Research project

Czech
Technical
University
in Prague

F4

Faculty of Nuclear Sciences and Physical Engineering
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Construction of realizations of Lie algebras

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Mathematical Physics

July 2016
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ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE
FAKULTA JADERNÁ A FYZIKÁLNĚ INŽENÝRSKÁ
PRAHA 1 - STARÉ MĚSTO, BŘEHOVÁ 7 - PSČ 115 19

Katedra: fyziky **Akademický rok:** 2015/2016

VÝZKUMNÝ ÚKOL

Posluchač: Bc. Daniel Gromada

Obor: Matematická fyzika

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Název úkolu (česky/anglicky):

Konstrukce realizací Lieových algeber / Construction of realizations of Lie algebras

Pokyny pro vypracování:

Z [1] nastudujte algebraickou metodu konstrukce neekvivalentních realizací dané Lieovy algebry založené na klasifikaci jejích podalgeber.

Z [2] nastudujte postup klasifikace Lieových podalgeber u vybrané třídy Lieovských algeber, jako jsou Poincarého, Galileovy nebo konformní Lieovy algebry.

Aplikujte postup z [2] na vybranou třídu, a poté zkonstruujte příslušné neekvivalentní realizace algoritmem z [1].

Součástí zadání výzkumného úkolu je jeho uložení na webové stránky katedry fyziky.

Literatura:

[1] Magazev A., Mikheyev V., Shirokov I. : Evaluation of composition function along with invariant vector fields on Lie group using structural constants of corresponding Lie algebra (Preprint arXiv:1312.0362), 2013

[2] Fushchych W., Barannyk L., Barannyk A. : Subgroup Analysis of the Galilei and Poincaré Groups and Reduction of Nonlinear Equations (Kiev: Naukova Dumka), in Russian, 1991

Datum zadání: 23.10.2015

Datum odevzdání: 24.06.2016

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Acknowledgement / Declaration

I would like to thank doc. Severin Pošta for offering me this topic, supervising my work and providing consultations. I am also grateful to dr. Maryna Nesterenko for reading draft of this work and offering me future collaboration.

I declare that I made this work by myself and used only the materials listed as references.

Abstrakt / Abstract

V práci jsou nejprve shrnuty důležité části teorie Lieových algeber a grup, poté se práce zabývá samotným problémem konstrukce realizací. Je zde popsán vztah klasifikace tranzitivních realizací a klasifikace podalgeber dané Lieovy algebry, dále je zde shrnuta metoda konstrukce lokálních tranzitivních realizací na základě [1]. Tato metoda je využita ke konstrukci realizací nerozložitelných Lieových algeber dimenze pět se čtyřrozměrným abelovským ideálem.

Klíčová slova: Lieovy algebry, realizace, podalgebry

Překlad titulu: Konstrukce realizací Lieových algeber

This work summarizes the important part of the theory of Lie algebras and groups and then it treats the problem of realizations construction itself. The relationship between classification of transitive realizations and classification of subalgebras of a given Lie algebra is described and the method for construction of local transitive realizations on the basis of [1] is summarized. This method is then used for construction of realizations of five-dimensional indecomposable Lie algebras with four-dimensional Abelian ideal.

Keywords: Lie algebras, realizations, subalgebras

/ Contents

Chapter 1
Introduction

This research project studies one of the most natural ways of representing the structure of an abstract Lie algebra—so called *realizations* by vector fields. The problem of classifying all realizations satisfying some property for a given Lie algebra was first considered by S. Lie himself. Since that time, few additional results were obtained by other authors. Nevertheless, all classification results were so far obtained only for low-dimensional Lie algebras or for small fixed number of variables. Currently, the most general result is probably the work of R. O. Popovych et al. [2].

This problem has variety of applications especially in group analysis of differential equations. For further references on applications of this theory, see e.g. [2–3].

Our aim is to use a method of construction of Lie algebra realizations proposed by I. V. Shirokov et al. in [1]. There is a special type of so called *transitive* realizations, whose classification is completely equivalent to subalgebra classification of the given Lie algebra. This well-known idea is described in [1] together with very simple formula for computing the explicit form of the realizations.

In our work, we tried to describe this method in clear, well arranged and more precise way. We have also decided to apply this method on Lie algebras of dimension five. Our interim results, summarized in Section A.2 of the appendix, contain classification of five-dimensional Lie algebras that have a four-dimensional commutative ideal. We plan to complete the classification of transitive realizations of all five-dimensional Lie algebras in the Master's thesis.

The structure of this paper is following. After this introduction, the most extensive chapter about Lie algebras and Lie groups follows. This chapter contains mostly the basic knowledge about those topics to recall the necessary definitions and theorems and clarify the notation. However, few subsections of Chapter 2 contain more recent findings closely related to the main topic (Subsections 2.3.1, 2.3.2, 2.6.2). In Chapter 3, the problem of realizations construction is discussed. As mentioned earlier, the explicit results are contained in the appendix. Finally, the last chapter is the conclusion, which contains not only summary of our work, but also several interesting questions that have arisen during the work and serve as a direction for further research.

Chapter 2
Lie groups and Lie algebras

In this chapter, necessary definitions and theorems about Lie groups, Lie algebras, and their connection is provided. The basic definitions and propositions of the group theory, Lie algebras, and differential geometry is presented briefly and without proofs. The relationship between Lie groups and Lie algebras and the topics closely connected to the main subject of this work are concerned more deeply. Most of the information were adopted from chapters 1 and 4 of [4]. Sections about Lie algebras are rather based on [5]. The information about Lie groups as a transformation groups and their action on manifolds come from [6].

2.1 Groups

In this section, we recall the very basic definitions concerning groups. Subsequently, we focus on the theory of group actions.

Definition 2.1. A *group* is an algebraic structure consisting of a set G and a binary operation on G that is associative, has a unity e, and an inverse g^{-1} for every element $g \in G$. The binary operation will be usually denoted as multiplication. A commutative group is called *Abelian*. A subset $H \subset G$ that forms a group with respect to the multiplication of G is called a *subgroup* of G and denoted $H \subset\subset G$.

Definition 2.2. A map $\varphi\colon G \to H$ between two groups preserving the group structure ($\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G$) is called a group *homomorphism*. A bijective homomorphism is called an *isomorphism* and an isomorphism of a group onto itself is called an *automorphism*. The set of all automorphisms of a given group G form a group with respect to composition and is denoted $\operatorname{Aut} G$.

Definition 2.3. Let G be a group and H its subgroup. Then we can define an equivalence relation on G as $g \sim \tilde{g}$ if there exists $h \in H$ such that $\tilde{g} = gh$. The equivalence classes $gH = \{gh \mid h \in H\}$ are called *left cosets*, the quotient set is denoted G/H. Analogically, for $g \sim \tilde{g} \Leftrightarrow \tilde{g} = hg$ we have classes Hg called *right cosets* and a quotient set $H \setminus G$. If left cosets coincide with right cosets, i.e. $gH = Hg$ for all $g \in G$, the subgroup H is called *normal*. The set of left or right cosets then possesses a group structure with respect to multiplication $g_1H \cdot g_2H = g_1g_2H$ and is called a *quotient group* of G by H.

Definition 2.4. The inverse image of unity by a homomorphism φ is called a *kernel* and denoted $\ker\varphi$. It can be easily shown that it is a normal subgroup in the domain of φ.

In the following text, we will use some important examples. The set of real numbers with respect to addition $(\mathbb{R}, +)$ form a group. For any set X, the set of all bijections of X (i.e. permutations if X is finite) form a group S_X called the *symmetric group* of X. For any vector space V, the set of all invertible linear transformations is a group called the *general linear group* of V and denoted $\mathrm{GL}(V)$.

Definition 2.5. Let G be a group. A *one-parameter subgroup* of G is a group homomorphism $\varphi\colon (\mathbb{R}, +) \to G$.

Definition 2.6. Let G be a group. An *action* of a group G on a set X is a map $\pi\colon G \times X \to X$ satisfying

$$\pi(e, x) = x \quad \text{and} \quad \pi(gh, x) = \pi(g, \pi(h, x)) \tag{1}$$

for all $g, h \in G$ and $x \in X$.

The action is often denoted as multiplication $\pi(g, x) = g \cdot x$. The defining relations then become $ex = x$ and $(gh)x = g(hx)$, so e has to be a (left) unity and the multiplication has to be associative. A set X with an action of a group G is called a *G-set*. Alternatively, one could consider a map $G \to S_X$, $g \mapsto \pi_g$, $\pi_g(x) = \pi(g, x) = gx$. The equations (1) are equivalent to the condition that this map is a homomorphism. Such homomorphism is sometimes called a *permutation representation*. The *kernel* of the action is the kernel of the corresponding permutation representation $\ker \pi = \{g \in G \mid \forall x \in X \; gx = x\}$. We say that the action is *effective* if the kernel contains only the unity.

An important example of a group action is an action on the group itself by left multiplication. This action is called *left translation* and denoted $L_g x = gx$ for $g, x \in G$. We also have *right translations* $R_g x = xg^{-1}$. The inversion is necessary to satisfy the associativity condition. Finally, we have the *conjugation action* $C_g = L_g \circ R_g$, $x \mapsto gxg^{-1}$. The left translation can be generalized for a coset space. Let H be a subgroup of G, then the left action of G on G/H is defined as $L_g(xH) = gxH$ for $g, x \in G$. Analogically, we can define the action of right multiplication of a group on the space of right cosets $H \setminus G$ as $R_g(Hx) = Hxg^{-1}$.

If we assumed that the group element acts "from the right side" instead of the left side, the associative law would look differently. A *right action* of a group G on a set X is a map $\pi\colon X \times G \to X$ satisfying for all $g, h \in G$ and $x \in X$

$$\pi(x, e) = x \quad \text{and} \quad \pi(x, gh) = \pi(\pi(x, g), h). \tag{2}$$

The action defined by the former definition is then called a *left action*. Nevertheless, in this chapter, we will use mostly left actions and call them just actions if there is no risk of misunderstanding.

Every assertion about left actions can be easily reformulated for the right actions since there is a one-to-one correspondence between left and right actions $g \cdot x = x \cdot g^{-1}$.

Lemma 2.7. The kernel of left multiplication of G on G/H is $\bigcap_{x \in G} xHx^{-1}$ and it is the largest normal subgroup contained in H.

Proof. Let us have $g \in \ker L$, so for every $x \in G$ we have $gxH = xH$, so choosing a representative $e \in H$ on the left hand side, there must exist an $h \in H$, such that $gx = xh$, so $g = xhx^{-1} \in xHx^{-1}$.

Conversely, let us take $y \in \bigcap_{x \in G} xHx^{-1}$, so for every $x \in G$ there is an $h \in H$ such that $y = xhx^{-1}$. Now we see that $yxH = xhx^{-1}xH = xH$, so $y \in \ker L$.

Finally, the subgroup is certainly normal because it is a kernel of a homomorphism. Take another normal subgroup K of G contained in H. Then $K = xKx^{-1} \subset xHx^{-1}$ for all $x \in G$, so $K \subset \bigcap_{x \in G} xHx^{-1}$. □

Definition 2.8. Let G be a group and X, Y G-sets. A map $\Phi\colon X \to Y$ is called *G-equivariant* or a *morphism* if

$$\Phi(gx) = g\Phi(x)$$

for all $x \in X$ and $g \in G$. If Φ is bijective, we call it *isomorphism*. An isomorphism of a G-set onto itself is called a *symmetry* of the action of G.

Definition 2.9. Let G and H be groups, X a G-set, and Y an H-set. A map $\Phi\colon X \to Y$ is called a *morphism* if there exists a homomorphism $\varphi\colon G \to H$, such that

$$\Phi(gx) = \varphi(g)\Phi(x).$$

If Φ is bijective, it is called a *similitude*. If a similitude between X and Y exists, then the sets are called *similar*.

Definition 2.10. Let G be a group and X a G-set. Then we can define an equivalence relation on X as $x \sim \tilde{x}$ if and only if there exists a $g \in G$ such that $\tilde{x} = gx$. The equivalence classes $Gx = \{gx \mid g \in G\}$ are called G*-orbits*. The quotient set of all orbits is denoted X/G. If there is only one orbit, so, for all $x, y \in X$, there exists $g \in G$ such that $y = gx$, we say that the action is *transitive* or that X is a *homogeneous space* of G. The map $\pi_x\colon G \to Gx\ g \mapsto gx$ is called the *orbit map*.

Definition 2.11. Let G be a group and X a G-set. For any $x \in X$ we denote $G_x = \{g \in G \mid gx = x\}$ the *stabilizer* of x. The action is called *free* if all stabilizers contain only unity. We say that a subset $Y \subset X$ is *stable* if it is invariant under the action of G. Equivalently, it means that Y is a union of stabilizers.

Lemma 2.12. Let G be a group, X an G-set, and $x \in X$. Then G_x is a subgroup of G and a map $\tilde{\pi}_x\colon G/G_x \to Gx\ gG_x \mapsto gx$ is an G-set isomorphism (for G acting on G/G_x by left multiplication).

Proof. By inspection. □

2.2 Lie algebras

Definition 2.13. Let V be a vector space over a field F and let $m\colon V \times V \to V$ be a bilinear operation. An ordered couple $A = (V, m)$ is called an *algebra*. If m is associative or commutative, then A is also called *associative* or *commutative*. If there exists $e \in V$ such that $m(e, x) = x = m(x, e)$ for all $x \in V$, then e is called a *unity* and A is called an algebra with unity.

Definition 2.14. An algebra $L = (V, [\cdot,\cdot])$ is called a *Lie algebra* if the corresponding operation is bilinear, antisymmetric and the so called *Jacobi identity*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \tag{3}$$

holds for all $x, y, z \in V$. The operation $[\cdot,\cdot]$ is called *Lie bracket*.

Lemma 2.15. Let $A = (V, \cdot)$ be an associative algebra. Then the operation of commutation defined for all $x, y \in V$ as

$$[x, y] = xy - yx \tag{4}$$

satisfies the axioms of a Lie bracket. Therefore, $L = (V, [\cdot,\cdot])$ forms a Lie algebra.

Proof. By inspection. □

The space of all linear operators acting on vector space V together with composition operation forms an associative algebra with unity. Therefore, linear operators with commutator form a Lie algebra. This Lie algebra is called the *general linear algebra* and denoted $\mathrm{gl}(V)$. This structure can be also expressed by means of matrices; Lie algebra of $n \times n$ matrices over a field F is denoted $\mathrm{gl}(n, F)$.

After bringing a definition of an algebraic structure it is natural to recall definition of substructure and morphisms.

Definition 2.16. A linear subspace $W \subset\subset V$ of a given algebra $A = (V, m)$ closed under the algebra operation also forms an algebra $B = (W, m)$ called a *subalgebra* and denoted $B \subset\subset A$. A linear map φ of two algebras $A = (V, m)$ and $\tilde{A} = (\tilde{V}, \tilde{m})$ preserving the algebraic structure, i.e. $\varphi(m(x, y)) = \tilde{m}(\varphi(x), \varphi(y))$ for all $x, y \in A$, is called an *algebraic homomorphism.* If it is also a bijection it is called an *isomorphism.* If $A = \tilde{A}$ it is called an *endomorphism.* If both conditions hold, it is called an *automorphism.* If there exists an isomorphism between algebras A and B, we say that A and B are *isomorphic* and denote $A \simeq B$.

For the sake of simplicity in notation, we will follow the common convention to denote identically any algebraic structure and its underlying set. However, it could cause an ambiguity of the symbols and notions defined above. If not stated otherwise, by $A \subset\subset B$ we will always mean that A is a subalgebra of B not only a subspace, and when talking about morphisms of A and B we will always mean morphism of algebras.

Definition 2.17. Let L be a Lie algebra, $x \in L$. A linear map $\mathrm{ad}_x \colon L \to L$ defined for all $y \in L$ as $\mathrm{ad}_x(y) = [x, y]$ is called an *adjoint map.*

Definition 2.18. Let A be an algebra. A linear map $D \colon A \to A$ is called *derivation* if it satisfies the Leibniz rule, i.e., for all $x, y \in A$,

$$D(m(x, y)) = m(Dx, y) + m(x, Dy). \tag{5}$$

Lemma 2.19. Linear space of all derivations of a given algebra A forms a subalgebra of $\mathrm{gl}(A)$

Proof. By inspection. □

The importance of such mappings will arise many times in this chapter. First observation is that the axiom of Jacobi identity is equivalent to requirement that all adjoint maps are derivations. In contrast, not all derivations correspond to an adjoint map. Those derivations that are adjoint maps are called *inner derivations* and form only a Lie subalgebra of the derivation algebra.

In linear algebra any regular operator P induces an automorphism of the algebra of linear operators through similarity transformation $A \mapsto PAP^{-1}$, which preserve not only the algebraic structure, but also important properties such as trace or spectrum. The same holds here. We can, for example, easily check that for a derivation D and an automorphism φ the map $\varphi D \varphi^{-1}$ is also a derivation.

One of the very important problems arising in the theory of (Lie) groups and (Lie) algebras is to find a way how to represent such structure by means of another concrete well-known mathematical structure. That is, to find a homomorphism to such structure. This homomorphism is then called a *representation.* A lot of information can be, however, lost by a homomorphism. It is, therefore, often desirable to ask only for *faithful* representations that are injective, so they do not lose any information.

In Section 2.1 we brought one example of a group representation—the permutation representation. In the theory of (Lie) groups and (Lie) algebras when talking about a representation we often mean another particular type of representation—a linear representation, that is a homomorphism to the group of regular linear operators $\mathrm{GL}(V)$ or to the algebra of linear operators $\mathrm{gl}(V)$, respectively. An example of such representation is the *adjoint representation* of a Lie algebra $\mathrm{ad} \colon L \to \mathrm{gl}(L)$, $x \mapsto \mathrm{ad}_x$.

In this work, we examine yet another type of representation— a *realization*, that is, a homomorphism to the Lie algebra of vector fields. A proper definition is provided in Chapter 3

2.3 Classification problems

In this section, we describe how a Lie algebra can be decomposed to several lower-dimensional algebras and, conversely, how bigger Lie algebras can be constructed of the smaller ones. This can be helpful if we look for the list of all non-isomorphic algebras or, given a particular Lie algebra, the list of its subalgebras. These particular problems are very important for our work and are discussed in separate subsections. Another classification problem is, of course, finding all representations or realizations, which is the main subject of this work and is discussed in Chapter 3.

Since two linear spaces are isomorphic if and only if they have the same dimension, it is clear that the structure of a given Lie algebra depend only on the dimension of underlying linear space and the Lie bracket. Because of bilinearity, the Lie bracket is determined by its "coordinates" in a chosen basis.

Definition 2.20. Let L be a finite-dimensional Lie algebra and $(e_1,\ldots,e_n)$ its (linear space) basis. Then the numbers C_{ij}^k such that

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k \tag{6}$$

are called the *structure constants* of L in the basis (e_i). If all structure constants are zero (equivalently, $[x,y]=0$ for all $x,y\in L$), the Lie algebra is called *Abelian*.

It is clear that the structure constants are antisymmetric in i,j, so it is sufficient to know them only for $i<j$.

In the following paragraphs, we bring the concept of direct sums and quotient spaces to the theory of Lie algebras. To make things clear, we recall these definitions for vector spaces at first.

Definition 2.21. Let V be a vector space and W its subspace. A *quotient space* V/W is a linear space of affine subspaces of V parallel to W.

$$V/W = \{x+W \mid x\in V\}, \quad (x+W)+(y+W) = (x+y)+W, \quad \alpha(x+W) = \alpha x + W.$$

Definition 2.22. Let L be a Lie algebra. A Lie subalgebra I satisfying $[I,L]\subset I$ (i.e. $[x,y]\in I$ for all $x\in I$ and $y\in L$) is called an *ideal* of L. The linear quotient space L/I with Lie bracket defined naturally as $[x+I,y+I]=[x,y]+I$ for all $x,y\in L$ form so called *quotient algebra*.

Theorem 2.23 (Isomorphism theorems)**.**

1. Let $\varphi\colon L_1\to L_2$ be a homomorphism. Then $\ker\varphi$ is an ideal of L_1 and
$$L_1/\ker\varphi \simeq \varphi(L_1).$$
2. Let I and J be ideals of a Lie algebra. Then $(I+J)/J \simeq I/(I\cap J)$.
3. Let I and J be ideals of a Lie algebra L such that $I\subset J$. Then J/I is an ideal of L/I and $(L/I)/(J/I)\simeq L/J$.

Proof. The definition of corresponding isomorphisms is straightforward:

$$\psi_1(x+\ker\varphi) = \varphi(x), \quad \psi_2(x+J) = x+I\cap J, \quad \psi_3(x+I+J/I) = x+J.$$

It is also clear that they are all homomorphisms and that they are surjective. One only needs to check that they are well-defined and injective, so it is really a one-to-one correspondence. □

Definition 2.24. Let W_1 and W_2 be subspaces of V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. We say that V is an *(internal) direct sum* of subspaces W_1 and W_2 and denote $V = W_1 \oplus W_2$.

Definition 2.25. Let W_1 and W_2 be vector spaces. A Cartesian product $W_1 \times W_2$ with coordinate-wise addition and scalar multiplication is called an *(external) direct sum* of spaces W_1 and W_2 and denoted $W_1 \oplus W_2$.

Remark 2.26. The first definition allows us to decompose a vector space to smaller parts, whereas the second does the opposite. The correspondence between these definitions is simple: an external direct sum $W_1 \oplus W_2$ can be decomposed to an internal direct sum $(W_1, 0) \oplus (0, W_2)$. Usually, it is clear from context which definition is used and we call it just a *direct sum.*

Now, it is time to generalize these definitions to Lie algebras.

Definition 2.27. Let L be a Lie algebra such that there are ideals $I, J \subset L$ satisfying $I + J = L$ and $I \cap J = \{0\}$. Then we say that L is an *(internal) direct sum* of ideals I and J and denote $L = I \oplus J$.

Definition 2.28. Let I and J be Lie algebras. An external direct sum of vector spaces I and J with Lie bracket defined for all $x_1, x_2 \in I$ and $y_1, y_2 \in J$ as

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2])$$

is called an *(external) direct sum* of Lie algebras I and J and is denoted $I \oplus J$.

It is evident that the Lie bracket is well-defined.

In contrast with vector spaces, there does not have to exist a nontrivial decomposition into direct sum of two ideals. Moreover, there does not have to exist any nontrivial ideal of a Lie algebra. Here we require two ideals that sum up to whole algebra. Particularly, it means that the algebra splits into two subspaces such that Lie bracket of elements belonging to different subspaces is zero. To remove the latter restriction, we present more general definition.

Definition 2.29. Let L be a Lie algebra, I its ideal, and S its subalgebra such that $I \cap S = \{0\}$ and $I + S = L$. Then we say that L is an *(internal) semidirect sum* of ideal I and subalgebra S and denote $L = I \inplus S$.

Definition 2.30. Let I and S be Lie algebras and D a derivation of S. An external direct sum of vector spaces I and S with Lie bracket defined for all $x_1, x_2 \in I$ and $s_1, s_2 \in S$ as

$$[(x_1, s_1), (x_2, s_2)] = ([x_1, x_2] + D(s_1)x_2 - D(s_2)x_1, [s_1, s_2])$$

is called an *(external) semidirect sum* of algebras I and S with respect to the derivation D and denoted $I \inplus_D S$.

Here, it is again evident that the defined Lie bracket is bilinear and antisymmetric. The Jacobi identity holds thanks to the derivation property of D.

Remark 2.31. According to the second Isomorphism theorem (2.23.2) is $I \oplus J/I \simeq J$. We can easily check that it also holds for a semidirect sum $I \inplus S/I \simeq S$. Particularly, S is an algebra of representatives of L/I classes. However, for a given Lie algebra L and its ideal I, there does not have to exist a subalgebra $S \subset\subset L$ such that $L = I \inplus S$. If it does exist, we say that L *splits* over I.

2.3.1 Classification of Lie algebras of a given dimension

The primary goal of this work is to classify realizations of five-dimensional Lie algebras. The problem of finding all Lie algebras of a given dimension was treated by G. M. Mubarakzyanov and solved for dimensions less or equal to five [7–8]. So far, we were able to bring the results for five-dimensional indecomposable Lie algebras with four-dimensional Abelian ideal. The relevant results of the Mubarakzyanov classification are summed up in Section A.1 of the appendix.

2.3.2 Classification of subalgebras of a given Lie algebra

This problem is considered, for example, in [9], where two methods of classification and a lot of examples and applications is provided. Let us begin with definition of the classification problem itself.

Definition 2.32. Let L be a Lie algebra and $G \subset\subset \operatorname{Aut} L$ be a group of automorphisms of L. Subalgebras S and S' are called *G-conjugate* if there is an automorphism $\varphi \in G$ such that $S' = \varphi(S)$.

The classification problem is to find all non-conjugated subalgebras with respect to a given group G. We are usually trying to find subalgebras with respect to all automorphisms, so called inner automorphisms or we are trying to find completely all subalgebras (i.e. $G = \{e\}$).

Universal classification methods were introduced in [10]. They are based on induction: using classification of lower-dimensional Lie algebras, we find the classification of a direct or semidirect sum. The authors also provided subalgebra classification for all Lie algebras of dimension not greater than four in [11].

We have decided to classify realizations of five-dimensional realization with respect to both weak and strong equivalence (see Chapter 3). In order to do that, we need to find all subalgebras and then classify them into equivalence classes with respect to the group of all automorphisms. Since we are not interested in classification with respect to inner automorphisms we cannot make use of the results [11] and proceed inductively. In order to obtain desired results, we can actually follow a much more primitive method, than those described in [10].

Finding the list of all subalgebras is very easy. We just need to take all subspaces of the given Lie algebra and eliminate those that are not closed under the Lie bracket. It is particularly easy in our case of Lie algebras with four-dimensional Abelian ideal $L = I \in\!\!\!\!+ S$, where I is the ideal spanned by e_1, e_2, e_3, e_4 and S spanned by e_5. The set of all subalgebras here consists of all subspaces J of the ideal I and subspaces of the form $\operatorname{span}(J \cup \{e_5 + ae_4 + be_3 + ce_2 + de_1\})$, where J is a subspace of I and a, b, c, d are coefficients such that $[J, e_5 + ae_4 + be_3 + ce_2 + de_1] \subset J$.

The classification of the $\operatorname{Aut} L$-conjugacy classes can be done simply by identifying those subalgebras that are $\operatorname{Aut} L$-conjugated in the list of all subalgebras. To do that we can use results presented in [12], where the groups of all automorphisms of Lie algebras of dimension less or equal to five are listed. To simplify the computation and, above all, to reduce amount of mistakes, we performed those computations using computer algebra program Mathematica.

The results obtained are listed in Section A.2 of the appendix.

2.4 The operator exponential

In this section, we recall the definition and basic properties of the exponential map. Then an application to the theory of Lie algebras is presented.

Definition 2.33. Let V be a finite-dimensional vector space, $L \in \operatorname{Lin} V$ a linear operator on V. Then the infinite sum

$$\exp(A) \equiv \mathrm{e}^A := \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{7}$$

is called the *exponential* of A. The map $\exp\colon \operatorname{Lin} V \to \operatorname{Lin} V$ $A \mapsto \exp(A)$ is called the *exponential map.*

It can be easily checked that the series is convergent for any linear operator A (generally for any bounded operator on an infinite-dimensional vector space). Since all norms on a finite-dimensional vector spaces are equivalent, the result does not depend on the norm chosen. Therefore, the operator exponential can be defined on an arbitrary finite-dimensional vector space, without need of defining a norm.

Now, let us summarize the basic properties of the exponential.

Lemma 2.34. Let V be a vector space and $A, B \in \operatorname{Lin} V$ linear operators on V. Then

1. if A and B commute, then $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B}$,
2. $\frac{\mathrm{d}\mathrm{e}^{tA}}{\mathrm{d}t} = A\mathrm{e}^{tA}$,
3. $B\mathrm{e}^A B^{-1} = \mathrm{e}^{BAB^{-1}}$,
4. $\det \mathrm{e}^A = \mathrm{e}^{\operatorname{Tr} A}$,
5. $\mathrm{e}^{A^t} = (\mathrm{e}^A)^t$.

From 1., it follows that $\mathrm{e}^A\mathrm{e}^{-A} = \mathrm{e}^{-A}\mathrm{e}^A = I$, so the exponential is invertible. The exponential map can be, therefore, treated as a map from the Lie algebra $\mathrm{gl}(V)$ of linear operators on V to the (Lie) group $\mathrm{GL}(V)$ of invertible linear operators on V. In the following sections, we bring a generalization of such map.

Definition 2.35. The smooth map $\mathbb{R} \to \mathrm{GL}(V)$ $t \mapsto \mathrm{e}^{tA}$ is called a *one-parameter subgroup of* $\mathrm{GL}(V)$ *generated by* A. The operator A is called the *infinitesimal generator* of the subgroup.

From the uniqueness of solution of linear differential equations, it follows that the exponential $t \mapsto \mathrm{e}^{tA}$ is determined by the differential equation 2. in the previous lemma with the initial condition $\mathrm{e}^{0A} = I$. Two corollaries follow from this fact.

Lemma 2.36. All one-parameter subgroups of $\mathrm{GL}(V)$, i.e. group homomorphisms $\varphi\colon (\mathbb{R}, +) \to \mathrm{GL}(V)$, are generated by an operator $A \in \operatorname{Lin} V$ in the form $\varphi(t) = \mathrm{e}^{tA}$.

Proof. Denote $A = \varphi'(0)$. From the one-parameter group properties we get the initial condition $\varphi(0) = I$ and differential equation

$$\varphi'(t) = \lim_{s\to 0} \frac{1}{s}(\varphi(t+s) - \varphi(t)) = \lim_{s\to 0} \frac{1}{t}(\varphi(s) - I)\varphi(t) = \varphi'(0)\varphi(t) = A\varphi(t).$$

□

Theorem 2.37. Let V be a vector space and $v_0 \in V$. The differential equation $v'(t) = Av(t)$ with initial condition $v(0) = v_0$ has a unique solution $v(t) = \mathrm{e}^{tA}v_0$.

In the theory of Lie algebras the exponential map is important because it provides us a connection between derivations and automorphisms.

Theorem 2.38. Let (A, m) be a finite-dimensional algebra and $D\colon A \to A$ a homomorphism. Then e^{tD} is a group of automorphisms if and only if D is a derivation.

Proof. The assertion that e^{tD} is an automorphism means that, for all $x, y \in A$,

$$\sum_{k=0}^{\infty} \frac{D^k}{k!} m(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t^{i+j} m\left(\frac{D^i}{i!}x, \frac{D^j}{j!}y\right).$$

Comparison of the terms linear in t gives us $D(m(x,y)) = m(Dx,y) + m(x,Dy)$.

Now, let D be a derivation. We already know that e^{tD} is a bijection, so we only have to prove that it is a homomorphism. Using the general Leibniz rule we get

$$\begin{aligned}\mathrm{e}^D m(x,y) &= \sum_{k=0}^{\infty} \frac{D^k}{k!} m(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{k}{l} m(D^l x, D^{k-l} y) = \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{1}{l!(k-l)!} m(D^l x, D^{k-l} y) = m\left(\sum_{i=0}^{\infty} \frac{D^i}{i!} D^i x, \sum_{j=0}^{\infty} \frac{D^j}{j!} y\right) = \\ &= m(\mathrm{e}^D x, \mathrm{e}^D y).\end{aligned}$$ □

Note that although we can find all one-parameter groups of automorphisms by exponentiating derivations, there may exist *discrete* automorphisms that does not belong to any one-parameter group and hence cannot be expressed in the form e^D.

In the case of Lie algebras, we have found a particular class of derivations called inner derivations represented by adjoint maps. Automorphisms of the form $\mathrm{e}^{\mathrm{ad}_a}$ corresponding to inner derivations are called *inner automorphisms*.

2.5 Notions of differential geometry

Definition 2.39. A *topological manifold* M of dimension n is a Hausdorff space with countable base of topology that is locally homeomorphic to $\mathbb{R}^n$. The corresponding homeomorphisms are called *coordinate charts* and the set of such homeomorphisms covering the manifold is called an *atlas*. On a set covered by two charts, a *transition map* is defined as a composition of one chart and the inversion of the other chart. The manifold is called *smooth* if all transition maps are smooth (i.e. all derivatives exist). A map of two manifolds $\Phi\colon M \to N$ is called *smooth* if its coordinate representation is smooth, i.e. for all coordinate charts φ of M and all coordinate charts ψ of N the composition $\psi\circ\Phi\circ\varphi^{-1} \in C^\infty(\mathbb{R})$. The algebra of all smooth functions $f\colon M \to \mathbb{R}$ with pointwise multiplication is denoted $C^\infty(M)$. A bijection Φ of smooth manifolds such that both Φ and Φ^{-1} are smooth is called a *diffeomorphism* of manifolds. If a diffeomorphism between two manifolds exists, the manifolds are called *diffeomorphic*, denoted $M \simeq N$. The set of all diffeomorphisms of M onto itself is denoted $\mathrm{Diff}\, M$.

Definition 2.40. Let M be a smooth manifold. A *tangent vector* at point $p \in M$ is a linear map $X\colon C^\infty(M) \to \mathbb{R}$ satisfying Leibniz rule and dependent only on the function values in a neighbourhood of p, i.e., for all $f,g \in C^\infty(M)$,

$$X(fg) = Xf\, g(p) + f(p)\, Xg,$$

$$f|_U = g|_U \quad \text{for } U \text{ a neighbourhood of } p \Rightarrow Xf = Xg.$$

Remark 2.41. An equivalent definition of a tangent vector is following. Let us define an equivalence of smooth curves (i.e. maps $\gamma\colon\mathbb{R}\to M$) satisfying $\gamma(0) = p$

$$\gamma \sim \tilde\gamma \Leftrightarrow (f\circ\gamma)'(0) = (f\circ\tilde\gamma)'(0).$$

Then each equivalence class represented by curve γ defines a linear map $X\colon C^\infty(M)\to\mathbb{R}$ as $Xf = (f\circ\gamma)'(0)$. We denote $X = \dot\gamma(0)$. Replacing the zero by an arbitrary $t \in \mathrm{Dom}(\gamma)$ we can define a tangent vector $\dot\gamma(t)$ at point $\gamma(t)$.

Any tangent vector can be represented in a given coordinate chart ψ as a linear combination of partial derivatives $Xf = \sum_{i=1}^n X^i \partial_i (f \circ \psi^{-1})(\psi(p))$. If the coordinates are denoted x^i, we write $X = \sum_{i=1}^n X^i \partial_{x^i}|_{x=\psi(p)} = \sum_{i=1}^n X^i \partial_{x^i}|_p$. Therefore, a *tangent space* T_pM of tangent vectors on a manifold M at point p is a linear space of dimension n equal to the dimension of the manifold. Using the definition of curve derivative, we can express the basis tangent vectors as $\partial_{x^i}|_p = \partial_i \psi^{-1}(\psi(p))$.

Definition 2.42. A disjoint union of tangent spaces at all points of a given manifold M is called a *tangent bundle* and denoted TM. A section of a tangent bundle, i.e. a map $X\colon M \to TM$, $p \mapsto X_p \in T_pM$, is called a *vector field*. A set of all vector fields on M is denoted $\operatorname{Vect} M$.

The coordinate representation is naturally the same as for tangent vectors, except for the coefficients being smoothly dependent on the position on the manifold: $X = \sum_{i=1}^n X^i(x^1, \dots, x^n)\partial_{x^i}$, where $\partial_{x_i} = \partial_i \psi^{-1}$. Thus, the tangent bundle can be, at least locally, treated not only as a infinite-dimensional vector space, but also as a finite-dimensional C^∞-module. Following lemma brings us an alternative definition of vector fields.

Lemma 2.43. Let M be a manifold. Let $X \in \operatorname{Vect} M$, then X, taken as a map $C^\infty(M) \to C^\infty(M)$ such that $(Xf)(p) = X_p f$ for all $p \in M$, is a derivation of $C^\infty(M)$ algebra. Conversely, every derivation of $C^\infty(M)$ defines a vector field on M.

Definition 2.44. Let M and N be smooth manifolds and $\Phi\colon M \to N$ a smooth map. A *derivative* or a *differential* or a *tangent map* of Φ at $p \in M$ is a linear map $\mathrm{d}\Phi_p\colon T_pM \to T_{\Phi(p)}N$ satisfying for all $X \in T_pM$ and $f \in C^\infty(N)$

$$(\mathrm{d}\Phi_p X)f = X(f \circ \Phi).$$

Naturally, we can define the derivative as a map $\mathrm{d}\Phi\colon TM \to TN$ as $\mathrm{d}\Phi X_p = \mathrm{d}\Phi_p X_p$. In our work, we will use mostly the latter, simpler, notation. If Φ is bijective (so it is a diffeomorphism actually), the derivative induces a map of vector fields $\Phi_*\colon \operatorname{Vect} M \to \operatorname{Vect} N$ called *pushforward* as

$$(\Phi_* X)_q = \mathrm{d}\Phi\, X_{\Phi^{-1}(q)}.$$

Remark 2.45. This definition of derivative corresponds to the definition of ordinary calculus. If we take coordinate functions φ on M and ψ on N and denote the coordinates x^i, $i = 1, \dots, m$ and y^j, $j = 1, \dots, n$ and denote coordinate expressions $X = \sum X^i \partial_{x^i}$, $\tilde{f} = f \circ \psi^{-1} \in C^\infty(\mathbb{R}^n)$, $\tilde{\Phi} = \psi \circ \Phi \circ \varphi^{-1}\colon \mathbb{R}^m \to \mathbb{R}^n$, we have

$$(\mathrm{d}\Phi\, X)f(\Phi(p)) = \sum_i X^i \partial_{x^i}(\tilde{f} \circ \tilde{\Phi})(x) = \sum_{i,j} X^i \frac{\partial \tilde{f}}{\partial y^j} \frac{\partial \tilde{\Phi}^j}{\partial x^i} = \left(\sum_{i,j} \frac{\partial \tilde{\Phi}^j}{\partial x^i} X^i \partial_{y^j} \right) f(p).$$

Thus, the matrix of the derivative is indeed the Jacobi matrix. We can, therefore, use the local theorems of ordinary calculus here too. For instance, if the map Φ is regular, i.e. the differential $\mathrm{d}\Phi$ as a linear map is regular, we have that Φ is a local diffeomorphism and so we can define the pushforward Φ_* at least locally.

Remark 2.46. Another consequence is that the chain rule can be used here as in ordinary calculus. This property can be expressed in a compact form $\mathrm{d}(\Phi_1 \circ \Phi_2) = \mathrm{d}\Phi_1\, \mathrm{d}\Phi_2$ for suitable maps Φ_1 and Φ_2. Particularly, if we take $\mathbb{R}$ for the first manifold, where $T_t\mathbb{R} = \{xD_t \mid x \in \mathbb{R}\}$, where D_t is the operator of differentiation at point t, we have for $\gamma\colon \mathbb{R} \to M$

$$(\mathrm{d}\gamma\, D_t)f = D_t(f \circ \gamma) = (f \circ \gamma)'(t) = \dot{\gamma}(t)f,$$

so this definition of diffeomorphism differentiantion corresponds to the former definition for curves: $\dot\gamma(t) = \mathrm{d}\gamma\, D_t$. Using the chain rule, we see that for $\Phi\colon M \to N$ and $\gamma\colon \mathbb{R} \to M$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(\gamma(t))) = \mathrm{d}(\Phi\circ\gamma)D_t = \mathrm{d}\Phi\,\mathrm{d}\gamma\, D_t = \mathrm{d}\Phi\,\dot\gamma(t).$$

Definition 2.47. Let M be a manifold. The dual space to the tangent space at point $p \in M$ is called the *cotangent space* at p and is denoted T^*_pM. Disjoint union of all cotangent spaces is called the *cotangent bundle* and denoted T^*M. Elements of T^*_pM are called *cotangent vectors* or *tangent covectors* at p and sections of T^*M are called *differential one-forms* on M.

If we take a point $p \in M$, coordinates x^i in the neighbourhood of p, we have a basis $\partial_{x^i}|_p$ of the tangent space T_pM and we can construct the dual basis for T^*_pM. The dual basis elements are, in fact, derivatives of the coordinate functions and that is why they are denoted $\mathrm{d}x^i|_p\colon T_pM \to \mathbb{R} \simeq T_{\psi^i(p)}\mathbb{R}$ or $\mathrm{d}x^i\colon TM \to \mathbb{R}$ for the corresponding one-forms.

Thus, any covector $\omega \in T^*_pM$ can be represented as $\omega = \sum_i \omega_i \mathrm{d}x^i|_p$. Now we can also represent any linear map $L\colon T_pM \to T_qN$ as $L = \sum_{i,j} L^j_i \partial_{y^j}|_q\, \mathrm{d}x^i|_p$, where x^i are the coordinates on M in the neighbourhood of $p \in M$ and y^j are the coordinates in the neighbourhood of $q \in N$. In particular, we have $\mathrm{d}\Phi_p = \sum_{i,j} \frac{\partial\Phi^j}{\partial x^i}\Big|_p \partial_{y^j}|_q\, \mathrm{d}x^i|_p$, where Φ^j is the coordinate expression of Φ.

Theorem 2.48. Let M and N be manifolds, $\Phi\colon M \to N$. Let $X_1, X_2 \in \operatorname{Vect} M$, $Y_1, Y_2 \in \operatorname{Vect} N$ be vector fields such that, for all $p \in M$,

$$(Y_1)_{\Phi(p)} = \mathrm{d}\Phi\,(X_1)_p, \quad (Y_2)_{\Phi(p)} = \mathrm{d}\Phi\,(X_2)_p.$$

Then, for all $p \in M$, we also have

$$[Y_1, Y_2]_{\Phi(p)} = \mathrm{d}\Phi\,[X_1, X_2]_p.$$

Proof. Let $f \in C^\infty(N)$. By assumption, we have

$$X_1(f\circ\Phi)(p) = \mathrm{d}\Phi\,(X_1)_p = (Y_1)_{\Phi(p)} = Y_1 f(\Phi(p)) = ((Y_1 f)\circ\Phi)(p),$$

so $X_1(f\circ\Phi) = Y_1 f\circ\Phi$. The same holds for X_2 and Y_2. Finally,

$$\begin{aligned}\mathrm{d}\Phi\,[X_1, X_2]_p f &= X_1(X_2(f\circ\Phi))(p) - X_2(X_1(f\circ\Phi))(p) = \\ &= X_1(Y_2 f\circ\Phi)(p) - X_2(Y_1 f\circ\Phi)(p) = \\ &= Y_1Y_2 f(\Phi(p)) - Y_2Y_1 f(\Phi(p)) = [Y_1, Y_2]_{\Phi(p)}.\end{aligned}$$

□

Corollary 2.49. Let $\Phi\colon M \to N$ be a diffeomorphism. Then Φ_* is an isomorphism of Lie algebras $\operatorname{Vect} M$ and $\operatorname{Vect} N$.

Definition 2.50. Let M be a smooth manifold. A smooth map $F\colon U \to M$, $U = U^\circ \subset \mathbb{R}\times M$ such that $F(0,p)$ is defined and equal to identity for all $p \in M$, $F(t,\cdot)$ is a diffeomorphism for all $t \in \mathbb{R}$, and $F(t+s,p) = F(t, F(s,p))$ for $t, s \in \mathbb{R}$ and $p \in M$ if at least one side is defined is called a *flow*. It can be understood as a group homomorphism $(\mathbb{R}, +) \to \operatorname{Diff}(M)$ $t \mapsto F_t$ and it is, therefore, also called a *one-parameter group of diffeomorphisms* on M.

According to the definition, a flow has not to be defined for all $t \in \mathbb{R}$. However, if there exist $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon)\times M$ lies in the domain of definition of the flow, then the domain can be extended to whole $\mathbb{R}\times M$ thanks to the group property as follows

$$F_t = F^N_{t/N}, \quad N > \frac{t}{\varepsilon}.$$

Definition 2.51. Let M be a smooth manifold and F a flow. The flow, for each point $p \in M$, defines a curve $F_p(t) = F(t,p)$. A vector field $X \in \operatorname{Vect} M$ satisfying

$$X = \dot{F}_p(0), \quad \text{i.e.} \quad Xf(p) = (f \circ F_p)'(0) \quad \text{for all } f \in C^\infty(M)$$

is called an *infinitesimal generator* of the flow F. Conversely, for a given vector field $X \in \operatorname{Vect} M$ there is at every point $p \in M$ uniquely defined *integral curve* F_p satisfying $\dot{F}_p(t) = X_{F_p(t)}$. The map $F(t,p) = F_p(t)$ is indeed a flow satisfying the equation above.

The uniqueness of the integral curve follows from the uniqueness of the solution of differential equations.

Definition 2.52. Let M and N be manifolds. Then any smooth map $\Phi\colon M \to N$ defines a homomorphism $\Phi^*\colon C^\infty(N) \to C^\infty(M)$

$$\Phi^*(f) = f \circ \Phi \quad \text{i.e.} \quad (\Phi^* f)(p) = f(\Phi(p))$$

for $f \in C^\infty(N)$ and $p \in M$ called *pullback*.

A pullback can be defined also for differential one-forms (generally for differential k-forms) or, if Φ is a diffeomorphism, for vector fields as well (generally for any tensor fields). Analogically, we could define a pushforward of any tensor fields by a diffeomorphism Φ. In our work, however, Φ_* will always mean a pushforward of vector fields and Φ^* a pullback of smooth functions.

If Φ is a diffeomorphism, then Φ^* is an algebra isomorphism (so if $\Phi \in \operatorname{Diff} M$, then $\Phi^* \in \operatorname{Aut} C^\infty(M)$). Consequently, for $X \in \operatorname{Vect} M$ as a derivation of $C^\infty(M)$ algebra $(\Phi^*)^{-1} \circ X \circ \Phi^*\colon C^\infty(M) \to C^\infty(M)$ is a derivation of $C^\infty(M)$ as well. Moreover, it is the pushforward of X.

Lemma 2.53. Let M be a smooth manifold, $\Phi \in \operatorname{Diff} M$, and $X \in \operatorname{Vect} M$.

1. $\Phi_* X = (\Phi^{-1})^* \circ X \circ \Phi^*$,
2. if F is the flow of the vector field X, then $\tilde{F}$ defined as $\tilde{F}(t,p) = \Phi(F(t,\Phi^{-1}(p)))$, i.e. $\tilde{F}_t = \Phi \circ F_t \circ \Phi^{-1}$, is the flow of the vector field $\Phi_* X$.

Proof. Take arbitrary $f \in C^\infty(M)$ and $p \in M$, then

$$\begin{aligned}(((\Phi^{-1})^* X \Phi^*) f)(p) &= ((\Phi^{-1})^*(X(\Phi^* f)))(p) = (X(f \circ \Phi))(\Phi^{-1}(p)) = \\ &= X_{\Phi^{-1}(p)}(f \circ \Phi) = \mathrm{d}\Phi\, X_{\Phi^{-1}(p)} f = (\Phi_* X)_p f = ((\Phi_* X) f)(p).\end{aligned}$$

As for the second proposition, $\tilde{F}$ is indeed a flow since

$$\tilde{F}_{t+s} = \Phi \circ F_{t+s} \circ \Phi^{-1} = \Phi \circ F_t \circ F_s \circ \Phi^{-1} = \Phi \circ F_t \circ \Phi^{-1} \circ \Phi \circ F_s \circ \Phi^{-1} = \tilde{F}_t \circ \tilde{F}_s.$$

Finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{F}(t,p)\bigg|_{t=0} = \mathrm{d}\Phi\ \frac{\mathrm{d}}{\mathrm{d}t} F(t,\Phi^{-1}(p))\bigg|_{t=0} = \mathrm{d}\Phi\, X_{\Phi^{-1}(p)} = (\Phi_* X)_p.$$

Theorem 2.54. Let M be a smooth manifold, $X, Y \in \operatorname{Vect} M$. Let F be the flow of the vector field X, denote $F(t,p) = F_t(p)$ and consider following parameter-dependent vector field

$$Y(t) = (F_t^{-1})_* Y.$$

Then $Y(t)$ satisfies following differential equation

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = [X, Y(t)].$$

Proof. For an arbitrary flow F generated by a vector field X, a function $f \in C^\infty(M)$, and a point $p \in M$, the map $t \mapsto f(F_t(p)) = (F_t^* f)(p)$ is an ordinary smooth function $\mathbb{R} \to \mathbb{R}$ and can be approximated by a Taylor polynomial

$$(F_t^* f)(p) = (f \circ F_t)(p) = \\ = (f \circ F_0)(p) + t\, \frac{\mathrm{d}}{\mathrm{d}t}(f \circ F_t)(p)\Big|_{t=0} + O(t^2) = f(p) + t(Xf)(p) + O(t^2).$$

Here, we have

$$\begin{aligned} Y(t)f &= (F_t^* \circ Y \circ (F_t^{-1})^*)f = F_t^* Y(f - tXf + O(t^2)) = \\ &= Yf + tXYf + O(t^2) - t(YXf + tXYXf + O(t^2)) + O(t^2) = \\ &= Yf + t[X,Y]f + O(t^2) \end{aligned}$$

Consequently, since $Y(t+s) = F_{t+s}^* \circ Y \circ (F_{t+s}^{-1})^* = F_s^* \circ Y(t) \circ (F_s^{-1})^* = (\Phi_s^{-1})_* Y(t)$, we have

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t}\Big|_{t=s} = (F_s^{-1})_* \frac{\mathrm{d}Y(t)}{\mathrm{d}t}\Big|_{t=0} = (F_s^{-1})_*[X,Y] = [(F_s^{-1})_* X, (F_s^{-1})_* Y] = [X, Y(s)].$$

□

The tangent vector $\mathrm{d}Y_p(t)/\mathrm{d}t|_{t=0} = [X,Y]_p$ from the previous theorem is called the *Lie derivative* of Y along X in p. The idea is that we want to differentiate a vector field Y as a function of t along an integral curve $F_p(t)$ of another vector field X. To calculate the derivative, it is necessary to *pull back* the tangent vectors $Y_{p'}$ in the neighbourhood of p into the point p. The pullback of a vector field is computed as inverse of the pushforward.

Corollary 2.55. The flows of given vector fields $X, Y \in \operatorname{Vect} M$ commute if and only if $[X,Y] = 0$.

Proof. The vector $Y(t)$ from the previous theorem generates the flow $F_s^{Y(t)} = (F_t^X)^{-1} \circ F_s^Y \circ F_t^X$. If $[X,Y] = 0$, then $Y(t)$ is constant and $F^{Y(t)} = F^Y$, hence the flows commute. Conversely, if the flows commute, $Y(t)$ has to be constant. □

Lemma 2.56. Let X, Y be vector fields on manifold M and F^X and F^Y their flows. If $[X,Y] = 0$, then for any $\alpha, \beta \in \mathbb{R}$ the linear combination $\alpha X + \beta Y$ generates the flow $F_t = F_{at}^X \circ F_{bt}^Y$.

Proof. Since commuting vector fields have commuting flows, we see that F is indeed a flow as well. Finally, we show that $\alpha X + \beta Y$ is its infinitesimal generator, i.e. $\alpha X_p + \beta Y_p$ is the tangent vector of $F_t(p)$ for $t = 0$, by means of Taylor approximation.

$$(f \circ F_{at}^X \circ F_{bt}^Y)(p) = (F_{bt}^Y)^{-1}(f(p) + btYf(p) + O(t^2)) = f(p) + atXf(p) + btYf(p) + O(t^2).$$

□

2.5.1 Distributions and Frobenius theorem

An integral curve of a given vector field can be interpreted as a one-dimensional submanifold generated by the vector field. This subsection describes the same proces for general dimension.

Definition 2.57. Let M and N be manifolds. A smooth map $\Phi\colon M \to N$ such that it is injective and $\mathrm{d}\Phi_p$ is also injective for every $p \in M$ is called an *immersion*. The manifold M is called an *immersed submanifold*.

Definition 2.58. Let M be a manifold. A function P assigning an n-dimensional subspace $P_p \subset\subset T_pM$ to every $p \in M$ is called an *n-dimensional distribution* on M. The distribution is called *smooth* if every point $p_0 \in M$ has a neighbourhood $U \subset M$ such that there exist n linearly independent vector fields $X_1, \ldots, X_n \in \operatorname{Vect} U$ that form a basis $((X_1)_p, \ldots, (X_n)_p)$ of P_p at every point $p \in U$. These vector fields are called a *local basis* of P at p_0.

Definition 2.59. Let M be a manifold and P an n-dimensional distribution on M. An n-dimensional manifold N immersed to M by $\Phi\colon N \to M$ such that $\mathrm{d}\Phi(T_qN) = P_{\Phi(q)}$ for every $q \in N$ is called an *integral manifold* of P.

So, if an integral manifold exists, then its tangent spaces are essentially the subspaces P_p. If an integral manifold exists for every point in M, then these submanifolds form so called *foliation* of M

Definition 2.60. A smooth distribution is called *involutive* if, for every point $p \in M$ and a local basis $(X_1, \ldots, X_n) \in \operatorname{Vect} U$ in p, there are functions $f^k_{ij} \in C^\infty(M)$ such that $[X_i, X_j] = \sum_k f^k_{ij} X_k$.

It can be easily checked that the property of being involutive is independent on the chosen basis.

Theorem 2.61 (Frobenius)**.** Let M be a manifold, P an n-dimensional involutive distribution on M and $p \in M$. Then there exist a system of coordinates $(x^1, \ldots, x^m)$ in a neighbourhood of p such that the distribution has a local basis $(\partial_{x^1}, \ldots, \partial_{x^n})$ at p. So, the manifold M is locally foliated by so called *level submanifolds* $x^i = \text{const.}$ for $i = n+1, \ldots, m$.

Proof. At first, we show that there exist a basis $(X_1, \ldots, X_n)$ of P such that $[X_i, X_j] = 0$. Let $(y^1, \ldots, y^m)$ be coordinates on M such that p lies in the origin and let $Y_i = \sum_{j=1}^m Y_i^j(y)\partial_{y^j}$ be a basis of P. Since Y_i are linearly independent, the matrix $Y_i^j(y)$ has to have n linearly independent rows. Without loss of generality, let us assume that these are the first n rows. Then we can choose a new basis of the form X_i such that $X_i^j = \delta_i^j$ for $j \leq n$, so $X_i = \partial_{y^i} + \sum_{j=n+1}^m X_i^j(y)\partial_{y^j}$. Now, we easily see that, in the coordinate basis ∂_{y^j}, we have $0 = [X_i, X_k]^j = f^l_{ik} X^j_l = f^j_{ik}$ for $j \leq n$.

Denote $\psi\colon U \to \mathbb{R}^m$ the coordinate chart corresponding to coordinates y^j. Now, we define the coordinates x^j. Let $F_i(t)\colon U \to M$ be the flows of the vector fields X_i. Define the coordinate chart $\varphi\colon U \to \mathbb{R}^m$ as $\varphi^{-1}(x^1, \ldots, x^m) = F_i(x^1) \cdots F_i(x^m)\psi^{-1}(0, \ldots, 0, x_{n+1}, \ldots, x_m)$. The derivative of $\varphi \circ \psi^{-1}$ at zero is the identity, so it is a local diffeomorphism in a neighbourhood of zero, so φ is a well defined coordinate chart in the neighbourhood of p. Since X_i commute, the flows $\Phi_i(t)$ commute as well and using this property we can easily check that $\Phi_i(t)$ act on U as a translation in the coordinates x^i. Hence, $X_i = \partial_{x^i}$. □

It is clear that, at least locally, the integral manifold is defined uniquely.

2.6 Lie groups

Definition 2.62. A group G is called a *Lie group* if it is also a smooth manifold and both multiplication and inversion are smooth maps.

Lemma 2.63. Connected Lie group G is generated by any open set.

Proof. Let H be generated by an open set U in G. H is a union of open sets hU, so H is also open. Then $G \setminus H$ is a union of left cosets gH, $g \notin H$, which are also open. Therefore, H is also closed and hence, if G is connected, $H = G$.

Definition 2.64. A map φ of two Lie groups is called *homomorphism* if it preserves both group and manifold structures, i.e. it is a smooth group homomorphism. A bijective homomorphism whose inversion is also a homomorphism (which is not trivially satisfied since inversion of a smooth map has not to be smooth) is called a Lie group *isomorphism*. Similarly, Lie group automorphism and other morphisms can be defined.

Definition 2.65. A *one-parameter subgroup* of a Lie group G is a (smooth) homomorphism $\varphi\colon (\mathbb{R},+) \to G$.

2.6.1 The Lie algebra of a Lie group

Since every topological group is homogeneous (through the left translation), it is sufficient to examine a neighbourhood of unity to examine its local properties. An important role here is played by the tangent space at unity. This space can be given a structure of a Lie algebra through so called left-invariant vector fields.

Lemma 2.66. Let G be a Lie group. If a transformation $T\colon G \to G$ commutes with every left translation, then $T = R_{T(e)^{-1}}$.
Proof. We have $T(g) = (T \circ L_g)(e) = (L_g \circ T)(e) = gT(e) = R_{T(e)^{-1}}(g)$. □

Definition 2.67. Let G be a Lie group. A vector field $X \in \operatorname{Vect} G$ is called *left-invariant* if $L_{g*}X = X$ for all $g \in G$. Analogically, X is *right-invariant* if $R_{g*}X = X$.

Lemma 2.68. Let G be a Lie group.

1. Left invariant vector fields form a Lie subalgebra of $\operatorname{Vect} G$.
2. Any left-invariant vector field $X \in \operatorname{Vect} G$ is determined by its value at unity X_e. We can compute $X_g = \mathrm{d}L_g\, X_e$ for any $g \in G$.
3. For any tangent vector $a \in T_e(G)$, there is a unique one-parameter subgroup φ_a of G such that $\dot\varphi_a(0) = a$. The corresponding left-invariant vector field $X_g = \mathrm{d}L_g\, a$ is the infinitesimal generator of the one-parameter group F_a, $F_a(t,g) = g\varphi_a(t)$ of right translations by $\varphi_a(t)$.

Proof. The first proposition is clear from the homomorphism property of pushforward L_{g*}. The second proposition follows directly from the definition of left-invariance

$$X_g = (L_{g*}X)_g = \mathrm{d}L_g\, X_{L_g^{-1}(g)} = \mathrm{d}L_g\, X_e$$

Finally, let F be the flow of the left-invariant vector field X corresponding to a tangent vector a. The subgroup φ_a has to be an integral curve of X, i.e. $\varphi(t) = F(t,e)$, since

$$\dot\varphi_a(t) = \frac{\mathrm{d}}{\mathrm{d}s}(\varphi_a(t)\varphi_a(s))\bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}(L_{\varphi_a(t)}\varphi_a(s))\bigg|_{s=0} = \\ = \mathrm{d}L_{\varphi_a(t)}\, \dot\varphi_a(0) = \mathrm{d}L_{\varphi_a(t)}\, a = X_{\varphi_a(t)}.$$

Hence, it is uniquely defined and obeys the group property. From Lemma 2.53 we see that F commutes with left translation. Therefore, according to Lemma 2.66, F has to be right translation $F(t,g) = R^{-1}_{F(t,e)}g = g\varphi(t) = \Phi_X(t,g)$. □

Definition 2.69. The one-parameter group φ_a from the previous lemma is called the *exponential* and denoted $\varphi_a(t) = \exp(ta) = \mathrm{e}^{ta}$. The map $\exp\colon T_eG \to G$ $a \mapsto \exp(a)$ is called the *exponential map*.

Remark 2.70. Since $\exp(0) = e$ and the differential of exp at 0 is the identity, the exponential map is a local diffeomorphism, mapping a neighbourhood of zero vector onto a neighbourhood of unity. Therefore, according to Lemma 2.63, if G is connected, it has to be generated by $\exp(T_eG)$.

Definition 2.71. The Lie algebra of left-invariant vector fields on G is called the *Lie algebra of* G and denoted $\operatorname{Lie} G$ or $\mathfrak{g}$.

This definition is correct thanks to Lemma 2.68. Moreover, this lemma brings us important isomorphism between the Lie algebra $\mathfrak{g}$ of G and the tangent space T_eG. Through this isomorphism, we can endow T_eG with the Lie algebra structure as $[a, b] = [X, Y]_e$ for $a, b \in T_eG$ and X, Y the corresponding left-invariant vector fields.

Remark 2.72. Instead of the left-invariant vector fields, we could have defined the Lie algebra of G by means of the right-invariant vector fields. We have, for example, $\tilde{X}_g = \mathrm{d}R_g^{-1}\,\tilde{X}_e$ for any right-invariant vector field $\tilde{X}$. We can consider the inversion diffeomorphism $i\colon g \mapsto g^{-1}$, which provides us the relationship between left and right-invariant vector fields. For any $a \in \mathfrak{g}$, we have

$$\mathrm{d}i\, a = \frac{\mathrm{d}}{\mathrm{d}t} i\mathrm{e}^{ta}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-at}\bigg|_{t=0} = -a$$

Since $R_g = iL_g i^{-1}$, we have

$$\tilde{X}_g = \mathrm{d}R_g^{-1}\,a = \mathrm{d}i\,\mathrm{d}L_g\,\mathrm{d}i\,a = -\mathrm{d}i\,X_g,$$

so the $\operatorname{Vect} G$ automorphism i_* is actually, in addition, an isomorphism of left-invariant vector fields onto the right-invariant. Finally, for tangent vectors a, b and $[a, b]$ of $\mathfrak{g}$, corresponding left-invariant vector fields $X, Y, [X, Y]$ and right-invariant vector fields $\tilde{X}, \tilde{Y}, \widetilde{[X, Y]}$ we have

$$[\tilde{X}, \tilde{Y}]_e = [-\mathrm{d}i\,a, -\mathrm{d}ib] = \mathrm{d}i[a, b] = -\widetilde{[X, Y]}_e.$$

Definition 2.73. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. For every $g \in G$, we define the *adjoint map* $\mathrm{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}$ as

$$\mathrm{Ad}_g = R_{g*}|_{\mathfrak{g}} = C_{g*}|_{\mathfrak{g}}.$$

The second equation holds thanks to the left-invariance and commutativity of left and right translations: $L_{g*}R_{g*}X = R_{g*}L_{g*}X = R_{g*}X$ for $X \in \mathfrak{g}$. The same equation also proves that the image is left-invariant as well.

Through the isomorphism mentioned above, we can define the adjoint map for tangent vectors $\mathrm{Ad}_g\colon T_eG \to T_eG$ as $\mathrm{Ad}_g\,a = (\mathrm{Ad}_g\,X)_e$, where X is the left-invariant vector field corresponding to a. From now on, we will identify the Lie algebra $\mathfrak{g}$ of a Lie group G with the space of tangent vectors T_eG and denote its elements by small letters $a, b, \ldots$ In proofs of some of the following theorems, it will be, however, necessary to return back to the left-invariant vector fields.

Lemma 2.74. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then, for all $g \in G$ and $a \in \mathfrak{g}$, we have

$$\mathrm{Ad}_g\,a = \mathrm{d}C_g\,a, \tag{8}$$

$$g\mathrm{e}^a g^{-1} = \mathrm{e}^{\mathrm{Ad}_g\,a}. \tag{9}$$

Proof. Let X be the corresponding left-invariant vector field. Then

$$\mathrm{Ad}_g\,a = (\mathrm{Ad}_g\,X)_e = (C_{g*}X)_e = \mathrm{d}C_g\,X_{C_g(e)} = \mathrm{d}C_g\,X_e = \mathrm{d}C_g\,a.$$

According to Lemma 2.68 $e^a = F_1(e)$, where F_t is the flow of the vector field X. According to Lemma 2.53 the flow of $\mathrm{Ad}_g\,X = C_{g*}X$ is $C_g \circ F_t \circ C_g^{-1}$, so

$$\mathrm{e}^{\mathrm{Ad}_g\,a} = (C_g \circ F_1 \circ C_g^{-1})(e) = C_g(F_1(e)) = g\mathrm{e}^a g^{-1}.$$

□

The preceding lemma gave us the connection between adjoint maps and inner automorphisms of the Lie group. Exponentiation of this relation gives us the connection between adjoint maps and inner automorphism of the Lie algebra.

Theorem 2.75. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, $a \in \mathfrak{g}$. The linear map $\mathrm{ad}_a\colon \mathfrak{g} \to \mathfrak{g}$ is the infinitesimal generator of the one-parameter group $\mathrm{Ad}_{\mathrm{e}^{ta}}\colon \mathfrak{g} \to \mathfrak{g}$. In other words, $\mathrm{Ad}_{\mathrm{e}^{ta}}$ is a one-parameter group of inner automorphisms. Thus, the following equations hold

$$\mathrm{Ad}_{\mathrm{e}^a} = \mathrm{e}^{\mathrm{ad}_a}, \tag{10}$$

$$\mathrm{e}^a \mathrm{e}^b \mathrm{e}^{-a} = \exp(\mathrm{Ad}_{\mathrm{e}^a} b) = \exp(\mathrm{e}^{\mathrm{ad}_a} b). \tag{11}$$

Proof. Let us take $a, b \in \mathfrak{g}$ and X, Y the corresponding left-invariant vector fields. Since, according to Lemma 2.68, the flow F corresponding to the left-invariant vector field X acts as right translation $F_t = R^{-1}_{\mathrm{e}^{ta}}$, we can define a parameter-dependent vector field $Y(t)$ as in Theorem 2.54 $Y(t) = R_{\mathrm{e}^{ta}*} Y = \mathrm{Ad}_{\mathrm{e}^{ta}} Y$ satisfying

$$\mathrm{ad}_a b(t) = [a, b(t)] = [X, Y(t)]_e = \frac{\mathrm{d}Y_e(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}_{\mathrm{e}^{ta}} b),$$

so according to Theorem 2.37 we have $\mathrm{Ad}_{\mathrm{e}^{ta}} b = \mathrm{e}^{t\,\mathrm{ad}_a} b$.

The second equation follows from equation (9) in the previous lemma. □

Remark 2.76. As we remarked in 2.45, the definition of differentiation in case of linear spaces corresponds to the ordinary differentiantion. The Lie algebra of a linear Lie group as a tangent space at unity is, therefore, formed by all possible derivatives of one-parameter operators. In the case of general Lie group $\mathrm{GL}(V)$ the Lie algebra consists of all linear operators on V. Indeed, a derivative of a parametrized operator is of course an operator; conversely, any linear operator L is a derivative of a curve $\gamma(t) = I + tL$ in $\mathrm{GL}(V)$. The exponential defined in the theory of Lie groups corresponds to the operator exponential since it satisfies the same differential equation $\frac{\mathrm{d}\mathrm{e}^{ta}}{\mathrm{d}t} = a\mathrm{e}^{ta}$. Finally, we can compute the Lie bracket for $a, b \in \mathrm{Lie}(\mathrm{GL}(V))$

$$\begin{aligned}[a, b] = \mathrm{ad}_a b = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{\mathrm{e}^{ta}} b\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}C_{\mathrm{e}^{ta}} b\bigg|_{t=0} = \\ = \frac{\partial^2}{\partial t \partial s} \mathrm{e}^{ta} \mathrm{e}^{sb} \mathrm{e}^{-ta}\bigg|_{t,s=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{ta} b e^{-ta}\bigg|_{t=0} = ab - ba.\end{aligned}$$

As a consequence, the Lie algebra of $\mathrm{GL}(V)$ is $\mathrm{gl}(V)$.

The general linear Lie group and Lie algebra are particularly important thanks to the following theorem. We do not bring its proof because it would require to build a lot of extra theory we do not need.

Theorem 2.77 (Ado)**.** Every finite-dimensional Lie algebra $\mathfrak{g}$ can be embedded in matrices. That is, for every finite-dimensional Lie algebra $\mathfrak{g}$ over a field F of characteristic zero, there exists $n \in \mathbb{N}$ and a monomorphism (injective homomorphism) $\varphi\colon \mathfrak{g} \to \mathrm{gl}(n, F)$.

Definition 2.78. A Lie subgroup of the general linear group is called a *linear group*.

2.6.2 Lie group coordinates

Now, we proceed to examination of the local properties of Lie groups. As we remarked in 2.70, exp is a local diffeomorphism between Lie algebra and Lie group. We can therefore locally define the inversion of exponential map—the logarithm $\ln\colon G \to \mathfrak{g}$. If we fix a basis of $\mathfrak{g}$, say $(e_1, \ldots, e_n)$, it defines us coordinates on the Lie algebra $\mathfrak{g}$, which induce *logarithmic coordinates* on G through relation $x^i = e^i \ln g$, where $(e^1, \ldots, e^n)$ is the dual basis. The element g_x with coordinates $x = (x^i)$ can be, therefore, written as

$$g_x = \exp\left(\sum_{i=1}^{n} x^i e_i\right). \tag{12}$$

One could, however, also exponentiate multiples of the basis elements at first and then take their product. Generally, consider a linear decomposition of $\mathfrak{g}$ to a direct sum of linear subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_m$. Then the map $\Phi\colon \mathfrak{g} \to G$, $a \mapsto \prod_{k=1}^{m} \exp(a_k)$, where a_k is a projection of a onto $\mathfrak{g}_k$, is a local diffeomorphism of the neighbourhood of zero onto the neighbourhood of unity too since $\Phi(0) = e$ and

$$\mathrm{d}\Phi_0\, e_i = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(te_i)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\exp(te_i)\bigg|_{t=0} = e_i,$$

where e_i is a basis such that each basis element belongs to one of the subspaces $\mathfrak{g}_k$, so the derivative at zero is again identity. Such map Φ is called *canonical* and the induced coordinates with coordinate functions $e^i \circ \Phi^{-1}$, where e^i is the dual basis to e_i, are called *canonical coordinates*.

For a trivial decomposition $\mathfrak{g} = \mathfrak{g}_1$ we get the logarithmic coordinates, which are also called *first canonical coordinates*. On the contrary, if we have $m = n$, so each of the subspaces is one-dimensional, we get the *second canonical coordinates*

$$g_x = \prod_{i=1}^{n} \exp(x^i e_i). \tag{13}$$

Let us now look on the coordinate expression of the basic Lie group structures. Let $\psi^1, \ldots, \psi^n$ be arbitrary local coordinate functions of G in the neighbourhood of unity. Denote $g_x = \psi^{-1}(x)$ as in the preceding text. Then we can define a coordinate expression of the multiplication $m(g,h) = gh$ as $M\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $M^i(x,y) = \psi^i(g_x, g_y)$. This map is called *composition function*.

Now, the basis vectors of $\operatorname{Vect} M$ module can be written as $\partial_{x^i} = \partial_i \psi^{-1} = \partial_{x^i} g_x$.

We can also explicitly express the form of left-invariant vector fields on G. Let $e_1, \ldots, e_n$ be a basis of the Lie algebra $\mathfrak{g}$ of G (as an algebra of tangent vectors). Then the corresponding basis of the algebra of left-invariant vector fields has the form $(X_i)_g = \mathrm{d}L_g\, e_i$. So, the coordinate expression is

$$X_i^a(x) = \frac{\partial \psi^a(L_{g_x}(g_y))}{\partial y^j}\bigg|_{y=0} e_i^j = \frac{\partial M^a(x,y)}{\partial y^j}\bigg|_{y=0} \delta_i^j = \frac{\partial M^a(x,y)}{\partial y^i}\bigg|_{y=0}. \tag{14}$$

In the rest of this subsection, we describe computation of left-invariant vector fields in the second canonical coordinates as was proposed by I. V. Shirokov in [13].

At first, let us take a look on the differential $\mathrm{d}L_g$ and compute its components at unity

$$\begin{aligned}\left[(\mathrm{d}L_{g_y})_e\right]^i_j &= \mathrm{d}x^i\, \mathrm{d}L_{g_y}\, \partial_{x^j}|_{x=0} = \mathrm{d}x^i\, \mathrm{d}L_{g_y}\, \partial_{x^i} g_x|_{x=0} = \\ &= \frac{\partial}{\partial x^j}\left(\psi^i(L_{g_y}(g_x))\right)\bigg|_{x=0} = \frac{\partial M^i(y,x)}{\partial x^j}\bigg|_{x=0} = X^i_j(y).\end{aligned} \tag{15}$$

In the following calculations, we omit the index e at $\mathrm{d}L_{g_y}$. So, the differential $\mathrm{d}L_{g_y}$ is a matrix, whose columns is formed by the left-invariant vector fields. This relation can be inverted

$$[\mathrm{d}L_{g_y^{-1}}]_i^j = [\mathrm{d}L_{g_y}^{-1}]_i^j = \omega_i^j(y), \tag{16}$$

where ω^j are the dual one-forms corresponding to left-invariant vector fields X_j. These are also called *left-invariant one-forms.*

Now, choose the second canonical coordinates such that

$$g_x = g_n(x^n)\cdots g_1(x^1) \quad \text{where} \quad g_k(t) = \mathrm{e}^{te_k}. \tag{17}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} g_k(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{te_k} = (X_k)_{\mathrm{e}^{te_k}} = \mathrm{d}L_{g_k(t)}\, e_k,$$

so

$$\begin{aligned}\partial_{x^k} = \partial_{x^k} g_x = \partial_{x_k}\left(L_{g_n(x^n)\cdots g_{k+1}(x^{k+1})} R_{g_1(x^1)^{-1}\cdots g_{k-1}(x^{k-1})^{-1}} g_k(x^k)\right) = \\ = \mathrm{d}L_{g_n(x^n)\cdots g_{k+1}(x^{k+1})}\, \mathrm{d}R_{g_1(x^1)^{-1}\cdots g_{k-1}(x^{k-1})^{-1}}\, \mathrm{d}L_{g_k(x^k)}\, e_k.\end{aligned} \tag{18}$$

Finally, since left translations commute with right translations, we can formulate the expression for the $\mathrm{d}L_{g_x}^{-1}$ in terms of the adjoint map using equation (8) as

$$\begin{aligned}[\mathrm{d}L_{g_x}^{-1}]_i^j = \mathrm{d}x^j\, \mathrm{d}L_{g_x^{-1}} \partial_{x^i} = \mathrm{d}x^j\ \mathrm{Ad}_{g_1(x^1)^{-1}} \cdots \mathrm{Ad}_{g_{i-1}(x^{i-1})^{-1}}\, e_i = \\ = \left[\exp(-x^1\,\mathrm{ad}_{e_1})\cdots\exp(-x^{i-1}\,\mathrm{ad}_{e_{i-1}})\right]_i^j.\end{aligned} \tag{19}$$

So, in order to calculate the coordinates of left-invariant vector fields, it is sufficient to calculate the inversion of the matrix whose elements is given by the formula (19).

2.6.3 Local Lie groups

In this work, we inspect Lie algebras, not Lie groups and our aim is to find their realizations by local vector fields. The local properties of Lie groups are, therefore, the main properties of our interest, so we can proceed in a coordinate way as in the previous subsection. Hence, it is convenient to bring formal definition of a local Lie group.

Definition 2.79. Let M be a manifold, U an open domain in M, e a point in U, V a neighbourhood of e and $m\colon V\times V \to U$ a smooth map called and denoted as *multiplication* satisfying $ex = xe = x$ for all $x \in V$, $(xy)z = x(yz)$ for all $x, y, z, xy, yz \in V$ and that the local *inversion* $i\colon W \to W$ $x \mapsto x^{-1}$ defined by the relation $xx^{-1} = x^{-1}x = e$ for all $x \in W$, $W \subset V$ neighbourhood of e is also a smooth map. Then the tuple (U, V, e, m) is called a *local Lie group.* Any local Lie group (U_1, V_1, e, m_1), such that $U_1 \subset U$, $V_1 \subset V$ and $m_1 = m|_{V_1}$ is called a *restriction* of the original local Lie group.

There can be defined an equivalence of the local Lie groups: two Lie groups are equivalent if they have a common restriction. Since the size of the neighbourhood of a local Lie group is irrelevant, we often identify the equivalent local Lie groups and by the term *local Lie group* we mean the equivalence class. In this sense, we also define the notion of local Lie group homomorphism.

Definition 2.80. Let G_1 and G_2 be local Lie groups, (U_1, V_1, m_1) and (U_2, V_2, m_2) their restrictions. A smooth map $\Phi\colon U_1 \to U_2$ satisfying $\Phi(V_1) \subset V_2$, $\Phi(m_1(x, i_1(y))) = m_2(\Phi(x), i_2(\Phi(y)))$ for all $x \in V_1$, $y \in W_1$, where W_1 is the domain of the first inversion i_1, is called a *homomorphism* of local Lie groups G_1 and G_2. Analogically, one could define local Lie group isomorphism or automorphism. Local Lie groups are called isomorphic if there exists a local Lie group isomorphism between them. A *local homomorphism* of Lie groups is a homomorphism of them as local Lie groups.

We are able to construct Lie algebras of Lie groups as tangent spaces at unity and we are able to construct morphisms of Lie algebras by differentiating morphisms of Lie groups. The following theorems prove that we are, at least locally, able to do it the other way around.

Lemma 2.81. Let G be a local Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Then there exists a Lie subgroup $H \subset\subset G$, such that $\mathfrak{h}$ is its Lie algebra.

Proof. If we interpret $\mathfrak{h}$ as an algebra of left-invariant vector fields on G, then it is also an involutive distribution on G. Denote H the integral manifold at unity. If we show that H is a subalgebra, then it will be clear that its left-invariant vector fields are in $\mathfrak{h}$, so $\mathfrak{h}$ is its Lie algebra. Take $g \in H$, $L_{g^{-1}}$ is a diffeomorphism of G preserving the left-invariant vector field, so $L_{g^{-1}}(H)$ has to be the integral submanifold and, since $e \in L_{g^{-1}}(H)$, it is an integral submanifold at unity and from uniqueness it is equal to H on a neighbourhood of unity. So, $g^{-1}h \in H$ for all g, h in a neighbourhood of unity in H. □

Lemma 2.82. Let G be a local Lie group, H its subgroup, and $\mathfrak{g}$, $\mathfrak{h}$ the corresponding Lie algebras. Then H is normal in G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Proof. The subgroup is normal if and only if for all $g, h \in G$ it holds that $h \in H \Leftrightarrow ghg^{-1} \in H$. The elements of local Lie group in the neighbourhood of unity can be uniquely represented by elements of the corresponding Lie algebra. Thus, the condition can be equivalently represented as $\mathrm{e}^b \in H \Leftrightarrow \mathrm{e}^a \mathrm{e}^b \mathrm{e}^{-a} = \exp(\mathrm{e}^{\mathrm{ad}_a} b) \in H$ for sufficiently small $a, b \in \mathfrak{g}$, where we used Theorem 2.75 to rewrite the expression. This can be expressed in terms of the Lie algebra elements themself $b \in \mathfrak{h} \Leftrightarrow \mathrm{e}^{\mathrm{ad}_a} b \in \mathfrak{h}$. Finally, it is sufficient to show that this is equivalent to the implication $b \in \mathfrak{h} \Rightarrow [a, b] \in \mathfrak{h}$. The "if" direction is trivial. To prove the opposite direction, let us assume, that $b \in \mathfrak{h}$, then

$$\mathrm{e}^{\mathrm{ad}_a} b = b + [a, b] + [a, [a, b]] + \ldots \in \mathfrak{h},$$

so, since b on the right-hand side lies in $\mathfrak{h}$, we have

$$\mathfrak{h} \ni [a, b] + [a, [a, b]] + \ldots = \mathrm{e}^{\mathrm{ad}_a} [a, b],$$

so $[a, b] \in \mathfrak{h}$. □

Theorem 2.83. Let G_1 and G_2 be local Lie groups, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ its Lie algebras. For every homomorphism $\varphi \colon \mathfrak{g}_1 \to \mathfrak{g}_2$, there exists unique local Lie group homomorphism Φ such that $\varphi = \mathrm{d}\Phi$.

Proof. Take a homomorphism $\varphi \colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Its graph $\Gamma := \{(a, \varphi(a)) \mid a \in \mathfrak{g}_1\}$ is a Lie subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. We can check that $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie algebra of $G_1 \times G_2$. Using Lemma 2.81, we can find a subalgebra $H \subset\subset G_1 \times G_2$, such that $\mathfrak{h}$ is its Lie algebra. The projection on G_1 restricted on H $\pi_1 \colon H \to G_1$ is a homomorphism, whose derivative at unity is regular mapping $(a, \varphi(a)) \mapsto a$. Hence π_1 is a local isomorphism at unity. Denote $\pi_2 \colon H \to G_2$ the second projection, then $\mathrm{d}\pi_2$ at unity maps $(a, \varphi(a)) \mapsto \varphi(a)$. So, $\pi_1^{-1} \circ \pi_2$ is the homomorphism we are looking for. □

Remark 2.84. It is clear that we have a one-to-one correspondence not only between homomorphisms, but also between isomorphisms of local Lie groups and their Lie algebras because a smooth map has a regular differential if and only if it is a local diffeomorphism. Consequently, given a group of automorphisms of a Lie group G, we can construct the corresponding group of automorphisms of its Lie algebra $\mathfrak{g}$ and vice versa. In particular, according to Lemma 2.74 and Theorem 2.75 the differentiation maps bijectively the group of inner automorphisms of a Lie group onto the group of inner automorphisms of its Lie algebra.

Theorem 2.85. For a given Lie algebra $\mathfrak{g}$, there exists, up to isomorphisms, unique local Lie group G such that $\mathfrak{g}$ is its Lie algebra.

Proof. According to the theorem of Ado 2.77, there exists a vector space V such that $\mathfrak{g} \in \mathrm{gl}(V)$. Using the preceding lemma, we find the local Lie group $G \subset\subset \mathrm{GL}(V)$. The uniqueness follows from Theorem 2.83 (identical map of the Lie algebra induces an isomorphism of its different local Lie groups). □

In fact, these theorems hold globally: for a given Lie algebra $\mathfrak{g}$, there exists, up to isomorphisms, unique simply connected Lie group G such that $\mathfrak{g}$ is its Lie algebra. Thus, the notion of a local Lie group is actually redundant since we see that every local Lie group can be uniquely extended to a simply connected Lie group. Nevertheless, this stronger proposition is harder to prove and the local approach can simplify our further considerations.

2.6.4 Lie group action on a manifold

Let G be a Lie group and M a manifold. By an action of G on M, we always mean a *smooth* action, that is, the map $\pi\colon G \times M \to M$ is always considered smooth. The G-set M is then called a *G-manifold*.

Lemma 2.86. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, M a manifold, and π an action of G on M. Then the mapping $\mathfrak{g} \to \mathrm{Vect}\, M$

$$a = X_e \mapsto \hat{X}, \quad \hat{X}_p = -\mathrm{d}\pi_p\, a = \frac{\mathrm{d}}{\mathrm{d}t}(\pi_p \mathrm{e}^{-ta})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-ta}\cdot p)\bigg|_{t=0}, \tag{20}$$

where X is the left-invariant vector field corresponding to a and π_p is the orbit map of point p, is a homomorphism.

Proof. At first, we show that negative right-invariant vector fields satisfy the assumptions of Theorem 2.48. Let $\tilde{X}$ be a right invariant field such that $\tilde{X}_e = a$, then

$$\mathrm{d}\pi_p\,(-\tilde{X}_g) = -\mathrm{d}\pi_p\,\mathrm{d}R_g^{-1}\,\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{ta}\bigg|_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t}((\pi_p \circ R_g^{-1})\mathrm{e}^{ta})\bigg|_{t=0} = \\ = -\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{ta}gp)\bigg|_{t=0} = -\mathrm{d}\pi_{gp}\,a = \hat{X}_{gp} = \hat{X}_{\pi_g(p)}.$$

Now, we have according to Theorem 2.48

$$[\hat{X},\hat{Y}]_p = \mathrm{d}\pi_p[-\tilde{X},-\tilde{Y}]_e = \mathrm{d}\pi_p[b,a] = \widehat{[X,Y]}_p.$$

□

Definition 2.87. The vector field $\hat{X}$ of the previous lemma is called the *fundamental vector field* of the action π corresponding to the vector field X. The homomorphism $X \mapsto \hat{X}$ will be denoted as π_*.

For the map $g \mapsto \pi_g$, which we called permutation representation in Section 2.1, we have $\pi_G \subset \mathrm{Diff}\, M \subset S_M$ because we demand the action π to be smooth. Therefore, the action of a group on a manifold can be understood as a generalization of the notion of a global flow or one-parameter group of diffeomorphisms. However, it is not one-parameter anymore. To complete the analogy, we bring the following definition corresponding to the notion of a local flow.

Definition 2.88. Let G be a Lie group and M a manifold. By a *local action* of G on M we mean a smooth map $\pi\colon W \to M$, where W is an open subset of $G \times M$ such that $\{e\} \times M \subset W$ satisfying $\pi(e,p) = p$ for all $p \in M$ and $\pi(g, \pi(h,p)) = \pi(gh, p)$ for all g and h for which both sides are well-defined.

If we consider an action of a local Lie group, we automatically mean a local action. In the case of local Lie groups, one also need to consider locally the properties of the action. For example, an action π of a local Lie group G is called *(locally) transitive* if, for every point $p \in M$, there is a neighbourhood U of p such that, for all $q \in U$, there exists $g \in G$ such that $q = \pi(p, g)$.

It seems natural that a local action is described by the *infinitesimal action*, that is, the differential $\mathrm{d}\pi_p$ for every $p \in M$. The role of the *infinitesimal generators* is played by the images of Lie algebra $\mathfrak{g}$ of G through the mapping $\mathrm{d}\pi_p$—the fundamental vector fields. From the uniqueness of the flow for a given vector field, we see that the local action is uniquely defined by the infinitesimal action. Nevertheless, the existence of a local action corresponding to an infinitesimal action is not so trivial to prove.

Theorem 2.89. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, M a smooth manifold, and $\varphi\colon \mathfrak{g} \to \operatorname{Vect} M$ a homomorphism. Then there exists a local action π of G on M such that $\varphi = \pi_*$.

Proof. The map φ induce a distribution $\varphi(\mathfrak{g})$ on $\operatorname{Vect} M$. The homomorphism property implies that the distribution is involutive. Recalling the Frobenius theorem 2.61 we can define the action π as translation in the adapted coordinates x^i. Finally, we can check that the fundamental vector fields are indeed ∂_{x^i}. □

It is again useful to be able to express these structures in coordinates. Let us have local coordinates of a given Lie group G in a neighbourhood of unity $x^1, \ldots, x^n$ and local coordinates of a given manifold M $q^1, \ldots, q^m$. Then, for arbitrary action π of G on M, we can define its coordinate representation as a function $\Pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. Let $\mathfrak{g}$ be a Lie algebra of G and $e_1, \ldots, e_n$ its basis. Then we can define a basis of the algebra of fundamental vector fields as $(\hat{X}_i)_p = \mathrm{d}\pi_p\, e_i$. The coordinates of these vector fields are

$$\hat{X}_i^a(q) = -\left.\frac{\partial \Pi^a(x,q)}{\partial x^j}\right|_{x=0} e_i^j = -\left.\frac{\partial \Pi^a(x,q)}{\partial x^i}\right|_{x=0}. \tag{21}$$

Morphisms, in the case of action on manifold, are defined in the same way as in the case of ordinary action except we demand them to be smooth. Isomorphisms have to be diffeomorphisms.

Now, we present a stronger variant of Lemma 2.12 for a Lie group action on manifold. The homogeneous space G/G_p can be given the structure of differentiable manifold induced by the quotient map $g \mapsto gG_p$. It holds that there exists a unique smooth structure on G/G_p such that the quotient map is smooth. For us, it is sufficient to bring the construction just locally. Let $\mathfrak{h}$ be the Lie group of stabiliser G_p. It is a subalgebra of $\mathfrak{g}$ consisting of vector fields whose integral curves act on p trivially, so it is the kernel of $\mathrm{d}\pi_p$ at unity. Let V be a vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$, so $\mathfrak{g}$ is a direct sum of vector spaces $\mathfrak{g} = V \oplus \mathfrak{h}$. Denote $(e_1, \ldots, e_k)$ the basis of V and $(e_{k+1}, \ldots, e_n)$ the basis of $\mathfrak{h}$. According to Subsection 2.6.2, we can choose canonical coordinates on G in the neighbourhood of unity by relation

$$g_{(t^1,\ldots,t^n)} = \exp\left(\sum_{i=1}^{k} t^i e_i\right) \exp\left(\sum_{i=k+1}^{n} t^i e_i\right).$$

We can define coordinates on G/G_p as $\bar{g}_{t^1,\ldots,t^k} = g_{t^1,\ldots,t^k,0,\ldots,0}G_p$. It is evident that the quotient map is smooth with respect to these coordinates since its coordinate representation is $(t^1,\ldots,t^n) \mapsto (t^1,\ldots,t^k)$. It is also clear that the left action of G on G/G_p is smooth as well. We can again introduce the map $\tilde{\pi}_p\colon G/G_p \to M\ gG_p \mapsto gp$, that was examined by Lemma 2.12. It is injective (mapping bijectively G/G_p onto pG) and its differential at unity $\mathrm{d}\tilde{\pi}_p$ is restriction of $\mathrm{d}\pi_p$ at unity on a complement of its kernel, so it is injective as well. Therefore, we locally proved the following theorem.

Theorem 2.90. Let π be an action of G on the manifold M, $p \in M$. Then the quotient map induces a structure of a smooth manifold on G/G_p. This manifold is immersed in M as the orbit of p by $\tilde{\pi}_p\colon G/G_p \to M\ gG_p \mapsto gp$.

In particular, if π is transitive, then the map $\tilde{\pi}_p$ is a diffeomorphism.

Corollary 2.91. Let π be a transitive action of G on the manifold M, $p \in M$. Then the G-spaces M and G/G_p are isomorphic. In particular, it means that if π is transitive and free, then it is isomorphic to the left translations of G.

Remark 2.92. Thus, an interesting question is: what is the fundamental vector field for the action of left translations. For $a \in \mathfrak{g}$ and X the corresponding left-invariant vector field, we have

$$\hat{X}_g = -\mathrm{d}_1 L(e,g)\, a = -\mathrm{d}R_{g^{-1}}\, a = -\tilde{X}_g = \mathrm{d}i\, X,$$

where $\tilde{X}$ is the right-invariant vector field corresponding to a and i is the group inversion.

Remark 2.93. We have formulated this section for left actions of a group on a manifold for the sake of consistence with the rest of this chapter, where we considered left actions only. Another reason is that it is formulated this way essentially in every literature. However, if we consider right actions, things get a little simpler in some ways.

The particular changes are following. The fundamental vector field is defined without the minus sign: $\hat{X}_p = \mathrm{d}\pi_p\, a$, where $\pi_p(g) = \pi(p,g)$. We will again denote $\hat{X} = \pi_* a$. Thus, the coordinate expression is

$$\hat{X}^a_i(q) = \left.\frac{\partial \Pi^a(q,x)}{\partial x^j}\right|_{x=0}. \tag{22}$$

Theorem 2.90 would state that the right coset space $G_p\backslash G$ is immersed to the manifold M. Therefore, the corollary is that every transitive right action on a manifold M is isomorphic to the right action of right multiplication on $G_p\backslash G$. The fundamental vector fields of right translations are left-invariant vector fields (this was basically already stated in Lemma 2.68.3).

Chapter 3
Classification of realizations

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra and M a manifold. A *realization* of $\mathfrak{g}$ on the manifold M is a homomorphism $R: \mathfrak{g} \to \operatorname{Vect} M$. The realization is called *faithful* if it is injective.

Definition 3.2. Let $\mathfrak{g}$ be a Lie algebra and M_1 and M_2 manifolds. Let A be a subgroup of $\operatorname{Aut}(\mathfrak{g})$. Realizations $R_1: \mathfrak{g} \to \operatorname{Vect} M_1$ and $R_2: \mathfrak{g} \to \operatorname{Vect} M_2$ are called *A-equivalent* if there exist an automorphism $\alpha \in A$ and a diffeomorphism $\Phi: M_1 \to M_2$ such that $R_2(\alpha(x)) = \Phi_* R_1(x)$ for all $x \in \mathfrak{g}$. If the group of automorphisms A contains only the identity, the realizations are called *strongly* equivalent. In contrast, $\operatorname{Aut}(\mathfrak{g})$-equivalent realizations are called *weakly* equivalent.

The tangent map Φ_* can be viewed as a coordinate change, so strong equivalence is equality up to a coordinate change.

In this work we consider realizations only locally, i.e. the vector fields realizing the Lie algebra are defined only locally. Since every manifold is, by definition, locally diffeomorphic to $\mathbb{R}^n$, we can take a neighborhood of zero $U \subset \mathbb{R}^n$ for the manifold M. Note that we could, without loss of generality, fix the neighborhood of zero or even take the whole set $\mathbb{R}^n$ to be our manifold since all neighborhoods in $\mathbb{R}^n$ can be transformed one onto another by a diffeomorphism.

This definition was adopted from [2]. Other authors may take more general definition considering the Lie algebra of vector fields to be a derivation of the algebra of formal power series over a field F of characteristic zero $\operatorname{Der} F[[x]]$ (see for example [3]).

The classification of realizations is usually performed with respect to the weak equivalence. It is reasonable beacuse even in this case the list is usually quite long. The classification is, however, harder to perform in this case because one needs to know explicitly the whole automorphism group to exclude the equivalent realizations and to prove that two realizations are equivalent. Nevertheless, it is not completely clear what is the reasonable definition of the classification problem for applications. This is probably going to be one of the subjects of study in the authors Master's thesis. We have, therefore, decided to provide classification with respect to both the strong and the weak equivalence.

To classify realizations of a given Lie algebra in a most direct way, it is neccesary to solve a complicated system of non-linear partial differential equations. Another way is to proceed inductively. To classify realizations of a given Lie algebra, we classify the realization of its subalgebras at first. Nevertheless, it still requires solving a system of partial differential equations. By this method all realizations of Lie algebras of dimension not greater than four was obtained in [2]. To test the inequivalence with respect to the weak equivalence a new method of so called megaideals (subalgebras invariant with respect to all, not only inner automorphisms) was invented.

Nevertheless, there is also another way how to search for and classify realizations. There is a one-to-one correspondence between Lie algebra subalgebras and so called transitive realizations. For the general case of realization by vectors of $\operatorname{Der} K[[x]]$ the theorem was formulated and proven by Guillemin and Sternberg [14]. In our case the correspondence follows very simply from the theory of Lie algebra actions described in section 2.6.4.

A simple purely algebraic method for computation the explicit form of the corresponding realizations was proposed in [1] by I. V. Shirokov et al. Eventually, if we want to find all transitive realizations of a given Lie algebra, the need of complete subalgebra classification turns out to be the hardest part of the task. The subsequent computation is very simple and can be performed by a computer. The proposed method was already used to classify the realizations of low-dimensional Poincaré algebras [15] and Galilei algebras [16]. We were so far able to classify the transitive realizations of all five-dimensional indecomposable Lie algebras with four-dimensional Abelian ideal.

On the other hand, the classification of non-transitive realizations is very hard to perform and there is no effective algorithm as for the transitive case. The transitive realizations will also be the subject of our further research.

3.1 Realizations, group actions and subalgebras

In this section the correspondence between realizations and actions of the corresponding Lie group will be stated. Then the methods of classification of realizations will be described.

The important thing to be noticed is that a realization $R\colon \mathfrak{g} \to \operatorname{Vect} M$ can be equivalently described by a local group action. According to Theorem 2.85, there exists (up to isomorphisms) unique local Lie group G for a given Lie algebra $\mathfrak{g}$. In addition, according to Theorem 2.89 and Remark 2.93, there exists a unique local right action π of G on M such that $\pi_* = R$. Thus, we can construct the realizations as fundamental vector fields of various actions.

As one would expect, there is a kind of one-to-one correspondence between local actions and realizations. To state the proposition precisely, we generalize the definition of similitude 2.9. Because we use right actions in this chapter, we will formulate the definition for right action as well. Let G_1 and G_2 be isomorphic local Lie groups and $\mathfrak{g}$ their Lie algebra. We say that actions $\pi^{(1)}$ and $\pi^{(2)}$ of G_1 and G_2 acting on M_1 and M_2, respectively, are *A-similar* for A a subgroup of $\operatorname{Aut}\mathfrak{g}$ if there is a diffeomorphism $\Phi\colon M_1 \to M_2$ and an isomorphism $\varphi\colon G_1 \to G_2$ such that $\mathrm{d}\varphi \in A$ sattisfying the similtude relation

$$\Phi(\pi^{(1)}(p,g)) = \pi^{(2)}(\Phi(p),\varphi(g)). \tag{1}$$

Proposition 3.3. Let G_1 and G_2 be local Lie groups of Lie algebra $\mathfrak{g}$. Local actions of G_1 and G_2 on a manifold M_1 and M_2 are A-similar if and only if the corresponding realizations of the Lie algebra $\mathfrak{g}$ are A-equivalent.

Proof. Let $\pi^{(1)}$ and $\pi^{(2)}$ be local actions of G_1 and G_2 on M_1 and M_2 corresponding to realizations R_1 and R_2, respectively. Let Φ be a diffeomorphism $M_1 \to M_2$ and φ an isomorphism $G_1 \to G_2$ such that $\mathrm{d}\varphi \in A$. Then for all $p \in M_1$, $a \in \mathfrak{g}$, and $t \in (-\varepsilon,\varepsilon)$ we have

$$\mathrm{d}\Phi R_1(\mathrm{d}L_{\mathrm{e}^{ta}}\, a)_p = \mathrm{d}\Phi\, \mathrm{d}\pi^{(1)}_p\, \mathrm{d}L_{\mathrm{e}^{ta}}\, a = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\pi^{(1)}(p,\mathrm{e}^{ta})),$$

$$R_2(\mathrm{d}\varphi\, \mathrm{d}L_{\mathrm{e}^{ta}}\, a)_{\Phi(p)} = \mathrm{d}\pi^{(2)}_{\Phi(p)}\, \mathrm{d}\varphi \mathrm{d}L_{\mathrm{e}^{ta}}\, a = \frac{\mathrm{d}}{\mathrm{d}t}\pi^{(2)}(\Phi(p),\varphi(\mathrm{e}^{ta})),$$

so we have $\Phi_* R_1(a) = R_2(\alpha(a))$ for all $p \in M_1$ and $a \in \mathfrak{g}$ if and only if $\Phi(\pi^{(1)}(p,g)) = \pi^{(2)}(\Phi(p),\varphi(g))$ for all $p \in M_1$ and $g \in G_1$, where $\alpha = \mathrm{d}\varphi \in A$. □

Proposition 3.4. A realization is faithful if and only if the corresponding local action is effective.

Proof. Let R be a realization of $\mathfrak{g}$ on M, G the corresponding local Lie group, and π the corresponding local action. The realization R is unfaithful if and only if

$$0 = R(\mathrm{d}L_{e^{ta}}\, a)_p = \frac{\mathrm{d}}{\mathrm{d}t}\pi(p, \mathrm{e}^{ta})$$

for all $a \in \mathfrak{g}$, $p \in M$, and $t \in (-\varepsilon, \varepsilon)$, which is equivalent to $\pi(p, g) = 0$ for all $g \in G$ and $p \in M$, meaning π is ineffective. □

3.1.1 Classification of the transitive realizations

In the following lemma, we define and characterize transitive realizations. In the rest of this subsection the correspondence with subalgebras of the concerned Lie algebra will be stated and the explicit computation of the transitive actions based on [1] will be described.

Lemma 3.5. Let $\mathfrak{g}$ be a Lie algebra and R its realization on a manifold M. Let G be a local Lie group of $\mathfrak{g}$ and π a local action on M corresponding to R. The action π is transitive if and only if $\{R(x)_p \mid x \in \mathfrak{g}\} = T_pM$ for all $p \in M$, i.e. $R_p\colon \mathfrak{g} \to T_pM$ $x \mapsto R(x)_p$ is surjective at every $p \in M$. If so, we call R *transitive* as well.

Proof. The map R_p is in fact identical to $\mathrm{d}\pi_p$. Therefore, it is surjective if and only if π_p is "locally surjective", i.e. there exists a neighbourhood V of p such that $\pi_p(G) \supset V$. Equivalently, for every $q \in V$ there is a $g \in G$ such that $q = \pi(p, g)$. □

We can reformulate this lemma for local realizations.

Lemma 3.6. Let $\mathfrak{g}$ be a Lie algebra and R its realization on U a neighborhood of zero in $\mathbb{R}^n$, let G be local Lie group of $\mathfrak{g}$ and π a local action on U corresponding to R. The action π is transitive if and only if $\{R(x)_0 \mid x \in \mathfrak{g}\} = T_0M$, i.e. R at zero is surjective.

According to Corollary 2.91 and Remark 2.93, every transitive action is isomorphic to the right multiplication of G acting on $G_p \setminus G$, $p \in M$. Hence, any transitive action of G is, up to isomorphisms, described by the subgroup H representing a stabilizer of a given point in G.

As a consequence of Proposition 3.3, we can move to Lie algebras. We see that every strong class of transitive realizations of a Lie algebra $\mathfrak{g}$ is determined by a subalgebra $\mathfrak{h}$. Moreover, if we look for a weaker classification of transitive realizations, we do not have to go through all subalgebras of $\mathfrak{g}$. The following proposition tells us that the realization class is determined by the subalgebra class uniquely.

Proposition 3.7. Let $\mathfrak{g}$ be a Lie algebra and $A \subset\subset \operatorname{Aut}\mathfrak{g}$. Transitive realizations correspond to A-conjugate subalgebras of $\mathfrak{g}$ if and only if they are A-equivalent.

Proof. Denote G a local Lie group of the Lie algebra $\mathfrak{g}$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ arbitrary subalgebras of $\mathfrak{g}$, and H_1 and H_2 the corresponding subgroups. Denote $\pi^{(1)}$ and $\pi^{(2)}$ the right multiplication of G acting on $H_1 \setminus G$ and $H_2 \setminus G$, respectively, and R_1, R_2 the corresponding realizations. Let φ be an automorphism of G, so $\mathrm{d}\varphi$ is an automorphism of $\mathfrak{g}$ and let $\Phi\colon H_1 \setminus G \to H_2 \setminus G$ be a local diffeomorphism at H_1 (such that $\Phi(H_1) = H_2$).

According to Proposition 3.3, the realizations R_1 and R_2 are equivalent with respect to the Lie algebra automorphism $\mathrm{d}\varphi$ and diffeomorphism Φ if and only if

$$\Phi(H_1gh) = \Phi(H_1g)\varphi(h)$$

for all $g, h \in G_1$.

If the Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are A-equivalent, so there is an $\alpha \in A$ such that $\mathfrak{h}_2 = \alpha(\mathfrak{h}_1)$, we just have to choose φ and Φ such that $\mathrm{d}\varphi = \alpha$ and $\Phi(H_1 g) = \varphi(H_1 g) = H_2\varphi(g)$. Conversely, if the realizations R_1 and R_2 are equivalent, so the equation above is sattisfied, we have for every $h \in H_1$

$$H_2 = \Phi(H_1) = \Phi(H_1 h) = \Phi(H_1)\varphi(h) = H_2\varphi(h),$$

so $\varphi(H_1) = H_2$. Thus, the Lie groups H_1 and H_2 as well as the Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are A-equivalent. □

Proposition 3.8. A transitive realization of a Lie algebra $\mathfrak{g}$ is faithful if and only if the corresponding subalgebra of $\mathfrak{g}$ does not contain any non-trivial ideal of $\mathfrak{g}$.

Proof. According to Proposition 3.4, the realization is faithful if and only if the corresponding action is effective. Here, we consider the right multiplication of G on $H \setminus G$. The kernel of this action is, according to Lemma 2.7, the largest normal subgroup contained in H. The action is, therefore, effective if and only if H does not contain nontrivial normal subgroup. This is, according to Lemma 2.82, equivalent to the proposition that $\mathfrak{h}$ does not contain any nontrivial ideal. □

The choice $\mathfrak{h} = \{0\}$, so $H = E$ corresponds to transitive free action, which is according to Corollary 2.91 and Remark 2.93 isomorphic to the action of G on itself by right multiplication. The corresponding realization on $M \simeq E \setminus G \simeq G$ is by left-invariant vector fields. Such realization is called *generic*. Other realizations are then just a restriction of the generic realization on a submanifold $H \setminus G$. To compute them locally, it is sufficient to choose suitable coordinates on G.

As was shown in [1], the most convenient are the second canonical coordinates. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ its complement. We choose a basis $(e_1, \ldots, e_m)$ of $\mathfrak{g}$ such that $(e_1, \ldots, e_{n-m})$ is a basis of $\mathfrak{h}$ and $(e_{n-m+1}, \ldots, e_m)$ is a basis of $\mathfrak{p}$. Then the group element g_x with coordinates x^i is computed as

$$g_x = \prod_{i=1}^{n} \exp(x^i e_i).$$

We can also define induced coordinates on the subgroup H and the quotient group $H \setminus G$ as

$$h_y = \prod_{a=1}^{n-m} \exp(y^\beta e_\beta) \quad \bar{g}_q = H g_{(0,q)} = H \prod_{a=1}^{m} \exp(q^a e_{n-m+a}).$$

Notice that the representatives of a class in $H \setminus G$ are determined by the last m coordinates: $\bar{g}_q = Hg(y, q)$ for all y. Therefore, coordinates of product of the action of right multiplication $\bar{g}_q \cdot g_z$ are the last m coordinates of product of a representative, say $g_{(0,q)}$, and g_z:

$$\Pi^a(q, z) = M^{n-m+a}((0, q), z).$$

Recall equations (14) and (22) of chapter 2 saying, that the components of left-invariant vector fields or fundamental vector fields are given by partial derivatives of the composition function or the action function, respectively. Thanks to the preceding relation (2), the components have to be identical in these coordinates. Therefore, for a generic realization

$$R^{\mathrm{gen}}(e_i)_{g_{(q,y)}} = X_i(q, y) = \sum_{a=1}^{m} X_i^a(q) \frac{\partial}{\partial q^a} + \sum_{\beta=1}^{n-m} X_i^\beta(q, y) \frac{\partial}{\partial y^\beta} \tag{2}$$

on $M \simeq G$, which can be computed in those coordinates by an algorithm described in Section 2.6.2, we have constructed a realization

$$R(e_i)_{\bar{g}_q} = \hat{X}_i(q) = \sum_{a=1}^{m} X_i^a(q) \frac{\partial}{\partial q^a} \tag{3}$$

on $M \simeq H \setminus G$.

As we mentioned earlier, this computation is very simple and can be performed purely by some computer program. We used a script in Mathematica kindly provided by supervisor doc. Severin Pošta who used it earlier in a joint work [16]. The results of classification of transitive realizations of five-dimensional indecomposable Lie algebras with four-dimensional Abelian ideal are listed in Section A.2 of the appendix.

Chapter 4
Conclusion

Our aim, in this research project, was to study possible algortithms for construction realizations of Lie algebras. We focused on the work of I. V. Shirokov et al. [1, 13]. We summarized his method and made more clear, what kind of realizations are classificated by this method, providing rigorous description of relationship with classification of subalgebras.

The main result of this work is both weak and strong classification of transitive realizations of all five-dimensional indecomposable Lie algebras with four-dimensional Abelian ideal. As an intermediate result, we obtained lists of subalgebras of these Lie algebras and classification of these subalgebras with respect to group of all automorphisms, which is a useful result by itself and it was actually the most difficult part of the computation.

Nevertheless, as the most interesting results, one may consider the questions that arose during the study of this topic. Namely, how to construct the non-transitive realizations, what is the reasonable classification problem for applications, how to construct realizations globally on a given non-trivial manifold, etc. Moreover, the so far most general classification of Lie algebra realizations [2] seems to have either a lot of non-transitive realizations missing or inaccurate definition of the classification problem. These are interesting problems for further research and the author would like to treat most of them in his Master's thesis. It would be also nice to complete the classification of subalgebras and realizations for all five-dimensional Lie algebras.

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List of used symbols

$\simeq$	Isomorphism (diffeomorphism) of algebraic (differential) structures	4, 10
$[\cdot,\cdot]$	Lie bracket	4
$\oplus$	Direct sum	7
$\subset\subset$	Algebraic substructure (vector subspace, subgroup or subalgebra)	2, 4
$\inplus$	Semidirect sum	7
A/B	Quotient space of A by B	2, 6
ad_x	Lie algebra adjoint map corresponding to x	5
Ad_g	Lie group adjoint map corresponding to g	17
$\mathrm{Aut}\,A$	Set of all automorphisms of the algebraic structure A	2, 4
$C^\infty(M)$	Algebra of smooth functions on a manifold M	10
C_g	Conjugation by g	3
$\mathrm{d}\Phi$	Derivative of map Φ	11
$\mathrm{Diff}\,M$	Group of all diffeomorphisms of a manifold M	10
e	Group unity	2
$\exp$, $\mathrm{e}^\bullet$	Exponential map	8, 16
Φ_*	Pushforward of a vector field by a diffeomorphism Φ	11
Φ^*	Pullback of a smooth function by a diffeomorphism Φ	13
$\mathfrak{g}, \mathfrak{h}, \ldots$	Lie algebra (corresponding to Lie group $G, H, \ldots$)	16
$G, H, \ldots$	(Lie) group	2, 15
G/H	Set of left cosets of G by H	2
$H \setminus G$	Set of right cosets of G by H	2
$\mathrm{gl}(V)$	General linear Lie algebra of V	4
$\mathrm{GL}(V)$	General linear group of V	2
G_x	Stabilizer of point x under action of group G	4
$\ker\varphi$	Kernel of a homomorphism φ	2
L_g	Left translation by g	3
$M, N, \ldots$	Manifold	10
$\pi_* X$	Fundamental vector field corresponding to X	22
$\mathbb{R}$	Set of real numbers	
R_g	Right translation by g^{-1}	3
S_X	Symmetric group of set X	2
TM	Tangent bundle on M	11
T_pM	Tangent space at $p \in M$	10
T^*M	Cotangent bundle on M	12
T_p^*M	Cotangent space at $p \in M$	12
$V, W, \ldots$	Vector space	
$\mathrm{Vect}\,M$	Lie algebra of vector fields on M	11
$X, Y, \ldots$	Vector field (or a tangent vector)	11
$\hat{X}$	Fundamental vector field corresponding to X	22

Appendix A
The classification results

A.1 Classification of Lie algebras

To express our results, we use the classification of real indecomposable Lie algebras of dimension not greater than five obtained by G. M. Mubarakzyanov in[8, 7]. With a few exceptions listed below, we use the same numbering and bases for the Lie algebras. For the sake of completeness, we quote those results in the following tables A.1 and A.2. We denote the one-dimensional Abelian Lie algebra $\mathfrak{g}_1$ and the two-dimensional non-Abelian Lie algebra $\mathfrak{g}_2$ with basis satisfying commutation relation $[e_1, e_2] = e_1$.

The exceptions in notation are related only to the range of the parameters. The formal expressions for the Lie brackets will always correspond to the tables. Sometimes, we get a Lie algebra from other family by substituting parameters out of their domain. For example, by $\mathfrak{g}_{3,4}^1$ we mean $\mathfrak{g}_{3,3}$. This will simplify the description of the subalgebra types in our tables. The second thing is that the range of parameters is chosen in a way that the Lie algebras with different parameter are nonisomorphic. For example, the Lie algebra $\mathfrak{g}_{3,4}^\alpha$ is well-defined also for $|\alpha| > 1$, but it is isomorphic to the cases with $|\alpha| < 1$. We will sometimes use these isomorphic representatives of the Lie algebra types.

Type	Nonzero commutation relations
$\mathfrak{g}_{3,1}$	$[e_2, e_3] = e_1$
$\mathfrak{g}_{3,2}$	$[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$
$\mathfrak{g}_{3,3}$	$[e_1, e_3] = e_1, [e_2, e_3] = e_2$
$\mathfrak{g}_{3,4}^\alpha$	$[e_1, e_3] = e_1, [e_2, e_3] = \alpha e_2; \quad -1 \leq \alpha < 1, \alpha \neq 0$
$\mathfrak{g}_{3,5}^\alpha$	$[e_1, e_3] = \alpha e_1 - e_2, [e_2, e_3] = e_1 + \alpha e_2; \quad \alpha \geq 0$
$\mathfrak{g}_{3,6}$	$[e_1, e_2] = e_1, [e_1, e_3] = 2e_2, [e_2, e_3] = e_3$
$\mathfrak{g}_{3,7}$	$[e_1, e_2] = e_3, [e_3, e_1] = e_2, [e_2, e_3] = e_1$
$\mathfrak{g}_{4,1}$	$[e_2, e_4] = e_1, [e_3, e_4] = e_2$
$\mathfrak{g}_{4,2}^\alpha$	$[e_1, e_4] = \alpha e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3; \quad \alpha \neq 0$
$\mathfrak{g}_{4,3}$	$[e_1, e_4] = e_1, [e_3, e_4] = e_2$
$\mathfrak{g}_{4,4}$	$[e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3$
$\mathfrak{g}_{4,5}^{\alpha,\beta}$	$[e_1, e_4] = e_1, [e_2, e_4] = \alpha e_2, [e_3, e_4] = \beta e_3; \quad -1 \leq \beta \leq \alpha \leq 1, \alpha\beta \neq 0$
$\mathfrak{g}_{4,6}^{\alpha,\beta}$	$[e_1, e_4] = \alpha e_1, [e_2, e_4] = \beta e_2 - e_3, [e_3, e_4] = e_2 + \beta e_3; \quad \alpha \neq 0, \beta \geq 0$
$\mathfrak{g}_{4,7}$	$[e_2, e_3] = e_1, [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$
$\mathfrak{g}_{4,8}^\alpha$	$[e_2, e_3] = e_1, [e_1, e_4] = (1 + \alpha)e_1, [e_2, e_4] = e_2, [e_3, e_4] = \alpha e_3; \quad \lvert\alpha\rvert \leq 1$
$\mathfrak{g}_{4,9}^\alpha$	$[e_2, e_3] = e_1, [e_1, e_4] = 2\alpha e_1, [e_2, e_4] = \alpha e_2 - e_3, [e_3, e_4] = e_2 + \alpha e_3; \quad \alpha \geq 0$
$\mathfrak{g}_{4,10}$	$[e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1$

Table A.1. Real three- and four-dimensional indecomposable Lie algebras [8].

Type	Nonzero commutation relations
$\mathfrak{g}_{5,1}$	$[e_3,e_5]=e_1, [e_4,e_5]=e_2$
$\mathfrak{g}_{5,2}$	$[e_2,e_5]=e_1, [e_3,e_5]=e_2, [e_4,e_5]=e_3$
$\mathfrak{g}_{5,7}^{\alpha,\beta,\gamma}$	$[e_1,e_5]=e_1, [e_2,e_5]=\alpha e_2, [e_3,e_5]=\beta e_3, [e_4,e_5]=\gamma e_4;$
	$-1\leq\gamma\leq\beta\leq\alpha\leq 1, \alpha\beta\gamma\neq 0$
$\mathfrak{g}_{5,8}^{\gamma}$	$[e_2,e_5]=e_1, [e_3,e_5]=e_3, [e_4,e_5]=\gamma e_4; \quad 0<\lvert\gamma\rvert\leq 1$
$\mathfrak{g}_{5,9}^{\beta,\gamma}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2, [e_3,e_5]=\beta e_3, [e_4,e_5]=\gamma e_4; \quad 0\neq\gamma\leq\beta$
$\mathfrak{g}_{5,10}$	$[e_2,e_5]=e_1, [e_3,e_5]=e_2, [e_4,e_5]=e_4$
$\mathfrak{g}_{5,11}^{\gamma}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2, [e_3,e_5]=e_2+e_3, [e_4,e_5]=\gamma e_4; \quad \gamma\neq 0$
$\mathfrak{g}_{5,12}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2, [e_3,e_5]=e_2+e_3, [e_4,e_5]=e_3+e_4$
$\mathfrak{g}_{5,13}^{\gamma,p,s}$	$[e_1,e_5]=e_1, [e_2,e_5]=\gamma e_2, [e_3,e_5]=pe_3-se_4, [e_4,e_5]=se_3+pe_4; \quad \lvert\gamma\rvert\leq 1, \gamma s\neq 0$
$\mathfrak{g}_{5,14}^{p}$	$[e_2,e_5]=e_2, [e_3,e_5]=pe_3-e_4, [e_4,e_5]=e_3+pe_4$
$\mathfrak{g}_{5,15}^{\gamma}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2, [e_3,e_5]=\gamma e_3, [e_4,e_5]=e_3+\gamma e_4; \quad -1\leq\gamma\leq 1$
$\mathfrak{g}_{5,16}^{p,s}$	$[e_1,e_5]=e_1, [e_2,e_5]=e_1+e_2, [e_3,e_5]=pe_3-se_4, [e_4,e_5]=se_3+pe_4; \quad s\neq 0$
$\mathfrak{g}_{5,17}^{p,q,s}$	$[e_1,e_5]=pe_1-e_2, [e_2,e_5]=e_1+pe_2,$
	$[e_3,e_5]=qe_3-se_4, [e_4,e_5]=se_3+qe_4; \quad s\neq 0$
$\mathfrak{g}_{5,18}^{p}$	$[e_1,e_5]=pe_1-e_2, [e_2,e_5]=e_1+pe_2,$
	$[e_3,e_5]=e_1+pe_3-e_4, [e_4,e_5]=e_2+e_3-pe_4; \quad p\geq 0$

Table A.2. Real five-dimensional indecomposable Lie algebras [7] with four-dimensional Abelian ideal.

A.2 Subalgebras classification and corresponding realizations

In this section subalgebra classification of the five-dimensional indecomposable Lie algebras with four-dimensional Abelian ideal and a list of corresponding realizations will be presented. For every Lie algebra, there will be a separate table. The tables are arranged in the following way. The column captioned “Subalgebra” (usually second) contains list of all subalgebras. In each row, there is a list of the subalgebra generators. The following column contains realizations corresponding to these subalgebras. As a complementary basis to the Lie subalgebra generators we usually take suitable subset of the Lie algebra basis $\{e_1,\dots,e_n\}$. The column captioned “Aut $\mathfrak{g}$ class’ brings subalgebra classification with respect to the whole group of automorphisms. Each subalgebra belongs to the class of equivalence in the corresponding row. The first column of the table states the type of the subalgebra.

The subalgebras are parametrized as follows. Letters a,b,c,d,e,f denote real parameters and ε is a discrete parameter (usually $\varepsilon\in\{-1,1\}$). We denote $\sigma_a := \operatorname{sgn} a$ the sign of parameter a, $\eta_{a_1,\dots,a_n} := \operatorname{sgn}(\lvert a_1\rvert+\ldots+\lvert a_n\rvert)$, i.e. $\eta_{a_1,\dots,a_n}$ is zero if all the parameters $a_1,\dots,a_n$ are zero and one otherwise, and $\bar\eta_{a_1,\dots,a_n} := (1-\eta_{a_1,\dots,a_n})$, which is one if all parameters are zero and zero otherwise.

Finally, we describe, how to read the tables, that is, how to obtain results of the considered classification problems. The list of all subalgebras is simply in the “Subalgebra” column. The strong classification of transitive realizations is in the “Corresponding realization” column. The classification of subalgebras with respect to the whole group of automorphisms is essentially in the “Aut $\mathfrak{g}$ class” column. Except the fact that one class can be listed more than once; nevertheless, it is always represented by the same representative. To get the weak classification of realizations, one needs to go through the “Corresponding realization” column and pick only one realization for each class by substituting the parameters of the class representative.

Every Lie algebra listed has only two families of faithful realizations since every Lie algebra with four-dimensional ideal has only two families of subalgebras not containing a non-trivial ideal of the Lie algebra. Namely, it is the zero subalgebra and a family of subalgebras generated by a vector of the form $e_5 + ae_4 + be_3 + ce_2 + de_1$ not lying in the four-dimensional ideal.

Table A.3. Subalgebras and realizations of $\mathfrak{g}_{5,1}$.

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$ class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + x_3\partial_1 + x_4\partial_2$	0
$\mathfrak{g}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_3\partial_1$	e_1
	$e_2 + ae_1$	$\partial_1,\ -a\partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + (x_2 - ax_3)\partial_1$	
	$e_3 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ -a\partial_2 - (b + x_4)\partial_1,\ \partial_3,\ \partial_4 + x_3\partial_2$	e_3
	$e_4 + ae_3 + be_2 + ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -a\partial_3 - (b + x_4)\partial_2 - (c + ax_4)\partial_1,\ \partial_4 + x_3\partial_1$	
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ -b\partial_3 - a\partial_4 + (-d + x_3)\partial_1 + (-c + x_4)\partial_2$	e_5
$2\mathfrak{g}_1$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3$	e_1, e_2
	$e_1,\ e_3 + ae_2$	$0,\ \partial_1,\ -a\partial_1,\ \partial_2,\ \partial_3 + x_2\partial_1$	e_1, e_3
	$e_2 + ae_1,\ e_4 + ae_3 + be_1$	$\partial_1,\ -a\partial_1,\ \partial_2,\ -a\partial_2 + -b\partial_1,\ \partial_3 + x_2\partial_1$	
	$e_1,\ e_4 + ae_3 + be_2$	$0,\ \partial_1,\ \partial_2,\ -a\partial_2 - (b + x_3)\partial_1,\ \partial_3$	e_1, e_4
	$e_2 + ae_1,\ e_3 + be_1$	$\partial_1,\ -a\partial_1,\ -(b + x_3)\partial_1,\ \partial_2,\ \partial_3 - ax_2\partial_1$	
	$e_2 + ae_1,\ e_4 + be_3 + ce_1;\ a \neq b$	$\partial_1,\ -a\partial_1,\ \partial_2,\ -b\partial_2 + (-c + (a - b)x_3)\partial_1,\ \partial_3 + x_2\partial_1$	
	$e_3 + ae_2 + be_1,\ e_4 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ -a\partial_2 - (b + x_3)\partial_1,\ -d\partial_1 - (c + x_3)\partial_2,\ \partial_3$	e_3, e_4
	$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ -b\partial_2 - a\partial_3 + (-c + x_3)\partial_1$	e_1, e_5
	$e_2 + ae_1,\ e_5 + be_4 + ce_3 + de_1$	$\partial_1,\ -a\partial_1,\ \partial_2,\ \partial_3,\ -c\partial_2 - b\partial_3 + (-d + x_2 - ax_3)\partial_1$	
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$0,\ 0,\ 0,\ \partial_1,\ \partial_2$	
	$e_1,\ e_2,\ e_4 + ae_3$	$0,\ 0,\ \partial_1,\ -a\partial_1,\ \partial_2$	e_1, e_3, e_4
	$e_1,\ e_3 + ae_2,\ e_4 + be_2$	$0,\ \partial_1,\ -a\partial_1,\ (-b - x_2)\partial_1,\ \partial_2$	e_1, e_2, e_3
	$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$\partial_1,\ -a\partial_1,\ (-b - x_2)\partial_1,\ (-c + ax_2)\partial_1,\ \partial_2$	
	$e_1,\ e_2,\ e_5 + ae_4 + be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ -b\partial_1 - a\partial_2$	e_1, e_2, e_5
$\mathfrak{g}_{3,1}$	$e_1,\ e_3 + ae_2,\ e_5 + be_4 + ce_2$	$0,\ \partial_1,\ -a\partial_1,\ \partial_2,\ -b\partial_2 + (-c + x_2)\partial_1$	e_1, e_3, e_5
	$e_2 + ae_1,\ e_4 + ae_3 + be_1,$ $e_5 + ce_3 + de_1$	$\partial_1,\ -a\partial_1,\ \partial_2,\ -b\partial_1 - a\partial_2,\ -c\partial_2 + (-d + x_2)\partial_1$	
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	e_1, e_2, e_3, e_4
$\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_5 + ae_4$	$0,\ 0,\ 0,\ \partial_1,\ -a\partial_1$	e_1, e_2, e_3, e_5
	$e_1,\ e_2,\ e_4 + ae_3,\ e_5 + be_3$	$0,\ 0,\ \partial_1,\ -a\partial_1,\ -b\partial_1$	

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$ class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + x_2\partial_1 + x_3\partial_2 + x_4\partial_3$	0
$\mathfrak{g}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_2\partial_1 + x_3\partial_2$	e_1
	$e_2 + ae_1$	$\partial_1,\ (-a - x_4)\partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_3\partial_2 - x_2(a + x_4)\partial_1$	e_2
	$e_3 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ -(a + x_4)\partial_2 - (b + ax_4 + \frac{1}{2}x_4^2)\partial_1,\ \partial_3,$ $\partial_4 - x_3(a + x_4)\partial_2 + (x_2 - bx_3 - ax_3x_4 - \frac{1}{2}x_3x_4^2)\partial_1$	e_3
	$e_4 + ae_3 + be_2 + ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -(a + x_4)\partial_3 - (b + ax_4 + \frac{1}{2}x_4^2)\partial_2 - (c + bx_4 + \frac{1}{2}ax_4^2 + \frac{1}{6}x_4^3)\partial_1,$ $\partial_4 + x_2\partial_1 + x_3\partial_2$	e_4
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ -a\partial_4 + (-d + x_2)\partial_1 + (-c + x_3)\partial_2 + (-b + x_4)\partial_3$	e_5
$2\mathfrak{g}_1$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3 + x_2\partial_1$	$e_1,\ e_2$
	$e_1,\ e_3 + ae_2$	$0,\ \partial_1,\ (-a - x_3)\partial_1,\ \partial_2,\ \partial_3 - x_2(a + x_3)\partial_1$	$e_1,\ e_3$
	$e_1,\ e_4 + ae_3 + be_2$	$0,\ \partial_1,\ \partial_2,\ -(a + x_3)\partial_2 - (b + ax_3 + \frac{1}{2}x_3^2)\partial_1,\ \partial_3 + x_2\partial_1$	$e_1,\ e_4$
	$e_2 + ae_1,\ e_3 + be_1$	$\partial_1,\ -(a + x_3)\partial_1,\ (-b + ax_3 + \frac{1}{2}x_3^2)\partial_1,\ \partial_2,\ \partial_3 + (-bx_2 + ax_2x_3 + \frac{1}{2}x_2x_3^2)\partial_1$	$e_2,\ e_3$
	$e_2 + ae_1,\ e_4 + be_3 + ce_1$	$\partial_1,\ -(a + x_3)\partial_1,\ \partial_2,\ -(b + x_3)\partial_2 + (-c + abx_3 + \frac{1}{2}(a + b)x_3^2 + \frac{1}{3}x_3^3)\partial_1,$ $\partial_3 - x_2(a + x_3)\partial_1$	$e_2,\ e_4 + \eta_a e_3$
	$e_3 + ae_2 + be_1,\ e_4 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ -(a + x_3)\partial_2 - (b + ax_3 + \frac{1}{2}x_3^2)\partial_1,$ $(-c + ax_3 + \frac{1}{2}x_3^2)\partial_2 + (-d + (b - c)x_3 + ax_3^2 + \frac{1}{3}x_3^3)\partial_1,\ \partial_3 + x_2\partial_1$	$e_3 + \sigma_{-a^2+b-c}e_1,$ e_4
	$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ -a\partial_3 + (-c + x_2)\partial_1 + (-b + x_3)\partial_2$	$e_1,\ e_5$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$0,\ 0,\ 0,\ \partial_1,\ \partial_2$	$e_1,\ e_2,\ e_3$
	$e_1,\ e_3 + ae_2,\ e_4 + be_2$	$0,\ \partial_1,\ (-a - x_2)\partial_1,\ (-b + ax_2 + \frac{1}{2}x_2^2)\partial_1,\ \partial_2$	$e_1,\ e_3,\ e_4$
	$e_1,\ e_2,\ e_4 + ae_3$	$0,\ 0,\ \partial_1,\ -(a + x_2)\partial_1,\ \partial_2$	$e_1,\ e_2,\ e_4$
	$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$\partial_1,\ -(a + x_2)\partial_1,\ (-b + ax_2 + \frac{1}{2}x_2)\partial_1,\ (-c + bx_2 - \frac{1}{2}ax_2^2 - \frac{1}{6}x_2^3)\partial_1,\ \partial_2$	$e_2,\ e_3,\ e_4$
$\mathfrak{g}_{3,1}$	$e_1,\ e_2,\ e_5 + ae_4 + be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ -a\partial_2 + (-b + x_2)\partial_1$	$e_1,\ e_2,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}_{4,1}$	$e_1,\ e_2,\ e_3,\ e_5 + ae_4$	$0,\ 0,\ 0,\ \partial_1,\ -a\partial_1$	$e_1,\ e_2,\ e_3,\ e_5$

Table A.4. Subalgebras and realizations of $\mathfrak{g}_{5,2}$.

Table A.5. Realizations of $\mathfrak{g}_{5,7}^{\alpha,\beta,\gamma}$, $\alpha\beta\gamma\neq 0$; $\tilde{a}(1-\alpha)=0$, $\tilde{b}(1-\beta)=0$, $\tilde{c}(1-\gamma)=0$, $\hat{a}(\beta-\alpha)=0$, $\hat{b}(\gamma-\alpha)=0$, $\bar{a}(\gamma-\beta)=0$.

Type	Subalgebra	Realization	Aut $\mathfrak{g}$ class ($\alpha=\beta=\gamma=1$)	Aut $\mathfrak{g}$ class ($\alpha=\beta=1$, $\gamma\neq 1$)
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5+x_1\partial_1+\alpha x_2\partial_2+\beta x_3\partial_3+\gamma x_4\partial_4$	0	0
$\mathfrak{g}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4+\alpha x_1\partial_1+\beta x_2\partial_2+\gamma x_3\partial_3$	e_1	e_1
	e_2+ae_1	$\partial_1,\ -a\mathrm{e}^{(1-\alpha)x_4}\partial_1,\ \partial_2,\ \partial_3,\ \partial_4+x_1\partial_1+\beta x_2\partial_2+\gamma x_3\partial_3$		
	$e_3+ae_2+be_1$	$\partial_1,\ \partial_2,\ -b\mathrm{e}^{(1-\beta)x_4}\partial_1-a\mathrm{e}^{(\alpha-\beta)x_4}\partial_2,\ \partial_3,$ $\partial_4+x_1\partial_1+\alpha x_2\partial_2+\gamma x_3\partial_3$		
	$e_4+ae_3+be_2+ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -c\mathrm{e}^{(1-\gamma)x_4}\partial_1-b\mathrm{e}^{(\alpha-\gamma)x_4}\partial_2-a\mathrm{e}^{(\beta-\gamma)x_4}\partial_3,$ $\partial_4+x_1\partial_1+\alpha x_2\partial_2+\beta x_3\partial_3$		$e_4+\eta_{a,b,c}e_1$
	$e_5+ae_3+be_3+ce_2+de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,$ $(-d+x_1)\partial_1+(-c+\alpha x_2)\partial_2+(-b+\beta x_3)\partial_3+(-a+\gamma x_4)\partial_4$	e_5	e_5
$2\mathfrak{g}_1$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3+\beta x_1\partial_1+\gamma x_2\partial_2$	$e_1,\ e_2$	$e_1,\ e_2$
	$e_1,\ e_3+ae_2$	$0,\ \partial_1,\ -a\mathrm{e}^{(\alpha-\beta)x_3}\partial_1,\ \partial_2,\ \partial_3+\alpha x_1\partial_1+\gamma x_2\partial_2$		
	$e_1,\ e_4+ae_3+be_2$	$0,\ \partial_1,\ \partial_2,\ -b\mathrm{e}^{(\alpha-\gamma)x_3}\partial_1-a\mathrm{e}^{(\beta-\gamma)x_3}\partial_2,\ \partial_3+\alpha x_1\partial_1+\beta x_2\partial_2$		$e_1,\ e_4+\eta_{a,b}e_2$
	$e_2+ae_1,\ e_3+be_1$	$\partial_1,\ -a\mathrm{e}^{(1-\alpha)x_3}\partial_1,\ -b\mathrm{e}^{(1-\beta)x_3}\partial_1,\ \partial_2,\ \partial_3+x_1\partial_1+\gamma x_2\partial_2$		$e_1,\ e_2$
	$e_2+ae_1,\ e_4+be_3+ce_1$	$\partial_1,\ -a\mathrm{e}^{(1-\alpha)x_3}\partial_1,\ \partial_2,\ -c\mathrm{e}^{(1-\gamma)x_3}\partial_1-b\mathrm{e}^{(\beta-\gamma)x_3}\partial_2,$ $\partial_3+x_1\partial_1+\beta x_2\partial_2$		$e_1,\ e_2+\eta_{b,c}e_2$
	$e_3+ae_2+be_1,$ $e_4+ce_2+de_1$	$\partial_1,\ \partial_2,\ -b\mathrm{e}^{(1-\beta)x_3}\partial_1-a\mathrm{e}^{(\alpha-\beta)x_3}\partial_2,$ $-d\mathrm{e}^{(1-\gamma)x_3}\partial_1-c\mathrm{e}^{(\alpha-\gamma)x_3}\partial_2,\ \partial_3+x_1\partial_1+\alpha x_2\partial_2$		$e_1,\ e_4+\eta_{c,d}\bar{\eta}_b e_2$
$\mathfrak{g}_2$	$e_1,\ e_5+ae_4+be_3+ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ (-c+\alpha x_1)\partial_1+(-b+\beta x_2)\partial_2+(-a+\gamma x_3)\partial_3$	$e_1,\ e_5$	$e_1,\ e_5$
	$e_2+\tilde{a}e_1,$ $e_5+be_4+ce_3+de_1$	$\partial_1,\ -\tilde{a}\partial_1,\ \partial_2,\ \partial_3,\ (-c+\beta x_2)\partial_2+(-b+\gamma x_3)\partial_3+(-d+x_1)\partial_1$		
	$e_3+\hat{a}e_2+\tilde{b}e_1,$ $e_5+ce_4+de_2+ee_1$	$\partial_1,\ \partial_2,\ -\tilde{b}\partial_1-\hat{a}\partial_2,\ \partial_3,$ $(-c+\gamma x_3)\partial_3+(-e+x_1)\partial_1+(-d+\alpha x_2)\partial_2$		
	$e_4+\bar{a}e_3+\hat{b}e_2+\tilde{c}e_1,$ $e_5+de_3+ee_2+fe_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -\tilde{c}\partial_1-\bar{b}\partial_2-\hat{a}\partial_3,$ $(-f+x_1)\partial_1+(-d+\beta x_3)\partial_3+(-e+\alpha x_2)\partial_2$		$e_4,\ e_5$

Table A.6. Realizations of $\mathfrak{g}_{5,7}^{\alpha,\beta,\gamma}$, $\alpha\beta\gamma \neq 0$; $\tilde{a}(1-\alpha)=0$, $\tilde{b}(1-\beta)=0$, $\tilde{c}(1-\gamma)=0$, $\hat{a}(\beta-\alpha)=0$, $\hat{b}(\gamma-\alpha)=0$, $\bar{a}(\gamma-\beta)=0$.

Type	Subalgebra		Aut $\mathfrak{g}$ class ($\alpha=\beta=\gamma=1$)	Aut $\mathfrak{g}$ class ($\alpha=\beta=1$, $\gamma\neq 1$)
$3\mathfrak{g}_1$	$e_1,\, e_2,\, e_3$	$0,\, 0,\, 0,\, \partial_1,\, \partial_2+\gamma x_1\partial_1$	$e_1,\, e_2,\, e_3$	$e_1,\, e_2,\, e_3$
	$e_1,\, e_2,\, e_4+ae_3$	$0,\, 0,\, \partial_1,\, -a\mathrm{e}^{(\beta-\gamma)x_2}\partial_1,\, \partial_2+\beta x_1\partial_1$		$e_1,\, e_2,\, e_4+\eta_a e_3$
	$e_1,\, e_3+ae_2,\, e_4+be_2$	$0,\, \partial_1,\, -a\mathrm{e}^{(\alpha-\beta)x_2}\partial_1,\, -b\mathrm{e}^{(\alpha-\gamma)x_2}\partial_1,\, \partial_2+\alpha x_1\partial_1$		$e_1,\, e_2,\, e_4+\eta_b e_3$
	$e_2+ae_1,\, e_3+be_1,\, e_4+ce_1$	$\partial_1,\, -a\mathrm{e}^{(1-\alpha)x_2}\partial_1,\, -b\mathrm{e}^{(1-\beta)x_2}\partial_1,\, -c\mathrm{e}^{(1-\gamma)x_2}\partial_1,\, \partial_2+x_1\partial_1$		$e_1,\, e_2,\, e_4+\eta_c e_3$
$\mathfrak{g}_{3,4}^{\alpha}$	$e_1,\, e_2,\, e_5+ae_4+be_3$	$0,\, 0,\, \partial_1,\, \partial_2,\, (-b+\beta x_1)\partial_1+(-a+\gamma x_2)\partial_2$	$e_1,\, e_2,\, e_5$	$e_1,\, e_2,\, e_5$
$\mathfrak{g}_{3,4}^{\beta}$	$e_1,\, e_3+\hat{a}e_2,\, e_5+be_4+ce_2$	$0,\, \partial_1,\, -\hat{a}\partial_1,\, \partial_2,\, (-b+\gamma x_2)\partial_2+(-c+\alpha x_1)\partial_1$		
$\mathfrak{g}_{3,4}^{\gamma}$	$e_1,\, e_4+\bar{a}e_3+\hat{b}e_2,$ $e_5+ce_3+de_2$	$0,\, \partial_1,\, \partial_2,\, -\hat{b}\partial_1-\bar{a}\partial_2,\, (-c+\beta x_2)\partial_2+(-d+\alpha x_1)\partial_1$		$e_1,\, e_4,\, e_5$
$\mathfrak{g}_{3,4}^{\beta/\alpha}$	$e_2+\tilde{a}e_1,\, e_3+\tilde{b}e_1,$ $e_5+ce_4+de_1$	$\partial_1,\, -\tilde{a}\partial_1,\, -\tilde{b}\partial_1,\, \partial_2,\, (-c+\gamma x_2)\partial_2+(-d+x_1)\partial_1$		$e_1,\, e_2,\, e_5$
$\mathfrak{g}_{3,4}^{\gamma/\alpha}$	$e_2+\tilde{a}e_1,\, e_4+\bar{a}e_3+\tilde{c}e_1,$ $e_5+de_3+ee_1$	$\partial_1,\, -\tilde{a}\partial_1,\, \partial_2,\, -\tilde{c}\partial_1-\bar{a}\partial_2,\, (-e+x_1)\partial_1+(-d+\beta x_2)\partial_2$		$e_1,\, e_4,\, e_5$
$\mathfrak{g}_{3,4}^{\gamma/\beta}$	$e_3+\hat{a}e_2+\tilde{b}e_1,$ $e_4+\hat{b}e_2+\tilde{c}e_1,$ $e_5+de_2+ee_1$	$\partial_1,\, \partial_2,\, -\tilde{b}\partial_1-\hat{a}\partial_2,\, -\tilde{c}\partial_1-\hat{b}\partial_2,\, (-e+x_1)\partial_1+(-d+\alpha x_2)\partial_2$		
$4\mathfrak{g}_1$	$e_1,\, e_2,\, e_3,\, e_4$	$0,\, 0,\, 0,\, 0,\, \partial_1$	$e_1,\, e_2,\, e_3,\, e_4$	$e_1,\, e_2,\, e_3,\, e_4$
$\mathfrak{g}_{4,5}^{\alpha,\beta}$	$e_1,\, e_2,\, e_3,\, e_5+ae_4$	$0,\, 0,\, 0,\, \partial_1,\, (-a+\gamma x_1)\partial_1$	$e_1,\, e_2,\, e_3,\, e_5$	$e_1,\, e_2,\, e_3,\, e_5$
$\mathfrak{g}_{4,5}^{\alpha,\gamma}$	$e_1,\, e_2,\, e_4+\bar{a}e_3,\, e_5+be_3$	$0,\, 0,\, \partial_1,\, -\bar{a}\partial_1,\, (-b+\beta x_1)\partial_1$		$e_1,\, e_2,\, e_4,\, e_5$
$\mathfrak{g}_{4,5}^{\beta,\gamma}$	$e_1,\, e_3+\hat{a}e_2,\, e_4+\hat{b}e_2,$ e_5+ce_2	$0,\, \partial_1,\, -\hat{a}\partial_1,\, -\hat{b}\partial_1,\, (-c+\alpha x_1)\partial_1$		
$\mathfrak{g}_{4,5}^{\beta/\alpha,\gamma/\alpha}$	$e_2+\tilde{a}e_1,\, e_3+\tilde{b}e_1,\, e_4+\tilde{c}e_1,$ e_5+de_1	$\partial_1,\, -\tilde{a}\partial_1,\, -\tilde{b}\partial_1,\, -\tilde{c}\partial_1,\, (-d+x_1)\partial_1$		

Subalgebra	Aut $\mathfrak{g}$ class ($\alpha = 1$, $\beta = \gamma = -1$)	Aut $\mathfrak{g}$ class ($\alpha = 1$, $\beta = \gamma \neq \pm 1$)	Aut $\mathfrak{g}$ class ($\alpha = 1$, $\beta \neq \gamma \neq 1$)	Aut $\mathfrak{g}$ class ($-1 = \gamma$, $\beta = -\alpha \neq 1$)	Aut $\mathfrak{g}$ class ($-1 < \gamma < \beta < \alpha < 1$)
0	0	0	0	0	0
e_1	e_1	e_1	e_1	e_1	e_1
$e_2 + ae_1$				$e_2 + \eta_a e_1$	$e_2 + \eta_a e_1$
$e_3 + ae_2 + be_1$	$e_1 + \eta_{a,b} e_3$	$e_3 + \eta_{a,b} e_1$	$e_3 + \eta_{a,b} e_1$	$e_2 + \eta_a e_3 + \eta_b e_4$	$e_3 + \eta_a e_2 + \eta_b e_1$
$e_4 + ae_3 + be_2 + ce_1$	$e_1 + \eta_{b,c} e_3$	$e_3 + \eta_{b,c} e_1$	$e_4 + \eta_a e_3 + \eta_{b,c} e_1$	$e_1 + \eta_a e_2 + \eta_b e_3 + \eta_c e_4$	$e_4 + \eta_a e_3 + \eta_b e_2 + \eta_c e_3$
$e_5 + ae_4 + be_3 + ce_2 + de_1$	e_5	e_5	e_5	e_5	e_5
e_1, e_2	e_1, e_2	e_1, e_2	e_1, e_2	e_1, e_2	e_1, e_2
$e_1, e_3 + ae_2$	$e_1, e_4 + \eta_a e_2$	$e_1, e_4 + \eta_a e_2$	$e_1, e_3 + \eta_a e_2$	$e_1, e_3 + \eta_a e_2$	$e_1, e_3 + \eta_a e_2$
$e_1, e_4 + ae_3 + be_2$	$e_1, e_4 + \eta_b e_2$	$e_1, e_4 + \eta_b e_2$	$e_1, e_4 + \eta_a e_3 + \eta_b e_2$	$e_1, e_4 + \eta_a e_3 + \eta_b e_2$	$e_1, e_4 + \eta_a e_3 + \eta_b e_2$
$e_2 + ae_1, e_3 + be_1$	$e_1, e_4 + \eta_b e_2$	$e_1, e_4 + \eta_b e_2$	$e_1, e_3 + \eta_b e_2$	$e_2 + \eta_a e_1, e_3 + \eta_b e_1$	$e_2 + \eta_a e_1, e_3 + \eta_b e_1$
$e_2 + ae_1, e_4 + be_3 + ce_1$	$e_1, e_4 + \eta_c e_2$	$e_1, e_4 + \eta_c e_2$	$e_1, e_4 + \eta_b e_3 + \eta_c e_2$	$e_3 + \eta_a e_4, e_1 + \eta_b e_2 + \eta_c e_4$	$e_2 + \eta_a e_1, e_4 + \eta_b e_3 + \eta_c e_1$
$e_3 + ae_2 + be_1$, $e_4 + ce_2 + de_1$	$e_1 + \eta_{ad-bc} e_3$, $e_2 + \eta_{a,b,c,d} e_4$	$e_3 + \eta_{a,b,c,d} e_1$, $e_4 + \eta_{ad-bc} e_2$	$e_3 + \eta_{a,b} e_1$, $e_4 + \eta_{ad-bc} e_2 + \eta_{c,d} e_1$	$e_2 + \eta_a e_3 + \eta_b e_4$, $e_1 + \eta_c e_3 + \eta_d e_4$ if $abd = 0$; $e_2 + e_3 + e_4$, $e_1 + \frac{bc}{ad} e_3 + e_4$ if $abd \neq 0$	$e_3 + \eta_a e_2 + \eta_b e_1$, $e_4 + \eta_c e_2 + \eta_d e_1$ if $abd = 0$; $e_3 + e_2 + e_1$, $e_4 + \frac{bc}{ad} e_2 + e_1$ if $abd \neq 0$
$e_1, e_5 + ae_4 + be_3 + ce_2$	e_1, e_5	e_1, e_5	e_1, e_5	e_1, e_5	e_1, e_5
$e_2 + \tilde{a} e_1$, $e_5 + be_4 + ce_3 + de_1$				e_2, e_5	e_2, e_5
$e_3 + \hat{a} e_2 + \tilde{b} e_1$, $e_5 + ce_4 + de_2 + ee_1$	e_3, e_5	e_3, e_5	e_3, e_5	e_2, e_5	e_3, e_5
$e_4 + \bar{a} e_3 + \hat{b} e_2 + \tilde{c} e_1$, $e_5 + de_3 + ee_2 + fe_1$			e_4, e_5	e_1, e_5	e_4, e_5

Table A.7. Subalgebras of $\mathfrak{g}_{5,7}^{\alpha,\beta,\gamma}$, $\alpha\beta\gamma \neq 0$; $\tilde{a}(1-\alpha) = 0$, $\tilde{b}(1-\beta) = 0$, $\tilde{c}(1-\gamma) = 0$, $\hat{a}(\beta-\alpha) = 0$, $\hat{b}(\gamma-\alpha) = 0$, $\bar{a}(\gamma-\beta) = 0$.

Table A.8. Subalgebras of $\mathfrak{g}_{5,7}^{\alpha,\beta,\gamma}$, $\alpha\beta\gamma\neq0$; $\tilde a(1-\alpha)=0$, $\tilde b(1-\beta)=0$, $\tilde c(1-\gamma)=0$, $\hat a(\beta-\alpha)=0$, $\hat b(\gamma-\alpha)=0$, $\bar a(\gamma-\beta)=0$.

Subalgebra	Aut $\mathfrak g$ class ($\alpha=1$, $\beta=\gamma=-1$)	Aut $\mathfrak g$ class ($\alpha=1$, $\beta=\gamma\neq\pm1$)	Aut $\mathfrak g$ class ($\alpha=1$, $\beta\neq\gamma\neq1$)	Aut $\mathfrak g$ class ($-1=\gamma$, $\beta=-\alpha\neq1$)	Aut $\mathfrak g$ class ($-1<\gamma<\beta<\alpha<1$)
$e_1,\,e_2,\,e_3$	$e_1,\,e_2,\,e_3$	$e_1,\,e_2,\,e_3$	$e_1,\,e_2,\,e_3$	$e_1,\,e_2,\,e_3$	$e_1,\,e_2,\,e_3$
$e_1,\,e_2,\,e_4+ae_3$			$e_1,\,e_2,\,e_4+\eta_ae_3$	$e_1,\,e_2,\,e_4+\eta_ae_3$	$e_1,\,e_2,\,e_4+\eta_ae_3$
$e_1,\,e_3+ae_2,\,e_4+be_2$	$e_1,\,e_2+\eta_{a,b}e_4,\,e_3$	$e_1,\,e_3,\,e_4+\eta_{a,b}e_2$	$e_1,\,e_3+\eta_ae_2,$ $e_4+\eta_be_2$	$e_1,\,e_2+\eta_ae_3,$ $e_4+\eta_be_2$	$e_1,\,e_3+\eta_ae_2,$ $e_4+\eta_be_2$
$e_2+ae_1,\,e_3+be_1,\,e_4+ce_1$	$e_1,\,e_2+\eta_{b,c}e_4,\,e_3$	$e_1,\,e_3,\,e_4+\eta_{b,c}e_2$	$e_1,\,e_3+\eta_be_2,$ $e_4+\eta_ce_2$	$e_2+\eta_ae_1,\,e_3+\eta_be_1,$ $e_4+\eta_ce_1$	$e_2+\eta_ae_1,\,e_3+\eta_be_1,$ $e_4+\eta_ce_1$
$e_1,\,e_2,\,e_5+ae_4+be_3$	$e_1,\,e_2,\,e_5$	$e_1,\,e_2,\,e_5$	$e_1,\,e_2,\,e_5$	$e_1,\,e_2,\,e_5$	$e_1,\,e_2,\,e_5$
$e_1,\,e_3+\hat ae_2,\,e_5+be_4+ce_2$	$e_1,\,e_3,\,e_5$	$e_1,\,e_3,\,e_5$	$e_1,\,e_3,\,e_5$	$e_1,\,e_3,\,e_5$	$e_1,\,e_3,\,e_5$
$e_1,\,e_4+\bar ae_3+\hat be_2,$ $e_5+ce_3+de_2$			$e_1,\,e_4,\,e_5$	$e_1,\,e_4,\,e_5$	$e_1,\,e_4,\,e_5$
$e_2+\tilde ae_1,\,e_3+\tilde be_1,$ $e_5+ce_4+de_1$			$e_1,\,e_3,\,e_5$	$e_2,\,e_3,\,e_5$	$e_2,\,e_3,\,e_5$
$e_2+\tilde ae_1,\,e_4+\bar ae_3+\tilde ce_1,$ $e_5+de_3+ee_1$	$e_1,\,e_2,\,e_5$	$e_3,\,e_4,\,e_5$	$e_1,\,e_4,\,e_5$	$e_2,\,e_4,\,e_5$	$e_2,\,e_4,\,e_5$
$e_3+\hat ae_2+\tilde be_1,$ $e_4+\hat be_2+\tilde ce_1,$ $e_5+de_2+ee_1$			$e_3,\,e_4,\,e_5$	$e_1,\,e_2,\,e_5$	$e_3,\,e_4,\,e_5$
$e_1,\,e_2,\,e_3,\,e_4$	$e_1,\,e_2,\,e_3,\,e_4$	$e_1,\,e_2,\,e_3,\,e_4$	$e_1,\,e_2,\,e_3,\,e_4$	$e_1,\,e_2,\,e_3,\,e_4$	$e_1,\,e_2,\,e_3,\,e_4$
$e_1,\,e_2,\,e_3,\,e_5+ae_4$	$e_1,\,e_2,\,e_3,\,e_5$	$e_1,\,e_2,\,e_3,\,e_5$	$e_1,\,e_2,\,e_3,\,e_5$	$e_1,\,e_2,\,e_3,\,e_5$	$e_1,\,e_2,\,e_3,\,e_5$
$e_1,\,e_2,\,e_4+\bar ae_3,\,e_5+be_3$			$e_1,\,e_2,\,e_4,\,e_5$	$e_1,\,e_2,\,e_4,\,e_5$	$e_1,\,e_2,\,e_4,\,e_5$
$e_1,\,e_3+\hat ae_2,\,e_4+\hat be_2,$ e_5+ce_2		$e_1,\,e_3,\,e_4,\,e_5$	$e_1,\,e_3,\,e_4,\,e_5$	$e_1,\,e_2,\,e_4,\,e_5$	$e_1,\,e_3,\,e_4,\,e_5$
$e_2+\tilde ae_1,\,e_3+\tilde be_1,\,e_4+\tilde ce_1,$ e_5+de_1				$e_1,\,e_2,\,e_3,\,e_5$	$e_2,\,e_3,\,e_4,\,e_5$

Table A.9. Realizaions of $\mathfrak{g}_{5,8}^{\gamma}$, $\gamma \neq 0$.

Type	Subalgebra	Realizations
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + x_2\partial_1 + x_3\partial_3 + \gamma x_4\partial_4$
$\mathfrak{g}_1$	e_3	$\partial_1,\ \partial_2,\ 0,\ \partial_3,\ \partial_4 + x_2\partial_1 + \gamma x_3\partial_3$
	$e_4 + ae_3$	$\partial_1,\ \partial_2,\ \partial_3,\ -a\mathrm{e}^{(1-\gamma)x_4}\partial_3,\ \partial_4 + x_2\partial_1 + x_3\partial_3$
	$e_1 + ae_3 + be_4$	$-a\mathrm{e}^{x_4}\partial_2 - b\mathrm{e}^{\gamma x_4}\partial_3,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + (-a\mathrm{e}^{x_4}x_1 + x_2)\partial_2 + (-b\mathrm{e}^{\gamma x_4}x_1 + \gamma x_3)\partial_3$
	$e_2 + ae_1 + be_3 + ce_4$	$\partial_1,\ -b\mathrm{e}^{x_4}\partial_2 - c\mathrm{e}^{\gamma x_4}\partial_3 - (a + x_4)\partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_2\partial_2 + \gamma x_3\partial_3$
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ -c\partial_2 + (-d + x_2)\partial_1 + (-b + x_3)\partial_3 + (-a + \gamma x_4)\partial_4$
$2\mathfrak{g}_1$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3 + x_1\partial_1 + \gamma x_2\partial_2$
	$e_1,\ e_3 + ae_2$	$0,\ \partial_1,\ -a\mathrm{e}^{-x_3}\partial_1,\ \partial_2,\ \partial_3 + \gamma x_2\partial_2$
	$e_1,\ e_4 + ae_3 + be_2$	$0,\ \partial_1,\ \partial_2,\ -b\mathrm{e}^{-\gamma x_3}\partial_1 - a\mathrm{e}^{(1-\gamma)x_3}\partial_2,\ \partial_3 + x_2\partial_2$
	$e_2 + ae_1,\ e_3 + be_1$	$\partial_1,\ -(a + x_3)\partial_1,\ -b\mathrm{e}^{-x_3}\partial_1,\ \partial_2,\ \partial_3 + \gamma x_2\partial_2$
	$e_2 + ae_1,\ e_4 + be_3 + ce_1$	$\partial_1,\ -(a + x_3)\partial_1,\ \partial_2,\ -c\mathrm{e}^{-\gamma x_3}\partial_1 - b\mathrm{e}^{(1-\gamma)x_3}\partial_2,\ \partial_3 + x_2\partial_2$
	$e_3 + ae_2 + be_1,\ e_4 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ -a\mathrm{e}^{-x_3}\partial_2 - \mathrm{e}^{-x_3}(b + ax_3)\partial_1,\ -c\mathrm{e}^{-\gamma x_3}\partial_2 - \mathrm{e}^{-\gamma x_3}(d + cx_3)\partial_1,\ \partial_3 + x_2\partial_1$
	$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ -c\partial_1 + (-b + x_2)\partial_2 + (-a + \gamma x_3)\partial_3$
$\mathfrak{g}_2$	$e_3,\ e_5 + ae_4 + be_2 + ce_1$	$\partial_1,\ \partial_2,\ 0,\ \partial_3,\ -b\partial_2 + (-c + x_2)\partial_1 + (-a + \gamma x_3)\partial_3$
	$e_4 + \tilde{c}e_3,\ e_5 + ae_3 + be_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -\tilde{c}\partial_3,\ -b\partial_2 + (-d + x_2)\partial_1 + (-a + x_3)\partial_3$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$0,\ \partial_1,\ -a\mathrm{e}^{-x_2}\partial_1,\ -b\mathrm{e}^{-\gamma x_2}\partial_1,\ \partial_2$
	$e_1,\ e_2,\ e_4 + ae_3$	$0,\ 0,\ 0,\ \partial_1,\ \partial_2 + \gamma x_1\partial_1$
	$e_1,\ e_3 + ae_2,\ e_4 + be_2$	$0,\ 0,\ \partial_1,\ -a\mathrm{e}^{(1-\gamma)x_2}\partial_1,\ \partial_2 + x_1\partial_1$
	$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$\partial_1,\ -(a + x_2)\partial_1,\ -b\mathrm{e}^{-x_2}\partial_1,\ -c\mathrm{e}^{-\gamma x_2}\partial_1,\ \partial_2$
$\mathfrak{g}_{3,1}$	$e_1,\ e_2,\ e_5 + ae_4 + be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ (-b + x_1)\partial_1 + (-a + \gamma x_2)\partial_2$
$\mathfrak{g}_1 \oplus \mathfrak{g}_2$	$e_1,\ e_3,\ e_5 + ae_4 + be_2$	$0,\ \partial_1,\ 0,\ \partial_2,\ -b\partial_1 + (-a + \gamma x_2)\partial_2$
	$e_1,\ e_4 + \tilde{c}e_3,\ e_5 + ae_3 + be_2$	$0,\ \partial_1,\ \partial_2,\ -\tilde{c}\partial_2,\ -b\partial_1 + (-a + x_2)\partial_2$
$\mathfrak{g}_{3,4}^{\gamma}$	$e_3,\ e_4,\ e_5 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ 0,\ 0,\ -a\partial_2 + (-b + x_2)\partial_1$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$
$\mathfrak{g}_{4,3}$	$e_1,\ e_2,\ e_3,\ e_5 + ae_4$	$0,\ 0,\ 0,\ \partial_1,\ (-a + \gamma x_1)\partial_1$
	$e_1,\ e_2,\ e_4 + \tilde{c}e_3,\ e_5 + ae_3$	$0,\ 0,\ \partial_1,\ -\tilde{c}\partial_1,\ (-a + x_1)\partial_1$
$\mathfrak{g}_1 \oplus \mathfrak{g}_{3,4}^{\gamma}$	$e_1,\ e_3,\ e_4,\ e_5 + ce_2$	$0,\ \partial_1,\ 0,\ 0,\ -a\partial_1$

Table A.10. Subalgebras of $\mathfrak{g}^{\gamma}_{5,8}$, $\gamma \neq 0$; $\tilde{c}(1-\gamma)=0$.

Subalgebra	Aut $\mathfrak{g}$ class ($\gamma = 1$)	Aut $\mathfrak{g}$ class ($\gamma = -1$)	Aut $\mathfrak{g}$ class ($\gamma \neq 1$)
0	0	0	0
e_3	e_3	e_3	e_3
$e_4 + ae_3$		$e_3 + \eta_a e_4$	$e_4 + \eta_a e_3$
$e_1 + ae_3 + be_4$	$e_1 + \eta_{a,b} e_3$	$e_1 + \eta_{a,b} e_3 + \eta_{ab} e_4$	$e_1 + \eta_a e_3 + \eta_b e_4$
$e_2 + ae_1 + be_3 + ce_4$	$e_2 + \eta_{b,c} e_3$	$e_2 + \eta_{b,c} e_3 + \eta_{bc} e_4$	$e_2 + \eta_b e_3 + \eta_c e_4$
$e_5 + ae_4 + be_3 + ce_2 + de_1$	e_5	e_5	e_5
$e_1,\ e_2$	$e_1,\ e_2$	$e_1,\ e_2$	$e_1,\ e_2$
$e_1,\ e_3 + ae_2$	$e_1,\ e_3 + \eta_a e_2$	$e_1,\ e_3 + \eta_a e_2$	$e_1,\ e_3 + \eta_a e_2$
$e_1,\ e_4 + ae_3 + be_2$	$e_1,\ e_3 + \eta_b e_2$	$e_1,\ e_4 + \eta_a e_3 + \eta_b e_2$	$e_1,\ e_4 + \eta_a e_3 + \eta_b e_2$
$e_2 + ae_1,\ e_3 + be_1$	$e_2,\ e_3 + \eta_b e_1$	$e_2,\ e_3 + \eta_b e_1$	$e_2,\ e_3 + \eta_b e_1$
$e_2 + ae_1,\ e_4 + be_3 + ce_1$	$e_2,\ e_3 + \eta_c e_1$	$e_2,\ e_4 + \eta_b e_3 + \eta_c e_1$	$e_2,\ e_4 + \eta_b e_3 + \eta_c e_1$
$e_3 + ae_2 + be_1,\ e_4 + ce_2 + de_1$	$e_3 + \eta_{a,c} e_2 + \bar{\eta}_{ac}\eta_{b,d} e_1,$ $e_4 + \eta_{bc-ad} e_1$	$e_3 + \eta_a \bar{\eta}_d e_2 + \bar{\eta}_a(\eta_b \bar{\eta}_c + \eta_c) e_1,$ $e_4 + \bar{\eta}_{a,c}\eta_{bd} e_1$ if $ac = 0$ $e_3 + e_2 + \left\lvert \frac{bc-ad}{ac} \right\rvert e_1,\ e_4 + e_2$ if $ac \neq 0$	$e_3 + \eta_a e_2 + \bar{\eta}_a \eta_b e_1,\ e_4 + \eta_c e_2 + \bar{\eta}_c \eta_d e_1$ if $ac = 0$ $e_3 + e_2 + \frac{bc-ad}{ac} e_1,\ e_4 + e_2$ if $ac \neq 0$
$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$e_1,\ e_5$	$e_1,\ e_5$	$e_1,\ e_5$
$e_3,\ e_5 + ae_4 + be_2 + ce_1$	$e_3,\ e_5$	$e_3,\ e_5$	$e_3,\ e_5$
$e_4 + \tilde{c}e_3,\ e_5 + ae_3 + be_2 + de_1$			$e_4,\ e_5$
$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$
$e_1,\ e_2,\ e_4 + ae_3$		$e_1,\ e_2,\ e_3 + \eta_a e_4$	$e_1,\ e_2,\ e_4 + \eta_a e_3$
$e_1,\ e_3 + ae_2,\ e_4 + be_2$	$e_1,\ e_3 + \eta_{a,b} e_2,\ e_4$	$e_1,\ e_3 + \eta_{a,b} e_2,\ e_4 + \eta_{ab} e_2$	$e_1,\ e_3 + \eta_a e_2,\ e_4 + \eta_b e_2$
$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$e_2,\ e_3,\ e_4 + \eta_{b,c} e_1$	$e_2,\ e_3 + \eta_{b,c} e_1,\ e_4 + \eta_{bc} e_1$	$e_2,\ e_3 + \eta_b e_1,\ e_4 + \eta_c e_1$
$e_1,\ e_2,\ e_5 + ae_4 + be_3$	$e_1,\ e_2,\ e_5$	$e_1,\ e_2,\ e_5$	$e_1,\ e_2,\ e_5$
$e_1,\ e_3,\ e_5 + ae_4 + be_2$	$e_1,\ e_3,\ e_5$	$e_1,\ e_3,\ e_5$	$e_1,\ e_3,\ e_5$
$e_1,\ e_4 + \tilde{c}e_3,\ e_5 + ae_3 + be_2$	$e_1,\ e_4,\ e_5$		$e_1,\ e_4,\ e_5$
$e_3,\ e_4,\ e_5 + ae_2 + be_1$	$e_3,\ e_4,\ e_5$	$e_3,\ e_4,\ e_5$	$e_3,\ e_4,\ e_5$
$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$
$e_1,\ e_2,\ e_3,\ e_5 + ae_4$	$e_1,\ e_2,\ e_3,\ e_5$	$e_1,\ e_2,\ e_3,\ e_5$	$e_1,\ e_2,\ e_3,\ e_5$
$e_1,\ e_2,\ e_4 + \tilde{c}e_3,\ e_5 + ae_3$			$e_1,\ e_2,\ e_4,\ e_5$
$e_1,\ e_3,\ e_4,\ e_5 + ce_2$	$e_1,\ e_3,\ e_4,\ e_5$	$e_1,\ e_3,\ e_4,\ e_5$	$e_1,\ e_3,\ e_4,\ e_5$

Table A.11. Realizations of $\mathfrak{g}_{5,9}^{\beta,\gamma}$, $\beta\gamma \neq 0$; $\tilde{b}(1-\beta)=0$, $\tilde{c}(1-\gamma)=0$, $\bar{a}(\beta-\gamma)=0$.

Type	Subalgebra		Aut $\mathfrak{g}$ class ($\beta=\gamma=1$)
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5+x_2\partial_2+(x_1+x_2)\partial_1+\beta x_3\partial_3+\gamma x_4\partial_4$	0
$\mathfrak{g}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4+x_1\partial_1+\beta x_2\partial_2+\gamma x_3\partial_3$	e_1
	e_4+ae_1	$\partial_1,\ \partial_2,\ \partial_3,\ -a\mathrm{e}^{(1-\gamma)x_4}\partial_1,\ \partial_4+x_2\partial_2+(x_1+x_2)\partial_1+\beta x_3\partial_3$	e_3
	$e_3+ae_1+be_4$	$\partial_1,\ \partial_2,\ -a\mathrm{e}^{(1-\beta)x_4}\partial_1-b\mathrm{e}^{(-\beta+\gamma)x_4}\partial_3,\ \partial_3,\ \partial_4+x_2\partial_2+(x_1+x_2)\partial_1+\gamma x_3\partial_3$	
	$e_2+ae_1+be_3+ce_4$	$\partial_1,\ -b\mathrm{e}^{(-1+\beta)x_4}\partial_2-c\mathrm{e}^{(\gamma-1)x_4}\partial_3-(a+x_4)\partial_1,\ \partial_2,\ \partial_3,\ \partial_4+x_1\partial_1+\beta x_2\partial_2+\gamma x_3\partial_3$	e_2
	$e_5+ae_4+be_3+ce_2+de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ (-c+x_2)\partial_2+(-d+x_1+x_2)\partial_1+(-b+\beta x_3)\partial_3+(-a+\gamma x_4)\partial_4$	e_5
$2\mathfrak{g}_1$	$e_1,\ e_4$	$0,\ \partial_1,\ \partial_2,\ 0,\ \partial_3+x_1\partial_1+\beta x_2\partial_2$	$e_1,\ e_3$
	$e_1,\ e_3+ae_4$	$0,\ \partial_1,\ -a\mathrm{e}^{(-\beta+\gamma)x_3}\partial_2,\ \partial_2,\ \partial_3+x_1\partial_1+\gamma x_2\partial_2$	$e_1,\ e_3$
	$e_1,\ e_2+ae_3+be_4$	$0,\ -a\mathrm{e}^{(-1+\beta)x_3}\partial_1-b\mathrm{e}^{(\gamma-1)x_3}\partial_2,\ \partial_1,\ \partial_2,\ \partial_3+\beta x_1\partial_1+\gamma x_2\partial_2$	$e_1,\ e_2$
	$e_3+ae_1,\ e_4+be_1$	$\partial_1,\ \partial_2,\ -a\mathrm{e}^{(1-\beta)x_3}\partial_1,\ -b\mathrm{e}^{(1-\gamma)x_3}\partial_1,\ \partial_3+x_2\partial_2+(x_1+x_2)\partial_1$	$e_3,\ e_4$
	$e_2+ae_3+be_1,\ e_4+ce_1$	$\partial_1,\ -a\mathrm{e}^{(-1+\beta)x_3}\partial_2-(b+x_3)\partial_1,\ \partial_2,\ -c\mathrm{e}^{(1-\gamma)x_3}\partial_1,\ \partial_3+x_1\partial_1+\beta x_2\partial_2$	$e_2,\ e_3$
	$e_2+ae_4+be_1,\ e_3+ce_4+de_1$	$\partial_1,\ -a\mathrm{e}^{(\gamma-1)x_3}\partial_2-(b+x_3)\partial_1,\ -d\mathrm{e}^{(1-\beta)x_3}\partial_1-c\mathrm{e}^{(-\beta+\gamma)x_3}\partial_2,\ \partial_2,\ \partial_3+x_1\partial_1+\gamma x_2\partial_2$	
$\mathfrak{g}_2$	$e_1,\ e_5+ae_4+be_3+ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ (-c+x_1)\partial_1+(-b+\beta x_2)\partial_2+(-a+\gamma x_3)\partial_3$	$e_1,\ e_5$
	$e_4+\tilde{c}e_1,\ e_5+ae_3+be_2+de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -\tilde{c}\partial_1,\ (-b+x_2)\partial_2+(-a+\beta x_3)\partial_3+(-d+x_1+x_2)\partial_1$	$e_3,\ e_5$
	$e_3+\bar{a}e_4+\tilde{b}e_1,\ e_5+ce_4+de_2+ee_1$	$\partial_1,\ \partial_2,\ -\tilde{b}\partial_1-\bar{a}\partial_3,\ \partial_3,\ (-d+x_2)\partial_2+(-e+x_1+x_2)\partial_1+(-c+\gamma x_3)\partial_3$	
$3\mathfrak{g}_1$	$e_1,\ e_3,\ e_4$	$0,\ \partial_1,\ 0,\ 0,\ \partial_2+x_1\partial_1$	$e_1,\ e_3,\ e_4$
	$e_1,\ e_2+ae_3,\ e_4$	$0,\ -a\mathrm{e}^{(-1+\beta)x_2}\partial_1,\ \partial_1,\ 0,\ \partial_2+\beta x_1\partial_1$	$e_1,\ e_2,\ e_3$
	$e_1,\ e_2+ae_4,\ e_3+be_4$	$0,\ -a\mathrm{e}^{(\gamma-1)x_2}\partial_1,\ -b\mathrm{e}^{(-\beta+\gamma)x_2}\partial_1,\ \partial_1,\ \partial_2+\gamma x_1\partial_1$	
	$e_2+ae_1,\ e_3+be_1,\ e_4+ce_1$	$\partial_1,\ (-a-x_2)\partial_1,\ -b\mathrm{e}^{(1-\beta)x_2}\partial_1,\ -c\mathrm{e}^{(1-\gamma)x_2}\partial_1,\ \partial_2+x_1\partial_1$	$e_2,\ e_3,\ e_4$
$\mathfrak{g}_{3,4}^{\gamma}$	$e_1,\ e_4,\ e_5+ae_2+be_3$	$0,\ \partial_1,\ \partial_2,\ 0,\ (-a+x_1)\partial_1+(-b+\beta x_2)\partial_2$	$e_1,\ e_3,\ e_5$
$\mathfrak{g}_{3,4}^{\beta}$	$e_1,\ e_3+\bar{a}e_4,\ e_5+be_4+ce_2$	$0,\ \partial_1,\ -\bar{a}\partial_2,\ \partial_2,\ (-c+x_1)\partial_1+(-b+\gamma x_2)\partial_2$	
$\mathfrak{g}_{3,2}$	$e_1,\ e_2+\tilde{b}e_3+\tilde{c}e_4,\ e_5+ae_3+de_4$	$0,\ -\tilde{b}\partial_1-\tilde{c}\partial_2,\ \partial_1,\ \partial_2,\ (-a+\beta x_1)\partial_1+(-b+\gamma x_2)\partial_2$	$e_1,\ e_2,\ e_5$
$\mathfrak{g}_{3,4}^{\gamma/\beta}$	$e_3+\tilde{b}e_1,\ e_4+\tilde{c}e_1,\ e_5+ae_2+de_1$	$\partial_1,\ \partial_2,\ -\tilde{b}\partial_1,\ -\tilde{c}\partial_1,\ (-a+x_2)\partial_2+(-d+x_1+x_2)\partial_1$	$e_3,\ e_4,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}_{4,2}^{\beta}$	$e_1,\ e_2+\tilde{c}e_4,\ e_3+\bar{a}e_4,\ e_5+be_4$	$0,\ -\tilde{c}\partial_1,\ -\bar{a}\partial_1,\ \partial_1,\ (-b+\gamma x_1)\partial_1$	$e_1,\ e_2,\ e_3,\ e_5$
$\mathfrak{g}_{4,2}^{\gamma}$	$e_1,\ e_2+\tilde{b}e_3,\ e_4,\ e_5+ae_3$	$0,\ -\tilde{b}\partial_1,\ \partial_1,\ 0,\ (-a+\beta x_1)\partial_1$	
$\mathfrak{g}_{4,5}^{\beta,\gamma}$	$e_1,\ e_3,\ e_4,\ e_5+ae_2$	$0,\ \partial_1,\ 0,\ 0,\ (-a+x_1)\partial_1$	$e_1,\ e_3,\ e_4,\ e_5$

Table A.12. Subalgebras of $\mathfrak{g}_{5,9}^{\beta,\gamma}$, $\beta\gamma \neq 0$; $\tilde{b}(1-\beta)=0$, $\tilde{c}(1-\gamma)=0$, $\bar{a}(\beta-\gamma)=0$.

Subalgebra	Aut $\mathfrak{g}$ class ($\gamma \neq \beta = 1$)	Aut $\mathfrak{g}$ class ($\beta = \gamma \neq 1$)	Aut $\mathfrak{g}$ class ($1 \neq \beta \neq \gamma \neq 1$)
0	0	0	0
e_1	e_1	e_1	e_1
$e_4 + ae_1$	$e_4 + \eta_a e_1$	$e_3 + \eta_a e_1$	$e_4 + \eta_a e_1$
$e_3 + ae_1 + be_4$	$e_3 + \eta_b e_4$	$e_3 + \eta_a e_1$	$e_3 + \eta_a e_1 + \eta_b e_4$
$e_2 + ae_1 + be_3 + ce_4$	$e_2 + \eta_c e_4$	$e_2 + \eta_{b,c} e_3$	$e_2 + \eta_b e_3 + \eta_c e_4$
$e_5 + ae_4 + be_3 + ce_2 + de_1$	e_5	e_5	e_5
$e_1,\ e_4$	$e_1,\ e_4$	$e_1,\ e_3$	$e_1,\ e_4$
$e_1,\ e_3 + ae_4$	$e_1,\ e_3 + \eta_a e_4$	$e_1,\ e_3$	$e_1,\ e_3 + \eta_a e_4$
$e_1,\ e_2 + ae_3 + be_4$	$e_1,\ e_2 + \eta_b e_4$	$e_1,\ e_2 + \eta_{a,b} e_3$	$e_1,\ e_2 + \eta_a e_3 + \eta_b e_4$
$e_3 + ae_1,\ e_4 + be_1$	$e_3,\ e_4 + \eta_b e_1$	$e_3,\ e_4 + \eta_{a,b} e_1$	$e_3 + \eta_a e_1,\ e_4 + \eta_b e_1$
$e_2 + ae_3 + be_1,\ e_4 + ce_1$	$e_2,\ e_4 + \eta_c e_1$	$e_2 + \eta_a e_4,\ e_3 + \eta_c e_1$	$e_2 + \eta_a e_3,\ e_4 + \eta_c e_1$
$e_2 + ae_4 + be_1,\ e_3 + ce_4 + de_1$	$e_2 + \bar{\eta}_c \eta_a e_4,\ e_3 + \eta_c e_4$		$e_2 + \eta_a e_4,\ e_3 + \eta_c e_4 + \eta_d e_1$ if $ad = 0$ $e_2 + e_4,\ e_3 - \frac{c}{da} e_4 + e_1$ if $ad \neq 0$
$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$e_1,\ e_5$	$e_1,\ e_5$	$e_1,\ e_5$
$e_4 + \tilde{c} e_1,\ e_5 + ae_3 + be_2 + de_1$	$e_3,\ e_5$	$e_3,\ e_5$	$e_3,\ e_5$
$e_3 + \bar{a} e_4 + \tilde{b} e_1,\ e_5 + ce_4 + de_2 + ee_1$	$e_4,\ e_5$		$e_2,\ e_5$
$e_1,\ e_3,\ e_4$	$e_1,\ e_3,\ e_4$	$e_1,\ e_3,\ e_4$	$e_1,\ e_3,\ e_4$
$e_1,\ e_2 + ae_3,\ e_4$	$e_1,\ e_2,\ e_4$	$e_1,\ e_2 + \eta_a e_4,\ e_3$	$e_1,\ e_2 + \eta_a e_3,\ e_4$
$e_1,\ e_2 + ae_4,\ e_3 + be_4$	$e_1,\ e_2 + \bar{\eta}_b \eta_a e_4,\ e_3 + \eta_b e_4$	$e_1,\ e_2 + \eta_a e_4,\ e_3$	$e_1,\ e_2 + \eta_a e_4,\ e_3 + \eta_b e_4$
$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$e_2,\ e_3,\ e_4 + \eta_c e_1$	$e_2,\ e_3,\ e_4 + \eta_{b,c} e_1$	$e_2,\ e_3 + \eta_b e_1,\ e_4 + \eta_c e_1$
$e_1,\ e_4,\ e_5 + ae_2 + be_3$	$e_1,\ e_4,\ e_5$	$e_1,\ e_3,\ e_5$	$e_1,\ e_4,\ e_5$
$e_1,\ e_3 + \bar{a} e_4,\ e_5 + be_4 + ce_2$	$e_1,\ e_3,\ e_5$		$e_1,\ e_3,\ e_5$
$e_1,\ e_2 + \tilde{b} e_3 + \tilde{c} e_4,\ e_5 + ae_3 + de_4$	$e_1,\ e_2,\ e_5$	$e_1,\ e_2,\ e_5$	$e_1,\ e_2,\ e_5$
$e_3 + \tilde{b} e_1,\ e_4 + \tilde{c} e_1,\ e_5 + ae_2 + de_1$	$e_3,\ e_4,\ e_5$	$e_3,\ e_4,\ e_5$	$e_3,\ e_4,\ e_5$
$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$
$e_1,\ e_2 + \tilde{c} e_4,\ e_3 + \bar{a} e_4,\ e_5 + be_4$	$e_1,\ e_2,\ e_3,\ e_5$	$e_1,\ e_2,\ e_3,\ e_5$	$e_1,\ e_2,\ e_3,\ e_5$
$e_1,\ e_2 + \tilde{b} e_3,\ e_4,\ e_5 + ae_3$	$e_1,\ e_2,\ e_4,\ e_5$		$e_1,\ e_2,\ e_4,\ e_5$
$e_1,\ e_3,\ e_4,\ e_5 + ae_2$	$e_1,\ e_3,\ e_4,\ e_5$	$e_1,\ e_3,\ e_4,\ e_5$	$e_1,\ e_3,\ e_4,\ e_5$

Table A.13. Subalgebras and realizations of $\mathfrak{g}_{5,10}$.

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$ class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5+\partial_1 x_2+\partial_2 x_3+\partial_4 x_4$	0
$\mathfrak{g}_1$	e_4	$\partial_1,\ \partial_2,\ \partial_3,\ 0,\ \partial_4+\partial_1 x_2+\partial_2 x_3$	e_4
	e_1+ae_4	$-a\mathrm{e}^{x_4}\partial_3,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4+\partial_1 x_2+\partial_3(-a\mathrm{e}^{x_4}x_1+x_3)$	$e_1+\eta_a e_4$
	$e_2+ae_1+be_4$	$\partial_1,\ -b\mathrm{e}^{x_4}\partial_3-\partial_1(a+x_4),\ \partial_2,\ \partial_3,\ \partial_4+\partial_3(-b\mathrm{e}^{x_4}x_2+x_3)-\partial_1 x_2(a+x_4)$	$e_2+\eta_b e_4$
	$e_3+ae_2+be_1+ce_4$	$\partial_1,\ \partial_2,\ -c\mathrm{e}^{x_4}\partial_3-\partial_2(a+x_4)-\partial_1(b+ax_4+\frac{1}{2}x_4^2),\ \partial_3,\ \partial_4+\partial_1 x_2+\partial_3 x_3$	$e_3+\eta_c e_4$
	$e_5+ae_4+be_3+ce_2+de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ -b\partial_3+\partial_1(-d+x_2)+\partial_2(-c+x_3)+\partial_4(-a+x_4)$	e_5
$2\mathfrak{g}_1$	$e_1,\ e_4$	$0,\ \partial_1,\ \partial_2,\ 0,\ \partial_3+\partial_1 x_2$	$e_1,\ e_4$
	$e_2+ae_1,\ e_4$	$\partial_1,\ -\partial_1(a+x_3),\ \partial_2,\ 0,\ \partial_3-\partial_1 x_2(a+x_3)$	$e_2,\ e_4$
	$e_3+ae_2+be_1,\ e_4$	$\partial_1,\ \partial_2,\ -\partial_2(a+x_3)-\partial_1(b+ax_3+\frac{1}{2}x_3^2),\ 0,\ \partial_3+\partial_1 x_2$	$e_3,\ e_4$
	$e_1+ae_4,\ e_2+be_4$	$-a\mathrm{e}^{x_3}\partial_2,\ \mathrm{e}^{x_3}\partial_2(-b+ax_3),\ \partial_1,\ \partial_2,\ \partial_3+\partial_2(-b\mathrm{e}^{x_3}x_1+x_2+a\mathrm{e}^{x_3}x_1x_3)$	$e_1+\eta_a e_4,\ e_2+\bar\eta_a\eta_b e_4$
	$e_1+ae_4,\ e_3+be_2+ce_4$	$-a\mathrm{e}^{x_3}\partial_2,\ \partial_1,\ -\partial_1(b+x_3)+\mathrm{e}^{x_3}\partial_2(-c+abx_3+\frac{1}{2}ax_3^2),\ \partial_2,$ $\partial_3+\partial_2(-a\mathrm{e}^{x_3}x_1+x_2)$	$e_1+\eta_a e_4,\ e_3+\bar\eta_a\eta_c e_4$
	$e_2+ae_1+be_4,$ $e_3+ce_1+de_4$	$\partial_1,\ -b\mathrm{e}^{x_3}\partial_2-\partial_1(a+x_3),\ \mathrm{e}^{x_3}\partial_2(-d+bx_3)+\partial_1(-c+ax_3+\frac{1}{2}x_3^2),\ \partial_2,$ $\partial_3+\partial_2 x_2$	$e_2+\eta_b e_4,\ e_3$ if $ab+d=0$ $e_2+\frac{b}{ab+d}e_4,\ e_3+e_4$ if $ab+d\neq 0$
	$e_1,\ e_5+ae_4+be_3+ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ -b\partial_2+\partial_1(-c+x_2)+\partial_3(-a+x_3)$	$e_1,\ e_5$
$\mathfrak{g}_2$	$e_4,\ e_5+ae_3+be_2+ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ 0,\ -a\partial_3+\partial_1(-c+x_2)+\partial_2(-b+x_3)$	$e_4,\ e_5$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_4$	$0,\ 0,\ \partial_1,\ 0,\ \partial_2$	$e_1,\ e_2,\ e_4$
	$e_1,\ e_3+ae_2,\ e_4$	$0,\ \partial_1,\ -\partial_1(a+x_2),\ 0,\ \partial_2$	$e_1,\ e_3,\ e_4$
	$e_2+ae_1,\ e_3+be_1,\ e_4$	$\partial_1,\ -(a+x_2)\partial_1,\ (a^2-b+ax_2+\frac{1}{2}x_2^2)\partial_1,\ 0,\ \partial_2$	$e_2,\ e_3,\ e_4$
	$e_1+ae_4,\ e_2+be_4,\ e_3+ce_4$	$-a\mathrm{e}^{x_2}\partial_1,\ \mathrm{e}^{x_2}\partial_1(-b+ax_2),\ \mathrm{e}^{x_2}\partial_1(-c+bx_2-\frac{1}{2}ax_2^2),\ \partial_1,\ \partial_2+\partial_1 x_1$	$e_1+\eta_a e_4,\ e_2+\bar\eta_a\eta_b e_4,$ $e_3+\bar\eta_a\bar\eta_b\eta_c e_4$
$\mathfrak{g}_{3,1}$	$e_1,\ e_2,\ e_5+ae_4+be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ -b\partial_1+\partial_2(-a+x_2)$	$e_1,\ e_2,\ e_5$
$\mathfrak{g}_1\oplus\mathfrak{g}_2$	$e_1,\ e_4,\ e_5+be_3+ce_2$	$0,\ \partial_1,\ \partial_2,\ 0,\ -b\partial_2+\partial_1(-c+x_2)$	$e_1,\ e_4,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}_{4,1}$	$e_1,\ e_2,\ e_3,\ e_5+ae_4$	$0,\ 0,\ 0,\ \partial_1,\ \partial_1(-a+x_1)$	$e_1,\ e_2,\ e_3,\ e_5$
$\mathfrak{g}_{4,3}$	$e_1,\ e_2,\ e_4,\ e_5+ae_3$	$0,\ 0,\ \partial_1,\ 0,\ -a\partial_1$	$e_1,\ e_2,\ e_4,\ e_5$

Table A.14. Subalgebras and realizations of $\mathfrak{g}^{\gamma}_{5,11}$, $\gamma \neq 0$; $\tilde{c}(1-\gamma)=0$, $\tilde{\eta}_x = \operatorname{sgn}|x|$ if $\gamma \neq 1$, $\tilde{\eta}_x = 0$ if $\gamma = 1$.

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$ class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + (x_1+x_2)\partial_1 + x_3\partial_3 + (x_2+x_3)\partial_2 + \gamma x_4\partial_4$	0
$\mathfrak{g}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_2\partial_2 + (x_1+x_2)\partial_1 + \gamma x_3\partial_3$	e_1
	$e_4 + ae_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -a\mathrm{e}^{(1-\gamma)x_4}\partial_1,\ \partial_4 + (x_1+x_2)\partial_1 + x_3\partial_3 + \partial_2(x_2+x_3$	$e_4 + \tilde{\eta}_a e_1$
	$e_2 + ae_1 + be_4$	$\partial_1,\ -b\mathrm{e}^{(\gamma-1)x_4}\partial_3 + (-a-x_4)\partial_1,\ \partial_2,\ \partial_3,$ $\partial_4 + x_2\partial_2 + (-b\mathrm{e}^{(\gamma-1)x_4\partial_3}x_2 + \gamma x_3) + (x_1 - x_2(a+x_4)\partial_1)$	$e_2 + \eta_b e_4$
	$e_3 + ae_2 + be_1 + ce_4$	$\partial_1,\ \partial_2,\ -c\mathrm{e}^{(\gamma-1)x_4}\partial_3 + (-a-x_4)\partial_2 + (-b - ax_4 - \frac{1}{2}x_4^2)\partial_1,\ \partial_3,$ $\partial_4 + x_2\partial_2 + (x_1+x_2)\partial_1 + \gamma x_3\partial_3$	$e_3 + \tilde{\eta}_c e_4$
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,$ $(-d+x_1+x_2)\partial_1 + (-b+x_3)\partial_3 + (-c+x_2+x_3)\partial_2 + (-a+\gamma x_4)\partial_4$	e_5
$2\mathfrak{g}_1$	$e_1,\ e_4$	$0,\ \partial_1,\ \partial_2,\ 0,\ \partial_3 + x_2\partial_2 + (x_1+x_2)\partial_1$	$e_1,\ e_4$
	$e_1,\ e_2 + ae_4$	$0,\ -a\mathrm{e}^{(\gamma-1)x_3}\partial_2,\ \partial_1,\ \partial_2,\ \partial_3 + x_1\partial_1 + (-a\mathrm{e}^{(\gamma-1)x_3\partial_2}x_1 + \gamma x_2)$	$e_1,\ e_2 + \eta_a e_4$
	$e_1,\ e_3 + ae_2 + be_4$	$0,\ \partial_1,\ -b\mathrm{e}^{(\gamma-1)x_3}\partial_2 + (-a-x_3)\partial_1,\ \partial_2,\ \partial_3 + x_1\partial_1 + \gamma x_2\partial_2$	$e_1,\ e_3 + \tilde{\eta}_b e_4$
	$e_2 + ae_1,\ e_4 + be_1$	$\partial_1,\ (-a-x_3)\partial_1,\ \partial_2,\ -b\mathrm{e}^{(1-\gamma)x_3}\partial_1,\ \partial_3 + x_2\partial_2 + (x_1 - ax_2 - x_2x_3)\partial_1$	$e_2,\ e_4 + \tilde{\eta}_b e_1$
	$e_3 + ae_2 + be_1,\ e_4 + ce_1$	$\partial_1,\ \partial_2,\ (-a-x_3)\partial_2 + (-b - ax_3 - \frac{1}{2}x_3^2)\partial_1,\ -c\mathrm{e}^{(1-\gamma)x_3}\partial_1,$ $\partial_3 + x_2\partial_2 + (x_1+x_2)\partial_1$	$e_3,\ e_4 + \tilde{\eta}_c e_1$
	$e_2 + ae_1 + be_4,\ e_3 + ce_1 + de_4$	$\partial_1,\ -b\mathrm{e}^{(\gamma-1)x_3}\partial_2 + (-a-x_3)\partial_1,\ \mathrm{e}^{(\gamma-1)x_3}(-d+bx_3)\partial_2 + (-c+ax_3+\frac{1}{2}x_3^2)\partial_1,$ $\partial_2,\ \partial_3 + x_1\partial_1 + \gamma x_2\partial_2$	$e_2 + \frac{b}{ab+d}e_4,\ e_3 + e_4$ if $ab+d \neq 0$ and $\gamma \neq 1$ $e_2 + \eta_b e_4,\ e_3$ otherwise
$\mathfrak{g}_2$	$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ (-b+x_2)\partial_2 + (-c+x_1+x_2)\partial_1 + (-a+\gamma x_3)\partial_3$	$e_1,\ e_5$
	$e_4 + \tilde{c}e_1,\ e_5 + ae_3 + be_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -\tilde{c}\partial_1,\ (-a+x_3)\partial_3 + (-b+x_2+x_3)\partial_2 + (-d+x_1+x_2)\partial_1$	$e_4,\ e_5$
$3\mathfrak{g}_1$	$e_1,\ e_2 + ae_4,\ e_3 + be_4$	$0,\ -a\mathrm{e}^{(\gamma-1)x_2}\partial_1,\ \mathrm{e}^{(\gamma-1)x_2}(-b+ax_2)\partial_1,\ \partial_1,\ \partial_2 + \gamma x_1\partial_1$	$e_1,\ e_2 + \eta_a e_4,$ $e_3 + (1-\tilde{\eta}_a)\tilde{\eta}_b e_4$
	$e_1,\ e_2,\ e_4$	$0,\ 0,\ \partial_1,\ 0,\ \partial_2 + x_1\partial_1$	$e_1,\ e_2,\ e_4$
	$e_1,\ e_3 + ae_2,\ e_4$	$0,\ \partial_1,\ (-a-x_2)\partial_1,\ 0,\ \partial_2 + x_1\partial_1$	$e_1,\ e_3,\ e_4$
	$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$\partial_1,\ (-a-x_2)\partial_1,\ (-b+ax_2+\frac{1}{2}x_2^2)\partial_1,\ -c\mathrm{e}^{(1-\gamma)x_2}\partial_1,\ \partial_2 + x_1\partial_1$	$e_2,\ e_3,\ e_4 + \tilde{c}e_1$
$\mathfrak{g}_{3,2}$	$e_1,\ e_4,\ e_5 + ae_3 + be_2$	$0,\ \partial_1,\ \partial_2,\ 0,\ (-a+x_2)\partial_2 + (-b+x_1+x_2)\partial_1$	$e_1,\ e_4,\ e_5$
$\mathfrak{g}^{\gamma}_{3,4}$	$e_1,\ e_2 + \tilde{c}e_4,\ e_5 + ae_4 + be_3$	$0,\ -\tilde{c}\partial_2,\ \partial_1,\ \partial_2,\ (-b+x_1)\partial_1 + (-a+\gamma x_2 - \tilde{c}x_1)\partial_2$	$e_1,\ e_2 + \tilde{\eta}_{\tilde{c}}e_4.\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}_{4,4}$	$e_1,\ e_2,\ e_3 + \tilde{c}e_4,\ e_5 + ae_4$	$0,\ 0,\ -\tilde{c}\partial_1,\ \partial_1,\ (-a+\gamma x_1)\partial_1$	$e_1,\ e_2,\ e_3,\ e_5$
$\mathfrak{g}^{\gamma}_{4,2}$	$e_1,\ e_2,\ e_4,\ e_5 + ae_3$	$0,\ 0,\ \partial_1,\ 0,\ (-a+x_1)\partial_1$	$e_1,\ e_2,\ e_4,\ e_5$

Table A.15. Subalgebras and realizations of $\mathfrak{g}_{5,12}$.

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$ class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5+(x_1+x_2)\partial_1+(x_2+x_3)\partial_2+x_4\partial_4+(x_3+x_4)\partial_3$	0
$\mathfrak{g}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4+(x_1+x_2)\partial_1+x_3\partial_3+(x_2+x_3)\partial_2$	e_1
	e_2+ae_1	$\partial_1,\ -(a+x_4)\partial_1,\ \partial_2,\ \partial_3,\ \partial_4+x_3\partial_3+(x_2+x_3)\partial_2+(x_1-ax_2+x_2x_4)\partial_1$	e_2
	$e_3+ae_2+be_1$	$\partial_1,\ \partial_2,\ -(a+x_4)\partial_2-(b+ax_4+\frac{1}{2}x_4^2)\partial_1,\ \partial_3,$ $\partial_4+x_3\partial_3+(x_2-ax_3-x_3x_4)\partial_2+(x_1+x_2-bx_3-ax_3x_4-\frac{1}{2}x_3x_4^2)\partial_1$	e_3
	$e_4+ae_3+be_2+ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -(a+x_4)\partial_3-(b+ax_4+\frac{1}{2}x_4^2)\partial_2-(c+bx_4+\frac{1}{2}ax_4^2+\frac{1}{6}x_4^3)\partial_1,$ $\partial_4+(x_1+x_2)\partial_1+x_3\partial_3+(x_2+x_3)\partial_2$	e_4
	$e_5+ae_4+be_3+ce_2+de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,$ $(-d+x_1+x_2)\partial_1+(-c+x_2+x_3)\partial_2+(-a+x_4)\partial_4+(-b+x_3+x_4)\partial_3$	e_5
$2\mathfrak{g}_1$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3+x_2\partial_2+(x_1+x_2)\partial_1$	$e_1,\ e_2$
	$e_1,\ e_3+ae_2$	$0,\ \partial_1,\ -(a+x_3)\partial_1,\ \partial_2,\ \partial_3+x_2\partial_2+(x_1-ax_2-x_2x_3)\partial_1$	$e_1,\ e_3$
	$e_1,\ e_4+ae_3+be_2$	$0,\ \partial_1,\ \partial_2,\ -(a+x_3)\partial_2-(b+ax_3+\frac{1}{2}x_3^2)\partial_1,\ \partial_3+x_2\partial_2+(x_1+x_2)\partial_1$	$e_1,\ e_4$
	$e_2+ae_1,\ e_3+be_1$	$\partial_1,\ -(a+x_3)\partial_1,\ (-b+ax_3+\frac{1}{2}x_3^2)\partial_1,\ \partial_2,$ $\partial_3+x_2\partial_2+(x_1-bx_2+ax_2x_3+\frac{1}{2}x_2x_3^2)\partial_1$	$e_2,\ e_3$
	$e_2+ae_1,\ e_4+be_3+ce_1$	$\partial_1,\ -(a+x_3)\partial_1,\ \partial_2,\ -(b+x_3)\partial_2+(-c+abx_3+\frac{1}{2}(a+b)x_3^2+\frac{1}{3}x_3^3)\partial_1,$ $\partial_3+x_2\partial_2+(x_1-ax_2-x_2x_3)\partial_1$	$e_2,\ e_4+(b-a)e_3$
	$e_3+ae_2+be_1,$ $e_4+ce_2+de_1$	$\partial_1,\ \partial_2,\ -(a+x_3)\partial_2-(b+ax_3+\frac{1}{2}x_3^2)\partial_1,$ $(-c+ax_3+\frac{1}{2}x_3^2)\partial_2+(-d+(b-c)x_3+ax_3^2+\frac{1}{3}x_3^3)\partial_1,\ \partial_3+x_2\partial_2+(x_1+x_2)\partial_1$	$e_3,\ e_4+(a^2-b+c)e_2$
$\mathfrak{g}_2$	$e_1,\ e_5+ae_4+be_3+ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ (-c+x_1+x_2)\partial_1+(-a+x_3)\partial_3+(-b+x_2+x_3)\partial_2$	$e_1,\ e_5$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$0,\ 0,\ 0,\ \partial_1,\ \partial_2+x_1\partial_1$	$e_1,\ e_2,\ e_3$
	$e_1,\ e_2,\ e_4+ae_3$	$0,\ 0,\ \partial_1,\ -(a+x_2)\partial_1,\ \partial_2+x_1\partial_1$	$e_1,\ e_2,\ e_4$
	$e_1,\ e_3+ae_2,\ e_4+be_2$	$0,\ \partial_1,\ -(a+x_2)\partial_1,\ (-b+ax_2+\frac{1}{2}x_2^2)\partial_1,\ \partial_2+x_1\partial_1$	$e_1,\ e_3,\ e_4$
	$e_2+ae_1,\ e_3+be_1,\ e_4+ce_1$	$\partial_1,\ -(a+x_2)\partial_1,\ (-b+ax_2+\frac{1}{2}x_2^2)\partial_1,\ (-c+bx_2-\frac{1}{2}ax_2^2-\frac{1}{6}x_2^3)\partial_1,\ \partial_2+x_1\partial_1$	$e_2,\ e_3,\ e_4$
$\mathfrak{g}_{3,2}$	$e_1,\ e_2,\ e_5+ae_4+be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ (-a+x_2)\partial_2+(-b+x_1+x_2)\partial_1$	$e_1,\ e_2,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}_{4,4}$	$e_1,\ e_2,\ e_3,\ e_5+ae_4$	$0,\ 0,\ 0,\ \partial_1,\ (-a+x_1)\partial_1$	$e_1,\ e_2,\ e_3,\ e_5$

Table A.16. Realizations of $\mathfrak{g}_{5,13}^{\gamma,p,s}$, $|\gamma|\le 1$, $\gamma s\neq 0$; $\tilde{c}(1-\gamma)=0$, $\mathrm{cs}(x;a)=a\cos x+b\sin x$

Type	Subalgebra	Corresponding realization
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5+x_1\partial_1+\gamma x_2\partial_2+(-sx_3+px_4)\partial_4+(px_3+sx_4)\partial_3$
$\mathfrak{g}_1$	e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4+\gamma x_1\partial_1+(-sx_2+px_3)\partial_3+(px_2+sx_3)\partial_2$
	e_2+ae_1	$\partial_1,\ -a\mathrm{e}^{(1-\gamma)x_4}\partial_1,\ \partial_2,\ \partial_3,\ \partial_4+x_1\partial_1+(-sx_2+px_3)\partial_3+(px_2+sx_3)\partial_2$
	$e_3+ae_2+be_1$	$\partial_1,\ \partial_2,\ -\frac{b\mathrm{e}^{(1-p)x_4}}{\cos(sx_4)}\partial_1-\frac{a\mathrm{e}^{(\gamma-p)x_4}}{\cos(sx_4)}\partial_2+\tan(sx_4)\partial_3,\ \partial_3,$
		$\partial_4+(p+s\tan(sx_4))x_3\partial_3+(\gamma x_2-\frac{asx_3\mathrm{e}^{(\gamma-p)x_4}}{\cos(sx_4)})\partial_2+(x_1-\frac{bsx_3\mathrm{e}^{(1-p)x_4}}{\cos(sx_4)})\partial_1$
	$e_4+ae_3+be_2+ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \frac{c\mathrm{e}^{(1-p)x_4}}{\mathrm{cs}(sx_4;-1,a)}\partial_1+\frac{b\mathrm{e}^{(\gamma-p)x_4}}{\mathrm{cs}(sx_4;-1,a)}\partial_2+\frac{\mathrm{cs}(sx_4;a,1)}{\mathrm{cs}(sx_4;-1,a)}\partial_3,$
		$\partial_4+\frac{\mathrm{cs}(sx_4;p+as,s-ap)}{\mathrm{cs}(sx_4;1,-a)}x_3\partial_3+\left(\gamma x_2+\frac{bsx_3\mathrm{e}^{(\gamma-p)x_4}}{\mathrm{cs}(sx_4;1,-a)}\right)\partial_2+\left(x_1+\frac{csx_3\mathrm{e}^{(1-p)x_4}}{\mathrm{cs}(sx_4;1,-a)}\right)\partial_1$
	$e_5+ae_4+be_3+ce_2+de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ (-d+x_1)\partial_1+(-c+\gamma x_2)\partial_2+(-a-sx_3+px_4)\partial_4+(-b+px_3+sx_4)\partial_3$
$\mathfrak{g}_2$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3+(-sx_1+px_2)\partial_2+(px_1+sx_2)\partial_1$
	$e_1,\ e_3+ae_2$	$0,\ \partial_1,\ -\frac{a\mathrm{e}^{(\gamma-p)x_3}}{\cos(sx_3)}\partial_1+\tan(sx_3)\partial_2,\ \partial_2,\ \partial_3+(p+s\tan(sx_3))x_2\partial_2+\left(\gamma x_1-\frac{asx_2\mathrm{e}^{(\gamma-p)x_3}}{\cos(sx_3)}\right)$
	$e_1,\ e_4+ae_3+be_2$	$0,\ \partial_1,\ \partial_2,\ b\mathrm{e}^{((\gamma-p)x_3}/\,\mathrm{cs}(sx_3;-1,a)\partial_1+\frac{a\,\mathrm{cs}(sx_3;1/a)}{\mathrm{cs}(sx_3;-1,a)}\partial_2,$
		$\partial_3+\frac{\mathrm{cs}(sx_3;p+as,s-ap)\sin(sx_3)}{\mathrm{cs}(sx_3;1,-a)}x_2\partial_2+\left(\gamma x_1+\frac{bsx_2\mathrm{e}^{(\gamma-p)x_3}}{\mathrm{cs}(sx_3;1,-a)}\right)\partial_1$
	$e_2+ae_1,\ e_3+be_1$	$\partial_1,\ -a\mathrm{e}^{(1-\gamma)x_3}\partial_1,\ -\frac{b\mathrm{e}^{(1-p)x_3}}{\cos(sx_3)}\partial_1+\tan(sx_3)\partial_2,\ \partial_2,\ \partial_3+x_2(p+s\tan(sx_3))\partial_2+\left(x_1-\frac{bsx_2\mathrm{e}^{(1-p)x_3}}{\cos(sx_3)}\right)\partial_1$
	$e_2+ae_1,\ e_4+be_3+ce_1$	$\partial_1,\ -a\mathrm{e}^{(1-\gamma)x_3}\partial_1,\ \partial_2,\ c\mathrm{e}^{(1-p)x_3}\,\mathrm{cs}(sx_3;-1,b)\partial_1+\frac{\mathrm{cs}(sx_3;b,1)}{\mathrm{cs}(sx_3;-1,b)}\partial_2,$
		$\partial_3+x_2\frac{\mathrm{cs}(sx_3;p+bs,s-bp)}{\mathrm{cs}(sx_3;1,-b)}\partial_2+\left(x_1+\frac{\mathrm{cs}\,x_2\mathrm{e}^{(1-p)x_3}}{\mathrm{cs}(sx_3;1,-b)}\right)\partial_1$
	$e_3+ae_2+be_1,$ $e_4+ce_2+de_1$	$\partial_1,\ \partial_2,\ -\mathrm{e}^{(\gamma-p)x_3}\,\mathrm{cs}(sx_3;a,c)\partial_2-\mathrm{e}^{(1-p)x_3}\,\mathrm{cs}(sx_3;b,d)\partial_1,$ $\mathrm{e}^{(\gamma-p)x_3}\,\mathrm{cs}(sx_3;-c,a)\partial_2+\mathrm{e}^{(1-p)x_3}\,\mathrm{cs}(sx_3;-d,b)\partial_1,\ \partial_3+x_1\partial_1+\gamma x_2\partial_2$
$\mathfrak{g}_2$	$e_1,\ e_5+ae_4+be_3+ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ (-c+\gamma x_1)\partial_1+(-a-sx_2+px_3)\partial_3+(-b+px_2+sx_3)\partial_2$
	$e_2,\ e_5+be_4+ce_3+de_1$	$\partial_1,\ 0,\ \partial_2,\ \partial_3,\ (-d+x_1)\partial_1+(-b-sx_2+px_3)\partial_3+(-c+px_2+sx_3)\partial_2$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$0,\ 0,\ \tan(sx_2)\partial_1,\ \partial_1,\ \partial_2+x_1(p+s\tan(sx_2))\partial_1$
	$e_1,\ e_2,\ e_4+ae_3$	$0,\ 0,\ \partial_1,\ \frac{\mathrm{cs}(sx_2;a,1)}{\mathrm{cs}(sx_2;-1,a)},\ \partial_2+x_1\frac{\mathrm{cs}(sx_2;p+as,s-ap)}{\mathrm{cs}(sx_2;1,-a)}\partial_1$
	$e_1,\ e_3+ae_2,\ e_4+be_2$	$0,\ \partial_1,\ -a\mathrm{e}^{(\gamma-p)x_2}\,\mathrm{cs}(sx_2;a,b)\partial_1,\ \mathrm{e}^{(\gamma-p)x_2}\,\mathrm{cs}(sx_2;-b,a)\partial_1,\ \partial_2+\gamma x_1\partial_1$
	$e_2+ae_1,\ e_3+be_1,\ e_4+ce_1$	$\partial_1,\ -a\mathrm{e}^{(1-\gamma)x_2}\partial_1,\ -\mathrm{e}^{(1-p)x_2}\,\mathrm{cs}(sx_2;b,c)\partial_1,\ \mathrm{e}^{(1-p)x_2}\,\mathrm{cs}(sx_2;-c,b)\partial_1,\ \partial_2+x_1\partial_1$
$\mathfrak{g}_{3,4}^{\gamma}$	$e_1,\ e_2,\ e_5+ae_4+be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ (-a-sx_1+px_2)\partial_2+(-b+px_1+sx_2)\partial_1$
$\mathfrak{g}_{3,5}^{p/s}$	$e_3,\ e_4,\ e_5+ae_2+be_1$	$\partial_1,\ \partial_2,\ 0,\ 0,\ (-b+x_1)\partial_1+(-a+\gamma x_2)\partial_2$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$
$\mathfrak{g}_{4,6}^{1/s,p/s}$	$e_1,\ e_3,\ e_4,\ e_5+ae_2$	$0,\ \partial_1,\ 0,\ 0,\ (-a+\gamma x_1)\partial_1$
$\mathfrak{g}_{4,6}^{\gamma/s,p/s}$	$e_2+\tilde{c},\ e_3,\ e_4,\ e_5+ae_1$	$\partial_1,\ -\tilde{c}\partial_1,\ 0,\ 0,\ (-a+x_1)\partial_1$

Table A.17. Subalgebras of $\mathfrak{g}_{5,13}^{\gamma,p,s}$, $|\gamma| \leq 1$, $\gamma s \neq 0$; $\tilde{c}(1-\gamma) = 0$

Type	Subalgebra	Aut $\mathfrak{g}$ class ($\gamma = 1$)	Aut $\mathfrak{g}$ class ($\gamma = -1$, $p = 0$)	Aut $\mathfrak{g}$ class ($\gamma \neq 1$)
0	0	0	0	0
$\mathfrak{g}_1$	e_1	e_1	e_1	e_1
	$e_2 + ae_1$		$e_1 + \eta_a e_2$	$e_2 + \eta_a e_1$
	$e_3 + ae_2 + be_1$	$e_3 + \eta_{a,b} e_1$	$e_3 + \eta_{ab} e_2 + \eta_{a,b} e_1$	$e_3 + \eta_a e_2 + \eta_b e_1$
	$e_4 + ae_3 + be_2 + ce_1$	$e_3 + \eta_{b,c} e_1$	$e_3 + \eta_{bc} e_2 + \eta_{b,c} e_1$	$e_3 + \eta_b e_2 + \eta_c e_1$
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	e_5	e_5	e_5
$\mathfrak{g}_2$	e_1, e_2	e_1, e_2	e_1, e_2	e_1, e_2
	e_1, $e_3 + ae_2$	e_1, $e_3 + \eta_a e_2$	e_1, $e_2 + \eta_a e_3$	e_1, $e_3 + \eta_a e_2$
	e_1, $e_4 + ae_3 + be_2$	e_1, $e_3 + \eta_b e_2$	e_1, $e_2 + \eta_b e_3$	e_1, $e_3 + \eta_b e_2$
	$e_2 + ae_1$, $e_3 + be_1$	e_1, $e_3 + \eta_b e_1$	$e_2 + \eta_a e_1$, $e_3 + \eta_b e_1$	$e_2 + \eta_a e_1$, $e_3 + \eta_b e_1$
	$e_2 + ae_1$, $e_4 + be_3 + ce_1$	e_1, $e_3 + \eta_c e_1$	$e_2 + \eta_a e_1$, $e_3 + \eta_c e_1$	$e_2 + \eta_a e_1$, $e_3 + \eta_c e_1$
	$e_3 + ae_2 + be_1$, $e_4 + ce_2 + de_1$	$e_3 + \eta_{ad-bc} e_2$, $e_4 + \eta_{a,b,c,d} e_1$	$e_3 + \eta_{a,c}\eta_{b,d} e_2 + \eta_{a,b,c,d} e_1$, e_4 if $ad = bc$ $e_3 + e_2 + \frac{ab+cd}{ad-bc} e_1$, $e_4 + e_1$ otherwise	$e_3 + \eta_{a,c} e_2 + \eta_{b,d} e_1$, e_4 if $ad = bc$ $e_3 + e_2 + \frac{ab+cd}{ad-bc} e_1$, $e_4 + e_1$ otherwise
$\mathfrak{g}_2$	e_1, $e_5 + ae_4 + be_3 + ce_2$	e_1, e_5	e_1, e_5	e_1, e_5
	e_2, $e_5 + be_4 + ce_3 + de_1$			e_2, e_5
$3\mathfrak{g}_1$	e_1, e_2, e_3	e_1, e_2, e_3	e_1, e_2, e_3	e_1, e_2, e_3
	e_1, e_2, $e_4 + ae_3$			
	e_1, $e_3 + ae_2$, $e_4 + be_2$	e_1, $e_3 + \eta_{a,b} e_2$, e_4	e_1, $e_3 + \eta_{a,b} e_2$, e_4	e_1, $e_3 + \eta_{a,b} e_2$, e_4
	$e_2 + ae_1$, $e_3 + be_1$, $e_4 + ce_1$	e_1, $e_3 + \eta_{b,c} e_2$, e_4	$e_1 + \eta_a e_2$, $e_3 + \eta_{b,c} e_2$, e_4	$e_2 + \eta_a e_1$, $e_3 + \eta_{b,c} e_2$, e_4
$\mathfrak{g}_{3,4}^{\gamma}$	e_1, e_2, $e_5 + ae_4 + be_3$	e_1, e_2, e_5	e_1, e_2, e_5	e_1, e_2, e_5
$\mathfrak{g}_{3,5}^{p/s}$	e_3, e_4, $e_5 + ae_2 + be_1$	e_3, e_4, e_5	e_3, e_4, e_5	e_3, e_4, e_5
$4\mathfrak{g}_1$	e_1, e_2, e_3, e_4	e_1, e_2, e_3, e_4	e_1, e_2, e_3, e_4	e_1, e_2, e_3, e_4
$\mathfrak{g}_{4,6}^{1/s,p/s}$	e_1, e_3, e_4, $e_5 + ae_2$	e_1, e_3, e_4, e_5	e_1, e_3, e_4, e_5	e_1, e_3, e_4, e_5
$\mathfrak{g}_{4,6}^{\gamma/s,p/s}$	$e_2 + \tilde{c}$, e_3, e_4, $e_5 + ae_1$			e_2, e_3, e_4, e_5

Table A.18. Realizations of $\mathfrak{g}^p_{5,14}$; $\mathrm{cs}(x;a,b)=a\cos x+b\sin x$

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$-class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5+x_2\partial_1+(px_3+x_4)\partial_3+(-x_3+px_4)\partial_4$	0
$\mathfrak{g}_1$	e_3	$\partial_1,\ \partial_2,\ \tan x_4\partial_3,\ \partial_3,\ \partial_4+x_2\partial_1+x_3(p+\tan x_4)\partial_3$	e_3
	e_4+ae_3	$\partial_1,\ \partial_2,\ \partial_3,\ \frac{\mathrm{cs}(x_4;a,1)}{\mathrm{cs}(x_4;-1,a)}\partial_3,\ \partial_4+x_2\partial_1+x_3\frac{(a+p)\cos x_4+(1-ap)\sin x_4}{\mathrm{cs}(x_4;1,-a)}\partial_3$	
	$e_1+ae_3+be_4$	$\mathrm{e}^{px_4}\,\mathrm{cs}(x_4;-b,a)\partial_3-\mathrm{e}^{px_4}\,\mathrm{cs}(x_4;a,b)\partial_2,\ \partial_1,\ \partial_2,\ \partial_3,$ $\partial_4+(-x_1\mathrm{e}^{px_4}\,\mathrm{cs}(x_4;a,b)+px_2+x_3)\partial_2+(x_1\mathrm{e}^{px_4}\,\mathrm{cs}(x_4;-b,a)\partial_3-x_2+px_3)$	$e_1+\eta_{a,b}e_3$
	$e_2+ae_1+be_3+ce_4$	$\partial_1,\ \mathrm{e}^{px_4}\,\mathrm{cs}(x_4;-c,b)\partial_3-\mathrm{e}^{px_4}\,\mathrm{cs}(x_4;b,c)\partial_2-(a+x_4)\partial_1,\ \partial_2,\ \partial_3,$ $\partial_4+(px_2+x_3)\partial_2+(-x_2+px_3)\partial_3$	$e_2+\eta_{b,c}e_3$
	$e_5+ae_4+be_3+ce_2+de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ -c\partial_2+(-d+x_2)\partial_1+(-b+px_3+x_4)\partial_3+(-a-x_3+px_4)\partial_4$	e_5
$2\mathfrak{g}_1$	$e_3,\ e_4$	$\partial_1,\ \partial_2,\ 0,\ 0,\ \partial_3+x_2\partial_1$	$e_3,\ e_4$
	$e_1+ae_4,\ e_3$	$-\frac{a\mathrm{e}^{px_3}}{\cos x_3}\partial_2,\ \partial_1,\ \tan x_3\partial_2,\ \partial_2,\ \partial_3+\left(-\frac{ax_1\mathrm{e}^{px_3}}{\cos x_3}+px_2+x_2\tan x_3\right)\partial_2$	$e_1+\eta_a e_4,\ e_3$
	$e_1+ae_3,\ e_4+be_3$	$\frac{a\mathrm{e}^{px_3}}{\mathrm{cs}(x_3;-1,b)}\partial_2,\ \partial_1,\ \partial_2,\ \frac{\mathrm{cs}(x_3;b,1)}{\mathrm{cs}(x_3;-1,b)}\partial_2,\ \partial_3+\frac{-ax_1\mathrm{e}^{px_3}+x_2\,\mathrm{cs}(x_3;b+p,bp-1)}{\mathrm{cs}(x_3;1,-b)}\partial_2$	
	$e_2+ae_1+be_4,\ e_3$	$\partial_1,\ -\frac{b\mathrm{e}^{px_3}}{\cos x_3}\partial_2-(a+x_3)\partial_1,\ \tan x_3\partial_2,\ \partial_2,\ \partial_3+x_2(p+\tan x_3)\partial_2$	$e_1+\eta_b e_4,\ e_3$
	$e_1+ae_2+be_3,\ e_4+ce_3$	$-\frac{a}{1+ax_3}\partial_1+\frac{b\mathrm{e}^{px_3}}{(1+ax_3)\,\mathrm{cs}(x_3;-1,c)}\partial_2,\ \partial_1,\ \partial_2,\ \frac{\mathrm{cs}(x_3;c,1)}{\mathrm{cs}(x_3;-1,c)}\partial_2,$ $\partial_3-\frac{ax_1}{1+ax_3}\partial_1+\frac{-bx_1\mathrm{e}^{px_3}+x_2(1+ax_3)\,\mathrm{cs}(x_3;c+p,cp-1)}{(1+ax_3)\,\mathrm{cs}(x_3;1,-c)}$	$\bar\eta_a e_1+\eta_a e_2+\eta_b e_4,\ e_3$
	$e_1+ae_3+be_4,$ $e_2+ce_3+de_4$	$\mathrm{e}^{px_3}\,\mathrm{cs}(x_3;-b,a)\partial_2-\mathrm{e}^{px_3}\,\mathrm{cs}(x_3;a,b)\partial_1,$ $\mathrm{e}^{px_3}(\mathrm{cs}(x_3;-d,c)+x_3\,\mathrm{cs}(x_3;b,-a))\partial_2+\mathrm{e}^{px_3}(\mathrm{cs}(x_3;-c,d)+x_3\,\mathrm{cs}(x_3;a,b))\partial_1,\ \partial_1,\ \partial_2,$ $\partial_3+(px_1+x_2)\partial_1+(-x_1+px_2)\partial_2$	$e_1+\eta_{a,b}e_3,\ e_2+\bar\eta_{a,b}\eta_{c,d}e_3$ if $ad=bc$; $e_1+\frac{a^2+b^2}{bc-ad}e_4,$ e_2+e_3 otherwise
	$e_1,\ e_5+ae_4+be_3+ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ -c\partial_1+(-b+px_2+x_3)\partial_2+(-a-x_2+px_3)\partial_3$	$e_1,\ e_5$
$3\mathfrak{g}_1$	$e_1+ae_4,\ e_2+be_4,\ e_3$	$-\frac{a\mathrm{e}^{px_2}}{\cos x_2}\partial_1,\ -\frac{(b-ax_2)\mathrm{e}^{px_2}}{\cos x_2}\partial_1,\ \tan x_2\partial_1,\ \partial_1,\ \partial_2+x_1(p+\tan x_2)\partial_1$	$e_1+\eta_a e_4,\ e_2+\bar\eta_a\eta_b e_4,\ e_3$
	$e_1+ae_3,\ e_2+be_3,\ e_4+ce_3$	$\frac{a\mathrm{e}^{px_2}}{\mathrm{cs}(x_2;-1,c)}\partial_1,\ \frac{(b-ax_2)\mathrm{e}^{px_2}}{\mathrm{cs}(x_2;-1,c)}\partial_1,\ \partial_1,\ \frac{\mathrm{cs}(x_2;c,1)}{\mathrm{cs}(x_2;-1,c)}\partial_1,\ \partial_2+\frac{x_1\,\mathrm{cs}(x_2;c+p,cp-1)}{\mathrm{cs}(x_2;1,-c)}\partial_1$	$e_1+\eta_a e_4,\ e_2+\bar\eta_a\eta_b e_4,\ e_3$
	$e_1,\ e_3,\ e_4$	$0,\ \partial_1,\ 0,\ 0,\ \partial_2$	$e_1,\ e_3,\ e_4$
	$e_2+ae_1,\ e_3,\ e_4$	$\partial_1,\ (-a-x_2)\partial_1,\ 0,\ 0,\ \partial_2$	$e_2,\ e_3,\ e_4$
$\mathfrak{g}_{3,1}$	$e_1,\ e_2,\ e_5+ae_4+be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ (-b+px_1+x_2)\partial_1+(-a-x_1+px_2)\partial_2$	$e_1,\ e_2,\ e_5$
$\mathfrak{g}^p_{3,5}$	$e_3,\ e_4,\ e_5+ae_2+be_1$	$\partial_1,\ \partial_2,\ 0,\ 0,\ -a\partial_2+(-b+x_2)\partial_1$	$e_3,\ e_4,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}_1\oplus\mathfrak{g}^p_{3,5}$	$e_1,\ e_3,\ e_4,\ e_5+ae_2$	$0,\ \partial_1,\ 0,\ 0,\ -a\partial_1$	$e_1,\ e_3,\ e_4,\ e_5$

Table A.19. Realizations of $\mathfrak{g}_{5,15}^{\gamma}$, $\gamma \in \mathbb{R}$, $\tilde{c}(1-\gamma) = \tilde{c}_1(1-\gamma) = \tilde{c}_2(1-\gamma) = 0$.

subalgebra	corresponding realization
0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + x_2\partial_2 + (x_1+x_2)\partial_1 + \gamma x_4\partial_4 + (\gamma x_3 + x_4)\partial_3$
e_1	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_1\partial_1 + \gamma x_3\partial_3 + (\gamma x_2 + x_3)\partial_2$
$e_3 + ae_1$	$\partial_1,\ \partial_2,\ -a\mathrm{e}^{(1-\gamma)x_4}\partial_1,\ \partial_3,\ \partial_4 + x_2\partial_2 + \gamma x_3\partial_3 + (x_1 + x_2 - a\mathrm{e}^{(1-\gamma)x_4}x_3)\partial_1$
$e_2 + ae_3 + be_1$	$\partial_1,\ -a\mathrm{e}^{(\gamma-1)x_4}\partial_2 + (-b - x_4)\partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_1\partial_1 + \gamma x_3\partial_3 + (\gamma x_2 + x_3)\partial_2$
$e_4 + ae_3 + be_2 + ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ -b\mathrm{e}^{(1-\gamma)x_4}\partial_2 + (-a - x_4)\partial_3 - \mathrm{e}^{(1-\gamma)x_4}(c + bx_4)\partial_1,\ \partial_4 + x_2\partial_2 + (x_1+x_2)\partial_1 + \gamma x_3\partial_3$
$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ (-c + x_2)\partial_2 + (-d + x_1 + x_2)\partial_1 + (-b + \gamma x_3 + x_4)\partial_3 + (-a + \gamma x_4)\partial_4$
$e_1,\ e_3$	$0,\ \partial_1,\ 0,\ \partial_2,\ \partial_3 + x_1\partial_1 + \gamma x_2\partial_2$
$e_1,\ e_2 + ae_3$	$0,\ -a\mathrm{e}^{(\gamma-1)x_3}\partial_1,\ \partial_1,\ \partial_2,\ \partial_3 + \gamma x_2\partial_2 + (\gamma x_1 + x_2)\partial_1$
$e_1,\ e_4 + ae_3 + be_2$	$0,\ \partial_1,\ \partial_2,\ -b\mathrm{e}^{(1-\gamma)x_3}\partial_1 + (-a - x_3)\partial_2,\ \partial_3 + x_1\partial_1 + \gamma x_2\partial_2$
$e_2 + ae_1,\ e_3 + be_1$	$\partial_1,\ (-a - x_3)\partial_1,\ -b\mathrm{e}^{(1-\gamma)x_3}\partial_1,\ \partial_2,\ \partial_3 + \gamma x_2\partial_2 + (x_1 - b\mathrm{e}^{(1-\gamma)x_3}x_2)\partial_1$
$e_3 + ae_1,\ e_4 + be_2 + ce_1$	$\partial_1,\ \partial_2,\ -a\mathrm{e}^{(1-\gamma)x_3}\partial_1,\ -b\mathrm{e}^{(1-\gamma)x_3}\partial_2 - \mathrm{e}^{(1-\gamma)x_3}(c + (b-a)x_3)\partial_1,\ \partial_3 + x_2\partial_2 + (x_1 + x_2)\partial_1$
$e_2 + ae_3 + be_1,\ e_4 + ce_3 + de_1$	$\partial_1,\ -a\mathrm{e}^{(\gamma-1)x_3}\partial_2 + (-b - x_3)\partial_1,\ \partial_2,\ -d\mathrm{e}^{(1-\gamma)x_3}\partial_1 + (-c - x_3)\partial_2,\ \partial_3 + x_1\partial_1 + \gamma x_2\partial_2$
$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ (-c + x_1)\partial_1 + (-b + \gamma x_2 + x_3)\partial_2 + (-a + \gamma x_3)\partial_3$
$e_3 + \tilde{c}e_1,\ e_5 + ae_4 + be_2 + de_1$	$\partial_1,\ \partial_2,\ -\tilde{c}\partial_1,\ \partial_3,\ (-b + x_2)\partial_2 + (-a + \gamma x_3)\partial_3 + (-d + x_1 + x_2 - \tilde{c}x_3)\partial_1$
$e_1,\ e_2,\ e_3$	$0,\ 0,\ 0,\ \partial_1,\ \partial_2 + \gamma x_1\partial_1$
$e_1,\ e_3,\ e_4 + ae_2$	$0,\ \partial_1,\ 0,\ -a\mathrm{e}^{(1-\gamma)x_2}\partial_1,\ \partial_2 + x_1\partial_1$
$e_1,\ e_2 + ae_3,\ e_4 + be_3$	$0,\ -a\mathrm{e}^{(\gamma-1)x_2}\partial_1,\ \partial_1,\ (-b - x_2)\partial_1,\ \partial_2 + \gamma x_1\partial_1$
$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$\partial_1,\ (-a - x_2)\partial_1,\ -b\mathrm{e}^{(1-\gamma)x_2}\partial_1,\ \mathrm{e}^{(1-\gamma)x_2}(-c + bx_2)\partial_1,\ \partial_2 + x_1\partial_1$
$e_1,\ e_3,\ e_5 + ae_4 + be_2$	$0,\ \partial_1,\ 0,\ \partial_2,\ (-b + x_1)\partial_1 + (-a + \gamma x_2)\partial_2$
$e_1,\ e_2 + \tilde{c}e_3,\ e_5 + ae_4 + be_3$	$0,\ -\tilde{c}\partial_1,\ \partial_1,\ \partial_2,\ (-a + \gamma x_2)\partial_2 + (-b + x_2 + \gamma x_1)\partial_1$
$e_3 + \tilde{c}_1e_1,\ e_4 + \tilde{c}_1e_2 + \tilde{c}_2e_1,$ $e_5 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ -\tilde{c}_1\partial_1,\ -\tilde{c}_2\partial_1 - \tilde{c}_1\partial_2,\ (-a + x_2)\partial_2 + (-b + x_1 + x_2)\partial_1$
$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$
$e_1,\ e_2,\ e_3,\ e_5 + ae_4$	$0,\ 0,\ 0,\ \partial_1,\ (-a + \gamma x_1)\partial_1$
$e_1,\ e_3,\ e_4 + \tilde{c}e_2,\ e_5 + ae_2$	$0,\ \partial_1,\ 0,\ -\tilde{c}\partial_1,\ (-a + x_1)\partial_1$

Table A.20. Realizations of $\mathfrak{g}_{5,15}^{\gamma}$, $\gamma \in \mathbb{R}$, $\tilde{c}(1-\gamma) = \tilde{c}_1(1-\gamma) = \tilde{c}_2(1-\gamma) = 0$, $\tilde{\varepsilon} \in \mathbb{R}$ if $\gamma = 1$, $\tilde{\varepsilon} \in \{-1, 0, 1\}$ otherwise, $d \in \mathbb{R}$ if $\gamma = 0$, $d = 0$ otherwise.

Type	Subalgebra	Aut $\mathfrak{g}$ class. $\gamma = 1$	Aut $\mathfrak{g}$ classification $\gamma = -1$	Aut $\mathfrak{g}$ classification $(\gamma+1)(\gamma-1) \neq 0$
0	0	0	0	0
$\mathfrak{g}_1$	e_1	e_1	e_1	e_1
	$e_3 + ae_1$		$e_1 + \eta_a e_3$	$e_3 + \eta_a e_1$
	$e_2 + ae_3 + be_1$	e_2	$e_2 + \eta_a e_3$	$e_2 + \eta_a e_3$
	$e_4 + ae_3 + be_2 + ce_1$		$e_2 + \eta_b e_4 + \bar{\eta}_b \eta_c e_3$	$e_4 + \eta_b e_2 + \bar{\eta}_b \eta_c e_1$
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	e_5	e_5	e_5
$2\mathfrak{g}_1$	$e_1,\ e_3$	$e_1,\ e_3$	$e_1,\ e_3$	$e_1,\ e_3$
	$e_1,\ e_2 + ae_3$	$e_1,\ e_2$	$e_1,\ e_2 + \eta_a e_3$	$e_1,\ e_2 + \eta_a e_3$
	$e_1,\ e_4 + ae_3 + be_2$	$e_1,\ e_4$	$e_3,\ e_2 + \eta_b e_4$	$e_1,\ e_4 + \eta_b e_2$
	$e_2 + ae_1,\ e_3 + be_1$		$e_2,\ e_3 + \eta_b e_1$	$e_2,\ e_3 + \eta_b e_1$
	$e_3 + ae_1,\ e_4 + ae_2 + ce_1$	$e_1,\ e_2$	$e_1 + \eta_a e_3,\ e_2 + \eta_a e_4 + \bar{\eta}_a \eta_c e_3$	$e_3 + \eta_a e_1,\ e_4 + \eta_a e_2 + \bar{\eta}_a \eta_c e_1$
	$e_3 + ae_1,\ e_4 + be_2 + ce_1$; $a \neq b$	$e_1,\ e_4$	$e_3 + \frac{a}{b-a} e_1, e_4 + \frac{b}{b-a} e_2$ if $a \in [-b, b]$ or $b = 0$;	$e_3 + \frac{a}{b-a} e_1, e_4 + \frac{b}{b-a} e_2$
			$e_3 + \frac{b}{a-b} e_1,\ e_4 + \frac{a}{a-b} e_2$ otherwise	
	$e_2 + ae_3 + be_1,\ e_4 + ce_3 + de_1$	$e_2,\ e_4$	$e_2 + ade_3,\ e_4 + \eta_a e_1$	$e_2 + ade_3 + \eta_a \bar{\eta}_d e_3,\ e_4 + \eta_{ad} e_1$
$\mathfrak{g}_2$	$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$e_1,\ e_5$	$e_1,\ e_5$	$e_1,\ e_5$
	$e_3 + \tilde{c} e_1,\ e_5 + ae_4 + be_2 + de_1$			$e_3,\ e_5$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$	$e_1,\ e_2,\ e_3$
	$e_1,\ e_3,\ e_4 + ae_2$		$e_1,\ e_2 + \eta_a e_4,\ e_3$	$e_1,\ e_3,\ e_4 + \eta_a e_2$
	$e_1,\ e_2 + ae_3,\ e_4 + be_3$	$e_1,\ e_2,\ e_4$	$e_1,\ e_2 + \eta_a e_3,\ e_4$	$e_1,\ e_2 + \eta_a e_3,\ e_4$
	$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$		$e_1 + \eta_b e_3,\ e_2 + \bar{\eta}_b \eta_c e_3,\ e_4$	$e_2,\ e_3 + \eta_b e_1,\ e_4 + \bar{\eta}_b \eta_c e_1$
$\mathfrak{g}_{3,4}^{\gamma}$	$e_1,\ e_3,\ e_5 + ae_4 + be_2$	$e_1,\ e_3,\ e_5$	$e_1,\ e_3,\ e_5$	$e_1,\ e_3,\ e_5$
$\mathfrak{g}_{3,2}$	$e_1,\ e_2 + \tilde{c} e_3,\ e_5 + ae_4 + be_3$	$e_1,\ e_2,\ e_5$	$e_1,\ e_2,\ e_5$	$e_1,\ e_2,\ e_5$
$\mathfrak{g}_{3,2}$ ($\mathfrak{g}_{3,1}$)	$e_3 + \tilde{c}_1 e_1,\ e_4 + \tilde{c}_1 e_2 + \tilde{c}_2 e_1,$ $e_5 + ae_2 + be_1$			$e_3,\ e_4,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}_{4,2}^{\gamma}$	$e_1,\ e_2,\ e_3,\ e_5 + ae_4$	$e_1,\ e_2,\ e_3,\ e_5$	$e_1,\ e_2,\ e_3,\ e_5$	$e_1,\ e_2,\ e_3,\ e_5$
$\mathfrak{g}_{4,2}^{1/\gamma}$ ($\mathfrak{g}_{4,3}$)	$e_1,\ e_3,\ e_4 + \tilde{c} e_2,\ e_5 + ae_2$			$e_1,\ e_3,\ e_4,\ e_5$

Table A.21. Realizations of $\mathfrak{g}^{p,s}_{5,16}$, $s \neq 0$; $\operatorname{cs}(x; a, b) = a\cos x + b\sin x$.

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$-class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + x_2\partial_2 + (x_1+x_2)\partial_1 + (-sx_3+px_4)\partial_4 + (px_3+sx_4)\partial_3$	0
$\mathfrak{g}_1$	e_3	$\partial_1,\ \partial_2,\ \tan(sx_4)\partial_3,\ \partial_3,\ \partial_4 + x_2\partial_2 + (x_1+x_2)\partial_1 + (p+s\tan(sx_4))x_3\partial_3$	e_3
	$e_4 + ae_3$	$\partial_1,\ \partial_2,\ \partial_3,\ \frac{a\operatorname{cs}(sx_4;a,1)}{\operatorname{cs}(sx_4;-1,a)}\partial_3,\ \partial_4 + x_2\partial_2 + (x_1+x_2)\partial_1 + x_3\frac{\operatorname{cs}(sx_4;p+as,s-ap)}{\operatorname{cs}(sx_4;1,-a)}\partial_3$	
	$e_1 + ae_3 + be_4$	$\mathrm{e}^{(p-1)x_4}\operatorname{cs}(sx_4;-b,a)\partial_3 - \mathrm{e}^{(p-1)x_4}\operatorname{cs}(sx_4;a,b)\partial_2,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_1\partial_1 +$ $(x_1\mathrm{e}^{(p-1)x_4}\operatorname{cs}(sx_4;-b,a) - sx_2 + px_3)\partial_3 + (-x_1\mathrm{e}^{(p-1)x_4}\operatorname{cs}(sx_4;a,b) + px_2 + sx_3)\partial_2$	$e_1 + \eta_{a,b}e_3$
	$e_2 + ae_1 + be_3 + ce_4$	$\partial_1,\ \mathrm{e}^{(p-1)x_4}\operatorname{cs}(sx_4;-c,b)\partial_3 - \mathrm{e}^{(p-1)x_4}\operatorname{cs}(sx_4;b,c)\partial_2 - (a+x_4)\partial_1,\ \partial_2,\ \partial_3,$ $\partial_4 + x_1\partial_1 + (-sx_2+px_3)\partial_3 + (px_2+sx_3)\partial_2$	$e_2 + \eta_{b,c}e_3$
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,$ $(-c+x_2)\partial_2 + (-d+x_1+x_2)\partial_1 + (-a-sx_3+px_4)\partial_4 + (-b+px_3+sx_4)\partial_3$	e_5
$2\mathfrak{g}_1$	$e_3,\ e_4$	$\partial_1,\ \partial_2,\ 0,\ 0,\ \partial_3 + x_2\partial_2 + (x_1+x_2)\partial_1$	$e_3,\ e4$
	$e_1 + ae_4,\ e_3$	$-\frac{a\mathrm{e}^{(p-1)x_3}}{\cos(sx_3)}\partial_2,\ \partial_1,\ \tan(sx_3)\partial_2,\ \partial_2,\ \partial_3 + x_1\partial_1 + \left(-\frac{ax_1\mathrm{e}^{(p-1)x_3}}{\cos(sx_3)} + (p+s\tan(sx_3))x_2\right)\partial_2$	$e_3,\ e_1 + \eta_a e_4$
	$e_1 + ae_3,\ e_4 + be_3$	$\partial_1,\ -\frac{b\mathrm{e}^{(p-1)x_3}}{\cos(sx_3)}\partial_2 + (-a-x_3)\partial_1,\ \tan(sx_3)\partial_2,\ \partial_2,\ \partial_3 + x_1\partial_1 + x_2(p+s\tan(sx_3))\partial_2$	
	$e_2 + ae_1 + be_4,\ e_3$	$\frac{a\mathrm{e}^{(p-1)x_3}}{\operatorname{cs}(sx_3;-1,b)}\partial_2,\ \partial_1,\ \partial_2,\ \frac{\operatorname{cs}(sx_3;b,1)}{\operatorname{cs}(sx_3;-1,b)}\partial_2,\ \partial_3 + x_1\partial_1 + \frac{x_2\operatorname{cs}(sx_3;p+bs,s-bp) - a\mathrm{e}^{(p-1)x_3}}{\operatorname{cs}(sx_3;1,-b)}\partial_2$	$e_3,\ e_2 + \eta_b e_4$
	$e_1 + ae_2 + be_3,\ e_4 + ce_3$	$-\frac{a}{1+ax_3}\partial_1 + \frac{b\mathrm{e}^{(p-1)x_3}}{(1+ax_3)\operatorname{cs}(sx_3;-1,c)}\partial_2,\ \partial_1,\ \partial_2,\ \frac{\operatorname{cs}(sx_3;c,1)}{\operatorname{cs}(sx_3;-1,c)}\partial_2,$ $\partial_3 + \left(1 - \frac{a}{1+ax_3}\right)x_1\partial_1 + \frac{x_2(1+ax_3)\operatorname{cs}(sx_3;p+cs,s-cp) - bx_1\mathrm{e}^{(p-1)x_3}}{(1+ax_3)\operatorname{cs}(sx_3;1,-c)}\partial_2$	$\bar\eta_a e_1 + \eta_a e_2 + \eta_b e_4,\ e_3$
	$e_1 + ae_3 + be_4,$ $e_2 + ce_3 + de_4$	$\mathrm{e}^{(p-1)x_3}\operatorname{cs}(sx_3;-b,a)\partial_2 - \mathrm{e}^{(p-1)x_3}\operatorname{cs}(sx_3;a,b)\partial_1,\ -\mathrm{e}^{(p-1)x_3}(\operatorname{cs}(sx_3;c,d) -$ $x_3\operatorname{cs}(sx_3;a,b))\partial_1 + \mathrm{e}^{(p-1)x_3}(\operatorname{cs}(sx_3;c,-d) + x_3\operatorname{cs}(sx_3;-a,b))\partial_2,\ \partial_1,\ \partial_2,$ $\partial_3 + (-sx_1+px_2)\partial_2 + (px_1+sx_2)\partial_1$	$e_1 + \frac{a^2+b^2}{ad-bc}e_3,\ e_2 + e_4$ if $ad \neq bc$; $e_1 + \eta_{a,b}e_3,$ $e_2 + \bar\eta_{a,b}\eta_{c,d}e_4$ otherwise
$\mathfrak{g}_2$	$e_1,\ e_5 + ae_4 + be_3 + ce_2$	$0,\ \partial_1,\ \partial_2,\ \partial_3,\ (-c+x_1)\partial_1 + (-a-sx_2+px_3)\partial_3 + (-b+px_2+sx_3)\partial_2$	$e_1,\ e_5$
$3\mathfrak{g}_1$	$e_1 + ae_4,\ e_2 + be_4,\ e_3$	$-\frac{a\mathrm{e}^{(p-1)x_2}}{\cos(sx_2)}\partial_1,\ -\frac{(b-ax_2)\mathrm{e}^{(p-1)x_2}}{\cos(sx_2)}\partial_1,\ \tan(sx_2)\partial_1,\ \partial_1,\ \partial_2 + x_1(p+s\tan(sx_2))\partial_1$	$e_1 + \eta_a e_4,\ e_2 + \bar\eta_a\eta_b e_4,\ e_3$
	$e_1 + ae_3,\ e_2 + be_3,\ e_4 + ce_3$	$0,\ \partial_1,\ 0,\ 0,\ \partial_2 + x_1\partial_1$	
	$e_1,\ e_3,\ e_4$	$\partial_1,\ (-a-x_2)\partial_1,\ 0,\ 0,\ \partial_2 + x_1\partial_1$	$e_1,\ e_3,\ e_4$
	$e_2 + ae_1,\ e_3,\ e_4$	$\frac{a\mathrm{e}^{(p-1)x_2}}{\operatorname{cs}(sx_2;-1,c)}\partial_1,\ \frac{(b-ax_2)\mathrm{e}^{(p-1)x_2}}{\operatorname{cs}(sx_2;-1,c)}\partial_1,\ \partial_1,\ \frac{\operatorname{cs}(sx_2;c,1)}{\operatorname{cs}(sx_2;-1,c)}\partial_1,\ \partial_2 + x_1\frac{\operatorname{cs}(sx_2;p+cs,s-cp)}{\operatorname{cs}(sx_2;1,-c)}\partial_1$	$e_2,\ e_3,\ e_4$
$\mathfrak{g}_{3,2}$	$e_1,\ e_2,\ e_5 + ae_4 + be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ (-a-sx_1+px_2)\partial_2 + (-b+px_1+sx_2)\partial_1$	$e_1,\ e_2,\ e_5$
$\mathfrak{g}^{p/s}_{3,5}$	$e_3,\ e_4,\ e_5 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ 0,\ 0,\ (-a+x_2)\partial_2 + (-b+x_1+x_2)\partial_1$	$e_3,\ e_4,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$
$\mathfrak{g}^{1/s,p/s}_{4,6}$	$e_1,\ e_3,\ e_4,\ e_5 + ae_2$	$0,\ \partial_1,\ 0,\ 0,\ (-a+x_1)\partial_1$	$e_1,\ e_3,\ e_4,\ e_5$

Table A.22. Realizations of $\mathfrak{g}_{5,17}^{p,q,s}$, $s \neq 0$; $\bar{c}$ and $\bar{d}$ are nonzero iff $p = q$ and $s = \pm 1$, $\mathrm{cs}(x; a, b) = a\cos x + b\sin x$.

Type	Subalgebra	Corresponding realization
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + (px_1 + x_2)\partial_1 + (-x_1 + px_2)\partial_2 + (-sx_3 + qx_4)\partial_4 + (qx_3 + sx_4)\partial_3$
$\mathfrak{g}_1$	e_1	$\tan x_4\partial_1,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + x_1(p + \tan x_4)\partial_1 + (-sx_2 + qx_3)\partial_3 + (qx_2 + sx_3)\partial_2$
	$e_2 + ae_1$	$\partial_1,\ \frac{\mathrm{cs}(x_4;a,1)}{\mathrm{cs}(x_4;-1,a)}\partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + \frac{\mathrm{cs}(x_4;a+p,1-ap)}{\mathrm{cs}(x_4;1,-a)}\partial_1 + (-sx_2 + qx_3)\partial_3 + (qx_2 + sx_3)\partial_2$
	$e_3 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ -\frac{\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;b,a)}{\cos(sx_4)}\partial_1 - \frac{\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;a,-b)}{\cos(sx_4)}\partial_2 + \tan(sx_4)\partial_3,\ \partial_3,$ $\partial_4 + x_3(q + s\tan(sx_4))\partial_3 + \left(px_1 + x_2 - \frac{sx_3\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;b,a)}{\cos(sx_4)}\right)\partial_1 + (-x_1 + px_2 + \frac{sx_3\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;-a,b)}{\cos(sx_4)}\partial_2)$
	$e_4 + ae_3 + be_2 + ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \frac{\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;-b,c)}{\mathrm{cs}(sx_4;1,-a)}\partial_2 + \frac{\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;c,b)}{\mathrm{cs}(sx_4;-1,a)}\partial_1 + \frac{\mathrm{cs}(sx_4;a,1)}{\mathrm{cs}(sx_4;-1,a)}\partial_3,$ $\partial_4 + \frac{x_3\,\mathrm{cs}(sx_4;q+as,s-aq)}{\mathrm{cs}(sx_4;1,-a)}\partial_3 + \left(px_1 + x_2 + \frac{sx_3\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;c,b)}{\mathrm{cs}(sx_4;1,-a)}\right)\partial_1 + \left(-x_1 + px_2 + \frac{sx_3\mathrm{e}^{(p-q)x_4}\,\mathrm{cs}(x_4;c,b)}{\mathrm{cs}(sx_4;1,-a)}\right)\partial_2$
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ (-d + px_1 + x_2)\partial_1 + (-c - x_1 + px_2)\partial_2 + (-a - sx_3 + qx_4)\partial_4 + (-b + qx_3 + sx_4)\partial_3$
$\mathfrak{g}_2$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3 + (-sx_1 + qx_2)\partial_2 + (qx_1 + sx_2)\partial_1$
	$e_1,\ e_3 + ae_2$	$\tan x_3\partial_1,\ \partial_1,\ -\frac{a\mathrm{e}^{(p-q)x_3}}{\cos x_3\cos(sx_3)}\partial_1 + \tan(sx_3)\partial_2,\ \partial_2,\ \partial_3 + x_2(q + s\tan(sx_3))\partial_2 + \left(px_1 + x_1\tan x_3 - \frac{asx_2\mathrm{e}^{(p-q)x_3}}{\cos x_3\cos(sx_3)}\right)\partial_1$
	$e_1,\ e_4 + ae_3 + be_2$	$\tan x_3\partial_1,\ \partial_1,\ \partial_2,\ \frac{b\mathrm{e}^{(p-q)x_3}}{\cos x_3\,\mathrm{cs}(sx_3;-1,a)}\partial_1 + \frac{\mathrm{cs}(sx_3;a,1)}{\mathrm{cs}(sx_3;-1,a)}\partial_2,$ $\partial_3 + \frac{x_2\,\mathrm{cs}(sx_3;q+as,s-aq)}{\mathrm{cs}(sx_3;1,-a)}\partial_2 + \left(p + \tan x_3 + \frac{bsx_2\mathrm{e}^{(p-q)x_3}}{\cos x_3\,\mathrm{cs}(sx_3;1,-a)}\right)$
	$e_2 + ae_1,\ e_3 + be_1$	$\partial_1,\ \frac{\mathrm{cs}(x_3;a,1)}{\mathrm{cs}(x_3;-1,a)}\partial_1,\ \frac{b\mathrm{e}^{(p-q)x_3}}{\cos(sx_3)\,\mathrm{cs}(x_3;-1,a)}\partial_1 + \tan(sx_3)\partial_2,\ \partial_2,$ $\partial_3 + x_2(q + s\tan(sx_3))\partial_2 + \frac{x_1\cos(sx_3)\,\mathrm{cs}(x_3;a+p,1-ap) - bsx_2\mathrm{e}^{(p-q)x_3}}{\cos(sx_3)\,\mathrm{cs}(x_3;1,-a)}\partial_1$
	$e_2 + ae_1,\ e_4 + be_3 + ce_1$	$\partial_1,\ \frac{\mathrm{cs}(x_3;a,1)}{\mathrm{cs}(x_3;-1,a)}\partial_1,\ \partial_2,\ -\frac{c\mathrm{e}^{(p-q)x_3}}{\mathrm{cs}(x_3;1,-a)\,\mathrm{cs}(sx_3;1,-b)}\partial_1 + \frac{\mathrm{cs}(sx_3;b,1)}{\mathrm{cs}(sx_3;-1,b)}\partial_2,$ $\partial_3 + \frac{x_2\,\mathrm{cs}(sx_3;q+bs,s-bq)}{\mathrm{cs}(sx_3;1,-b)}\partial_2 + \frac{x_1\,\mathrm{cs}(x_3;a+p,1-ap)\,\mathrm{cs}(sx_3;1,-b) + csx_2\mathrm{e}^{(p-q)x_3}}{\mathrm{cs}(x_3;1,-a)\,\mathrm{cs}(sx_3;1,-b)}\partial_1$
	$e_3 + ae_2 + be_1,\ e_4 + ce_2 + de_1$	$\partial_1,\ \partial_2,$ $-\mathrm{e}^{(p-q)x_3}(\sin x_3\,\mathrm{cs}(sx_3;a,c) + \cos x_3\,\mathrm{cs}(sx_3;b,d))\partial_1 + \mathrm{e}^{(p-q)x_3}(-\cos x_3\,\mathrm{cs}(sx_3;a,c) + \sin x_3\,\mathrm{cs}(sx_3;b,d))\partial_2,$ $-\mathrm{e}^{(p-q)x_3}(\sin x_3\,\mathrm{cs}(sx_3;c,-a) + \cos x_3\,\mathrm{cs}(sx_3;d,-b))\partial_1 - \mathrm{e}^{(p-q)x_3}\cos(x_3\,\mathrm{cs}(sx_3;c,-a) + \sin x_3\,\mathrm{cs}(sx_3,-d,b))\partial_2,$ $\partial_3 + (px_1 + x_2)\partial_1 + (-x_1 + px_2)\partial_2$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$0,\ 0,\ \tan(sx_2)\partial_1,\ \partial_1,\ \partial_2 + x_1(q + \tan(sx_2))\partial_1$
	$e_1,\ e_2,\ e_4 + ae_3$	$0,\ 0,\ \partial_1,\ \frac{\mathrm{cs}(sx_2;a,1)}{\mathrm{cs}(sx_2;-1,a)}\partial_1,\ \partial_2 + \frac{\mathrm{cs}(sx_2;q+as,s-aq)}{\mathrm{cs}(sx_2;1,-a)}\partial_1$
	$e_1,\ e_3 + ae_2,\ e_4 + be_2$	$\tan x_2\partial_1,\ \partial_1,\ -\frac{\mathrm{e}^{(p-q)x_2}\,\mathrm{cs}(sx_2;a,b)}{\cos x_2}\partial_1,\ -\frac{\mathrm{e}^{(p-q)x_2}\,\mathrm{cs}(sx_2;b,-a)}{\cos(x_2)}\partial_1,\ \partial_2 + x_1(p + \tan x_2)\partial_1$
	$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$\partial_1,\ \frac{\mathrm{cs}(x_2;a,1)}{\mathrm{cs}(x_2;-1,a)}\partial_1,\ \frac{\mathrm{e}^{(p-q)x_2}\,\mathrm{cs}(sx_2;b,c)}{\mathrm{cs}(x_2;-1,a)}\partial_1,\ \frac{\mathrm{e}^{(p-q)x_2}\,\mathrm{cs}(sx_2;-c,b)}{\mathrm{cs}(x_2;1,-a)}\partial_1,\ \partial_2 + \frac{x_1\,\mathrm{cs}(x_2;a+p,1-ap)}{\mathrm{cs}(x_2;1,-a)}\partial_1$
$\mathfrak{g}_{3,5}^{p}$	$e_1,\ e_2,\ e_5 + ae_4 + be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ (-a - sx_1 + qx_2)\partial_2 + (-b + qx_1 + sx_2)\partial_1$
$\mathfrak{g}_{3,5}^{q/s}$	$e_3 + \bar{c}e_2 + \bar{d}e_1,$ $e_4 + s\bar{d}e_2 - s\bar{c}e_1,$ $e_5 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ -d\partial_1 - c\partial_2,\ cs\partial_1 - ds\partial_2,\ (-a - x_1 + px_2)\partial_2 + (-b + px_1 + x_2)\partial_1$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$

Type	Subalgebra	Aut $\mathfrak{g}$ class ($p=q$, $s=\pm1$)	Aut $\mathfrak{g}$ class ($p=q=0$, $s\neq\pm1$)	Aut $\mathfrak{g}$ class (otherwise)
0	0	0	0	0
$\mathfrak{g}_1$	e_1	e_1	e_1	e_1
	e_2+ae_1			
	$e_3+ae_2+be_1$		$e_1+\eta_{a,b}e_3$	$e_3+\eta_{a,b}e_1$
	$e_4+ae_3+be_2+ce_1$		$e_1+\eta_{b,c}e_3$	$e_3+\eta_{b,c}e_1$
	$e_5+ae_4+be_3+ce_2+de_1$	e_5	e_5	e_5
$\mathfrak{g}_2$	e_1, e_2	e_1, e_2	e_1, e_2	e_1, e_2
	e_1, e_3+ae_2	e_1, e_3	e_1, $e_3+\eta_a e_2$	e_1, $e_3+\eta_a e_2$
	e_1, $e_4+ae_3+be_2$		e_1, $e_3+\eta_b e_2$	e_1, $e_3+\eta_b e_2$
	e_2+ae_1, e_3+be_1		e_1, $e_3+\eta_b e_2$	e_1, $e_3+\eta_b e_2$
	e_2+ae_1, $e_4+be_3+ce_1$		e_1, $e_3+\eta_c e_2$	e_1, $e_3+\eta_c e_2$
	$e_3+ae_2+be_1$, $e_4+ce_2+de_1$	e_1, e_2 if $a=0$ and $b=c$; e_1, e_3 otherwise	e_1, e_2 if $a=b=c=d=0$; e_1, e_2+e_4 if $bc=ad$, a, b, c or d is nonzero; $e_3+(A\pm\sqrt{A^2-1})e_1$, e_4+e_2 otherwise*	e_3, e_4 if $a=b=c=d=0$; e_3, e_4+e_2 if $bc=ad$, a, b, c or d is nonzero; $e_3+(A\pm\sqrt{A^2-1})e_1$, e_4+e_2 otherwise*
$3\mathfrak{g}_1$	e_1, e_2, e_3	e_1, e_2, e_3	e_1, e_2, e_3	e_1, e_2, e_3
	e_1, e_2, e_4+ae_3			
	e_1, e_3+ae_2, e_4+be_2		e_1, e_3, $e_2+\eta_{a,b}e_4$	e_1, e_3, $e_4+\eta_{a,b}e_2$
	e_2+ae_1, e_3+be_1, e_4+ce_1		e_1, e_3, $e_2+\eta_{b,c}e_4$	e_1, e_3, $e_4+\eta_{b,c}e_2$
$\mathfrak{g}_{3,5}^{p}$	e_1, e_2, $e_5+ae_4+be_3$	e_1, e_2, e_5	e_1, e_2, e_5	e_1, e_2, e_5
$\mathfrak{g}_{3,5}^{q/s}$	$e_3+\bar{c}e_2+\bar{d}e_1$, $e_4+s\bar{d}e_2-s\bar{c}e_1$, $e_5+ae_2+be_1$			e_3, e_4, e_5
$4\mathfrak{g}_1$	e_1, e_2, e_3, e_4	e_1, e_2, e_3, e_4	e_1, e_2, e_3, e_4	e_1, e_2, e_3, e_4

Table A.23. Subalgebras of $\mathfrak{g}_{5,17}^{p,q,s}$, $s\neq 0$; $\bar{c}$ and $\bar{d}$ are nonzero iff $p=q$ and $s=\pm1$. *) $A=\frac{a^2+b^2+c^2+d^2}{2(bc-ad)}$. It holds, that $A^2\geq 1$. The sign can be chosen such that the absolute value of the coefficient $A\pm\sqrt{A^2-1}$ is greater or equal to one.

Table A.24. Realizations of $\mathfrak{g}_{5,18}^{p}$, $p \geq 0$; $\mathrm{cs}(x; a, b) = a\cos x + b\sin x$.

Type	Subalgebra	Corresponding realization	Aut $\mathfrak{g}$ class
0	0	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,\ \partial_5 + \partial_1(px_1 + x_2 + x_3) + \partial_2(-x_1 + px_2 + x_4) + \partial_3(px_3 + x_4) + \partial_4(-x_3 + px_4)$	0
$\mathfrak{g}_1$	e_1	$\tan x_4\partial_1,\ \partial_1,\ \partial_2,\ \partial_3,\ \partial_4 + \partial_2(px_2 + x_3) + (px_1 + (x_1 + x_2)\tan x_4 + x_3)\partial_1 + \partial_3(-x_2 + px_3)$	e_1
	$e_2 + ae_1$	$\partial_1,\ \frac{\mathrm{cs}(x_4;a,1)}{\mathrm{cs}(x_4;-1,a)}\partial_1,\ \partial_2,\ \partial_3,$ $\partial_4 + \partial_2(px_2 + x_3) + \partial_3(-x_2 + px_3) + \frac{\mathrm{cs}(x_4;(p+a)x_1+x_2-ax_3),(1-ap)x_1-ax_2-x_3)}{\mathrm{cs}(x_4,1,-a)}\partial_1$	
	$e_3 + ae_2 + be_1$	$\partial_1,\ \partial_2,\ \tan x_4\partial_3 - (b + a\tan x_4 + x_4)\partial_1 + (-a + (b + x_4)\tan x_4)\partial_2,\ \partial_3,\ \partial_4 + x_3(p + \tan x_4)\partial_3 +$ $(px_1 + x_2 - bx_3 - ax_3\tan x_4 - x_3x_4)\partial_1 + (-x_1 + px_2 + x_3 - ax_3 + x_3(b + x_4)\tan x_4)\partial_2$	e_3
	$e_4 + ae_3 + be_2 + ce_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \frac{\mathrm{cs}(x_4;a,1)}{\mathrm{cs}(x_4;-1,a)}\partial_3 - \frac{\mathrm{cs}(x_4;b+x_4,c+ax_4)}{\mathrm{cs}(x_4;1,-a)}\partial_1 + \frac{\mathrm{cs}(x_4;-b-x_4,c+ax_4)}{\mathrm{cs}(x_4;1,-a)}\partial_2,$ $\partial_4 + \frac{x_3\,\mathrm{cs}(x_4;a+p,1-ap)}{\mathrm{cs}(x_4;1,-a)}\partial_3 + \frac{\mathrm{cs}(x_4;px_1+x_2+(1+c)x_3+ax_3x_4,-apx_1-ax_2+(b-a)x_3+x_3x_4)}{\mathrm{cs}(x_4;1,-a)}\partial_1 +$ $\frac{\mathrm{cs}(x_4;-x_1+px_2+x_3+bx_3x_4,ax_1-apx_2-(a+c)x_3-ax_3x_4)}{\mathrm{cs}(x_4;1,-a)}\partial_2$	
	$e_5 + ae_4 + be_3 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4,$ $\partial_1(-d + px_1 + x_2 + x_3) + \partial_2(-c - x_1 + px_2 + x_4) + \partial_3(-b + px_3 + x_4) + \partial_4(-a - x_3 + px_4)$	e_5
$\mathfrak{g}_2$	$e_1,\ e_2$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_3 + \partial_1(px_1 + x_2) + \partial_2(-x_1 + px_2)$	$e_1,\ e_2$
	$e_1,\ e_3 + ae_2$	$\tan x_3\partial_1,\ \partial_1,\ -\frac{a}{\cos^2 x_3}\partial_1 + \tan x_3\partial_2,\ \partial_2,$ $\partial_3 + x_2(p + \tan x_3)\partial_2 + \frac{px_1+(1-2a)x_2+x_1\sin(2x_3)+(px_1+x_2)\cos(2x_3)}{2\cos^2 x_3}\partial_1$	$e_1,\ e_3$
	$e_1,\ e_4 + ae_3 + be_2$	$\tan x_3\partial_1,\ \partial_1,\ \partial_2,\ \frac{\mathrm{cs}(x_3;a,1)}{\mathrm{cs}(x_3;-1,a)}\partial_2 + \frac{b+x_3}{(-1+a\tan x_3)\cos^2 x_3}\partial_1,$ $\partial_3 + \frac{x_2\,\mathrm{cs}(x_3;a+p,1-ap)}{\mathrm{cs}(x_3;1,-a)}\partial_2 + \frac{(a-p)x_1+(a-2b)x_2-2x_2x_3+\mathrm{cs}(2x_3;-(a+p)x_1-ax_2,(ap-1)x_1-x_2)}{2(-1+a\tan x_3)\cos^2 x_3}\partial_1$	$e_1,\ e_4 + ae_3$
	$e_2 + ae_1,\ e_3 + be_1$	$\partial_1,\ \frac{\mathrm{cs}(x_3;a,1)}{\mathrm{cs}(x_3;-1,a)}\partial_1,\ \tan x_3\partial_2 + \frac{b+x_3}{(-1+a\tan x_3)\cos^2 x_3}\partial_1,\ \partial_2,$ $\partial_3 + x_2(p + \tan x_3)\partial_2 + \frac{-(a+p)x_1+(a+2b)x_2+2x_2x_3+\mathrm{cs}(2x_3;-(p+a)x_1+ax_2,(ap-1)x_1+x_2)}{2(-1+a\tan x_3)\cos^2 x_3}$	$e_1,\ e_4 - ae_3$
	$e_2 + ae_1,\ e_4 + be_3 + ce_1$	$\partial_1,\ \frac{\mathrm{cs}(x_3;a,1)}{\mathrm{cs}(x_3;-1,a)}\partial_1,\ \partial_2,\ \frac{\mathrm{cs}(x_3;b,1)}{\mathrm{cs}(x_3;-1,b)}\partial_2 - \frac{(c+(b-a)x_3)}{\mathrm{cs}(x_3;1,-a)\,\mathrm{cs}(x_3;1,-b)}\partial_1,\ \partial_3 + \frac{\mathrm{cs}(x_3;b+p,1-bp)}{\mathrm{cs}(x_3;1,-b)}\partial_2 +$ $\frac{(a-b+p+abp)x_1+(1+ab+2c)x_2+2(b-a)x_2x_3+\mathrm{cs}(2x_3;(a+b+p-bp)x_1+(1-b)x_2,(1-bp-ab-ap)x_1-(a+b)x_2)}{2\,\mathrm{cs}(x_3;1,-a)\,\mathrm{cs}(x_3;1,-b)}$	$e_1,\ e_2$ if $a = b \wedge ab = -1$; $e_1,\ e_3$ if $a = b \wedge ab \neq -1$; $e_1,\ e_4 + \frac{1+ab}{a-b}e_3$ if $a \neq b$
	$e_3 + ae_2 + be_1,$ $e_4 + ce_2 + de_1$	$\partial_1,\ \partial_2,\ \frac{1}{2}(d - a + \mathrm{cs}(2x_3; -a - d, b - c))\partial_2 + \frac{1}{2}(-b - c - 2x_3 + \mathrm{cs}(2x_3; -b + c, -a - d))\partial_1,$ $\frac{1}{2}(a - d + \mathrm{cs}(2x_3; -a - d, b - c))\partial_1 + \frac{1}{2}(-b - c - 2x_3 + \mathrm{cs}(2x_3; b - c, a + d))\partial_2,$ $\partial_3 + \partial_1(px_1 + x_2) + \partial_2(-x_1 + px_2)$	$e_3,\ e_4 +$ $\sqrt{(a + d)^2 + (c - b)^2}e_1$
$3\mathfrak{g}_1$	$e_1,\ e_2,\ e_3$	$0,\ 0,\ \tan x_2\partial_1,\ \partial_1,\ \partial_2 + x_1(p + \tan x_2)\partial_1$	$e_1,\ e_2,\ e_3$
	$e_1,\ e_2,\ e_4 + ae_3$	$0,\ 0,\ \partial_1,\ \frac{\mathrm{cs}(x_2;a,1)}{\mathrm{cs}(x_2;-1,a)}\partial_1,\ \partial_2 + \frac{x_1\,\mathrm{cs}(x_2;a+p,1-ap)}{\mathrm{cs}(x_2;1,-a)}\partial_1$	
	$e_1,\ e_3 + ae_2,\ e_4 + be_2$	$\tan x_2\partial_1,\ \partial_1,\ (-a - (b + x_2)\tan x_2)\partial_1,\ (-b - x_2 + a\tan x_2)\partial_1,\ \partial_2 + x_1(p + \tan x_2)\partial_1$	$e_1,\ e_3,\ e_4$
	$e_2 + ae_1,\ e_3 + be_1,\ e_4 + ce_1$	$\partial_1,\ \frac{\mathrm{cs}(x_2;a,1)}{\mathrm{cs}(x_2;-1,a)}\partial_1,\ \frac{\mathrm{cs}(x_2;-b-x_2,-c+ax_2)}{\mathrm{cs}(x_2;1,-a)}\partial_1,\ \frac{\mathrm{cs}(x_2;-c+ax_2,b+x_2)}{\mathrm{cs}(x_2;1,-a)}\partial_1,\ \partial_2 + \frac{x_1\,\mathrm{cs}(x_2;a+p,1-ap)}{\mathrm{cs}(x_2;1,-a)}\partial_1$	
$\mathfrak{g}_{3,5}^{p}$	$e_1,\ e_2,\ e_5 + ae_4 + be_3$	$0,\ 0,\ \partial_1,\ \partial_2,\ \partial_1(-b + px_1 + x_2) + \partial_2(-a - x_1 + px_2)$	$e_1,\ e_2,\ e_5$
$4\mathfrak{g}_1$	$e_1,\ e_2,\ e_3,\ e_4$	$0,\ 0,\ 0,\ 0,\ \partial_1$	$e_1,\ e_2,\ e_3,\ e_4$