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F4

Faculty of Nuclear Sciences and Physical Engineering
Department of Physics

Classification of realizations of low-dimensional Lie algebras

Daniel Gromada
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Supervisor: Doc. Ing. Severin Pošta, PhD.

ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE
FAKULTA JADERNÁ A FYZIKÁLNĚ INŽENÝRSKÁ
PRAHA 1 - STARÉ MĚSTO, BŘEHOVÁ 7 - PSČ 115 19

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Název práce: (anglicky) Classification of realizations of low dimensional Lie algebras

Pokyny pro vypracování:

1. Z [1,4] nastudujte metodu konstrukce neekvivalentních realizací Lieovy algebry na základě klasifikace jejích podalgeber. Formulujte, jakou klasifikační úlohu lze touto metodou vyřešit. Ověřte funkčnost metody na výsledcích z [2].
2. Diskutujte různé druhy ekvivalence realizací (ekvivalence až na změnu souřadnic, ekvivalence vůči grupě automorfismů) a ozřejměte, v čem se jednotlivé přístupy ke klasifikaci realizací liší a kde je výhodné je použít.
3. Zpracujte klasifikaci vůči změně souřadnic pro algebry dimenze 5. Na vybranou třídu se pokuste aplikovat i ekvivalenci vůči zvolené grupě automorfismů. Při klasifikaci využijte výsledky článku [3].

Doporučená literatura:

[1] A. A. Magazev, V. V. Mikheyev and I. V. Shirokov: Computation of composition functions and invariant vector fields in terms of structure constants of associated Lie algebras, SIGMA 11, 06617 (2015)

[2] R. O. Popovych, V. M. Boyko, M. O. Nesterenko and M. W. Lutfullin: Realizations of real lowdimensional Lie algebras, J. Phys. A 36, 7337 (2003)

[3] D. J. Fisher, R. J. Gray and P. E. Hydon: Automorphisms of real Lie algebras of dimension five or less, J. Phys. A 46, 225204 (2013)

[4] M. Nesterenko, S. Pošta, O. Vaneeva: Realizations of Galilei algebras, J. Phys. A 49, 115203 (2016)

Jméno a pracoviště vedoucího diplomové práce:

doc. Ing. Severin Pošta, Ph.D., Katedra matematiky, Fakulta jaderná a fyzikálně inženýrská, ČVUT v Praze

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ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE Fakulta jaderná a fyzikálně inženýrská -1-

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/ Declaration

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

Nemám závažný důvod proti použití tohoto školního díla ve smyslu § 60 Zákona č. 121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

V Nymburce dne 4. 5. 2017

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Abstrakt / Abstract

Diplomová práce se zabývá metodami klasifikace realizací Lieových algeber pomocí vektorových polí. Práce nejprve shrnuje základy Lieovy teorie, poté popisuje vztah mezi klasifikací podalgeber a tranzitivních lokálních relaizací a prezentuje metodu I. V. Širokova pro konstrukci explicitního tvaru těchto realizací. Tento vztah a výpočetní metoda jsou zobecněny na případ regulárních realizací. Dále je rigorózně formulována rozumná klasifikační úloha pro obecné realizace a představen algoritmus na řešení této úlohy. Tento algoritmus je nakonec využit ke klasifikaci realizací nerozložitelných nilpotentních Lieových algeber dimenze pět. Cenným meziproduktem tohoto výpočtu je rovněž klasifikace podalgeber vzhledem ke grupám vnitřních automorfismů a všech automorfismů.

Klíčová slova: Lieovy algebry, realizace, vektorová pole, podalgebry

Překlad titulu: Klasifikace realizací Lieových algeber nízké dimenze

The thesis studies methods of classification of Lie algebra realizations by vector fields. After summarizing basics of the Lie theory, the correspondence between classification of subalgebras and transitive local realizations is described and a method of explicit construction due to I. V. Shirokov et al. is presented. This correspondence and method of computation is generalized to the case of regular local realizations. A reasonable classification problem for general realizations is rigorously formulated and an algorithm for construction of such classification is presented. This algorithm is used to classify realizations of five-dimensional nilpotent indecomposable Lie algebras. A valuable byproduct of this computation is also a classification of subalgebras with respect to group of inner automorphisms and group of all automorphisms.

Keywords: Lie algebras, realizations, vector fields, subalgebras

/ Contents

Chapter 1
Introduction

In the second half of the nineteenth century a Norwegian mathematician Sophus Lie started to investigate *continuous groups of transformations* similar to the Abelian theory of solving algebraic equations. He found out that all the classical methods of solving differential equations are a special case of a procedure based on invariance of the differential equation with respect to some group of symmetries [21–22].

Lie's work created a new branch of mathematics that is still a subject of contemporary research. His concept of continuous groups of transformations was generalized to the abstract notion of so called *Lie group*. One of the fundamental results stating that the group of transformations can be described by so called infinitesimal transformations gave rise to the abstract structure of a *Lie algebra*. Nowadays, the theory of Lie groups and Lie algebras has a wide range of applications not only in the theory of differential equations. Besides applications in mathematics it plays central role in the modern physical theories describing the fundamental symmetries of spacetime that every physical theory should respect.

One kind of problems that are studied in the theory of Lie groups and Lie algebras are classification problems. One can, for example, try to find list of all possible Lie groups or algebras, or to classify possible *representations* of the abstract structure by linear operators on a vector space. The aim of this work is to classify *realizations* of Lie algebras by vector fields.

On one hand, a realization by vector fields is just a special type of a Lie algebra representation mapping the Lie algebra elements on a special type of linear operators in the algebra of smooth functions on some manifold. On the other hand, this task relates to the Lie's original concept of local (Lie) groups of transformations since classification of Lie algebra realizations means classification of possible Lie algebras of infinitesimal generators of the transformation groups that realize the abstract Lie algebra structures.

Hence, it is not a surprise that this problem was for the first time considered by Lie himself [5] and that it finds a wide range of applications in the theory of differential equations [21–22] and many other areas of mathematics and physics (see, for example, [26] and references therein).

There are several possible approaches to formulate the classification problem. One possibility is to fix a manifold M and then classify all possible subalgebras of $\operatorname{Vect} M$. Two-dimensional Lie algebras of vector fields on a circle were classified by Spichak [29]. However, in case of more complicated manifolds this problem would be very hard to solve.

Much more reasonable approach is to consider realizations only locally and ignore the global structure of the manifold. Therefore, we can try to fix a number of variables m and classify all Lie algebras of (local) vector fields in m variables. This problem was solved by Lie for realizations in one and two complex and real variables [5]. More recent discussion on this topic is available, for example, in [8].

However, we will focus on another type of problem. Given a Lie algebra, we are going to find all its realizations. Number of publications was devoted to this problem as well. One of the most general result was obtained by Popovych et al. in [26], where realizations of all Lie algebras of dimension less or equal to four were classified.

Our goal was to continue in this work an classify realizations of some Lie algebras of dimension five. Besides that, there are also several theoretical questions our work deals with. First of all, the classification problem in [26] is not described in much detail. It is, therefore, necessary to describe, what the local classification of realizations of a given Lie algebra mean. Secondly, the realizations in [26] were constructed directly by solving the corresponding partial differential equations. However, very easy algorithm for construction of *transitive* realizations were introduced by Shirokov et al. [15]. We are going to try to generalize this algorithm to be able to use it for our classification.

In Chapter 2 we describe the basic ideas of the Lie theory. The basics of the Lie group theory are summarized very briefly and the we focus on the theory of local Lie groups and their action on manifolds.

In Chapter 3 we summarize methods and known results regarding properties and classification of Lie algebras that are needed for performing the method of classification of Lie algebra realizations.

The main theoretical results of this work are described in Chapter 4. Firstly, we bring formal definition of realizations and their equivalence and describe the correspondence between realizations of Lie algebras and actions of Lie groups. Then describe the well known correspondence between classification of subalgebras of a given Lie algebra and classification of its transitive realizations [1, 4, 12, 15]. Our main result is generalization of this correspondence to the case of regular realizations and formulating a classification algorithm based on this result.

These theoretical results were applied to classify realizations of five-dimensional Lie algebras. So far, we were able to classify realizations of all nilpotent indecomposable five-dimensional Lie algebras. Such computation involves obtaining results of other classification problems, which are interesting by themselves. Firstly, the computation of groups of inner automorphisms and all automorphisms (which is not our original result, see [6]), and secondly classification of subalgebras of the considered Lie algebras with respect to the groups of inner and all automorphisms. The resulting classification including those interim results are listed in the Appendix A.

Chapter 2
Lie group action and vector fields

In this chapter the basic theory closely connected to the topic of the thesis is described and several important more recent results in this area are presented. The basic concept are groups of transformations that were firstly studied by Sophus Lie, who laid the foundations of the theory in the end of nineteenth century in his work [14]. In particular, we focus on the theory of Lie algebras of vector fields, Frobenius theorem, and Lie group action on a manifold. This chapter is based on [27, 9]. Most of the theory connected to this topic is summarized in [21]; unfortunately, a lot of the basic theorems are left without proof here. The author have already touched the topic of realizations by vector fields in his research project [10] and described the foundations of Lie theory more deeply there. Some parts of the text are, therefore, adopted from the research project.

Firstly, we would like to clarify some notions and notation. In the whole thesis, *smooth* means infinitely differentiable. By a *manifold* we mean a smooth manifold, that is, a topological manifold such that all transition functions are smooth. The algebra of all smooth functions defined on a manifold M is denoted $C^\infty(M)$. By a *vector field* we mean a (smooth) section of a tangent bundle or, equivalently, a derivation of $C^\infty(M)$. The Lie algebra of vector fields on M is denoted $\operatorname{Vect} M$.

Taking a smooth map $\Phi\colon M \to N$, we define a *derivative* at $p \in M$ as a linear map $\mathrm{d}\Phi_p\colon T_pM \to T_{\Phi(p)}N$. The index p is usually omitted, so we have $\mathrm{d}\Phi\colon TM \to TN$. For Φ a diffeomorphism, the pushforward of vector fields is denoted $\Phi_*\colon \operatorname{Vect} M \to \operatorname{Vect} N$. For a curve $\gamma\colon \mathbb{R} \to M$, the tangent vector at $t \in \mathbb{R}$ is denoted $\dot\gamma(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}s}\Big|_{s=t}$. The group of all diffeomorphisms $M \to M$ is denoted $\operatorname{Diff} M$.

For a (Lie) group G we denote the left action of left multiplication as $L_g h = gh$ and the right action of right multiplication $R_g h = hg$. The left action of conjugation will be denoted $C_g h = ghg^{-1}$. For a subgroup H of G, we denote $G/H = \{gH\}_{g\in G}$ the *left coset space* and $H \setminus G = \{Hg\}_{g\in G}$ the *right coset space*. The left multiplication on G/H and right multiplication on $H \setminus G$ is again denoted L_g and R_g respectively.

As the title of this chapter and the first paragraph indicate, we are going to describe the correspondence between vector fields and Lie group action. Establishing connection between continuous groups of transformations and vector fields describing the direction of "infinitesimal transformations" was the basic idea of the Lie theory. Generalization of these notions led to establishing connection between abstract structures called Lie group and Lie algebra. Going back to transformation groups means examining the action of a Lie group on a manifold and its relationship with vector fields. This relationship arises as generalization of the following consideration.

Definition 2.1. Let M be a smooth manifold. A smooth map $F\colon U \to M$, $U = U^\circ \subset \mathbb{R} \times M$ such that $F(0,p)$ is defined and equal to identity for all $p \in M$, $F(t,\cdot)$ is a diffeomorphism for all $t \in \mathbb{R}$, and $F(t+s,p) = F(t,F(s,p))$ for $t,s \in \mathbb{R}$ and $p \in M$ if at least one side is defined is called a *flow*.

Definition 2.2. Let M be a smooth manifold and F a flow. For each point $p \in M$, the flow defines a curve $F_p(t) = F(t, p)$. A vector field $X \in \operatorname{Vect} M$ satisfying

$$X = \dot{F}_p(0), \quad \text{i.e.} \quad Xf(p) = (f \circ F_p)'(0) \quad \text{for all } f \in C^\infty(M) \tag{1}$$

is called an *infinitesimal generator* of the flow F. Conversely, for a given vector field $X \in \operatorname{Vect} M$ there is at every point $p \in M$ uniquely defined *integral curve* F_p satisfying $\dot{F}_p(t) = X_{F_p(t)}$. The map $F(t, p) = F_p(t)$ is called the *flow of the vector field* X and it is indeed a flow satisfying the equation above.

Those propositions have several implications. Firstly, given a vector field X, the manifold can be divided into the integral curves (if $X_p = 0$ the curve is constant, so it is actually a point) that are immersed submanifolds of dimension one or zero. Moreover, the defining condition of a flow means that it is a local group homomorphism. Local because U does not have to be the whole $\mathbb{R} \times M$. So, the flow can be understood as a one-parameter group of local diffeomorphisms or as a local action of one-parameter group $(\mathbb{R}, +)$ on the manifold M. The integral curves are then orbits of this local action. (See Sections 2.3, 2.5 for precise definitions of the local notions.)

2.1 Frobenius' theorem

In this section, we generalize the first part. We give a condition for collection of vector fields to define a foliation by immersed submanifolds.

Definition 2.3. Let M be a manifold. A function P assigning an n-dimensional subspace $P_p \subset T_pM$ to every $p \in M$ is called an *n-dimensional distribution* on M. The distribution is called *smooth* if every point $p_0 \in M$ has a neighbourhood $U \subset M$ such that there exist n linearly independent vector fields $X_1, \ldots, X_n \in \operatorname{Vect} U$ that form a basis $((X_1)_p, \ldots, (X_n)_p)$ of P_p at every point $p \in U$. These vector fields are called a *local basis* of P at p_0.

Definition 2.4. Let M be a manifold and P an n-dimensional distribution on M. An n-dimensional manifold N immersed to M by $\Phi\colon N \to M$ such that $\mathrm{d}\Phi(T_qN) = P_{\Phi(q)}$ for every $q \in N$ is called an *integral manifold* of P.

So, if an integral manifold exists, then its tangent spaces are essentially the subspaces P_p. If an integral manifold exists for every point in M, then these submanifolds form so called *foliation* of M

Definition 2.5. A smooth distribution is called *involutive* if, for every point $p \in M$ and a local basis $(X_1, \ldots, X_n) \in \operatorname{Vect} U$ in p, there are functions $f_{ij}^k \in C^\infty(M)$ such that $[X_i, X_j] = \sum_k f_{ij}^k X_k$.

It can be easily checked that the property of being involutive is independent on the chosen basis.

Theorem 2.6 (Frobenius). Let M be a manifold, P an n-dimensional involutive distribution on M and $p \in M$. Then there exists a system of coordinates $(x^1, \ldots, x^m)$ in a neighbourhood of p such that the distribution has a local basis $(\partial_{x^1}, \ldots, \partial_{x^n})$ at p. So, the manifold M is locally foliated by so called *level submanifolds* $x^i = \text{const.}$ for $i = n+1, \ldots, m$.

Proof. At first, we show that there exists a basis $(X_1, \ldots, X_n)$ of P such that $[X_i, X_j] = 0$. Let $(y^1, \ldots, y^m)$ be coordinates on M such that p lies in the origin and let $Y_i = \sum_{j=1}^m Y_i^j(y)\partial_{y^j}$ be a basis of P. Since Y_i are linearly independent, the matrix $Y_i^j(y)$ has to have n linearly independent rows. Without loss of generality, let us assume that these are the first n rows. Then we can choose a new basis (X_i) such that $X_i^j = \delta_i^j$ for

$j \le n$, so $X_i = \partial_{y^i} + \sum_{j=n+1}^{m} X_i^j(y)\partial_{y^j}$. Now, we easily see that, in the coordinate basis ∂_{y^j}, we have $0 = [X_i, X_k]^j = f_{ik}^l X_l^j = f_{ik}^j$ for $j \le n$.

Denote $\psi: U \to \mathbb{R}^m$ the coordinate chart corresponding to coordinates y^j. Now, we define the coordinates x^j. Let $F_i(t): U \to M$ be the flows of the vector fields X_i. Define the coordinate chart $\varphi: U \to \mathbb{R}^m$ as $\varphi^{-1}(x^1, \ldots, x^m) = F_i(x^1) \cdots F_i(x^m)\psi^{-1}(0, \ldots, 0, x_{n+1}, \ldots, x_m)$. The derivative of $\varphi \circ \psi^{-1}$ at zero is the identity, so it is a local diffeomorphism in a neighbourhood of zero, so φ is a well defined coordinate chart in a neighbourhood of p. Since X_i commute, the flows $\Phi_i(t)$ commute as well and using this property we can easily check that $\Phi_i(t)$ act on U as a translation in the coordinates x^i. Hence, $X_i = \partial_{x^i}$. □

At least locally, it is evident that the integral submanifolds are defined uniquely. It can be proven that, globally, there exist unique maximal integral submanifolds that foliate M.

2.2 Lie group and its Lie algebra

In this section, we briefly recall the definition of a Lie group and the construction of the corresponding Lie algebra.

Definition 2.7. A group G is called a *Lie group* if it is also a smooth manifold and both multiplication and inversion are smooth maps.

Lemma 2.8. Connected Lie group G is generated by any open set.

Proof. Let H be generated by an open set U in G. H is a union of open sets hU, so H is also open. Then $G \setminus H$ is a union of left cosets gH, $g \notin H$, which are also open. Therefore, H is also closed and hence, if G is connected, $H = G$.

Definition 2.9. A map φ of two Lie groups is called *homomorphism* if it preserves both group and manifold structures, that is, if it is a smooth group homomorphism. A bijective homomorphism whose inversion is also a homomorphism (which is not trivially satisfied since inversion of a smooth map has not to be smooth) is called a Lie group *isomorphism*. Similarly, Lie group automorphism and other morphisms can be defined.

Definition 2.10. A *one-parameter subgroup* of a Lie group G is a (smooth) homomorphism $\varphi: (\mathbb{R}, +) \to G$.

Lemma 2.11. Let G be a Lie group. A transformation $T: G \to G$ that commutes with every left translation acts as right translation $T = R_{T(e)}$.

Proof. We have $T(g) = (T \circ L_g)(e) = (L_g \circ T)(e) = gT(e) = R_{T(e)}(g)$. □

Definition 2.12. Let G be a Lie group. A vector field $X \in \operatorname{Vect} G$ is called *left-invariant* if $L_{g*}X = X$ for all $g \in G$. Analogically, X is *right-invariant* if $R_{g*}X = X$.

Lemma 2.13. Let G be a Lie group.

1. Left invariant vector fields form a Lie subalgebra of $\operatorname{Vect} G$.
2. Any left-invariant vector field $X \in \operatorname{Vect} G$ is determined by its value at unity X_e by relation $X_g = \mathrm{d}L_g\, X_e$ for $g \in G$.
3. For any tangent vector $a \in T_eG$, there is a unique one-parameter subgroup φ_a of G such that $\dot{\varphi}_a(0) = a$. The corresponding left-invariant vector field $X_g = \mathrm{d}L_g\, a$ is the infinitesimal generator of a one-parameter group F, $F_t = R_{\varphi_a(t)}$.

Proof. The first proposition is clear from the homomorphism property of pushforward L_{g*}. The second proposition follows directly from the definition of left-invariance

$$X_g = (L_{g*}X)_g = \mathrm{d}L_g\, X_{L_g^{-1}(g)} = \mathrm{d}L_g\, X_e$$

Finally, let F be the flow of the left-invariant vector field X corresponding to a tangent vector a. The subgroup φ_a has to be an integral curve of X, i.e., $\varphi(t) = F(t, e)$, since

$$\dot{\varphi}_a(t) = \frac{\mathrm{d}}{\mathrm{d}s}(\varphi_a(t)\varphi_a(s))\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}(L_{\varphi_a(t)}\varphi_a(s))\Big|_{s=0} = \\ = \mathrm{d}L_{\varphi_a(t)}\,\dot{\varphi}_a(0) = \mathrm{d}L_{\varphi_a(t)}\,a = X_{\varphi_a(t)}.$$

Hence, it is uniquely defined and obeys the group property. Since $X = L_{g*}X$, we have $F_t = L_g \circ F_t \circ L_g^{-1}$ for the flow, so it commutes with left translation and hence, from Lemma 2.11, is right translation. □

Definition 2.14. Let G be a Lie group. According to Lemma 2.13.2 there is an isomorphism between the Lie algebra of left-invariant vector fields and the tangent space T_eG. This isomorphism defines a Lie bracket on T_eG as $[a, b] = [X, Y]_e$, where X, Y are the left-invariant vector fields corresponding to $a, b \in T_eG$. The Lie algebra $(T_eG, [\cdot, \cdot])$ is called the *Lie algebra corresponding to the Lie group* G and denoted $\mathfrak{g}$. The one-parameter subgroup $\varphi_a \colon \mathbb{R} \to G$ from 2.13.3 corresponding to $a \in \mathfrak{g}$ is called the *exponential* and denoted $\varphi_a(t) = \exp(ta) = \mathrm{e}^{ta}$. It defines a map $\exp\colon \mathfrak{g} \to G$ $a \mapsto \exp(a)$ called the *exponential map*.

Remark 2.15. Since $\exp(0) = e$ and the differential of exp at 0 is the identity, the exponential map is a local diffeomorphism, mapping a neighbourhood of zero vector onto a neighbourhood of unity.

From the commutativity of left and right translations, we have for invariant vector field $X \in \operatorname{Vect} G$ the following relation $L_{g*}R_{g^{-1}*}X = R_{g^{-1}*}L_{g*}X = R_{g^{-1}*}X$, so $C_{g*}X = R_{g^{-1}*}X$ and it is left invariant. Thus it defines a map on the Lie algebra $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ called the *adjoint map* or *adjoint representation* (being indeed a representation of G on the vector space $\mathfrak{g}$).

Lemma 2.16. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then, for all $g \in G$ and $a \in \mathfrak{g}$, we have

$$\mathrm{Ad}_g\, a = \mathrm{d}C_g\, a, \tag{2}$$

$$g\mathrm{e}^a g^{-1} = \mathrm{e}^{\mathrm{Ad}_g a}. \tag{3}$$

Proof. Let X be the corresponding left-invariant vector field. Then

$$\mathrm{Ad}_g\, a = (\mathrm{Ad}_g X)_e = (C_{g*}X)_e = \mathrm{d}C_g\, X_{C_g(e)} = \mathrm{d}C_g\, X_e = \mathrm{d}C_g\, a.$$

According to Lemma 2.13 $\mathrm{e}^a = F_1(e)$, where F_t is the flow of the vector field X, so the flow of $\mathrm{Ad}_g X = C_{g*}X$ is $C_g \circ F_t \circ C_g^{-1}$, so

$$\mathrm{e}^{\mathrm{Ad}_g a} = (C_g \circ F_1 \circ C_g^{-1})(e) = C_g(F_1(e)) = g\mathrm{e}^a g^{-1}.$$

□

Theorem 2.17. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, $a \in \mathfrak{g}$. The linear map $\mathrm{ad}_a \colon \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_a\, b = [a, b]$ is the infinitesimal generator of the one-parameter group $\mathrm{Ad}_{\mathrm{e}^{ta}} \colon \mathfrak{g} \to \mathfrak{g}$. Thus, the following equations hold

$$\mathrm{Ad}_{\mathrm{e}^a} = \mathrm{e}^{\mathrm{ad}_a}, \tag{4}$$

$$\mathrm{e}^a\mathrm{e}^b\mathrm{e}^{-a} = \exp(\mathrm{Ad}_{\mathrm{e}^a} b) = \exp(\mathrm{e}^{\mathrm{ad}_a} b). \tag{5}$$

Proof. Let us take $a, b \in \mathfrak{g}$ and X, Y the corresponding left-invariant vector fields. Since, according to Lemma 2.13, the flow F corresponding to the left-invariant vector field X acts as right translation $F_t = R_{e^{ta}}$, we can compute the Lie bracket $[X, Y]$ as a Lie derivative of Y along X, so

$$[X,Y] = \left.\frac{\mathrm{d}R_{e^{ta}*}Y}{\mathrm{d}t}\right|_{t=0}$$

Using this relation for $Y(t) := R_{e^{ta}*}Y = \mathrm{Ad}_{e^{ta}}\,Y$ and defining $b(t) = Y(t)_e$, we have

$$\mathrm{ad}_a\,b(t) = [a, b(t)] = [X, Y(t)]_e = \frac{\mathrm{d}Y(t)_e}{\mathrm{d}t} = b'(t).$$

This is a differential equation for $b(t)$ that has a unique solution $b(t) = e^{t\,\mathrm{ad}_a}b$.

The second equation follows from equation (3) in the previous lemma. □

Remark 2.18. A special case of a Lie group is the *general linear* Lie group $\mathrm{GL}(V)$ of all invertible linear operators on a given vector space V. Its Lie algebra is $\mathrm{gl}(V)$ the *general linear* Lie algebra consisting of all operators of V. Indeed, a derivative of a parametrized operator is of course an operator; conversely, any linear operator L is a derivative of a curve $\gamma(t) = I + tL$ in $\mathrm{GL}(V)$. The exponential defined in the theory of Lie groups corresponds to the operator exponential since it satisfies the same differential equation $\frac{\mathrm{d}e^{ta}}{\mathrm{d}t} = ae^{ta}$. Finally, we can compute the Lie bracket for $a, b \in \mathrm{gl}(V)$

$$\begin{aligned}[a,b] = \mathrm{ad}_a\,b &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{e^{ta}}\,b\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}C_{e^{ta}}\,b\right|_{t=0} = \\ &= \left.\frac{\partial^2}{\partial t\partial s}e^{ta}e^{sb}e^{-ta}\right|_{t,s=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}e^{ta}be^{-ta}\right|_{t=0} = ab - ba.\end{aligned}$$

The general linear Lie group and Lie algebra are particularly important thanks to the theorem of Ado.

Theorem 2.19 (Ado)**.** Every finite-dimensional Lie algebra $\mathfrak{g}$ can be embedded in matrices. That is, for every finite-dimensional Lie algebra $\mathfrak{g}$ over a field F of characteristic zero, there exists $n \in \mathbb{N}$ and a monomorphism (injective homomorphism) $\varphi\colon \mathfrak{g} \to \mathrm{gl}(n, F)$.

Definition 2.20. A Lie subgroup of the general linear group is called a *linear group*.

Remark 2.21. Another special case of a Lie group is the group of all diffeomorphisms $\mathrm{Diff}\,M$ of a given manifold M. Since a derivative of a one-parameter group of diffeomorphisms is a vector field, we can see that the Lie algebra corresponding to $\mathrm{Diff}\,M$ is the Lie algebra of all vector fields $\mathrm{Vect}\,M$ on M. (The group of all diffeomorphisms is isomorphic to the group of the corresponding pullbacks acting on functions, which is a linear group. Their derivatives act on functions as vector fields and from Remark 2.18 we see that the Lie bracket is indeed defined as a commutator.)

2.3 Local Lie groups

Definition 2.22. Let M be a manifold, U a domain in M, e a point in U, V a neighbourhood of e and $m\colon V \times V \to U$ a smooth map called and denoted as *multiplication* satisfying $ex = xe = x$ for all $x \in V$, $(xy)z = x(yz)$ for all $x, y, z, xy, yz \in V$ and that the local *inversion* $i\colon V \to V$ $x \mapsto x^{-1}$ defined by the relation $xx^{-1} = x^{-1}x = e$ is also a smooth map. Then the tuple (U, V, e, m) is called a *local Lie group*. Any local Lie group (U_1, V_1, e, m_1), such that $U_1 \subset U$, $V_1 \subset V$ and $m_1 = m|_{V_1\times V_1}$ is called a *restriction* of the original local Lie group.

Remark 2.23. There can be defined an equivalence of the local Lie groups: two Lie groups are equivalent if they have a common restriction. Since the size of the neighbourhood of a local Lie group is irrelevant, we often identify the equivalent local Lie groups and by the term *local Lie group* we mean the equivalence class. All notions concerning local Lie groups should be formulated only locally, that is, they should not depend on the choice of the representative of the equivalence class.

Definition 2.24. Let G_1 and G_2 be local Lie groups, (U_1, V_1, m_1) and (U_2, V_2, m_2) their restrictions. A smooth map $\Phi\colon U_1 \to U_2$ satisfying $\Phi(V_1) \subset V_2$, $\Phi(m_1(x, i_1(y))) = m_2(\Phi(x), i_2(\Phi(y)))$ for all $x \in V_1$, $y \in W_1$, where W_1 is the domain of the first inversion i_1, is called a *homomorphism* of the local Lie groups G_1 and G_2. Analogically, one could define local Lie group isomorphism or automorphism. Local Lie groups are called isomorphic if there exists a local Lie group isomorphism between them. A *local homomorphism* of Lie groups is a homomorphism between them taken as local Lie groups.

Definition 2.25. Let (U, V, e, m) be a restriction of a local Lie group G, W a submanifold embedded in U. If $e \in W$, $i(W \cap V) \subset W$, and $m(W \cap V, W \cap V) \subset W$, then $H = (W, W \cap V, e, m|_{(W\cap V)^2})$ forms a local Lie group and is said to be a *local Lie subgroup* of G.

Remark 2.26. In a sufficiently small neighbourhood of unity, local Lie subgroup is always closed. Indeed, if we restrict the neighbourhoods U and V such that W is a slice in U and V, then W is also closed in U and V.

Remark 2.27. Even in the case of local Lie groups it makes sense to construct left- and right-coset spaces and quotient groups since it is again a local notion. Considering a local Lie group $G = (U, V, e, m)$ and its subgroup $H = (W, V \cap W, e, m|_{(V\cap W)^2})$, we can define an equivalence $g \sim \tilde{g}$ for $g, \tilde{g} \in V$ if there exists $h_1, \ldots, h_n \in V \cap W$ such that $\tilde{g} = gh_1 \cdots h_n$. For a restriction $G_1 = (U_1, V_1, e, m_1 = m|_{V_1^2})$ of $G = (U, V, e, m)$ and a restriction of the subgroup $H_1 = (W \cap U_1, W \cap V_1, e, m|_{(W\cap V_1)^2})$, it can be shown that the equivalence classes gH, gH_1 satisfy $gH_1 = gH \cap V_1$ (thanks to the property that a connected Lie group is generated by arbitrarily small neighbourhood of unity).

From now on, we will not distinguish between classes and representatives of local Lie groups and will treat them as ordinary Lie groups keeping on mind that we should always consider only elements from “small neighbourhood of unity”.

We are going to show that Lie algebras are completely equivalent to local Lie groups. We show that for every Lie algebra there exists a unique local Lie group, that there is a one to one correspondence between morphisms of Lie algebras and local Lie groups and so on. A lot of these statements can be actually formulated for global connected or simply connected Lie groups. Moreover, every local Lie group can be uniquely extended to a simply connected global Lie group, so the notion of the local Lie group is actually redundant. Nevertheless, in the whole thesis, we work with Lie groups only locally, so the local approach in this section allow us to simplify the proofs and our future considerations.

Theorem 2.28. Let G_1, G_2 be Lie groups and $\alpha\colon G_1 \to G_2$ an homomorphism. Then $\mathrm{d}\alpha$ is a homomorphism of the corresponding Lie algebras.

Proof. Denote $\mathfrak{g}_1$ the Lie algebra corresponding to G_1. Take $a \in \mathfrak{g}_1$, then $\alpha(\mathrm{e}^{ta})$ is a one-parameter subgroup of G_2 that is generated by $\mathrm{d}\alpha(\mathrm{e}^{ta})/\mathrm{d}t|_{t=0} = \mathrm{d}\alpha\, a$, so $\alpha(\mathrm{e}^{ta}) = \mathrm{e}^{t\,\mathrm{d}\alpha\, a}$. Similarly as in remark 2.18, we can write

$$\mathrm{d}\alpha[a, b] = \frac{\partial^2}{\partial t \partial s}\alpha(\mathrm{e}^{ta}\mathrm{e}^{sb}\mathrm{e}^{-ta}) = \frac{\partial^2}{\partial t \partial s}\mathrm{e}^{t\,\mathrm{d}\alpha\, a}\mathrm{e}^{s\,\mathrm{d}\alpha\, b}\mathrm{e}^{-t\,\mathrm{d}\alpha\, a} = [\mathrm{d}\alpha\, a, \mathrm{d}\alpha\, b].$$

□

Remark 2.29. A map is locally injective or surjective in a point if and only if its differential in the point is injective or surjective. So, for example, isomorphisms of Lie groups correspond to isomorphisms of Lie algebras.

Theorem 2.30. Let G be a local Lie group and H its Lie subgroup. Then $\mathfrak{h} = T_eH = \{a \in \mathfrak{g} \mid \mathrm{e}^{ta} \in H$ for all t in some neighbourhood of zero$\} \subset \mathfrak{g}$ is the Lie algebra of H.

Proof. Denote $\mathfrak{h} = T_eH$ the Lie algebra of H as a tangent space of Lie algebra of H at unity. The identity $\iota\colon H \to G$ is a homomorphism. The tangent space $\widetilde{T_eH}$ as a subspace of T_eG is formally defined as image of T_eH by differential of the inclusion map $\mathrm{d}\iota$, which is an injective homomorphism. So, we indeed have that $\tilde{\mathfrak{h}} := \widetilde{T_eH} \subset \mathfrak{g}$ is also a Lie algebra of H. The second equation follows from the fact that homomorphic image of one-parameter subgroup is a one-parameter subgroup, so the exponential $\mathfrak{h} \to H$ coincide with restriction of the exponential $\mathfrak{g} \to G$. $\square$

We are able to construct Lie algebras of Lie groups as tangent spaces at unity and we are able to construct morphisms of Lie algebras by differentiating morphisms of Lie groups. Now, we formulate the opposite direction.

Lemma 2.31. Let G be a local Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Then there exists a Lie subgroup $H \subset G$, such that $\mathfrak{h}$ is its Lie algebra.

Proof. If we interpret $\mathfrak{h}$ as an algebra of left-invariant vector fields on G, then it is also an involutive distribution on G. Denote H the integral manifold at unity. If we show that H is a subalgebra, then it will be clear that its left-invariant vector fields are in $\mathfrak{h}$, so $\mathfrak{h}$ is its Lie algebra. Take $g \in H$, $L_{g^{-1}}$ is a diffeomorphism of G preserving the left-invariant vector field, so $L_{g^{-1}}(H)$ has to be the integral submanifold and, since $e \in L_{g^{-1}}(H)$, it is an integral submanifold at unity and from uniqueness it is equal to H on a neighbourhood of unity. So, $g^{-1}h \in H$ for all g, h in a neighbourhood of unity in H. $\square$

Lemma 2.32. Let G be a local Lie group, H its subgroup, and $\mathfrak{g}$, $\mathfrak{h}$ the corresponding Lie algebras. Then H is normal in G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Proof. The subgroup is normal if and only if for all $g, h \in G$ it holds that $h \in H \Leftrightarrow ghg^{-1} \in H$. The elements of local Lie group in the neighbourhood of unity can be uniquely represented by elements of the corresponding Lie algebra. Thus, the condition can be equivalently represented as $\mathrm{e}^b \in H \Leftrightarrow \mathrm{e}^a\mathrm{e}^b\mathrm{e}^{-a} = \exp(\mathrm{e}^{\mathrm{ad}_a} b) \in H$ for sufficiently small $a, b \in \mathfrak{g}$, where we used Theorem 2.17 to rewrite the expression. This can be expressed in terms of the Lie algebra elements themself $b \in \mathfrak{h} \Leftrightarrow \mathrm{e}^{\mathrm{ad}_a} b \in \mathfrak{h}$. Finally, it is sufficient to show that this is equivalent to the implication $b \in \mathfrak{h} \Rightarrow [a, b] \in \mathfrak{h}$. The "if" direction is trivial. To prove the opposite direction, let us assume, that $b \in \mathfrak{h}$, then

$$\mathrm{e}^{\mathrm{ad}_a} b = b + [a, b] + [a, [a, b]] + \ldots \in \mathfrak{h},$$

so, since b on the right-hand side lies in $\mathfrak{h}$, we have

$$\mathfrak{h} \ni [a, b] + [a, [a, b]] + \ldots = \mathrm{e}^{\mathrm{ad}_a}[a, b],$$

so $[a, b] \in \mathfrak{h}$. $\square$

Theorem 2.33. Let G_1 and G_2 be local Lie groups, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ its Lie algebras. For every homomorphism $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$, there exists unique local Lie group homomorphism Φ such that $\varphi = \mathrm{d}\Phi$.

Proof. Take a homomorphism $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Its graph $\mathfrak{h} := \{(a, \varphi(a)) \mid a \in \mathfrak{g}_1\}$ is a Lie subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. We can check that $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie algebra of $G_1 \times G_2$. Using Lemma 2.31, we can find a subalgebra $H \subset G_1 \times G_2$, such that $\mathfrak{h}$ is its Lie algebra.

The projection $\pi_1\colon H \to G_1$ of H to G_1 is a homomorphism, whose derivative at unity is a regular mapping $(a, \varphi(a)) \mapsto a$. Hence π_1 is a local isomorphism at unity. Denote $\pi_2\colon H \to G_2$ the second projection, then $\mathrm{d}\pi_2$ at unity maps $(a, \varphi(a)) \mapsto \varphi(a)$. So, $\pi_1^{-1} \circ \pi_2$ is the homomorphism we are looking for. □

Theorem 2.34. For a given Lie algebra $\mathfrak{g}$, there exists, up to isomorphisms, unique local Lie group G such that $\mathfrak{g}$ is its Lie algebra.

Proof. Using the theorem of Ado 2.19, we can assume that $\mathfrak{g}$ is a subalgebra of $\mathrm{gl}(V)$ for some vector space V. Using the preceding lemma, we find the local Lie group $G \subset \mathrm{GL}(V)$. The uniqueness follows from Theorem 2.33 (identical map of the Lie algebra induces an isomorphism of its different local Lie groups). □

2.4 Canonical coordinates

Since we are going to work with Lie groups only locally, it is useful to define some coordinates on Lie groups.

As we remarked in 2.15, exp is a local diffeomorphism between Lie algebra and Lie group. We can therefore locally define the inversion of exponential map—the logarithm $\ln\colon G \to \mathfrak{g}$. If we fix a basis of $\mathfrak{g}$, say $(e_1, \ldots, e_n)$, it defines us coordinates on the Lie algebra $\mathfrak{g}$, which induce *logarithmic coordinates* on G through relation $x^i = e^i \ln g$, where $(e^1, \ldots, e^n)$ is the dual basis. The element g_x with coordinates $x = (x^i)$ can be, therefore, written as

$$g_x = \exp\left(\sum_{i=1}^{n} x^i e_i\right). \tag{6}$$

One could, however, also exponentiate multiples of the basis elements at first and then take their product. Generally, consider a linear decomposition of $\mathfrak{g}$ to a direct sum of linear subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_m$. Then the map $\Phi\colon \mathfrak{g} \to G$, $a \mapsto \prod_{k=1}^{m} \exp(a_k)$, where a_k is a projection of a onto $\mathfrak{g}_k$, is a local diffeomorphism of a neighbourhood of zero onto a neighbourhood of unity too since $\Phi(0) = e$ and

$$\mathrm{d}\Phi_0\, e_i = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(te_i)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\exp(te_i)\bigg|_{t=0} = e_i,$$

where (e_i) is a basis such that each basis element belongs to one of the subspaces $\mathfrak{g}_k$, so the derivative at zero is again identity. Such map Φ is called *canonical* and the induced coordinates with coordinate functions $e^i \circ \Phi^{-1}$, where (e^i) is the dual basis to (e_i), are called *canonical coordinates.*

For a trivial decomposition $\mathfrak{g} = \mathfrak{g}_1$ we get the logarithmic coordinates, which are also called *first canonical coordinates.* On the contrary, if we have $m = n$, so each of the subspaces is one-dimensional, we get the *second canonical coordinates*

$$g_x = \prod_{i=1}^{n} \exp(x^i e_i). \tag{7}$$

Let us now look on the coordinate expression of the basic Lie group structures. Let $\psi^1, \ldots, \psi^n$ be arbitrary local coordinate functions of G in the neighbourhood of unity. Denote $g_x = \psi^{-1}(x)$ as in the preceding text. Then the basis vectors of $\operatorname{Vect} M$ module can be written as $\partial_{x^i} = \partial_i \psi^{-1} = \partial_{x^i} g_x$.

Now, we can define a coordinate expression of the multiplication $m(g,h) = gh$ as $M\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $M^i(x,y) = \psi^i(m(g_x, g_y)) = \psi^i(g_x g_y)$. This map is called the *composition function.*

We can also explicitly express the form of left-invariant vector fields on G. Let $e_1, \ldots, e_n$ be a basis of the Lie algebra $\mathfrak{g}$ of G (as an algebra of tangent vectors). Then the corresponding basis of the algebra of left-invariant vector fields has the form $(X_i)_g = \mathrm{d}L_g\, e_i$. So, the coordinate expression is

$$X_i^a(x) = \left.\frac{\partial \psi^a(L_{g_x}(g_y))}{\partial y^j}\right|_{y=0} e_i^j = \left.\frac{\partial M^a(x,y)}{\partial y^i}\right|_{y=0}. \tag{8}$$

In the rest of this subsection, we describe computation of left-invariant vector fields in the second canonical coordinates as was proposed by I. V. Shirokov in [28].

At first, let us have a look on the differential $\mathrm{d}L_g$ and compute its components at unity

$$\begin{aligned}\left[(\mathrm{d}L_{g_y})_e\right]^i_j &= \mathrm{d}x^i\, \mathrm{d}L_{g_y}\, \partial_{x^j}|_{x=0} = \mathrm{d}x^i\, \mathrm{d}L_{g_y}\, \partial_{x^i} g_x|_{x=0} = \\ &= \frac{\partial}{\partial x^j}\left(\psi^i(L_{g_y}(g_x))\right)\Big|_{x=0} = \left.\frac{\partial M^i(y,x)}{\partial x^j}\right|_{x=0} = X_j^i(y).\end{aligned} \tag{9}$$

In the following calculations, we omit the index e at $\mathrm{d}L_{g_y}$. So, the differential $\mathrm{d}L_{g_y}$ is a matrix, whose columns is formed by the left-invariant vector fields. This relation can be inverted

$$[\mathrm{d}L_{g_y^{-1}}]^j_i = [\mathrm{d}L^{-1}_{g_y}]^j_i = \omega^j_i(y), \tag{10}$$

where ω^j are the dual one-forms corresponding to left-invariant vector fields X_j. These are also called *left-invariant one-forms.*

Now, choose the second canonical coordinates such that

$$g_x = g_n(x^n)\cdots g_1(x^1) \quad \text{where} \quad g_k(t) = \mathrm{e}^{te_k}. \tag{11}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} g_k(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{te_k} = (X_k)_{\mathrm{e}^{te_k}} = \mathrm{d}L_{g_k(t)}\, e_k,$$

so

$$\begin{aligned}\partial_{x^k} &= \partial_{x^k} g_x = \partial_{x_k}\left(L_{g_n(x^n)\cdots g_{k+1}(x^{k+1})} R_{g_1(x^1)^{-1}\cdots g_{k-1}(x^{k-1})^{-1}} g_k(x^k)\right) = \\ &= \mathrm{d}L_{g_n(x^n)\cdots g_{k+1}(x^{k+1})}\, \mathrm{d}R_{g_1(x^1)^{-1}\cdots g_{k-1}(x^{k-1})^{-1}}\, \mathrm{d}L_{g_k(x^k)}\, e_k.\end{aligned} \tag{12}$$

Finally, since left translations commute with right translations, we can formulate the expression for the $\mathrm{d}L^{-1}_{g_x}$ in terms of the adjoint map using equation (2) as

$$\begin{aligned}[\mathrm{d}L^{-1}_{g_x}]^j_i &= \mathrm{d}x^j\, \mathrm{d}L_{g_x^{-1}} \partial_{x^i} = \mathrm{d}x^j\ \mathrm{Ad}_{g_1(x^1)^{-1}} \cdots \mathrm{Ad}_{g_{i-1}(x^{i-1})^{-1}}\, e_i = \\ &= \left[\exp(-x^1\, \mathrm{ad}_{e_1}) \cdots \exp(-x^{i-1}\, \mathrm{ad}_{e_{i-1}})\right]^j_i.\end{aligned} \tag{13}$$

So, in order to calculate the coordinates of left-invariant vector fields, it is sufficient to calculate the inversion of the matrix whose elements is given by the formula (13).

2.5 Lie group action on a manifold

Let G be a Lie group and M a manifold. Although an action of a group is usually considered as a left action, in this case it is convenient to consider right actions. We will also always assume smoothness of the action. So, by an *action* of G on M, we always mean a smooth right action $\pi\colon M \times G \to M$. A manifold equipped with an action of a Lie group G is called a *G-manifold.*

We recall the definition of action morphisms.

Definition 2.35. Let G_1 and G_2 be Lie groups, π_1 and π_2 their actions on manifolds M_1 and M_2. A smooth map $\Phi\colon M_1 \to M_2$ is called a *morphism* if there exists a homomorphism $\varphi\colon G_1 \to G_2$, such that

$$\Phi(\pi_1(p, g)) = \pi_2(\Phi(p), \varphi(g)). \tag{14}$$

If Φ is bijective, it is called a *similitude*. It is called an *isomorphism* if, in addition, $G_1 = G_2$ and φ is identity.

Lemma 2.36. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, M a manifold, and π an action of G on M. Then the mapping $\mathfrak{g} \to \operatorname{Vect} M$

$$a = X_e \mapsto \hat{X}, \quad \hat{X}_p = \mathrm{d}\pi_p\, a = \frac{\mathrm{d}}{\mathrm{d}t}(\pi_p \mathrm{e}^{-ta})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(p \cdot \mathrm{e}^{-ta})\bigg|_{t=0}, \tag{15}$$

where X is the left-invariant vector field corresponding to a tangent vector $a \in \mathfrak{g} = T_e G$ and π_p is the orbit map of point p, is a homomorphism.

Proof. Let us have $a, b \in \mathfrak{g}$, X, Y the corresponding left-invariant vector fields. Then

$$\mathrm{d}\pi_p\, X_g = \mathrm{d}\pi_p\, \mathrm{d}L_g\, \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{ta}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} pg\, \mathrm{e}^{ta}\bigg|_{t=0} = \mathrm{d}\pi_{pg}\, a = \hat{X}_{pg} = \hat{X}_{\pi_p(g)}.$$

Similarly, $\hat{Y}_{\pi_p(g)} = \mathrm{d}\,\pi_p Y_g$, so $[\hat{X}, \hat{Y}]_{\pi_p(g)} = \mathrm{d}\pi_p\, [X, Y]_g$. Therefore,

$$[\hat{X}, \hat{Y}]_p = [\hat{X}, \hat{Y}]_{\pi_p(e)} = \mathrm{d}\pi_p\, [X, Y]_e = \mathrm{d}\pi_p\, [a, b] = [\widehat{X, Y}]_p.$$

□

Definition 2.37. The vector field $\hat{X}$ of the previous lemma is called the *fundamental vector field* of the action π corresponding to the vector field X. The homomorphism $X \mapsto \hat{X}$ will be denoted as π_*.

For the map $g \mapsto \pi_g$, where $\pi_g(p) = \pi(p, g)$, we have $\pi_G \subset \operatorname{Diff} M$ because we demand the action π to be smooth. Therefore, the action of a group on a manifold can be understood as a generalization of the notion of one-parameter group of diffeomorphisms, i.e., a global flow. However, it is not one-parameter anymore. To complete the analogy, we bring the following definition corresponding to the notion of a local flow.

Definition 2.38. Let G be a Lie group and M a manifold. By a *local action* of G on M we mean a smooth map $\pi\colon W \to M$, where W is an open subset of $G \times M$ such that $\{e\} \times M \subset W$ satisfying $\pi(e, p) = p$ for all $p \in M$ and $\pi(g, \pi(h, p)) = \pi(gh, p)$ for all g and h for which both sides are well-defined.

If we consider an action of a local Lie group, we automatically mean a local action. In the case of local Lie groups, one also need to consider locally the properties of the action. For example, an action π of a local Lie group G is called *(locally) transitive* if for every point $p \in M$ there is a neighbourhood U of p such that for all $q \in U$ there exists $g \in G$ such that $q = \pi(p, g)$. Formally, if we understand local Lie groups as equivalence classes, we should define the actions of local Lie groups as equivalence classes as well. Two local actions of a global Lie group G on a manifold M are equivalent if there exists a local action of G on M that is their common restriction.

It can be shown that a stabilizer of a given point for a local action of a (local) Lie group G is a local subgroup of G. The action is free if the stabilizer is trivial. Note that in the case of local action of a global Lie group the stabilizer is again only a local subgroup, not global.

It seems natural that a local action is described by the *infinitesimal action*, that is, the differential $\mathrm{d}\pi_p$ for every $p \in M$. The role of the *infinitesimal generators* is played by the images of Lie algebra $\mathfrak{g}$ of G through the mapping $\mathrm{d}\pi_p$—the fundamental vector fields.

Theorem 2.39. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, M a smooth manifold, and $\varphi\colon \mathfrak{g} \to \operatorname{Vect} M$ a homomorphism. Then there exists unique local action π of G on M such that $\varphi = \pi_*$. This action is effective if and only if $\varphi = \pi_*$ is an isomorphism.

Proof. According to Remark 2.21 the Lie group corresponding to $\operatorname{Vect} M$ is $\operatorname{Diff} M$. According to Theorem 2.33, Lie algebra homomorphism φ defines a unique homomorphism of local Lie groups $\pi\colon G \to \operatorname{Diff} M$, which is an isomorphism if and only if φ is an isomorphism. ☐

It is again useful to be able to express these structures in coordinates. Let us have local coordinates of a given Lie group G in a neighbourhood of unity $x^1, \ldots, x^n$ and local coordinates of a given manifold M $q^1, \ldots, q^m$. Then, for arbitrary action π of G on M, we can define its coordinate representation as a function $\Pi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Let $\mathfrak{g}$ be a Lie algebra of G and $e_1, \ldots, e_n$ its basis. Then the algebra of fundamental vector fields is generated by $(\hat{X}_i)_p = \mathrm{d}\pi_p\, e_i$. The coordinates of these vector fields are

$$\hat{X}_i^a(q) = \left.\frac{\partial \Pi^a(q,x)}{\partial x^i}\right|_{x=0}. \tag{16}$$

Given a group G acting on a set X and a point $x \in X$, the stabilizer G_x of x is a subgroup of G. The space of right cosets $G_x \setminus G$ with right multiplication of G is isomorphic to the orbit xG of x through map $G_x g \mapsto xg$. This can be generalized to the case of Lie groups.

Given a smooth action of a Lie group G on a manifold M, the right coset space $G_p \setminus G$ for $p \in M$ can be given a structure of a manifold through the quotient map $g \mapsto G_p g$. It holds that there exists a unique smooth structure on $G_p \setminus G$ such that the quotient map is smooth. For us, it is sufficient to bring the construction just locally. Let $\mathfrak{h}$ be the Lie algebra of stabilizer G_p. It is a subalgebra of $\mathfrak{g}$ consisting of vector fields whose integral curves act on p trivially, so it is the kernel of $\mathrm{d}\pi_p$ at unity. Let V be a vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$, so $\mathfrak{g}$ is a direct sum of vector spaces $\mathfrak{g} = V \oplus \mathfrak{h}$. Denote $(e_1, \ldots, e_k)$ the basis of V and $(e_{k+1}, \ldots, e_n)$ the basis of $\mathfrak{h}$. We can define second canonical coordinates on G in the neighbourhood of unity by relation

$$g_{(t^1,\ldots,t^n)} = \exp(t^n e_n)\, \exp(t^{n-1} e_{n-1}) \cdots \exp(t^1 e_1). \tag{17}$$

We can define coordinates on $G_p \setminus G$ as $\bar{g}_{t^1,\ldots,t^k} = G_p g_{t^1,\ldots,t^k,0,\ldots,0}$. It is evident that the quotient map is smooth with respect to these coordinates since its coordinate representation is $(t^1, \ldots, t^n) \mapsto (t^1, \ldots, t^k)$. It is also clear that the right action of G on $G_p \setminus G$ is smooth as well. We can introduce the map $\tilde{\pi}_p\colon G_p \setminus G \to M$ $G_p g \mapsto pg$. It is injective (mapping bijectively $G_p \setminus G$ onto pG) and its differential at unity $\mathrm{d}\tilde{\pi}_p$ is restriction of $\mathrm{d}\pi_p$ at unity on a complement of its kernel, so it is injective as well. Therefore, we locally proved the following theorem.

Theorem 2.40. Let π be an action of G on the manifold M, $p \in M$. Then the quotient map induces a structure of a smooth manifold on $G_p \setminus G$. This manifold is immersed in M as the orbit of p by $\tilde{\pi}_p\colon G_p \setminus G \to M$ $G_p g \mapsto pg$.

In particular, if π is transitive, then the map $\tilde{\pi}_p$ is a diffeomorphism.

Corollary 2.41. Let π be a transitive action of G on the manifold M, $p \in M$. Then the G-spaces M and $G_p \setminus G$ are isomorphic. In particular, it means that if π is transitive and free, then it is isomorphic to the right translations of G.

Remark 2.42. An interesting question is, what are the fundamental vector fields for the action of right translations. The right action of right multiplication and the left action of left multiplication taken as a map $G \times G \to G$ are in fact both identical to simple multiplication. The difference is only in the notation. Moreover, the orbit map of the right multiplication is the left multiplication and vice versa. So, for $a \in \mathfrak{g}$ and X the corresponding left-invariant vector field, we have

$$\hat{X}_g = \mathrm{d}L_g\, a = X_g,$$

so the fundamental vector fields of right translations are the left-invariant vector fields.

As we mentioned above, we proved Theorem 2.40 only locally. To formulate the local version of this proposition properly it is convenient to introduce the notion of "locally defined action". The local action, as defined in by 2.38, is defined on the whole manifold M at least for group elements very close to unity. Consequently, it defines global fundamental vector fields on M that realize the Lie algebra corresponding to the Lie group. However, the action of right multiplication on a local Lie group G or a right coset space $H \setminus G$ is not only local (because the group is local) but also locally defined, because the manifold is also "only local". Formally, we should again define the locally defined actions as some equivalence classes.

Definition 2.43. Let G be a (local) Lie group, M a manifold $p \in M$. Let U_1 and U_2 be neighbourhoods of p. Then local actions of G on U_1 and U_2 are locally equivalent at p if there is a neighbourhood $V \subset U_1 \cap U_2$ of p and an action of G on V that is their common restriction. The classes of equivalence are called *locally defined actions at p*. A locally defined action is called *faithful* or *transitive* if there exists its representative that is faithful or transitive.

Definition 2.44. Let G_1 and G_2 be local Lie groups, π_1 and π_2 their locally defined actions at $p_1 \in M_1$ and $p_2 \in M_2$. A morphism of their representatives $\Phi\colon U_1 \subset M_1 \to U_2 \subset M_2$ is called a *morphism* of the locally defined actions if $\Phi(p_1) = p_2$. In the same manner we define a *similitude* and an *isomorphism*.

We will now formulate the local version of Corollary 2.41 using this notion, which will be important in Chapter 4.

Theorem 2.45. Let π be a locally defined transitive action of G on the manifold M at $p \in M$. Then the quotient map induces a structure of a smooth manifold on $G_p \setminus G$. The action π is isomorphic to right multiplication on $G_p \setminus G$ (locally in the neighbourhood of unity class G_p) through $\tilde{\pi}_p\colon G_p p \mapsto pg$.

Finally, since every transitive action is, according to this theorem, isomorphic to right multiplication of a local Lie group G on some coset space $H \setminus G$. It will be convenient to be able to express the fundamental vector fields of such actions explicitly. As was mentioned in Remark 2.42, the fundamental vector fields corresponding to trivial subgroup $H = E$ are left-invariant vector fields. Those can be explicitly computed in second canonical coordinates by simple formula described in Section 2.4 as was suggested in [28]. Fundamental vector fields corresponding to non-trivial subgroup H are essentially just projection of the left-invariant vector fields on the submanifold $H \setminus G$.

The explicit computation is described in [15]. It is convenient to use second canonical coordinates of the form (17). We can see that the right multiplication acts as

$$\bar{g}_{t^1,\ldots,t^k} g_{x^1,\ldots,x^n} = G_p g_{t^1,\ldots,t^k,y^{k+1},\ldots,y^n} g_{x^1,\ldots,x^n} = \bar{g}_{M^1((t,y),x),\ldots,M^k((t,y),x)} \tag{18}$$

for arbitrary $y^{k+1},\ldots,y^n$, where M is the composition function. So, the coordinate expression for the action is

$$\Pi^i(t,x) = M^i((t,y),x) \tag{19}$$

for $i = 1, \ldots, k$. Therefore, according to equation (16) the coordinates of the fundamental vector fields are

$$\hat{X}_i^a(t) = \left.\frac{\partial \Pi^a(t,x)}{\partial x^i}\right|_{x=0} = \left.\frac{\partial M^a((t,y),x)}{\partial x^i}\right|_{x=0} = X_i^a(t,y), \tag{20}$$

where X_i are the left-invariant vector fields. So, the first k coordinates of the left-invariant vector fields depend only on the first k coordinates of the position and coincide with the coordinates of the fundamental vector fields.

Chapter 3
Classification problems of Lie algebras

In this chapter, we mention the important results regarding classification of Lie algebras. The goal of this work is to classify realizations of five-dimensional Lie algebras. Firstly, we have to know the list of all five-dimensional Lie algebras. Secondly, as we show in Chapter 4, we need classification of subalgebras of the Lie algebras with respect to some groups of automorphisms. The following sections are devoted to these problems.

Since both those problems—finding the groups of automorphisms and classification of subalgebras—are very interesting by themselves, we publish our results together with the classification of realizations in the Section A.2 of the appendix.

The description of general theory is based on [27, 2]. The important classification results are cited in the text.

3.1 Classification of Lie algebras of a given dimension

The problem of finding all Lie algebras of a given dimension was treated by G. M. Mubarakzyanov and solved for dimensions less or equal to five [16–17]. The relevant results of the Mubarakzyanov classification are summed up in Section A.1 of the appendix.

3.2 Groups of automorphisms of Lie algebras

Given a Lie algebra $\mathfrak{g}$, all its automorphisms form a group that will be denoted $\operatorname{Aut}\mathfrak{g}$. Given a basis (e_i) of $\mathfrak{g}$, the condition for a linear operator $\alpha\colon\mathfrak{g}\to\mathfrak{g}$ to be an automorphism $\alpha[x,y]=[\alpha(x),\alpha(y)]$ can be rewritten as

$$\sum_j \alpha^i_j c^j_{kl} = \sum_{j,m} c^i_{jm}\alpha^j_k\alpha^m_l, \tag{1}$$

where c^i_{jk} are the structure constants of the Lie algebra and α^i_j are the matrix elements of α in the basis (e_i). Solving this equation for the matrix elements α^i_j gives the general form of an automorphism of $\mathfrak{g}$.

The groups of automorphisms of all Lie algebras of dimension less or equal to five were computed in [6]. For every Lie algebra we have been studying in our work, we have computed independently the group of all automorphisms and checked the correctness of those results.

Please note that when formulating the automorphisms in a form of matrices, we interpret the upper index as row index and the lower index as column index. Some authors use the opposite convention.

Automorphisms of algebras are closely connected their derivations.

Theorem 3.1. Let (A, m) be a finite-dimensional algebra and $D\colon A \to A$ a homomorphism. Then e^{tD} (as an ordinary matrix exponential) is a group of automorphisms if and only if D is a derivation.

Proof. The assertion that e^{tD} is an automorphism means that, for all $x, y \in A$,

$$\sum_{k=0}^{\infty} \frac{D^k}{k!} m(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t^{i+j} m\left(\frac{D^i}{i!}x, \frac{D^j}{j!}y\right).$$

Comparison of the terms linear in t gives us $D(m(x,y)) = m(Dx,y) + m(x,Dy)$.

Now, let D be a derivation. Since exponential of an operator is always invertible, we only have to prove the homomorphism property. Using the general Leibniz rule we get

$$\begin{aligned}
\mathrm{e}^D m(x,y) &= \sum_{k=0}^{\infty} \frac{D^k}{k!} m(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{k}{l} m(D^l x, D^{k-l} y) = \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{1}{l!\,(k-l)!} m(D^l x, D^{k-l} y) = m\left(\sum_{i=0}^{\infty} \frac{D^i}{i!} D^i x, \sum_{j=0}^{\infty} \frac{D^j}{j!} y\right) = \\
&= m(\mathrm{e}^D x, \mathrm{e}^D y).
\end{aligned}$$

□

In case of Lie algebras, it follows that the Lie algebra of derivations $\operatorname{Der}\mathfrak{g}$ is the Lie algebra of $\operatorname{Aut}\mathfrak{g}$ as a Lie group. Note however that the map $\exp\colon \operatorname{Der}\mathfrak{g} \to \operatorname{Aut}\mathfrak{g}$ might not be surjective. Moreover, $\exp(\mathfrak{g})$ might not generate $\operatorname{Aut}\mathfrak{g}$. In other words, there may exist elements $\alpha \in \operatorname{Aut}\mathfrak{g}$ that are not of the form e^D for some $D \in \operatorname{Der}\mathfrak{g}$ and there might also exist so called *discrete* automorphisms that cannot be expressed as a product of automorphisms of such form.

An important subalgebra of $\operatorname{Der}\mathfrak{g}$ is the subalgebra of *inner derivations* $\mathrm{ad}_\mathfrak{g} = \{\mathrm{ad}_x \mid x \in \mathfrak{g}\}$, where $\mathrm{ad}_x\colon \mathfrak{g} \to \mathfrak{g}$ is the *adjoint representation* of $x \in \mathfrak{g}$ mapping $y \mapsto [x,y]$. The corresponding Lie group generated by elements of the form $\mathrm{e}^{\mathrm{ad}_x}$, $x \in \mathfrak{g}$ is denoted $\operatorname{Int}\mathfrak{g}$ and its elements are called *inner automorphisms* of the Lie algebra.

Consider the local Lie group G corresponding to $\mathfrak{g}$. Then the generators of $\operatorname{Int}\mathfrak{g}$ can be expressed as Ad_g, $g \in G$ since $\mathrm{e}^{t\,\mathrm{ad}_a} = \mathrm{Ad}_{\mathrm{e}^{ta}}$.

Note that some authors define the group of inner automorphisms as generated only by exponentials of such elements ad_x that are nilpotent. This definition is, of course, stronger and does not correspond to the adjoint representation of the corresponding connected Lie groups as indicated above.

Since $\operatorname{Int}\mathfrak{g}$ is generated by curves, it has to be path-connected and hence connected. Therefore, it is generated by any open subset. In particular, by domain of second canonical coordinates

$$(t^1, \ldots, t^n) \mapsto \alpha(t^1, \ldots, t^n) := \mathrm{e}^{t^1\,\mathrm{ad}_{e_1}} \mathrm{e}^{t^2\,\mathrm{ad}_{e_2}} \cdots \mathrm{e}^{t^n\,\mathrm{ad}_{e_n}}, \tag{2}$$

where (e_n) is a basis of $\mathfrak{g}$. Thus, $\operatorname{Int}\mathfrak{g}$ is generated by $\mathrm{e}^{t\,\mathrm{ad}_{e_i}}$.

This is also the easiest way, how to explicitly compute the group of inner automorphisms. Very often it holds that the automorphisms of the form (2) are closed with respect to composition, so they already form the whole group of the inner automorphisms. According to [6] there are only two exceptions in dimension less or equal to five.

The group of inner automorphisms is, of course, a subgroup of $\operatorname{Aut}\mathfrak{g}$ (but not necessarily a Lie subgroup, i.e., it does not have to be an embedded submanifold). Moreover, it is a normal subgroup. Taking a Lie algebra automorphism, denote α the corresponding local Lie group automorphism. Then, differentiating the homomorphism property $\alpha \circ C_g = C_{\alpha(g)} \circ \alpha$, we get $\mathrm{d}\alpha \operatorname{Ad}_g = \operatorname{Ad}_g \mathrm{d}\alpha$, which proves the statement. We can, therefore, compute the quotient group $\operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g}$, whose elements are called *outer automorphisms*.

3.3 Classification of subalgebras of a given Lie algebra

For a given algebra we can also look for a set of its subalgebras. Finding such list is quite easy. A harder task is to divide the subalgebras into classes with respect to the following equivalence.

Definition 3.2. Let $\mathfrak{g}$ be a Lie algebra and $A \subset \operatorname{Aut}\mathfrak{g}$ be a group of automorphisms of $\mathfrak{g}$. Subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ are called *A-conjugate* if there is an automorphism $\alpha \in A$ such that $\mathfrak{h}' = \alpha(\mathfrak{h})$.

Universal classification methods were introduced in [25]. They are based on induction: using classification of lower-dimensional Lie algebras, we find the classification of a direct or semidirect sum. The authors also provided subalgebra classification for all Lie algebras of dimension not greater than four with respect to inner automorphisms in [24]. A classification of subalgebras of physically interesting Poincaré and Galilei Lie algebras together with the related theory is available in [7].

As we are going to describe in Chapter 4, we are going to need a classification of subalgebras with respect to groups of all and inner automorphisms. In this particular work, we were able to obtain results for all five-dimensional indecomposable nilpotent Lie algebras.

To obtain our classification, we used much simpler method than [25].

At first, we find all subalgebras of a given Lie algebra. This task is very simple and can be done very quickly by computer. We just list all subspaces of the Lie algebra and for each we check whether it is closed with respect to Lie bracket. Then we express the action of the groups of automorphisms on these subalgebras and divide them into the conjugacy classes. Practically, the easiest way to do that is to guess the answer and then check by computer, whether it is right.

We believe that such straightforward computation is in our case better that the inductive approach proposed in [25] for several reasons. Firstly, using computer algebra system makes it about equally exacting and time-consuming. Secondly, it does not depend on correctness of several previous results, so it is more reliable. And thirdly, the results for classification of four-dimensional Lie algebras is available only with respect to inner automorphisms, not with respect to the group of all automorphisms.

Chapter 4
Classification of realizations

In this chapter the main theoretical results of this work are contained. Those results are also the subject of an article in preparation, whose preprint is available on arXiv [11].

As we have mentioned in the introduction, our goal was to continue in the work [26], which classifies realizations of all Lie algebras up to dimension four, and try to obtain realizations classification of five-dimensional Lie algebras. In order to do that, we developed a simple algortithm that leads to such classification.

As we indicated in Section 2.5, there is a close relationship between realizations of Lie algebras by vector fields and actions of the corresponding Lie group. This relationship is studied in more detail in Section 4.2.

An important class of realizations are so called *transitive* realizations. Theoretical results on classification of transitive local realizations, together with powerful methods of explicit computation are already available in the literature [1, 4, 12, 15]. In particular, it can be shown that classification of all transitive realizations is equivalent to classification of subalgebras. These results are summarized in Section 4.3.

On the basis of those results, our algorithm is formulated. The key correspondence between realizations and subalgebras was generalized to the case of *regular* realizations in Section 4.6. The main result formulated in Theorem 4.19 is followed by a simple example illustrating the practical computation.

When dealing with general realizations it is actually not so clear how to choose the classification problem to get reasonable results. As we show in Section 4.7 one of possible classification problems could be to find a system of local realizations such that every realization is at each point of a dense subset of its domain locally equivalent to some realization of this system. Such a system will be called a *complete system of local realizations* (see Definition 4.24) and we show that such a system can be chosen to consist of regular local realizations.

Finally, in Section 4.8, we formulate an algorithm that solves the established classification problem (that is, to find the complete system of local realizations) and Theorem 4.32 summarizing this algorithm is followed by an example illustrating this procedure.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra and M a manifold. A *realization* of $\mathfrak{g}$ on the manifold M is a homomorphism $R\colon \mathfrak{g} \to \operatorname{Vect} M$. The realization is called *faithful* if it is injective.

This definition was adopted from [26]. Another approach is to consider a derivation of the algebra of formal power series over a field F of characteristic zero $\operatorname{Der} F[[x]]$ instead of vector fields (see for example [4]). This definition is more general regarding the arbitrary field F, on the other hand it corresponds to local analytic realizations only.

By restricting the realizing vector fields on an open subset U of M we get a *restriction* of a realization on U, which we will denote $R|_U$. The manifold, where the realizing vector fields are defined is called the *domain* of the realization and denoted $\operatorname{Dom} R$.

The realizations are often considered only locally. This means that we specify a point in the manifold M and consider the realizing vector fields only in a small neighbourhood of this point. This is equivalent to considering the realizations on a neighbourhood of zero at $\mathbb{R}^m$. Formally, the notion of local realization can be defined as an equivalence class, as in the case of Lie group actions (cf. Definition 2.43). The notion of a local realization is introduced to simplify the classification problem leaving aside the global structure of the manifold and the realizing vector fields.

Definition 4.2. Let $\mathfrak{g}$ be a Lie algebra, M a manifold $p \in M$. Let U_1 and U_2 be neighbourhoods of p. Then realizations R_1 and R_2 of $\mathfrak{g}$ defined on U_1 and U_2, respectively, *locally coincide* at p if there is a neighbourhood $V \subset U_1 \cap U_2$ of p such that $R_1|_V = R_2|_V$. The classes of locally coincident realizations at a specified point p are called *local realizations at* p. A local realization is *faithful* if every its representative is faithful.

We will usually not strictly distinguish between local realizations and their representatives. For a given global realization R defined on a manifold M we denote $R|_p$ the corresponding local realization at $p \in M$. For a given local realization R at $p \in M$ and U a neighbourhood of p we denote $R|_U$ the corresponding representative defined on U.

4.1 Realizations equivalence

As, for example, in the case of representations by matrices, it does not make sense to find the list of literally all the representations. We usually introduce some equivalence between representations and classify the equivalence classes. The definition of realization equivalence is an analogy of action isomorphism and similitude (cf. Definitions 2.35, 2.44).

Definition 4.3. Let $\mathfrak{g}$ be a Lie algebra and M_1 and M_2 manifolds. Let A be a subgroup of $\mathrm{Aut}(\mathfrak{g})$. Realizations $R_1\colon \mathfrak{g} \to \mathrm{Vect}(M_1)$ and $R_2\colon \mathfrak{g} \to \mathrm{Vect}(M_2)$ are called A-*equivalent* if there exist an automorphism $\alpha \in A$ and a diffeomorphism $\Phi\colon M_1 \to M_2$ such that $R_2(\alpha(x)) = \Phi_* R_1(x)$ for all $x \in \mathfrak{g}$. If we do not consider automorphisms (so $A = \{\mathrm{id}\}$) the realizations are called just equivalent (or *strongly* equivalent in case we need to emphasise that we do not consider automorphisms). For inner automorphisms $A = \mathrm{Int}\,\mathfrak{g}$ we will write shortly Int-equivalent and for all automorphisms $A = \mathrm{Aut}\,\mathfrak{g}$ we will say Aut-equivalent.

Definition 4.4. Local realization R_1 at $p_1 \in M_1$ is A-equivalent to local realization R_2 at $p_2 \in M_2$ if there exist their representatives defined in neighbourhoods $U_1 \ni p_1$, $U_2 \ni p_2$ that are A-equivalent and the corresponding diffeomorphism $\Phi\colon U_1 \to U_2$ satisfies $\Phi(p_1) = p_2$.

For a given global or local realization R we denote $\bar{R}$ the corresponding equivalence class. It should be clear from context which group of automorphisms A we consider.

Since faithfulness of the realization or the dimension of the realizing manifold is invariant under the equivalence, we can assign those characteristics to the classes. In particular, in case of local realizations, the global structure of the manifold is irrelevant and the dimension is the only characteristic of the manifold, so we often refer to a *class of local realizations in* m *variables.*

Every class of local realizations in m variables $\bar{R}$ of an n-dimensional Lie algebra $\mathfrak{g}$ can be represented by a realization defined in a neighbourhood of zero in $\mathbb{R}^m$, so it is determined by $n \cdot m$ functions ξ_i^j, $i = 1, \ldots, n$, $j = 1, \ldots, m$ defined in some neighbourhood of zero at $\mathbb{R}^m$ that form one of the representatives $R_0 \in \bar{R}$

$$R_0(e_i) = \sum_{j=1}^{m} \xi_i^j(x^1, \ldots, x^m)\partial_{x^j}. \tag{1}$$

It means that for all representatives R defined a neighbourhood of a point $p \in M$ there exist coordinates $(x^1, \ldots, x^m)$ in some neighbourhood of p such that the coordinate expression of R coincides with (1).

This also illustrates the connection with the definition of realizations by derivations of formal power series. In this case, we have formal power series instead of the functions ξ_i^j and the equivalence is provided by formal coordinate change preserving zero (since all the power series are centered at zero). Therefore, there is a one-to-one correspondence between local realizations at zero that have an analytic representative and formal realizations that have a convergent representative and also between the corresponding classes. In the case of so called transitive realizations (see Section 4.3), it can be shown that all classes of local realizations have an analytic representative and all classes of formal realizations have a convergent representative.

4.2 Realizations and group actions

The fundamental result of the Lie theory described in Chapter 2 is that local transformations are completely defined by vector fields representing the infinitesimal transformations. That is, for a local Lie group and its action on a manifold we can find the fundamental vector fields that essentially form a subalgebra of the corresponding Lie algebra. The map $\pi_*\colon \mathfrak{g} \to \operatorname{Vect} M$ is actually a realization of $\mathfrak{g}$ on M. On the other hand, according to Theorem 2.34, every Lie algebra uniquely defines a local Lie group and, according to Theorem 2.39, for every realization there exists, up to isomorphism, a unique local action of the corresponding Lie group.

Moreover, A-classes of realizations correspond to A-classes of local actions in the following sense. Let G a local Lie group and $\mathfrak{g}$ its Lie algebra. We say that actions $\pi^{(1)}$ and $\pi^{(2)}$ of G acting on M_1 and M_2, respectively, are *A-similar* for A a subgroup of $\operatorname{Aut} G$ if there is a diffeomorphism $\Phi\colon M_1 \to M_2$ and an isomorphism $\varphi \in A$ satisfying

$$\Phi(\pi^{(1)}(p, g)) = \pi^{(2)}(\Phi(p), \varphi(g)). \tag{2}$$

Lemma 4.5. Let G be a local Lie group and $\mathfrak{g}$ its Lie algebra. Local actions of G on manifolds M_1 and M_2 are A-similar if and only if the corresponding realizations of the Lie algebra $\mathfrak{g}$ are $\mathrm{d}A$-equivalent, where $\mathrm{d}A = \{\mathrm{d}\varphi \mid \varphi \in A\} \subset \operatorname{Aut} \mathfrak{g}$.

Proof. Let $\pi^{(1)}$ and $\pi^{(2)}$ be actions of G on M_1 and M_2 and denote the corresponding realizations $R_1 = \pi_*^{(1)}$ and $R_2 = \pi_*^{(2)}$. Denote $\tilde{\pi}^{(i)}\colon G \to \operatorname{Diff} M_i$ $\tilde{\pi}^{(i)}(g)(p) = \pi^{(i)}(p, g)$. The condition (2) can be rewritten as

$$\Phi \circ \tilde{\pi}^{(1)}(g) \circ \Phi^{-1} = \pi^{(2)}(\varphi(g)).$$

Note that the expression on both sides of the equation is a local Lie group homomorphism in g mapping $G \to \operatorname{Diff} M_2$. Differentiation of this equation in g leads to

$$\Phi_* R_1(a) = \Phi^* \circ R_1(a) \circ (\Phi^{-1})^* = R_2(\mathrm{d}\varphi\, a),$$

which means that R_1 is equivalent to R_2 through diffeomorphism Φ and automorphism $\mathrm{d}\varphi$. This proves the left-right implication and thanks to Theorem 2.33 it also proves the opposite direction. $\square$

We can do the same for local realizations that correspond to locally defined actions (Definition 2.43). We can again define the A-similitude for locally defined actions by generalizing Definition 2.44. Then A-classes of local realization correspond to A-classes of locally defined actions.

4.3 Classification of transitive realizations

Definition 4.6. Let R be a realization of a Lie algebra $\mathfrak{g}$ on a manifold M. The *rank* of the realization R at point $p \in M$ is the rank of the linear map $R_p\colon \mathfrak{g} \to T_pM$, $v \mapsto R(v)_p$, that is, $\operatorname{rank} R_p = \dim R(\mathfrak{g})_p = \dim\{R(v)_p \mid v \in \mathfrak{g}\}$. If the function $p \mapsto \operatorname{rank} R_p$ is locally constant at p_0 we say that R is *regular* at p_0. Local realizations at p are called *regular* if their representatives are regular at p.

Lemma 4.7. Let R be a realization of a Lie algebra $\mathfrak{g}$ on a manifold M. Let G be a local Lie group of $\mathfrak{g}$ and π a local action on M corresponding to R. The local action π is (locally) transitive if and only if the rank of R is constant and equal to $m = \dim M$, i.e., the map $R_p\colon \mathfrak{g} \to T_pM$ is surjective for every $p \in M$. A locally defined action is (locally) transitive if the corresponding local realization in m variables has rank m. Realizations satisfying this property are called *transitive* as well.

Proof. The map R_p is in fact identical to $\mathrm{d}\pi_p$. Therefore, it is surjective if and only if π_p is "locally surjective", i.e., there exists a neighbourhood V of p such that $\pi_p(G) \supset V$. Equivalently, for every $q \in V$ there is a $g \in G$ such that $q = \pi(p, g)$. $\square$

According to Theorem 2.45, every locally defined transitive action at $p \in M$ of a local Lie group G is isomorphic to the right multiplication of G acting on $G_p \setminus G$. Hence, any locally defined transitive action of G is, up to isomorphisms, described by the subgroup H representing a stabilizer of a given point in G. On the other hand, for a given subgroup H we can easily construct such an action as the right multiplication of G on $H \setminus G$.

In the language of Lie algebras it means that every strong class of local transitive realizations of a Lie algebra $\mathfrak{g}$ is uniquely determined by a subalgebra $\mathfrak{h}$.

This correspondence can be formulated purely algebraically without need of introducing (local) Lie groups and their action. For a local realization R at $p \in M$ of a Lie algebra $\mathfrak{g}$ the corresponding subalgebra $\mathfrak{h} \subset \mathfrak{g}$ can be defined as the kernel of the linear map $\mathfrak{g} \to T_pM$, $a \mapsto R(a)_p$. On the other hand, to prove that every subalgebra defines uniquely a strong class of local realizations is not so simple. In the case of realizations by formal power series over general field F, the correspondence was proven by Guillemin and Sternberg [12]. Later, Blattner [1] came with even more abstract proof of correspondence between subalgebras and certain classes of representations. These works are valuable not only because they are very general in the definition of realization, but they are also obtained purely algebraically.

Moreover, we can easily see that A-classes of local realizations correspond to A-conjugacy classes of subalgebras. Indeed, taking a local realization R and the corresponding subalgebra $\mathfrak{h} = \ker R_p$, the subalgebra that corresponds to $R \circ \alpha$ for $\alpha \in A$ is simply $\ker R_p \circ \alpha = \alpha(\ker R_p) = \alpha(\mathfrak{h})$. So, we have the following.

Lemma 4.8. Let $\mathfrak{g}$ be a Lie algebra and $A \subset \operatorname{Aut} \mathfrak{g}$ a group of automorphisms. Local transitive realizations correspond to A-conjugate subalgebras of $\mathfrak{g}$ if and only if they are A-equivalent.

Finally, the faithfulness of the realization can be characterized by property of the subalgebra. From the theory of group actions we know that the kernel of an action of right multiplication on $H \setminus G$ is the largest normal subgroup contained in H. Using Lemma 2.32 we can transfer this relation to Lie algebras and realizations and formulate the following propositions.

Lemma 4.9. The kernel of a local transitive realization of $\mathfrak{g}$ is the largest ideal contained in the corresponding subgroup of $\mathfrak{g}$.

Lemma 4.10. A transitive realization of a Lie algebra $\mathfrak{g}$ is faithful if and only if the corresponding subalgebra of $\mathfrak{g}$ does not contain any non-trivial ideal of $\mathfrak{g}$.

An algebraic proof of these propositions is presented for example in [4]

To sum up, classification of all local transitive realizations with respect to a group of automorphisms A is equivalent to classification of subalgebras with respect to A. The corresponding representatives of those classes are fundamental vector fields of right multiplication of G on $H \setminus G$, where G is a local Lie group corresponding to the Lie algebra $\mathfrak{g}$ and H is its subgroup corresponding to a subalgebra $\mathfrak{h}$ considered. The explicit form of those realizations can be computed by algorithm of Shirokov et al. [15] described at the end of Section 2.5.

This method was already used to classify transitive realizations of low-dimensional Poincaré algebras [19] and Galilei algebras [20].

Remark 4.11. In [14] Lie conjectured that any local transitive realization can be expressed (after a suitable change of coordinates) by entire functions of coordinates and exponentials of linear functions in coordinates (over $\mathbb{C}$). Over $\mathbb{R}$ (or arbitrary other field) it can be reformulated as follows. Any local transitive realization can be expressed in certain coordinates (i.e., any class has such representative in $\mathbb{R}^m$) as functions of coordinates that are a solution of some differential equation with constant coefficients.

For certain types of realizations, this conjecture was proven by Draisma [3] but it was not proven generally (and Draisma believes that it is generally not true). The Shirokov's method does not prove or disprove this conjecture. Nevertheless, we are able to formulate weaker proposition. Every realization constructed by the Shirokov's method is a rational function of functions of coordinates that are a solution of a differential equation with constant coefficients. So, over $\mathbb{C}$ it would be a rational function of exponentials.

4.4 Inner automorphisms

In this section, we describe, how inner automorphisms act on transitive local realizations, which will be important for classification of regular local realizations.

Lemma 4.12. Let G be a (global) Lie group and $\mathfrak{g}$ its Lie algebra. Let H be a subgroup of G and take $g \in G$. Then (global) realization on $H \setminus G$ by fundamental vector fields of right multiplication by G is strongly equivalent to realization by fundamental vector fields on the manifold $\tilde{H} \setminus G$, $\tilde{H} = g^{-1}Hg$.

Proof. Take the action π of right multiplication corresponding to the first realization. Then H is the stabilizer of the class corresponding to unity $\bar{e} = H$. We can easily see that $\tilde{H} = g^{-1}Hg$ is the stabilizer of a point $\bar{e}g = Hg = \bar{g}$. According to Theorem 2.40 π is isomorphic to the right multiplication on $\tilde{H} \backslash G$, which is the action that corresponds to the second realization. This action isomorphism being also a manifold diffeomorphism provides the equivalence. □

This means that globally Lemma 4.8 does not hold. For different conjugated subgroups, which correspond to subalgebras that are inequivalent with respect to strong equivalence but equivalent with respect to inner automorphisms, we get strongly equivalent realizations. This lemma holds locally because we cannot consider translations as coordinate changes for local realizations, which are defined only by local behaviour in a given point. In terms of local realizations, we can formulate the following lemma.

Lemma 4.13. Let R be a transitive realization of $\mathfrak{g}$ on M, $p \in M$. Then for every $q \in M$ the local realizations $R|_p$ and $R|_q$ are equivalent with respect to inner automorphisms. Conversely, for every neighbourhood U of p there exists a neighbourhood of unity V in the group of inner automorphisms such that for every $\alpha \in V$ there exists $q \in U$ such that $R|_p$ and $R|_q$ correspond to α-conjugated subalgebres and hence are α-equivalent.

Proof. Denote $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ the subalgebras corresponding to local realizations $R|_p$ and $R|_q$. Choose a local Lie group G and denote H, $\tilde{H}$ the corresponding subgroups. Denote π the corresponding action of G on M. From transitivity of the realization there exist $g \in G$ such that $q = \pi(p, g)$, so $\tilde{H} = g^{-1}Hg$. Thus, $\tilde{\mathfrak{h}} = \mathrm{Ad}_{g^{-1}}\,\mathfrak{h}$, so the realizations are equivalent with respect to this inner automorphism.

The second proposition is just a local version of Lemma 4.12. □

Convention. When presenting concrete examples we will use $x_1, \ldots, x_m$ as coordinates with lower indices to simplify notation and avoid confusion with powers. The partial derivatives are denoted just ∂_i instead of ∂_{x_i}.

Example 4.1. Take a Lie algebra $\mathfrak{g}_{3,1} = \mathrm{span}\{e_1, e_2, e_3\}$, $[e_2, e_3] = e_1$. All one-dimensional subalgebras and the corresponding realizations in a neighbourhood of $(0,0) \in \mathbb{R}^2$ are following.

$$\begin{aligned}
\mathrm{span}\{e_1\}: &\qquad e_1 \mapsto 0, \quad e_2 \mapsto \partial_1, \quad e_3 \mapsto \partial_2 \\
\mathrm{span}\{e_2 - ae_1\}: &\qquad e_1 \mapsto \partial_1, \quad e_2 \mapsto (a - x_2)\partial_1, \quad e_3 \mapsto \partial_2 \\
\mathrm{span}\{e_3 - be_2 - ae_1\}: &\qquad e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto (a + x_2)\partial_1 + b\partial_2
\end{aligned}$$

All the subalgebras in the second row are equivalent with respect to inner automorphisms and, for fixed b, all the realizations in the third row are equivalent with respect to inner automorphisms. In other words, classification of subalgebras and realizations with respect to inner automorphisms is obtained by removing the parameter a (setting $a := 0$).

All these realizations are, of course, mutually inequivalent with respect to strong equivalence as local realizations. However, if we consider them as global realizations on $\mathbb{R}^2$, then all realizations in the second row are strongly equivalent and this equivalence is provided by simple translation in x_2. The same holds for the last row for fixed b.

In other words, all inequivalent local transitive realizations with respect to strong equivalence are obtained by restricting one of the following realizations on $\mathbb{R}^2$.

$$e_1 \mapsto 0, \quad e_2 \mapsto \partial_1, \quad e_3 \mapsto \partial_2,$$

$$e_1 \mapsto \partial_1, \quad e_2 \mapsto -x_2\partial_1, \quad e_3 \mapsto \partial_2,$$

$$e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto x_2\partial_1 + b\partial_2.$$

At the same time, this is classification of local realizations in a neighbourhood of $(0,0) \in \mathbb{R}^2$ with respect to inner automorphisms.

4.5 Topology of subalgebra and realization systems

In the following section we are going to construct new realizations by interpreting parameters of transitive realizations as new coordinates. This is, of course, possible only in the case when the transitive realizations depend “smoothly” on those parameters.

We are going to say that a function $F\colon \mathbb{R}^s \to \operatorname{Vect} M$ is smooth if a vector field $X^F \in \operatorname{Vect}(M \times \mathbb{R}^s)$ defined as $X^F_{(p,x)} = F(x)_p$ for $p \in M$ and $x \in \mathbb{R}^s$ is smooth. As indicated for example in [18], Theorem 46.11, such a definition corresponds to compact open topology on $\operatorname{Vect} M$. For further references on the vector fields topology, see e.g. [13]. Nevertheless, we will not use any special properties of such topology here.

This induces a topology and the notion of smoothness on the space of realizations. We also get a topology on the space of local realizations and spaces of A-classes of local realizations as a topological quotient spaces. Note, however, that those quotient spaces may not be even Hausdorff.

Nevertheless, we can again induce the notion of smooth map. A map between two quotient spaces will be called smooth if it is locally a quotient of a smooth map. We formulate it precisely in the following definition.

Definition 4.14. Let M_1 and M_2 be manifolds and let $\bar{M}_1$ and $\bar{M}_2$ be their quotient spaces. A map $\bar{\Phi}\colon \bar{M}_1 \to \bar{M}_2$ will be called *smooth* in $\bar{x}_0 \in \bar{M}_1$ if there exists $x_0 \in \bar{x}_0$, its neighbourhood U, and a map $\Phi\colon U \to M_2$ such that for all $x \in U$ $\Phi(x) \in \bar{\Phi}(\bar{x})$, where $\bar{x}$ is the class corresponding to x. A smooth bijection, whose inversion is smooth as well, will be called a *diffeomorphism*. A smooth injection, whose inversion is smooth as well, will be called an *embedding*.

Remark 4.15. This definition is compatible with the quotient topology in a sense that ever smooth map is continuous, so every diffeomorphism is a homeomorphism. A composition of such smooth maps is smooth.

Let $\mathscr{S}_m$ be the space of all subalgebras of codimension m of a given Lie algebra $\mathfrak{g}$. This is an affine subvariety of the Grassmannian $\operatorname{Gr}(\mathfrak{g}, n-m)$, $n = \dim \mathfrak{g}$. A map $F\colon \mathbb{R}^s \to \mathscr{S}_m$ will be called smooth if it is smooth as a map $\mathbb{R}^s \to \operatorname{Gr}(\mathfrak{g}, n-m)$. A smooth map to the space of subalgebra A-classes is again defined in sense of Definition 4.14.

Lemma 4.16. Let $\mathfrak{g}$ be a Lie algebra, $\bar{\mathscr{S}}_m$ the space of all A-classes of subalgebras of codimension m in $\mathfrak{g}$ and $\bar{\mathscr{T}}_m$ the space of all A-classes of local transitive realizations in m variables. Then $\bar{\mathscr{S}}_m$ is diffeomorphic to $\bar{\mathscr{T}}_m$.

Proof. It is sufficient to prove this proposition for space of subalgebras $\mathscr{S}_m$ and space of strong classes $\bar{\mathscr{T}}_m$. Then we only “factor” both sides.

In Section 4.3 we showed that there is a bijection between these two sets. It is clear that the map $\mathscr{T}_m \to \mathscr{S}_m$, $R \mapsto \ker R_p$ is smooth. Therefore, the same holds for the map of classes. To show the smoothness of the inverse, we can make use of the Shirokov’s computation that smoothly depends on the choice of the subalgebra. ☐

Therefore, the space of strong classes of transitive realizations actually is an homeomorphic image of an algebraic variety and therefore it is a Hausdorff space. The space of general A-classes, however, does not have to be.

4.6 Classification of regular realizations

In this section we characterize the classification problem for regular realizations. Classification of regular realizations is very important also for general realizations since we have the following lemma.

Lemma 4.17. Let R be a realization of $\mathfrak{g}$ on M. Then the regular points of R form an open dense set in M. Hence, the set of singular points is nowhere dense.

Proof. The set is obviously open. Choose a point $p_0 \in M$ and its neighborhood U. We find a regular point $p \in U$.

Denote $r := \max_{p\in U} \operatorname{rank} R_p$. We can easily construct a continuous function $f\colon U \to \mathbb{R}$ such that $\operatorname{rank} R_p = r$ if and only if $F(p) \neq 0$ as sum of squares of some minors of the linear map R_p. Preimage of $\mathbb{R} \setminus 0$ is open, nonempty, and contain points, where $\operatorname{rank} R_p$ is locally constant and equal to r. $\square$

Now, we generalize the correspondence between transitive realizations and subalgebras to the case of regular realizations.

Let us have a local realization R at $p \in U$ such that R has a constant rank r on U. Then $R(\mathfrak{g})$ forms an involutive r-dimensional distribution so, according to the Frobenius theorem 2.6, we can choose coordinates $x^1, \ldots, x^m$ on a neighbourhood U of p, such that U is foliated by integral submanifolds given by equations $x^i = \text{const}$ for $i = r+1, \ldots, m$. These integral submanifolds are also the orbits of the action π corresponding to the realization R. The basis elements $e_1, \ldots, e_n$ of $\mathfrak{g}$ are, therefore, realized by vector fields $X_i = R(e_i)$ of the form

$$X_i = \sum_{a=1}^{r} X_i^a(x^1, \ldots, x^m)\partial_{x^a}. \tag{3}$$

This realization induces an $(m-r)$-parameter set of realizations parametrized by $x^{r+1}, \ldots, x^m$ on the submanifolds $pG \simeq G_p \setminus G$. These realizations are transitive, so they are equivalent to the realizations found by the algorithm described in Section 4.3.

So, a regular realization R in a given point defines a unique transitive realization on the integral submanifold of the point. We will call this realization a *transitive restriction* of R. This relation obviously does not break by applying a diffeomorphism or an automorphism. A local diffeomorphism of the whole neighbourhood U induces a diffeomorphism of the orbits. An orbit of an action does not change by composing it with an automorphism of the group. To sum up:

Proposition 4.18. Let $\mathfrak{g}$ be a Lie algebra, A a subgroup of $\operatorname{Aut}\mathfrak{g}$. Then a class of A-equivalence of local realizations of $\mathfrak{g}$ with constant rank r uniquely defines an A-class of local transitive realizations of $\mathfrak{g}$ in r variables.

To find all non-transitive regular realizations, we can proceed the other way around. Every local regular realization in m variables is of the form

$$R(e_j)_{x^1,\ldots,x^m} = R^{(x^{r+1},\ldots,x^m)}(e_j)_{x^1,\ldots,x^r}, \tag{4}$$

where $R^{(a^1,\ldots,a^{m-r})}$ is an $(m-r)$-parameter set of transitive realizations.

Now, we formulate the main theorem. In the formulation we use the term *local smooth s-parameter set of subalgebra classes with codimension r*. By that we mean a smooth map $U \to \bar{\mathscr{S}}_r$, where U is a neighborhood of $0 \in \mathbb{R}^s$ considering its values only locally at zer at zeroo (as in Def. 4.2). By a *class* of such maps we mean a class of equivalence up to “regular reparametrization”, that is, S and S' are equivalent if and only if there exists a local diffeomorphism $\Psi\colon \mathbb{R}^s \to \mathbb{R}^s$, $\Psi(0) = 0$ such that $S' = S \circ \Psi$.

Theorem 4.19. Let $\bar{\mathscr{S}}_r$ the system of all Int-classes of subalgebras with codimension r. For $\bar{\mathfrak{h}} \in \bar{\mathscr{S}}_r$ denote $\bar{R}^{\bar{\mathfrak{h}}}$ the corresponding Int-class of transitive realizations. Then there is a bijection between Int-classes of regular local realizations of $\mathfrak{g}$ in m variables with rank r and classes of local smooth $(m-r)$-parameter sets of Int-classes of subalgebras with codimension r. For any such $(m-r)$-parameter set $S\colon V \to \bar{\mathscr{S}}_r$, we define a local realization $R \in \mathrm{Vect}(U \subset \mathbb{R}^m)$ at zero as follows

$$R(e_j)_{x^1,\ldots,x^m} = R^{S(x^{r+1},\ldots,x^m)}(e_j)_{x^1,\ldots,x^r}, \tag{5}$$

where the representatives $R^{S(x^{r+1},\ldots,x^m)}$ are local realizations at $0 \in \mathbb{R}^r$ chosen to be smooth in the variables $x^{r+1},\ldots,x^m$.

Proof. At first, we prove that the map is well-defined. The smoothness of S implies that we have indeed defined a smooth vector fields in sense of Section 4.5. Next, we have to show that those vector fields do not depend on the choice of representative S and representatives of the realizations R^S.

Assume we chose another representatives for both R^S and S, say $R'^{(S\circ\Psi)(x^{r+1},\ldots,x^m)} = \Phi_*^{(x^{r+1},\ldots,x^m)} R^{(S\circ\Psi)(x^{r+1},\ldots,x^m)} \circ \alpha$, where $\Phi^{(x^{r+1},\ldots,x^m)}$ is a smoothly parametrized set of diffeomorphisms. Then the resulting realization would be

$$R'(e_j)_{x^1,\ldots,x^m} = \Phi_*^{(x^{r+1},\ldots,x^m)} R^{(S\circ\Psi)(x^{r+1},\ldots,x^m)}(\alpha(e_j))_{x^1,\ldots,x^r} = \tilde{\Phi}_* R(\alpha(e_j))_{x^1,\ldots,x^m},$$

where $\tilde{\Phi}\colon \mathbb{R}^m \to \mathbb{R}^m$ is a local diffeomorphism defined as

$$\tilde{\Phi}(x^1,\ldots,x^m) = (\Phi^{(x^{r+1},\ldots,x^m)}(x^1,\ldots,x^r), \Psi(x^{r+1},\ldots,x^m)).$$

The surjectivity of such a map follows from the Frobenius theorem as was described above.

To prove the injectivity, lets assume that realizations R_1 and R_2 of the form (5) corresponding to local maps S_1 and S_2 are equivalent so $\Phi_* R_1 = R_2 \circ \alpha$.

The diffeomorphism Φ must preserve the integral submanifolds $x^j = \text{const}$ for $j > r$, so $\Phi^j(x^1,\ldots,x^r,x_0^{r+1},\ldots,x_0^m)$ has to be constant in $x^1,\ldots,x^r$ for $j > r$. Hence, we can denote

$$\Phi(x^1,\ldots,x^m) = \begin{pmatrix} \tilde{\Phi}_1^{(x^{r+1},\ldots,x^m)}(x^1,\ldots,x^r) \\ \Phi_2(x^{r+1},\ldots,x^m) \end{pmatrix},$$

where $\tilde{\Phi}_1^{(x^{r+1},\ldots,x^m)}\colon \mathbb{R}^r \to \mathbb{R}^r$ is an $(m-r)$-parameter set of local diffeomorphisms and Φ_2 is a local diffeomorphism of $\mathbb{R}^{m-r}$. Note that although $\tilde{\Phi}_1^{(0,\ldots,0)}(0,\ldots,0) = 0$, generally $\tilde{\Phi}_1^{(x^{r+1},\ldots,x^m)}(0,\ldots,0)$ does not have to be zero, so these local diffeomorphisms at zero translate the point zero. So, denote

$$\Phi_1^{(x^{r+1},\ldots,x^m)}(x^1,\ldots,x^r) = \tilde{\Phi}_1^{(x^{r+1},\ldots,x^m)}(x^1,\ldots,x^r) - \tilde{\Phi}_1^{(x^{r+1},\ldots,x^m)}(0,\ldots,0)$$

We can write

$$\Phi_* R_1(e_j)_{x^1,\ldots,x^m} = \Phi_{1*}^{(x^{r+1},\ldots,x^m)} R^{(S_1\circ\Phi_2)(x^{r+1},\ldots,x^m)}(\alpha^{(x^{r+1},\ldots,x^m)}(e_j))_{x^1,\ldots,x^r},$$

where $\alpha^{(x^{r+1},\ldots,x^m)}$ is the inner automorphism corresponding (in sense of Lemma 4.13) to translation $x \mapsto x + \tilde{\Phi}_1^{(x^{r+1},\ldots,x^m)}(0,\ldots,0)$. So, the equivalence means that

$$\Phi_{1*}^{x^{r+1},\ldots,x^m} R^{(S_1\circ\Phi_2)(x^{r+1},\ldots,x^m)} \circ \alpha^{(x^{r+1},\ldots,x^m)} = R^{S_2(x^{r+1},\ldots,x^m)} \circ \alpha,$$

so $R^{(S_1\circ\Phi_2)(x^{r+1},\ldots,x^m)}$ is equivalent to $R^{S_2(x^{r+1},\ldots,x^m)}$, which holds if and only if the corresponding classes of subalgebras coincide, so $S_1 \circ \Phi_2 = S_2$, which means that S_1 is equivalent to S_2. $\square$

Remark 4.20. The meaning of the map S is that for a realization R of the form (5), the Int-class of subalgebras $S(x^{r+1},\ldots,x^m)$ corresponds to the transitive restriction of R in $(x^1,\ldots,x^m)$, where $x^1,\ldots,x^r$ are arbitrary and determine only the class representative. Therefore, outer automorphisms act on R only by modifying these subalgebra classes. For $\alpha\in\operatorname{Aut}\mathfrak{g}$ we have

$$R(\alpha(e_j))_{x^1,\ldots,x^m} = R^{(\bar\alpha\circ S)(x^{r+1},\ldots,x^m)}(e_j)_{x^1,\ldots,x^r}, \tag{6}$$

where $\bar\alpha\in\operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g}$ is the corresponding class of α. Therefore, regular realizations corresponding to S_1 and S_2 are Aut-equivalent if and only if $S_1=\bar\alpha\circ S_2$.

Remark 4.21. If the map S is constant, the resulting realization is of the form

$$R(e_j)_{x^1,\ldots,x^m} = R^{\mathfrak{h}_0}(e_j)_{x^1,\ldots,x^r},$$

so it has the same form as the original transitive realization. It is just formally defined on larger manifold. Such a regular realization will be called *trivial extension* of the transitive realization.

Lemma 4.22. The kernel of a realization of the form (5) is the largest common ideal contained in representatives of subalgebra classes $S(x^{r+1},\ldots,x^m)$ for $(x^{r+1},\ldots,x^m)$ in a small neighbourhood of zero. Thus, the realization is faithful if and only if the corresponding subalgebras does not contain a common non-trivial ideal.

Proof. Follows from Lemma 4.9. □

Example 4.2. Let us try to classify all regular realizations of two-dimensional Abelian Lie algebra $2\mathfrak{g}_1=\operatorname{span}\{e_1,e_2\}$. In this case, there are no non-trivial inner automorphisms, so Int-equivalence is the same as strong equivalence. There is of course only one zero-dimensional subalgebra that corresponds to transitive realization

$$e_1\mapsto\partial_1,\quad e_2\mapsto\partial_2. \tag{7}$$

Since the set $\mathscr{S}_2$ contains only one element, all regular realizations with rank two are obtained as trivial extension of this realization.

The set of one-dimensional subalgebras form a circle and can be parametrized by $\varphi\in[0,2\pi)$ as $\mathfrak{h}_\varphi=\operatorname{span}\{\cos\varphi\, e_1+\sin\varphi\, e_2\}$. The corresponding transitive realizations are following

$$R^{\mathfrak{h}_\varphi}(e_1)_{x_1}=\sin\varphi\,\partial_1,\quad R^{\mathfrak{h}_\varphi}(e_2)_{x_1}=\cos\varphi\,\partial_1. \tag{8}$$

Therefore, the set of all regular local realizations with rank one consists of following realizations

$$R^f(e_1)_{x_1,\ldots,x_m}=\sin f(x_2,\ldots,x_m)\,\partial_1,\quad R^f(e_1)_{x_1,\ldots,x_m}=-\cos f(x_2,\ldots,x_m)\,\partial_1, \tag{9}$$

where $f\colon\mathbb{R}^{m-1}\to\mathbb{R}$ are arbitrary functions. Such realizations R_f and R_g are equivalent if and only if there exists a local diffeomorphism $\Psi\colon\mathbb{R}^{m-1}\to\mathbb{R}^{m-1}$ at zero such that $g=f\circ\Psi$.

Now let us try to do this classification with respect to all automorphisms. We only have to describe, when R^f and R^g are Aut-equivalent. In this case, $\operatorname{Aut}(2\mathfrak{g}_1)$ consists of all invertible linear maps. One of such map is rotation, which acts as a rotation also on the set $\mathscr{S}_1=\{\mathfrak{h}_\varphi\}$. Thus, realizations R_f and R_g, where f and g differ by a constant, are Aut-equivalent. We can therefore fix, for example, $f(0,\ldots,0)=0$.

For a general automorphism $\alpha\in\operatorname{Aut}(2\mathfrak{g}_1)=\operatorname{GL}(2\mathfrak{g}_1)$, the subalgebra $\alpha(\mathfrak{h}_{f(x_2,\ldots,x_m)})$ is generated by

$$\begin{pmatrix}\alpha_{11}&\alpha_{12}\\ \alpha_{21}&\alpha_{22}\end{pmatrix}\begin{pmatrix}\cos f(x_2,\ldots,x_m)\\ \sin f(x_2,\ldots,x_m)\end{pmatrix}$$

Such an automorphism preserves the property $f(0,\ldots,0)=0$ if and only if $\alpha_{11}=1$ and $\alpha_{21}=0$. Fixing these entries, α is an automorphism if and only if $\alpha_{22}\neq 0$.

So, the conclusion is that every regular local realization with rank one is Aut-equivalent to R_f, where $f(0,\ldots,0)=0$. Such realizations R_f and R_g are Aut-equivalent if and only if

$$\cos(g\circ\Psi)=\frac{\cos f+\alpha_{12}\sin f}{(\cos f+\alpha_{12}\sin f)^2+(\alpha_{22}\sin f)^2}, \tag{10}$$

$$\sin(g\circ\Psi)=\frac{\alpha_{22}\sin f}{(\cos f+\alpha_{12}\sin f)^2+(\alpha_{22}\sin f)^2}, \tag{11}$$

where $\alpha_{12},\alpha_{22}\in\mathbb{R}$, $\alpha_{22}\neq 0$, Ψ is a local diffeomorphism in $m-1$ variables.

Example 4.3. One should pay attention to the fact that the space $\bar{\mathscr{S}}_m$ does not have to be Hausdorff. Take a non-commutative two-dimensional Lie algebra $\mathfrak{g}_2=\operatorname{span}\{e_1,e_2\}$, $[e_1,e_2]=e_1$. All one-dimensional subspaces of the form $\mathfrak{h}_2^a:=\operatorname{span}\{e_2+ae_1\}$ are equivalent to $\mathfrak{h}_2^0=\operatorname{span}\{e_2\}$ with respect to inner automorphisms. Therefore, there are only two Int-classes of one-dimensional subalgebras represented by $\mathfrak{h}_1:=\operatorname{span}\{e_1\}$ and $\mathfrak{h}_2^0$. This, however, does not mean that there are no non-constant smooth curves $S\colon\mathbb{R}\to\bar{\mathscr{S}}_1$. We can take, for example, map $S(x)=\overline{\operatorname{span}\{e_1+xe_2\}}$ that is evidently smooth despite it is equal to $\bar{\mathfrak{h}}_1$ in zero and $\bar{\mathfrak{h}}_2^0$ everywhere else. Therefore, it leads to a new regular local realization not equivalent to trivial extension of the transitive ones.

Remark 4.23. Similar procedure was suggested in [20] with reference to a private letter from Shirokov (the authors suggested replacing the parameters by variables, not functions of variables), but it was not discussed what kind of new realizations are obtained or whether the list of realizations is complete.

4.7 Classification problem

In this section, we are going to discuss, what is the reasonable classification problem for (possibly general) Lie algebra realizations. Our definition is inspired by article [26], which is probably the so far most extensive result on realizations classification, where the authors claim to classify all realizations of all Lie algebras of dimension less or equal to four. Nevertheless, they do not specify precisely what is the actual classification problem they were solving. They only formulate Definition 4.3 and mention that they are working locally. Their result is not a complete classification of all local realizations (nor classification of some suitable subclass as regular or transitive realizations). In this section, we formulate the problem clearly and rigorously.

Sets of (local) realizations will be denoted by capital script letters $\mathscr{R},\mathscr{T},\ldots$ The sets of corresponding A-classes will be denoted with bar $\bar{\mathscr{R}}=\{\bar{R}\mid R\in\mathscr{R}\}$.

Let $\bar{\mathscr{R}}_{\rm all}$ be the system of all classes of local realizations of a given Lie algebra $\mathfrak{g}$ with respect to a given group of automorphisms $A\subset\operatorname{Aut}\mathfrak{g}$. A complete classification of local realizations would mean to find this system or, more precisely, to find a set of representatives $\mathscr{R}_{\rm all}$ that would contain precisely one representative of every class in $\bar{\mathscr{R}}_{\rm all}$. We prefer to choose representatives defined in a neighborhood at $0\in\mathbb{R}^m$. Every global realization would be at every point locally equivalent to a realization from our list. However, such classification would be very hard to perform. Nevertheless, the situation will get much more simple if we only require that every global realization is equivalent to a local realization from our list at every point *from a dense subset.*

Definition 4.24. Let $\mathfrak{g}$ be a Lie algebra, $A\subset\operatorname{Aut}\mathfrak{g}$ a group of its automorphisms. Let $\bar{\mathscr{R}}$ be a system of local realizations classes of $\mathfrak{g}$. We will say that $\bar{\mathscr{R}}$ is *complete* if for every realization R on any manifold M there is a point $p\in M$ such that $\bar{R}|_p\in\bar{\mathscr{R}}$.

Note that this condition consequently means that for every realization R on any manifold M there is a dense subset of points $p \in M$ such that $\bar{R}|_p \in \bar{\mathscr{R}}$ since the realization R can be restricted to arbitrary open subset where the point has to exist as well.

We can reformulate this condition for the corresponding set of representatives. A set of local realizations $\mathscr{R}$ is called *complete* if the corresponding system $\bar{\mathscr{R}}$ is complete. It can be easily seen that such a set $\mathscr{R}$ is complete if and only if and only if for every realization R on any manifold M there is a point $p \in M$ and a local realization $R' \in \mathscr{R}$ such that $R|_p$ is A-equivalent to R'.

If we consider a complete system of local realizations $\mathscr{R}$ such that all elements are defined in a neighbour of zero in $\mathbb{R}^m$, then the completeness means following. For every realization R on any manifold M there exist a point $p \in M$ (in fact a dense set of such points), coordinates with origin at this point, and a realization $R_0 \in \mathscr{R}$ such that the coordinate expression for R coincides with R_0 (up to automorphisms).

Example 4.4. Let us take two dimensional non-commutative Lie algebra $\mathfrak{g}_2 = \operatorname{span}\{e_1, e_2\}$, $[e_1, e_2] = e_1$. It can be shown that the complete system of local realizations in one variable of $\mathfrak{g}_2$ can be chosen to contain only zero realization and

$$R(e_1)_x = \frac{\mathrm{d}}{\mathrm{d}x}, \quad R(e_2)_x = x\frac{\mathrm{d}}{\mathrm{d}x}. \tag{12}$$

In [29] Spichak classified realizations of $\mathfrak{g}_2$ on circle. He used weaker definition of realization and equivalence, but we can use his results as an example of global realizations

$$R_n(e_1)_\vartheta = (\cos(n\vartheta) - 1)\frac{\mathrm{d}}{\mathrm{d}\vartheta}, \quad R_n(e_2)_\vartheta = \left(\cos(n\vartheta) - 1 - \frac{1}{n}\sin(n\vartheta)\right)\frac{\mathrm{d}}{\mathrm{d}\vartheta}, \tag{13}$$

where $\vartheta \in [0, 2\pi)$ parametrizes the circle and $n \in \mathbb{N}$. The completeness of the system $\mathscr{R} = \{0, R\}$ means that for all the realizations R_n there is a dense subset of the circle such that for all points of this subset the realization is locally equivalent to R (or zero). Indeed, for all $\vartheta_0 \neq \frac{k\pi}{n}$, $k \in \{0, 1, \ldots, n-1\}$, $R_n|_{\vartheta_0}$ is equivalent to R through transformation

$$\vartheta \mapsto x = \frac{1}{n}\left(\cot\left(\frac{n\vartheta}{2}\right) - \cot\left(\frac{n\vartheta_0}{2}\right)\right). \tag{14}$$

Now, we are going to formulate a condition for a subsystem of a complete system of local realizations to stay complete.

Definition 4.25. Let $\bar{\mathscr{R}}$ be a system of classes of local realizations of a Lie algebra $\mathfrak{g}$ with respect to a group of automorphisms $A \subset \operatorname{Aut}\mathfrak{g}$, $\bar{\mathscr{R}}' \subset \bar{\mathscr{R}}$. We will say that $\bar{\mathscr{R}}'$ is a *sufficient subsystem* of $\bar{\mathscr{R}}$ if for all classes $\bar{R} \in \bar{\mathscr{R}}$ and all their representatives $R \in \bar{R}$ there exists $q \in \operatorname{Dom} R$ such that $\bar{R}|_q \in \bar{\mathscr{R}}'$.

Lemma 4.26. Let $\bar{\mathscr{R}}$ be a complete system of local realizations classes of a Lie algebra $\mathfrak{g}$ with respect to $A \subset \operatorname{Aut}\mathfrak{g}$. A subsystem $\bar{\mathscr{R}}' \subset \bar{\mathscr{R}}$ is complete if and only if $\bar{\mathscr{R}}'$ is sufficient.

Proof. The left-right implication follows directly from the definition of completeness of the system $\bar{\mathscr{R}}'$.

Now let us take a realization R on a manifold M. From completeness of $\bar{\mathscr{R}}$ there is a point $p \in M$ such that $\bar{R}|_p \in \mathscr{R}$. But from the definition of sufficient subsystem, taking R as a representative of $R|_p \in \mathscr{R}$, there exists $q \in M$ such that $R|_q \in \mathscr{R}'$ □

Again, we transfer this condition to the corresponding sets of representatives. For a set of local realizations $\mathscr{R}$, its subset $\mathscr{R}' \subset \mathscr{R}$ is called *sufficient* with respect to a group $A \subset \operatorname{Aut}\mathfrak{g}$ if $\bar{\mathscr{R}}'$ is sufficient subsystem of $\bar{\mathscr{R}}$.

Proposition 4.27. Let $\mathscr{R}$ be a set of local realizations of a Lie algebra $\mathfrak{g}$. A subset $\mathscr{R}' \subset \mathscr{R}$ is sufficient with respect to $A \subset \operatorname{Aut}\mathfrak{g}$ if for all local realizations $R \in \mathscr{R}$ at some $p \in \operatorname{Dom} R$ and for every neighbourhood U of p there exists $q \in U$ and a realization $R' \in \mathscr{R}'$ such that $R|_q$ is A-equivalent to R'.

Proof. Take sets of local realizations $\mathscr{R}' \subset \mathscr{R}$ and denote $\bar{\mathscr{R}}'$ and $\bar{\mathscr{R}}$ the corresponding systems of A-classes of local realizations.

First we are going to prove the left-right implication, so assume that $\bar{\mathscr{R}}'$ is sufficient. Take $R \in \mathscr{R}$ a local realization at p, U a neighbourhood of p. Then $\overline{R|_U} \in \bar{R} \in \bar{\mathscr{R}}$. From completeness of $\bar{\mathscr{R}}'$ there exists a $q \in \operatorname{Dom} R|_U = U$ such that $\overline{(R|_U)|_q} = \bar{R}|_q \in \bar{\mathscr{R}}'$, so there exists $R' \in \mathscr{R}'$ such that $R|_q$ is A-equivalent to R'.

For the right-left implication, we have to prove that $\mathscr{R}'$ is sufficient assuming the condition on the right hand side. So, take $\bar{R} \in \bar{\mathscr{R}}$, $R \in \bar{R}$. Choose a representative $R_0 \in \bar{R}$ that is contained in $\mathscr{R}$, so R_0 is A-equivalent to R. This means that there exists $U_0 \subset \operatorname{Dom} R_0$ and $U \subset \operatorname{Dom} R$, $\alpha \in A$ and $\Phi\colon U_0 \to U$ a diffeomorphism such that $R|_U \circ \alpha = \Phi_* R_0|_{U_0}$. By assumption there exists $q_0 \in U_0$ and $R' \in \mathscr{R}'$ such that R' is equivalent to $R_0|_{q_0} = (R_0|_{U_0})|_{q_0}$, which is also equivalent to $(R|_U)_q = R|_q$, where $q = \Phi(q_0) \in U$. Therefore, $R|_q$ is equivalent to R' and hence $\bar{R}|_q \in \bar{\mathscr{R}}'$. □

We claim that the reasonable classification problem is to find a “small” complete system of local realizations. It is also more or less the classification problem that was solved in [26]. Note however that it is, for example, not true that an intersection of two complete systems of classes would be complete. As we are going to show in Example 4.6, complete systems do not have to have the least element or a minimal element with respect to ordering by set inclusion.

From Lemma 4.17 it follows that system of all regular realizations is complete. Moreover, we can formulate the following lemma.

Lemma 4.28. Let $\mathscr{R}$ be a complete system and $\mathscr{R}'$ its subsystem containing all regular realizations of $\mathscr{R}$. Then $\mathscr{R}'$ is complete.

Proof. According to Lemma 4.26 we have to show that for every class of singular realizations of $\bar{R} \in \bar{\mathscr{R}}$ and every representative $R \in \bar{R}$ there exists a $q \in \operatorname{Dom} R$ such that $\bar{R}|_q \in \bar{\mathscr{R}}'$.

Choosing an open set $U \subset \operatorname{Dom} R$ of regular points of R we can define restriction $R|_U$, which is a regular realization, and from completeness of $\mathscr{R}$ there is point $q \in U$ such that $\bar{R}|_q \in \bar{\mathscr{R}}$ and since it is regular, it belongs to $\bar{\mathscr{R}}'$. □

Consequently, looking for a classification of realizations, one can deal with regular realizations only. Looking for complete system instead of classification of all regular realizations or even completely all realizations simplify the result as we are going to show on simple examples.

Example 4.5. Let us take the one dimensional Lie algebra $\mathfrak{g}_1$ spanned by one element e_1. It is evident that any realization R on any manifold M is of the form $R_p(e_1) = X_p$, where X is an arbitrary vector field on M. In particular, any local realization at zero in $\mathbb{R}^m$ is of the form

$$R_{x^1,\ldots,x^m}(e_1) = \sum_{j=1}^{m} f^j(x^1,\ldots,x^m)\partial_{x^j}, \tag{15}$$

where f^j are arbitrary smooth functions. The classification [26], however, state that there is only one class of equivalence for every $m \in \mathbb{N}$ (except for the only unfaithful realization, which is zero) represented by realization

$$R_{x^1,\dots,x^m}(e_1) = \partial_{x^1}. \tag{16}$$

A general realization of the form (15) taken in the neighbourhood of zero can be transformed into the realization given by (16) if and only if $f^j(0) \neq 0$ for some j. So, this is not the solution of classification with respect to Definition 4.4.

Moreover, we can see that this problem does not have a reasonable solution, because realizations of the form (15) can be equivalent only if the sets $\{x \mid f(x) = 0\}$ are diffeomorphic, so we would have to classify such functions, but there are too many different possibilities. And the situation is, of course, even more complicated for less trivial Lie algebras.

Nevertheless, it actually is true that for every realization R of $\mathfrak{g}_1$ on some manifold M there is a point $p \in M$ such that R is locally equivalent to ∂_{x^1} in p. So, the realization (16) together with zero realization form the complete system of realizations of the one-dimensional Lie algebra. (Actually, they form a classification of all regular realizations.)

Now, if we are interested, for example, in analytic global realizations of $\mathfrak{g}_1$ on a given manifold M, we can construct them as analytic continuations of local realizations defined in some open subset of M that are equivalent to (16).

For example, take M to be a line $\mathbb{R}$. One instance of a global realization on $\mathbb{R}$ is $R_x(e_1) = \frac{\mathrm{d}}{\mathrm{d}x}$, $x \in \mathbb{R}$. Now, by applying a diffeomorphism $y = \Phi(x) = \exp x$, which maps $\mathbb{R} \to \mathbb{R}^+$, on this realization, we get

$$R'_y(e_1) = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y} = \mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}y} = y\frac{\mathrm{d}}{\mathrm{d}y}, \quad y \in \mathbb{R}^+,$$

whose analytic continuation on $\mathbb{R}$ is $R''_x(e_1) = x\frac{\mathrm{d}}{\mathrm{d}x}$, $x \in \mathbb{R}$, which is not equivalent to the former realization R.

Example 4.6. Now, consider two-dimensional Abelian Lie algebra $2\mathfrak{g}_1$ and no automorphisms $A = \{\mathrm{id}\} = \mathrm{Int}(2\mathfrak{g}_1)$, which was already examined in Example 4.2, where we presented classification of all regular realizations. The result was that every regular realization with rank one in m variables is equivalent to following

$$R^f(e_1)_{x_1,\dots,x_m} = \sin f(x_2,\dots,x_m)\,\partial_1, \quad R^f(e_2)_{x_1,\dots,x_m} = -\cos f(x_2,\dots,x_m)\,\partial_1,$$

where $f\colon \mathbb{R}^{m-1} \to \mathbb{R}$ are arbitrary functions defined locally in a neighborhood of zero. Here it can be shown, that the sufficient subsystem of those realizations are formed by following

$$R_1^a(e_1)_{x_1,\dots,x_m} = \partial_1, \quad R_1^a(e_2)_{x_1,\dots,x_m} = a\partial_1, \tag{17}$$

$$R_2^a(e_1)_{x_1,\dots,x_m} = \partial_1, \quad R_2^a(e_2)_{x_1,\dots,x_m} = (a + x_2)\partial_1, \tag{18}$$

where a is a real parameter. So, we were able to get rid of the general function f. However, for Lie algebras of higher dimension we will already not be able to avoid families of realizations parametrized by functions.

Note that we could restrict the parameter a for the second realization R_2 to any dense subset of $\mathbb{R}$. Therefore, there is no minimal complete system of realizations classes for $2\mathfrak{g}_1$ with respect to strong equivalence.

4.8 Sufficient subsystems of regular realizations

In this section we present an algorithm for construction of complete system of local realizations.

First of all, we need to parametrize the subalgebra classes properly. That is, to find a decomposition of $\bar{\mathscr{S}}_r$ to a finite disjoint union $\bigcup_i S_r^i(D_i)$, where $S_r^i\colon D_i \to \bar{\mathscr{S}}_r$ is embedding of D_i a domain in $\mathbb{R}^{s_i}$. Such a parametrization will be called *proper* if for every smooth map $S\colon U \to \bar{\mathscr{S}}_r$, where U is a neighborhood of zero in $\mathbb{R}^s$, there exists $x \in U$ and its neighborhood V such that $S(V) \subset S_r^i(D_i)$ for some i.

Proposition 4.29. Considering a proper parametrization, the regular local realizations corresponding to local $\bar{\mathscr{S}}_r$-valued maps $S_r^i \circ F$, where $F\colon \mathbb{R}^{m-r} \to \mathbb{R}^{s_i}$, form a sufficient subsystem of all regular local realizations with rank r.

Proof. For a general map $S\colon \mathbb{R}^{m-r} \to \bar{\mathscr{S}}_r$ we find the corresponding neighborhood V. Since $S(V) \subset S_r^i(D_i)$ for some i, there exists $F\colon \mathbb{R}^{m-r} \to \mathbb{R}^{s_i}$ such that $S = S_r^i \circ F$ on V. $\square$

Finally, we can try to simplify the map F. We formulate the result in Lemma 4.30. Since its formulation is rather complicated, we illustrate it on an example.

Example 4.7. Take a three-dimensional Abelian Lie algebra $3\mathfrak{g}_1 = \operatorname{span}\{e_1, e_2, e_3\}$. There are no inner automorphisms. Every one-dimensional subspace is a subalgebra. Therefore, we can present the following proper parametrization of the set of one-dimensional subalgebras $\mathscr{S}_2$ and compute the corresponding transitive realizations.

$$\begin{aligned} \operatorname{span}\{e_1\} &\qquad e_1 \mapsto 0, \quad e_2 \mapsto \partial_1, \quad e_3 \mapsto \partial_2 \\ \operatorname{span}\{e_2 - ae_1\} &\qquad e_1 \mapsto \partial_1, \quad e_2 \mapsto a\partial_1, \quad e_3 \mapsto \partial_2 \\ \operatorname{span}\{e_3 - ae_2 - be_1\} &\qquad e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto a\partial_1 + b\partial_2 \end{aligned}$$

According to Proposition 4.29, the following realizations form a sufficient subsystem of local realizations in m variables with rank two

$$\begin{aligned} &e_1 \mapsto 0, \quad e_2 \mapsto \partial_1, \quad e_3 \mapsto \partial_2, \\ &e_1 \mapsto \partial_1, \quad e_2 \mapsto f(x_3, \ldots, x_m)\partial_1, \quad e_3 \mapsto \partial_2, \\ &e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto f(x_3, \ldots, x_m)\partial_1 + g(x_3, \ldots, x_m)\partial_2, \end{aligned}$$

where f and g are arbitrary local functions in $m-2$ variables. We will try to find a sufficient subsystem of the family of realizations in the last row.

Several cases can take place here. At first, if both f and g are constant in some neighborhood of zero, then we get a trivial extension of the original transitive realization only. Secondly, one of the functions may be locally constant at zero, while the other might not be. Then it means there is a point $(\varepsilon_3, \ldots, \varepsilon_m)$ in a neighborhood of zero, where the first function, for example f, is locally constant equal to a, while the second function g has a non-zero partial derivative with respect to some coordinate x_i, $i \geq 3$. Without lost of generality, we can assume $\frac{\partial g}{\partial x_3}\Big|_{(\varepsilon_3, \ldots, \varepsilon_m)} \neq 0$ (otherwise we can change the order of coordinates in the first place). Now we present a change of coordinates

$$x_3 \mapsto y_3 := g(x_3 - \varepsilon_3, \ldots, x_m - \varepsilon_m) - c,$$

$$x_i \mapsto y_i := x_i - \varepsilon_i, \quad i > 3$$

where $c = g(\varepsilon_3, \ldots, \varepsilon_m)$. In those coordinates, the realization taken in the neighborhood of $(\varepsilon_3, \ldots, \varepsilon_m)$, which has new coordinates $y_i = 0$, has the form

$$e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto a\partial_1 + (c + y_3)\partial_2.$$

Analogically, we obtain a realization

$$e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto (c + y_3)\partial_1 + b\partial_2.$$

Finally, both functions might not be constant. Then the function f can again be transformed into $a + y_3$. The function g might then depend only on y_3, so we get

$$e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto (a + y_3)\partial_1 + \tilde{g}(y_3)\partial_2$$

or it can depend on other variables as well (if $m \geq 4$), so it can be transformed into y_4, so we get

$$e_1 \mapsto \partial_1, \quad e_2 \mapsto \partial_2, \quad e_3 \mapsto (a + y_3)\partial_1 + (b + y_4)\partial_2.$$

Lemma 4.30. Let $\mathfrak{g}$ be a Lie algebra, $\bar{\mathscr{S}}_r$ be the system of all Int-classes of $\mathfrak{g}$ subalgebras with codimension r. Let D be an open domain in $\mathbb{R}^s$, $S\colon D \to \bar{\mathscr{S}}_r$ a smooth map. Then the system of realizations of $\mathfrak{g}$ of the form

$$R(e_j)_{x^1,\ldots,x^m} = R^{(S\circ F)(x^{r+1},\ldots,x^m)}(e_j)_{x^1,\ldots,x^r}, \tag{19}$$

where $F\colon \mathbb{R}^{m-r} \to D$ is a smooth function, and $R^{(S\circ F)(x^{r+1},\ldots,x^m)}$ are transitive local realizations at $0 \in \mathbb{R}^r$ corresponding to the subalgebra class $(S\circ F)(x^{r+1},\ldots,x^m)$ chosen in a way that they smoothly depend on the coordinates, has a sufficient subsystem with respect to strong equivalence (so with respect to Int-equivalence as well) consisting of the following local realizations at $0 \in \mathbb{R}^m$

$$R(e_j)_{x^1,\ldots,x^m} = R^{S(c^1+f^1(x^{r+1},\ldots,x^m),\ldots,c^s+f^s(x^{r+1},\ldots,x^m))}(e_j)_{x^1,\ldots,x^r}, \tag{20}$$

where $(c^1,\ldots,c^s) \in D$ are constant numbers and f^j are local functions mapping $f^j(0,\ldots,0) = 0$ that are of the following form. Set $l_0 = 0$, then for all $j > 0$ either $f^j(x^{r+1},\ldots,x^m) = x^{l_j}$, where $l_j = l_{j-1} + 1$, or $f^j(x^{r+1},\ldots,x^m)$ depend only on first l_j variables, where $l_j = l_{j-1}$. That is, either f^j is equal to a "new" variable or it depends only on "already used" variables.

Proof. We have to find a suitable local diffeomorphism $\Psi\colon \mathbb{R}^{m-r} \to \mathbb{R}^{m-r}$ mapping a point in an arbitrarily small neighbourhood of zero onto zero such that $F^j \circ \Psi = c^j + f^j$ on an even smaller neighbourhood of this point.

Let us start with F^1. If it is locally constant at zero (i.e., there is a neighbourhood of zero such that F^1 is constant on this neighbourhood), then it does not need to be transformed. We just have to restrict ourselves on this neighbourhood. Otherwise, there exists at every neighbourhood of zero a point x_1 and an index $j \in \{1,\ldots,m-r\}$ such that $\left.\frac{\partial F^1}{\partial x^{r+j}}\right|_{x_1} \neq 0$. Without loss of generality, assume $j = 1$ (otherwise, apply diffeomorphism changing the order of variables at first). We can apply a diffeomorphism Ψ_1 mapping

$$x^{r+1} \mapsto y^{r+1} = F^1(x^{r+1} - x_1^{r+1},\ldots,x^m - x_1^m) - c_1^1,$$
$$x^{r+j} \mapsto y^{r+j} = x^{r+j} - x_1^{r+j}, \quad j > 1$$

where $c_1^1 = F^1(x_1^{r+1},\ldots,x_1^m)$, so $(F^1 \circ \Psi_1)(y^{r+1},x^{r+2},\ldots,x^m) = c_1^1 + y^{r+1}$.

Then we proceed by induction. Assume, we have found a diffeomorphism Ψ_k, such that $(F^i \circ \Psi_k)(x^{r+1},\ldots,x^m) = c_k^i + f_k^i(x^{r+1},\ldots,x^m)$ for all $i \leq k$ on some neighbourhood of zero. Then we examine $F^{k+1} \circ \Psi_k$. If it is locally constant at zero, then nothing has to be done, so $\Psi_{k+1} = \Psi_k$. Second possibility is that F^{k+1} depends only on $x^{r+1},\ldots,x^{l_k-1}$. Then, again, nothing has to be done. If it has non-zero partial derivative with respect to x^j, $j \geq l_k$ in a point x_{k+1} arbitrarily close to x_k, without loss of generality let $j = l_k$, then we can introduce a diffeomorphism $\tilde{\Psi}_{k+1}$ sending

$$x^{r+k+1} \mapsto y^{r+k+1} = F^{k+1}(x^{r+1} - x^{r+1}_{k+1}, \ldots, x^m - x^m_{k+1}) - c^{k+1}_{k+1},$$

$$x^{r+j} \mapsto y^{r+j} = x^{r+j} - x^{r+j}_{k+1}, \quad j \neq k,$$

where $c^{k+1}_{k+1} = F^{k+1}(x^{r+1}_{k+1}, \ldots, x^m_{k+1})$. Then if we define $\Psi_{k+1} := \tilde{\Psi}_{k+1} \circ \Psi_k$, we have $(F^{k+1} \circ \Psi_{k+1})(y^{r+1}, \ldots, y^m) = c^{k+1}_{k+1} + y^{k+1}$. The form of $F^i \circ \Psi_{k+1}$ for $i < k+1$ remains the same, only the constants c^i_k change to new c^i_{k+1} because of the translation. □

The functions $\vec{c} + \vec{f}$ in Lemma 4.30 were constructed in such a way that they are mutually inequivalent with respect to change of coordinates, so the set of realizations constructed by this lemma contains mutually Int-inequivalent realizations. Moreover, small translations in the variables $x^1, \ldots, x^r$ cause only a transformation of the whole realization by some inner automorphism. Small translations in the variables $x^{r+1}, \ldots, x^m$ do either nothing (if R does not depend on those variables) or essentially only changes those parameters c^j for which $f^j(x^{r+1}, \ldots, x^m) = x^{r+l_j}$. So, we have the following proposition.

Proposition 4.31. The system of regular local realizations described by Lemma 4.30 contain mutually inequivalent realizations with respect to inner automorphisms. All sufficient subsystems of this system with respect to inner automorphisms are obtained by restricting those parameters c^j such that $f^j(x^{r+1}, \ldots, x^m) = x^{r+l_j}$ to a dense subset of their domain of definition.

Finally, we can summarize the algorithm for construction of complete system of realizations into the following theorem.

Theorem 4.32. Let

$$\bar{\mathscr{S}}_m = \bigcup_i \bar{S}^i_m(D_i), \quad D_i \subset \mathbb{R}^{s_i} \tag{21}$$

be a proper parametrization of classes of $\mathfrak{g}$ subalgebras with codimension m. Let $\mathscr{S}_m = \bigcup_i S^i_m(D_i)$ be such set of their representatives that S^i_m are smooth. Let $\mathscr{T}_m = \bigcup_i R^{i\,(D_i)}_m$ be the corresponding local transitive realizations. Let $\mathscr{R}^i_m$ be the sets of regular realizations constructed as in Lemma 4.30 from s_i-parameter sets $R^{i\,(D_i)}_m$ of transitive realizations. Then the system of local realizations $\mathscr{R} = \bigcup_m \left(\mathscr{T}_m \cup \bigcup_i \mathscr{R}^i_m\right)$ is complete with respect to inner automorphisms and contains mutually Int-inequivalent local realizations.

Remark 4.33. If we are doing classification with respect to strong equivalence, all we have to do in the end is to apply the inner automorphisms to the complete system we have found. This essentially mean putting back the parameters we have eliminated when doing the classification with respect to inner automorphisms.

Remark 4.34. If we are doing classification with respect to all automorphism, we make use of Remark 4.20. Often it only means to "remove the unnecessary parameters" using automorphisms. The situation is more complicated, if we have realizations parametrized by functions.

Finally, we are going to illustrate the presented algorithm for construction of complete system of realizations, on a more complicated example. We illustrate classification with respect to all strong, Int-, and Aut-equivalence.

Example 4.8. Let us consider a Lie algebra $\mathfrak{g} = \mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1 = \operatorname{span}\{e_1, e_2, e_3, e_4\}$, $[e_1, e_2] = e_1$. We are going to find a complete system of realizations with rank three.

The groups of automorphisms expressed in the basis (e_1, e_2, e_3, e_4) are following

$$\operatorname{Int}\mathfrak{g} = \left\{ \begin{pmatrix} \tilde{t}_1 & t_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, \tilde{t}_1 \in \mathbb{R}^+, t_2 \in \mathbb{R} \right\} \tag{22}$$

$$\operatorname{Aut}\mathfrak{g} = \left\{ \begin{pmatrix} t_1 & t_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t_3 & t_5 & t_6 \\ 0 & t_4 & t_7 & t_8 \end{pmatrix} \middle| \, \begin{array}{c} t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8 \in \mathbb{R}, \\ t_1 \neq 0, \ t_5 t_8 - t_6 t_7 \neq 0 \end{array} \right\} \tag{23}$$

Of course, all one-dimensional subspaces are also one-dimensional subalgebras. In Table 4.1 we classify the one-dimensional subalgebras. In the first column, parametrization of all subalgebras is presented. In the second column we choose an Int-representative for each subalgebra. In the third column we choose the Aut-representatives. We denote $\sigma_x = \operatorname{sgn} x$ the sign of x. In the last column we express a condition for the subalgebra to be an ideal.

	Subalgebras	Int-classes	Aut-classes	Is ideal?
$\mathfrak{h}_1$	e_1	e_1	e_1	Yes.
$\mathfrak{h}_2^c$	$e_3 + ce_1$	$e_3 + \sigma_c e_1$	$e_4 + \lvert\sigma_c\rvert e_1$	If $c = 0$.
$\mathfrak{h}_3^{a,c}$	$e_4 + ae_3 + ce_1$	$e_4 + ae_3 + \sigma_c e_1$	$e_4 + \lvert\sigma_c\rvert e_1$	If $c = 0$.
$\mathfrak{h}_4^{a,b,c}$	$e_2 + ae_3 + be_4 + ce_1$	$e_2 + ae_3 + be_4$	e_2	No.

Table 4.1. One-dimensional subalgebras of $\mathfrak{g}$

The Shirokov's computation leads to classification of transitive realizations with respect to strong equivalence listed in Table 4.2. In the second column, a realization corresponding to subalgebra in the first column is presented. The entry consists of images of the basis elements e_1, e_2, e_3 and e_4. In the last column the negation of last column of Table 4.1 expresses a condition for the realization to be faithful.

	Subalgebras	Realizations	Is faithful?
$\mathfrak{h}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3$	No.
$\mathfrak{h}_2^c$	$e_3 + ce_1$	$\partial_1,\ x_1\partial_1 + \partial_2,\ -c\mathrm{e}^{x_2}\partial_1,\ \partial_3$	If $c \neq 0$.
$\mathfrak{h}_3^{a,c}$	$e_4 + ae_3 + ce_1$	$\partial_1,\ x_1\partial_1 + \partial_2,\ \partial_3,\ -c\mathrm{e}^{x_2}\partial_1 - a\partial_3$	If $c \neq 0$.
$\mathfrak{h}_4^{a,b,c}$	$e_2 + ae_3 + be_4 + ce_1$	$\partial_1,\ (x_1 - c)\partial_1 - a\partial_2 - b\partial_3,\ \partial_2,\ \partial_3$	Yes.

Table 4.2. Classification of local transitive realizations of $\mathfrak{g}$ in three variables

The classification with respect to inner or all automorphisms is obtained easily by substituting the parameters of subalgebras by parameters of the chosen representatives. By doing so in case of inner automorphisms and applying Lemma 4.30, we get a complete system of realizations of $\mathfrak{g}$ with respect to inner automorphisms and list it in Table 4.3. The function $f\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary local function satisfying $f(0) = 0$. All realizations are Int-inequivalent.

Now, let us make the classification with respect to strong equivalence. As we mentioned in Remark 4.33, we just have to put back the eliminated parameters. That is, instead of listing realizations corresponding, for example, to $\mathfrak{h}_4^{a,b+x_4,0}$, we list realizations corresponding to all $\mathfrak{h}_4^{a,b+x_4,c}$. The result is in Table 4.4. All realizations are strongly inequivalent.

Subalgebra	Realization
$\mathfrak{h}_1$	$0,\ \partial_1,\ \partial_2,\ \partial_3$
$\mathfrak{h}_2^0$	$\partial_1,\ x_1\partial_1+\partial_2,\ 0,\ \partial_3$
$\mathfrak{h}_2^1$	$\partial_1,\ x_1\partial_1+\partial_2,\ -\mathrm{e}^{x_2}\partial_1,\ \partial_3$
$\mathfrak{h}_2^{-1}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \mathrm{e}^{x_2}\partial_1,\ \partial_3$
$\mathfrak{h}_3^{a,0}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -a\partial_3$
$\mathfrak{h}_3^{a+x_4,0}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -(a+x_4)\partial_3$
$\mathfrak{h}_3^{a,1}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -\mathrm{e}^{x_2}\partial_1-a\partial_3$
$\mathfrak{h}_3^{a+x_4,1}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -\mathrm{e}^{x_2}\partial_1-(a+x_4)\partial_3$
$\mathfrak{h}_3^{a,-1}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ \mathrm{e}^{x_2}\partial_1-a\partial_3$
$\mathfrak{h}_3^{a+x_4,-1}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ \mathrm{e}^{x_2}\partial_1-(a+x_4)\partial_3$
$\mathfrak{h}_4^{a,b,0}$	$\partial_1,\ x_1\partial_1-a\partial_2-b\partial_3,\ \partial_2,\ \partial_3$
$\mathfrak{h}_4^{a,b+x_4,0}$	$\partial_1,\ x_1\partial_1-a\partial_2-(b+x_4)\partial_3,\ \partial_2,\ \partial_3$
$\mathfrak{h}_4^{a+x_4,b+f(x_4),0}$	$\partial_1,\ x_1\partial_1-(a+x_4)\partial_2-(b+f(x_4))\partial_3,\ \partial_2,\ \partial_3$
$\mathfrak{h}_4^{a+x_4,b+x_5,0}$	$\partial_1,\ x_1\partial_1-(a+x_4)\partial_2-(b+x_5)\partial_3,\ \partial_2,\ \partial_3$

Table 4.3. Complete system of realizations of $\mathfrak{g}$ with rank three with respect to inner automorphisms

Subalgebra	Realization
$\mathfrak{h}_1$	$0,\ \partial_1,\ \partial_2,\ \partial_3$
$\mathfrak{h}_2^c$	$\partial_1,\ x_1\partial_1+\partial_2,\ -c\mathrm{e}^{x_2}\partial_1,\ \partial_3$
$\mathfrak{h}_3^{a,c}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -c\mathrm{e}^{x_2}\partial_1-a\partial_3$
$\mathfrak{h}_3^{a+x_4,c}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -c\mathrm{e}^{x_2}\partial_1-(a+x_4)\partial_3$
$\mathfrak{h}_4^{a,b,c}$	$\partial_1,\ (x_1-c)\partial_1-a\partial_2-b\partial_3,\ \partial_2,\ \partial_3$
$\mathfrak{h}_4^{a,b+x_4,c}$	$\partial_1,\ (x_1-c)\partial_1-a\partial_2-(b+x_4)\partial_3,\ \partial_2,\ \partial_3$
$\mathfrak{h}_4^{a+x_4,b+f(x_4),c}$	$\partial_1,\ (x_1-c)\partial_1-(a+x_4)\partial_2-(b+f(x_4))\partial_3,\ \partial_2,\ \partial_3$
$\mathfrak{h}_4^{a+x_4,b+x_5,c}$	$\partial_1,\ (x_1-c)\partial_1-(a+x_4)\partial_2-(b+x_5)\partial_3,\ \partial_2,\ \partial_3$

Table 4.4. Complete system of realizations of $\mathfrak{g}$ with rank three with respect to strong equivalence

Finally, let us do the classification with respect to all automorphisms. As we mentioned in Remark 4.34, the first step is to remove the unnecessary parameters. For example, all subalgebras $\mathfrak{h}_3^{a,c}$ for $c \neq 0$ are Aut-equivalent to $\mathfrak{h}_3^{0,1}$. Therefore, realizations corresponding to $\mathfrak{h}_3^{a,c}$ and $\mathfrak{h}_3^{a+x_4,c}$ are equivalent to the realizations corresponding to $\mathfrak{h}_3^{0,1}$ and $\mathfrak{h}_3^{x_4,1}$. The result of such a process is listed in Table 4.5. In the last column, we list the number of the realization in the classification [26], p. 7345 (only in case it is faithful).

The realization corresponding to $\mathfrak{h}_4^{0,x_4,0}$ is Aut-equivalent to realization $\mathfrak{h}_4^{x_4,0,0}$, so we can remove it from the table. Furthermore, realizations corresponding to $\mathfrak{h}_4^{x_4,f(x_4),0}$ may be, for different f, also equivalent. All other realizations listed in Table 4.5 are Aut-inequivalent.

To find the condition for the realizations corresponding to $\mathfrak{h}_4^{x_4,f(x_4),0}$ to be equivalent, we have to examine, how automorphisms act on this subalgebra-valued function. The equation

$$\alpha_{t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8}(\mathfrak{h}_4^{x_4,f(x_4),0}) = \mathfrak{h}_4^{\tilde{x}_4,\tilde{f}(\tilde{x}_4),0},$$

where α is an automorphism of $\mathfrak{g}$ parametrized as in (23), together with conditions $\tilde{x}_4(x_4=0)=0$ and $f(0)=0$, leads to constraint $t_2,t_3,t_4=0$ and then is equivalent to

$$\tilde{x}_4 = t_5x_4 + t_6 f(x_4), \quad \tilde{f}(\tilde{x}_4) = t_7x_4 + t_8 f(x_4), \tag{24}$$

Subalgebra	Realization	No. in [26]
$\mathfrak{h}_1$	$0,\ \partial_1,\ \partial_2,\ \partial_3$	
$\mathfrak{h}_3^{0,0}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ 0$	
$\mathfrak{h}_3^{x_4,0}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -x_4\partial_3$	7
$\mathfrak{h}_3^{0,1}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -\mathrm{e}^{x_2}\partial_1$	6
$\mathfrak{h}_3^{x_4,1}$	$\partial_1,\ x_1\partial_1+\partial_2,\ \partial_3,\ -\mathrm{e}^{x_2}\partial_1-x_4\partial_3$	5
$\mathfrak{h}_4^{0,0,0}$	$\partial_1,\ x_1\partial_1,\ \partial_2,\ \partial_3$	4
$\mathfrak{h}_4^{0,x_4,0}$	$\partial_1,\ x_1\partial_1-x_4\partial_3,\ \partial_2,\ \partial_3$	3
$\mathfrak{h}_4^{x_4,f(x_4),0}$	$\partial_1,\ x_1\partial_1-x_4\partial_2-f(x_4)\partial_3,\ \partial_2,\ \partial_3$	3
$\mathfrak{h}_4^{x_4,x_5,0}$	$\partial_1,\ x_1\partial_1-x_4\partial_2-x_5\partial_3,\ \partial_2,\ \partial_3$	2

Table 4.5. Complete system of realizations of $\mathfrak{g}$ with rank three with respect to all automorphisms

Therefore, realizations corresponding to $\mathfrak{h}_4^{x_4,f(x_4),0}$ and $\mathfrak{h}_4^{x_4,\tilde{f}(x_4),0}$, where $f(0)=\tilde{f}(0)=0$ are equivalent if and only if

$$\tilde{f}(t_5x+t_6f(x))=t_7x+t_8f(x), \tag{25}$$

where $t_5t_8-t_6t_7\neq 0$ (cf. [26] p. 7351).

Remark 4.35. In [26] the authors list only faithful realizations. The reason is that the unfaithful ones can be constructed very easily from the faithful ones. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}^{(S\circ F)(x^{r+1},\dots,x^m)}$ parametrized set of subalgebras, and R the corresponding realization. Let $\mathfrak{i}$ be the largest ideal of $\mathfrak{g}$ contained in all the subalgebras $\mathfrak{h}^{(S\circ F)(x^{r+1},\dots,x^m)}$. According to Lemma 4.22 $\mathfrak{i}$ is the kernel of $R\colon\mathfrak{g}\to\operatorname{Vect}\mathbb{R}^m$, so the unfaithful realization R is determined by a faithful realization $\tilde{R}$ of $\mathfrak{g}/\mathfrak{i}$.

So, given a Lie algebra $\mathfrak{g}$, one can construct all its unfaithful realizations R by finding all ideals $\mathfrak{i}$ of $\mathfrak{g}$ and then extend the faithful realizations $\tilde{R}$ of $\mathfrak{g}/\mathfrak{i}$ on the whole Lie algebra $\mathfrak{g}$. It is clear that such realizations R_1 and R_2 are A-equivalent if and only if the ideals are A-conjugated $\mathfrak{i}_2=\alpha(\mathfrak{i}_1)$ and the corresponding realizations on $\mathfrak{g}/\mathfrak{i}_1\simeq\mathfrak{g}/\mathfrak{i}_2$ are A-equivalent.

Very often the ideals of the Lie algebras are spanned by several generating elements, say $\mathfrak{g}=\operatorname{span}\{e_1,\dots,e_n\}$, $\mathfrak{i}=\operatorname{span}\{e_1,\dots,e_k\}$. Also, it is very often that the commutation relations of $\mathfrak{g}/\mathfrak{i}$ induced by canonical commutations relations of $\mathfrak{g}$ coincide with canonical commutation relations of the Lie algebra type of $\mathfrak{g}/\mathfrak{i}$. In those cases the "table entry" is constructed very easily. It consists of k zeros representing the generators of the ideal and then the corresponding realization of the Lie algebra $\mathfrak{g}/\mathfrak{i}=\operatorname{span}\{\bar{e}_{k+1},\dots,\bar{e}_n\}$.

This is the case also in our example. We have two Aut-classes of one-dimensional ideals $\mathfrak{i}_1=\operatorname{span}\{e_1\}$ and $\mathfrak{i}_2=\operatorname{span}\{e_4\}$. The quotient algebra of the first one is the three dimensional Abelian Lie algebra $\mathfrak{g}/\mathfrak{i}_1\simeq 3\mathfrak{g}_1$. The complete system of faithful realizations of $3\mathfrak{g}_1=\operatorname{span}\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3\}$ in three variables consists of single realization $(\tilde{e}_1,\tilde{e}_2,\tilde{e}_3)\mapsto(\partial_1,\partial_2,\partial_3)$. Since any linear map provides the isomorphism, we can immediately write the first row of Table 4.5. Similarly, we have $\mathfrak{g}/\mathfrak{i}_2\simeq\mathfrak{g}_2\oplus\mathfrak{g}_1=\operatorname{span}\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3\}$, $[\tilde{e}_1,\tilde{e}_2]=\tilde{e}_1$. The isomorphism is provided by $e_i+\mathfrak{i}\mapsto\tilde{e}_i$, $i=1,2,3$. Therefore, we can immediately write the second row of Table 4.5 by looking to [26] and copying the only listed realization of $\mathfrak{g}_2\oplus\mathfrak{g}_1$ in three variables and adding zero that realizes the last generating element e_4 of $\mathfrak{g}_2\oplus 2\mathfrak{g}_1$.

Chapter 5
Conclusion

Our motivation for this work was to study a new method for construction of Lie algebra realizations introduced by Shirokov et al. [15] and try to use it on some class of Lie algebras as was recently done in [19–20]. This method allows to easily construct local transitive realizations (i.e. Lie algebras of vector fields that integrate to a transitive transformation group), whose classification is equivalent to classification of subalgebras of the given Lie algebra.

Our goal was to try to use this method to obtain not only classification of transitive realizations but to solve a more general classification problem and obtain results similar to [26]. One of the difficulties that appeared during work on this problem was that the classification problem solved in [26] was not very clearly defined. One of the important results of this work can be considered the definition of the *complete system* of local realizations of a given Lie algebra that provides formal definition of the considered classification problem.

A remark of Shirokov published in [20] stating that replacing a parameter in a parameter-dependent realization by a new variable can be used to construct new realizations led to the most important result of this thesis, which is the generalization of the correspondence between subalgebra classification and transitive realizations classification to the case of regular realizations. We formulated Theorem 4.19 stating that local regular realizations (i.e. local realizations with constant rank) are in a correspondence with parametrizations of subalgebras (i.e. maps from $\mathbb{R}^k$ to the set of all subalgebras).

Since complete system of local realizations consists of regular realizations, we could make use of this result to solve the original classification problem. Thus, the next outcome of this thesis is a formulation of an algorithm for construction of complete system of local realizations.

All those procedures were illustrated on simple examples. The reader may have noticed that we often used Lie algebras with a weak Lie algebra structure for those illustrations (very often we used Abelian Lie algebras). The reason for that is not that the procedures would not work for more complicated Lie algebras. Quite the contrary. Since Lie algebras with weaker structure have less inner automorphisms, they have Int-classes of subalgebras with more parameters and one has to deal with problems like families of realizations parametrized by functions.

We wanted to use this algorithm to produce an original classification result and since realizations of all Lie algebras of dimension less than five were classified in [26], we have focused on the five-dimensional ones. The last result of this thesis follows in appendix and it is a classification of realizations of all five-dimensional nilpotent indecomposable Lie algebras. We compute the complete systems of realizations for these Lie algebras with respect to groups of all automorphisms (as in [26]).

The computation consists of several complicated steps, from which some of them are interesting by themselves for other applications (namely determining the groups of automorphisms and classifying the subalgebras with respect to these groups) and are published in the appendix as well. There was no special reason for choosing the family of nilpotent algebras to start the classification with. Our goal for the future is to continue in the computations and complete the classification for all (or at least indecomposable) five-dimensional Lie algebras.

The author have already obtained some partial results also for other Lie algebras. In the research project [10] the subalgebras of all five-dimensional indecomposable Lie algebras with four-dimensional Abelian ideal were classified with respect to the group of all automorphisms and the corresponding transitive realizations were computed. The author have also classified subalgebras of many of those Lie algebras with respect to inner automorphisms. Nevertheless, the subsequent computation, that is, determining which realizations parametrized by functions are equivalent, as well as checking for errors in all the computations and also putting the results together in reasonable tables are all very uneasy and time-consuming work. That is the reason why we were able to publish only the results for the nilpotent Lie algebras.

References

[1] Robert J. Blattner. Induced and Produced Representations of Lie Algebras. *Transactions of the American Mathematical Society*. 1969, 144 457-474. doi:10.2307/1995292.

[2] Nicolas Bourbaki. *Lie Groups and Lie algebras, Chapters 1–3*. Paris: Hermann, 1975. ISBN 2-7056-5826-2.

[3] Jan Draisma. On a conjecture of Sophus Lie. In: *Differential Equations and the Stokes Phenomenon*. Singapore: World Scientific, 2002. 65–87.

[4] Jan Draisma. Transitive Lie Algebras of Vector Fields: An Overview. *Qualitative Theory of Dynamical Systems*. 2012, 11 (1), 39–60. doi:10.1007/s12346-011-0062-9.

[5] Friedrich Engel, and Poul Heegaard. *Sophus Lie: Gesammelte Abhandlungen,* vol. 5. Liepzig: B. G. Teubner, 1924.

[6] David J. Fisher, Robert J. Gray, and Peter E. Hydon. Automorphisms of real Lie algebras of dimension five or less. *Journal of Physics A: Mathematical and Theoretical*. 2013, 46 (22), 225204. doi:10.1088/1751-8113/46/22/225204.

[7] Wilhelm I. Fushchyk, Leonid F. Barannyk, and Anatoliy F. Barannyk. *Subgroup Analysis of the Galilei and Poincaré Groups and Reduction of Nonlinear Equations*. Kiev: Naukova Dumka, 1991. ISBN 5-12-001326-0.

[8] Artemio González-López, Niky Kamran, and Peter J. Olver. Lie Algebras of Vector Fields in the Real Plane. *Proceedings of the London Mathematical Society*. 1992, s3-64 (2), 339–368. doi:10.1112/plms/s3-64.2.339.

[9] Vladimir V. Gorbatsevich, Arkadii L. Onishchik, and Èrnest B. Vinberg. *Lie Groups and Lie Algebras I: Foundations of Lie Theory Lie Transformation Groups*. Berlin Heidelberg: Springer-Verlag, 1993. ISBN 0-387–18697-2.

[10] Daniel Gromada. *Construction of realizations of Lie algebras*. Research project, FNSPE, CTU. Prague, 2016.
`http://physics.fjfi.cvut.cz/publications/mf/2016/vu_mf_16_Gromada.pdf`.

[11] Daniel Gromada, and Severin Pošta. *On classification of Lie algebra realizations*. 2017. arXiv:1703.00808.

[12] Victor W. Guillemin, and Shlomo Sternberg. An algebraic model of transitive differential geometry. *Bulletin of the American Mathematical Society*. 1964, 70 16–47. doi:10.1090/S0002-9904-1964-11019-3 .

[13] Saber Jafarpour, and Andrew Lewis. *Time-Varying Vector Fields and Their Flows*. Springer International Publishing, 2014. ISBN 978-3-319-10139-2.

[14] Sophus Lie, and Friedrich Engel. *Theorie der Transformationsgruppen*. Leipzig: 1888, 1890, 1893.

[15] Alexey A. Magazev, Vitaly V. Mikheyev, and Igor V. Shirokov. Computation of Composition Functions and Invariant Vector Fields in Terms of Structure Constants of Associated Lie Algebras. *SIGMA*. 2015, 11 (066), doi:10.3842/SIGMA.2015.066.

[16] Gamir M. Mubarakzyanov. Classification of real structures of Lie algebras of fifth order. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*. 1963, (3), 99–106. mi.mathnet.ru/ivm2184.

[17] Gamir M. Mubarakzyanov. On solvable Lie algebras. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*. 1963, (1), 114–123. mi.mathnet.ru/ivm2141.

[18] James R. Munkers. *Topology*. Upper Saddle River: Prentice-Hall, 2000. ISBN 0-13-181629-2.

[19] Maryna Nesterenko. The Poincaré algebras $p(1, 1)$ and $p(1, 2)$: realizations and deformations. *Journal of Physics: Conference Series*. 2015, 621 (1), 012009. doi:10.1088/1742-6596/621/1/012009.

[20] Maryna Nesterenko, Severin Pošta, and Olena Vaneeva. Realizations of Galilei algebras. *Journal of Physics A: Mathematical and Theoretical*. 2016, 49 (11), 115203. doi:10.1088/1751-8113/49/11/115203.

[21] Peter J. Olver. *Applications of Lie Groups to Differential Equations*. New York: Springer-Verlag, 1993. ISBN 978-0-387-95000-6.

[22] Lev V. Ovsiannikov. *Group Analysis of Differential Equations* . New York: Academic Press, 1982. ISBN 978-0-12-531680-4.

[23] Jiří Patera, Robert T. Sharp, Pavel Winternitz, and Hans Zassenhaus. Invariants of real low dimension Lie algebras. *Journal of Mathematical Physics*. 1976, 17 (6), 986-994. doi:10.1063/1.522992.

[24] Jiří Patera, and Pavel Winternitz. Subalgebras of real three- and four-dimensional Lie algebras. *Journal of Mathematical Physics*. 1977, 18 (7), 1449–1455. doi:10.1063/1.523441.

[25] Jiří Patera, Pavel Winternitz, and Hans Zassenhaus. Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group. *Journal of Mathematical Physics*. 1975, 16 (8), 1597–1614. doi:10.1063/1.522729.

[26] Roman O. Popovych, Vyacheslav M. Boyko, Maryna O. Nesterenko, and Maxim W. Lutfullin. Realizations of real low-dimensional Lie algebras. *Journal of Physics A: Mathematical and General*. 2003, 36 (26), 7337. doi:10.1088/0305-4470/36/26/309.

[27] Claudio Procesi. *Lie Groups: An Approach through Invariants and Representations*. New York: Springer-Verlag, 2007. ISBN 978-0-387-26040-2.

[28] Igor V. Shirokov. Constructing lie algebras of first-order differential operators. *Russian Physics Journal*. 1997, 40 (6), 525–530. doi:10.1007/BF02766382.

[29] Stanislav V. Spichak. Preliminary Classification of Realizations of Two-Dimensional Lie Algebras of Vector Fields on a Circle. In: *Sixth Workshop Group Analysis of Differential Equations and Integrable Systems*. Nicosia: Department of Mathematics and Statistics, University of Cyprus, 2013. 212–218. ISBN 978-9963-700-63-9.
`http://www.mas.ucy.ac.cy/~symmetry/GADEISVI.pdf`.

List of used symbols

$\oplus$	Direct sum	
$[\cdot,\cdot]$	Lie bracket, commutator	
ad_x	Lie algebra adjoint representation of x	
Ad_g	Lie group adjoint representation of g	6
A^{T}	Transposition of matrix A	
$\mathrm{Aut}\,\mathfrak{g}$	Group of all automorphisms of a Lie algebra $\mathfrak{g}$	16
$C^\infty(M)$	Algebra of smooth functions on a manifold M	3
C_g	Conjugation multiplication by g	3
$\mathbb{C}$	Set of complex numbers	
$\mathrm{Der}\,A$	Derivation of an algebra A	
$\mathrm{d}\Phi$	Derivative of map Φ	3
$\mathrm{Diff}\,M$	Group of all diffeomorphisms of a manifold M	3
$\mathrm{Dom}\,R$	Domain of a realization R	19
e	Group unity	
$\exp$, $\mathrm{e}^\bullet$	Exponential map	6
Φ_*	Pushforward of a vector field by a diffeomorphism Φ	3
$\mathfrak{g},\mathfrak{h},\ldots$	Lie algebra (corresponding to Lie group $G,H,\ldots$)	6
$G,H,\ldots$	(Lie) group	
G/H	Set of left cosets of G by H	3
$H\setminus G$	Set of right cosets of G by H	3
$\mathrm{gl}(V)$	General linear algebra of V	7
$\mathrm{GL}(V)$	General linear group of V	7
$\mathrm{Gr}(V,k)$	Grassmanian – set of all k-dimensional subspaces of a vector space V	25
I	Identity matrix	
$\mathrm{Int}\,\mathfrak{g}$	Group of inner automorphisms of a Lie algebra $\mathfrak{g}$	17
$\ker\varphi$	Kernel of homomorphism φ	
L_g	Left multiplication by g	3
$M,N,\ldots$	Manifold	
$\mathbb{N}$	Set of natural numbers $\{1,2,\ldots\}$	
π_*X	Fundamental vector field corresponding to X	12
$\mathbb{R}$	Set of real numbers	
$\mathrm{rank}\,R_p$	Rank of a realization R at point p	22
R_g	Right multiplication by g	3
σ_x	Sign of x (1 if positive, 0 if zero, -1 if negative)	
span	Linear span of a set of vectors	
TM	Tangent bundle on M	
T_pM	Tangent space at $p\in M$	
$V,W,\ldots$	Vector space	
$\mathrm{Vect}\,M$	Lie algebra of vector fields on M	3
$X,Y,\ldots$	Vector field (or a tangent vector)	
$\hat{X}$	Fundamental vector field corresponding to X	12

Appendix A
The classification results

A.1 Classification of Lie algebras

To express our results, we use the classification of real indecomposable Lie algebras of dimension not greater than five obtained by G. M. Mubarakzyanov in [16–17]. Unfortunately, the numbering of the Lie algebras is not completely consistent in the literature. Since the original work of Mubarakzyanov was not translated to English, many authors use the numbering chosen in [23].

The subalgebras including the defining canonical commutation relations that are relevant for our results are listed in Table A.1. The first number n in the subalgebra name $\mathfrak{g}_{n,k}$ stands for the dimension of the Lie algebra. The basis elements are always denoted $\{e_1, \dots, e_n\}$. We follow the numbering and bases chosen in [23]. The only difference here with the Mubarakzyanov notation is in the algebra $\mathfrak{g}_{5,3}$, where two of the generating elements are switched. The reason for our choice was to get triangular shape of automorphism matrices.

Type	Nonzero commutation relations
$\mathfrak{g}_1$	
$\mathfrak{g}_{3,1}$	$[e_2, e_3] = e_1$
$\mathfrak{g}_{4,1}$	$[e_2, e_4] = e_1, [e_3, e_4] = e_2$
$\mathfrak{g}_{5,1}$	$[e_3, e_5] = e_1, [e_4, e_5] = e_2$
$\mathfrak{g}_{5,2}$	$[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$
$\mathfrak{g}_{5,3}$	$[e_3, e_4] = e_2, [e_3, e_5] = e_1, [e_4, e_5] = e_3$
$\mathfrak{g}_{5,4}$	$[e_2, e_4] = e_1, [e_3, e_5] = e_1$
$\mathfrak{g}_{5,5}$	$[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2$
$\mathfrak{g}_{5,6}$	$[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$

Table A.1. Real nilpotent indecomposable Lie algebras up to dimension five.

A.2 Subalgebras and realizations classification

In this section our classification results for five-dimensional indecomposable nilpotent Lie algebras will be presented. For every Lie algebra we give groups of inner and all automorphisms, classification of subalgebras and complete system of faithful realizations.

The automorphisms are presented in matrix form (as linear operators). Implicitly we assume the domain of all parameters is $\mathbb{R}$. If needed, additional constraints on their domain are listed.

The classification of subalgebras is presented in a table with five columns. We present classification with respect to both groups of automorphisms. The classification with respect to inner automorphisms is available in the third column. The subalgebras listed are parametrized by parameters a, b, c, d, whose domain is implicitly assumed to be $\mathbb{R}$ unless the parameter occurs in a denominator of a fraction (then maximal domain is assumed) or other condition is explicitly stated. All the subalgebras are inequivalent with respect to inner automorphisms. The subalgebras (or subalgebra classes) are given name in the second row. In the first row, the type of the subalgebra is written.

The classification with respect to all automorphisms is available in the fourth column of this table. Again, all subalgebras in this column are mutually inequivalent with respect to the group of all automorphisms. Every subalgebra from the third column is Aut-equivalent to the representative listed in the fourth column in the corresponding row. Finally, the last column shows the type of the corresponding quotient algebra. For every Aut-class $\mathfrak{h}$ we write either dash (–) if the subalgebra $\mathfrak{h}$ is not an ideal or the Lie algebra type of the quotient $\mathfrak{g}/\mathfrak{h}$.

The classification of realizations is usually presented as the last table. For every considered Lie algebra we bring the complete system of faithful local realizations with respect to the group of all automorphisms. The representatives of all the local realization classes are chosen to be defined at zero. As follows from the theory in Chapter 4, complete system of realizations consists of regular realizations only. Every regular local realizations in m variables with rank r is characterized by $(m-r)$-parameter set of subalgebras. In the first column of the table we list all possible subalgebra parametrizations ($\mathfrak{h}_0$ always stands for zero subalgebra) and for each the corresponding regular realization is computed and the images of the generators $e_1, \ldots, e_5$ are written in the second column.

The realizations often depend on some functions f, g. Implicitly we assume that those are arbitrary real functions in one or two variables defined in a neighbourhood of zero mapping zero to zero (i.e. $f(0) = 0$, $g(0) = 0$). The realizations depending on those functions may be equivalent for different functions. From the theory it follows that they are equivalent if and only if the corresponding subalgebra parametrizations are equivalent with respect to outer automorphisms in sense of Remark 4.20. If such case takes place, we list those equivalences in a separate table. Usually we try to express the function transformations explicitly, but in some cases only implicit expressions are provided.

As an example, how to read the tables, we can have a look on the Lie algebra $\mathfrak{g}_{5,1}$. As we see in Table A.2, it has five parametrized families of one-dimensional subalgebra classes with respect to inner automorphisms, whose representatives are denoted as $\mathfrak{h}_{1,1}$, $h^a_{1,2}$, $\mathfrak{h}^a_{1,3}$, $\mathfrak{h}^{a,b}_{1,4}$, and $\mathfrak{h}^{a,b}_{1,5}$. All of them are, of course, type $\mathfrak{g}_1$. All subalgebras of the first two families are equivalent and as a representative we can choose $\mathfrak{h}_{1,1} = \operatorname{span}\{e_1\}$. The next two are also mutually equivalent and all the subalgebras in the last family are also equivalent. From these three Aut-classes of subalgebras, only the first one is an ideal of the Lie algebra, and the quotient algebra is of the type $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$.

Using the information in the last column, we can easily construct classification of unfaithful Lie algebras as we indicated in Remark 4.35. For example, all realizations with kernel $\mathfrak{h}_{1,1}$ can be constructed by extending all faithful realizations of $\mathfrak{g}/\mathfrak{h}_{1,1} \simeq \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$.

The faithful realizations of $\mathfrak{g}_{5,1}$ are listed in Table A.4. Some realizations here may be equivalent. All possible equivalences are listed in Table A.3. For example, the family of realizations on the fifth row corresponding to curve $\mathfrak{h}^{x_5, f(x_5)}_{1,4}$ in the set of subalgebras contain some mutually equivalent realizations. Realization parametrized by function f is equivalent with all realizations parametrized by function $\tilde{f}$ defined as $\tilde{f}(x) = bx + cf(ax)$.

Lie algebra $\mathfrak{g}_{5,1}$

$$\operatorname{Int}\mathfrak{g}=\left\{\begin{pmatrix}1&0&t_3&0&t_1\\0&1&0&t_3&t_2\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\right\},$$

$$\operatorname{Aut}\mathfrak{g}=\left\{\begin{pmatrix}as_1&as_2&t_5&t_6&t_1\\as_3&as_4&t_7&t_8&t_2\\0&0&s_1&s_2&t_3\\0&0&s_3&s_4&t_4\\0&0&0&0&a\end{pmatrix}\;\middle|\;\begin{matrix}a\neq 0,\\ s_1s_4\neq s_2s_3\end{matrix}\right\}$$

Type	Name	Int-class	Aut-class	$\mathfrak{g}/\mathfrak{h}$ type
$\mathfrak{g}_1$	$\mathfrak{h}_{1,1}$	e_1	e_1	$\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$
	$\mathfrak{h}^{a}_{1,2}$	e_2-ae_1		
	$\mathfrak{h}^{a}_{1,3}$	e_3-ae_2	e_3	–
	$\mathfrak{h}^{a,b}_{1,4}$	$e_4-ae_3-be_1$		
	$\mathfrak{h}^{a,b}_{1,5}$	$e_5-ae_4-be_3$	e_5	–
$2\mathfrak{g}_1$	$\mathfrak{h}_{2,1}$	$e_1,\,e_2$	$e_1,\,e_2$	$3\mathfrak{g}_1$
	$\mathfrak{h}^{a}_{2,2}$	$e_1,\,e_3-ae_2$	$e_1,\,e_3$	$\mathfrak{g}_{3,1}$
	$\mathfrak{h}^{a,b}_{2,3}$	$e_2-ae_1,\,e_4-ae_3-be_1$		
	$\mathfrak{h}^{a}_{2,4}$	$e_1,\,e_4-ae_3$	$e_1,\,e_4$	–
	$\mathfrak{h}^{a}_{2,5}$	$e_1-ae_2,\,e_4$		
	$\mathfrak{h}^{a,b}_{2,6}$	$e_2-ae_1,\,e_3-be_4;\ ab\neq 1$		
	$\mathfrak{h}^{a,b}_{2,7}$	$e_1,\,e_5-ae_4-be_3$	$e_1,\,e_5$	–
	$\mathfrak{h}^{a,b,c}_{2,8}$	$e_2-ae_1,\,e_5-be_4-ce_3$		
	$\mathfrak{h}^{a,b,c}_{2,9}$	$e_3-ae_2,\,e_4-be_2-ce_1$	$e_3,\,e_4$	–
$3\mathfrak{g}_1$	$\mathfrak{h}_{3,1}$	$e_1,\,e_2,\,e_3$	$e_1,\,e_2,\,e_3$	$2\mathfrak{g}_1$
	$\mathfrak{h}^{a}_{3,2}$	$e_1,\,e_2,\,e_4-ae_3$		
	$\mathfrak{h}^{a,b}_{3,3}$	$e_1,\,e_2,\,e_5-ae_4-be_3$	$e_1,\,e_2,\,e_5$	$2\mathfrak{g}_1$
	$\mathfrak{h}^{a}_{3,4}$	$e_1,\,e_3-ae_2,\,e_4$	$e_1,\,e_3,\,e_4$	–
	$\mathfrak{h}^{a,b}_{3,5}$	$e_2-ae_1,\,e_3,\,e_4-be_1$		
$\mathfrak{g}_{3,1}$	$\mathfrak{h}^{a,b}_{3,6}$	$e_1,\,e_3-ae_2,\,e_5-be_4$	$e_1,\,e_3,\,e_5$	–
	$\mathfrak{h}^{a,b,c}_{3,7}$	$e_2-ae_1,\,e_4-ae_3-be_1,\,e_5-ce_3$		
$4\mathfrak{g}_1$	$\mathfrak{h}_{4,1}$	$e_1,\,e_2,\,e_3,\,e_4$	$e_1,\,e_2,\,e_3,\,e_4$	$\mathfrak{g}_1$
$\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$	$\mathfrak{h}^{a}_{4,2}$	$e_1,\,e_2,\,e_3,\,e_5-ae_4$	$e_1,\,e_2,\,e_3,\,e_5$	$\mathfrak{g}_1$
	$\mathfrak{h}^{a,b}_{4,3}$	$e_1,\,e_2,\,e_4-ae_3,\,e_5-be_3$		

Table A.2. Subalgebras of $\mathfrak{g}_{5,1}$.

$$\mathfrak{h}_{2,9}^{F(x_4,x_5,\dots)}\sim\mathfrak{h}_{2,9}^{\tilde F(x_4,x_5,\dots)};\quad \left(A\otimes(A^{\mathrm{T}})^{-1}\right)\begin{pmatrix}0\\F_1\\F_3\\F_2\end{pmatrix}=\begin{pmatrix}0\\\tilde F_1\\\tilde F_3\\\tilde F_2\end{pmatrix},A\in\mathrm{GL}(2,\mathbb{R})$$

$$\mathfrak{h}_{1,4}^{x_5,f(x_5)} \sim \mathfrak{h}_{1,4}^{x_5,bx_5+cf(ax_5)};\ a,c \neq 0$$
$$\mathfrak{h}_{1,5}^{x_5,f(x_5)} \sim \mathfrak{h}_{1,5}^{x_5,\tilde{f}(x_5)};$$
$$\tilde{f}(ax+bf(x)) = cx+df(x),\ ad \neq bc$$
$$\mathfrak{h}_{2,3}^{x_4,f(x_4)} \sim \mathfrak{h}_{2,3}^{x_4,bx_4+cf(ax_4)};\ a,c \neq 0$$
$$\mathfrak{h}_{2,6}^{x_4,f(x_4)} \sim \mathfrak{h}_{2,6}^{x_4,af(ax_4)};\ a \neq 0$$
$$\mathfrak{h}_{2,8}^{x_4,f(x_4),g(x_4)} \sim \mathfrak{h}_{2,8}^{x_4,bf(ax_4),abg(ax_4)};\ a,b \neq 0$$
$$\mathfrak{h}_{2,8}^{x_4,f(x_4),x_5} \sim \mathfrak{h}_{2,8}^{x_4,bf(ax_4),x_5};\ a,b \neq 0$$
$$\mathfrak{h}_{2,8}^{x_4,x_5,g(x_4,x_5)} \sim \mathfrak{h}_{2,8}^{x_4,x_5,a/b\,g(ax_4,bx_5)};\ a,b \neq 0$$
$$\mathfrak{h}_{3,5}^{x_3,f(x_3)} \sim \mathfrak{h}_{3,5}^{x_3,bx_3+cf(ax_3)};\ a,c \neq 0$$
$$\mathfrak{h}_{3,7}^{x_3,f(x_3),ax_3} \sim \mathfrak{h}_{3,7}^{x_3,cx_3+df(x_3),ex_3};\ b,d \neq 0$$
$$\mathfrak{h}_{3,7}^{x_3,f(x_3),g(x_3)} \sim \mathfrak{h}_{3,7}^{x_3,bf(ax_3),cx_3+dg(x_3)};\ a,b,d \neq 0$$
$$\mathfrak{h}_{3,7}^{x_3,f(x_3),x_4} \sim \mathfrak{h}_{3,7}^{x_3,bf(ax_3),x_4};\ a,b \neq 0$$
$$\mathfrak{h}_{3,7}^{x_3,x_4,g(x_3,x_4)} \sim \mathfrak{h}_{3,7}^{x_3,x_4,cx_3+dg(ax_3,bx_4)};\ a,b,d \neq 0$$

Table A.3. Subalgebra parametrization equivalence for $\mathfrak{g}_{5,1}$

Subalgebra	Realization
$\mathfrak{h}_0$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ x_3\partial_1 + x_4\partial_2 + \partial_5$
$\mathfrak{h}_{1,2}^{x_5}$	$\partial_1,\ x_5\partial_1,\ \partial_2,\ \partial_3,\ (x_2 + x_5x_3)\,\partial_1 + \partial_4$
$\mathfrak{h}_{1,4}^{0,0}$	$\partial_1,\ \partial_2,\ \partial_3,\ -x_4\partial_2,\ x_3\partial_1 + \partial_4$
$\mathfrak{h}_{1,4}^{0,x_5}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_5\partial_1 - x_4\partial_2,\ x_3\partial_1 + \partial_4$
$\mathfrak{h}_{1,4}^{x_5,f(x_5)}$	$\partial_1,\ \partial_2,\ \partial_3,\ (f(x_5) + x_5x_4)\,\partial_1 - x_4\partial_2 + x_5\partial_3,\ x_3\partial_1 + \partial_4$
$\mathfrak{h}_{1,4}^{x_5,x_6}$	$\partial_1,\ \partial_2,\ \partial_3,\ (x_6 + x_5x_4)\,\partial_1 - x_4\partial_2 + x_5\partial_3,\ x_3\partial_1 + \partial_4$
$\mathfrak{h}_{1,5}^{0,0}$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ x_3\partial_1 + x_4\partial_2$
$\mathfrak{h}_{1,5}^{x_5,f(x_5)}$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ x_3\partial_1 + x_4\partial_2 + f(x_5)\partial_3 + x_5\partial_4$
$\mathfrak{h}_{1,5}^{x_5,x_6}$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ x_3\partial_1 + x_4\partial_2 + x_6\partial_3 + x_5\partial_4$
$\mathfrak{h}_{2,3}^{x_4,f(x_4)}$	$\partial_1,\ x_4\partial_1,\ \partial_2,\ f(x_4)\partial_1 + x_4\partial_2,\ x_2\partial_1 + \partial_3$
$\mathfrak{h}_{2,3}^{x_4,x_5}$	$\partial_1,\ x_4\partial_1,\ \partial_2,\ x_5\partial_1 + x_4\partial_2,\ x_2\partial_1 + \partial_3$
$\mathfrak{h}_{2,5}^{x_4}$	$x_4\partial_1,\ \partial_1,\ \partial_2,\ -x_3\partial_1,\ x_4x_2\partial_1 + \partial_3$
$\mathfrak{h}_{2,6}^{x_4,f(x_4)}$	$\partial_1,\ x_4\partial_1,\ (-1 + x_4f(x_4))x_3\partial_1 + f(x_4)\partial_2,\ \partial_2,\ x_4x_2\partial_1 + \partial_3$
$\mathfrak{h}_{2,6}^{x_4,x_5}$	$\partial_1,\ x_4\partial_1,\ (-1 + x_4x_5)x_3\partial_1 + x_5\partial_2,\ \partial_2,\ x_4x_2\partial_1 + \partial_3$
$\mathfrak{h}_{2,8}^{x_4,f(x_4),g(x_4)}$	$\partial_1,\ x_4\partial_1,\ \partial_2,\ \partial_3,\ (x_2 + x_4x_3)\,\partial_1 + g(x_4)\partial_2 + f(x_4)\partial_3$
$\mathfrak{h}_{2,8}^{x_4,f(x_4),x_5}$	$\partial_1,\ x_4\partial_1,\ \partial_2,\ \partial_3,\ (x_2 + x_4x_3)\,\partial_1 + x_5\partial_2 + f(x_4)\partial_3$
$\mathfrak{h}_{2,8}^{x_4,x_5,g(x_4,x_5)}$	$\partial_1,\ x_4\partial_1,\ \partial_2,\ \partial_3,\ (x_2 + x_4x_3)\,\partial_1 + g(x_4,x_5)\partial_2 + x_5\partial_3$
$\mathfrak{h}_{2,8}^{x_4,x_5,x_6}$	$\partial_1,\ x_4\partial_1,\ \partial_2,\ \partial_3,\ (x_2 + x_4x_3)\,\partial_1 + x_6\partial_2 + x_5\partial_3$
$\mathfrak{h}_{2,9}^{0,0,0}$	$\partial_1,\ \partial_2,\ -x_3\partial_1 + 0\partial_2,\ -x_3\partial_2,\ \partial_3$
$\mathfrak{h}_{2,9}^{0,f(x_4),x_4}$	$\partial_1,\ \partial_2,\ -x_3\partial_1,\ x_4\partial_1 + (f(x_4) - x_3)\,\partial_2,\ \partial_3$
$\mathfrak{h}_{2,9}^{0,x_5,x_4}$	$\partial_1,\ \partial_2,\ -x_3\partial_1,\ x_4\partial_1 + (x_5 - x_3)\,\partial_2,\ \partial_3$
$\mathfrak{h}_{2,9}^{x_4,f(x_4),g(x_4)}$	$\partial_1,\ \partial_2,\ -x_3\partial_1 + x_4\partial_2,\ g(x_4)\partial_1 + (f(x_4) - x_3)\,\partial_2,\ \partial_3$
$\mathfrak{h}_{2,9}^{x_4,x_5,g(x_4)}$	$\partial_1,\ \partial_2,\ -x_3\partial_1 + x_4\partial_2,\ g(x_4)\partial_1 + (x_5 - x_3)\,\partial_2,\ \partial_3$
$\mathfrak{h}_{2,9}^{x_4,f(x_4,x_5),x_5}$	$\partial_1,\ \partial_2,\ -x_3\partial_1 + x_4\partial_2,\ x_5\partial_1 + (f(x_4,x_5) - x_3)\,\partial_2,\ \partial_3$
$\mathfrak{h}_{2,9}^{x_4,x_5,x_6}$	$\partial_1,\ \partial_2,\ -x_3\partial_1 + x_4\partial_2,\ x_6\partial_1 + (x_5 - x_3)\,\partial_2,\ \partial_3$
$\mathfrak{h}_{3,5}^{x_3,f(x_3)}$	$\partial_1,\ x_3\partial_1,\ -x_2\partial_1,\ (f(x_3) - x_3x_2)\,\partial_1,\ \partial_2$
$\mathfrak{h}_{3,5}^{x_3,x_4}$	$\partial_1,\ x_3\partial_1,\ -x_2\partial_1,\ (x_4 - x_3x_2)\,\partial_1,\ \partial_2$
$\mathfrak{h}_{3,7}^{x_3,f(x_3),g(x_3)}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ f(x_3)\partial_1 + x_3\partial_2,\ x_2\partial_1 + g(x_3)\partial_2$
$\mathfrak{h}_{3,7}^{x_3,f(x_3),x_4}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ f(x_3)\partial_1 + x_3\partial_2,\ x_2\partial_1 + x_4\partial_2$
$\mathfrak{h}_{3,7}^{x_3,x_4,g(x_3,x_4)}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ x_4\partial_1 + x_3\partial_2,\ x_2\partial_1 + g(x_3,x_4)\partial_2$
$\mathfrak{h}_{3,7}^{x_3,x_4,x_5}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ x_4\partial_1 + x_3\partial_2,\ x_2\partial_1 + x_5\partial_2$

Table A.4. Complete system of faithful realizations of $\mathfrak{g}_{5,1}$.

Lie algebra $\mathfrak{g}_{5,2}$

$$\operatorname{Int}\mathfrak{g} = \left\{ \begin{pmatrix} 1 & t_4 & \frac{1}{2}t_4^2 & \frac{1}{6}t_4^3 & t_1 \\ 0 & 1 & t_4 & \frac{1}{2}t_4^2 & t_2 \\ 0 & 0 & 1 & t_4 & t_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$
$$\operatorname{Aut}\mathfrak{g} = \left\{ \begin{pmatrix} ab^3 & b^2t_7 & bt_6 & t_5 & t_1 \\ 0 & ab^2 & bt_7 & t_6 & t_2 \\ 0 & 0 & ab & t_7 & t_3 \\ 0 & 0 & 0 & a & t_4 \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \middle| \; a, b \neq 0 \right\}$$

Type	Name	Int-class	Aut-class	$\mathfrak{g}/\mathfrak{h}$ type
$\mathfrak{g}_1$	$\mathfrak{h}_{1,1}$	e_1	e_1	$\mathfrak{g}_{4,1}$
	$\mathfrak{h}_{1,2}$	e_2	e_2	–
	$\mathfrak{h}_{1,3}^{a}$	$e_3 - ae_1$	e_3	–
	$\mathfrak{h}_{1,4}^{a,b}$	$e_4 - ae_2 - be_1$	e_4	–
	$\mathfrak{h}_{1,5}^{a}$	$e_5 - ae_4$	e_5	–
$2\mathfrak{g}_1$	$\mathfrak{h}_{2,1}$	$e_1,\, e_2$	$e_1,\, e_2$	$3\mathfrak{g}_1$
	$\mathfrak{h}_{2,2}$	$e_1,\, e_3$	$e_1,\, e_3$	–
	$\mathfrak{h}_{2,3}^{a}$	$e_1,\, e_4 - ae_2$	$e_1,\, e_4$	–
	$\mathfrak{h}_{2,4}^{a}$	$e_1,\, e_5 - ae_4$	$e_1,\, e_5$	–
	$\mathfrak{h}_{2,5}^{a}$	$e_2,\, e_3 - ae_1$	$e_2,\, e_3$	–
	$\mathfrak{h}_{2,6}^{a,b}$	$e_2,\, e_4 - ae_3 - be_1;$	$e_2,\, \lvert\sigma_a\rvert e_3 + e_4$	–
	$\mathfrak{h}_{2,7}^{a,b,c}$	$e_3 - ae_1,\, e_4 - be_2 - ce_1$	$e_3 + \sigma_{b-a}e_1,\, e_4$	–
$3\mathfrak{g}_1$	$\mathfrak{h}_{3,1}$	$e_1,\, e_2,\, e_3$	$e_1,\, e_2,\, e_3$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,2}$	$e_1,\, e_2,\, e_4$	$e_1,\, e_2,\, e_4$	–
	$\mathfrak{h}_{3,3}^{a}$	$e_1,\, e_3,\, e_4 - ae_2$	$e_1,\, e_3,\, e_4$	–
	$\mathfrak{h}_{3,4}^{a,b}$	$e_2,\, e_3 - ae_1,\, e_4 - be_1$	$e_2.\;e_3,\, e_4$	–
$\mathfrak{g}_{3,1}$	$\mathfrak{h}_{3,5}^{a}$	$e_1,\, e_2,\, e_5 - ae_4$	$e_1,\, e_2,\, e_5$	–
$4\mathfrak{g}_1$	$\mathfrak{h}_{4,1}$	$e_1,\, e_2,\, e_3,\, e_4$	$e_1,\, e_2,\, e_3,\, e_4$	$\mathfrak{g}_1$
$\mathfrak{g}_{4,1}$	$\mathfrak{h}_{4,2}^{a}$	$e_1,\, e_2,\, e_3,\, e_5 - ae_4$	$e_1,\, e_2,\, e_3,\, e_5$	$\mathfrak{g}_1$

Table A.5. Subalgebras of $\mathfrak{g}_{5,2}$. We denote $\sigma_x = \operatorname{sgn} x$.

$\mathfrak{h}_{1,4}^{x_5,f(x_5)} \sim \mathfrak{h}_{1,4}^{x_5,a^3f(x_5/a^2)};\ a \neq 0$	$\mathfrak{h}_{2,7}^{1+x_4,f(x_4),g(x_4)} \sim \mathfrak{h}_{2,7}^{1+x_4,f(x_4),-g(x_4)}$
$\mathfrak{h}_{2,6}^{1+x_4,f(x_4)} \sim \mathfrak{h}_{2,6}^{1+x_4,f(x_4)+ax_4}$	$\mathfrak{h}_{2,7}^{1+x_4,x_5,g(x_4,x_5)} \sim \mathfrak{h}_{2,7}^{1+x_4,x_5,-g(x_4,x_5)}$
$\mathfrak{h}_{2,7}^{x_4,x_4,f(x_4)} \sim \mathfrak{h}_{2,7}^{x_4,x_4,a^3f(x_4/a^2)}$	$\mathfrak{h}_{3,4}^{x_3,f(x_3)} \sim \mathfrak{h}_{3,4}^{x_3,a^3f(x_3/a^2)};\ a \neq 0$

Table A.6. Subalgebra parametrization equivalence for $\mathfrak{g}_{5,2}$

Subalgebra	Realization
$\mathfrak{h}_0$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ x_2\partial_1+x_3\partial_2+x_4\partial_3+\partial_5$
$\mathfrak{h}_{1,2}$	$\partial_1,\ -x_4\partial_1,\ \partial_2,\ \partial_3,\ -x_2x_4\partial_1+x_3\partial_2+\partial_4$
$\mathfrak{h}_{1,3}^{0}$	$\partial_1,\ \partial_2,\ \frac{x_4^2}{2}\partial_1-x_4\partial_2,\ \partial_3,\ \left(x_2-\frac{1}{2}x_3x_4^2\right)\partial_1-x_3x_4\partial_2+\partial_4$
$\mathfrak{h}_{1,3}^{x_5}$	$\partial_1,\ \partial_2,\ \left(x_5-\frac{x_4^2}{2}\right)\partial_1-x_4\partial_2,\ \partial_3,\ \left(x_2+x_5x_3-\frac{1}{2}x_3x_4^2\right)\partial_1-x_3x_4\partial_2+\partial_4$
$\mathfrak{h}_{1,4}^{0,0}$	$\partial_1,\ \partial_2,\ \partial_3,\ -\frac{x_4^3}{6}\partial_1-\frac{x_4^2}{2}\partial_2-x_4\partial_3,\ x_2\partial_1+x_3\partial_2+\partial_4$
$\mathfrak{h}_{1,4}^{0,x_5}$	$\partial_1,\ \partial_2,\ \partial_3,\ \left(x_5-\frac{x_4^3}{6}\right)\partial_1-\frac{x_4^2}{2}\partial_2-x_4\partial_3,\ x_2\partial_1+x_3\partial_2+\partial_4$
$\mathfrak{h}_{1,4}^{x_5,f(x_5)}$	$\partial_1,\ \partial_2,\ \partial_3,\ \left(f(x_5)+x_5x_4-\frac{x_4^3}{6}\right)\partial_1+\left(x_5-\frac{x_4^2}{2}\right)\partial_2-x_4\partial_3,\ x_2\partial_1+x_3\partial_2+\partial_4$
$\mathfrak{h}_{1,4}^{x_5,x_6}$	$\partial_1,\ \partial_2,\ \partial_3,\ \left(x_6+x_5x_4-\frac{x_4^3}{6}\right)\partial_1+\left(x_5-\frac{x_4^2}{2}\right)\partial_2-x_4\partial_3,\ x_2\partial_1+x_3\partial_2+\partial_4$
$\mathfrak{h}_{1,5}^{0}$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ x_2\partial_1+x_3\partial_2+x_4\partial_3$
$\mathfrak{h}_{1,5}^{x_5}$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ x_2\partial_1+x_3\partial_2+x_4\partial_3+x_5\partial_4$
$\mathfrak{h}_{2,5}^{0}$	$\partial_1,\ -x_3\partial_1,\ \frac{x_3^2}{2}\partial_1,\ \partial_2,\ \frac{1}{2}x_2x_3^2\partial_1+\partial_3$
$\mathfrak{h}_{2,5}^{x_4}$	$\partial_1,\ -x_3\partial_1,\ \left(x_4+\frac{x_3^2}{2}\right)\partial_1,\ \partial_2,\ \frac{1}{2}x_2\left(2x_4+x_3^2\right)\partial_1+\partial_3$
$\mathfrak{h}_{2,6}^{0,0}$	$\partial_1,\ -x_3\partial_1,\ \partial_2,\ \frac{x_3^3}{3}\partial_1-x_3\partial_2,\ -x_2x_3\partial_1+\partial_3$
$\mathfrak{h}_{2,6}^{0,x_4}$	$\partial_1,\ -x_3\partial_1,\ \partial_2,\ \left(x_4+\frac{x_3^3}{3}\right)\partial_1-x_3\partial_2,\ -x_2x_3\partial_1+\partial_3$
$\mathfrak{h}_{2,6}^{1,0}$	$\partial_1,\ -x_3\partial_1,\ \partial_2,\ \left(-\frac{x_3^2}{2}+\frac{x_3^3}{3}\right)\partial_1+(1-x_3)\,\partial_2,\ -x_2x_3\partial_1+\partial_3$
$\mathfrak{h}_{2,6}^{1,x_4}$	$\partial_1,\ -x_3\partial_1,\ \partial_2,\ \left(x_4-\frac{x_3^2}{2}+\frac{x_3^3}{3}\right)\partial_1+(1-x_3)\,\partial_2,\ -x_2x_3\partial_1+\partial_3$
$\mathfrak{h}_{2,6}^{1+x_4,f(x_4)}$	$\partial_1,\ -x_3\partial_1,\ \partial_2,\ \left(f(x_4)-\frac{(1+x_4)x_3^2}{2}+\frac{x_3^3}{3}\right)\partial_1+(1+x_4-x_3)\,\partial_2,\ -x_2x_3\partial_1+\partial_3$
$\mathfrak{h}_{2,6}^{1+x_4,x_5}$	$\partial_1,\ -x_3\partial_1,\ \partial_2,\ \left(x_5-\frac{(1+x_4)x_3^2}{2}+\frac{x_3^3}{3}\right)\partial_1+(1+x_4-x_3)\,\partial_2,\ -x_2x_3\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{0,0,0}$	$\partial_1,\ \partial_2,\ -\frac{x_3^2}{2}\partial_1-x_3\partial_2,\ \frac{x_3^3}{3}\partial_1+\frac{x_3^2}{2}\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{0,0,x_4}$	$\partial_1,\ \partial_2,\ -\frac{x_3^2}{2}\partial_1-x_3\partial_2,\ \left(x_4+\frac{x_3^3}{3}\right)\partial_1+\frac{x_3^2}{2}\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{x_4,x_4,g(x_4)}$	$\partial_1,\ \partial_2,\ \left(x_4-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,\ \left(g(x_4)+\frac{x_3^3}{3}\right)\partial_1+\left(x_4+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{x_4,x_4,x_5}$	$\partial_1,\ \partial_2,\ \left(x_4-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,\ \left(x_5+\frac{x_3^3}{3}\right)\partial_1+\left(x_4+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1,0,0}$	$\partial_1,\ \partial_2,\ \left(1-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,\ \left(-x_3+\frac{x_3^3}{3}\right)\partial_1+\frac{x_3^2}{2}\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1,0,x_4}$	$\partial_1,\ \partial_2,\ \left(1-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,\ \left(x_4-x_3+\frac{x_3^3}{3}\right)\partial_1+\frac{x_3^2}{2}\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1,x_4,g(x_4)}$	$\partial_1,\ \partial_2,\ \left(1-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,\ \left(g(x_4)+(x_4-1)x_3+\frac{x_3^3}{3}\right)\partial_1+\left(x_4+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1,x_4,x_5}$	$\partial_1,\ \partial_2,\ \left(1-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,\ \left(x_5+(x_4-1)x_3+\frac{x_3^3}{3}\right)\partial_1+\left(x_4+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1+x_4,f(x_4),g(x_4)}$	$\partial_1,\ \partial_2,\ \left(1+x_4-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,$
	$\left(g(x_4)+(f(x_4)-1-x_4)x_3+\frac{x_3^3}{3}\right)\partial_1+\left(f(x_4)+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1+x_4,f(x_4),x_5}$	$\partial_1,\ \partial_2,\ \left(1+x_4-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,$
	$\left(x_5+(f(x_4)-1-x_4)x_3+\frac{x_3^3}{3}\right)\partial_1+\left(f(x_4)+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1+x_4,x_5,g(x_4,x_5)}$	$\partial_1,\ \partial_2,\ \left(1+x_4-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,$
	$\left(g(x_4,x_5)+(x_5-1-x_4)x_3+\frac{x_3^3}{3}\right)\partial_1+\left(x_5+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{2,7}^{1+x_4,x_5,x_6}$	$\partial_1,\ \partial_2,\ \left(1+x_4-\frac{x_3^2}{2}\right)\partial_1-x_3\partial_2,\ \left(x_6+(x_5-1-x_4)x_3+\frac{x_3^3}{3}\right)\partial_1+\left(x_5+\frac{x_3^2}{2}\right)\partial_2,\ x_2\partial_1+\partial_3$
$\mathfrak{h}_{3,4}^{0,0}$	$\partial_1,\ -x_2\partial_1,\ \frac{x_2^2}{2}\partial_1,\ -\frac{x_2^3}{6}\partial_1,\ \partial_2$
$\mathfrak{h}_{3,4}^{0,x_3}$	$\partial_1,\ -x_2\partial_1,\ \frac{x_2^2}{2}\partial_1,\ \left(x_3-\frac{x_2^3}{6}\right)\partial_1,\ \partial_2$
$\mathfrak{h}_{3,4}^{x_3,f(x_3)}$	$\partial_1,\ -x_2\partial_1,\ \left(x_3+\frac{x_2^2}{2}\right)\partial_1,\ \left(f(x_3)-x_3x_2-\frac{x_2^3}{6}\right)\partial_1,\ \partial_2$
$\mathfrak{h}_{3,4}^{x_3,x_4}$	$\partial_1,\ -x_2\partial_1,\ \left(x_3+\frac{x_2^2}{2}\right)\partial_1,\ \left(x_4-x_3x_2-\frac{x_2^3}{6}\right)\partial_1,\ \partial_2$

Table A.7. Complete system of faithful realizations of $\mathfrak{g}_{5,2}$.

Lie algebra $\mathfrak{g}_{5,3}$

$$\operatorname{Int}\mathfrak{g}=\left\{\begin{pmatrix}1&0&t_3&\frac{1}{2}t_3^2&t_1\\0&1&t_2&t_1+t_2t_3&-\frac{1}{2}t_2^2\\0&0&1&t_3&-t_2\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\right\},$$

$$\operatorname{Aut}\mathfrak{g}=\left\{\begin{pmatrix}Du_5&Dt_5&t_3u_5-t_5u_3&t_1&u_1\\Du_4&Dt_4&t_3u_4-t_4u_3&t_2&u_2\\0&0&D&t_3&u_3\\0&0&0&t_4&u_4\\0&0&0&t_5&u_5\end{pmatrix}\middle|\ D:=t_4u_5-t_5u_4\neq 0\right\}$$

Type	Name	Int-class	Aut-class	$\mathfrak{g}/\mathfrak{h}$ type
$\mathfrak{g}_1$	$\mathfrak{h}_{1,1}$	e_1	e_1	$\mathfrak{g}_{4,1}$
	$\mathfrak{h}_{1,2}^a$	e_2-ae_1		
	$\mathfrak{h}_{1,3}$	e_3	e_3	–
	$\mathfrak{h}_{1,4}^a$	e_4-ae_1	e_4	–
	$\mathfrak{h}_{1,5}^{a,b}$	$e_5-ae_4-be_2$		
$2\mathfrak{g}_1$	$\mathfrak{h}_{2,1}$	$e_1,\ e_2$	$e_1,\ e_2$	$\mathfrak{g}_{3,1}$
	$\mathfrak{h}_{2,2}$	$e_1,\ e_3$	$e_1,\ e_3$	–
	$\mathfrak{h}_{2,3}^a$	$e_2-ae_1,\ e_3$		
	$\mathfrak{h}_{2,4}^a$	$e_1-ae_2,\ e_4$	$e_1,\ e_4$	–
	$\mathfrak{h}_{2,5}^a$	$e_1,\ e_5-ae_4;\ a\neq 0$		
	$\mathfrak{h}_{2,6}^{a,b}$	$e_2-ae_1,\ e_5-be_4;\ ab\neq 1$		
	$\mathfrak{h}_{2,7}^{a,b}$	$e_1-ae_2,\ e_5-ae_4-be_2$	$e_1,\ e_5$	–
	$\mathfrak{h}_{2,8}^a$	$e_2,\ e_4-ae_1;$		
$3\mathfrak{g}_1$	$\mathfrak{h}_{3,1}$	$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,2}$	$e_1,\ e_2,\ e_4$	$e_1,\ e_2,\ e_4$	–
	$\mathfrak{h}_{3,3}^a$	$e_1,\ e_2,\ e_5-ae_4$		
$\mathfrak{g}_{3,1}$	$\mathfrak{h}_{3,4}^{a,b}$	$e_1-ae_2,\ e_3,\ e_5-ae_4-be_2$	$e_1,\ e_3,\ e_5$	–
	$\mathfrak{h}_{3,5}^a$	$e_2,\ e_3,\ e_4-ae_1$		
$\mathfrak{g}_1\oplus\mathfrak{g}_{3,1}$	$\mathfrak{h}_{4,1}$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$\mathfrak{g}_1$
	$\mathfrak{h}_{4,2}^a$	$e_1,\ e_2,\ e_3,\ e_5-ae_4$		

Table A.8. Subalgebras of $\mathfrak{g}_{5,3}$.

$\mathfrak{h}_{1,5}^{x_5,f(x_5)}\sim\mathfrak{h}_{1,5}^{x_5,bx_5+cf(ax_5)};\ a\neq 0, c>0$	$\mathfrak{h}_{2,7}^{x_4,f(x_4)}\sim\mathfrak{h}_{2,7}^{x_4,bx_4+cf(ax_4)};\ a\neq 0, c>0$
$\mathfrak{h}_{2,6}^{x_4,f(x_4)}\sim\mathfrak{h}_{2,6}^{x_4,af(ax_4)+ax_4};\ a\neq 0$	$\mathfrak{h}_{3,4}^{x_3,f(x_3)}\sim\mathfrak{h}_{3,4}^{x_3,bx_3+cf(ax_3)};\ a\neq 0, c>0$

Table A.9. Subalgebra parametrization equivalence for $\mathfrak{g}_{5,3}$

Subalgebra	Realization
$\mathfrak{h}_0$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_2+\partial_4,\, x_3\partial_1+\frac{1}{2}x_4^2\partial_2+x_4\partial_3+\partial_5$
$\mathfrak{h}_{1,2}^{x_5}$	$\partial_1,\, x_5\partial_1,\, \partial_2,\, x_5x_2\partial_1+\partial_3,\, \left(x_2+\frac{x_5x_3^2}{2}\right)\partial_1+x_3\partial_2+\partial_4$
$\mathfrak{h}_{1,3}$	$\partial_1,\, \partial_2,\, -x_4\partial_1-x_3\partial_2,\, \partial_3,\, -x_3x_4\partial_1-\frac{1}{2}x_3^2\partial_2+\partial_4$
$\mathfrak{h}_{1,4}^{0}$	$\partial_1,\, \partial_2,\, \partial_3,\, -\frac{x_4^2}{2}\partial_1+x_3\partial_2-x_4\partial_3,\, x_3\partial_1+\partial_4$
$\mathfrak{h}_{1,4}^{x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, \left(x_5-\frac{x_4^2}{2}\right)\partial_1+x_3\partial_2-x_4\partial_3,\, x_3\partial_1+\partial_4$
$\mathfrak{h}_{1,5}^{0,0}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_2+\partial_4,\, x_3\partial_1+\frac{x_4^2}{2}\partial_2+x_4\partial_3$
$\mathfrak{h}_{1,5}^{0,x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_2+\partial_4,\, x_3\partial_1+\left(x_5+\frac{x_4^2}{2}\right)\partial_2+x_4\partial_3$
$\mathfrak{h}_{1,5}^{x_5,f(x_5)}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_2+\partial_4,\, x_3\partial_1+\left(f(x_5)+\frac{x_4^2}{2}\right)\partial_2+x_4\partial_3+x_5\partial_4$
$\mathfrak{h}_{1,5}^{x_5,x_6}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_2+\partial_4,\, x_3\partial_1+\left(x_6+\frac{x_4^2}{2}\right)\partial_2+x_4\partial_3+x_5\partial_4$
$\mathfrak{h}_{2,3}^{x_4}$	$\partial_1,\, x_4\partial_1,\, \left(-x_4x_2-x_3\right)\partial_1,\, \partial_2,\, -\frac{1}{2}x_2\left(x_4x_2+2x_3\right)\partial_1+\partial_3$
$\mathfrak{h}_{2,6}^{x_4,f(x_4)}$	$\partial_1,\, x_4\partial_1,\, \partial_2,\, x_4x_2\partial_1+\partial_3,\, \left(x_2+\frac{x_4x_3^2}{2}\right)\partial_1+x_3\partial_2+f(x_4)\partial_3$
$\mathfrak{h}_{2,6}^{x_4,x_5}$	$\partial_1,\, x_4\partial_1,\, \partial_2,\, x_4x_2\partial_1+\partial_3,\, \left(x_2+\frac{x_4x_3^2}{2}\right)\partial_1+x_3\partial_2+x_5\partial_3$
$\mathfrak{h}_{2,7}^{x_4,f(x_4)}$	$x_4\partial_1,\, \partial_1,\, \partial_2,\, x_2\partial_1+\partial_3,\, \left(f(x_4)+x_4x_2+\frac{x_3^2}{2}\right)\partial_1+x_3\partial_2+x_4\partial_3$
$\mathfrak{h}_{2,7}^{x_4,x_5}$	$x_4\partial_1,\, \partial_1,\, \partial_2,\, x_2\partial_1+\partial_3,\, \left(x_5+x_4x_2+\frac{x_3^2}{2}\right)\partial_1+x_3\partial_2+x_4\partial_3$
$\mathfrak{h}_{3,4}^{x_3,f(x_3)}$	$x_3\partial_1,\, \partial_1,\, -x_2\partial_1,\, \partial_2,\, \left(f(x_3)-\frac{x_2^2}{2}\right)\partial_1+x_3\partial_2$
$\mathfrak{h}_{3,4}^{x_3,x_4}$	$x_3\partial_1,\, \partial_1,\, -x_2\partial_1,\, \partial_2,\, \left(x_4-\frac{x_2^2}{2}\right)\partial_1+x_3\partial_2$

Table A.10. Complete system of realizations of $\mathfrak{g}_{5,3}$.

Lie algebra $\mathfrak{g}_{5,4}$

$$\operatorname{Int}\mathfrak{g} = \left\{ \begin{pmatrix} 1 & t_1 & t_2 & t_3 & t_4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$$\operatorname{Aut}\mathfrak{g} = \left\{ \begin{pmatrix} a & \vec{t} \\ 0 & B \end{pmatrix} \,\middle|\, a \neq 0,\ B \in \mathbb{R}^{4,4}, B^{\mathrm{T}}\Omega B = a\Omega \right\},$$

where $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the matrix of the standard symplectic form.

Type	Name	Int-class	Aut-class	$\mathfrak{g}/\mathfrak{h}$ type
$\mathfrak{g}_1$	$\mathfrak{h}_{1,1}$	e_1	e_1	$4\mathfrak{g}_1$
	$\mathfrak{h}_{1,2}$	e_2	e_2	–
	$\mathfrak{h}_{1,3}^{a}$	$e_3 - ae_2$		
	$\mathfrak{h}_{1,4}^{a,b}$	$e_4 - ae_3 - be_2$		
	$\mathfrak{h}_{1,5}^{a,b,c}$	$e_5 - ae_4 - be_3 - ce_2$		
$2\mathfrak{g}_1$	$\mathfrak{h}_{2,1}$	$e_1,\ e_2$	$e_1,\ e_2$	$3\mathfrak{g}_1$
	$\mathfrak{h}_{2,2}^{a}$	$e_1,\ e_3 - ae_2$		
	$\mathfrak{h}_{2,3}^{a,b}$	$e_1,\ e_4 - ae_3 - be_2$		
	$\mathfrak{h}_{2,4}^{a,b,c}$	$e_1,\ e_5 - ae_4 - be_3 - ce_2$		
	$\mathfrak{h}_{2,5}$	$e_2,\ e_3$	$e_2,\ e_3$	–
	$\mathfrak{h}_{2,6}^{a}$	$e_2,\ e_5 - ae_3$		
	$\mathfrak{h}_{2,7}^{a,b}$	$e_3 - ae_2,\ e_4 + ae_5 - be_2$		
	$\mathfrak{h}_{2,8}^{a,b,c}$	$e_4 - ae_3 - be_2,\ e_5 - ce_3 - ae_2$		
$3\mathfrak{g}_1$	$\mathfrak{h}_{3,1}$	$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,2}^{a}$	$e_1,\ e_2,\ e_5 - ae_3$		
	$\mathfrak{h}_{3,3}^{a,b}$	$e_1,\ e_3,\ e_4 - ae_2$		
	$\mathfrak{h}_{3,4}^{a,b}$	$e_1,\ e_3 - ae_2,\ e_4 + ae_5 - be_2$		
	$\mathfrak{h}_{3,5}^{a,b,c}$	$e_1,\ e_4 - ae_3 - be_2,\ e_5 - ce_3 - ae_2$		
$\mathfrak{g}_{3,1}$	$\mathfrak{h}_{3,6}^{a,b}$	$e_1,\ e_2,\ e_4 - ae_5 - be_3$	$e_1,\ e_2,\ e_4$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,7}^{a,b}$	$e_1,\ e_3 - ae_2,\ e_4 - be_2;\ a \neq 0$		
	$\mathfrak{h}_{3,8}^{a,b,c}$	$e_1,\ e_3 - ae_2,\ e_5 - be_4 - ce_2;\ ab \neq -1$		
	$\mathfrak{h}_{3,9}^{a,b,c,d}$	$e_1,\ e_4 - ae_3 - be_2,\ e_5 - ce_3 - de_2;\ a \neq d$		
$\mathfrak{g}_1 \oplus \mathfrak{g}_{3,1}$	$\mathfrak{h}_{4,1}$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$\mathfrak{g}_1$
	$\mathfrak{h}_{4,2}^{a}$	$e_1,\ e_2,\ e_3,\ e_5 - ae_4$		
	$\mathfrak{h}_{4,3}^{a,b}$	$e_1,\ e_2,\ e_4 - ae_3,\ e_5 - be_3$		
	$\mathfrak{h}_{4,4}^{a,b,c}$	$e_1,\ e_3 - ae_2,\ e_4 - be_2,\ e_5 - ce_2$		

Table A.11. Subalgebras of $\mathfrak{g}_{5,4}$.

Subalgebra	Realization
$\mathfrak{h}_0$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, x_3\partial_1 + \partial_5$
$\mathfrak{h}_{1,5}^{0,0,0}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, x_3\partial_1$
$\mathfrak{h}_{1,5}^{0,0,x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, (x_3 + x_5x_4)\,\partial_1 + x_5\partial_2$
$\mathfrak{h}_{1,5}^{0,x_5,g(x_5)}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, (x_3 + g(x_5)x_4)\,\partial_1 + g(x_5)\partial_2 + x_5\partial_3$
$\mathfrak{h}_{1,5}^{0,x_5,x_6}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, (x_3 + x_6x_4)\,\partial_1 + x_6\partial_2 + x_5\partial_3$
$\mathfrak{h}_{1,5}^{x_5,f(x_5),g(x_5)}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, (x_3 + g(x_5)x_4)\,\partial_1 + g(x_5)\partial_2 + f(x_5)\partial_3 + x_5\partial_4$
$\mathfrak{h}_{1,5}^{x_5,f(x_5),x_6}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, (x_3 + x_6x_4)\,\partial_1 + x_6\partial_2 + f(x_5)\partial_3 + x_5\partial_4$
$\mathfrak{h}_{1,5}^{x_5,x_6,g(x_5,x_6)}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, (x_3 + g(x_5,x_6)x_4)\,\partial_1 + g(x_5,x_6)\partial_2 + x_6\partial_3 + x_5\partial_4$
$\mathfrak{h}_{1,5}^{x_5,x_6,x_7}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4,\, (x_3 + x_7x_4)\,\partial_1 + x_7\partial_2 + x_6\partial_3 + x_5\partial_4$
$\mathfrak{h}_{2,8}^{0,0,0}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1,\, x_3\partial_1$
$\mathfrak{h}_{2,8}^{0,0,x_4}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1,\, x_3\partial_1 + x_4\partial_3$
$\mathfrak{h}_{2,8}^{0,x_4,g(x_4)}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + x_4\partial_2,\, x_3\partial_1 + g(x_4)\partial_3$
$\mathfrak{h}_{2,8}^{0,x_4,x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + x_4\partial_2,\, x_3\partial_1 + x_5\partial_3$
$\mathfrak{h}_{2,8}^{x_4,f(x_4),g(x_4)}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + f(x_4)\partial_2 + x_4\partial_3,\, x_3\partial_1 + x_4\partial_2 + g(x_4)\partial_3$
$\mathfrak{h}_{2,8}^{x_4,f(x_4),x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + f(x_4)\partial_2 + x_4\partial_3,\, x_3\partial_1 + x_4\partial_2 + x_5\partial_3$
$\mathfrak{h}_{2,8}^{x_4,x_5,g(x_4,x_5)}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + x_5\partial_2 + x_4\partial_3,\, x_3\partial_1 + x_4\partial_2 + g(x_4,x_5)\partial_3$
$\mathfrak{h}_{2,8}^{x_4,x_5,x_6}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_2\partial_1 + x_5\partial_2 + x_4\partial_3,\, x_3\partial_1 + x_4\partial_2 + x_6\partial_3$

Table A.12. Complete system of realizations of $\mathfrak{g}_{5,4}$.

In this case it would be very complicated to express the parametrization equivalence explicitly, so we only express the action of $\operatorname{Aut}\mathfrak{g}$ on the subalgebra families $\mathfrak{h}_{1,5}$ and $\mathfrak{h}_{2,8}$, which defines the reparametrizations implicitly.

$$
\begin{aligned}
\mathfrak{h}_{1,5}^{a,b,c} \sim \mathfrak{h}_{1,5}^{\tilde{a},\tilde{b},\tilde{c}} \qquad &\Leftrightarrow \qquad \exists B \in \widetilde{\mathrm{Sp}}(4) \quad B \begin{pmatrix} -c \\ -b \\ -a \\ 1 \end{pmatrix} = \begin{pmatrix} -\tilde{c} \\ -\tilde{b} \\ -\tilde{a} \\ 1 \end{pmatrix} \\
\mathfrak{h}_{2,8}^{a,b,c} \sim \mathfrak{h}_{2,8}^{\tilde{a},\tilde{b},\tilde{c}} \qquad &\Leftrightarrow \qquad \exists B \in \widetilde{\mathrm{Sp}}(4) \quad \exists \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \\
&\qquad \begin{pmatrix} s_1B & s_2B \\ s_3B & s_4B \end{pmatrix} \begin{pmatrix} -b \\ -a \\ 1 \\ 0 \\ -a \\ -c \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\tilde{b} \\ -\tilde{a} \\ 1 \\ 0 \\ -\tilde{a} \\ -\tilde{c} \\ 0 \\ 1 \end{pmatrix},
\end{aligned}
$$

where $\widetilde{\mathrm{Sp}}(4) = \{B \in \mathbb{R}^{4,4} \mid \exists a \in \mathbb{R}\ B^{\mathrm{T}}\Omega B = a\Omega\}$. The parameters $a, b, c, \tilde{a}, \tilde{b}, \tilde{c}$ should be replaced by the corresponding functions used in table A.12.

Lie algebra $\mathfrak{g}_{5,5}$

$$\operatorname{Int}\mathfrak{g} = \left\{ \begin{pmatrix} 1 & t_4 & t_3 + \frac{1}{2}t_4^2 & t_2 & t_1 \\ 0 & 1 & t_4 & 0 & t_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$$\operatorname{Aut}\mathfrak{g} = \left\{ \begin{pmatrix} ab^2 & at_4 - t_3t_8 + bt_7 & t_6 & t_5 & t_1 \\ 0 & ab & t_7 & t_3b & t_2 \\ 0 & 0 & a & 0 & t_3 \\ 0 & 0 & t_8 & b^2 & t_4 \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \middle| \; a, b \neq 0 \right\}$$

Type	Name	Int-class	Aut-class	$\mathfrak{g}/\mathfrak{h}$ type
$\mathfrak{g}_1$	$\mathfrak{h}_{1,1}$	e_1	e_1	$\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$
	$\mathfrak{h}_{1,2}$	e_2	e_2	–
	$\mathfrak{h}_{1,3}^{a}$	$e_4 - ae_2$	e_4	–
	$\mathfrak{h}_{1,4}^{a}$	$e_3 - ae_4$	e_3	–
	$\mathfrak{h}_{1,5}^{a,b}$	$e_5 - ae_4 - be_3$	e_5	–
$2\mathfrak{g}_1$	$\mathfrak{h}_{2,1}$	$e_1,\, e_2$	$e_1,\, e_2$	$3\mathfrak{g}_1$
	$\mathfrak{h}_{2,2}^{a}$	$e_1,\, e_4 - ae_2$	$e_1,\, e_4$	$\mathfrak{g}_{3,1}$
	$\mathfrak{h}_{2,3}^{a}$	$e_1,\, e_3 - ae_4$	$e_1,\, e_3$	–
	$\mathfrak{h}_{2,4}^{a,b}$	$e_1,\, e_5 - ae_4 - be_3$	$e_1,\, e_5$	–
	$\mathfrak{h}_{2,5}$	$e_2,\, e_4$	$e_2,\, e_4$	–
	$\mathfrak{h}_{2,6}^{a}$	$e_2,\, e_3 - ae_4$	$e_2,\, e_3$	–
	$\mathfrak{h}_{2,7}^{a,b}$	$e_4 - ae_2 - be_1,\, e_5 - ae_3$	$e_4,\, e_5$	–
$3\mathfrak{g}_1$	$\mathfrak{h}_{3,1}$	$e_1,\, e_2,\, e_4$	$e_1,\, e_2,\, e_4$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,2}^{a}$	$e_1,\, e_2,\, e_3 - ae_4$	$e_1,\, e_2,\, e_3$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,3}^{a}$	$e_1,\, e_4 - ae_2,\, e_5 - ae_3$	$e_1,\, e_4,\, e_5$	–
$\mathfrak{g}_{3,1}$	$\mathfrak{h}_{3,4}^{a,b}$	$e_1,\, e_2,\, e_5 - ae_4 - be_3$	$e_1,\, e_2,\, e_5$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,5}^{a}$	$e_1,\, e_3,\, e_4 - ae_2$	$e_1,\, e_3,\, e_4$	–
	$\mathfrak{h}_{3,6}^{a,b}$	$e_1,\, e_4 - ae_2,\, e_5 - be_3;\ a \neq b$	$e_1,\, e_4 - e_2,\, e_5$	–
$\mathfrak{g}_1 \oplus \mathfrak{g}_{3,1}$	$\mathfrak{h}_{4,1}$	$e_1,\, e_2,\, e_3,\, e_4$	$e_1,\, e_2,\, e_3,\, e_4$	$\mathfrak{g}_1$
	$\mathfrak{h}_{4,2}^{a}$	$e_1,\, e_2,\, e_4,\, e_5 - ae_3$	$e_1,\, e_2,\, e_4,\, e_5$	$\mathfrak{g}_1$
$\mathfrak{g}_{4,1}$	$\mathfrak{h}_{4,3}^{a,b}$	$e_1,\, e_2,\, e_3 - ae_4,\, e_5 - be_4$	$e_1,\, e_2,\, e_3,\, e_5$	$\mathfrak{g}_1$

Table A.13. Subalgebras of $\mathfrak{g}_{5,5}$.

$\mathfrak{h}_{1,5}^{f(x_5),x_5} \sim \mathfrak{h}_{1,5}^{bx_5+cf(ax_5),x_5};\ a, c \neq 0$	$\mathfrak{h}_{2,7}^{x_4,f(x_4)} \sim \mathfrak{h}_{2,7}^{x_4,bf(ax_4)};\ a \neq 0, c > 0$

Table A.14. Subalgebra parametrization equivalence for $\mathfrak{g}_{5,5}$

Subalgebra	Realization
$\mathfrak{h}_0$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, x_2\partial_1 + x_3\partial_2 + \partial_5$
$\mathfrak{h}_{1,2}$	$\partial_1,\, -x_4\partial_1,\, \partial_2,\, x_2\partial_1 + \partial_3,\, -x_2x_4\partial_1 + \partial_4$
$\mathfrak{h}_{1,3}^{0}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1,\, x_2\partial_1 + x_3\partial_2 + \partial_4$
$\mathfrak{h}_{1,3}^{x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, (x_3 + x_5x_4)\,\partial_1 + x_5\partial_2,\, x_2\partial_1 + x_3\partial_2 + \partial_4$
$\mathfrak{h}_{1,4}^{0}$	$\partial_1,\, \partial_2,\, \left(-x_3 - \frac{x_4^2}{2}\right)\partial_1 - x_4\partial_2,\, \partial_3,\, x_2\partial_1 + \partial_4$
$\mathfrak{h}_{1,4}^{x_5}$	$\partial_1,\, \partial_2,\, \left(-x_3 - \frac{x_4^2}{2}\right)\partial_1 - x_4\partial_2 + x_5\partial_3,\, \partial_3,\, x_2\partial_1 + \partial_4$
$\mathfrak{h}_{1,5}^{0,0}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, x_2\partial_1 + x_3\partial_2$
$\mathfrak{h}_{1,5}^{x_5,0}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, x_2\partial_1 + x_3\partial_2 + x_5\partial_4$
$\mathfrak{h}_{1,5}^{f(x_5),x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, (x_2 + x_5x_4)\,\partial_1 + x_3\partial_2 + x_5\partial_3 + f(x_5)\partial_4$
$\mathfrak{h}_{1,5}^{x_6,x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, (x_2 + x_5x_4)\,\partial_1 + x_3\partial_2 + x_5\partial_3 + x_6\partial_4$
$\mathfrak{h}_{2,5}$	$\partial_1,\, -x_3\partial_1,\, \partial_2,\, x_2\partial_1,\, -x_2x_3\partial_1 + \partial_3$
$\mathfrak{h}_{2,6}^{0}$	$\partial_1,\, -x_3\partial_1,\, \frac{1}{2}\left(-2x_2 + x_3^2\right)\partial_1,\, \partial_2,\, \partial_3$
$\mathfrak{h}_{2,6}^{x_4}$	$\partial_1,\, -x_3\partial_1,\, \frac{1}{2}\left(-2x_2 + x_3^2\right)\partial_1 + x_4\partial_2,\, \partial_2,\, \partial_3$
$\mathfrak{h}_{2,7}^{0,0}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1,\, x_2\partial_1 + x_3\partial_2$
$\mathfrak{h}_{2,7}^{0,x_4}$	$\partial_1,\, \partial_2,\, \partial_3,\, (x_4 + x_3)\,\partial_1,\, x_2\partial_1 + x_3\partial_2$
$\mathfrak{h}_{2,7}^{x_4,f(x_4)}$	$\partial_1,\, \partial_2,\, \partial_3,\, (f(x_4) + x_3)\,\partial_1 + x_4\partial_2,\, x_2\partial_1 + x_3\partial_2 + x_4\partial_3$
$\mathfrak{h}_{2,7}^{x_4,x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, (x_5 + x_3)\,\partial_1 + x_4\partial_2,\, x_2\partial_1 + x_3\partial_2 + x_4\partial_3$

Table A.15. Complete system of realizations of $\mathfrak{g}_{5,5}$.

Lie algebra $\mathfrak{g}_{5,6}$

$$\operatorname{Int}\mathfrak{g} = \left\{ \begin{pmatrix} 1 & t_4 & \frac{1}{2}t_4^2 - t_3 & \frac{1}{6}t_4^3 - t_3t_4 + t_2 & t_1 \\ 0 & 1 & t_4 & \frac{1}{2}t_4^2 & t_2 \\ 0 & 0 & 1 & t_4 & t_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$$\operatorname{Aut}\mathfrak{g} = \left\{ \begin{pmatrix} a^5 & a^2(t_7 + t_4 a) & -t_3a^2 + t_6 a + t_7 t_4 & t_5 & t_1 \\ 0 & a^4 & t_7 a & t_6 & t_2 \\ 0 & 0 & a^3 & t_7 & t_3 \\ 0 & 0 & 0 & a^2 & t_4 \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \middle| \, a \neq 0 \right\}$$

Type	Name	Int-class	Aut-class	$\mathfrak{g}/\mathfrak{h}$ type
$\mathfrak{g}_1$	$\mathfrak{h}_{1,1}$	e_1	e_1	$\mathfrak{g}_{4,1}$
	$\mathfrak{h}_{1,2}$	e_2	e_2	–
	$\mathfrak{h}_{1,3}$	e_3	e_3	–
	$\mathfrak{h}_{1,4}^a$	$e_4 - ae_2$	e_4	–
	$\mathfrak{h}_{1,5}^a$	$e_5 - ae_4$	e_5	–
$2\mathfrak{g}_1$	$\mathfrak{h}_{2,1}$	$e_1,\, e_2$	$e_1,\, e_2$	$\mathfrak{g}_{3,1}$
	$\mathfrak{h}_{2,2}$	$e_1,\, e_3$	$e_1,\, e_3$	–
	$\mathfrak{h}_{2,3}^a$	$e_1,\, e_4 - ae_2$	$e_1,\, e_4$	–
	$\mathfrak{h}_{2,4}^a$	$e_1,\, e_5 - ae_4$	$e_1,\, e_5$	–
	$\mathfrak{h}_{2,5}$	$e_2,\, e_3$	$e_2,\, e_3$	–
	$\mathfrak{h}_{2,6}^a$	$e_2,\, e_4 - ae_3$	$e_2,\, e_4$	–
$3\mathfrak{g}_1$	$\mathfrak{h}_{3,1}$	$e_1,\, e_2,\, e_3$	$e_1,\, e_2,\, e_3$	$2\mathfrak{g}_1$
	$\mathfrak{h}_{3,2}$	$e_1,\, e_2,\, e_4$	$e_1,\, e_2,\, e_4$	–
$\mathfrak{g}_{3,1}$	$\mathfrak{h}_{3,3}^a$	$e_1,\, e_2,\, e_5 - ae_4$	$e_1,\, e_2,\, e_5$	–
	$\mathfrak{h}_{3,4}^a$	$e_1,\, e_3,\, e_4 - ae_2$	$e_1,\, e_3,\, e_4$	–
$\mathfrak{g}_1 \oplus \mathfrak{g}_{3,1}$	$\mathfrak{h}_{4,1}$	$e_1,\, e_2,\, e_3,\, e_4$	$e_1,\, e_2,\, e_3,\, e_4$	$\mathfrak{g}_1$
$\mathfrak{g}_{4,1}$	$\mathfrak{h}_{4,2}^a$	$e_1,\, e_2,\, e_3,\, e_5 - ae_4$	$e_1,\, e_2,\, e_3,\, e_5$	$\mathfrak{g}_1$

Table A.16. Subalgebras of $\mathfrak{g}_{5,6}$.

Subalgebra	Realization
$\mathfrak{h}_0$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, \left(x_2 + \frac{x_4^2}{2}\right)\partial_1 + x_3\partial_2 + x_4\partial_3 + \partial_5$
$\mathfrak{h}_{1,2}$	$\partial_1,\, -x_4\partial_1,\, \partial_2,\, x_2\partial_1 + \partial_3,\, \frac{1}{2}\left(x_3^2 - 2x_2x_4\right)\partial_1 + x_3\partial_2 + \partial_4$
$\mathfrak{h}_{1,3}$	$\partial_1,\, \partial_2,\, \left(-x_3 - \frac{x_4^2}{2}\right)\partial_1 - x_4\partial_2,\, \partial_3,\, \left(x_2 - \frac{1}{2}x_3\left(x_3 + x_4^2\right)\right)\partial_1 - x_3x_4\partial_2 + \partial_4$
$\mathfrak{h}_{1,4}^0$	$\partial_1,\, \partial_2,\, \partial_3,\, \left(x_3 - \frac{x_4^3}{6}\right)\partial_1 - \frac{x_4^2}{2}\partial_2 - x_4\partial_3,\, x_2\partial_1 + x_3\partial_2 + \partial_4$
$\mathfrak{h}_{1,4}^{x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, \left(x_3 + x_5x_4 - \frac{x_4^3}{6}\right)\partial_1 + \left(x_5 - \frac{x_4^2}{2}\right)\partial_2 - x_4\partial_3,\, x_2\partial_1 + x_3\partial_2 + \partial_4$
$\mathfrak{h}_{1,5}^0$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, \left(x_2 + \frac{x_4^2}{2}\right)\partial_1 + x_3\partial_2 + x_4\partial_3$
$\mathfrak{h}_{1,5}^{x_5}$	$\partial_1,\, \partial_2,\, \partial_3,\, x_3\partial_1 + \partial_4,\, \left(x_2 + \frac{x_4^2}{2}\right)\partial_1 + x_3\partial_2 + x_4\partial_3 + x_5\partial_4$
$\mathfrak{h}_{2,5}$	$\partial_1,\, -x_3\partial_1,\, \frac{1}{2}\left(-2x_2 + x_3^2\right)\partial_1,\, \partial_2,\, \frac{1}{2}x_2\left(-x_2 + x_3^2\right)\partial_1 + \partial_3$
$\mathfrak{h}_{2,6}^0$	$\partial_1,\, -x_3\partial_1,\, \partial_2,\, \left(x_2 + \frac{1}{3}x_3^3\right)\partial_1 - x_3\partial_2,\, -x_2x_3\partial_1 + \partial_3$
$\mathfrak{h}_{2,6}^{x_4}$	$\partial_1,\, -x_3\partial_1,\, \partial_2,\, \left(x_2 + \frac{1}{6}x_3^2\left(-3x_4 + 2x_3\right)\right)\partial_1 + \left(x_4 - x_3\right)\partial_2,\, -x_2x_3\partial_1 + \partial_3$

Table A.17. Complete system of realizations of $\mathfrak{g}_{5,6}$.