S_G

Using the Theorem 3.9 we solve the system of contraction equations S_G. First of all we choose the appropriate sequence of assumptions. These assumptions are chosen in order to enforce as many nonzero's among the contraction parameters as possible while $\varepsilon_{(010)(100)}$ is considered nonzero. The solutions of S_G are found in seven consecutive steps. Let us note that if there were any nonzero parameters in solution, it was always possible to renormalize them to 1, i.e. to find strongly equivalent contraction matrix with units on their positions. Other parameters which are arbitrary complex numbers are denoted by a, b, c, d, e, f.

Now, we list now the seven steps and then make more detailed explanation.

1. $\mathcal{R}^0 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_G) \mid \varepsilon_{(010)(100)} \neq 0,\ \varepsilon^{\circ}_{(001)(101)} \neq 0\}$

$\mathcal{S}^0:\ \varepsilon_{(010)(100)A}\varepsilon^{\circ}_{(001)(101)A} = 0 \quad \forall A \in SL_G$

2. $\mathcal{R}^1 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_G \cup \mathcal{S}^0) \mid \varepsilon_{(010)(100)} \neq 0,\ \varepsilon^{\bullet}_{(111)(110)} \neq 0\}$

$\mathcal{S}^1:\ \varepsilon_{(010)(100)A}\varepsilon^{\bullet}_{(111)(110)A} = 0 \quad \forall A \in SL_G$

3. $\mathcal{R}^2 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_G \cup \mathcal{S}^0 \cup \mathcal{S}^1) \mid \varepsilon_{(010)(100)} \neq 0,\ \varepsilon^{\circ}_{(001)(010)} \neq 0,\ \varepsilon_{(101)(110)} \neq 0\}$

$\mathcal{S}^2:\ \varepsilon_{(010)(100)A}\varepsilon^{\circ}_{(001)(010)A}\varepsilon_{(101)(110)A} = 0 \quad \forall A \in SL_G$

4. $\mathcal{R}^3 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_G \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2) \mid \varepsilon_{(010)(100)} \neq 0,\ \varepsilon^{\circ}_{(001)(010)} \neq 0\}$

$\mathcal{S}^3:\ \varepsilon_{(010)(100)A}\varepsilon^{\circ}_{(001)(010)A} = 0 \quad \forall A \in SL_G$

5. $\mathcal{R}^4 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_G \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3) \mid \varepsilon_{(010)(100)} \neq 0,\ \varepsilon_{(111)(101)} \neq 0\}$

$\mathcal{S}^4:\ \varepsilon_{(010)(100)A}\varepsilon_{(111)(101)A} = 0 \quad \forall A \in SL_G$

6. $\mathcal{R}^5 = \{\varepsilon \in \mathcal{R}(\mathsf{S}_G \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3 \cup \mathcal{S}^4) \mid \varepsilon_{(010)(100)} \neq 0\}$

$\mathcal{S}^5:\ \varepsilon_{(010)(100)A} = 0 \quad \forall A \in SL_G$

7. $\mathcal{R}^6 = \mathcal{R}(\mathsf{S}_G \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3 \cup \mathcal{S}^4 \cup \mathcal{S}^5)$

Step 1. The explicit solution of the system S_G under the assumptions $\varepsilon_{(010)(100)} \neq 0$ and $\varepsilon^{\circ}_{(001)(101)} \neq 0$ is given as one parametric matrix. All parameters which do not allow the zero value can be renormalized to 1. Thus, we have $\mathcal{R}^0_{nor} = \{\varepsilon^0_1\}$, where

$$\varepsilon^0_1 = \begin{pmatrix} 0 & 1 & 1 & a & a & 1 & a \\ 1 & 0 & b & c & cb & c & b \\ 1 & b & 0 & 1 & b & 1 & b \\ a & c & 1 & 0 & ca & c & a \\ a & cb & b & ca & 0 & c & ba \\ 1 & c & 1 & c & c & 0 & 1 \\ a & b & b & a & ba & 1 & 0 \end{pmatrix}.$$

Step 2. The non–equivalence system $\mathcal{S}^0$ has 24 equations. Solutions of S_G and $\mathcal{S}^0$ under the assumptions $\varepsilon_{(010)(100)} \neq 0$ and $\varepsilon^{\bullet}_{(111)(110)} \neq 0$ are described by 3 parametric matrices. After normalization we have $\mathcal{R}^1_{nor} = \{\varepsilon^1_1, \varepsilon^1_2, \varepsilon^1_3\}$, where

$$\varepsilon^1_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \varepsilon^1_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & a & a & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & a & 0 \\ a & 1 & a & 1 & a & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \varepsilon^1_3 = \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & a & 1 & 1 & 1 \\ 0 & a & 0 & a & 0 & 0 & 1 \\ 0 & a & a & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Step 3. The solutions of S_G and 36 non–equivalence conditions $\mathcal{S}^0 \cup \mathcal{S}^1$ under the assumptions $\varepsilon_{(010)(100)} \neq 0, \varepsilon^\circ_{(001)(010)} \neq 0$, and $\varepsilon_{(101)(110)} \neq 0$ are given by two parametric matrices $\mathcal{R}^2 = \{\varepsilon^2_1, \varepsilon^2_2\}$. These assumptions were chosen in order to separate the solution ε^2_1 which is a consequence of three–term equations. Let us note that solutions of this type do not appear if only two–term equations are considered.

$$\varepsilon^2_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 1 & b & \frac{1}{2}ab + \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & b & 0 & 0 & 0 & 1 \\ 0 & a & \frac{1}{2}ab + \frac{1}{2} & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \varepsilon^2_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Step 4. Assumptions $\varepsilon_{(010)(100)} \neq 0, \varepsilon^\circ_{(001)(010)} \neq 0$ lead to the solutions $\mathcal{R}^3_{nor} = \{\varepsilon^3_1, \varepsilon^3_2\}$,

$$\varepsilon^3_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 \\ 1 & a & b & c & d & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \varepsilon^3_2 = \begin{pmatrix} 0 & a & 0 & 0 & 0 & 1 & b \\ a & 0 & 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & c & 0 & 0 & 0 & 0 & 1 \\ b & d & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Step 5. Equations $\mathcal{S}^3$ are satisfied identically due to $\mathcal{S}^4$. Thus, we evaluate the solutions $\mathcal{R}^4_{nor} = \{\varepsilon^4_1, \varepsilon^4_2\}$ of S_G and $\mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^4$ under assumptions $\varepsilon_{(010)(100)} \neq 0$ and $\varepsilon_{(111)(101)} \neq 0$:

$$\varepsilon^4_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -a & a & b \\ 0 & 0 & 1 & 0 & 0 & 0 & c \\ 0 & 0 & -a & 0 & 0 & 0 & c \\ 0 & 0 & a & 0 & 0 & 0 & 1 \\ 0 & 1 & b & c & c & 1 & 0 \end{pmatrix}, \quad \varepsilon^4_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -a & a & 0 \\ 0 & 1 & -1 & 0 & -b & 0 & b \\ 0 & 0 & -a & -b & 0 & a & b \\ 0 & 1 & a & 0 & a & 0 & 1 \\ 0 & 1 & 0 & b & b & 1 & 0 \end{pmatrix}.$$

Step 6. There are six parametric solutions of S_G with assumption $\varepsilon_{(010)(100)} \neq 0$, inequivalent to those listed above. After renormalization of nonzero parameters we have the set

$$\mathcal{R}^5_{nor} = \{\varepsilon^5_1, \varepsilon^5_2, \varepsilon^5_3, \varepsilon^5_4, \varepsilon^5_5, \varepsilon^5_6\}$$

$$\varepsilon_1^5 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&a&0&b&c\\ 0&0&a&0&0&d&e\\ 0&0&0&0&0&0&0\\ 0&0&b&d&0&0&1\\ 0&0&c&e&0&1&0 \end{pmatrix},\ \varepsilon_2^5 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&a\\ 0&0&0&0&0&0&b\\ 0&0&0&0&0&0&c\\ 0&0&0&0&0&0&d\\ 0&0&0&0&0&0&1\\ 0&a&b&c&d&1&0 \end{pmatrix},\ \varepsilon_3^5 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&a&b\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&a&0&0&0&0&1\\ 0&b&0&0&0&1&0 \end{pmatrix},$$

$$\varepsilon_4^5 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&a&0\\ 0&0&0&0&0&b&0\\ 0&0&0&0&0&c&0\\ 0&0&0&0&0&d&0\\ 0&a&b&c&d&0&1\\ 0&0&0&0&0&1&0 \end{pmatrix},\ \varepsilon_5^5 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&a&b\\ 0&0&0&0&a&0&1\\ 0&0&0&0&b&1&0 \end{pmatrix},\ \varepsilon_6^5 = \begin{pmatrix} 0&a&0&0&0&0&0\\ a&0&0&0&0&0&0\\ 0&0&0&b&0&0&0\\ 0&0&b&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{pmatrix}.$$

Step 7. Finally, the non–equivalence system $\mathcal{S}^5 = \left\{\varepsilon_{(010)(100)A} = 0 \,|\, \forall A \in SL_G\right\}$ enforces zero value of all unmarked contraction parameters and ensures fulfilment of all previous non–equivalence systems. Due to these 12 zero contraction parameters, all two–term equations are satisfied and three–term equations are reduced to $\varepsilon^{\circ}_{(001)(100)A}\varepsilon^{\bullet}_{(011)(010)A} = 0,\ \forall A \in SL_G$. Corresponding solutions are $\mathcal{R}^6_{nor} = \mathcal{R}^6 = \{\varepsilon_1^6, \varepsilon_2^6, \varepsilon_3^6, \varepsilon_4^6, \varepsilon_5^6\}$, where

$$\varepsilon_1^6 = \begin{pmatrix} 0&a&b&c&d&e&f\\ a&0&0&0&0&0&0\\ b&0&0&0&0&0&0\\ c&0&0&0&0&0&0\\ d&0&0&0&0&0&0\\ e&0&0&0&0&0&0\\ f&0&0&0&0&0&0 \end{pmatrix},\ \varepsilon_2^6 = \begin{pmatrix} 0&0&a&0&0&0&b\\ 0&0&0&0&0&0&0\\ a&0&0&0&0&0&c\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ b&0&c&0&0&0&0 \end{pmatrix},\ \varepsilon_3^6 = \begin{pmatrix} 0&0&0&a&0&b&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ a&0&0&0&0&c&0\\ 0&0&0&0&0&0&0\\ b&0&0&c&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$\varepsilon_4^6 = \begin{pmatrix} 0&a&0&0&b&0&0\\ a&0&0&0&c&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ b&c&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},\ \varepsilon_5^6 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&a&0&0\\ 0&0&0&0&0&0&b\\ 0&0&0&0&0&c&0\\ 0&a&0&0&0&0&0\\ 0&0&0&c&0&0&0\\ 0&0&b&0&0&0&0 \end{pmatrix}.$$

We have solved the contraction system S_G up to the equivalence. Solutions are collected in the sets $\mathcal{R}^k_{nor},\ k = 0, 1, 2, \ldots, 6$ and the contraction matrices from different sets are inequivalent. Since equivalent solutions have the same number of zeros $\nu(\varepsilon)$, we discuss when elements of contraction matrices vanish and divide each set $\mathcal{R}^k_{nor}$ into subsets according to $\nu(\varepsilon)$. Let us note that, besides all possible combinations of zero and nonzero parameters, the case $a = -1/b$ has to be considered for the solution ε_1^2. In these subsets we collect all solutions with the same support. In fact, there was always only one solution with the given support. For example, solutions ε_4^5 with $a = b = c = 0$ and ε_5^5 with $b = 0$ have the same support, but they also represent the same solution only with different notation of parameter. Using the symmetry group we compare projections of solutions with different supports and choose only inequivalent ones as in example 5.2. All 89 normalized representatives of equivalence classes of solutions are listed in Appendix A.2.

Let us note that the solution without zeros, i.e. ε_1^0, where $a, b, c, d \neq 0$, is strongly equivalent to the trivial solution $\varepsilon^{0,1}$ which has all relevant contraction parameters equal to 1. Thus, any contraction matrix without zeros has a form of the normalization matrix α (3.28).

6.4 Higher–order identities

Since any solution without zeros has the form of normalization matrix $\alpha_{ij} = \frac{a_i a_j}{a_{i+j}}, (ij) \in \mathcal{I}$, any continuous contraction matrix is a limit of normalization matrices. Thus, any higher–order identity can be deduced from the identities which hold for normalization matrix. For example the equation

$$\alpha_{(001)(100)}\alpha_{(011)(010)} = \frac{a_{(001)}a_{(100)}}{a_{(101)}}\frac{a_{(011)}a_{(010)}}{a_{(001)}} = \frac{a_{(011)}a_{(110)}}{a_{(101)}}\frac{a_{(010)}a_{(100)}}{a_{(110)}} = \alpha_{(011)(110)}\alpha_{(010)(100)} \tag{6.15}$$

is evidently satisfied for any normalization matrix. However, considering the contraction matrix ε_2^4 we get $0 = -b$ and thus (6.15) is violated for any $b \neq 0$. Therefore, the equation (6.15) represents 2nd order identity.

Applying the symmetry group SL_G to (6.15), we can write the 24-point orbit of 2nd order identities in the form

$$\varepsilon^{\circ}_{(001)(100)}\varepsilon^{\bullet}_{(011)(010)} = \varepsilon_{(011)(110)}\varepsilon_{(010)(100)}, \quad \forall A \in SL_G. \tag{6.16}$$

Note that the action is effective for all 24 elements of the symmetry group SL_G.

We have found a set of all 57 second order identities. These identities are divided into 5 orbits. Their representatives and the number of the resulting identities under the action of SL_G are written in Table 6.3. For each solution of the system S_G we were able to decide whether it is continuous or discrete. Any discrete contraction violated at least one of 57 identities listed in Table 6.3. For the remaining solutions — the continuous ones — we explicitly found a continuous path of the form (3.34). Among 89 solutions there were 50 continuous solutions, and 36 discrete ones. Remaining 3 solutions were continuous only for a special value of its parameters, otherwise they were discrete.

Table 6.3: Orbits of 2nd order identities for the Gell-Mann grading

Representative equation	Number of equations
$\varepsilon^{\circ}_{(001)(100)}\varepsilon^{\bullet}_{(011)(010)} = \varepsilon_{(011)(110)}\varepsilon_{(010)(100)}$	*24*
$\varepsilon^{\circ}_{(001)(100)}\varepsilon^{\bullet}_{(101)(100)} = \varepsilon_{(110)(100)}\varepsilon_{(010)(100)}$	*12*
$\varepsilon_{(111)(010)}\varepsilon_{(011)(100)} = \varepsilon_{(011)(110)}\varepsilon_{(010)(100)}$	*12*
$\varepsilon_{(111)(100)}\varepsilon_{(011)(100)} = \varepsilon_{(110)(100)}\varepsilon_{(010)(100)}$	*6*
$\varepsilon^{\circ}_{(001)(101)}\varepsilon^{\circ}_{(001)(100)} = \varepsilon^{\circ}_{(001)(111)}\varepsilon^{\circ}_{(001)(110)}$	*3*

6.5 Identification of contracted Lie algebras

All 89 solutions of the system of contraction equations S_G for the Gell–Mann graded $\mathrm{sl}(3,\mathbb{C})$ are divided into 14 groups according to the number of zeros ν among the 21 relevant contraction parameters. The number of contraction matrices in these groups are summarized in the following table:

Number of zeros ν	0	6	9	11	12	13	14	15	16	17	18	19	20	21
Number of solutions	1	2	1	3	2	1	2	9	12	18	23	11	3	1

Contraction matrices are denoted $\varepsilon^{\nu,i}$, where the second index i is numbering solutions with the same number of zeros ν. The contracted Lie algebra given by solution $\varepsilon^{\nu,i}$ is denoted $\mathcal{G}_{\nu,i}$.

As always, there are two trivial solutions: $\varepsilon^{21,1}$ (with 21 zeros) corresponding to the 8–dimensional abelian Lie and $\varepsilon^{0,1}$ (without zeros) corresponds to the initial Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Among the remaining 87 nontrivial solutions, 8 solutions depend on one nonzero complex parameter a and two depend on two nonzero complex parameters a, b. The corresponding parametric families of Lie algebras — the parametric Lie algebras — are denoted by $\mathcal{G}_{\nu,i}(a)$, $\mathcal{G}_{\nu,i}(a,b)$. Each of these parametric Lie algebras will be counted as one algebra.

During the identification it turned out that among all 87/10 (behind slash being the number of parametric cases) nontrivial contractions:

- 66/8 algebras allow the central decomposition. Further, only non–abelian parts $\mathcal{G}'_{\nu,i}$ (with the dimensions lower than 8) of these algebras $\mathcal{G}_{\nu,i}$ were investigated.

- 7/1 algebras are decomposable into the direct sum of two non–abelian indecomposable ideals.

Thus, after the decompositions there are 94/10 indecomposable Lie algebras. These algebras are divided according to their dimensions as follows:

Dimension	3	5	6	7	8
Number of algebras	22/1	12/2	10	29/5	21/2

The computation of the derived, lower and upper central series revealed that there are 54/1 nilpotent, 33/9 solvable (non–nilpotent) and 7 non–solvable Lie algebras. Values of numerical invariants $\mathrm{inv}(\mathcal{G})$ (4.69) divided these algebras into 28 classes of nilpotent, 17 classes of solvable and 4 classes of non–solvable Lie algebras. The Levi decomposition was nontrivial only in 3 cases, two of them with abelian radical.

We have found 26 isomorphic algebras — which can be omitted — in the classes of nilpotent, 16/4 isomorphic algebras in the classes of nilpotent and 3 isomorphic algebras in the classes of non–solvable Lie algebras. Omitting these algebras we get only one algebra in each $\mathrm{inv}(\mathcal{G})$–class. Thus, all algebras are now identified up to ranges of parameters for the parametric algebras. There are 28/1 nilpotent, 17/5 solvable and 4 non–solvable non–isomorphic indecomposable Lie algebras. These indecomposable Lie algebras are listed together with their invariant characteristics in Appendix B.2. All resulting decomposable Lie algebras can be written as their direct sums.

There are only 4 mutually non–isomorphic decomposable Lie algebras among the graded contractions of Gell–Mann graded $\mathrm{sl}(3,\mathbb{C})$:

Non–solvable (discrete contraction)	$\mathcal{G}_{9,1} \cong \mathcal{G}'_{18,8} \oplus \mathcal{G}'_{18,8} \oplus 2\mathcal{A}_1$
Solvable (discrete contractions)	$\mathcal{G}_{13,1} \cong \mathcal{G}'_{19,2} \oplus \mathcal{G}'_{19,2} \oplus 2\mathcal{A}_1$
	$\mathcal{G}_{18,2} \cong \mathcal{G}'_{19,2} \oplus \mathcal{G}'_{20,1} \oplus 2\mathcal{A}_1$
Nilpotent (continuous contraction)	$\mathcal{G}_{17,18} \cong \mathcal{G}'_{20,1} \oplus \mathcal{G}'_{20,1} \oplus 2\mathcal{A}_1$

where $\mathcal{A}_1$ stands for one–dimensional abelian Lie algebra. Remaining three decomposable algebras are isomorphic to those listed above:

$$\mathcal{G}_{17,2}(a) \cong \mathcal{G}_{13,1}, \qquad \mathcal{G}_{19,1} \cong \mathcal{G}_{19,11} \cong \mathcal{G}_{17,18}.$$

For the purpose of completeness, we also list all isomorphisms among the indecomposable Lie algebras:

Not–solvable $\quad \mathcal{G}_{18,21} \cong \mathcal{G}_{18,8},$

Solvable

$\mathcal{G}_{19,8} \cong \mathcal{G}_{19,6} \cong \mathcal{G}_{19,2}$, $\mathcal{G}_{18,14} \cong \mathcal{G}_{18,13} \cong \mathcal{G}_{18,11} \cong \mathcal{G}_{18,9}$, $\mathcal{G}_{17,10} \cong \mathcal{G}_{17,9} \cong \mathcal{G}_{17,7} \cong \mathcal{G}_{17,6}$,
$\mathcal{G}_{17,8}(a) \cong \mathcal{G}_{17,11}(4a)$, $\mathcal{G}_{16,3}(a) \cong \mathcal{G}_{16,5}(4a) \cong \mathcal{G}_{16,4}(4a)$,

Nilpotent

$\mathcal{G}_{20,3} \cong \mathcal{G}_{20,2} \cong \mathcal{G}_{20,1}$, $\mathcal{G}_{19,10} \cong \mathcal{G}_{19,4}$, $\mathcal{G}_{19,9} \cong \mathcal{G}_{19,7} \cong \mathcal{G}_{19,5} \cong \mathcal{G}_{19,3}$,
$\mathcal{G}_{18,20} \cong \mathcal{G}_{18,19} \cong \mathcal{G}_{18,7}$, $\mathcal{G}_{18,17} \cong \mathcal{G}_{18,16}$, $\mathcal{G}_{18,22} \cong \mathcal{G}_{18,4} \cong \mathcal{G}_{16,12}$,
$\mathcal{G}_{17,14} \cong \mathcal{G}_{17,12}$, $\mathcal{G}_{18,15} \cong \mathcal{G}_{18,12} \cong \mathcal{G}_{18,10} \cong \mathcal{G}_{18,6}$, $\mathcal{G}_{18,3} \cong \mathcal{G}_{18,1}$,
$\mathcal{G}_{17,17} \cong \mathcal{G}_{17,13}$, $\mathcal{G}_{16,10} \cong \mathcal{G}_{16,8}$, $\mathcal{G}_{16,11} \cong \mathcal{G}_{16,7}$.

Let us note that isomorphic graded contraction were always of the same type, i.e. all discrete or all continuous.

Table 6.4 provides the overview of the number of contracted Lie algebras for the Gell–Mann graded $\mathrm{sl}(3,\mathbb{C})$. Lie algebras are divided there according to the dimension of their non–abelian parts and their types.

Table 6.4: The number of nontrivial graded contractions of the Gell-Mann graded $\mathrm{sl}(3,\mathbb{C})$

Dimension of non–abelian part	Solvable		Nilpotent		Non–solvable		Total
	Indec.	Dec.	Indec.	Dec.	Indec.	Dec.	
3	1		1		1		3
4							
5	2		2				4
6	2	2	3	1	1	1	10
7	6		10				16
8	6		12		2		20
							53

Including two trivial contractions we have obtained 55 non–isomorphic contracted Lie algebras as the graded contractions of the Gell–Mann graded Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Among them there are 4 one–parametric and 2 two–parametric families of Lie algebras. From all these contracted Lie algebras 20 are discrete contractions, 32 continuous contractions and 3 parametric algebras represent continuous contractions for a special value of the parameter, otherwise they are discrete. We used the invariant functions to determine the ranges of the parameters for all one–parametric algebras. These functions are given in Appendix C.2.

6.6 Example of identification

In this section we will demonstrate the whole identification procedure on a concrete example. Let us consider an one–parametric solution

$$\varepsilon^{17,2} = \begin{pmatrix} 0 & a & 0 & 1 & 1 & 1 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.17}$$

of S_G. In order to get the commutation relations of the corresponding contracted Lie algebra $\mathcal{G}_{17,2}(a)$, we have to multiply the elements of the commutation Table 6.1 of $\mathrm{sl}(3,\mathbb{C})$ by the corresponding elements of the matrix $(\varepsilon^{17,2})^*$ according to (4.1). Matrix $(\varepsilon^{17,2})^*$ is formed from $\varepsilon^{17,2}$ by doubling its first row and then its first column. Thus, the nonzero commutation relations of Lie algebra $\mathcal{G}_{17,2}(a)$ are

$$\begin{array}{llll} [e_1, e_3] = -2ae_6, & [e_1, e_5] = e_7, & [e_1, e_6] = -2e_3, & [e_1, e_7] = e_5, \\ [e_2, e_3] = ae_6, & [e_2, e_5] = -2e_7, & [e_2, e_6] = e_3, & [e_2, e_7] = -2e_5, \end{array} \tag{6.18}$$

where the parameter a runs through $\mathbb{C} \setminus \{0\}$. It is in fact one–parametric continuum of Lie algebras, thus we have to investigate it carefully for each possible value of the complex parameter a.

At first, we compute the center and the derived algebra of $\mathcal{G}_{17,2}(a)$. These are both independent of the value of the parameter $a \neq 0$

$$C(\mathcal{G}_{17,2}(a)) = \mathrm{span}_{\mathbb{C}}\{e_4, e_8\}, \qquad D(\mathcal{G}_{17,2}(a)) = \mathrm{span}_{\mathbb{C}}\{e_3, e_5, e_6, e_7\}. \tag{6.19}$$

Since $C(\mathcal{G}_{17,2}(a)) \nsubseteq D(\mathcal{G}_{17,2}(a))$, we choose the complement $\mathcal{A} = \mathrm{span}_{\mathbb{C}}\{e_4, e_8\}$ of the derived algebra in the center and separate it as a maximal central component of $\mathcal{G}_{17,2}(a)$. Thus, we get

$$\mathcal{G}_{17,2}(a) = \mathrm{span}_{\mathbb{C}}\{e_4, e_8\} \oplus \mathrm{span}_{\mathbb{C}}\{e_1, e_2, e_3, e_5, e_6, e_7\}. \tag{6.20}$$

From now on we continue with non–abelian part $\mathcal{G}'_{17,2}(a) = \mathrm{span}_{\mathbb{C}}\{e_1, e_2, e_3, e_5, e_6, e_7\}$ only. The six–dimensional Lie algebra $\mathcal{G}'_{17,2}(a)$ has the following commutation table

$\mathcal{G}'_{17,2}(a)$	e_1	e_2	e_3	e_5	e_6	e_7
e_1	0	0	$-2ae_6$	e_7	$-2e_3$	e_5
e_2	0	0	ae_6	$-2e_7$	e_3	$-2e_5$
e_3	$2ae_6$	$-ae_6$	0	0	0	0
e_5	$-e_7$	$2e_7$	0	0	0	0
e_6	$2e_3$	e_3	0	0	0	0
e_7	$-e_5$	$2e_5$	0	0	0	0

In order to determine the decomposability of $\mathcal{G}'_{17,2}(a)$ we have to find the centralizer $C_R(\mathrm{ad}(\mathcal{G}'_{17,2}(a)))$ of its adjoint representation $\mathrm{ad}(\mathcal{G}'_{17,2}(a))$ in $R = \mathrm{gl}(6, \mathbb{C})$. For any $a \neq 0$ we get $C_R(\mathrm{ad}(\mathcal{G}'_{17,2}(a))) = \mathrm{span}_{\mathbb{C}}\{b_1, b_2\}$, where

$$b_1 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}. \tag{6.21}$$

The Jacobson radical (4.14) of $C_R(\mathrm{ad}(\mathcal{G}'_{17,2}(a)))$ is formed by zero matrix only and, therefore, $\mathcal{G}'_{17,2}(a)$ is decomposable for any $a \neq 0$. We choose a new basis $(b_1 + b_2 = \mathbf{1}_6, b_2)$ in $C_R(\mathrm{ad}(\mathcal{G}'_{17,2}(a)))$ and factorize the minimal polynomial $t(t+3)$ of b_2. The equation $P_1 t + P_2(t+3) = 1$ is fulfilled for $P_2 = -P_1 = 1/3$ and thus, the matrix

$$E = -\frac{1}{3} b_2 \tag{6.22}$$

is an idempotent in $C_R(\mathrm{ad}(\mathcal{G}'_{17,2}(a)))$. The eigen–subspaces of the idempotent E corresponding to the eigenvalues 0 and 1

$$\mathcal{G}'_{17,2}(a)^0 = \mathrm{span}_{\mathbb{C}}\{2e_1 + e_2, e_3, e_6\}, \qquad \mathcal{G}'_{17,2}(a)^1 = \mathrm{span}_{\mathbb{C}}\{e_1 + 2e_2, e_5, e_7\} \tag{6.23}$$

form ideals in $\mathcal{G}'_{17,2}(a)$. Thus, we have decomposed Lie algebra $\mathcal{G}'_{17,2}(a)$ into the direct sum of two new three–dimensional Lie algebras

$$\mathcal{G}'_{17,2}(a) = \mathcal{G}'_{17,2}(a)^0 \oplus \mathcal{G}'_{17,2}(a)^1. \tag{6.24}$$

Now, we continue with the identification of these two three–dimensional Lie algebras. Having changed the notation of their basis elements, we get their commutation relations in the following form

$\mathcal{G}'_{17,2}(a)^0$	x_1	x_2	x_3
x_1	0	$-3ax_3$	$-3x_2$
x_2	$3ax_3$	0	0
x_3	$3x_2$	0	0

${\mathcal{G}'_{17,2}}^1$	x_1	x_2	x_3
x_1	0	$-3x_3$	$-3x_2$
x_2	$3x_3$	0	0
x_3	$3x_2$	0	0

The commutation relations of $\mathcal{G}'_{17,2}(a)^1$ are independent of the parameter a and thus, $\mathcal{G}'_{17,2}(a)^1$ represents only one Lie algebra ${\mathcal{G}'_{17,2}}^1$.

The centralizers of the adjoint representations of algebras $\mathcal{G}'_{17,2}(a)^0$ and ${\mathcal{G}'_{17,2}}^1$ are one–dimensional (formed only by the multiples of the unit matrix) and therefore, these algebras are indecomposable. We continue in the identification procedure with the computation of numerical invariants. For both algebras we get (independently of the values of $a \neq 0$)

$$DS = (3,2,0), \quad CS = (3,2), \quad US = (0), \quad \dim_{(\alpha,\beta,\gamma)} = [4,3,1,2,0,1], \quad \tau = 1. \tag{6.25}$$

Thus, both these algebras are solvable, non–nilpotent, with zero center and belong into the same inv–class labelled by numbers (6.25). This class contains six more algebras (given by other contractions). During the search for isomorphisms it appears that all algebras in this class are isomorphic. In fact, one can see that rescaling the basis vectors of $\mathcal{G}'_{17,2}(a)^0$, i.e. taking a new basis $(\frac{1}{\sqrt{a}}x_1, x_2, \sqrt{a}x_3)$ in $\mathcal{G}'_{17,2}(a)^0$, we get the same commutation relations as for ${\mathcal{G}'_{17,2}}^1$. Thus, $\mathcal{G}'_{17,2}(a)^0$ belongs for any $a \neq 0$ into the same isomorphism class and, therefore, represents also only one Lie algebra. We choose the representative $\mathcal{G}'_{19,2}$ of this isomorphism class, usually according to the name of the graded contraction with indecomposable non–abelian part and the lowest number of zeros.

All parts of the Lie algebra $\mathcal{G}_{17,2}(a)$ are now identified and we conclude that

$$\mathcal{G}_{17,2}(a) \cong \mathcal{G}_{17,2}(1) \cong \mathcal{G}'_{19,2} \oplus \mathcal{G}'_{19,2} \oplus 2\mathcal{A}_1, \tag{6.26}$$

$$\mathcal{G}_{17,2}(a) = \mathrm{span}_{\mathbb{C}}\{2e_1 + e_2, e_3, e_6\} \oplus \mathrm{span}_{\mathbb{C}}\{e_1 + 2e_2, e_5, e_7\} \oplus \mathrm{span}_{\mathbb{C}}\{e_4, e_8\}. \tag{6.27}$$

For the listing of the representative $\mathcal{G}'_{19,2}$ in appendix we determine its nilradical. We use the form of ${\mathcal{G}'_{17,2}}^1$. Since the derived algebra of any solvable Lie algebra is a nilpotent ideal, we immediately have that $N({\mathcal{G}'_{17,2}}^1) = \mathrm{span}_{\mathbb{C}}\{x_2, x_3\}$. Finally, we simplify the commutation relations of ${\mathcal{G}'_{17,2}}^1$ by choice of basis $(y_1, y_2, y_3) = (x_2 + x_3, x_2 - x_3, \frac{1}{3}x_1)$ starting with vectors from nilradical. Thus, we get the nonzero commutation relations of $\mathcal{G}'_{19,2}$

$$[y_1, y_3] = y_1, \qquad [y_2, y_3] = -y_2. \tag{6.28}$$

Comparing these commutation relations with the list of Lie algebras in [57] we see that our algebra $\mathcal{G}'_{19,2}$ is the solvable three–dimensional algebra denoted by $\mathcal{A}_{3,4}$.

Chapter 7

Cartan graded contractions of $\mathrm{sl}(3,\mathbb{C})$

This chapter is devoted to the determination of all contractions of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$ which preserve the Cartan grading. The solution of this task was already given in [1, 13]. We apply our method in order to obtain and eventually improve the results of [1, 13].

7.1 Cartan grading of $\mathrm{sl}(3,\mathbb{C})$

The most famous grading is probably the Cartan grading — also known as root space decomposition [28]. This grading plays a crucial role in the classification theory of simple Lie algebras. The Cartan grading, as a fine grading, decomposes $\mathrm{sl}(n,\mathbb{C})$ into n^2-n one–dimensional grading subspaces (root spaces) and one $(n-1)$–dimensional abelian subalgebra (Cartan subalgebra). The bases of these grading subspaces are formed by $n\times n$ matrices $E_{ij}, i\neq j$ with unit on position i,j and zeros elsewhere. The basis of the abelian subalgebra is formed by $n\times n$ diagonal traceless matrices.

The Cartan grading of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$ has a form:

$$\begin{aligned}\Gamma_C : \mathrm{sl}(3,\mathbb{C}) &= L_{01}\oplus L_{10}\oplus L_{01}\oplus L_{11}\oplus L_{-1-1}\oplus L_{0-1}\oplus L_{-10},\\ &= \mathrm{span}_{\mathbb{C}}\{e_1,e_2\}\oplus \mathbb{C}e_3\oplus \mathbb{C}e_4\oplus \mathbb{C}e_5\oplus \mathbb{C}e_6\oplus \mathbb{C}e_7\oplus \mathbb{C}e_8, \end{aligned} \tag{7.1}$$

where the basis $(e_1,\dots,e_8)$ is in the defining 3–dimensional representation formed by matrices

$$\begin{aligned} e_1 &= \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \quad e_3 = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \quad e_4 = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\\ e_5 &= \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \quad e_6 = \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \quad e_7 = \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}, \quad e_8 = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}. \end{aligned} \tag{7.2}$$

The commutation relations of $\mathrm{sl}(3,\mathbb{C})$ corresponding to this basis are in Table 7.1.

Table 7.1: Commutation relations of the Cartan graded sl$(3,\mathbb{C})$

sl$(3,\mathbb{C})$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	0	$2e_3$	$-e_4$	e_5	$-e_6$	e_7	$-2e_8$
e_2	0	0	$-e_3$	$2e_4$	e_5	$-e_6$	$-2e_7$	e_8
e_3	$-2e_3$	e_3	0	e_5	0	$-e_7$	0	e_1
e_4	e_4	$-2e_4$	$-e_5$	0	0	e_8	e_2	0
e_5	$-e_5$	$-e_5$	0	0	0	e_1+e_2	e_3	$-e_4$
e_6	e_6	e_6	e_7	$-e_8$	$-(e_1+e_2)$	0	0	0
e_7	$-e_7$	$2e_7$	0	$-e_2$	$-e_3$	0	0	e_6
e_8	$2e_8$	$-e_8$	$-e_1$	0	e_4	0	$-e_6$	0

The index set I for the Cartan grading of sl$(3,\mathbb{C})$ is formed by 7 ordered pairs (i,j), where $i,j = -1,0,1$, with exception of the pairs $(1,-1),(-1,1)$. The operation $\diamond$ is the componentwise addition, where $1+1=-1$ and $(-1)+(-1)=1$. Thus, the index set I is the subset of the additive abelian group $\mathbb{Z}_3\times\mathbb{Z}_3$, where 2 is congruent to -1.

The symmetry group $\Delta_{\Gamma_C}(\mathrm{Aut}(\Gamma_C))$ of the Cartan grading, described in [34, 36], is isomorphic to the subgroup GL_C of $GL(2,\mathbb{Z}_3)$ which consists of the following 12 matrices:

$$GL_C: \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}, \begin{pmatrix}1&1\\0&-1\end{pmatrix}, \begin{pmatrix}-1&-1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\-1&-1\end{pmatrix}, \begin{pmatrix}-1&0\\1&1\end{pmatrix},$$
$$\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \begin{pmatrix}1&1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&-1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&1\end{pmatrix}. \tag{7.3}$$

The action of the symmetry $\pi_A \in \Delta_{\Gamma_C}(\mathrm{Aut}(\Gamma_C))$ corresponding to the matrix $A\in GL_C$ on the index $(i,j)\in I$ is given by the right multiplication of the vector (i,j) by the matrix A.

Let us note that the ordering $\mathcal{O}$ is given by the order of grading subspaces in (7.1).

7.2 Contraction system for the Cartan grading

Before approaching the construction of the system of contraction equations S_C for the Cartan grading of sl$(3,\mathbb{C})$, we describe the orbits of the action of the symmetry group $GL_C \cong \Delta_{\Gamma_C}(\mathrm{Aut}(\Gamma_C))$ on the sets I, I_u^2, I_u^3.

There are two orbits in the set of grading indices $I = \mathbb{Z}_3^2 \setminus \{(1,-1,),(-1,1)\}$, one orbit with only one point $(0,0)$ and the six–point orbit represented by point $(1,0)$. It corresponds to the fact that the symmetry group $\mathrm{Aut}(\Gamma_C)$ leaves the Cartan subalgebra invariant while the root spaces are permuted.

The set of unordered pairs of grading indices I_u^2 consists of six orbits. Three orbits are formed by irrelevant pairs of grading indices: two 6–point orbits represented by points $((1,0)(1,0))$, $((1,0)(1,1))$ and 1–point orbit represented by point $((0,0)(0,0))$. Remaining three orbits are formed by relevant pairs $\mathcal{I}$ of grading indices:

- 6–point orbit represented by $((1,0)(0,1))$,
- 6–point orbit represented by $((1,0)(0,0))$, corresponding relevant contraction parameters will be marked by superscript $\circ$,
- 3–point orbit represented by $((1,0)(-1,0))$, corresponding relevant contraction parameters will be marked by superscript $\bullet$.

The irrelevant parameters are zeros and thus, the explicit form of the contraction matrix ε with respect to chosen ordering $\mathcal{O}$ is:

$$\varepsilon = \begin{pmatrix} 0 & \varepsilon^{\circ}_{(00)(10)} & \varepsilon^{\circ}_{(00)(01)} & \varepsilon^{\circ}_{(00)(11)} & \varepsilon^{\circ}_{(00)(-1-1)} & \varepsilon^{\circ}_{(00)(0-1)} & \varepsilon^{\circ}_{(00)(-10)} \\ \varepsilon^{\circ}_{(00)(10)} & 0 & \varepsilon_{(10)(01)} & 0 & \varepsilon_{(10)(-1-1)} & 0 & \varepsilon^{\bullet}_{(10)(-10)} \\ \varepsilon^{\circ}_{(00)(01)} & \varepsilon_{(10)(01)} & 0 & 0 & \varepsilon_{(01)(-1-1)} & \varepsilon^{\bullet}_{(01)(0-1)} & 0 \\ \varepsilon^{\circ}_{(00)(11)} & 0 & 0 & 0 & \varepsilon^{\bullet}_{(11)(-1-1)} & \varepsilon_{(11)(0-1)} & \varepsilon_{(11)(-10)} \\ \varepsilon^{\circ}_{(00)(-1-1)} & \varepsilon_{(10)(-1-1)} & \varepsilon_{(01)(-1-1)} & \varepsilon^{\bullet}_{(11)(-1-1)} & 0 & 0 & 0 \\ \varepsilon^{\circ}_{(00)(0-1)} & 0 & \varepsilon^{\bullet}_{(01)(0-1)} & \varepsilon_{(11)(0-1)} & 0 & 0 & \varepsilon_{(0-1)(-10)} \\ \varepsilon^{\circ}_{(00)(-10)} & \varepsilon^{\bullet}_{(10)(-10)} & 0 & \varepsilon_{(11)(-10)} & 0 & \varepsilon_{(0-1)(-10)} & 0 \end{pmatrix}. \tag{7.4}$$

Let us note that during the action of GL_C on the contraction matrix ε, the marked contraction parameters are transformed to contraction parameters with the same mark.

Each contraction equation $e((i,j)(k,l)(m,n)) \in \mathsf{S}_C$ has a form

$$\begin{gathered} \varepsilon_{(i,j)(k+m,l+n)}\varepsilon_{(k,l)(m,n)}[x_{ij}[x_{kl},x_{mn}]] + \text{ cyclically } = 0, \\ \forall x_{ij} \in L_{ij}, \forall x_{kl} \in L_{kl}, \forall x_{mn} \in L_{mn}, \end{gathered} \tag{7.5}$$

where the word "cyclically" means that the two remaining terms are obtained from the first one by permutation of its grading indices. Since $[x_{ij}[x_{kl},x_{mn}]] \in L_{i+k+m,j+l+n}$ and $L_{1-1} = L_{-11} = \{0\}$, the equations is fulfilled for any unordered triplet $((i,j)(k,l)(m,n))$ for which $i+k+m = 1$ and $j+l+n = -1$ or $i+k+m = -1$ and $j+l+n = 1$. Moreover, if there are two grading indices equal in the unordered triplet $((i,j)(k,l)(m,n)) \in I_u^3$, then the corresponding equation is identically fulfilled (since all grading subspaces are abelian subalgebras).

There are 13 orbits in the set I_u^3 of unordered triplets of grading indices with respect to the action of the symmetry group GL_C. The representatives and the number of points in these orbits are summarized in Table 7.2. Since all grading subspaces form abelian subalgebras of sl$(3,\mathbb{C})$, only five orbits can lead to nontrivial contraction equations.

Table 7.2: Representatives and number of points in orbits of action GL_C on I_u^3

Trivial equations				Nontrivial equations	
(1,0)(1,0)(1,1)	12	(1,0)(0,1)(1,1)	6	(1,0)(0,1)(-1,0)	12
(1,0)(1,0)(0,1)	12	(1,0)(0,0)(0,0)	6	(1,0)(0,1)(0,0)	6
(1,0)(1,0)(1,0)	6	(1,0)(1,1)(0,0)	6	(1,0)(-1,0)(0,0)	3
(1,0)(1,0)(0,0)	6	(0,0)(0,0)(0,0)	1	(1,0)(0,1)(-1,-1)	2
(1,0)(1,0)(-1,0)	6				

Considering the point $(1,0)(0,1)(1,1)$ we get trivial equations. Thus, there are only four orbits leading to nontrivial contraction equations. We write the contraction equation corresponding to the representative point $(1,0)(0,1)(-1,0)$ of 12–point orbit

$$\varepsilon_{(10)(-11)}\varepsilon_{(01)(-10)}[e_3,[e_4,e_8]]+\overset{\circ}{\varepsilon}_{(00)(01)}\overset{\bullet}{\varepsilon}_{(10)(-10)}[e_4,[e_8,e_3]]+\varepsilon_{(11)(-10)}\varepsilon_{(10)(01)}[e_8,[e_3,e_4]]=0.$$

Since $[e_4,e_8]=0$, the first term containing the irrelevant contraction parameter vanishes. Therefore, we get equation

$$\overset{\circ}{\varepsilon}_{(00)(01)}\overset{\bullet}{\varepsilon}_{(10)(-10)}=\varepsilon_{(11)(-10)}\varepsilon_{(10)(01)}. \tag{7.6}$$

Since the index of the term on the left–hand side in (7.6), i.e. $[(00)(01)][(10)(-10)]$, belongs into 12–point orbit in $\mathcal{I}_u^2$, while the index of the right–hand side lies in 6–point orbit, the action of GL_C on the equation (7.6) generates 12 linearly independent equations S_c^1.

Since the point $(0,0)(1,0)(0,1)$ contains the index of the two–dimensional grading subspace (Cartan subalgebra) L_{00}, there are two corresponding contraction equations

$$\overset{\circ}{\varepsilon}_{(00)(11)}\varepsilon_{(10)(01)}[e_i,[e_3,e_4]]+\varepsilon_{(10)(01)}\overset{\circ}{\varepsilon}_{(00)(01)}[e_3,[e_4,e_i]]+\varepsilon_{(10)(01)}\overset{\circ}{\varepsilon}_{(00)(10)}[e_4,[e_i,e_3]]=0,$$

where $i=1,2$. Using Table 7.1 with the commutation relations of sl$(3,\mathbb{C})$ we get

$$(\overset{\circ}{\varepsilon}_{(00)(11)}\varepsilon_{(10)(01)}+\varepsilon_{(10)(01)}\overset{\circ}{\varepsilon}_{(00)(01)}-2\varepsilon_{(10)(01)}\overset{\circ}{\varepsilon}_{(00)(10)})e_5=0,$$
$$(\overset{\circ}{\varepsilon}_{(00)(11)}\varepsilon_{(10)(01)}-2\varepsilon_{(10)(01)}\overset{\circ}{\varepsilon}_{(00)(01)}+\varepsilon_{(10)(01)}\overset{\circ}{\varepsilon}_{(00)(10)})e_5=0.$$

By summing and subtracting these equations we obtain the following two terms equation:

$$\underbrace{\varepsilon^{\circ}_{(00)(11)}\varepsilon_{(10)(01)}}_{a} = \underbrace{\varepsilon^{\circ}_{(00)(01)}\varepsilon_{(10)(01)}}_{b} = \underbrace{\varepsilon^{\circ}_{(00)(10)}\varepsilon_{(10)(01)}}_{c}. \tag{7.7}$$

A stability subgroup of the point $(0,0)(1,0)(0,1)$ is a cyclic group $\{1, X\}$, where $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in GL_C$. The action of X on the equation (7.7) transforms c into b while a is preserved. In fact, the indices of b and c have stability subgroups of order 1 and lie in the same 12–point orbit in $\mathcal{I}^2_u$ while the index of a has a stability subgroup of order 2 and thus, belongs into the 6–point orbit. Therefore, using the action of GL_C we get from (7.7) 12 linearly independent equations S^2_c generated by equation $a = c$.

Three–point orbit with the representative point $(1,0)(-1,0)(0,0)$ generates the third part S^3_c of the system of contraction equations:

$$\varepsilon^{\bullet}_{(10)(-10)}\varepsilon^{\circ}_{(00)(-10)}[e_3,[e_8,e_i]] + \varepsilon^{\bullet}_{(10)(-10)}\varepsilon^{\circ}_{(00)(10)}[e_8,[e_i,e_3]] + \varepsilon_{(00)(00)}\varepsilon^{\bullet}_{(10)(-10)}[e_i,[e_3,e_8]] = 0,$$

for $i = 1,2$. Since $[e_3, e_8] = e_1$ and L_{00} is abelian, third term with the irrelevant contraction parameter vanishes. Evaluating the rest of the commutators we get only one two–term equation

$$\varepsilon^{\bullet}_{(10)(-10)}\varepsilon^{\circ}_{(00)(-10)} = \varepsilon^{\bullet}_{(10)(-10)}\varepsilon^{\circ}_{(00)(10)}. \tag{7.8}$$

This equation is invariant under the stability subgroup $G_1 = \{1, -1, Y, -Y\}$, where $Y = \left(\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$, of the point $(1,0)(-1,0)(0,0)$ and thus, the action of GL_C on (7.8) generates only three different equations. These equations are linearly independent, because the indices of the left–hand side and the right–hand side of (7.8) lie in the same 6–point orbit and $\mathcal{I}^2_u$ and their stability subgroup $\{1, Y\}$ belongs into the stability subgroup G_1.

The last part of the system S^4_c is generated from the equation corresponding to the representative point $(1,0)(0,1)(-1,-1)$ of 2–point orbit

$$\varepsilon^{\bullet}_{(10)(-10)}\varepsilon_{(01)(-1-1)}[e_3,[e_4,e_6]]+\varepsilon^{\bullet}_{(01)(0-1)}\varepsilon_{(10)(-1-1)}[e_4,[e_6,e_3]]+\varepsilon^{\bullet}_{(11)(-1-1)}\varepsilon_{(10)(01)}[e_6,[e_3,e_4]]=0.$$

Using commutation relations we get

$$(\varepsilon^{\bullet}_{(10)(-10)}\varepsilon_{(01)(-1-1)} - \varepsilon^{\bullet}_{(11)(-1-1)}\varepsilon_{(10)(01)})e_1 + (\varepsilon^{\bullet}_{(01)(0-1)}\varepsilon_{(10)(-1-1)} - \varepsilon^{\bullet}_{(11)(-1-1)}\varepsilon_{(10)(01)})e_2 = 0,$$

which leads to the following two–term equations:

$$\underbrace{\varepsilon^{\bullet}_{(10)(-10)}\varepsilon_{(01)(-1-1)}}_{a} = \underbrace{\varepsilon^{\bullet}_{(01)(0-1)}\varepsilon_{(10)(-1-1)}}_{b} = \underbrace{\varepsilon^{\bullet}_{(11)(-1-1)}\varepsilon_{(10)(01)}}_{c}. \tag{7.9}$$

The indices of terms a, b, c belong into one 6–point orbit in $\mathcal{I}_u^2$ and the action of matrix $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ transforms a into b while c is preserved. Thus, the S_c^4 consists of four linearly independent equations generated from $a = c$.

The whole system of contraction equations S_C consists of the following four subsystems:

$$\mathsf{S}_C^1 : \varepsilon^{\bullet}_{(10)(-10)A}\varepsilon^{\circ}_{(00)(01)A} = \varepsilon_{(11)(-10)A}\varepsilon_{(10)(01)A}, \qquad \forall A \in GL_C, \quad 12 \text{ equations}, \tag{7.10}$$

$$\mathsf{S}_C^2 : \varepsilon^{\circ}_{(00)(11)A}\varepsilon_{(10)(01)A} = \varepsilon^{\circ}_{(00)(10)A}\varepsilon_{(10)(01)A}, \qquad \forall A \in GL_C, \quad 12 \text{ equations}, \tag{7.11}$$

$$\mathsf{S}_C^3 : \varepsilon^{\bullet}_{(10)(-10)A}\varepsilon^{\circ}_{(00)(-10)A} = \varepsilon^{\bullet}_{(10)(-10)A}\varepsilon^{\circ}_{(00)(10)A}, \qquad \forall A \in GL_C, \quad 3 \text{ equations}, \tag{7.12}$$

$$\mathsf{S}_C^4 : \varepsilon^{\bullet}_{(10)(-10)A}\varepsilon_{(01)(-1-1)A} = \varepsilon^{\bullet}_{(11)(-1-1)A}\varepsilon_{(10)(01)A}, \qquad \forall A \in GL_C, \quad 4 \text{ equations}, \tag{7.13}$$

where the abbreviation $\varepsilon_{(ij)(kl)A} = \varepsilon_{(ij)A(kl)A}$ has been used. Let us note that the indices of the terms from different subsystems belong into different orbits in $\mathcal{I}_u^2$ and, therefore, these 31 equations are linearly independent.

7.3 Finding the solution of S_C

In order to solve the system of contraction equations S_C for the Cartan grading of $\mathrm{sl}(3, \mathbb{C})$ we construct the sequence of assumptions and then use Theorem 3.9. We solve the system S_C in four steps. There are two 6–point orbits and one 3–point orbit of relevant contraction parameters. We choose $\varepsilon_{(10)(01)}$ from 6–point orbit and we assume that $\varepsilon_{(10)(01)} \neq 0$ in first three steps, since the number of points in the orbit determines the number of independent equations in the non–equivalence system in the last step. For the first step we extend the assumption such that the term $\varepsilon^{\bullet}_{(11)(-1-1)A}\varepsilon_{(10)(01)A}$ of the equation (7.13) is nonzero. This assumption together with S_C^4 ensures that all contraction parameters marked by $\bullet$ are nonzero and the corresponding non–equivalence system in the next step will ensure that S_C^4 is satisfied. All steps with the assumptions and the non–equivalence systems are listed below.

1. $\mathcal{R}^0 = \left\{\varepsilon \in \mathcal{R}(\mathsf{S}_C) \mid \varepsilon_{(10)(01)} \neq 0,\, \varepsilon^{\bullet}_{(11)(-1-1)} \neq 0\right\}$

   $\mathcal{S}^0 :\ \varepsilon_{(10)(01)A}\varepsilon^{\bullet}_{(11)(-1-1)A} = 0 \quad \forall A \in GL_C$
2. $\mathcal{R}^1 = \left\{\varepsilon \in \mathcal{R}(\mathsf{S}_C \cup \mathcal{S}^0) \mid \varepsilon_{(10)(01)} \neq 0,\, \varepsilon_{(11)(-10)} \neq 0\right\}$

   $\mathcal{S}^1 :\ \varepsilon_{(10)(01)A}\varepsilon_{(11)(-10)A} = 0 \quad \forall A \in GL_C$
3. $\mathcal{R}^2 = \left\{\varepsilon \in \mathcal{R}(\mathsf{S}_C \cup \mathcal{S}^0 \cup \mathcal{S}^1) \mid \varepsilon_{(10)(01)} \neq 0, \right\}$

   $\mathcal{S}^2 :\ \varepsilon_{(10)(01)A} = 0 \quad \forall A \in GL_C$
4. $\mathcal{R}^3 = \mathcal{R}(\mathsf{S}_C \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \mathcal{S}^2)$

In order to list the solution of separate steps, we renormalize all nonzero parameters of these solutions to 1, i.e. we find a strongly equivalent contraction matrix with units in the places of nonzero parameters. This renormalization was successful in all cases. Other parameters in solutions which are arbitrary complex numbers are denoted by a, b, c, d, e, f. We describe solutions of these four steps in detail.

Step 1. The explicit solution of the system S_C under the assumptions $\varepsilon_{(10)(01)} \neq 0$ and $\varepsilon^{\bullet}_{(11)(-1-1)} \neq 0$ is given as one parametric matrix, i.e. $\mathcal{R}^0_{nor} = \{\varepsilon^0_1\}$, where

$$\varepsilon^0_1 = \begin{pmatrix} 0 & a & a & a & a & a & a \\ a & 0 & 1 & 0 & 1 & 0 & 1 \\ a & 1 & 0 & 0 & 1 & 1 & 0 \\ a & 0 & 0 & 0 & 1 & a & a \\ a & 1 & 1 & 1 & 0 & 0 & 0 \\ a & 0 & 1 & a & 0 & 0 & a \\ a & 1 & 0 & a & 0 & a & 0 \end{pmatrix}.$$

Step 2. The non–equivalence system $\mathcal{S}^0$ has 6 equations and ensures that S^4_C is satisfied. The solutions of S_C and $\mathcal{S}^0$ under the assumptions $\varepsilon_{(10)(01)} \neq 0$ and $\varepsilon_{(11)(-10)} \neq 0$ are described by two parametric matrices. Since all parameters of these matrices are nonzero, after the renormalization we get $\mathcal{R}^1_{nor} = \{\varepsilon^1_1, \varepsilon^1_2\}$, where

$$\varepsilon^1_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^1_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Step 3. Solutions of S_C and 12 non–equivalence conditions $\mathcal{S}^0 \cup \mathcal{S}^1$ under the assumptions $\varepsilon_{(10)(01)} \neq 0$ are given by eight parametric matrices $\mathcal{R}^2_{nor} = \{\varepsilon^2_1, \varepsilon^2_2, \varepsilon^2_3, \varepsilon^2_4, \varepsilon^2_5, \varepsilon^2_6, \varepsilon^2_7, \varepsilon^2_8\}$, where

$$\varepsilon^2_1 = \begin{pmatrix} 0 & a & a & a & b & c & d \\ a & 0 & 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \varepsilon^2_2 = \begin{pmatrix} 0 & a & a & a & a & a & b \\ a & 0 & 1 & 0 & c & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & c & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \varepsilon^2_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & b \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon^2_4 = \begin{pmatrix} 0 & a & a & a & b & b & b \\ a & 0 & 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & c \\ b & 0 & 0 & 0 & 0 & c & 0 \end{pmatrix}, \varepsilon^2_5 = \begin{pmatrix} 0 & a & a & a & a & b & a \\ a & 0 & 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & c & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & c & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \varepsilon^2_6 = \begin{pmatrix} 0 & a & a & a & a & a & a \\ a & 0 & 1 & 0 & b & 0 & 0 \\ a & 1 & 0 & 0 & c & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon_7^2 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&a&b&0\\ 0&0&0&0&0&0&0\\ 0&0&a&0&0&0&0\\ 0&0&b&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},\ \varepsilon_8^2 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&a\\ 0&1&0&0&0&b&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&b&0&0&0&c\\ 0&a&0&0&0&c&0 \end{pmatrix}.$$

Step 4. Finally, the non–equivalence system $\mathcal{S}^2 = \left\{\varepsilon_{(10)(01)A} = 0 \,|\, \forall A \in GL_C\right\}$ enforces a zero value for the whole orbit of unmarked contraction parameters and ensures the fulfilment of all previous non–equivalence systems. Due to this 6 zeros, the systems S_C^2 and S_C^4 are satisfied automatically and the solutions of remaining 15 equations are $\mathcal{R}^3_{nor} = \{\varepsilon_1^3, \varepsilon_2^3, \varepsilon_3^3, \varepsilon_4^3, \varepsilon_5^3\}$, where

$$\varepsilon_1^3 = \begin{pmatrix} 0&a&b&c&d&e&f\\ a&0&0&0&0&0&0\\ b&0&0&0&0&0&0\\ c&0&0&0&0&0&0\\ d&0&0&0&0&0&0\\ e&0&0&0&0&0&0\\ f&0&0&0&0&0&0 \end{pmatrix}, \varepsilon_2^3 = \begin{pmatrix} 0&0&0&a&a&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ a&0&0&0&b&0&0\\ a&0&0&b&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \varepsilon_3^3 = \begin{pmatrix} 0&0&a&0&0&a&0\\ 0&0&0&0&0&0&0\\ a&0&0&0&0&b&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ a&0&b&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$\varepsilon_4^3 = \begin{pmatrix} 0&a&0&0&0&0&a\\ a&0&0&0&0&0&b\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ a&b&0&0&0&0&0 \end{pmatrix},\ \varepsilon_5^3 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&a\\ 0&0&0&0&0&b&0\\ 0&0&0&0&c&0&0\\ 0&0&0&c&0&0&0\\ 0&0&b&0&0&0&0\\ 0&a&0&0&0&0&0 \end{pmatrix}.$$

All solutions of S_C are divided into four sets $\mathcal{R}^0_{nor}, \mathcal{R}^1_{nor}, \mathcal{R}^2_{nor}, \mathcal{R}^3_{nor}$ and the contraction matrices from different sets are inequivalent. As in previous chapters we discussed when the elements of the contraction matrices vanish and divided each set $\mathcal{R}^k_{nor}$ into subsets according to the number of zeros $\nu(\varepsilon)$. In these subsets we collected all solutions with the same support. Since all matrices with the same support were either strongly equivalent, or a special case of one of them, there was always only one inequivalent solution with the given support. Using the symmetry group we compare the projections of solutions with different supports and choose only inequivalent ones as in example 5.2. All 47 normalized representatives of the equivalence classes of solutions are listed in Appendix A.3.

7.4 Continuous and discrete solutions

The only solution without zeros listed in sets $\mathcal{R}^k_{nor}$ is the solution ε^0_1 where $a \neq 0$. This solution can be renormalized to its own projection. Therefore, every solution without zeros has a form of the normalization matrix (3.28) and we can use the normalization matrix for constructing higher–order identities as in previous chapters. However, in the case of Cartan grading, higher–order identities follow immediately from the system of contraction equations. Dividing the equation (7.11) by $\varepsilon_{(10)(01)A}$ we have

$$\varepsilon^{\circ}_{(00)(11)A} = \varepsilon^{\circ}_{(00)(10)A}, \qquad \forall A \in GL_C. \tag{7.14}$$

Since the action of matrix $Z = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ transforms the left–hand side of (7.14) into its right–hand side and the stability subgroup of the equation (7.14) is $\{1, Z\}$, we get 6–point orbit of first order identities. These identities can be written in the following form

$$\varepsilon^{\circ}_{(00)(10)} = \varepsilon^{\circ}_{(00)(01)} = \varepsilon^{\circ}_{(00)(11)} = \varepsilon^{\circ}_{(00)(-1-1)} = \varepsilon^{\circ}_{(00)(0-1)} = \varepsilon^{\circ}_{(00)(-10)}. \tag{7.15}$$

Thus, we have the necessary condition for continuous solutions: the contraction matrix ε is a continuous contraction if all its relevant contraction parameters in the first row (marked by $\circ$) have the same value.

This condition also appeared to be a sufficient condition, since for all solutions satisfying (7.15) we explicitly found a continuous path of the form (3.34). Among all 47 solutions of S_C there are 17 continuous solutions, and 26 discrete ones. Remaining 4 solutions are continuous only for a special value (namely unit) of their parameters, otherwise they were discrete.

7.5 Identification of contracted Lie algebras

There are 47 inequivalent contraction matrices solving the system of contraction equations S_C for the Cartan graded sl$(3, \mathbb{C})$. These contraction matrices are divided into 12 groups according to the number of zeros ν among 15 relevant contraction parameters. The number of contraction matrices in these group are summarized in the following table:

Number of zeros ν	0	4	6	7	8	9	10	11	12	13	14	15
Number of solutions	1	1	1	2	3	4	4	6	12	9	3	1

The contracted Lie algebra corresponding to the contraction matrix $\varepsilon^{\nu,i}$ is denoted $\mathcal{T}_{\nu,i}$, where the second index i is numbering contraction matrices with the same number of zeros ν.

There are two trivial solutions: $\varepsilon^{15,1}$ (with 15 zeros) corresponding to the 8–dimensional abelian Lie algebra and $\varepsilon^{0,1}$ (without zeros) corresponding to the initial Lie algebra sl$(3, \mathbb{C})$.

Among the remaining 45 nontrivial solutions, there are 21 parametric contraction matrices. Nontrivial contraction matrices are divided according to the number of their parameters as follows:

Number of parameters	0	1	2	3	4	5
Number of solutions	24	9	6	4	1	1

These parameters in contraction matrices are denoted by a, b, c, d, f and all are nonzero complex numbers. The parametric families of Lie algebras corresponding to the parametric solutions will be counted as one algebra and their number among the investigated cases will be written behind the slash.

We have to identify 45/21 non–abelian Lie algebras. The results of the direct decomposition are following:

- 34/16 algebras allow the central decomposition. Further, only non–abelian parts $\mathcal{T}'_{\nu,i}$ (with dimensions lower than 8) of these algebras $\mathcal{T}_{\nu,i}$ were investigated.
- 11/5 algebras are decomposable into the direct sum of two non–abelian indecomposable ideals.
- 2/2 algebras are decomposable into the direct sum of three non–abelian indecomposable ideals.

After the decomposition we continued in the identification procedure with 60/23 indecomposable Lie algebras. These algebras are divided according to their dimensions as follows:

Dimension	2	3	4	5	6	7	8
Number of algebras	13/5	17/4	0	8/3	6/4	7/3	9/4

We computed the numerical invariants $\mathrm{inv}(\mathcal{T})$ and found out that there are 35/23 solvable non–nilpotent Lie algebras divided into 15 classes, 22 nilpotent Lie algebras divided into 8 classes and 3 non–solvable Lie algebras in 3 classes. Thus, non–solvable Lie algebras are now separated. The Levi decomposition is nontrivial only for two of them, in both cases the radical is non–abelian.

During the search for isomorphisms, we have found 19/11 isomorphic algebras in the classes of solvable and 14 isomorphic algebras in the classes of nilpotent Lie algebras. By omitting these algebras we get only one algebra in each class of the nilpotent Lie algebras and we conclude that there are 8 nilpotent and 3 non–solvable indecomposable Lie algebras.

Among the classes of solvable non–nilpotent Lie algebras, there remains one class containing two algebras. These algebras are two–parametric families $\mathcal{T}'_{9,2}(a,b)$ and $\mathcal{T}'_{9,3}(a,b)$. In order to distinguish among them we use the invariant function φ. The tabulation of invariant functions for more than one–parametric Lie algebras is usually too laborious. Fortunately, it is not necessary in this case. Since every invariant function has only a finite number of points for which its value is different from the general value (the value in the last column of the table with the given invariant function), it is sufficient to show that $\varphi_{\mathcal{T}'_{9,2}(a,b)}$ and $\varphi_{\mathcal{T}'_{9,3}(a,b)}$ have different general values. The computation shows that these general values are independent of the parameters a, b and $\varphi_{\mathcal{T}'_{9,2}(a,b)}(x) = 25$ while $\varphi_{\mathcal{T}'_{9,3}(a,b)}(x) = 27$. Therefore, Lie algebras $\mathcal{T}'_{9,2}(a,b)$ and $\mathcal{T}'_{9,3}(a,b)$ are not isomorphic and we conclude that there are 16/12 solvable non–nilpotent indecomposable Lie algebras. These indecomposable Lie algebras are listed together with their invariant characteristics in Appendix B.3. All algebras are now identified up to the ranges of their parameters.

The decomposable Lie algebras are given as the direct sums of the indecomposable ones and the abelian Lie algebra $\mathcal{A}_1$. There are 9 non–isomorphic decomposable Lie algebras. Eight of them are solvable (non–nilpotent) and appear as discrete graded contractions:

$$\begin{aligned} &\mathcal{T}_{10,4} \cong \mathcal{T}'_{14,2} \oplus \mathcal{T}'_{11,4}, && \mathcal{T}_{12,4}(a) \cong \mathcal{T}'_{14,1} \oplus \mathcal{T}'_{14,1} \oplus \mathcal{T}'_{14,2} \oplus \mathcal{A}_1, \\ &\mathcal{T}_{11,3}(a,b,c) \cong \mathcal{T}'_{13,3}(a) \oplus \mathcal{T}'_{13,3}(\tfrac{c}{b}) \oplus 2\mathcal{A}_1, && \mathcal{T}_{12,6} \cong \mathcal{T}'_{14,1} \oplus \mathcal{T}'_{13,6} \oplus \mathcal{A}_1, \\ &\mathcal{T}_{11,5}(a,b) \cong \mathcal{T}'_{12,1}(a,-b) \oplus \mathcal{T}'_{14,2}, && \mathcal{T}_{13,1}(a) \cong \mathcal{T}'_{14,1} \oplus \mathcal{T}'_{14,1} \oplus 4\mathcal{A}_1, \\ &\mathcal{T}_{12,3}(a,b) \cong \mathcal{T}'_{14,1} \oplus \mathcal{T}'_{13,3}(b) \oplus 3\mathcal{A}_1, && \mathcal{T}_{13,4} \cong \mathcal{T}'_{14,1} \oplus \mathcal{T}'_{14,2} \oplus 3\mathcal{A}_1, \end{aligned}$$

and one is nilpotent and appears as a continuous contraction:

$$\mathcal{T}_{13,8} \cong \mathcal{T}'_{14,2} \oplus \mathcal{T}'_{14,2} \oplus 2\mathcal{A}_1.$$

Other decomposable results are isomorphic to those listed above:

$$\begin{aligned} &\mathcal{T}_{12,5}(a) \cong \mathcal{T}_{12,4}(a), && \mathcal{T}_{13,5} \cong \mathcal{T}_{13,4}, \\ &\mathcal{T}_{13,2}(a) \cong \mathcal{T}_{13,1}(a), && \mathcal{T}_{12,9} \cong \mathcal{T}_{13,8}. \end{aligned}$$

We also list the isomorphisms among the indecomposable Lie algebras. There are isomorphisms among the nilpotent algebras (left column) and two among the solvable algebras (right column):

$$\begin{aligned} &\mathcal{T}_{14,3} \cong \mathcal{T}_{14,2}, && \mathcal{T}_{12,2}(a,b) \cong \mathcal{T}_{12,1}(-a,-b), \\ &\mathcal{T}_{13,7} \cong \mathcal{T}_{13,6}, && \mathcal{T}_{11,2}(a,b,c) \cong \mathcal{T}_{11,1}(-a,c,b), \\ &\mathcal{T}_{12,12} \cong \mathcal{T}_{12,10}, && \end{aligned}$$

Let us note that isomorphic graded contractions were always of the same type, i.e. all discrete or all continuous.

The overview of the number of contracted Lie algebras for the Cartan grading of $\mathrm{sl}(3,\mathbb{C})$ is given in Table 7.3. Lie algebras are divided there according to the dimension of their non–abelian parts and their types.

Table 7.3: The number of nontrivial graded contractions of the Cartan graded sl$(3, \mathbb{C})$

Dimension of non–abelian part	Solvable		Nilpotent		Non–solvable		Total
	Indec.	Dec.	Indec.	Dec.	Indec.	Dec.	
2	1						1
3	1		1		1		3
4		1					1
5	2	2	1				5
6	3	1	1	1	1		7
7	4	2	2				8
8	5	2	3		1		11
							36

Including two trivial contractions we have obtained 38 non–isomorphic contracted Lie algebras as graded contractions of the Cartan graded Lie algebra sl$(3, \mathbb{C})$. Among them, there are 12 parametric families of Lie algebras, one of them with five parameters.

From all these 38 contracted Lie algebras, 21 are discrete contractions and 13 continuous contractions. Remaining 4 parametric algebras are continuous contractions for a special value of parameters otherwise they are discrete. In order to determine the ranges of parameters for one–parametric algebras, we have used the invariant functions. These functions are given in Appendix C.3.

7.6 Comparison of results

Having classified all contracted Lie algebras of the Cartan graded sl$(3, \mathbb{C})$, we compare them with the results in papers [1, 13]. We have found 38 different graded contractions which preserve the Cartan grading of sl$(3, \mathbb{C})$, while in the paper [1] only 34 different graded contractions were found. These contractions were denoted by C_k and classified according to their types, possible decomposition, series DS, CS, US and nilradicals. The comparison of results, based on this classification (extended by the number of formal invariants, τ whenever it was necessary), shows that the missing algebras are algebras $\mathcal{T}_{10,2}, \mathcal{T}_{12,8}$ and parametric families $\mathcal{T}_{9,2}(a, b), \mathcal{T}_{12,3}(1, b)$. Moreover, in [1] solely special cases, namely $C_{33} \cong \mathcal{T}_{11,3}(1, 1, 1)$, $C_{34} \cong \mathcal{T}_{13,3}(1)$, of parametric Lie algebras $\mathcal{T}_{11,3}(a, 1, c)$ and $\mathcal{T}_{13,3}(a)$ were found. Let us note that in the earlier paper [13] these two algebras C_{33} and C_{34} were omitted.

The correspondence between the graded contractions from [1], denoted by C_k, and our results, denoted by $\mathcal{T}_{i,j}$, is shown in Table 7.4.

Table 7.4: Comparison of graded contractions for Cartan graded sl$(3,\mathbb{C})$

Non–solvable		Solvable non–nilpotent		Nilpotent	
Indecomposable Lie algebras					
C_1	$\mathcal{T}_{0,1}$	C_3	$\mathcal{T}_{9,1}(a,b,c,d,f)$	C_8	$\mathcal{T}_{12,11}$
C_2	$\mathcal{T}_{4,1}$	C_4	$\mathcal{T}_{8,1}(a,b,c)$	C_9	$\mathcal{T}_{11,6}$
		C_5	$\mathcal{T}_{7,1}(a)$	C_{10}	$\mathcal{T}_{9,4}$
		C_6	$\mathcal{T}_{7,2}(a)$		
		C_7	$\mathcal{T}_{6,1}$		
Decomposable Lie algebras					
C_{11}	$\mathcal{T}_{12,7}$	C_{13}	$\mathcal{T}_{10,1}(a,b,c,d)$	C_{27}	$\mathcal{T}_{12,10}$
C_{12}	$\mathcal{T}_{8,3}$	C_{14}	$\mathcal{T}_{9,3}(a,b)$	C_{28}	$\mathcal{T}_{12,9}$
		C_{15}	$\mathcal{T}_{8,2}$	C_{29}	$\mathcal{T}_{13,6}$
		C_{16}	$\mathcal{T}_{11,1}(a,b,c)$	C_{30}	$\mathcal{T}_{13,8}$
		C_{17}	$\mathcal{T}_{10,3}(a)$	C_{31}	$\mathcal{T}_{14,2}$
		C_{18}	$\mathcal{T}_{12,1}(a,b)$	C_{32}	$\mathcal{T}_{15,1}$
		C_{19}	$\mathcal{T}_{11,4}$		
		C_{20}	$\mathcal{T}_{12,6}$		
		C_{21}	$\mathcal{T}_{11,5}(a,b)$		
		C_{22}	$\mathcal{T}_{10,4}$		
		C_{23}	$\mathcal{T}_{12,4}$		
		C_{24}	$\mathcal{T}_{13,4}$		
		C_{25}	$\mathcal{T}_{13,1}$		
		C_{26}	$\mathcal{T}_{14,1}$		
		C_{33}	$\mathcal{T}_{11,3}(1,1,1)$		
		C_{34}	$\mathcal{T}_{13,3}(1)$		

Chapter 8

$\mathrm{sl}(3,\mathbb{C})$ graded contractions summary

In this chapter we outline the situation for the last grading Γ_4 of $\mathrm{sl}(3,\mathbb{C})$. Then, we compare the graded contractions obtained from different gradings of $\mathrm{sl}(3,\mathbb{C})$. We also compare our results with results of [5, 76, 77] which were obtained as solutions of general two–term equations.

8.1 Γ_4 grading of $\mathrm{sl}(3,\mathbb{C})$

In contrast to all previously described gradings, the finest grading Γ_4 is known [34] only for the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. It decomposes $\mathrm{sl}(3,\mathbb{C})$ into eight one–dimensional grading subspaces

$$\mathrm{sl}(3,\mathbb{C}) = \bigoplus_{i=0}^{7} L_i, \qquad L_i = \mathrm{span}_{\mathbb{C}}\{e_{i+1}\}, \tag{8.1}$$

where the basis vectors e_i are usually represented by 3×3 complex matrices

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_8 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.2}$$

The commutation relations corresponding to the basis $(e_1, \ldots, e_8)$ are written in Table 8.1.

The index set I for grading Γ_4 coincides with the additive group $\mathbb{Z}_8$. The symmetry group $\Delta_{\Gamma_4}(\mathrm{Aut}(\Gamma_4))$ of the grading Γ_4, described in [34, 36], is isomorphic to the multiplicative group

$$G_4 = \{1, 3, 5, 7\}, \tag{8.3}$$

where the multiplication is considered modulo 8. The action of the symmetry group G_4 on the index set $I = \mathbb{Z}_8$ is also given as a multiplication modulo 8, i.e. if $\pi_a \in \Delta_{\Gamma_4}(\mathrm{Aut}(\Gamma_4))$

Table 8.1: Commutation relations of Γ_4 grading of $\mathrm{sl}(3,\mathbb{C})$

$\mathrm{sl}(3,\mathbb{C})$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	$-e_2$	$2e_3$	e_4	0	$-e_6$	$-2e_7$	e_8
e_2	e_2	0	e_4	e_5	$-3e_6$	$-2e_7$	0	e_1
e_3	$-2e_3$	$-e_4$	0	0	0	$-e_8$	e_1	0
e_4	$-e_4$	$-e_5$	0	0	$-3e_8$	e_1	e_2	$2e_3$
e_5	0	$3e_6$	0	$3e_8$	0	$3e_2$	0	$3e_4$
e_6	e_6	$2e_7$	e_8	$-e_1$	$-3e_2$	0	0	$-e_5$
e_7	$2e_7$	0	$-e_1$	$-e_2$	0	0	0	$-e_6$
e_8	$-e_8$	$-e_1$	0	$-2e_3$	$-3e_4$	e_5	e_6	0

is a permutation of the index set I corresponding to $a \in G_4$, then for any $i \in I$ we have $\pi_a(i) = (ia)_{\bmod 8}$.

8.2 Contraction system for grading Γ_4

We construct the system of contraction equations S_{Γ_4} for the Γ_4 grading of $\mathrm{sl}(3,\mathbb{C})$. Since the symmetry group $G_4 \cong \Delta_{\Gamma_4}(\mathrm{Aut}(\Gamma_4))$ has only four elements, we will deal with more orbits than in the case of other fine gradings. Even the set of the grading indices $I = \mathbb{Z}_7$ consists of four orbits. These orbits are $\{0\}$, $\{4\}$, $\{2,6\}$ and $\{1,3,5,7\}$. It reflects the fact that the grading subspaces L_0 and L_4 are preserved under the action of the symmetry group $\mathrm{Aut}(\Gamma_4)$ and the subspaces L_2, L_6 can be transformed only to each other.

The set of unordered pairs of grading indices I_u^2 splits under the action of G_4 into 15 orbits. Eight of these orbits are formed by relevant grading indices and the corresponding orbits of the relevant contraction parameters are marked in the contraction matrix ε by different superscripts

$$\varepsilon = \begin{pmatrix} 0 & \varepsilon_{01} & \varepsilon_{02}^{*} & \varepsilon_{03} & 0 & \varepsilon_{05} & \varepsilon_{06}^{*} & \varepsilon_{07} \\ \varepsilon_{01} & 0 & \varepsilon_{12}^{\circ} & \varepsilon_{13}^{\blacksquare} & \varepsilon_{14}^{\bullet} & \varepsilon_{15}^{\shortparallel} & 0 & \varepsilon_{17}^{\star} \\ \varepsilon_{02}^{*} & \varepsilon_{12}^{\circ} & 0 & 0 & 0 & \varepsilon_{25}^{\circ} & \varepsilon_{26}^{\diamond} & 0 \\ \varepsilon_{03} & \varepsilon_{13}^{\blacksquare} & 0 & 0 & \varepsilon_{34}^{\bullet} & \varepsilon_{35}^{\star} & \varepsilon_{36}^{\circ} & \varepsilon_{37}^{\shortparallel} \\ 0 & \varepsilon_{14}^{\bullet} & 0 & \varepsilon_{34}^{\bullet} & 0 & \varepsilon_{45}^{\bullet} & 0 & \varepsilon_{47}^{\bullet} \\ \varepsilon_{05} & \varepsilon_{15}^{\shortparallel} & \varepsilon_{25}^{\circ} & \varepsilon_{35}^{\star} & \varepsilon_{45}^{\bullet} & 0 & 0 & \varepsilon_{57}^{\blacksquare} \\ \varepsilon_{06}^{*} & 0 & \varepsilon_{26}^{\diamond} & \varepsilon_{36}^{\circ} & 0 & 0 & 0 & \varepsilon_{67}^{\circ} \\ \varepsilon_{07} & \varepsilon_{17}^{\star} & 0 & \varepsilon_{37}^{\shortparallel} & \varepsilon_{47}^{\bullet} & \varepsilon_{57}^{\blacksquare} & \varepsilon_{67}^{\circ} & 0 \end{pmatrix}. \tag{8.4}$$

There are 42 orbits in the set I_u^3, however, only 13 of them lead to nontrivial contraction equations. The representatives of these 13 orbits, the number of their points and the corresponding contraction equations are summarized in Table 8.2.

Table 8.2: Equations of S_{Γ_4}

Representative	Generating equation	Number of points
$(1,3,5)$	$3\varepsilon_{13}^{\cdot}\varepsilon_{45}^{\bullet} - 2\varepsilon_{15}^{\shortparallel}\varepsilon_{36}^{\circ} - \varepsilon_{01}\varepsilon_{35}^{\star} = 0$	4
$(0,1,2)$	$\varepsilon_{12}^{\circ}(2\varepsilon_{02}^{*} - \varepsilon_{01} - \varepsilon_{03}) = 0$	4
$(0,1,5)$	$\varepsilon_{15}^{\shortparallel}(2\varepsilon_{06}^{*} - \varepsilon_{01} - \varepsilon_{05}) = 0$	2
$(1,2,5)$	$2\varepsilon_{15}^{\shortparallel}\varepsilon_{26}^{\diamond} - \varepsilon_{12}^{\circ}\varepsilon_{35}^{\star} - \varepsilon_{25}^{\circ}\varepsilon_{17}^{\star} = 0$	2
$(1,2,6)$	$\varepsilon_{12}^{\circ}\varepsilon_{36}^{\circ} - \varepsilon_{01}\varepsilon_{26}^{\diamond} = 0$	4
$(1,2,7)$	$\varepsilon_{12}^{\circ}\varepsilon_{37}^{\shortparallel} - \varepsilon_{02}^{*}\varepsilon_{17}^{\star} = 0$	4
$(1,2,4)$	$\varepsilon_{12}^{\circ}\varepsilon_{34}^{\bullet} - \varepsilon_{14}^{\bullet}\varepsilon_{25}^{\circ} = 0$	4
$(1,4,7)$	$\varepsilon_{13}^{\cdot}\varepsilon_{47}^{\bullet} - \varepsilon_{14}^{\bullet}\varepsilon_{57}^{\cdot} = 0$	2
$(1,3,4)$	$\varepsilon_{17}^{\star}\varepsilon_{34}^{\bullet} - \varepsilon_{14}^{\bullet}\varepsilon_{35}^{\star} = 0$	2
$(0,1,3)$	$\varepsilon_{13}^{\cdot}(\varepsilon_{03} - \varepsilon_{01}) = 0$	2
$(0,1,4)$	$\varepsilon_{14}^{\bullet}(\varepsilon_{05} - \varepsilon_{01}) = 0$	4
$(0,1,7)$	$\varepsilon_{17}^{\star}(\varepsilon_{07} - \varepsilon_{01}) = 0$	2
$(0,2,6)$	$\varepsilon_{26}^{\diamond}(\varepsilon_{06}^{*} - \varepsilon_{02}^{*}) = 0$	1

Table 8.3: Assumptions and number of solutions for S_{Γ_4}

$\varepsilon_{12}^{\circ}\varepsilon_{36}^{\circ} \neq 0$	11	$\varepsilon_{12}^{\circ}\varepsilon_{47}^{\bullet} \neq 0$	9	$\varepsilon_{12}^{\circ}\varepsilon_{15}^{\shortparallel} \neq 0$	1
$\varepsilon_{12}^{\circ}\varepsilon_{37}^{\shortparallel} \neq 0$	3	$\varepsilon_{12}^{\circ}\varepsilon_{57}^{\cdot} \neq 0$	6	$\varepsilon_{12}^{\circ}\varepsilon_{13}^{\cdot} \neq 0$	1
$\varepsilon_{12}^{\circ}\varepsilon_{45}^{\bullet} \neq 0$	7	$\varepsilon_{12}^{\circ}\varepsilon_{05} \neq 0$	9	$\varepsilon_{12}^{\circ}\varepsilon_{14}^{\bullet} \neq 0$	1
$\varepsilon_{12}^{\circ}\varepsilon_{26}^{\diamond} \neq 0$	5	$\varepsilon_{12}^{\circ}\varepsilon_{01} \neq 0$	5	$\varepsilon_{12}^{\circ}\varepsilon_{67}^{\circ} \neq 0$	1
$\varepsilon_{12}^{\circ}\varepsilon_{35}^{\star} \neq 0$	1	$\varepsilon_{12}^{\circ}\varepsilon_{02}^{*} \neq 0$	2	$\varepsilon_{12}^{\circ} \neq 0$	1
$\varepsilon_{12}^{\circ}\varepsilon_{17}^{\star} \neq 0$	8	$\varepsilon_{12}^{\circ}\varepsilon_{06}^{*} \neq 0$	1	$\emptyset$	82
$\varepsilon_{12}^{\circ}\varepsilon_{25}^{\circ} \neq 0$	8	$\varepsilon_{12}^{\circ}\varepsilon_{07} \neq 0$	1		

The whole system of contraction equations S_{Γ_4} consists of 37 linearly independent equations generated by action of the group G_4 from those listed in Table 8.2.

Since there are only four symmetries in G_4, the solution of the contraction system S_{Γ_4} according to the Theorem 3.9 is more complicated than in previous cases. However, it is still feasible. We have solved this system in 20 steps and obtained 163 parametric matrices. The assumptions of these steps together with the number of obtained solutions in each step are in Table 8.3.

After discussing the vanishing contraction parameters in all 163 solutions of S_{Γ_4} and eliminating equivalent solutions we finished with 977 non–equivalent contractions of Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$. From these 977 contraction matrices two are trivial and 498 depend on at least one complex parameter. The number of contraction matrices according to the number of their parameters are given in the following table:

number of parameters	0	1	2	3	4	5
number of solutions	479	288	152	49	8	1

All contraction matrices are divided into sets according to the number ν of zeros among the relevant contraction parameter. The number of the matrices in these sets are summarized in the following table:

ν	0	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Solutions	1	2	1	3	6	11	27	51	98	134	157	167	159	109	42	8	1

Since the only solutions without zeros have the form of normalization matrix (3.28), it is easy to verify that the following equations hold for any solution without zeros

$$\varepsilon_{01} = \varepsilon^{*}_{02} = \varepsilon_{03} = \varepsilon_{05} = \varepsilon^{*}_{06} = \varepsilon_{07}, \qquad \varepsilon^{\shortparallel}_{15}\varepsilon^{\diamond}_{26} = \varepsilon^{\circ}_{12}\varepsilon^{\star}_{35}, \qquad \varepsilon^{\bullet}_{14}\varepsilon^{\bullet}_{45} = \varepsilon^{\bullet}_{34}\varepsilon^{\bullet}_{47}. \tag{8.5}$$

Moreover, for any of these equations there exists at least one solution with zeros which violates it. Therefore, equations (8.5) are higher–order identities. It also appears that every solution which satisfies these identities is a generalized Inönü–Wigner contraction, i.e. it is of the form (3.34). Finally, we conclude that among all 977 solutions of S_{Γ_4}, there are 195 continuous contractions and 713 discrete ones. Remaining 69 are continuous contractions only for a special value (unit) of their parameters, otherwise they are discrete.

8.3 Identification of contracted Lie algebras

The classification of all 977 graded contractions of the Γ_4-graded Lie algebra $\mathrm{sl}(3,\mathbb{C})$ seems to be unattainable at present, mainly because of the high number of parametric solutions. Therefore, we focus only on the results (Appendix A.4) which lead to non–solvable Lie algebras. We have 28/1 such contraction matrices. Among them there is only one parametric solution depending on one nonzero complex parameter a.

The solution without zero $\varepsilon^{0,1}$ is trivial and leads to the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Exploring the decomposability of the remaining 27/1 solutions we find that 13 solutions allow the central decomposition and 12/1 algebras are decomposable into the direct sum of two non–abelian ideals. Numerical invariants divide all non–solvable parts into 11 classes. With the exception of one class (containing two algebras distinguished by their radicals), all these classes contain isomorphic Lie algebras only. The representatives of these classes are always chosen to be non–abelian parts of some of 27 investigated algebras. Having determined the solvable parts of Lie algebras we have finished the identification of all 28 non–solvable contractions of Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$.

There are 23 non–isomorphic non–solvable graded contractions of Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$. In addition to the trivial contraction $\mathcal{L}_{0,1} = \mathrm{sl}(3,\mathbb{C})$ there are 12 other contractions with the indecomposable non–abelian parts (these are tabulated in Appendix B.4) and the following 10 decomposable Lie algebras

$$\begin{aligned}
&\mathcal{L}_{9,5} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathrm{sl}(2,\mathbb{C}) \oplus 2\mathcal{A}_1, && \mathcal{L}_{16,71} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{4,1} \oplus \mathcal{A}_1,\\
&\mathcal{L}_{13,96} \cong (2\mathcal{A}_1 \lhd \mathrm{sl}(2,\mathbb{C})) \oplus \mathcal{A}_{3,1}, && \mathcal{L}_{16,95} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{5,1},\\
&\mathcal{L}_{14,121}(a) \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{5,7}^{(\frac{1}{\sqrt{a}},\frac{-1}{\sqrt{a}},-1)}, && \mathcal{L}_{15,133} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{3,4} \oplus 2\mathcal{A}_1,\\
&\mathcal{L}_{15,88} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{5,3}, && \mathcal{L}_{15,160} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{5,4},\\
&\mathcal{L}_{15,134} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{5,8}^{(-1)}, && \mathcal{L}_{17,70} \cong \mathrm{sl}(2,\mathbb{C}) \oplus \mathcal{A}_{3,1} \oplus 2\mathcal{A}_1.
\end{aligned}$$

For the solvable parts of the Lie algebras we use the notation of [57], where the first index i is the dimension of the Lie algebra $\mathcal{A}_{i,j}$. The graded contractions are denoted by $\mathcal{L}_{i,j}$, where i is the number of zeros among the relevant contraction parameters in the corresponding contraction matrix. Let us note that all decomposable graded contractions listed above appeared as discrete contractions. For the completeness we also write the isomorphisms found among the non–solvable graded contractions of Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$

$$\mathcal{L}_{10,11} \cong \mathcal{L}_{10,4}, \quad \mathcal{L}_{14,83} \cong \mathcal{L}_{14,77}, \quad \mathcal{L}_{16,141} \cong \mathcal{L}_{16,95}, \quad \mathcal{L}_{17,71} \cong \mathcal{L}_{17,70}, \quad \mathcal{L}_{18,59} \cong \mathcal{L}_{18,47}.$$

Let us note that except $\mathcal{L}_{10,11}$ all these algebras are discrete contractions.

Table 8.4 summarizes the number of non–solvable Γ_4-graded contractions of $\mathrm{sl}(3,\mathbb{C})$. The number of algebras are divided there according to the dimension of non–abelian parts and the type of contraction. Together with one trivial we have obtained 23 non–isomorphic Lie algebras. One of them depends on one nonzero parameter.

Table 8.4: The number of nontrivial graded contractions of the Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$

Dimension of non–abelian part	Non–solvable		Total
	Indec.	Dec.	
3	1		1
5	1		1
6	2	3	5
7	1	1	2
8	7	6	13
			22

8.4 Comparison of results from different gradings

We have obtained the graded contractions from all four fine gradings of $\mathrm{sl}(3,\mathbb{C})$. There always arise two common Lie algebras, namely $8\mathcal{A}_1$ and $\mathrm{sl}(3,\mathbb{C})$, as results of the trivial contractions. Since there are common coarsenings for some of these four fine grading, it is expectable that there will be also common (isomorphic) Lie algebras among the corresponding non–trivial contractions. However, not all common results are the consequence of the existence of common coarsening.

The comparison of the resulting Lie algebras is based on our classification of the results. We have compared the numerical invariants inv (4.69) and we tried to find an isomorphism for algebras with the same values of these invariants. We also computed invariant functions whenever it was necessary for the proof of nonexistence of the isomorphism. This procedure was sufficient in all cases.

Table 8.5 shows common non–solvable Lie algebras among all graded contractions of $\mathrm{sl}(3,\mathbb{C})$. The isomorphic algebras are always on the same line. Let us note that all 5 non–solvable graded contractions from the Gell–Mann case were reobtained in the case of Γ_4. Let us also recall that there are no non–trivial non–solvable graded contractions in the case of Pauli grading.

The resulting solvable and nilpotent Lie algebras are compared in Tables 8.6 and 8.7.

Note that for these types of algebras we can compare only results from Cartan, Gell–Mann and Pauli grading. Let us also note that all 9 nilpotent graded contractions from Cartan case appear also in Gell–Mann case.

Table 8.5: Non–solvable graded contractions of sl$(3, \mathbb{C})$

Cartan	Gell–Mann	Γ_4
$\mathcal{T}_{12,7}$	$\mathcal{G}_{18,8}$	$\mathcal{L}_{18,47}$
	$\mathcal{G}_{12,2}$	$\mathcal{L}_{12,51}$
	$\mathcal{G}_{9,1}$	$\mathcal{L}_{9,5}$
	$\mathcal{G}_{6,2}$	$\mathcal{L}_{6,1}$
$\mathcal{T}_{4,1}$	$\mathcal{G}_{6,1}$	$\mathcal{L}_{6,2}$

Table 8.6: Solvable non–nilpotent graded contractions of sl$(3, \mathbb{C})$

Cartan	Gell–Mann	Pauli
$\mathcal{T}_{13,3}(1)$	$\mathcal{G}_{19,2}$	
	$\mathcal{G}_{15,2}(\frac{-1+\sqrt{3}i}{8}, \frac{-1-\sqrt{3}i}{2})$	$\mathcal{P}_{18,29}(1)$
$\mathcal{T}_{12,1}(-1,-1)$		$\mathcal{P}_{18,34}$
$\mathcal{T}_{9,1}(1, \sqrt{a}, \sqrt{b}, \sqrt{b}, \sqrt{a})$	$\mathcal{G}_{15,1}(a, b)$	
$\mathcal{T}_{9,1}(1,1,1,1,1)$	$\mathcal{G}_{15,1}(1,1)$	$\mathcal{P}_{12,2}(1,1)$
$\mathcal{T}_{7,2}(1)$	$\mathcal{G}_{11,2}$	
$\mathcal{T}_{6,1}$		$\mathcal{P}_{9,1}$

Table 8.7: Nilpotent graded contractions of $\mathrm{sl}(3,\mathbb{C})$

Cartan	Gell–Mann	Pauli
$\mathcal{T}_{14,2}$	$\mathcal{G}_{20,1}$	$\mathcal{P}_{23,1}$
$\mathcal{T}_{13,8}$	$\mathcal{G}_{17,18}$	$\mathcal{P}_{22,2}$
$\mathcal{T}_{13,6}$	$\mathcal{G}_{19,3}$	$\mathcal{P}_{22,3}$
$\mathcal{T}_{12,9}$	$\mathcal{G}_{18,16}$	$\mathcal{P}_{21,16}$
$\mathcal{T}_{12,10}$	$\mathcal{G}_{16,12}$	$\mathcal{P}_{21,12}$
$\mathcal{T}_{12,11}$	$\mathcal{G}_{18,23}$	$\mathcal{P}_{18,35}$
$\mathcal{T}_{11,6}$	$\mathcal{G}_{15,9}$	$\mathcal{P}_{20,33}$
$\mathcal{T}_{9,4}$	$\mathcal{G}_{15,5}$	$\mathcal{P}_{15,5}(1,1)$
$\mathcal{T}_{12,8}$	$\mathcal{G}_{18,6}$	
	$\mathcal{G}_{19,4}$	$\mathcal{P}_{22,7}$
	$\mathcal{G}_{18,7}$	$\mathcal{P}_{21,19}$
	$\mathcal{G}_{17,13}$	$\mathcal{P}_{20,1}$
	$\mathcal{G}_{18,18}$	$\mathcal{P}_{21,20}$
	$\mathcal{G}_{17,5}$	$\mathcal{P}_{20,39}$
	$\mathcal{G}_{18,5}$	$\mathcal{P}_{21,11}$
	$\mathcal{G}_{15,6}(\frac{-1+\sqrt{3}i}{2})$	$\mathcal{P}_{18,13}$
	$\mathcal{G}_{16,2}$	$\mathcal{P}_{19,41}$

8.5 Comparison of methods

There are three different approaches to graded contractions. In the original one, introduced by J. Patera and M. de Montigny in [17], graded contractions are examined independently of the detailed structure of the contracted Lie algebra. The only necessary information is the knowledge of the grading group and of the relevant contraction parameters. Contraction equations are considered in the generic two–term form only, i.e. in the form of (3.12). Let us note that only the relevant contraction parameters appear in these contraction equations.

For this method a computer program was also written [5]. We will refer to this method as non–generic case.

The second approach used by E. Weimar-Woods in [76] considers only the so called generic case — when all contraction parameters are relevant — it completely ignores specifications of the contracted Lie algebra. Here are also considered two–term equations only. These equations are based solely on the knowledge of the grading group. The advantages of this and the previous method lie in the possibility to study graded contractions of whole classes of Lie algebras simultaneously.

In comparison to our approach, both these methods have two important disadvantages. These disadvantages follow from the independence of the structure of the contracted Lie algebra. First, they can neither use the symmetries of the grading to simplify the computation, nor to classify solutions. Secondly, they exclusively use two–term contraction equations, but these are often too restrictive, and therefore, result in less solutions then our method.

We have investigated the difference in the number of solutions of these three methods as follows. We took all our solutions and tested whether they satisfy the systems of two–term equations. It is a simple test for non–generic case. However, in the generic case where irrelevant parameters are also present, we have to substitute our solution into equations and then solve the rest of equations for the irrelevant parameters. Some of these irrelevant parameters are even not defined in our approach since they correspond to the zero grading subspaces.

The number of solutions of the system of contraction equations for $\mathrm{sl}(3,\mathbb{C})$ from three different methods are summarized in Table 8.8. The column solutions contains the number of our solutions for each grading of $\mathrm{sl}(3,\mathbb{C})$. The columns yes (no) contain the number of our solutions which solve (do not solve) the system for the given method. The column par. contains the number of parametric solutions which solve the relevant system only for some special values of parameters (usually unity).

Since there are two–term equations only in the cases of Cartan and Pauli grading , our solutions correspond to the solutions from the non–generic case. However, in the case of Gell–Mann and Γ_4 grading, where three–term equations also appear, the non–generic method gives less solutions. Having done the the classification of results, we can say that in Gell–Mann case it means the loss of 10 unique (non–isomorphic) graded contractions and one one–parametric continuum. In Γ_4 case only 2 unique graded contractions among the non–solvable graded contractions were lost.

Table 8.8 clearly shows that the consideration of Generic case leads only to a part of solutions even if only two–term equations exist. Considering also the identification of results,

Table 8.8: Number of solutions of the contraction system for $\mathrm{sl}(3, \mathbb{C})$

Grading	Solutions	Non–generic case			Generic case		
		yes	par.	no	yes	par.	no
Cartan	47	47			26	21	
Gell–Mann	89	74	1	14	55	6	28
Pauli	188	188			42	7	139
Γ_4	977	555	88	334	193	197	587

where the structure of original Lie algebras plays a significant role, it seems to be more convenient to apply our method.

Conclusion

We have described the concept of graded contractions in detail. In Chapter 3 we have extended the original concept [17] from group gradings to arbitrary gradings. We have also used individual approach which takes into account the structure of the graded Lie algebra and takes the advantage of the grading symmetries. These symmetries simplified the construction and solution of the contraction system and also enabled us to classify its solutions. The concept of higher–order identities was retaken from [76] and modified for our purpose to discriminate discrete and continuous solutions. Using Theorem 3.9 from [40] we finished the Chapter 3 with an algorithm for computing the graded contractions. The algorithm was implemented in the computer system MAPLE 8 where all computations were done as well.

This algorithm produces all nonequivalent contraction matrices. However, demanded outcome of our work was the set of all unique (non–isomorphic) graded contractions. This problem was solved in Chapter 4, where the algorithm for the identification of finite–dimensional complex Lie algebras was given. This algorithm extends the known algorithm for the decompositions and nilradical from [66] by computation Casimir operators, numerical invariants and invariant functions. The invariant functions appeared to be a very useful tool for the identification of parametric families of Lie algebras, especially for nilpotent ones where no other general method is known. We have also mentioned the other possible invariants known from literature. This identification algorithm was also implemented in MAPLE 8.

In Chapters 5,6,7 and 8 we have investigated the graded contractions for all four fine gradings of $\mathrm{sl}(3,\mathbb{C})$. We have always described the grading, its symmetry group, the construction and the solution of the contraction system and the identification of resulting graded contractions. Except for the case of Γ_4 we have, using our two algorithms, found and identified all possible non–isomorphic graded contractions. The nonequivalent contraction matrices as well as non–isomorphic graded contractions with indecomposable non–abelian part are tabulated in Appendices. The examples of the identification are given in Chapters 5 and 6. Our results in Chapter 5 concerning the Pauli grading of $\mathrm{sl}(3,\mathbb{C})$ were already published [II, IV, V]. The publication about graded contractions of the Gell–Mann graded Lie algebra $\mathrm{sl}(3,\mathbb{C})$, content of the Chapter 6, is in preparation.

Since the graded contractions for the Cartan graded sl$(3, \mathbb{C})$ were already computed in the literature [1, 13], Chapter 7 contains the comparison of our and previously known results. We have found six new graded contractions which were omitted in [13] and [1].

Chapter 8 contains partial results of the graded contractions for the case of Γ_4-graded sl$(3, \mathbb{C})$. Because of the number of nonequivalent solutions being too large (977) — following from the low number of symmetries (4) — we have classified only non–solvable graded contractions. Moreover, the results from previous chapters were there compared and common graded contraction tabulated.

Since there are still two approaches to the concept of graded contractions [17, 76], we have also compared our method with them in Chapter 8. It appeared that our method, which takes into account the structure of sl$(3, \mathbb{C})$, produces more solutions than the other methods. Moreover, we could simplify the solution of the contraction system and the classification of the contracted Lie algebras (only nonequivalent contraction matrices are considered).

The graded contractions were originally introduced as an algebraical method for computing classical continuous contractions which are usually considered in physical applications. Therefore, we have also determined which graded contractions are continuous and which are discrete. Let us note that all continuous graded contractions of sl$(3, \mathbb{C})$ are generalized Inönü–Wigner contractions.

Unfortunately, the general solution of the graded contractions for the fixed grading (for example Pauli grading) of sl$(n, \mathbb{C})$, similarly as for Lie algebra so(n) in [39], seems to be unreachable, since even in Pauli's case for $n = 5$ three–term equations appear. The identification of the potential results is also almost impossible, the dimension being too high. However, the part of the contraction matrices can be obtained from those computed generally for generic case [76].

The possible application of our results is in computing representations of the resulting Lie algebras. Following the concept of the graded contractions of the representations [54], where the know-ledge of graded contraction is essential, one can contract the representations of sl$(3, \mathbb{C})$ to the representation of its graded contraction.

Appendix A: Contraction matrices

Appendix A contains the complete lists of non–equivalent solutions of contraction system for the Pauli, Gell–Mann and Cartan grading and the list of non–equivalent solutions corresponding to non–solvable graded contractions of Γ_4-graded Lie algebra $\mathrm{sl}(3,\mathbb{C})$. Except the last one, these lists are divided according to the number of zeros among the relevant contraction parameters. The solution $\varepsilon^{i,j}$ refers to the j–th solution in the relevant list of solutions with i zeros.

If it is not specified, parameters a, b, c, d, f in contraction matrices are arbitrary nonzero complex numbers. The subscript C or D denotes continuous or discrete solution, respectively. Zeros in contraction matrices are shown as dots.

A.1 Contraction matrices for Pauli grading

- Trivial solutions $\varepsilon^{0,1}$ and $\varepsilon^{24,1}$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

- Solution with 9 zeros $\varepsilon^{9,1}$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C$$

- Solutions with 12 zeros $\varepsilon^{12,1}, \varepsilon^{12,2}$

$$\begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & b & b & 1 \\ a & 1 & \cdot & \cdot & a & 1 & b & a \\ \cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & a & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_*$$

* $\mathrm{Re}\, b > 0 \vee (\mathrm{Re}\, b = 0 \wedge \mathrm{Im}\, b > 0)$; continuous for $a = 1, b = 1$, otherwise discrete.

- Solutions with 15 zeros $\varepsilon^{15,1},\dots,\varepsilon^{15,7}$

$$\begin{pmatrix} \cdot&\cdot&1&1&1&1&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&1&\cdot&\cdot&\cdot&1\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&1&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&1&1&\cdot&\cdot&1&1\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot&\cdot&a&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ a&1&\cdot&\cdot&b&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&b&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_{**} \begin{pmatrix} \cdot&\cdot&1&a&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&1&1&\cdot\\ a&1&\cdot&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&a&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ a&1&\cdot&\cdot&1&1&1&1\\ \cdot&1&\cdot&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

** Continuous for $a = 1, b = 1$, otherwise discrete.

- Solutions with 16 zeros $\varepsilon^{16,1},\dots,\varepsilon^{16,7}$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&a\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&a&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_\dagger \begin{pmatrix} \cdot&\cdot&a&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ a&1&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&a&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ a&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&1\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&1\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&1&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&1&1&\cdot&1\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

† Continuous for $a = 1$, otherwise discrete.

- Solutions with 17 zeros $\varepsilon^{17,1},\dots,\varepsilon^{17,17}$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&1&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&a&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ a&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_\ddagger \begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&a&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ a&1&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

‡ Continuous for $a = 1$, otherwise discrete.

• Solutions with 18 zeros $\varepsilon^{18,1},\ldots,\varepsilon^{18,36}$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&1&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&1&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&a&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&1&\cdot\\ a&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_*$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&1&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&a&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&a&\cdot&\cdot&1&1&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_* \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&a&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&a&\cdot&\cdot&1&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_*$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&a&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&1&1&\cdot\\ a&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_* \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

* Continuous for $a = 1$, otherwise discrete.

• Solutions with 19 zeros $\varepsilon^{19,1},\ldots,\varepsilon^{19,45}$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&1&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&1&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&1 \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&1&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&1 \\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&1&1&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

- Solutions with 20 zeros $\varepsilon^{20,1},\ldots,\varepsilon^{20,42}$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&1&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C \begin{pmatrix} \cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D \begin{pmatrix} \cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_D$$

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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D
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- Solutions with 21 zeros $\varepsilon^{21,1}, \ldots, \varepsilon^{21,21}$

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\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
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\begin{pmatrix} \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C
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\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
$$

$$
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
$$

- Solutions with 22 zeros $\varepsilon^{22,1}, \dots, \varepsilon^{22,7}$

$$
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
$$

$$
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
$$

- Solution with 23 zeros $\varepsilon^{23,1}$

$$
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
$$

A.2 Contraction matrices for Gell-Mann grading

- Trivial solutions $\varepsilon^{0,1}, \varepsilon^{21,1}$

$$
\begin{pmatrix} \cdot&1&1&1&1&1&1\\ 1&\cdot&1&1&1&1&1\\ 1&1&\cdot&1&1&1&1\\ 1&1&1&\cdot&1&1&1\\ 1&1&1&1&\cdot&1&1\\ 1&1&1&1&1&\cdot&1\\ 1&1&1&1&1&1&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}_C
$$

- Solutions with 6 zeros $\varepsilon^{6,1}, \varepsilon^{6,2}$

$$
\begin{pmatrix} \cdot&1&1&1&1&1&1\\ 1&\cdot&1&1&1&1&1\\ 1&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&1&1&\cdot&1&1\\ 1&1&\cdot&\cdot&1&\cdot&\cdot\\ 1&1&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}_C
\begin{pmatrix} \cdot&\cdot&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&1&1&\cdot&\cdot\\ 1&1&\cdot&1&1&1&1\\ 1&1&1&\cdot&1&1&1\\ 1&1&1&1&\cdot&1&1\\ \cdot&\cdot&1&1&1&\cdot&\cdot\\ \cdot&\cdot&1&1&1&\cdot&\cdot \end{pmatrix}_C
$$

- Solution with 9 zeros $\varepsilon^{9,1}$

$$
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&-1&1&\cdot&-1&1\\ \cdot&-1&\cdot&1&1&-1&\cdot\\ \cdot&1&1&\cdot&1&\cdot&1\\ \cdot&\cdot&1&1&\cdot&1&1\\ \cdot&-1&-1&\cdot&1&\cdot&1\\ \cdot&1&\cdot&1&1&1&\cdot \end{pmatrix}_D
$$

• Solutions with 11 zeros $\varepsilon^{11,1}, \varepsilon^{11,2}, \varepsilon^{11,3}$

$$\begin{pmatrix} \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

• Solutions with 12 zeros $\varepsilon^{12,1}, \varepsilon^{12,2}$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & 1 & -1 & 1 \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & \cdot & 1 & 1 \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & 1 & -1 & 1 & \cdot \\ \cdot & -1 & 1 & \cdot & -1 & \cdot & 1 \\ \cdot & \cdot & -1 & -1 & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}_D$$

• Solution with 13 zeros $\varepsilon^{13,1}$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & -1 & 1 \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}_D$$

• Solutions with 14 zeros $\varepsilon^{14,1}, \varepsilon^{14,2}$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & 1 & -1 & 1 \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & 1 & 1 & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

• Solutions with 15 zeros $\varepsilon^{15,1}, \ldots, \varepsilon^{15,9}$

$$\begin{pmatrix} \cdot & a & b & 1 & 1 & 1 & 1 \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_* \begin{pmatrix} \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & b & 1 & 1 & 1 & 1 \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_* \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot & \cdot & \frac{1}{2}(a{+}1) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & 1 & 1 & \cdot & \cdot \\ \frac{1}{2}(a{+}1) & a & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_\dagger \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & -1 & 1 \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

* Continuous for $a = b = 1$, otherwise discrete. † $a \neq 0, -1$, continuous for $a = 1$, otherwise discrete.

• Solutions with 16 zeros $\varepsilon^{16,1}, \ldots, \varepsilon^{16,12}$

$$\begin{pmatrix} \cdot & a & 1 & 1 & 1 & 1 & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & 1 & 1 & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & a & 1 & \cdot & 1 & 1 \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & 1 & 1 & 1 & 1 \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \frac{1}{2} & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&-1&1&1&\cdot&\cdot\\\cdot&-1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&1&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&1&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

- Solutions with 17 zeros $\varepsilon^{17,1},\ldots,\varepsilon^{17,18}$

$$\begin{pmatrix}\cdot&1&1&1&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&a&\cdot&1&1&1&\cdot\\a&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&1&1&1&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&1&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&1&1&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&1&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D$$

$$\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&1&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&a&\cdot&\cdot&\cdot&\cdot&\cdot\\a&\cdot&1&\cdot&1&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&1&1&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&1&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&a&1&\cdot&1&1\\\cdot&a&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&\frac{1}{2}&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\\frac{1}{2}&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&1&\cdot&1\\\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&1&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&-1&1&\cdot&\cdot&\cdot\\\cdot&-1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

- Solutions with 18 zeros $\varepsilon^{18,1},\ldots,\varepsilon^{18,23}$

$$\begin{pmatrix}\cdot&1&1&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&1&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&1&1&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&1&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&1&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

$$\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

- Solutions with 19 zeros $\varepsilon^{19,1},\ldots,\varepsilon^{19,11}$

$$\begin{pmatrix}\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&\cdot&\cdot&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D$$

$$\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_D \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C \begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}_C$$

- Solutions with 20 zeros $\varepsilon^{20,1}, \varepsilon^{20,2}, \varepsilon^{20,3}$

$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

A.3 Contraction matrices for Cartan grading

- Trivial solutions $\varepsilon^{0,1}, \varepsilon^{15,1}$

$$\begin{pmatrix} \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

- Solution with 4 zeros $\varepsilon^{4,1}$

$$\begin{pmatrix} \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}_C$$

- Solution with 6 zeros $\varepsilon^{6,1}$

$$\begin{pmatrix} \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

- Solutions with 7 zeros $\varepsilon^{7,1}, \varepsilon^{7,2}$

$$\begin{pmatrix} \cdot & 1 & 1 & 1 & 1 & 1 & a \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_* \begin{pmatrix} \cdot & a & 1 & a & 1 & a & 1 \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_*$$

* Continuous for $a = 1$, otherwise discrete.

- Solutions with 8 zeros $\varepsilon^{8,1}, \varepsilon^{8,2}, \varepsilon^{8,3}$

$$\begin{pmatrix} \cdot & 1 & 1 & 1 & b & c & a \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_\dagger \begin{pmatrix} \cdot & 1 & 1 & 1 & 1 & 1 & \cdot \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

† Continuous for $a = b = c = 1$, otherwise discrete.

- Solutions with 9 zeros $\varepsilon^{9,1}, \dots, \varepsilon^{9,4}$

$$\begin{pmatrix} \cdot & a & b & c & d & f & 1 \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_\ddagger \begin{pmatrix} \cdot & 1 & 1 & 1 & b & a & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & b & a \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

‡ Continuous for $a = b = c = d = f = 1$, otherwise discrete.

- Solutions with 10 zeros $\varepsilon^{10,1},\dots,\varepsilon^{10,4}$

$$\begin{pmatrix} \cdot & a & b & c & d & 1 & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & 1 & 1 & a & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & a & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

- Solutions with 11 zeros $\varepsilon^{11,1},\dots,\varepsilon^{11,6}$

$$\begin{pmatrix} \cdot & a & b & c & 1 & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & a & b & c & \cdot & 1 & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & a & b & c & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & a & b & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

- Solutions with 12 zeros $\varepsilon^{12,1},\dots,\varepsilon^{12,12}$

$$\begin{pmatrix} \cdot & a & b & 1 & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & a & b & \cdot & 1 & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & a & \cdot & b & 1 & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & a & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & a & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

- Solutions with 13 zeros $\varepsilon^{13,1},\dots,\varepsilon^{13,9}$

$$\begin{pmatrix} \cdot & a & 1 & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & a & \cdot & 1 & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & a & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

- Solutions with 14 zeros $\varepsilon^{14,1},\varepsilon^{14,2},\varepsilon^{14,3}$

$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_C$$

A.4 Contraction matrices for Γ_4 grading

- Trivial solution

$$\varepsilon^{0,1} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot \end{pmatrix}_C$$

• Nontrivial solutions

$$\varepsilon^{6,1} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \end{pmatrix}_C \qquad \varepsilon^{6,2} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \end{pmatrix}_C \qquad \varepsilon^{7,1} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}_C$$

$$\varepsilon^{8,2} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}_C \qquad \varepsilon^{8,3} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \end{pmatrix}_C \qquad \varepsilon^{9,5} = \begin{pmatrix} \cdot & 3 & \cdot & 3 & \cdot & 3 & \cdot & 3 \\ 3 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ 3 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \end{pmatrix}_D$$

$$\varepsilon^{9,6} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}_C \qquad \varepsilon^{10,4} = \begin{pmatrix} \cdot & -2 & 1 & 4 & \cdot & 4 & 1 & -2 \\ -2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -2 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{10,11} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}_C$$

$$\varepsilon^{12,50} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{12,51} = \begin{pmatrix} \cdot & 3 & \cdot & 3 & \cdot & 3 & \cdot & 3 \\ 3 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 3 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{13,55} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\varepsilon^{13,96} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{14,77} = \begin{pmatrix} \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{14,83} = \begin{pmatrix} \cdot & -2 & \cdot & 4 & \cdot & 4 & 1 & \cdot \\ -2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\varepsilon^{14,121} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & a & \cdot & 1 & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{15,88} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{15,134} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\varepsilon^{16,71} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{16,95} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{16,133} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\varepsilon^{16,141} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{16,160} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{17,70} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

$$\varepsilon^{17,71} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{18,47} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D \qquad \varepsilon^{18,59} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}_D$$

Appendix B: Graded contractions of $\mathrm{sl}(3,\mathbb{C})$

The lists of all contracted Lie algebras of the Pauli, Gell–Mann and Cartan graded $\mathrm{sl}(3,\mathbb{C})$ as well as the list of all non–solvable contracted Lie algebras of Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$ are presented. Only indecomposable non–abelian parts of contracted Lie algebras are tabulated. The structure of decomposable Lie algebras is found in previous chapters. Algebras are divided into classes according to the dimensions of the derived series (DS), the lower central series (CS) and the upper central series (US). For each of the listed Lie algebras we give its nonzero commutation relations, dimensions of algebras of generalized derivations $\dim_{(\alpha,\beta,\gamma)}$, number of formal invariants τ and the type of contraction (C–continuous, D–discrete). The Levi decomposition is given in the last column for non–solvable Lie algebras. For the non–solvable and the solvable non–nilpotent Lie algebras the nilradical is added. For the nilpotent Lie algebras Casimir operators are presented.

Parametric Lie algebras are written in general form on the first lines (ended by restriction on parameters) in tables. Solely the values of parameters for which some characteristics are different are presented on the following lines. Casimir operators are rewritten only if their order depends on the value of parameters. For specification of the ranges of parameters for one–parametric contractions, the following notations are used

$$\begin{aligned}
\mathbb{C}_0 &= \mathbb{C}\setminus\{0\},\\
\mathbb{C}_1 &= \{z\in\mathbb{C}\,|\,|z|<1\}\cup\{z\in\mathbb{C}\,|\,|z|=1\wedge \mathrm{Im}(z)\geq 0\}\,,\\
\mathbb{C}_{10} &= \{z\in\mathbb{C}\,|\,0<|z|<1\}\cup\{z\in\mathbb{C}\,|\,|z|=1\wedge \mathrm{Im}(z)\geq 0\}\,,\\
\mathbb{C}_{20} &= \left\{z\in\mathbb{C}\,|\,0<|z+1|<1\wedge \mathrm{Re}(z)\geq -\tfrac{1}{2}\right\}\cup\\
&\qquad \left\{z\in\mathbb{C}\,|\,|z+1|=1\wedge \mathrm{Re}(z)\geq -\tfrac{1}{2}\wedge \mathrm{Im}(z)>0\right\}.
\end{aligned}$$

We use the superscript $*$ for any of the listed sets if there are no isomorphisms among Lie algebras corresponding to different parameters in the given set.

For low–dimensional Lie algebras the alternative name (AN) is assigned according to the list of algebras from [57]. This name is also used for nilradicals and Levi decompositions.

B.1: Nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T	AN
(310)(310)(13)	$\mathcal{P}'_{23,1}$	$[e_2,e_3]=e_1$	$[6,6,3,5,3,4]$	1	e_1	C	$\mathcal{A}_{3,1}$
(420)(4210)(124)	$\mathcal{P}'_{22,1}$	$[e_2,e_4]=e_1,\ [e_3,e_4]=e_2$	$[7,7,3,4,2,5]$	2	$e_1,\ e_2^2-2e_1e_3$	C	$\mathcal{A}_{4,1}$
(510)(510)(15)	$\mathcal{P}'_{22,7}$	$[e_2,e_4]=e_1,\ [e_3,e_5]=e_1$	$[15,15,5,14,10,11]$	1	e_1	C	$\mathcal{A}_{5,4}$
(520)(520)(25)	$\mathcal{P}'_{22,3}$	$[e_3,e_5]=e_1,\ [e_4,e_5]=e_2$	$[13,13,7,9,7,11]$	3	$e_1,\ e_2,\ e_2e_3-e_1e_4$	C	$\mathcal{A}_{5,1}$
(520)(5210)(135)	$\mathcal{P}'_{21,7}$	$[e_2,e_4]=e_1,\ [e_3,e_5]=e_1,\ [e_4,e_5]=e_2$	$[10,11,5,7,4,7]$	1	e_1	C	$\mathcal{A}_{5,5}$
(530)(5320)(235)	$\mathcal{P}'_{21,2}$	$[e_3,e_4]=e_1,\ [e_3,e_5]=e_2,\ [e_4,e_5]=e_3$	$[10,10,5,4,4,11]$	3	$e_1,\ e_2,\ e_3^2+2e_1e_5-2e_2e_4$	C	$\mathcal{A}_{5,3}$
(620)(620)(26)	$\mathcal{P}'_{21,19}$	$[e_3,e_5]=e_1,\ [e_4,e_6]=e_1,\ [e_5,e_6]=e_2$	$[17,18,10,14,10,14]$	2	$e_1,\ e_2$	C	$\mathcal{A}_{6,4}$
(620)(6210)(146)	$\mathcal{P}'_{21,21}$	$[e_2,e_6]=e_1,\ [e_3,e_5]=e_1,\ [e_4,e_6]=e_2$	$[14,14,5,11,7,10]$	2	$e_1,\ e_2^2-2e_1e_4$	C	$\mathcal{A}_{6,12}$
(630)(630)(36)	$\mathcal{P}'_{21,16}$	$[e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3$	$[18,18,10,9,10,19]$	4	$e_1,\ e_2,\ e_3,\ e_1e_6-e_2e_5+e_3e_4$	C	$\mathcal{A}_{6,3}$
(630)(6310)(136)	$\mathcal{P}'_{20,1}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_5]=e_3,\ [e_4,e_6]=e_2$	$[11,10,4,6,4,8]$	2	$e_1,\ e_2e_3-e_1e_4$	C	$A_{6,14}^{(1)}$
	$\mathcal{P}'_{20,2}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_6]=e_3,\ [e_5,e_6]=e_2$	$[12,12,5,7,4,8]$	2	$e_1,\ e_3^2-2e_1e_4$	C	$\mathcal{A}_{6,13}$
(630)(6310)(246)	$\mathcal{P}'_{21,1}$	$[e_3,e_6]=e_1,\ [e_4,e_5]=e_2,\ [e_4,e_6]=e_3$	$[13,15,7,8,6,13]$	2	$e_1,\ e_2$	C	$\mathcal{A}_{6,7}$
	$\mathcal{P}'_{21,3}$	$[e_3,e_6]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3$	$[15,16,7,9,6,13]$	4	$e_1,\ e_2,\ e_3^2-2e_1e_5,\ e_2e_3-e_1e_4$	C	$\mathcal{A}_{6,1}$
(630)(6320)(246)	$\mathcal{P}'_{20,8}$	$[e_3,e_6]=e_1,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_4$	$[13,15,7,8,6,13]$	2	$e_1,\ e_2$	C	$\mathcal{A}_{6,9}$
(640)(64310)(1346)	$\mathcal{P}'_{19,2}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_5]=e_3,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_4$	$[10,9,4,5,3,8]$	2	$e_1,\ e_1e_4-e_2e_3$	C	$A_{6,18}^{(1)}$
(710)(710)(17)	$\mathcal{P}'_{21,20}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_7]=e_1$	$[28,28,7,27,21,22]$	1	e_1	C	
(720)(720)(27)	$\mathcal{P}'_{20,39}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_1$	$[19,19,11,15,11,15]$	3	$e_1,\ e_2,\ e_1^2e_4-e_1e_2e_5+e_2^2e_3$	C	
	$\mathcal{P}'_{21,11}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2$	$[21,22,11,18,14,18]$	3	$e_1,\ e_2,\ e_1e_5-e_2e_4$	C	
(720)(7210)(157)	$\mathcal{P}'_{20,40}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2$	$[19,20,7,16,11,14]$	1	e_1	C	
(730)(730)(37)	$\mathcal{P}'_{21,12}$	$[e_4,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	$[20,24,13,15,13,22]$	3	$e_1,\ e_2,\ e_3$	C	
(730)(7310)(137)	$\mathcal{P}'_{19,44}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_6]=e_3,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2$	$[13,13,5,9,5,9]$	1	e_1	D	
	$\mathcal{P}'_{20,42}$	$[e_2,e_6]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_7]=e_3$	$[14,14,5,9,5,10]$	3	$e_1,\ e_2^2-2e_1e_4,\ e_3^2-2e_1e_5$	D	
	$\mathcal{P}'_{19,1}$	$[e_2,e_6]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_2$	$[15,14,5,9,5,10]$	3	$e_1,\ e_3^2-2e_1e_4,\ e_1e_5-e_2e_3$	C	
(730)(7310)(147)	$\mathcal{P}'_{19,7}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_4$	$[16,16,7,11,7,11]$	1	e_1	C	
(730)(7310)(257)	$\mathcal{P}'_{20,19}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_1$	$[15,18,9,11,8,15]$	3	$e_1,\ e_2,\ 2e_1^2e_4-2e_1e_2e_5+e_2e_3^2$	D	
	$\mathcal{P}'_{20,5}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_2$	$[17,18,9,11,9,16]$	3	$e_1,\ e_2,\ e_3^2+2e_2e_4-2e_1e_5$	C	
	$\mathcal{P}'_{20,41}$	$[e_3,e_7]=e_1,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2$	$[18,20,10,13,9,16]$	3	$e_1,\ e_2,\ e_3^2-2e_1e_4$	C	
	$\mathcal{P}'_{20,14}$	$[e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_1,\ [e_6,e_7]=e_3$	$[19,20,10,13,9,16]$	3	$e_1,\ e_2,\ e_1e_4-e_2e_3$	C	

B.1: Nilpotent graded contractions for Pauli graded sl(3, ℂ)

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T
(730)(7320)(257)	$\mathcal{P}'_{19,19}$	$[e_3,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_4$	$[14,15,9,9,8,15]$	3	$e_1,e_2,e_1e_4^2+2(e_2^2e_3+e_1e_2e_7-e_1^2e_6)$	D
	$\mathcal{P}'_{20,9}$	$[e_3,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_1,\ [e_6,e_7]=e_4$	$[16,17,9,11,9,16]$	3	$e_1,\ e_2,\ e_4^2+2(e_2e_7-e_1e_6)$	D
	$\mathcal{P}'_{19,17}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_1,\ [e_6,e_7]=e_5$	$[16,18,9,12,8,15]$	3	$e_1,\ e_2,\ e_1^2e_4-e_1e_2e_5+e_2^2e_3$	C
(740)(7410)(147)	$\mathcal{P}'_{18,13}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_4,$ $[e_6,e_7]=e_3$	$[15,13,7,9,6,11]$	1	e_1	C
(740)(7410)(247)	$\mathcal{P}'_{19,8}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_4$	$[16,17,8,9,7,16]$	3	$e_1,\ e_2,\ e_3^2+2(e_2e_4-e_1e_5)$	C
(740)(7410)(357)	$\mathcal{P}'_{20,6}$	$[e_4,e_7]=e_1,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_3$	$[18,22,10,10,9,22]$	3	$e_1,\ e_2,\ e_3$	C
(740)(7420)(247)	$\mathcal{P}'_{18,9}$	$[e_3,e_5]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_3,$ $[e_6,e_7]=e_4$	$[11,14,7,6,6,15]$	3	$e_1,\ e_2,$ $e_1e_4^2-2(e_1^2e_6-e_1e_2e_7+e_2^2e_5)+e_2e_3^2$	D
	$\mathcal{P}'_{19,9}$	$[e_3,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_4$	$[12,15,7,7,6,15]$	3	$e_1,e_2,e_1e_4^2-2(e_1^2e_6-e_1e_2e_7)+e_2e_3^2$	D
	$\mathcal{P}'_{20,7}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_4$	$[13,17,7,8,6,15]$	3	$e_1,\ e_2,\ e_1e_4^2-2e_1e_2e_5+e_2e_3^2$	D
	$\mathcal{P}'_{19,4}$	$[e_3,e_5]=e_1,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_3$	$[15,15,7,7,6,15]$	3	$e_1,\ e_2,\ e_1e_7-e_2e_6+e_3e_4$	C
	$\mathcal{P}'_{19,5}$	$[e_3,e_7]=e_1,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_5$	$[16,18,7,9,6,15]$	3	$e_1,\ e_2,\ e_3^2-2e_1e_4$	C
	$\mathcal{P}'_{20,13}$	$[e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_4$	$[19,19,7,10,6,15]$	5	$e_1,\ e_2,\ e_3^2-2e_1e_5,\ e_4^2-2e_2e_6,$ $e_2e_3-e_1e_4$	C
(740)(7420)(357)	$\mathcal{P}'_{20,3}$	$[e_4,e_7]=e_1,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_5$	$[17,22,10,10,9,22]$	3	$e_1,\ e_2,\ e_3$	C
(740)(74310)(1357)	$\mathcal{P}'_{18,1}$	$[e_2,e_6]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_2,$ $[e_6,e_7]=e_5$	$[13,13,5,8,4,9]$	3	$e_1,\ e_3^2-2e_1e_4,\ e_2e_3-e_1e_5$	C
(7510)(754210)(12457)	$\mathcal{P}'_{17,1}$	$[e_2,e_7]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3,$ $[e_5,e_7]=e_4,\ [e_6,e_7]=e_5$	$[12,10,4,4,2,9]$	3	$e_1,\ e_2^2-2e_1e_3,\ e_1e_6-e_2e_5+e_3e_4$	C
(820)(820)(28)	$\mathcal{P}_{18,35}$	$[e_3,e_6]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_1,$ $[e_5,e_8]=e_2$	$[22,25,13,21,15,19]$	2	$e_1,\ e_2$	C
	$\mathcal{P}_{19,42}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_8]=e_2$	$[24,25,13,21,15,19]$	2	$e_1,\ e_2$	C
	$\mathcal{P}_{20,32}$	$[e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_8]=e_2,\ [e_6,e_8]=e_1$	$[26,26,13,22,18,22]$	4	$e_1,\ e_2,\ e_1e_4-e_2e_3,\ e_1e_5-e_2e_6$	C
	$\mathcal{P}_{20,38}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_8]=e_1,\ [e_7,e_8]=e_2$	$[28,29,14,25,19,23]$	2	$e_1,\ e_2$	C
(830)(830)(38)	$\mathcal{P}_{19,41}$	$[e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_1,\ [e_6,e_8]=e_3$	$[20,24,16,15,16,25]$	4	$e_1,\ e_2,\ e_3,$ $e_1(e_1e_7-e_2e_6-e_3e_8)+e_2e_3e_5-e_3^2e_4$	C
	$\mathcal{P}_{20,35}$	$[e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_8]=e_3$	$[23,28,16,19,16,25]$	4	$e_1,\ e_2,\ e_3,\ e_1^2e_6-e_1e_3e_5+e_2e_3e_4$	C
	$\mathcal{P}_{20,31}$	$[e_4,e_7]=e_1,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_2,\ [e_6,e_8]=e_3$	$[26,28,16,19,17,26]$	4	$e_1,\ e_2,\ e_3,\ e_1e_6-e_2e_4-e_3e_5$	C

B.1: Nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T
(830)(8310)(138)	$\mathcal{P}_{17,5}$	$[e_2,e_8]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_3$	[16,15,6,12,6,10]	2	$e_1,\ e_1e_5-e_2e_3$	D
	$\mathcal{P}_{18,3}$	$[e_2,e_7]=e_1,\ [e_3,e_8]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_2$	$[19,16,6,12,6,10]$	4	$e_1,\ e_1e_6-e_2e_3,\ e_2^2-2e_1e_4,\ e_3^2-2e_1e_5$	D
(830)(8310)(148)	$\mathcal{P}_{18,2}$	$[e_2,e_8]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_3,\ [e_6,e_8]=e_2$	$[18,17,6,13,8,12]$	2	$e_1,\ e_1e_5-e_2e_3$	C
(830)(8310)(158)	$\mathcal{P}_{19,3}$	$[e_2,e_8]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_3$	$[20,19,6,15,11,15]$	2	$e_1,\ e_1e_5-e_2e_3$	C
(830)(8310)(268)	$\mathcal{P}_{18,24}$	$[e_3,e_8]=e_1,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_6]=e_1,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_2$	$[19,20,11,14,10,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{19,24}$	$[e_3,e_8]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_1,\ [e_6,e_8]=e_2$	$[20,22,11,15,10,17]$	4	$e_1,\ e_2,\ 2e_1e_5-e_3^2,\ e_1^2e_4-e_1e_2e_6+e_2^2e_3$	D
	$\mathcal{P}_{19,23}$	$[e_3,e_8]=e_1,\ [e_4,e_6]=e_1,\ [e_5,e_7]=e_1,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_2$	$[20,22,11,15,11,18]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{19,31}$	$[e_3,e_8]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_1,\ [e_6,e_8]=e_3$	$[21,23,11,16,12,19]$	4	$e_1,\ e_2,\ e_1e_4-e_2e_3,\ 2(e_1^2e_5-e_1e_2e_6)+e_2e_3^2$	D
	$\mathcal{P}_{20,11}$	$[e_3,e_7]=e_1,\ [e_4,e_7]=e_3,\ [e_5,e_8]=e_1,\ [e_6,e_8]=e_2$	$[21,23,11,16,12,19]$	4	$e_1,\ e_2,\ e_1e_6-e_2e_5,\ 2e_1e_4-e_3^2$	D
	$\mathcal{P}_{20,17}$	$[e_3,e_8]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_8]=e_3,\ [e_6,e_8]=e_2$	$[23,25,11,18,13,20]$	4	$e_1,\ e_2,\ e_1e_6-e_2e_3,\ 2e_1e_5-e_3^2$	D
	$\mathcal{P}_{19,43}$	$[e_3,e_8]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_6]=e_1,\ [e_6,e_8]=e_3,\ [e_7,e_8]=e_2$	$[24,25,13,18,13,20]$	2	$e_1,\ e_2$	C
(830)(8320)(268)	$\mathcal{P}_{17,14}$	$[e_3,e_7]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_2,\ [e_5,e_8]=e_1,\ [e_7,e_8]=e_3$	$[16,19,11,13,10,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{18,20}$	$[e_3,e_6]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_1,\ [e_6,e_8]=e_3$	$[18,19,11,13,10,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{19,21}$	$[e_3,e_6]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_8]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_3$	$[20,20,11,14,12,19]$	4	$e_1,\ e_2,\ e_1e_4-e_2e_5,\ 2(e_1e_7-e_2e_6)+e_3^2$	D
	$\mathcal{P}_{18,18}$	$[e_3,e_7]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_6]=e_2,\ [e_5,e_7]=e_2,\ [e_6,e_8]=e_1,\ [e_7,e_8]=e_3$	$[20,22,11,15,11,18]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{19,18}$	$[e_3,e_7]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_6]=e_1,\ [e_5,e_8]=e_1,\ [e_7,e_8]=e_3$	$[22,24,11,17,13,20]$	2	$e_1,\ e_2$	D
(840)(840)(48)	$\mathcal{P}_{20,33}$	$[e_5,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4$	$[24,33,17,17,17,33]$	4	$e_1,\ e_2,\ e_3,\ e_4$	C
(840)(8410)(248)	$\mathcal{P}_{18,36}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_1,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_3,\ [e_6,e_8]=e_4,\ [e_7,e_8]=e_2$	$[18,22,10,13,9,18]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{19,45}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_3,\ [e_6,e_8]=e_4,\ [e_7,e_8]=e_2$	$[19,22,10,13,9,18]$	4	$e_1,\ e_2,\ 2e_1e_5-e_3^2,\ 2e_1e_6-e_4^2$	D

B.1: Nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T
(840)(8410)(258)	$\mathcal{P}_{18,10}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_1,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_8]=e_4,$ $[e_7,e_8]=e_3$	$[19,22,10,13,9,18]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{19,6}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_8]=e_4,\ [e_7,e_8]=e_3$	$[20,22,10,13,9,18]$	4	$e_1,\ e_2,\ e_1e_5-e_2e_3,\ 2e_1e_6-e_4^2$	D
(840)(8410)(368)	$\mathcal{P}_{19,25}$	$[e_4,e_8]=e_1,\ [e_5,e_7]=e_1,\ [e_5,e_8]=e_4,\ [e_6,e_7]=e_2,\ [e_6,e_8]=e_3$	$[20,27,13,14,12,25]$	4	$e_1,e_2,e_3,2e_1(e_1e_6-e_2e_5-e_3e_4)+e_2e_4^2$	D
	$\mathcal{P}_{19,13}$	$[e_4,e_8]=e_1,\ [e_5,e_7]=e_1,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_2,\ [e_6,e_8]=e_4$	$[21,27,13,14,12,25]$	4	$e_1,e_2,e_3,\ e_1(2e_1e_6-2e_2e_5-e_4^2)+2e_2e_3e_4$	C
	$\mathcal{P}_{20,12}$	$[e_4,e_8]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_8]=e_4,\ [e_7,e_8]=e_3$	$[22,28,13,15,12,25]$	4	$e_1,\ e_2,\ e_3,\ 2e_1e_6-e_4^2$	D
	$\mathcal{P}_{20,4}$	$[e_4,e_8]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,\ [e_7,e_8]=e_4$	$[23,28,13,15,12,25]$	4	$e_1,\ e_2,\ e_3,\ e_1e_5-e_2e_4$	C
(840)(8420)(248)	$\mathcal{P}_{16,4}$	$[e_3,e_5]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_3,$ $[e_5,e_8]=e_2,\ [e_6,e_7]=e_1,\ [e_6,e_8]=e_4$	$[12,16,9,8,8,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{17,16}$	$[e_3,e_5]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,$ $[e_6,e_8]=e_4,\ [e_7,e_8]=e_1$	$[14,16,9,8,8,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{17,15}$	$[e_3,e_7]=e_1,\ [e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_1,$ $[e_6,e_7]=e_2,\ [e_6,e_8]=e_4$	$[15,18,9,10,8,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{18,23}$	$[e_3,e_5]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_3,$ $[e_6,e_8]=e_4$	$[16,16,9,8,8,18]$	4	$e_1,\ e_2,\ 2(e_1e_6-e_2e_8)-e_4^2,$ $2(e_1e_7-e_2e_5)+e_3^2$	D
	$\mathcal{P}_{18,21}$	$[e_3,e_7]=e_1,\ [e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_1,$ $[e_6,e_8]=e_4$	$[16,18,9,10,8,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{18,22}$	$[e_3,e_8]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_1,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_4,$ $[e_7,e_8]=e_2$	$[16,18,9,10,8,17]$	4	$e_1,\ e_2,\ 2e_1e_5-e_3^2,$ $e_1(2e_1e_6-2e_2e_7-e_4^2)+2e_2^2e_3$	D
	$\mathcal{P}_{18,26}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_2,$ $[e_6,e_8]=e_4$	$[16,21,9,12,8,17]$	4	$e_1,\ e_2,\ 2e_1e_2e_6-e_1e_4^2-2e_2^2e_3,$ $2e_1(e_1e_4-e_2e_5)+e_2e_3^2$	D
	$\mathcal{P}_{17,2}$	$[e_3,e_7]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_1,$ $[e_5,e_8]=-e_3,\ [e_7,e_8]=e_6$	$[17,16,9,8,8,18]$	4	$e_1,\ e_2,\ e_1e_8-e_2e_7+e_3e_6,$ $2(e_1e_4-e_2e_5)-e_3^2$	C
	$\mathcal{P}_{19,22}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_6]=e_3,\ [e_7,e_8]=e_4$	$[17,19,9,10,8,18]$	4	$e_1,\ e_2,\ 2e_1e_5-e_3^2,\ 2(e_1e_8-e_2e_7)+e_4^2$	D
	$\mathcal{P}_{19,28}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_8]=e_4,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_1$	$[17,21,9,12,8,17]$	4	$e_1,e_2,2e_2e_5-e_4^2,2e_1(e_1e_4-e_2e_6)+e_2e_3^2$	D
(840)(8420)(258)	$\mathcal{P}_{16,3}(a)$	$[e_3,e_6]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_6]=e_2,$ $[e_5,e_7]=e_1,\ [e_6,e_8]=e_3,\ [e_7,e_8]=-ae_4,\ a\in\mathbb{C}_{10}$	$[16,19,9,11,8,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{17,9}$	$[e_3,e_6]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_1,$ $[e_6,e_8]=e_3,\ [e_7,e_8]=e_4$	$[16,19,9,11,8,17]$	2	$e_1,\ e_2$	D

B.1: Nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T
	$\mathcal{P}_{17,12}$	$[e_3,e_6]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_1,$ $[e_6,e_8]=e_3,\ [e_7,e_8]=e_4$	$[16,19,9,11,8,17]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{18,12}$	$[e_3,e_6]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_1,\ [e_6,e_8]=e_3,$ $[e_7,e_8]=e_4$	$[17,19,9,11,9,18]$	4	$e_1,\ e_2,\ e_1e_4-e_2e_5,$ $2e_1e_2e_8+e_1e_4^2-2e_2^2e_6+e_2e_3^2$	D
	$\mathcal{P}_{18,14}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_1,\ [e_6,e_8]=e_3,$ $[e_7,e_8]=e_4$	$[17,20,9,12,9,18]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{18,17}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_3,$ $[e_6,e_8]=e_4$	$[17,21,9,12,8,17]$	4	$e_1,\ e_2,\ e_1^2e_4+e_1e_2e_5-e_2^2e_3,$ $2e_1e_2e_6-e_1e_4^2-e_2e_3^2$	D
	$\mathcal{P}_{19,15}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4$	$[18,21,9,12,9,18]$	4	$e_1,e_2,\ e_1e_4-e_2e_5,\ 2e_1e_2e_6-e_1e_4^2-e_2e_3^2$	D
	$\mathcal{P}_{17,7}(a)$	$[e_3,e_8]=e_1,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,$ $[e_6,e_8]=e_3,\ [e_7,e_8]=-ae_4,\ a\in\mathbb{C}_0^*$	$[19,20,9,12,8,17]$	2	$e_1,\ e_2$	D
		$a=1$				C
	$\mathcal{P}_{18,6}$	$[e_3,e_8]=e_1,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_6,e_8]=e_3,$ $[e_7,e_8]=e_4$	$[19,21,9,13,8,17]$	4	$e_1,\ e_2,\ 2e_1e_6-e_3^2,$ $e_1^2e_5-e_1e_2e_4+e_2^2e_3$	D
	$\mathcal{P}_{18,7}$	$[e_3,e_8]=e_1,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_6]=e_1,\ [e_6,e_8]=e_3,$ $[e_7,e_8]=e_4$	$[20,22,10,13,9,18]$	2	$e_1,\ e_2$	D
	$\mathcal{P}_{18,25}(a)$	$[e_3,e_8]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_1,\ [e_5,e_8]=e_3,$ $[e_6,e_7]=-ae_2,\ [e_6,e_8]=e_4,\ a\in\mathbb{C}_1^*$	$[20,22,10,13,9,18]$	4	$e_1,\ e_2,\ e_1e_4-e_2e_3,$ $2e_1(e_1e_6+ae_2e_5-e_3e_4)+(1-a)e_2e_3^2$	D
		$a=1$			$e_1,\ e_2,\ e_1e_4-e_2e_3,\ e_1e_6+e_2e_5-e_3e_4$	C
		$a=-1$	$[22,22,10,13,9,18]$			
		$a=0$	$[21,23,10,14,9,18]$		$e_1,\ e_2,\ e_1e_4-e_2e_3,\ 2e_2e_6-e_4^2$	
(840)(8420)(368)	$\mathcal{P}_{18,19}$	$[e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_2,$ $[e_6,e_8]=e_4$	$[16,24,13,12,12,25]$	4	$e_1,\ e_2,\ e_3,\ 2e_1(e_1e_7+e_2e_4-e_3e_8)+$ $+2e_2(e_3e_6-e_2e_5)-e_3e_4^2$	D
	$\mathcal{P}_{18,16}$	$[e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_1,\ [e_5,e_8]=e_3,\ [e_6,e_7]=e_2,$ $[e_6,e_8]=e_4$	$[19,24,13,12,12,25]$	4	$e_1,\ e_2,\ e_3,\ 2e_1(e_1e_8-e_2e_6-e_3e_7)+$ $+e_1e_4^2+2e_2(e_2e_5-e_3e_4)$	D
	$\mathcal{P}_{19,20}$	$[e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_1,\ [e_6,e_8]=e_4$	$[19,25,13,13,12,25]$	4	$e_1,e_2,e_3,\ 2(e_1^2e_7-e_1e_3e_8+e_2e_3e_6)-e_3e_4^2$	D
	$\mathcal{P}_{19,11}$	$[e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_6]=e_1,\ [e_6,e_8]=e_3,\ [e_7,e_8]=e_4$	$[21,25,13,13,12,25]$	4	$e_1,\ e_2,\ e_3,\ 2(e_1e_8-e_2e_7+e_3e_5)+e_4^2$	D
	$\mathcal{P}_{19,10}$	$[e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_6,e_8]=e_3,\ [e_7,e_8]=e_4$	$[21,27,13,15,12,25]$	4	$e_1,\ e_2,\ e_3,\ e_1e_3e_5+e_2^2e_6-e_2e_3e_4$	C

B.1: Nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T
(840)(84310)(1368)	$\mathcal{P}_{16,1}(a)$	$[e_2,e_8]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_3,\ [e_5,e_6]=e_1,$	$[16,15,6,11,5,10]$	2	$e_1,\ e_1e_4-e_2e_3$	D
		$[e_5,e_8]=e_2,\ [e_6,e_7]=ae_3,\ [e_7,e_8]=e_4,$				
		$a\in\mathbb{C}_0^*$				
		$a=1$				C
	$\mathcal{P}_{17,4}$	$[e_2,e_8]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_3,\ [e_5,e_7]=e_3,$	$[16,15,6,11,5,10]$	4	$e_1,\ e_1e_4-e_2e_3,\ 2e_1e_5-e_3^2,$	D
		$[e_6,e_8]=e_2,\ [e_7,e_8]=e_4$			$2e_1e_6-e_2^2$	
(840)(84310)(1468)	$\mathcal{P}_{17,3}$	$[e_2,e_8]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_3,\ [e_5,e_6]=e_1,$	$[17,16,6,12,7,12]$	2	$e_1,\ e_1e_4-e_2e_3$	D
		$[e_6,e_8]=e_2,\ [e_7,e_8]=e_4$				
(840)(84310)(1568)	$\mathcal{P}_{18,4}$	$[e_2,e_8]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_3,\ [e_5,e_6]=e_1,$	$[19,18,6,14,10,15]$	2	$e_1,\ e_1e_4-e_2e_3$	D
		$[e_7,e_8]=e_4$				
(850)(8520)(258)	$\mathcal{P}_{15,5}(a,b)$	$[e_3,e_6]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_2,\ [e_4,e_8]=be_1,\ [e_5,e_7]=e_1,$	$[16,19,7,9,6,17]$	2	$e_1,\ e_2$	D
		$[e_5,e_8]=e_2,\ [e_6,e_7]=e_3,\ [e_6,e_8]=-ae_4,\ [e_7,e_8]=e_5,\ a\neq 0$				
		$a=-1,\ b=0$	$[17,19,7,9,6,17]$			
		$a=1,\ b=1$	$[18,19,7,9,6,17]$			C
	$\mathcal{P}_{17,8}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_3,$	$[16,20,7,10,6,17]$	4	$e_1,\ e_2,\ e_1^2e_4-e_1e_2e_5+e_2^2e_3,$	D
		$[e_6,e_8]=e_4,\ [e_7,e_8]=e_5$			$2e_1e_2e_6-e_1e_4^2-e_2e_3^2$	
	$\mathcal{P}_{17,10}$	$[e_3,e_7]=e_1,\ [e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_3,$	$[16,20,8,10,7,18]$	2	$e_1,\ e_2$	D
		$[e_6,e_8]=e_4,\ [e_7,e_8]=e_5$				
	$\mathcal{P}_{17,13}(a)$	$[e_3,e_6]=e_1,\ [e_3,e_7]=e_2,\ [e_4,e_8]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,$	$[17,19,8,9,7,18]$	4	$e_1,\ e_2,\ e_1e_5-e_2e_4,\ (1-a)e_2e_4^2$	D
		$[e_6,e_8]=-ae_4,\ [e_7,e_8]=e_5,\ a\in\mathbb{C}_{10}^*$			$+e_1e_3^2+2e_1(e_1e_7-e_2e_6-e_4e_5)$	
		$a=-1$	$[19,19,8,9,7,18]$			
	$\mathcal{P}_{18,11}$	$[e_3,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4,$	$[17,20,8,10,7,18]$	4	$e_1,\ e_2,\ e_1e_4-e_2e_5,$	D
		$[e_7,e_8]=e_5$			$2e_1e_2e_6-e_1e_4^2-e_2e_3^2$	
(850)(8520)(358)	$\mathcal{P}_{17,11}$	$[e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_1,\ [e_6,e_7]=e_3,$	$[16,24,10,9,9,25]$	4	$e_1,\ e_2,\ e_3,\ 2e_1(e_1e_7-e_2e_8+e_3e_4)$	D
		$[e_6,e_8]=e_4,\ [e_7,e_8]=e_5$			$+2e_2(e_2e_6-e_3e_5)-e_1e_5^2-e_2e_4^2$	
	$\mathcal{P}_{18,15}$	$[e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4,$	$[17,25,10,10,9,25]$	4	$e_1,\ e_2,\ e_3,\ 2e_1e_2e_8+e_1e_5^2$	D
		$[e_7,e_8]=e_5$			$-2e_2^2e_6+2e_2e_3e_5+e_2e_4^2$	
	$\mathcal{P}_{19,16}$	$[e_4,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_4,\ [e_6,e_8]=e_5,\ [e_7,e_8]=e_3$	$[18,26,10,11,9,25]$	4	$e_1,\ e_2,\ e_3,\ 2e_1e_2e_6-e_1e_5^2-e_2e_4^2$	D

B.1: Nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T
	$\mathcal{P}_{18,8}$	$[e_4,e_8]=e_1,\ [e_5,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,$ $[e_6,e_8]=e_4,\ [e_7,e_8]=e_5$	$[20,26,10,11,9,25]$	4	$e_1,\ e_2,\ e_3,$ $2e_1(e_1e_6-e_3e_5)-e_1e_4^2+2e_2e_3e_4$	C
	$\mathcal{P}_{19,14}$	$[e_4,e_8]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4,\ [e_7,e_8]=e_5$	$[22,26,10,11,9,25]$	4	$e_1,\ e_2,\ e_3,\ e_1e_5-e_2e_4$	C
(850)(8520)(468)	$\mathcal{P}_{19,12}$	$[e_5,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4,\ [e_7,e_8]=e_5$	$[22,32,13,12,12,33]$	4	$e_1,\ e_2,\ e_3,\ e_4$	C
(850)(8530)(358)	$\mathcal{P}_{18,5}$	$[e_4,e_7]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_6]=e_1,\ [e_5,e_8]=e_3,\ [e_6,e_8]=e_4,$ $[e_7,e_8]=e_5$	$[19,24,10,9,9,25]$	4	$e_1,\ e_2,\ e_3,$ $e_1e_8-e_2e_7-e_3e_6+e_4e_5$	C
(8510)(85310)(1358)	$\mathcal{P}_{17,6}$	$[e_2,e_7]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_5]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_8]=e_3,$ $[e_6,e_8]=e_4,\ [e_7,e_8]=-e_5$	$[16,11,4,6,4,10]$	2	$e_1,\ e_1e_8-e_2e_5+e_3e_4$	C
(8510)(854210)(12468)	$\mathcal{P}_{15,2}$	$[e_2,e_7]=e_1,\ [e_3,e_5]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_2,$ $[e_5,e_7]=e_3,\ [e_5,e_8]=e_4,\ [e_6,e_8]=e_3,\ [e_7,e_8]=-e_5$	$[13,10,4,5,3,9]$	2	$e_1,\ e_1e_8-e_2e_5+e_3e_4$	C
(8620)(865320)(23568)	$\mathcal{P}_{15,3}$	$[e_3,e_7]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_6]=-e_1,\ [e_4,e_8]=e_3,$ $[e_5,e_6]=e_2,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_4,\ [e_6,e_8]=e_5,\ [e_7,e_8]=e_6$	$[14,16,6,4,4,18]$	4	$e_1,\ e_2,\ 2e_1e_5+2e_2e_4-e_3^2,$ $e_1e_8-e_2e_7+e_3e_6-e_4e_5$	C

B.1: Solvable non–nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T	AN
(430)(43)(0)	$\mathcal{P}'_{21,9}$	$[e_1,e_4]=e_2,\ [e_2,e_4]=e_3,\ [e_3,e_4]=e_1$	$[6,4,1,3,0,1]$	2	$3\mathcal{A}_1$	D	$\mathcal{A}_{4,6}^{(\frac{\pm 2}{\sqrt{3}},\frac{\mp 1}{\sqrt{3}})}$
(530)(53)(0)	$\mathcal{P}'_{18,34}$	$[e_1,e_4]=e_2,\ [e_1,e_5]=e_3,\ [e_2,e_4]=e_3,\ [e_2,e_5]=e_1,\ [e_3,e_4]=e_1,\ [e_3,e_5]=e_2$	$[6,4,1,3,0,1]$	1	$3\mathcal{A}_1$	D	$\mathcal{A}_{5,33}^{(-1,-1)}$
(640)(643)(12)	$\mathcal{P}'_{20,16}$	$[e_2,e_6]=e_3,\ [e_3,e_6]=e_4,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_1$	$[10,11,3,6,2,7]$	4	$5\mathcal{A}_1$	D	
(740)(743)(12)	$\mathcal{P}'_{17,17}$	$[e_2,e_6]=e_3,\ [e_2,e_7]=e_4,\ [e_3,e_6]=e_4,\ [e_3,e_7]=e_2,$ $[e_4,e_6]=e_2,\ [e_4,e_7]=e_3,\ [e_5,e_6]=e_1$	$[11,12,4,7,3,8]$	3	$5\mathcal{A}_1$	D	
(750)(7543)(123)	$\mathcal{P}'_{19,29}$	$[e_2,e_7]=e_1,\ [e_3,e_7]=e_5,\ [e_4,e_7]=e_3,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_2$	$[12,13,3,7,2,8]$	5	$6\mathcal{A}_1$	D	
(760)(76)(0)	$\mathcal{P}'_{18,29}(a)$	$[e_1,e_7]=-ae_2,\ [e_2,e_7]=e_5,\ [e_3,e_7]=e_6,\ [e_4,e_7]=e_3,$ $[e_5,e_7]=e_1,\ [e_6,e_7]=e_4,\ a\in\mathbb{C}^*_{10}$	$[12,7,1,6,0,1]$	5	$6\mathcal{A}_1$	D	
		$a=-1$	$[18,7,1,6,0,1]$				
		$a=1$				C	

B.1: Solvable non–nilpotent graded contractions for Pauli graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T
(7630)(76)(0)	$\mathcal{P}'_{15,1}$	$[e_1,e_7]=-e_3,\ [e_2,e_7]=e_1,\ [e_3,e_7]=-e_2,\ [e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_6,$ $[e_5,e_6]=e_3,\ [e_5,e_7]=-e_4,\ [e_6,e_7]=e_5$	$[9,6,1,3,0,1]$	3	$\mathcal{A}_{6,3}$	C
(840)(843)(13)	$\mathcal{P}_{16,7}$	$[e_2,e_7]=e_1,\ [e_3,e_8]=e_1,\ [e_4,e_7]=e_5,\ [e_4,e_8]=e_6,\ [e_5,e_7]=e_6,\ [e_5,e_8]=e_4,$ $[e_6,e_7]=e_4,\ [e_6,e_8]=e_5$	$[14,15,5,10,5,10]$	4	$6\mathcal{A}_1$	D
(840)(843)(14)	$\mathcal{P}_{19,32}$	$[e_2,e_3]=e_1,\ [e_4,e_8]=e_1,\ [e_5,e_8]=e_6,\ [e_6,e_8]=e_7,\ [e_7,e_8]=e_5$	$[17,18,5,13,7,12]$	4	$\mathcal{A}_{3,1}\oplus 4\mathcal{A}_1$	D
(850)(853)(23)	$\mathcal{P}_{16,5}$	$[e_3,e_7]=e_1,\ [e_3,e_8]=e_2,\ [e_4,e_7]=e_6,\ [e_4,e_8]=e_5,\ [e_5,e_7]=e_4,\ [e_5,e_8]=e_6,$ $[e_6,e_7]=e_5,\ [e_6,e_8]=e_4$	$[13,20,7,9,6,17]$	4	$6\mathcal{A}_1$	D
(850)(853)(24)	$\mathcal{P}_{19,27}$	$[e_3,e_8]=e_5,\ [e_4,e_8]=e_3,\ [e_5,e_8]=e_4,\ [e_6,e_7]=e_1,\ [e_7,e_8]=e_2$	$[16,21,7,10,6,17]$	4	$4\mathcal{A}_1\oplus\mathcal{A}_{3,1}$	D
(850)(8543)(123)	$\mathcal{P}_{15,6}(a)$	$[e_2,e_8]=e_1,\ [e_3,e_7]=ae_1,\ [e_3,e_8]=-e_2,\ [e_4,e_7]=e_5,\ [e_4,e_8]=e_6,\ [e_5,e_7]=e_6,$ $[e_5,e_8]=e_4,\ [e_6,e_7]=e_4,\ [e_6,e_8]=e_5,\ a\in\mathbb{C}$	$[13,14,4,8,3,9]$	4	$6\mathcal{A}_1$	D
(850)(8543)(124)	$\mathcal{P}_{18,28}$	$[e_2,e_4]=e_1,\ [e_3,e_4]=e_2,\ [e_3,e_8]=e_1,\ [e_5,e_8]=e_6,\ [e_6,e_8]=e_7,\ [e_7,e_8]=e_5$	$[13,14,4,8,3,9]$	4	$\mathcal{A}_{4,1}\oplus 3\mathcal{A}_1$	D
(850)(8543)(134)	$\mathcal{P}_{18,27}$	$[e_2,e_3]=e_1,\ [e_3,e_8]=e_4,\ [e_4,e_8]=e_1,\ [e_5,e_8]=e_7,\ [e_6,e_8]=e_5,\ [e_7,e_8]=e_6$	$[15,16,5,10,4,10]$	4	$\mathcal{A}_{3,1}\oplus 4\mathcal{A}_1$	D
(860)(86)(0)	$\mathcal{P}_{12,2}(a,b)$	$[e_1,e_7]=e_3,\ [e_1,e_8]=ae_6,\ [e_2,e_7]=e_5,\ [e_2,e_8]=-e_4,\ [e_3,e_7]=e_6,\ [e_3,e_8]=ae_1,$ $[e_4,e_7]=-be_2,\ [e_4,e_8]=e_5,\ [e_5,e_7]=be_4,\ [e_5,e_8]=be_2,\ [e_6,e_7]=e_1,$ $[e_6,e_8]=ae_3,\ \mathrm{Re}(b)>0\ \vee(\mathrm{Re}(b)=0\wedge\mathrm{Im}(b)>0)$	$[12,7,1,6,0,1]$	4	$6\mathcal{A}_1$	D
		$a=b=\pm i$	$[18,7,1,6,0,1]$			
		$a=b=1$				C
(8630)(86)(0)	$\mathcal{P}_{9,1}$	$[e_1,e_7]=e_2,\ [e_1,e_8]=-e_3,\ [e_2,e_7]=e_3,\ [e_2,e_8]=e_1,\ [e_3,e_7]=-e_1,\ [e_3,e_8]=e_2,$ $[e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_5,\ [e_4,e_8]=e_6,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_6,$ $[e_5,e_8]=e_4,\ [e_6,e_7]=e_4,\ [e_6,e_8]=e_5$	$[9,3,1,3,0,1]$	2	$\mathcal{A}_{6,3}$	C
(8710)(87)(1)	$\mathcal{P}_{15,4}$	$[e_2,e_5]=e_1,\ [e_2,e_8]=e_4,\ [e_3,e_6]=e_1,\ [e_3,e_8]=e_7,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_6,$ $[e_5,e_8]=e_3,\ [e_6,e_8]=e_2,\ [e_7,e_8]=-e_5$	$[11,8,2,1,1,9]$	2	$\mathcal{P}'_{21,20}$	C
(8740)(87)(1)	$\mathcal{P}_{12,1}$	$[e_2,e_5]=e_1,\ [e_2,e_8]=e_3,\ [e_3,e_6]=e_1,\ [e_3,e_8]=-e_4,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_2,$ $[e_5,e_6]=e_2,\ [e_5,e_7]=e_4,\ [e_5,e_8]=-e_7,\ [e_6,e_7]=e_3,\ [e_6,e_8]=-e_5,\ [e_7,e_8]=e_6$	$[10,8,2,1,1,9]$	2	$\mathcal{P}'_{18,13}$	C

B.2: Non–solvable graded contractions for Gell-Mann graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T	Levi dec.
(3)(3)(0)	$\mathcal{G}'_{18,8}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=e_2,\ [e_2,e_3]=e_1$	$[3,0,1,0,0,1]$	1	$\{0\}$	D	$\mathrm{sl}(2,\mathbb{C})$
(6)(6)(0)	$\mathcal{G}'_{12,2}$	$[e_1,e_4]=e_2,\ [e_1,e_6]=-e_3,\ [e_2,e_4]=-e_1,\ [e_2,e_5]=-e_3,\ [e_3,e_5]=e_2,$ $[e_3,e_6]=e_1,\ [e_4,e_5]=e_6,\ [e_4,e_6]=-e_5,\ [e_5,e_6]=e_4$	$[7,0,2,0,0,2]$	2	$3\mathcal{A}_1$	D	$3\mathcal{A}_1 \triangleleft \mathrm{sl}(2,\mathbb{C})$
(8)(8)(0)	$\mathcal{G}_{6,2}$	$[e_1,e_6]=e_3,\ [e_1,e_8]=e_5,\ [e_2,e_7]=-e_4,\ [e_2,e_8]=e_5,\ [e_3,e_6]=4e_1-2e_2,$ $[e_3,e_7]=e_5,\ [e_3,e_8]=e_4,\ [e_4,e_6]=e_5,\ [e_4,e_7]=2e_1-4e_2,\ [e_4,e_8]=-e_3,$ $[e_5,e_6]=e_4,\ [e_5,e_7]=e_3,\ [e_5,e_8]=-2e_1-2e_2,\ [e_6,e_7]=e_8,\ [e_6,e_8]=e_7,$ $[e_7,e_8]=-e_6$	$[9,0,1,0,0,1]$	2	$5\mathcal{A}_1$	C	$5\mathcal{A}_1 \triangleleft \mathrm{sl}(2,\mathbb{C})$
(87)(87)(0)	$\mathcal{G}_{6,1}$	$[e_1,e_5]=e_1,\ [e_1,e_6]=e_2,\ [e_1,e_8]=e_1,\ [e_2,e_5]=e_2,\ [e_2,e_7]=e_1,$ $[e_2,e_8]=-e_2,\ [e_3,e_5]=-e_3,\ [e_3,e_6]=e_4,\ [e_3,e_8]=e_3,\ [e_4,e_5]=-e_4,$ $[e_4,e_7]=e_3,\ [e_4,e_8]=-e_4,\ [e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[9,0,1,0,0,1]$	2	$4\mathcal{A}_1$	C	$\mathcal{A}_{5,7}^{(1,-1,-1)} \triangleleft \mathrm{sl}(2,\mathbb{C})$

B.2: Nilpotent graded contractions for Gell-Mann graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T	AN
(310)(310)(13)	$\mathcal{G}'_{20,1}$	$[e_2,e_3]=e_1$	$[6,6,3,5,3,4]$	1	e_1	C	$\mathcal{A}_{3,1}$
(510)(510)(15)	$\mathcal{G}'_{19,4}$	$[e_2,e_4]=e_1,\ [e_3,e_5]=e_1$	$[15,15,5,14,10,11]$	1	e_1	C	$\mathcal{A}_{5,4}$
(520)(520)(25)	$\mathcal{G}'_{19,3}$	$[e_3,e_5]=e_1,\ [e_4,e_5]=e_2$	$[13,13,7,9,7,11]$	3	$e_1,\ e_2,\ e_2e_3-e_1e_4$	C	$\mathcal{A}_{5,1}$
(620)(620)(26)	$\mathcal{G}'_{18,7}$	$[e_3,e_5]=e_1,\ [e_4,e_6]=e_1,\ [e_5,e_6]=e_2$	$[17,18,10,14,10,14]$	2	$e_1,\ e_2$	C	$\mathcal{A}_{6,4}$
(630)(630)(36)	$\mathcal{G}'_{18,16}$	$[e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3$	$[18,18,10,9,10,19]$	4	$e_1,\ e_2,\ e_3,\ e_1e_6-e_2e_5+e_3e_4$	C	$\mathcal{A}_{6,3}$
(630)(6310)(136)	$\mathcal{G}'_{17,13}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_5]=e_3,\ [e_4,e_6]=e_2$	$[11,10,4,6,4,8]$	2	$e_1,\ e_2e_3-e_1e_4$	D	$\mathcal{A}_{6,14}^{(1)}$
(710)(710)(17)	$\mathcal{G}'_{18,18}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_7]=e_1$	$[28,28,7,27,21,22]$	1	e_1	C	
(720)(720)(27)	$\mathcal{G}'_{17,5}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_1$	$[19,19,11,15,11,15]$	3	$e_1,\ e_2,\ e_1^2e_4-e_1e_2e_5+e_2^2e_3$	C	
	$\mathcal{G}'_{18,5}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2$	$[21,22,11,18,14,18]$	3	$e_1,\ e_2,\ e_1e_5-e_2e_4$	C	

B.2: Nilpotent graded contractions for Gell-Mann graded sl$(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T
(730)(730)(37)	$\mathcal{G}'_{17,16}$	$[e_4,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_2,\ [e_5,e_7]=e_3$	$[19,24,13,15,13,22]$	3	$e_1,\ e_2,\ e_3$	C
	$\mathcal{G}'_{16,12}$	$[e_4,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	$[20,24,13,15,13,22]$	3	$e_1,\ e_2,\ e_3$	C
	$\mathcal{G}'_{17,12}$	$[e_4,e_7]=e_1,\ [e_5,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	$[22,24,13,15,13,22]$	3	$e_1,\ e_2,\ e_3$	C
	$\mathcal{G}'_{18,6}$	$[e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	$[25,25,13,16,13,22]$	5	$e_1,\ e_2,\ e_3,\ e_1e_5-e_2e_4,\ e_1e_6-e_3e_4$	C
(730)(7310)(147)	$\mathcal{G}'_{16,8}$	$[e_2,e_5]=e_1,\ [e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_2$	$[15,16,7,10,7,11]$	1	e_1	D
(740)(7410)(147)	$\mathcal{G}'_{15,6}(a)$	$[e_2,e_5]=(a+1)e_1,\ [e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_6]=-ae_4,$	$[15,13,7,9,6,11]$	1	e_1	D
		$[e_5,e_7]=e_3,\ [e_6,e_7]=e_2 \quad a\in\mathbb{C}^*_{20}$				
		$a=-\frac{1}{2}\cong a=1$	$[17,13,7,9,6,11]$			C
(740)(7410)(247)	$\mathcal{G}'_{16,7}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1,\ [e_5,e_6]=e_4,\ [e_5,e_7]=e_3,\ [e_6,e_7]=e_2$	$[15,17,8,9,7,16]$	3	$e_1,\ e_2,\ e_1e_5-e_3e_4$	D
(820)(820)(28)	$\mathcal{G}_{18,23}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_8]=e_1+e_2$	$[22,25,13,21,15,19]$	2	$e_1,\ e_2$	C
(830)(830)(38)	$\mathcal{G}_{15,3}$	$[e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_2,$	$[19,24,16,15,16,25]$	4	$e_1,\ e_2,\ e_3,\ e_1(e_1e_5-e_2e_7+e_3e_8)$	C
		$[e_6,e_8]=e_3,\ [e_7,e_8]=e_1$			$+e_2^2(e_5-e_4)-e_2e_3e_6+e_3^2e_4$	
	$\mathcal{G}_{16,2}$	$[e_4,e_6]=e_1,\ [e_4,e_8]=e_2,\ [e_5,e_7]=e_3,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_2,$	$[20,24,16,15,16,25]$	4	$e_1,\ e_2,\ e_3,\ e_1(e_2e_7-e_3e_8)$	C
		$[e_6,e_8]=e_3$			$+e_2^2(e_4-e_5)+e_2e_3e_6-e_3^2e_4$	
	$\mathcal{G}_{17,3}$	$[e_4,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_3,\ [e_6,e_8]=e_3,\ [e_7,e_8]=e_1$	$[21,25,16,16,16,25]$	4	$e_1,\ e_2,\ e_3,$	C
					$e_1(e_1e_5+e_2e_8-e_3e_7)+e_2e_3e_4$	
	$\mathcal{G}_{18,1}$	$[e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_5,e_8]=e_2,\ [e_6,e_8]=e_3$	$[22,28,16,19,16,25]$	4	$e_1,\ e_2,\ e_3,\ e_1(e_2e_6-e_3e_5)+e_2e_3e_4$	C
	$\mathcal{G}_{16,6}$	$[e_4,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_2,\ [e_6,e_8]=e_1,\ [e_7,e_8]=e_3$	$[26,28,16,19,16,25]$	4	$e_1,\ e_2,\ e_3,\ e_1^2e_5-e_1e_2e_6+e_2^2e_4$	C
	$\mathcal{G}_{17,4}$	$[e_4,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_8]=e_1,\ [e_7,e_8]=e_3$	$[27,30,17,21,17,26]$	4	$e_1,\ e_2,\ e_3,\ e_1e_5-e_2e_6$	C
(840)(840)(48)	$\mathcal{G}_{15,9}$	$[e_5,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4$	$[24,33,17,17,17,33]$	4	$e_1,\ e_2,\ e_3,\ e_4$	C
	$\mathcal{G}_{16,9}$	$[e_5,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_2,\ [e_6,e_8]=e_3,\ [e_7,e_8]=e_4$	$[25,33,17,17,17,33]$	4	$e_1,\ e_2,\ e_3,\ e_4$	C
	$\mathcal{G}_{17,15}$	$[e_5,e_7]=e_1,\ [e_6,e_7]=e_2,\ [e_6,e_8]=e_3,\ [e_7,e_8]=e_4$	$[27,33,17,17,17,33]$	4	$e_1,\ e_2,\ e_3,\ e_4$	C
(840)(8410)(148)	$\mathcal{G}_{15,4}$	$[e_2,e_6]=e_1,\ [e_3,e_7]=e_1,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_4,\ [e_5,e_8]=e_3,$	$[18,16,6,10,7,12]$	2	$e_1,\ e_1e_5-e_3e_4$	C
		$[e_6,e_8]=e_3,\ [e_7,e_8]=e_2$				
(850)(8520)(258)	$\mathcal{G}_{15,5}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1+e_2,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_5,$	$[18,19,7,9,6,17]$	2	$e_1,\ e_2$	C
		$[e_6,e_8]=e_4,\ [e_7,e_8]=e_3$				

B.2: Solvable non–nilpotent graded contractions for Gell-Mann graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	AN	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T
(320)(32)(0)	$\mathcal{G}'_{19,2}$	$[e_1,e_3]=e_1,\ [e_2,e_3]=-e_2$	$\mathcal{A}_{3,4}$	$[4,3,1,2,0,1]$	1	$2\mathcal{A}_1$	D
(530)(532)(12)	$\mathcal{G}'_{18,9}$	$[e_2,e_5]=e_1,\ [e_3,e_5]=e_3,\ [e_4,e_5]=-e_4$	$\mathcal{A}_{5,8}^{(-1)}$	$[8,9,3,5,2,6]$	3	$4\mathcal{A}_1$	D
(540)(54)(0)	$\mathcal{G}'_{17,11}(a)$	$[e_1,e_5]=e_2,\ [e_2,e_5]=e_1,\ [e_3,e_5]=e_4,\ [e_4,e_5]=ae_3,\ a\in\mathbb{C}^*_{10}$	$\mathcal{A}_{5,17}^{(\sqrt{a},0,0)}$	$[8,5,1,4,0,1]$	3	$4\mathcal{A}_1$	D
		$a=1$		$[12,5,1,4,0,1]$			
(640)(64)(0)	$\mathcal{G}'_{15,7}$	$[e_1,e_6]=e_2,\ [e_2,e_6]=e_1,\ [e_3,e_5]=e_1,\ [e_3,e_6]=e_4,\ [e_4,e_5]=e_2,\ [e_4,e_6]=e_3$		$[9,6,2,4,0,2]$	2	$\mathcal{A}_{5,1}$	D
(6510)(65)(1)	$\mathcal{G}'_{15,8}$	$[e_2,e_4]=e_1,\ [e_2,e_6]=e_5,\ [e_3,e_5]=e_1,\ [e_3,e_6]=e_4,\ [e_4,e_6]=e_3,\ [e_5,e_6]=e_2$		$[10,6,2,1,1,7]$	2	$\mathcal{A}_{5,4}$	D
(740)(742)(24)	$\mathcal{G}'_{17,6}$	$[e_3,e_7]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_7]=e_6,\ [e_6,e_7]=e_5$		$[16,19,7,10,6,15]$	5	$6\mathcal{A}_1$	D
(750)(754)(1)	$\mathcal{G}'_{12,1}$	$[e_2,e_6]=e_3,\ [e_2,e_7]=e_4,\ [e_3,e_6]=e_2,\ [e_3,e_7]=e_5,\ [e_4,e_6]=e_5,\ [e_4,e_7]=e_2,\ [e_5,e_6]=e_4,$		$[10,12,3,6,2,8]$	3	$5\mathcal{A}_1$	D
		$[e_5,e_7]=e_3,\ [e_6,e_7]=e_1$					
	$\mathcal{G}'_{14,1}$	$[e_2,e_7]=e_3,\ [e_3,e_7]=e_2,\ [e_4,e_6]=e_2,\ [e_4,e_7]=e_5,\ [e_5,e_6]=e_3,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_1$		$[11,12,3,6,2,8]$	3	$\mathcal{A}_1\oplus\mathcal{A}_{5,1}$	D
(750)(754)(12)	$\mathcal{G}'_{16,4}(a)$	$[e_2,e_7]=ae_3,\ [e_3,e_7]=e_2,\ [e_4,e_7]=e_5,\ [e_5,e_7]=e_4,\ [e_6,e_7]=e_1,\ a\in\mathbb{C}^*_{10}$		$[12,13,3,7,2,8]$	5	$6\mathcal{A}_1$	D
		$a=1$		$[16,13,3,7,2,8]$			
(7510)(75)(12)	$\mathcal{G}'_{14,2}$	$[e_2,e_4]=e_1,\ [e_2,e_7]=e_5,\ [e_3,e_5]=e_1,\ [e_3,e_7]=e_4,\ [e_4,e_7]=e_3,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_1$		$[12,12,3,3,2,8]$	1	$\mathcal{A}_1\oplus\mathcal{A}_{5,4}$	D
(760)(76)(0)	$\mathcal{G}'_{15,2}(a,b)$	$[e_1,e_7]=4ae_4,\ [e_2,e_7]=be_5,\ [e_3,e_7]=e_6,\ [e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3,\ a,b\neq 0$		$[12,7,1,6,0,1]$	5	$6\mathcal{A}_1$	D
		$a=\frac{1}{4}$ xor $b=1$ xor $b=4a$		$[16,7,1,6,0,1]$			
		$a=\frac{1}{4}\wedge b=1$		$[24,7,1,6,0,1]$			
		$a=b=1$					C
(840)(842)(25)	$\mathcal{G}_{17,1}$	$[e_3,e_5]=e_1,\ [e_3,e_8]=-e_1,\ [e_4,e_5]=e_2,\ [e_4,e_8]=e_2,\ [e_6,e_8]=e_7,\ [e_7,e_8]=e_6$		$[16,21,9,12,8,17]$	4	$2\mathcal{A}_1\oplus\mathcal{A}_{5,1}$	D
(850)(854)(12)	$\mathcal{G}_{16,1}(a)$	$[e_2,e_8]=ae_3,\ [e_3,e_8]=e_2,\ [e_4,e_7]=e_1,\ [e_4,e_8]=e_1,\ [e_5,e_7]=e_6,\ [e_6,e_7]=e_5,\ a\in\mathbb{C}_{10}$		$[13,14,4,8,3,9]$	4	$6\mathcal{A}_1$	D
(860)(86)(0)	$\mathcal{G}_{15,1}(a,b)$	$[e_1,e_8]=ae_2,\ [e_2,e_8]=e_1,\ [e_3,e_7]=be_4,\ [e_4,e_7]=e_3,\ [e_5,e_7]=e_6,\ [e_5,e_8]=e_6,$		$[12,7,1,6,0,1]$	4	$6\mathcal{A}_1$	D
		$[e_6,e_7]=e_5,\ [e_6,e_8]=e_5,\ a,b\neq 0$					
		$a=b=1$					C
(8620)(86)(0)	$\mathcal{G}_{11,2}$	$[e_1,e_7]=e_1,\ [e_1,e_8]=e_1,\ [e_2,e_3]=e_1,\ [e_2,e_7]=e_2,\ [e_3,e_8]=e_3,\ [e_4,e_7]=-e_4,$		$[10,2,1,2,0,1]$	2	$\mathcal{A}_{3,1}\oplus\mathcal{A}_{3,1}$	C
		$[e_4,e_8]=-e_4,\ [e_5,e_6]=e_4,\ [e_5,e_7]=-e_5,\ [e_6,e_8]=-e_6$					
	$\mathcal{G}_{11,1}$	$[e_1,e_8]=e_2,\ [e_2,e_8]=e_1,\ [e_3,e_6]=e_1,\ [e_3,e_8]=e_5,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_5,$		$[11,7,2,2,0,2]$	2	$\mathcal{G}'_{17,5}$	C
		$[e_5,e_6]=e_2,\ [e_5,e_7]=e_1,\ [e_5,e_8]=2e_3+2e_4,\ [e_6,e_8]=-e_7,\ [e_7,e_8]=-e_6$					
(8730)(87)(1)	$\mathcal{G}_{11,3}$	$[e_2,e_8]=2e_3,\ [e_3,e_8]=2e_2,\ [e_4,e_6]=e_1+e_3,\ [e_4,e_7]=e_2,\ [e_4,e_8]=e_5,$		$[12,10,2,3,1,9]$	2	$\mathcal{G}'_{17,16}$	C
		$[e_5,e_6]=e_2,\ [e_5,e_7]=e_3-e_1,\ [e_5,e_8]=e_4,\ [e_6,e_8]=e_7,\ [e_7,e_8]=e_6$					

B.3: Non–solvable graded contractions for Cartan graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T	Levi dec.
(3)(3)(0)	$\mathcal{T}'_{12,7}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=-2e_1,\ [e_2,e_3]=2e_2,$	$[3,0,1,0,0,1]$	1	$\{0\}$	D	$\mathrm{sl}(2,\mathbb{C})$
(65)(65)(0)	$\mathcal{T}'_{8,3}$	$[e_1,e_3]=e_1,\ [e_1,e_4]=e_1,\ [e_1,e_5]=e_2,\ [e_2,e_3]=e_2,\ [e_2,e_4]=-e_2,$ $[e_2,e_6]=e_1,\ [e_4,e_5]=2e_5,\ [e_4,e_6]=-2e_6,\ [e_5,e_6]=e_4$	$[6,0,1,0,0,1]$	0	$2\mathcal{A}_1$	D	$\mathcal{A}_{3,3}\lhd\mathrm{sl}(2,\mathbb{C})$
(87)(87)(0)	$\mathcal{T}_{4,1}$	$[e_1,e_5]=e_1,\ [e_1,e_6]=e_2,\ [e_1,e_8]=e_1,\ [e_2,e_5]=e_2,\ [e_2,e_7]=e_1,$ $[e_2,e_8]=-e_2,\ [e_3,e_5]=-e_3,\ [e_3,e_6]=e_4,\ [e_3,e_8]=e_3,\ [e_4,e_5]=-e_4,$ $[e_4,e_7]=e_3,\ [e_4,e_8]=-e_4,\ [e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[9,0,1,0,0,1]$	2	$4\mathcal{A}_1$	C	$\mathcal{A}_{5,7}^{(1,-1,-1)}\lhd\mathrm{sl}(2,\mathbb{C})$

B.3: Nilpotent graded contractions for Cartan graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Casimir operators	T	AN
(310)(310)(13)	$\mathcal{T}'_{14,2}$	$[e_2,e_3]=e_1$	$[6,6,3,5,3,4]$	1	e_1	C	$\mathcal{A}_{3,1}$
(520)(520)(25)	$\mathcal{T}'_{13,6}$	$[e_3,e_5]=e_1,\ [e_4,e_5]=e_2$	$[13,13,7,9,7,11]$	3	$e_1,\ e_2,\ e_2e_3-e_1e_4$	C	$\mathcal{A}_{5,1}$
(630)(630)(36)	$\mathcal{T}'_{12,9}$	$[e_4,e_5]=e_1,\ [e_4,e_6]=e_2,\ [e_5,e_6]=e_3$	$[18,18,10,9,10,19]$	4	$e_1,\ e_2,\ e_3,\ e_1e_6-e_2e_5+e_3e_4$	C	$\mathcal{A}_{6,3}$
(730)(730)(37)	$\mathcal{T}'_{12,10}$	$[e_4,e_6]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	$[20,24,13,15,13,22]$	3	$e_1,\ e_2,\ e_3$	C	
	$\mathcal{T}'_{12,8}$	$[e_4,e_7]=e_1,\ [e_5,e_7]=e_2,\ [e_6,e_7]=e_3$	$[25,25,13,16,13,22]$	5	$e_1,\ e_2,\ e_3,\ e_1e_5-e_2e_4,\ e_1e_6-e_3e_4$	C	
(820)(820)(28)	$\mathcal{T}_{12,11}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_2,\ [e_5,e_8]=e_1+e_2$	$[22,25,13,21,15,19]$	2	$e_1,\ e_2$	C	
(840)(840)(48)	$\mathcal{T}_{11,6}$	$[e_5,e_7]=e_1,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_3,\ [e_6,e_8]=e_4$	$[24,33,17,17,17,33]$	4	$e_1,\ e_2,\ e_3,\ e_4$	C	
(850)(8520)(258)	$\mathcal{T}_{9,4}$	$[e_3,e_6]=e_1,\ [e_4,e_7]=e_1+e_2,\ [e_5,e_8]=e_2,\ [e_6,e_7]=e_5,$ $[e_6,e_8]=e_4,\ [e_7,e_8]=e_3$	$[18,19,7,7,9,6,17]$	2	$e_1,\ e_2$	C	

B.3: Solvable non–nilpotent graded contractions for Cartan graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T	AN
(210)(21)(0)	$\mathcal{T}'_{14,1}$	$[e_1,e_2]=e_1$	$[2,3,1,1,0,1]$	0	$\mathcal{A}_1$	D	$\mathcal{A}_{2,1}$
(320)(32)(0)	$\mathcal{T}'_{13,3}(a)$	$[e_1,e_3]=ae_1,\ [e_2,e_3]=-e_2,\ a\in\mathbb{C}^*_{10}$	$[4,3,1,2,0,1]$	1	$2\mathcal{A}_1$	D	$\mathcal{A}_{3,5}^{(-a)}$
		$a=-1$	$[6,3,1,2,0,1]$				$\mathcal{A}_{3,3}$
(530)(53)(0)	$\mathcal{T}'_{12,1}(a,b)$	$[e_1,e_4]=be_1,\ [e_2,e_5]=ae_2,\ [e_3,e_4]=e_3,\ [e_3,e_5]=e_3,\ a,b\neq 0$	$[6,4,1,3,0,1]$	1	$3\mathcal{A}_1$	D	$\mathcal{A}_{5,33}^{(\frac{1}{a},\frac{1}{b})}$
(5310)(53)(0)	$\mathcal{T}'_{11,4}$	$[e_1,e_4]=e_1,\ [e_1,e_5]=e_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_2,\ [e_3,e_4]=e_3$	$[5,1,1,1,0,1]$	1	$\mathcal{A}_{3,1}$	D	$\mathcal{A}_{5,36}$
(640)(64)(0)	$\mathcal{T}'_{11,1}(a,b,c)$	$[e_1,e_5]=be_1,\ [e_1,e_6]=be_1,\ [e_2,e_6]=ce_2,\ [e_3,e_6]=-e_3,\ [e_4,e_5]=-ae_4,\ a,b,c\neq 0$	$[8,5,1,4,0,1]$	2	$4\mathcal{A}_1$	D	
		$c=-1$	$[10,5,1,4,0,1]$				
(6410)(64)(0)	$\mathcal{T}'_{10,3}(a)$	$[e_1,e_5]=ae_1,\ [e_2,e_6]=e_2,\ [e_3,e_4]=e_2,\ [e_3,e_5]=-e_3,\ [e_4,e_5]=e_4,$	$[7,2,1,2,0,1]$	0	$\mathcal{A}_1\oplus\mathcal{A}_{3,1}$	D	
		$[e_4,e_6]=e_4,\ a\in\mathbb{C}^*_0$					
		$a=-1$	$[8,2,1,2,0,1]$				
	$\mathcal{T}'_{10,2}(a)$	$[e_1,e_6]=-ae_1,\ [e_2,e_6]=e_2,\ [e_3,e_4]=e_2,\ [e_3,e_5]=e_3,\ [e_4,e_5]=-e_4,$	$[7,2,1,2,0,1]$	2	$\mathcal{A}_1\oplus\mathcal{A}_{3,1}$	D	
		$[e_4,e_6]=e_4,\ a\in\mathbb{C}^*_0$					
		$a=-1$	$[8,2,1,2,0,1]$				
(750)(75)(0)	$\mathcal{T}'_{10,1}(a,b,c,d)$	$[e_1,e_6]=ae_1,\ [e_1,e_7]=ae_1,\ [e_2,e_6]=-be_2,\ [e_3,e_6]=e_3,$	$[10,6,1,5,0,1]$	3	$5\mathcal{A}_1$	D	
		$[e_4,e_7]=ce_4,\ [e_5,e_7]=-de_5,\ a,b,c,d\neq 0$					
		$b=-1$ xor $c=-d$	$[12,6,1,5,0,1]$				
		$b=-1\wedge c=-d$	$[14,6,1,5,0,1]$				
(7510)(75)(0)	$\mathcal{T}'_{9,2}(a,b)$	$[e_1,e_6]=ae_1,\ [e_2,e_7]=-be_2,\ [e_3,e_7]=e_3,\ [e_4,e_5]=e_3,\ [e_4,e_6]=-e_4,$	$[9,3,1,3,0,1]$	1	$2\mathcal{A}_1\oplus\mathcal{A}_{3,1}$	D	
		$[e_5,e_6]=e_5,\ [e_5,e_7]=e_5,\ a,b\neq 0$					
		$a=-1$ xor $b=-1$	$[10,3,1,3,0,1]$				
		$a=b=-1$	$[11,3,1,3,0,1]$				
	$\mathcal{T}'_{9,3}(a,b)$	$[e_1,e_6]=-ae_1,\ [e_2,e_7]=-be_2,\ [e_3,e_6]=e_3,\ [e_3,e_7]=e_3,\ [e_4,e_5]=e_3,$	$[9,3,1,3,0,1]$	1	$2\mathcal{A}_1\oplus\mathcal{A}_{3,1}$	D	
		$[e_4,e_6]=e_4,\ [e_5,e_7]=e_5,\ a,b\neq 0$					
		$a=-1$ xor $b=-1$	$[10,3,1,3,0,1]$				
		$a=b=-1$	$[11,3,1,3,0,1]$				
(7520)(75)(0)	$\mathcal{T}'_{8,2}$	$[e_1,e_7]=e_1,\ [e_2,e_6]=e_2,\ [e_3,e_5]=e_1,\ [e_3,e_6]=-e_3,\ [e_4,e_5]=e_2,$	$[8,2,1,2,0,1]$	1	$\mathcal{A}_{5,1}$	D	
		$[e_4,e_7]=-e_4,\ [e_5,e_6]=e_5,\ [e_5,e_7]=e_5$					

B.3: Solvable non–nilpotent graded contractions for Cartan graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T
(860)(86)(0)	$\mathcal{T}_{9,1}(a,b,c,d,f)$	$[e_1,e_7]=ae_1,\ [e_1,e_8]=-ae_1,\ [e_2,e_7]=-be_2,\ [e_3,e_8]=-ce_3,$	$[12,7,1,6,0,1]$	4	$6\mathcal{A}_1$	D
		$[e_4,e_8]=de_4,\ [e_5,e_7]=fe_5,\ [e_6,e_7]=-e_6,\ [e_6,e_8]=e_6,\ a,b,c,d,f\neq 0$				
		$a=-1$ xor $b=-f$ xor $c=-d$	$[14,7,1,6,0,1]$			
		$(a=-1\wedge b=-f)$ xor $(a=-1\wedge c=-d)$ xor $(b=-f\wedge c=-d)$	$[16,7,1,6,0,1]$			
		$a=-1\wedge b=-f\wedge c=-d$	$[18,7,1,6,0,1]$			
		$a=b=c=d=f=1$				C
(8610)(86)(0)	$\mathcal{T}_{8,1}(a,b,c)$	$[e_1,e_7]=ae_1,\ [e_1,e_8]=ae_1,\ [e_2,e_7]=be_2,\ [e_3,e_8]=-ce_3,\ [e_4,e_7]=-e_4,$	$[11,4,1,4,0,1]$	2	$3\mathcal{A}_1\oplus\mathcal{A}_{3,1}$	D
		$[e_5,e_6]=e_4,\ [e_5,e_7]=-e_5,\ [e_5,e_8]=-e_5,\ [e_6,e_8]=e_6,\ a,b,c\neq 0$				
		$a=-1$ xor $b=-1$ xor $c=-1$	$[12,4,1,4,0,1]$			
		$a=b=-1$ xor $a=c=-1$ xor $b=c=-1$	$[13,4,1,4,0,1]$			
		$a=b=c=-1$	$[14,4,1,4,0,1]$			
		$a=b=c=1$				C
(8620)(86)(0)	$\mathcal{T}_{7,2}(a)$	$[e_1,e_7]=ae_1,\ [e_1,e_8]=ae_1,\ [e_2,e_3]=e_1,\ [e_2,e_7]=ae_2,\ [e_3,e_8]=ae_3,$	$[10,2,1,2,0,1]$	2	$\mathcal{A}_{3,1}\oplus\mathcal{A}_{3,1}$	D
		$[e_4,e_7]=-e_4,\ [e_4,e_8]=-e_4,\ [e_5,e_6]=e_4,\ [e_5,e_7]=-e_5,\ [e_6,e_8]=-e_6,\ a\in\mathbb{C}^*_{10}$				
		$a=1$				C
	$\mathcal{T}_{7,1}(a)$	$[e_1,e_7]=ae_1,\ [e_1,e_8]=ae_1,\ [e_2,e_7]=-e_2,\ [e_3,e_8]=-e_3,\ [e_4,e_6]=e_2,\ [e_4,e_8]=e_4,$	$[10,3,1,3,0,1]$	2	$\mathcal{A}_1\oplus\mathcal{A}_{5,1}$	D
		$[e_5,e_6]=e_3,\ [e_5,e_7]=e_5,\ [e_6,e_7]=-e_6,\ [e_6,e_8]=-e_6,\ a\in\mathbb{C}^*_0$				
		$a=-1$	$[11,3,1,3,0,1]$			
		$a=1$				C
(8630)(86)(0)	$\mathcal{T}_{6,1}$	$[e_1,e_7]=e_1,\ [e_2,e_8]=e_2,\ [e_3,e_7]=-e_3,\ [e_3,e_8]=-e_3,\ [e_4,e_5]=e_1,$	$[9,3,1,3,0,1]$	2	$\mathcal{A}_{6,3}$	C
		$[e_4,e_6]=e_2,\ [e_4,e_7]=e_4,\ [e_4,e_8]=e_4,\ [e_5,e_6]=e_3,\ [e_5,e_8]=-e_5,\ [e_6,e_7]=-e_6$				

B.4: Non–solvable graded contractions for Γ_4 graded $\mathrm{sl}(3,\mathbb{C})$

DS,CS,US	Name	Commutation relations	$\dim_{(\alpha,\beta,\gamma)}$	τ	Nilradical	T	Levi dec.
(3)(3)(0)	$\mathcal{L}'_{18,47}$	$[e_1,e_2]=e_3,\ [e_1,e_3]=-2e_1,\ [e_2,e_3]=2e_2,$	$[3,0,1,0,0,1]$	1	$\{0\}$		$\mathrm{sl}(2,\mathbb{C})$
(5)(5)(0)	$\mathcal{L}'_{14,77}$	$[e_1,e_4]=e_2,\ [e_1,e_5]=-e_1,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_2,\ [e_3,e_4]=e_5,$ $[e_3,e_5]=-2e_3,\ [e_4,e_5]=2e_4$	$[6,0,1,0,0,1]$	1	$2\mathcal{A}_1$	D	$2\mathcal{A}_1 \lhd \mathrm{sl}(2,\mathbb{C})$
(6)(6)(0)	$\mathcal{L}'_{12,51}$	$[e_1,e_4]=e_3,\ [e_1,e_6]=e_1,\ [e_2,e_5]=e_3,\ [e_2,e_6]=-e_2,\ [e_3,e_4]=e_2,$ $[e_3,e_5]=e_1,\ [e_4,e_5]=e_6,\ [e_4,e_6]=-e_4,\ [e_5,e_6]=e_5$	$[7,0,2,0,0,2]$	2	$3\mathcal{A}_1$	D	$3\mathcal{A}_1 \lhd \mathrm{sl}(2,\mathbb{C})$
(6)(6)(1)	$\mathcal{L}'_{13,55}$	$[e_2,e_3]=e_1,\ [e_2,e_4]=e_3,\ [e_2,e_6]=e_2,\ [e_3,e_5]=e_2,\ [e_3,e_6]=-e_3,$ $[e_4,e_5]=e_6,\ [e_4,e_6]=-2e_4,\ [e_5,e_6]=2e_5$	$[6,6,1,0,0,7]$	2	$2\mathcal{A}_1$	D	$\mathcal{A}_{3,1} \lhd \mathrm{sl}(2,\mathbb{C})$
(7)(7)(0)	$\mathcal{L}'_{10,4}$	$[e_1,e_5]=e_4,\ [e_1,e_7]=e_1,\ [e_2,e_5]=e_3,\ [e_2,e_7]=e_2,\ [e_3,e_6]=e_2,\ [e_3,e_7]=-e_3,$ $[e_4,e_6]=e_1,\ [e_4,e_7]=-e_4,\ [e_5,e_6]=e_7,\ [e_5,e_7]=-2e_5,\ [e_6,e_7]=2e_6$	$[11,0,1,0,0,1]$	1	$4\mathcal{A}_1$	D	$4\mathcal{A}_1 \lhd \mathrm{sl}(2,\mathbb{C})$
(8)(8)(0)	$\mathcal{L}_{6,1}$	$[e_1,e_7]=2e_4,\ [e_1,e_8]=-2e_1,\ [e_2,e_6]=2e_5,\ [e_2,e_8]=2e_2,\ [e_3,e_6]=3e_4,$ $[e_3,e_7]=3e_5,\ [e_4,e_6]=e_1,\ [e_4,e_7]=e_3,\ [e_4,e_8]=-e_4,\ [e_5,e_6]=e_3,$ $[e_5,e_7]=e_2,\ [e_5,e_8]=e_5,\ [e_6,e_7]=e_8,\ [e_6,e_8]=-e_6,\ [e_7,e_8]=e_7$	$[9,0,1,0,0,1]$	2	$5\mathcal{A}_1$	C	$5\mathcal{A}_1 \lhd \mathrm{sl}(2,\mathbb{C})$
	$\mathcal{L}_{7,1}$	$[e_1,e_7]=e_2,\ [e_1,e_8]=-e_1,\ [e_2,e_6]=e_1,\ [e_2,e_8]=e_2,\ [e_3,e_4]=e_2,$ $[e_3,e_5]=e_1,\ [e_4,e_5]=e_3,\ [e_4,e_6]=e_5,\ [e_4,e_8]=e_4,\ [e_5,e_7]=e_4,$ $[e_5,e_8]=-e_5,\ [e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[9,0,1,0,0,1]$	2	$\mathcal{A}_{5,3}$	C	$\mathcal{A}_{5,3} \lhd \mathrm{sl}(2,\mathbb{C})$
(8)(8)(1)	$\mathcal{L}_{8,3}$	$[e_2,e_4]=e_1,\ [e_2,e_6]=e_4,\ [e_2,e_8]=e_2,\ [e_3,e_5]=e_1,\ [e_3,e_7]=e_5,$ $[e_3,e_8]=-e_3,\ [e_4,e_7]=e_2,\ [e_4,e_8]=-e_4,\ [e_5,e_6]=e_3,\ [e_5,e_8]=e_5,$ $[e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[9,8,1,0,0,9]$	2	$\mathcal{A}_{5,4}$	C	$\mathcal{A}_{5,4} \lhd \mathrm{sl}(2,\mathbb{C})$
(8)(8)(1)	$\mathcal{L}_{9,6}$	$[e_1,e_6]=e_2,\ [e_1,e_8]=e_1,\ [e_2,e_7]=e_1,\ [e_2,e_8]=-e_2,\ [e_4,e_5]=e_3,\ [e_4,e_6]=e_5,$ $[e_4,e_8]=e_4,\ [e_5,e_7]=e_4,\ [e_5,e_8]=-e_5,\ [e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[10,8,1,0,0,9]$	2	$2\mathcal{A}_1 \oplus \mathcal{A}_{3,1}$	C	$(2\mathcal{A}_1 \oplus \mathcal{A}_{3,1}) \lhd \mathrm{sl}(2,\mathbb{C})$
(86)(86)(13)	$\mathcal{L}_{12,50}$	$[e_2,e_4]=e_1,\ [e_3,e_5]=e_1,\ [e_3,e_7]=e_5,\ [e_3,e_8]=-e_3,\ [e_5,e_6]=e_3,$ $[e_5,e_8]=e_5,\ [e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[11,11,3,5,3,10]$	2	$\mathcal{A}_{5,4}$	D	$\mathcal{A}_{5,4} \lhd \mathrm{sl}(2,\mathbb{C})$
(87)(87)(0)	$\mathcal{L}_{6,2}$	$[e_1,e_5]=e_1,\ [e_1,e_6]=e_2,\ [e_1,e_8]=e_1,\ [e_2,e_5]=e_2,\ [e_2,e_7]=e_1,$ $[e_2,e_8]=-e_2,\ [e_3,e_5]=-e_3,\ [e_3,e_6]=e_4,\ [e_3,e_8]=e_3,\ [e_4,e_5]=-e_4,$ $[e_4,e_7]=e_3,\ [e_4,e_8]=-e_4,\ [e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[9,0,1,0,0,1]$	2	$4\mathcal{A}_1$	C	$\mathcal{A}_{5,7}^{(1,-1,-1)} \lhd \mathrm{sl}(2,\mathbb{C})$
(87)(87)(0)	$\mathcal{L}_{8,2}$	$[e_1,e_6]=e_2,\ [e_1,e_8]=e_1,\ [e_2,e_7]=e_1,\ [e_2,e_8]=-e_2,\ [e_3,e_5]=e_1,$ $[e_3,e_6]=e_4,\ [e_3,e_8]=e_3,\ [e_4,e_5]=e_2,\ [e_4,e_7]=e_3,\ [e_4,e_8]=-e_4,$ $[e_6,e_7]=e_8,\ [e_6,e_8]=-2e_6,\ [e_7,e_8]=2e_7$	$[10,0,1,0,0,1]$	2	$\mathcal{A}_{5,1}$	C	$\mathcal{A}_{5,1} \lhd \mathrm{sl}(2,\mathbb{C})$

Appendix C: Invariant functions

Appendix C contains the invariant functions ψ, φ and φ^0 for one–parametric graded contractions of $\mathrm{sl}(3,\mathbb{C})$. The listing for any contraction contains its name, original restriction of parameter, isomorphism (if it was found), invariant functions and final restrictions of parameters. Combining isomorphisms and the invariant functions, the following sets of allowed parameters were found:

$$\begin{aligned}
\mathbb{C}_0 &= \mathbb{C} \setminus \{0\}, \\
\mathbb{C}_1 &= \{z \in \mathbb{C} \,|\, |z| < 1\} \cup \{z \in \mathbb{C} \,|\, |z| = 1 \wedge \mathrm{Im}(z) \geq 0\}, \\
\mathbb{C}_{10} &= \{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \cup \{z \in \mathbb{C} \,|\, |z| = 1 \wedge \mathrm{Im}(z) \geq 0\}, \\
\mathbb{C}_{20} &= \left\{z \in \mathbb{C} \,|\, 0 < |z+1| < 1 \wedge \mathrm{Re}(z) \geq -\tfrac{1}{2}\right\} \cup \\
&\quad \left\{z \in \mathbb{C} \,|\, |z+1| = 1 \wedge \mathrm{Re}(z) \geq -\tfrac{1}{2} \wedge \mathrm{Im}(z) > 0\right\}.
\end{aligned}$$

We use the superscript $*$ for any of the listed sets if there are no isomorphisms among Lie algebras corresponding to different parameters in the given set.

In the tables of the invariant functions the blank space stands for general complex number, different from all previously listed in given table, and $\pm\sqrt{a}$ denotes the roots of equation $a^2 = 1$. Let us note that the function φ was omitted, whenever its computation was too laborious and its knowledge was not necessary.

C.1 Pauli graded $\mathrm{sl}(3,\mathbb{C})$

- $\mathcal{P}_{16,3}(a),\ a \neq 0, \qquad \mathcal{P}_{16,3}(a) \cong \mathcal{P}_{16,3}(\frac{1}{a}) \qquad \longrightarrow \quad a \in \mathbb{C}_{10}$

$a \neq 0$

α	0	
$\psi(\alpha)$	19	16

$a = 1$

α	0	1	
$\varphi(\alpha)$	96	78	72

$a \neq 0, 1$

α	0	1	
$\varphi(\alpha)$	88	77	71

$a \neq 0$

α	1	0	
$\varphi^0(\alpha)$	22	18	16

- $\mathcal{P}_{17,7}(a),\ a \neq 0 \qquad \longrightarrow \quad a \in \mathbb{C}_0^*$

$a \neq 0$

α	0	1	
$\psi(\alpha)$	20	19	18

$a = 1$

α	0	1	-1	
$\varphi(\alpha)$	112	83	81	80

$a = -1$

α	0	1	-1	
$\varphi(\alpha)$	104	83	81	80

$a = \frac{1}{4} - \frac{\sqrt{7}i}{4}$

α	0	1	$-\frac{1}{4} + \frac{\sqrt{7}i}{4}$	
$\varphi(\alpha)$	104	82	82	80

$a = \frac{1}{4} + \frac{\sqrt{7}i}{4}$

α	0	1	$-\frac{1}{4} - \frac{\sqrt{7}i}{4}$	
$\varphi(\alpha)$	104	82	82	80

$a = \frac{1}{3}$

α	0	1	$-\frac{1}{3}$	
$\varphi(\alpha)$	104	83	81	80

$a \neq 0, \pm 1, \frac{1}{3}, \frac{1}{4} \pm \frac{\sqrt{7}i}{4}$

α	0	1	$-a$	$-\frac{1}{2} + \frac{1}{2a}$	
$\varphi(\alpha)$	104	82	81	81	80

$a = 1$

α	1	0	
$\varphi^0(\alpha)$	25	19	17

$a \neq 0, 1$

α	1	0	$1-a$	
$\varphi^0(\alpha)$	25	19	18	17

- $\mathcal{P}_{18,25}(a)$, $a \neq 0$, $\quad \mathcal{P}_{18,25}(a) \cong \mathcal{P}_{18,25}\left(\frac{1}{a}\right) \quad \longrightarrow \quad a \in \mathbb{C}_1^*$

$a = -1$

α	0	1	
$\psi(\alpha)$	22	22	19

$a = 1$

α	0	-1	1	
$\psi(\alpha)$	22	21	20	19

$a \neq 0, \pm 1$

α	0	1	$-a$	$-\frac{1}{a}$	
$\psi(\alpha)$	22	20	20	20	19

$a = 1$

α	0	1	-1	
$\varphi(\alpha)$	112	88	84	82

$a = -1$

α	0	1	-1	
$\varphi(\alpha)$	104	87	87	81

$a = \pm i$

α	0	1	i	$-i$	
$\varphi(\alpha)$	104	85	83	83	81

$a \neq 0, \pm 1, \pm i$

α	0	1	a	$-a$	$\frac{1}{a}$	$-\frac{1}{a}$	
$\varphi(\alpha)$	104	85	82	82	82	82	81

$a = 1$

α	1	0	2	
$\varphi^0(\alpha)$	28	22	20	19

$a = -1$

α	1	2	0	
$\varphi^0(\alpha)$	28	24	22	19

$a \neq 0, \pm 1$

α	1	0	2	$1-a$	$1-\frac{1}{a}$	
$\varphi^0(\alpha)$	28	22	20	20	20	19

$$\mathcal{P}_{18,25}(0) := \mathcal{P}_{19,26}$$

- $\mathcal{P}_{16,1}(a),\ a \neq 0 \qquad \longrightarrow \quad a \in \mathbb{C}^*_0$

$a \neq 0$

α	1	0	2	
$\psi(\alpha)$	16	15	15	14

$a = 1$

α	0	1	-1	5	
$\varphi(\alpha)$	104	77	73	71	70

$a = -\frac{1}{2}$

α	0	1	-1	3	
$\varphi(\alpha)$	88	75	71	71	70

$a = -\frac{1+2i}{10}$

α	0	1	-1	$1+2i$	
$\varphi(\alpha)$	88	75	71	71	70

$a = -\frac{1-2i}{10}$

α	0	1	-1	$1-2i$	
$\varphi(\alpha)$	88	75	71	71	70

$a \neq 0, 1, -\frac{1}{2}, -\frac{1\pm 2i}{10}$

α	0	1	-1	$\frac{4a+1-\sqrt{20a^2+4a+1}}{2a}$	$\frac{4a+1+\sqrt{20a^2+4a+1}}{2a}$	
$\varphi(\alpha)$	88	75	71	71	71	70

$a \neq 0$

α	1	
$\varphi^0(\alpha)$	25	11

- $\mathcal{P}_{17,13}(a),\ a \neq 0, \qquad \mathcal{P}_{17,13}(a) \cong \mathcal{P}_{17,13}\left(\frac{1}{a}\right) \qquad \longrightarrow \quad a \in \mathbb{C}^*_{10}$

$a = -1$

α	1	0	-1	
$\psi(\alpha)$	19	19	17	16

$a = 1$

α	-1	0	1	
$\psi(\alpha)$	19	19	17	16

$a \neq 0, \pm 1$

α	0	$-a$	$-\frac{1}{a}$	-1	1	
$\psi(\alpha)$	19	17	17	17	17	16

$a \neq 0$

α	
$\varphi^0(\alpha)$	15

- $\mathcal{P}_{18,29}(a),\ a \neq 0, \qquad \mathcal{P}_{18,29}(a) \cong \mathcal{P}_{18,29}\left(\frac{1}{a}\right) \qquad \longrightarrow \quad a \in \mathbb{C}^*_{10}$

$a=-1$

α	$-\frac{1-\sqrt{3}i}{2}$	$-\frac{1+\sqrt{3}i}{2}$	1	
$\psi(\alpha)$	19	19	18	7

$a=1$

α	-1	$\frac{1+\sqrt{3}i}{2}$	$\frac{1-\sqrt{3}i}{2}$	$-\frac{1-\sqrt{3}i}{2}$	$-\frac{1+\sqrt{3}i}{2}$	1	
$\psi(\alpha)$	13	13	13	13	13	12	7

$a\neq 0,\pm 1$

α	$-\frac{1-\sqrt{3}i}{2}$	$-\frac{1+\sqrt{3}i}{2}$	1	$\alpha^3=-a$			$\alpha^3=-\frac{1}{a}$			
$\psi(\alpha)$	13	13	12	10	10	10	10	10	10	7

$a=-1$

α	2	$\frac{1+\sqrt{3}i}{2}$	$\frac{1-\sqrt{3}i}{2}$	
$\varphi^0(\alpha)$	18	12	12	0

$a=1$

α	0	2	$\frac{1+\sqrt{3}i}{2}$	$\frac{1-\sqrt{3}i}{2}$	$\frac{3+\sqrt{3}i}{2}$	$\frac{3-\sqrt{3}i}{2}$	
$\varphi^0(\alpha)$	6	6	6	6	6	6	0

$a\neq 0,\pm 1$

α	2	$\frac{1+\sqrt{3}i}{2}$	$\frac{1-\sqrt{3}i}{2}$	$(\alpha-1)^3=-a$			$(\alpha-1)^3=-\frac{1}{a}$			
$\varphi^0(\alpha)$	6	6	6	3	3	3	3	3	3	0

• $\mathcal{P}_{15,6}(a),\ a\neq 0 \qquad \longrightarrow \quad a\in\mathbb{C}$

$a\neq 0$

α	0	1	
$\psi(\alpha)$	14	13	10

$a\neq 0$

α	0	1	-1	
$\varphi(\alpha)$	72	63	53	49

$a\neq 0$

α	1	2	0	
$\varphi^0(\alpha)$	13	8	5	4

$$\mathcal{P}_{15,6}(0):=\mathcal{P}_{16,6}$$

C.2 Gell–Mann graded $\mathrm{sl}(3,\mathbb{C})$

• $\mathcal{G}'_{17,11}(a),\ a\neq 0, \qquad \mathcal{G}'_{17,11}(a)\cong\mathcal{G}'_{17,11}\left(\frac{1}{a}\right) \qquad \longrightarrow \quad a\in\mathbb{C}^*_{10}$

$a=1$

α	-1	1	
$\psi(\alpha)$	13	12	5

$a=-1$

α	$-i$	i	-1	1	
$\psi(\alpha)$	9	9	9	8	5

$a\neq 0,\pm 1$

α	-1	1	$\frac{1}{\sqrt{a}}$	$-\frac{1}{\sqrt{a}}$	$\sqrt{a}$	$-\sqrt{a}$	
$\psi(\alpha)$	9	8	7	7	7	7	5

$a=1$

α	0	-2	
$\varphi(\alpha)$	40	24	20

$a=-1$

α	0	$-1-i$	$-1+i$	
$\varphi(\alpha)$	30	24	24	20

$a\neq 0,\pm1$

α	0	$-1+\sqrt{a}$	$-1-\sqrt{a}$	$-1-\frac{1}{\sqrt{a}}$	$-1+\frac{1}{\sqrt{a}}$	
$\varphi(\alpha)$	30	22	22	22	22	20

$a=1$

α	2	0	
$\varphi^0(\alpha)$	12	8	0

$a=-1$

α	$1+i$	$1-i$	0	2	
$\varphi^0(\alpha)$	4	4	4	4	0

$a\neq 0,\pm1$

α	0	2	$1-\sqrt{a}$	$1+\sqrt{a}$	$1-\frac{1}{\sqrt{a}}$	$1+\frac{1}{\sqrt{a}}$	
$\varphi^0(\alpha)$	4	4	2	2	2	2	0

• $\mathcal{G}'_{16,4}(a),\ a\neq 0,\qquad \mathcal{G}'_{16,4}(a)\cong\mathcal{G}'_{16,4}\left(\frac{1}{a}\right)\qquad\longrightarrow\quad a\in\mathbb{C}^*_{10}$

$a=1$

α	-1	1	0	
$\psi(\alpha)$	17	16	13	9

$a=-1$

α	0	$-i$	i	-1	1	
$\psi(\alpha)$	13	13	13	13	12	9

$a\neq 0,\pm1$

α	0	-1	1	$\frac{1}{\sqrt{a}}$	$-\frac{1}{\sqrt{a}}$	$\sqrt{a}$	$-\sqrt{a}$	
$\psi(\alpha)$	13	13	12	11	11	11	11	9

$a=1$

α	0	-1	1	-2	
$\varphi(\alpha)$	77	55	54	51	47

$a=-1$

α	0	$-i$	i	$-1-i$	$-1+i$	-1	1	
$\varphi(\alpha)$	63	49	49	49	49	49	48	45

$a=-\frac{1\pm\sqrt{3}i}{2}$

α	0	$-\frac{1+\sqrt{3}i}{2}$	$-\frac{1-\sqrt{3}i}{2}$	-1	1	$\frac{1+\sqrt{3}i}{2}$	$\frac{1-\sqrt{3}i}{2}$	$-\frac{3+\sqrt{3}i}{2}$	$-\frac{3-\sqrt{3}i}{2}$	
$\varphi(\alpha)$	63	49	49	49	48	47	47	47	47	45

$a=\frac{3\pm\sqrt{5}}{2}$

α	0	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1-\sqrt{5}}{2}$	-1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{3+\sqrt{5}}{2}$	$-\frac{3-\sqrt{5}}{2}$	
$\varphi(\alpha)$	63	49	49	49	48	47	47	47	47	45

$a=4,1/4$

α	0	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$	-2	2	-3	$-\frac{3}{2}$	
$\varphi(\alpha)$	63	50	49	49	47	47	47	47	47	45

$a \neq 0$

α	0	-1	1	$\pm\sqrt{a}$	$-1\pm\sqrt{a}$	$\pm\frac{1}{\sqrt{a}}$	$-1\pm\frac{1}{\sqrt{a}}$	
$\varphi(\alpha)$	63	49	48	47	47	47	47	45

$a = 1$

α	2	0	1	
$\varphi^0(\alpha)$	14	10	6	2

$a = -1$

α	2	0	1	$1-i$	$1+i$	
$\varphi^0(\alpha)$	6	6	6	6	6	2

$a \neq 0, \pm 1$

α	2	0	1	$1+\sqrt{a}$	$1-\sqrt{a}$	$1+\frac{1}{\sqrt{a}}$	$1-\frac{1}{\sqrt{a}}$	
$\varphi^0(\alpha)$	6	6	6	4	4	4	4	2

• $\mathcal{G}_{16,1}(a),\ a \neq 0, \qquad \mathcal{G}'_{16,1}(a) \cong \mathcal{G}'_{16,1}\left(\frac{1}{a}\right) \qquad \longrightarrow \quad a \in \mathbb{C}_{10}$

$a \neq 0$

α	0	-1	1	
$\psi(\alpha)$	14	14	13	10

$a \neq 0$

α	0	1	-1	
$\varphi(\alpha)$	88	65	63	55

$a \neq 0$

α	1	0	2	
$\varphi^0(\alpha)$	13	8	8	4

• $\mathcal{G}'_{15,6}(a),\ a \neq 0, -1, \qquad \longrightarrow \quad a \in \mathbb{C}^*_{20}$

$$\mathcal{G}'_{15,6}(a) \cong \mathcal{G}'_{15,6}\left(\frac{1}{a}\right) \cong \mathcal{G}'_{15,6}(-a-1) \cong \mathcal{G}'_{15,6}\left(\frac{-1}{a+1}\right) \cong \mathcal{G}'_{15,6}\left(\frac{-a}{a+1}\right) \cong \mathcal{G}'_{15,6}\left(\frac{a+1}{-a}\right)$$

$a = -\frac{1}{2}, -2, 1$

α	1	$-\frac{1}{2}$	-2	
$\psi(\alpha)$	17	15	15	13

$a = -\frac{1\pm\sqrt{3}i}{2}$

α	$-\frac{1+\sqrt{3}i}{2}$	$-\frac{1-\sqrt{3}i}{2}$	1	
$\psi(\alpha)$	16	16	15	13

$a \neq 0, \pm 1, -\frac{1}{2}, -2, -\frac{1\pm\sqrt{3}i}{2}$

α	1	a	$\frac{1}{a}$	$-1-a$	$-\frac{1}{1+a}$	$-\frac{a}{1+a}$	$-\frac{1+a}{a}$	
$\psi(\alpha)$	15	14	14	14	14	14	14	13

$a = -\frac{1}{2}, -2, 1$

α	0	1	
$\varphi(\alpha)$	84	59	56

$a = -\frac{1}{2} \pm \frac{\sqrt{6}}{4}, -5 \pm 2\sqrt{6}, 4 \pm 2\sqrt{6}$

α	0	1	5	
$\varphi(\alpha)$	77	59	57	56

$a \neq 0, \pm 1, \ldots$

α	0	1	
$\varphi(\alpha)$	77	59	56

$a = 1 \pm \sqrt{3}, -\frac{1\pm\sqrt{3}}{2}, -2 \pm \sqrt{3}$

α	1	
$\varphi^0(\alpha)$	13	12

$a \neq 0, -1, 1 \pm \sqrt{3}, -\frac{1\pm\sqrt{3}}{2}, -2 \pm \sqrt{3}$

α	
$\varphi^0(\alpha)$	12

C.3 Cartan graded $\mathrm{sl}(3,\mathbb{C})$

- $\mathcal{T}'_{13,3}(a),\ a\neq 0,\qquad \mathcal{T}'_{13,3}(a)\cong\mathcal{T}'_{13,3}(1/a)\qquad\longrightarrow\quad a\in\mathbb{C}^*_{10}$

$a=-1$

α	1	
$\psi(\alpha)$	6	3

$a=1$

α	-1	1	
$\psi(\alpha)$	5	4	3

$a\neq 0,\pm 1$

α	1	$-a$	$-\frac{1}{a}$	
$\psi(\alpha)$	4	4	4	3

$a=1$

α	0	
$\varphi(\alpha)$	9	7

$a\neq 0,1$

α	
$\varphi(\alpha)$	6

$a=-1$

α	2	
$\varphi^0(\alpha)$	6	0

$a=1$

α	0	2	
$\varphi^0(\alpha)$	2	2	0

$a\neq 0,\pm 1$

α	2	$1-a$	$1-\frac{1}{a}$	
$\varphi^0(\alpha)$	2	1	1	0

- $\mathcal{T}'_{10,3}(a),\ a\neq 0\qquad\longrightarrow\quad a\in\mathbb{C}^*_{0}$

$a=-1$

α	1	2	
$\psi(\alpha)$	8	3	2

$a=-2$

α	1	2	
$\psi(\alpha)$	7	4	2

$a\neq 0,-1,-2$

α	1	$-a$	2	
$\psi(\alpha)$	7	3	3	2

$a=1$

α	0	1	-1	
$\varphi(\alpha)$	36	32	21	19

$a=-1$

α	1	0	
$\varphi(\alpha)$	32	30	19

$a\neq 0,\pm 1$

α	1	0	$-a$	$-\frac{1}{a}$	
$\varphi(\alpha)$	31	30	20	20	19

$a=-1$

α	2	1	
$\varphi^0(\alpha)$	4	2	0

$a\neq 0,-1$

α	2	1	$1-\frac{1}{a}$	
$\varphi^0(\alpha)$	2	2	1	0

- $\mathcal{T}'_{10,2}(a),\ a\neq 0\qquad\longrightarrow\quad a\in\mathbb{C}^*_{0}$

$a=-1$

α	1	2	
$\psi(\alpha)$	8	3	2

$a=-\frac{1}{2}$

α	1	2	
$\psi(\alpha)$	7	4	2

$a\neq 0,-1,-\frac{1}{2}$

α	1	2	$-\frac{1}{a}$	
$\psi(\alpha)$	7	3	3	2

$a=1$

α	1	0	-1	
$\varphi(\alpha)$	31	30	22	17

$a=-1$

α	1	0	
$\varphi(\alpha)$	32	30	17

$a\neq 0,\pm 1$

α	1	0	$-a$	
$\varphi(\alpha)$	31	30	20	17

$a=-1$

α	2	1	
$\varphi^0(\alpha)$	3	2	0

$a \neq 0, -1$

α	2	1	$1-a$	
$\varphi^0(\alpha)$	2	2	1	0

- $\mathcal{T}_{7,2}(a),\ a \neq 0, \qquad \mathcal{T}_{7,2}(a) \cong \mathcal{T}_{7,2}\left(\frac{1}{a}\right) \qquad \longrightarrow \quad a \in \mathbb{C}^*_{10}$

$a \neq 0$

α	1	2	
$\psi(\alpha)$	10	3	2

$a=1$

α	0	1	-1	
$\varphi(\alpha)$	72	60	30	26

$a=-1$

α	1	0	
$\varphi(\alpha)$	58	56	26

$a \neq 0, \pm 1$

α	1	0	$-a$	$-\frac{1}{a}$	
$\varphi(\alpha)$	58	56	28	28	26

$a \neq 0$

α	1	2	
$\varphi^0(\alpha)$	2	2	0

- $\mathcal{T}_{7,1}(a),\ a \neq 0 \qquad \longrightarrow \quad a \in \mathbb{C}^*_{0}$

$a=1$

α	1	-1	2	
$\psi(\alpha)$	10	6	4	3

$a=-1$

α	1	-1	2	
$\psi(\alpha)$	11	5	4	3

$a=-2$

α	1	-1	2	
$\psi(\alpha)$	10	5	5	3

$a \neq 0, \pm 1, -2$

α	1	-1	-a	2	
$\psi(\alpha)$	10	5	4	4	3

$a=1$

α	0	1	-1	
$\varphi(\alpha)$	72	60	39	29

$a=-1$

α	0	1	-1	2	
$\varphi(\alpha)$	64	60	39	30	29

$a=\frac{1}{2}$

α	0	1	-1	2	-2	
$\varphi(\alpha)$	64	58	39	30	30	29

$a=2$

α	0	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	
$\varphi(\alpha)$	64	58	38	31	30	29

$a \neq 0, \pm 1, 2, \frac{1}{2}$

α	0	1	-1	$\frac{1}{a}$	$\frac{-1}{a}$	$1-\frac{1}{a}$	
$\varphi(\alpha)$	64	58	38	30	30	30	29

$a=1$

α	0	1	2	
$\varphi^0(\alpha)$	3	3	2	0

$a=-1$

α	2	1	0	
$\varphi^0(\alpha)$	4	3	2	0

$a \neq 0, \pm 1$

α	1	2	0	$1-\frac{1}{a}$	
$\varphi^0(\alpha)$	3	2	2	1	0

C.4 Γ_4-graded $\mathrm{sl}(3,\mathbb{C})$

- $\mathcal{L}^{(1)}_{14,121}(a) = \mathcal{A}_{5,7}^{(\frac{1}{\sqrt{a}},\frac{-1}{\sqrt{a}},-1)}$, $a \neq 0$, $\qquad \mathcal{L}^{(1)}_{14,121}(a) = \mathcal{L}^{(1)}_{14,121}(\frac{1}{a}) \qquad \longrightarrow \quad a \in \mathbb{C}^*_0$

$a = 1$

α	-1	1	
$\psi(\alpha)$	13	12	5

$a = -1$

α	-1	i	$-i$	1	
$\psi(\alpha)$	9	9	9	8	5

$a \neq 0, \pm 1$

α	-1	1	$\sqrt{a}$	$-\sqrt{a}$	$\frac{1}{\sqrt{a}}$	$\frac{-1}{\sqrt{a}}$	
$\psi(\alpha)$	9	8	7	7	7	7	5

$a = 1$

α	0	-2	
$\varphi(\alpha)$	40	24	20

$a = -1$

α	0	$-1-i$	$-1+i$	
$\varphi(\alpha)$	30	24	24	20

$a \neq 0, \pm 1$

α	0	$\sqrt{a}-1$	$-\sqrt{a}-1$	$\frac{1}{\sqrt{a}}-1$	$\frac{-1}{\sqrt{a}}-1$	
$\varphi(\alpha)$	30	22	22	22	22	20

$a = 1$

α	2	0	
$\varphi^0(\alpha)$	12	8	0

$a = -1$

α	0	2	$1+i$	$1-i$	
$\varphi^0(\alpha)$	4	4	4	4	0

$a \neq 0, \pm 1$

α	0	2	$1+\sqrt{a}$	$1-\sqrt{a}$	$1+\frac{1}{\sqrt{a}}$	$1-\frac{1}{\sqrt{a}}$	
$\varphi^0(\alpha)$	4	4	2	2	2	2	0

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