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Generalized stochastic processes with applications to financial markets

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Supervisor: Petr Jizba

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Zobecněné stochastické procesy a jejich využití na finančních trzích

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Supervisor: Petr Jizba

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Abstract: We introduce general concepts of the option pricing theory. Key notions like a completeness of the market and martingale measures are discussed. In particular, the Black-Scholes theory is introduced and its limitations and possible generalizations are discussed in detail. A theory of Levy processes and stochastic calculus for them are introduced. A key result - generalized Ito lemma - is presented. This theory is then applied for the option pricing in exp-Levy models. The concept of fractional derivatives and fractional differential equations is discussed in a connection with an anomalous diffusion and applied for the option pricing. A connection between quantum mechanics with non-self-adjoint Hamiltonians and the option pricing is established. Both Hamiltonian and path integral formulation are presented. The quantum field theory of forward interest rates is introduced and applied for the pricing of bond options.

Key words: option pricing, Levy processes, fractional processes, quantum finance

Název práce:
Zobecněné stochastické procesy a jejich využití na finančních trzích

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Abstrakt: Představíme obecnou teorii oceňování opcí a zavedeme důležité pojmy jako úplnost trhu a risk neutrální míry. Také představíme Black-Scholesova teorii a budeme diskutovat její nedostatky a možná zobecnění. Zavedeme teorii Levyho procesú a vybudujeme stochastický počet pro nespojité procesy. Klíčovým výsledkem je zobecněná verze Itova lemma. Tato teorie je poté aplikována na oceňování opcí v exp-Levy modelech. Koncept frakčních derivací je představen ve spojitosti s anomální difusí a frakční procesy jsou aplikovány při oceňování opcí. Demonstrujeme spojitost mezi teorií oceňování opcí a kvantovou mechanikou s nesamosdruženými Hamiltoniany. Odvodíme jak Hamiltonovskou formulaci teorie oceňování opcí tak i formulaci v řeči dráhových integrálů. Formulujeme kvantovou teorie pole úrokovćh sazeb a aplikujeme ji při oceňování opcí na dluhopisy.

Klíčová slova: oceňování opcí, Levyho procesy, frakční procesy, kvantové finance

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Introduction

The theory of stochastic processes forms a very important part of both theoretical and applied mathematics nowadays. Possible applications of stochastic processes range from an astrophysics, a quantum theory and diffusion related problems to topics like a weather forecasting, insurance claims and quantitative finance. The field of stochastic processes has been dominated by diffusion processes driven by Brownian dynamics for long time. The theory of Brownian motion has been developed by people like Einstein, Langevin, Wiener and many others. Today, the theory of Brownian diffusion is summarized by so called Ito calculus developed by Japanese mathematician Kiyosi Ito. However, a need to describe complex dynamical systems with non-trivial correlations led to the shift of a focus from Brownian diffusions to more general classes of stochastic processes.

The main objective of this thesis is to apply continuous-time processes beyond the framework of Ito calculus to the option pricing problem. The option pricing problem is one of the typical problems in quantitative finance. The first rigorous mathematical formula for the pricing of European options was derived by F. Black and M. Scholes in 1973. Their formula was praised for some time but it turned out to be deeply flawed. It fails miserably in situations when market is not in equilibrium, typically during crises. Huge swings in asset prices occur in such situations and Brownian dynamics cannot well describe such behaviour. This led to a development of new models that would be able to better describe this type of behaviour. We will focus on two possible generalizations in particular. A generalization based on discontinuous Levy processes and one based on processes describing an anomalous diffusion will be analysed.

In the first chapter we will introduce general ideas and concepts of the option pricing theory. The notions like a completeness of the market and martingale measures will be discussed. The theory will be presented in very general settings so it can be applied to a range of models. In particular, standard Black-Scholes theory will be presented, its limitations will be discussed and number of possible generalizations will be outlined. We will also discuss the empirical properties of asset returns, a self-similarity of financial markets and other related concepts. We will conclude that models based on Brownian motion are in many ways not optimal for description of financial markets.

In the second chapter, we will deal with Levy processes and their applications in the option pricing. Levy processes are processes with independent and stationary increments. The famous Levy-Ito decomposition states that every Levy process can be written as a superposition of Brownian diffusion, a drift and a pure jump component driven by Poison-type dynamics. The basic results about Levy processes are reviewed and the stochastic calculus for discontinuous processes is developed. The key result is generalized Ito lemma. This theory is then applied for the option pricing in exp-Levy models. This generalization of the standard theory is therefore based on adding jumps to a driving noise. These jumps are well-suited for describing huge price swings that can be observed on financial markets. We build a robust theory for the option pricing in exp-Levy models, derive a generalized version of the Black-Scholes equation, introduce a change of a measure technique for Levy models and discuss some numerical methods for calculating the option prices.

In the third chapter, a more physically motivated approach is discussed. A standard diffusion is described by Fokker-Planck equation. We introduce concepts of fractional derivatives, i.e. derivatives of non-integer order and fractional differential equations. Then we derive so called fractional Fokker-Plack equation from the continuous time random walk. The solutions of fractional F.-P. equation are called fractional processes and they play an important role in describing an anomalous diffusion and an anomalous transport nowadays. We obtain stable processes as solutions of a space fractional diffusion. The fractional processes are then applied to the option pricing.

There is a deep connection between a quantum theory and a theory of stochastic processes [19]. The similarity between Fokker-Planck and Schroedinger equations leads to a correspondence between standard diffusion and quantum mechanics. We use this correspondence in the chapter 4 to reformulate the option pricing problem to the framework of a quantum mechanics with non-self-adjoint Hamiltonians. Stochastic volatility models are discussed in particular and analysed in this framework. The path integral formulation is also presented in an analogy with a quantum mechanics.

The quantum field theory is also related with a theory of stochastic processes in some sense [1]. However, the quantum field theory often goes beyond the framework of stochastic calculus and describes systems with an infinite number of degrees of freedom and non-trivial correlation structures. We use the framework of QFT and formulate the quantum field theory of forward interest rates. Forward interest rates will be modelled as quantum strings with a finite rigidity. The standard stochastic model will be recovered in the case of an infinite rigidity. We will apply this theory for the pricing of European bond options.

Chapter 1

Option pricing problem

In this chapter we will first define the option pricing problem and state basic results of Black-Scholes theory. We will also discuss a limitation of Black-Scholes's approach and empirical properties of a time evolution of asset prices. The most important part of this chapter will be a generalization of methods used in Black-Scholes theory to a much broader set of models. In particular, this will be useful when we derive the option pricing formulas for exp-Levy models.

1.1 Black-Scholes option pricing

Derivatives are assets, of which prices are derived from the prices of some underlying assets. The pricing of derivatives is very important in practise but it is also a very interesting task from a mathematical point of view.

A put/call option is a derivative that gives us the right to sell/buy an underlying asset at certain time for an in-advance-agreed price. This price is usually called a strike price. We also distinguish between European and American options. European options give us a right to buy/sell only at given time, which is usually called the expiration time of an option. American options give us a right to do so in any time between a moment we bought the option till its expiration. There exist many other types of options, for example barrier or forward start options. These options are usually called exotic, see [15] for more details. Some of them will be discussed later in this thesis. However, we will be mainly interested in European put and call options.

1.1.1 Black-Scholes formula

We will state basic results and notions of Black-Scholes option pricing here, however, the reader is presumed to be already familiar with this theory.

The main assumption of Black-Scholes approach is, that the price S_t of an underlying risky asset is given by a stochastic equation

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \tag{1.1}$$

where W_t is a standard Brownian motion. The parameter μ is called a rate of return and σ is a volatility and it is a constant in Black-Scholes model.

Using Ito lemma, see appendix A, we can easily verify

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{1.2}$$

This process is called geometric Brownian motion. Clearly lnS_t is just Brownian motion with a linear drift. The exponential model is more desirable because we logically want a size of changes of a value of S_t to depend on a total value of S_t.

Further we assume an existence of risk-free assets S^0, which yield fixed interests

$$S^0(t) = S^0(0)e^{rt} \tag{1.3}$$

The parameter r is an interest rate.

Now we want to create a portfolio V given by a strategy $\phi = (\phi_t^0, \phi_t^1)$, with a value

$$V_t(\phi) = \phi_t^0 S_t^0 + \phi_t^1 S_t^1 \equiv \phi_t.S_t \tag{1.4}$$

Naturally we can work with portfolios composed of more than two components but we will for a simplicity work with only two here.

Changes of value of this portfolio should be only given by behaviour of $S_t = (S_t^0, S_t^1)$, where the time evolution of S^i is given by (1.2) and (1.3). Mathematically it means

$$\mathrm{d}V_t = \phi_t.\mathrm{d}S_t = \phi_t^0.\mathrm{d}S_t^0 + \phi_t^1.\mathrm{d}S_t^1 \tag{1.5}$$

where $\mathrm{d}S_t^1$ is given by (1.1) and $\mathrm{d}S_t^0 = re^{rt}\mathrm{d}t$. The strategy fulfilling this property is called self-financing.

We can also calculate $\mathrm{d}V_t$ from (1.4) by using Ito lemma. We obtain the following partial differential equation for a function $V = V(t,S)$ by the comparison with (1.5)

$$\phi_t^1 = \frac{\partial V}{\partial S}(t, S_t) \tag{1.6}$$

and

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{1.7}$$

This is the famous Black-Scholes equation. It governs a time evolution of the price of the self-financing portfolio.

We want to create a self-financing portfolio that will duplicate a price of the option. It means that we want to solve Black-Scholes equation with boundary conditions given by a particular option. We have the following boundary conditions for an European call option C with an expiration time T and a strike price K

1. $C(T,S) = (S-K)^+$
2. $C(t,0) = 0$
3. $C(t,S) \sim S$ for $S \to \infty$

Black-Scholes equation with these boundary conditions has a unique analytic solution. It is one of the reasons why Black-Scholes theory is so popular. The solution is given by

$$C(T-t,S) = S\Phi(g(t,S)) - Ke^{-tr}\Phi(g(t,S)-\sigma\sqrt{t}) \tag{1.8}$$

where Φ is a cumulative distribution function of a normal distribution $N(0,1)$ and

$$g(t,S) = \frac{\ln\frac{S}{K} + (r+(1/2)\sigma^2)t}{\sigma\sqrt{t}} \tag{1.9}$$

so we have derived Black-Scholes price of an European call option at time t

$$C^{BS}(S_t,K,\tau=T-t,\sigma) = S_t\Phi(g(\tau,S_t)) - Ke^{-\tau r}\Phi(g(\tau,S_t)-\sigma\sqrt{\tau}) \tag{1.10}$$

The self-financing strategy that should perfectly duplicate a price of this option is given by

$$\phi_t^1 = \frac{\partial C}{\partial S}(t,S_t) \tag{1.11}$$

and

$$\phi_t^0 = \frac{C(t,S_t) - \phi_t^1 S_t^1}{S_t^0} \tag{1.12}$$

So Black-Scholes price depends on four parameters σ, r, K, T. It is worth noticing that it does not depend on the parameter μ. This will not change even if we consider more general $\mu = \mu(t, S_t)$.

We can derive a right price similarly for European put options. However, there is a direct connection between a price of the call and put options. It is called the put-call parity. Let us denote a price of a put option by $P(t,S_t)$. Let us presume that both of them have the same strike price K and expiration time T, then the following relation holds

$$P(t,S_t) = C(t,S_t) - S_t + Ke^{-r(T-t)} \tag{1.13}$$

This follows from the fact that the portfolio given by

$$V_t = S_t + P(t,S_t) - C(t,S_t) \tag{1.14}$$

guarantees a risk-less profit K at the time T.

Change of measure

The option pricing method we presented above is quite general. However it usually leads to complicated differential equations, which must be solved numerically. We will see in the next chapter that in a case of the jump-diffusion models it furthermore leads to integro-differential equations.

Here we will present a very useful method for the option pricing. It can sometime give us an analytic solution even if we cannot solve corresponding Black-Scholes equation.

The main idea is to change an underlying probability measure P to the different probability measure Q. Q is chosen in the way that the discounted price $\hat{S}_t = e^{-rt}S_t$ will be a martingale under it. This means that $E(\hat{S}_t|\mathcal{F}_s) = \hat{S}_s$ will hold. See [5] or [16] for more details about martingales. We can determine the option price by calculating a conditional expectation with use of martingale property then. We will take a closer look at the change of a measure technique in much more general settings later in this chapter.

First we need the following Girsanov theorem.

Theorem 1.1. *Let us consider the process* $\mathrm{d}X_t = q(t, X_t)\mathrm{d}t + \mathrm{d}W_t$ *where* $t \in \langle 0, T\rangle$. *Let us assume that*

$$E\ (exp(\frac{1}{2}\int_0^T q^2(t,\omega)\mathrm{d}t)) < \infty \tag{1.15}$$

This is known as Novikov's condition. Now we define process M *by*

$$M_T(\omega) = \exp\,(-\int_0^T q\mathrm{d}W_t - \frac{1}{2}\int_0^T q^2\mathrm{d}t) \tag{1.16}$$

then under the probability measure Q *given by*

$$\mathrm{d}Q(\omega) = M_T(\omega)\mathrm{d}P(\omega) \tag{1.17}$$

is X *standard Brownian motion.*

This result is taken from [7].

It is clear from Radon-Nykodym's theorem [5], that P and Q are equivalent measures. We will call them equivalent martingale measures from now on.

With a use of Ito lemma we get

$$\mathrm{d}\hat{S}_t = \hat{S}_t((\mu - r)\mathrm{d}t + \sigma\mathrm{d}W_t) \tag{1.18}$$

if we denote $\widetilde{W}_t = W_t + \frac{\mu - r}{\sigma}t$ we get

$$\mathrm{d}\hat{S}_t = \sigma\hat{S}_t\mathrm{d}\widetilde{W}_t \tag{1.19}$$

Let Q be the probability measure given by Girsanov theorem, under which $\widetilde{W}_t$ is a standard Brownian motion.

We will consider the self-financing portfolio V_t now. We will denote $\hat{V}_t = e^{-rt}V_t$. We get after some calculations

$$\mathrm{d}\hat{V}_t = \phi_t^1\mathrm{d}\hat{S}_t = \sigma\phi_t^1\hat{S}_t\mathrm{d}\widetilde{W}_t \tag{1.20}$$

So $\hat{V}_t$ is a martingale under Q. We can write

$$V_t = E^Q(e^{-r(T-t)}V_T|\mathcal{F}_t) \tag{1.21}$$

So we have for the European options with payoff $h(S_T)$ at strike time T

$$V_t = E^Q(e^{-r(T-t)}h(S_T)|\mathcal{F}_t) = E^Q(e^{-r(T-t)}h(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)})|\mathcal{F}_t). \tag{1.22}$$

We get the following relation for European call options (so $h(x) = (x-K)^+$)

$$V_t = \int_{\mathbb{R}} e^{-r(T-t)}(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma y (T-t)^{1/2}} - K)^+ \varphi(y)\mathrm{d}y \tag{1.23}$$

where $\varphi(y)$ is a density of the distribution $N(0,1)$. If we now calculate this integral, we obtain the same result as in (1.10). So this approach is an equivalent alternative to Black-Scholes equation.

This idea of change of measure is very general and plays key role in derivative pricing. We will discuss it in more general settings later.

1.1.2 Models beyond Black-Scholes

We will just outline the main approaches to generalize Black-Scholes formula here. We will present some of these approaches later in further details.

Local and stochastic volatility models

An assumption of constant volatility σ is very restrictive in Black-Scholes theory. Furthermore we can say that it is also incorrect from empirical observations. The main approaches beyond a constant volatility are

1. stochastic volatility
$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t^1 \tag{1.24}$$
$$\sigma_t = f(Y_t), \quad \mathrm{d}Y_t = \alpha_t \mathrm{d}t + \beta_t \mathrm{d}W_t^2 \tag{1.25}$$
where σ_t is a positive process and we assume a correlation ρ between W^1, W^2

2. local volatility
$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma(t, S_t) S_t \mathrm{d}W_t \tag{1.26}$$

It is interesting that the price S_t itself is not Markov process in stochastic volatility models. This is caused by the assumption of two different driving noises W^1, W^2. However (S_t, σ_t) is a two-dimensional Markov diffusion. We can apply multidimensional Ito lemma and derive an equivalent of Black-Scholes equation for this model. This will be properly done in chapter 4 when we will discuss Hamiltonian formulation of the option pricing problem. The assumption of the stochastic volatility increases a dimension of the problem, which is very inconvenient for numerical solutions. The most famous stochastic volatility model is Heston model [17].

Black-Scholes equation the same form in local volatility models as in the standard case

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(t,S)S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{1.27}$$

however, a non-constant volatility makes this equation very complicated.

A local volatility can lead to heavy tailed distributions. Such behaviour is a desired property. We will discuss this in the next section.

Local volatility models are also very effective for fitting real prices. It was shown that any observed price evolution of a a call option can be fitted by the unique local volatility. More precisely, if we know the value $C_0(T,K)$ of a call option at time $t=0$ with a strike price $K>0$ and expiration time $T \in \langle 0, \hat{T}\rangle$, then there exists the unique local volatility

$$\sigma(T,K) = \sqrt{2\frac{\frac{\partial C_0}{\partial T} + Kr\frac{\partial C_0}{\partial K}}{K^2\frac{\partial^2 C_0}{\partial K^2}}} \tag{1.28}$$

which gives right prices $C_t(T,K)$. For more details see [18] .

An implied volatility is a notion closely connected to a local volatility. It is not obvious how to correctly determine the volatility of the given asset. There are many ways to do it. One of the most popular ones is to get it by inverting Black-Scholes formula. Black-Scholes price

$$C^{BS}(S_t, K, \tau = T-t, \sigma) = S_t\Phi(g(\tau, S_t)) - Ke^{-\tau r}\Phi(g(\tau, S_t) - \sigma\sqrt{\tau}) \tag{1.29}$$

in an increasing function of σ on $(0,\infty)$. So it can be inverted. Let $C_t^*(T,K)$ be a real observed option price. Then we will define an implied volatility $\Sigma_t(T,K) > 0$ by

$$C^{BS}(S_t, K, \tau = T-t, \Sigma_t(T,K)) = C_t^*(T,K) \tag{1.30}$$

A graph of the function Σ_t is called a implied volatility surface. An implied volatility as a function of a strike price forms a convex function known as the volatility smile.

In Black-Scholes world $\Sigma_t(T,K) = \sigma$ however it is well known that Σ_t is a non-constant function in both parameters.

So we can see that a fitting of a local volatility to the real prices is just a fitting of the local volatility to the implied volatility surfaces.

Beyond diffusion

All models discussed above work exclusively with continuous processes. However a continuity of sample paths is not necessarily a sound assumption.

The problem is that for the processes with the discontinuous sample paths Ito lemma does not work! The jumps also bring other problems like an incompleteness of the market. It means that in the models with jumps the prices of options cannot be perfectly duplicated by the risk-less portfolios.

The complete and incomplete markets will be discussed later in this chapter. The option pricing with non-continuous processes will be discussed in detail in the next chapter.

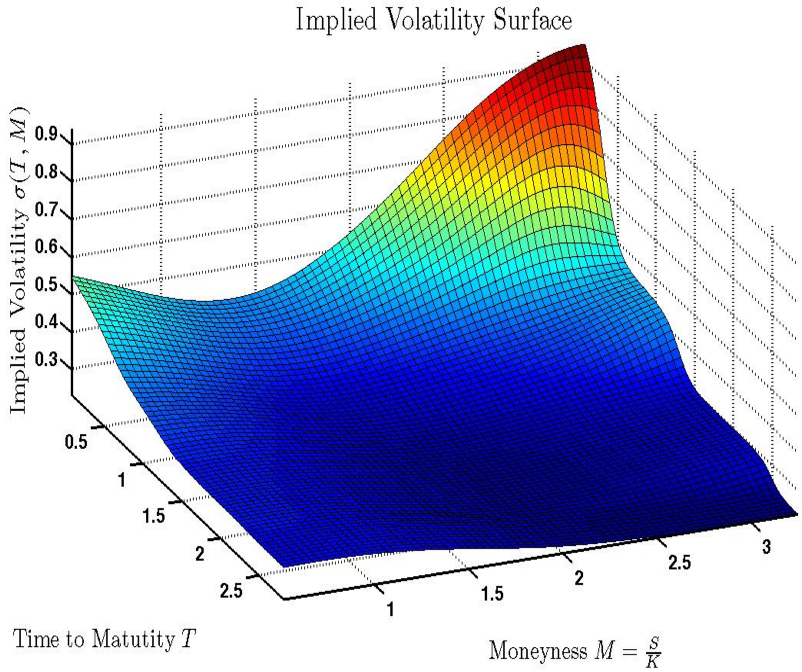

Figure 1.1: Implied volatility

1.2 Empirical properties

We will discuss the empirical properties of asset returns and how well they can be fitted with the different models in this section. This problematic has a long tradition in the financial econometrics, see for example [4], but we will only scratch a surface here.

We will start with the discussion of the shortcomings of a normal distribution. It has been more then 50 years since B. Mandelbrot proposed the usage of the heavy tailed distributions with a density

$$p(x) \sim \frac{1}{|x|^{1+\alpha}}$$

His motivation was mainly empirical. A normal distribution gives only the marginal probabilities to the extreme events. However these events are observed on the financial markets!

For example in Gaussian model the probability of a deviation 5σ in a unit of the time is

$$P(|\Delta X| > 5\sigma) = 2(1 - \Phi(5)) \approx 6.10^{-7} \tag{1.31}$$

and for 6σ many programs round this probability to zero. But deviations of these sizes are observed on financial markets. Especially during crises are such deviations regularly observed. Therefore heavy tail models are much more realistic then models based on normal distributions.

Even though we will work only with the continuous time processes in this thesis, the discrete time processes have a much longer tradition in econometrics. The returns r_t on the scale Δ defined as

$$r_t(\Delta) = S_{t+\Delta} - S_t$$

are usually analysed. However methods that work for the time lag Δ often do not work for the different time lag Δ^*. In other words these models are not stable under the time aggregation. There exists a number of interesting approaches to this problematic like well-known GARCH, ARCH models or entropic methods [40].

This implies that the continuous time processes are a more principal approach to the option pricing problem. However for analysis of asset price itself is discrete time approach unavoidable.

Let us now get back to discussion about the proper distributions for the modelling of financial markets. We will start with an overview of the properties that seems to be common to almost all markets and assets. It turns out that various assets on various markets all share some similar statistical properties. This is a very interesting result.

1. **Heavy tails:** The distribution of returns has heavy tails, usually tail index α is between two and five

2. **Absence of autocorrelations:** Autocorrelations of returns seem to vanish on larger scales then approximately 20 minutes.

3. **Gain/loss asymmetry:** The tails of the distribution are not same in $\pm\infty$

4. **Aggregation normality:** With the larger scales Δ distribution of returns is getting closer to normal.

5. **Volatility clustering:** The absolute returns $|r_t(\Delta)|$ are strongly and positively correlated, this autocorrelation has a very slow decay. In other words small changes are usually followed by small changes and big ones by big ones.

It is hard to fit all this properties into a tractable model. For example a volatility clustering implies that the returns are not independent. So any price process with an independent increments will not fulfil this property. However the model can give good predictions even if not all of these properties are fulfilled.

There is a huge amount of a theory around the estimations of parameters of the asset return models. Besides the regular statistical methods like MLE can be used for example the extreme value theory [15],[19].

Scaling and self-similarity

We know that behaviour of $r_t(\Delta)$ significantly depends on the scale Δ. It is interesting to analyse how will the behaviour of r_t vary on different scales.

By scaling we want to determine the statistical quantities invariant under time aggregation. We start with a definition of self-similarity.

Definition 1.1. *The stochastic process $(X_t, t \geq 0)$ is H - self similar for some $H > 0$ if all its finite dimension distributions satisfy the condition*

$$(T^H X_{t_1}, ..., T^H X_{t_n}) \stackrel{d}{=} (X_{Tt_1}, ..., X_{Tt_n})$$

for every $T > 0$.

The parameter H is usually called Hurst index. Clearly Brownian motion is self similar with $H = 1/2$.

We should also mention the notion of self-affinity. The process X_t is self-affine if $X_{ct} = c^H X_t + b_c(t)$. For example Brownian motion with drift is self-affine but not self-similar.

The density of a self similar process has clearly the form

$$p_t(x) = \frac{1}{t^H} p_1(\frac{x}{t^H}) \tag{1.32}$$

which implies

$$EX_t = t^H EX_1, \quad VarX_t = t^{2H} VarX_1 \tag{1.33}$$

But for every process X with independent and stationary increments and finite variance it also holds $VarX_t = tVarX_1$. So we can see that every self-similar Levy process (i.e. process with independent and stationary increments) has an infinite variance or is Brownian motion.

Further it is well known that the self-similar Levy processes with an infinite variance are α-stable. They are given by their characteristic function

$$\varphi_t(z) = \exp(-ct|z|^\alpha) \quad \alpha \in (0, 2\rangle, c > 0 \tag{1.34}$$

Hurst index is given by $H = 1/\alpha$ in this case. The stable processes will be further discussed in the next chapter. The importance of the stable distributions lies in their role in the generalized central theorem, see appendix B.

However the self-similarity has nothing to do with independent increments. The typical example of a self-similar process without independent increments is fractional Brownian motion.

Definition 1.2. *Stochastic process $(X_t, t \geq 0)$ is called a fractional Brownian motion (FBM) with a parameter $H \in (0, 1)$ if*

1. *$X_0 = 0$ a.s.*
2. *$X_t(\omega)$ is continuous a.s.*

3. $X_{t+h} - X_t \sim N(0, h^{2H})$

Clearly FBM is H-self similar and does not have independent increments because

$$EX_t(X_{t+h} - X_t) = \frac{1}{2}(-t^{2H} - h^{2H} + (t+h)^{2H}) \overset{H \neq \frac{1}{2}}{\neq} 0 \tag{1.35}$$

For its covariance we have

$$Cov(X_t, X_h) = \frac{1}{2}(t^{2H} + h^{2H} - |t-h|^{2H}) \tag{1.36}$$

So Hurst index H does not determine the process uniquely, except of case $H = 1/2$. We saw that there are two ways to generate self-similarity. The first is through the high variability and heavy tails. The other one is via strongly correlated increments. These effects are called Noah's and Joseph's.

We should also mention that there is a connection between Hurst index and the fractal dimension of the graph and of sample paths. Concretely if we denote the graph of the process by P we have $dim_B P = 2 - H$. So determining fractal dimension of a sample path is a way to also determine self-similarity index H. This can be particularly useful in finance [2],[8].

Important question now is :

Are the financial markets self similar?

Many authors use the following test. Let us assume that the log-price $X_t = \ln S_t$ has stationary increments. From a self-similarity follows $p_t(0) = \frac{1}{t^H} p_1(0)$. First we estimate an empirical density at zero $\hat{p}_t(0)$. This can be done from the histogram for example. Then we get the following relation

$$\ln \hat{p}_t(0) \simeq H \ln \frac{\Delta}{t} + \ln \hat{p}_\Delta(0) \tag{1.37}$$

Mantegna and Stanley applied this to SP 500 and obtained $H \simeq 0,55$. They concluded the α-stable model with $\alpha \simeq 1,75$. But we already know that there are more processes with the same Hurst index. Furthermore (1.37) is a necessary but not sufficient condition for the self-similarity. It is complicated from the discrete data we have to prove the self-similarity of the financial markets beyond doubt. However it is widely believed that the financial markets are self-similar.

1.3 Market completeness and martingale measures

In this section we will take a look at general results of the option pricing theory. Theory presented here is applicable to a very general set of models. However we present it mainly for the option pricing in exp-Levy models, which will be discussed in the next chapter.

1.3.1 Pricing rules and martingale measures

Let us consider the stochastic process X as random variable $X : \Omega \to A\langle 0, T\rangle$ where $A\langle 0, T\rangle$ is a suitable set of functions $f : \langle 0, T\rangle \to \mathbb{R}^n$. So the stochastic process X induce measure $P^X = P \circ X^{-1}$ on the path space $A\langle 0, T\rangle$. Let us consider processes X, Y which induce equivalent measures i.e.

$$P^X(A) = 0 \iff P^Y(A) = 0 \quad \forall A \in \mathcal{F} \tag{1.38}$$

Where $\mathcal{F}$ is a σ-algebra on the path space. We will denote $P^X = P$ and $P^Y = Q$ now.

Radon-Nykodym theorem now states that $\frac{\mathrm{d}Q}{\mathrm{d}P} = Z$ is a strictly positive random variable. Let us consider the filtration $\mathcal{F}_t$ on the path space. Then we can define process

$$Z_t = E(Z|\mathcal{F}_t) \tag{1.39}$$

Z_t is clearly P - martingale with $EZ_t = 1$.

For Z_t also following holds

$$\frac{\mathrm{d}Q}{\mathrm{d}P}|_{\mathcal{F}_t} = Z_t, \quad i.e. \;\; Q(A) = E^P(Z_t I_A) \;\; \forall A \in \mathcal{F}_t \tag{1.40}$$

So equivalent measures generate martingale process in this sense. The idea behind a change of measure technique is to find measure Q such that $\hat{S}_t Z_t$ will be P - martingale.

It is also worth noticing, that under different equivalent measures are the path properties, in other words " almost sure properties ", invariant.

Let us get back to the option pricing now. Let us consider the process $S : \langle 0, T\rangle \times \Omega \to \mathbb{R}^{n+1}$ to be a price process adapted to the filtration $\mathcal{F}_t$. S is the price vector so $S_t = (S_t^0, ..., S_t^n)$. We will only consider $S_t^0 = \exp(rt)$ as in Black-Scholes case. We will also use a notation $\hat{V}_t = \frac{V_t}{S_t^0}$.

We will now consider set $\mathcal{H}$ of the possible pay-offs of an option at time T. Technically $H \in \mathcal{H}$ can be any $\mathcal{F}_T$-measurable variable $H : \Omega \to \mathbb{R}$. We worked with $H = (S_T - K)^+$ in Black-Scholes theory. But we can also consider for example the path dependent options $H = h(S_t(\omega))$.

Now we want to specify the pricing rule $\Pi_t : H \in \mathcal{H} \to \Pi_t(H)$. We require it to fulfil following conditions

1. Π_t is adapted to $\mathcal{F}_t$
2. $H(\omega) \geq 0 \Rightarrow \Pi_t(H) \geq 0$
3. pricing is linear: $\Pi_t(\sum_{i=1}^n H_i) = \sum_{i=1}^n (\Pi_t(H_i))$

One of possible pay-offs is I_A where $A \in \mathcal{F} = \mathcal{F}_T$. For $A = \Omega$ is $I_\Omega = I$ and we get

$$\Pi_t(I) = e^{-r(T-t)} \tag{1.41}$$

We can define the measure Q now by

$$Q(A) = \frac{\Pi_0(I_A)}{\Pi_0(I)} = e^{r(T-t)}\Pi_t(I_A) \tag{1.42}$$

We can easily see that $0 \leq Q(A) \leq 1$ and for A, B disjoint we have $Q(A \cup B) = Q(A) + Q(B)$ from properties of Π_t. So Q is a probability measure.

Reversely let us have the probability measure Q. Pay-offs in the form $H = \sum_i c_i I_{A_i}$ are dense in $\mathcal{H}$ in the norm $\|.\|_{L^1}$. For every H in this form we have

$$\Pi_0(H) = e^{-rT} E^Q[H]$$

If we now assume that Π_t is continuous we can broaden this statement $\forall H \in \mathcal{H}$. More generally we get

$$\Pi_t(H) = e^{-r(T-t)} E^Q[H|\mathcal{F}_t] \tag{1.43}$$

Arbitrage free pricing

We have shown that an option pricing rule is equivalent to some new probability measure Q. We will show now that Q must be equivalent to the original measure P and the discounted price must be martingale under Q. Otherwise there will be an arbitrage opportunity.

Arbitrage means that there exists a self-financing strategy $\phi = (\phi^1, ..., \phi^n)$ for the trading of risky assets with prices $S = (S^1, ..., S^n)$ such that the resulting portfolio V_t guarantees a profit

$$P(V_T(\phi) - V_0(\phi) > 0) = 1 \tag{1.44}$$

We should note that not all self-financing strategies are admissible. We need to work only with strategies that are proper integrands. Admissible strategies in the case of continuous processes are given by Ito calculus, see appendix A. In the case of Levy - type processes only predictable strategies with caglad sample paths will be admissible. This will be further discussed in the next chapter.

Let us consider set A such that $P(A) = 0$ then

$$\Pi_0(I_A) = e^{-rT} Q(A) \stackrel{!}{=} 0 \tag{1.45}$$

otherwise that would be an arbitrage opportunity. So we have shown that P and Q are the equivalent measures.

$$P \sim Q \iff (P(A) = 0 \iff Q(A) = 0)$$

Obviously the payoff $H = S^i_T$ given by asset S^i is possible. So we have

$$S^i_t = \Pi_t(S^i) = e^{-r(T-t)} E^Q[S^i_T|\mathcal{F}_t] \tag{1.46}$$

So the discounted price $\hat{S}^i_t = e^{-rt} S_t$ is martingale under Q

$$E^Q[\hat{S}^i_T|\mathcal{F}_t] = \hat{S}^i_t \tag{1.47}$$

The stochastic integral with respect to martingale integrator has to be martingale (or at least local martingale) in every reasonable theory. This implies that integral $\int_0^t \phi \mathrm{d}\hat{S}$ is Q-martingale for " suitable " integrands ϕ. Further we have $\hat{V}_t = \int_0^t \phi \mathrm{d}\hat{S}$, so the discounted price of our portfolio is Q - martingale. This is analogical to equations (1.19), (1.20) in Black-Scholes theory.

For the portfolio V_t we now have

$$E^Q(V_t(\phi)) = E^Q(V_0 + \int_0^t \phi \mathrm{d}S) = V_0 \tag{1.48}$$

So we can write

$$Q(V_T(\phi) - V_0 = \int_0^T \phi \mathrm{d}S \ > 0) \neq 1 \tag{1.49}$$

and the same holds for P because $P \sim Q$. So there is not an arbitrage opportunity.

We have shown that defining of an arbitrage free pricing is equivalent to defining the measure $Q \sim P$ under which is the discounted price martingale.

We will end this section with two theorems summarizing our results.

Theorem 1.2. *Let us consider a market with the probability space* $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ *and an arbitrage free linear pricing rule* Π_t. *Then there exists a probability measure* Q *that* $P \sim Q$, *the discounted prices* $\hat{S}_t$ *are martingale under* Q *and* Π_t *can be written as*

$$\Pi_t(H) = e^{-r(T-t)} E^Q[H|\mathcal{F}_t]$$

Theorem 1.3. *Market* $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ *with the asset prices* S_t^i *is arbitrage free if and only if there exists an equivalent measure* Q *under which are the discounted prices* $\hat{S}_t^i$ *martingale.* Q *is called equivalent martingale measure.*

1.3.2 Market completeness

We consider the payoff H, then the self-financing strategy $\phi = (\phi_t^0, \phi_t^1)$ is called the perfect hedge of H if

$$H = V_0 + \int_0^T \phi_t^0 \mathrm{d}S_t^0 + \int_0^T \phi_t^1 \mathrm{d}S_t^1 \tag{1.50}$$

This strategy is unique because otherwise there would be an arbitrage opportunity. The price of H at $t = 0$ is obviously V_0.

However there is no guarantee that every payoff can be perfectly hedged. We will say that the market is complete if $\forall H \in \mathcal{H}$ there exists a perfect hedge.

For the complete market we get

$$\hat{H} = V_0 + \int_0^T \phi_t^1 \mathrm{d}\hat{S}_t^1 \tag{1.51}$$

This is analogical to the equation (1.20) in Black-Scholes case.

For " suitable" integrands ϕ is H Q-martingale so we have

$$E^Q[\hat{H}] = V_0$$

Because $\forall A \in \mathcal{F}$ is $I_A \in \mathcal{H}$ determines this Q uniquely . So the completeness of market implies the uniqueness of Q. The reverse implication also holds so we have the following result.

Theorem 1.4. *Market $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ with asset prices S_t^i is complete if and only if there exists only one equivalent martingale measure Q .*

This theorem is known as the second fundamental theorem of asset pricing.

There are some problems with the mathematical rigorosity if S is for example Levy process with infinitely many jumps. However for our purposes we will interpret this theorem as follows

$$\forall H \in \mathcal{H}, \quad H = EH + \int_0^T \phi_t^1 \mathrm{d}\hat{S}_t^1 \iff \exists_1 Q \text{ equivalent martingale measure}$$

where ϕ_t is self-financing strategy.

All diffusion models, in particular all local volatility models, defines complete markets. However the stochastic volatility models and jump models are usually incomplete. The jump models include the exp-Levy models and jump diffusion models which will be discussed in the next chapter.

It is very important to realize that in incomplete markets the options cannot be perfectly hedged. So options are not a redundant asset in incomplete markets. This seems to correspond to a reality much better then complete markets do. There is always some risk that cannot be hedged away when hedging in incomplete markets.

1.3.3 Pricing in incomplete markets

The problem with pricing in incomplete markets is that defining the stochastic dynamics driving the stock price does not determine the option price uniquely. We have to choose also one of possible equivalent martingale measures and typically there are infinitely many martingale measures available.

Minimal entropy approach One of possible approaches is to choose measure that is the " closest" one to the original measure P in some sense. Most common approach is minimal entropy approach. Its idea is to find such an equivalent martingale measure Q that it minimize a relative entropy of Q to P defined as

$$\varepsilon(Q, P) = E^Q(\ln(\frac{\mathrm{d}Q}{\mathrm{d}P})) \tag{1.52}$$

We will discuss this method for exp-Levy models in the next chapter.

Quadratic hedging The main idea of this approach is to minimize the hedge risk in mean square sense. In contract with previous method we will minimize the risk here over possible strategies not over martingale measures.

We assume a self-financing strategy (ϕ_t^0, ϕ_t^1). We want to minimize in some sense the following term

$$\inf_{\phi} E(|V_T(\phi) - H|^2) \tag{1.53}$$

where

$$V_T(\phi) = V_0 + \int_0^T \phi_t^0 r e^{rt} \mathrm{d}t + \int_0^T \phi_t \mathrm{d}S_t \tag{1.54}$$

however we need to specify under which measure we calculate the expected value in (1.53). We will choose a equivalent martingale measure Q, so we want to minimize

$$\inf_{\phi, V_0} E^Q(|\vartheta(V_0, \phi)|^2) \tag{1.55}$$

where under Q

$$\vartheta(V_0, \phi) = \hat{H} - \hat{V}_T = \hat{H} - V_0 - \int_0^T \phi_t \mathrm{d}\hat{S}_t \tag{1.56}$$

We will assume that $H \in L^2(\Omega, \mathcal{F}, Q)$ and $\hat{S}_t$ is L^2-integrable Q-martingale. Further we will consider only strategies fulfilling

$$E^Q[|\int_0^T \phi_t \mathrm{d}\hat{S}_t|^2] < \infty \tag{1.57}$$

These assumptions guarantee that random variables $A = V_0 + \int_0^T \phi_t \mathrm{d}\hat{S}_t$ fulfil $A \in L^2(\Omega, \mathcal{F}, Q)$. We will denote the set of these variables by $\mathcal{A}$. We can reformulate our task now because

$$\vartheta(V_0, \phi) = \inf_A \left\| \hat{H} - A \right\|^2_{L^2(Q)} \tag{1.58}$$

so we are just looking for the orthogonal projection of H into $\mathcal{A}$. If we assume that $\mathcal{A}$ is a closed subspace of $L^2(\Omega, \mathcal{F}, Q)$ we know that the OG projection exists. We will formulate this in the following theorem.

Theorem 1.5. *Let $\hat{S}_t$ be L^2-integrable Q-martingale, let $\hat{H}$ be L^2-integrable variable dependent on history of S_t. Then $\hat{H}$ can be represented as*

$$\hat{H} = E^Q \hat{H} + \int_0^T \phi_t^H \mathrm{d}\hat{S}_t + N^H \tag{1.59}$$

where ϕ_t^H is a L^2-integrable predictable strategy and N^H is orthogonal to every stochastic integral with respect to $\hat{S}_t$.

So integral $\int_0^T \phi_t^H \mathrm{d}\hat{S}_t$ is the wanted OG projection. N^H is a residual risk that cannot be hedged away. Clearly in the case of the complete markets $N^H = 0$

We omitted some technical mathematical assumptions in order to make idea behind quadratic hedging clearer, however this idea is very general and almost model independent. We take closer look at this method in exp-Levy models option pricing in the next chapter.

Generalized Black-Scholes equation We can obtain analogues of Black-Scholes equation even in incomplete models. All we need to do so is Ito lemma. However the exact form of the equation can depend on the concrete martingale measure we choose.

We will discuss the form of generalized Black-Scholes equation for stochastic volatility models in chapter 4 and for Levy models in chapter 2. However we will first need to derive generalized Ito lemma for Levy processes in order to derive it.

Merton approach We will know introduce simple incomplete model, we will choose one possible martingale measure in this case and obtain closed form solution for price of European call option. In Merton model price of risky asset is given by

$$S_t = S_0 \exp\left(\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right) \tag{1.60}$$

where N_t is Poison process with intensity λ independent of W_t and $Y_i \sim N(m, \delta^2)$ are iid variables.

This is in fact exp-Levy model and it is incomplete. So there exists a lot of equivalent martingale measures Q . Here we choose Q to only change the drift of Brownian motion as in Black-Scholes theory.

Under the new measure Q_M we will have

$$S_t = S_0 \exp\left(\mu^M t + \sigma W_t^M + \sum_{i=1}^{N_t} Y_i\right) \tag{1.61}$$

where

$$\mu^M = r - \frac{\sigma^2}{2} - \lambda E(e^{Y_i} - 1) = r - \frac{\sigma^2}{2} - \lambda(\exp\left(m + \frac{\delta^2}{2}\right) - 1) \tag{1.62}$$

So $\hat{S}_t$ is martingale under Q because $\ln S_t$ has independent increments and $E\hat{S}_t = 1$. The assumption of existence of the measure Q_M will be justified in the next chapter.

This choice of Q corresponds to the belief that jumps do not bring big premium risk. So we hedged away a diffusion component only. However this is not always an optimal choice because jumps usually do bring up significant risks.

For the European option with payoff $H(S_T)$ we get

$$C_t^M = e^{-r(T-t)} E^Q[H(S_t \exp\left(\mu^M \tau + \sigma W_\tau^M + \sum_{i=1}^{N_\tau} Y_i\right)] \tag{1.63}$$

where $\tau = T - t$. After some straightforward calculations we obtain

$$C^M(t, S) = \sum_{n \geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} C^{BS}(\tau, S_n, \sigma_n) \tag{1.64}$$

where $\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}$ and

$$S_n = Sexp(nm + \frac{n\delta^2}{2} - \lambda\tau(\exp{(m + \frac{\delta^2}{2})} - 1)) \tag{1.65}$$

and Black-Scholes price is given by

$$C^{BS}(\tau, S, \sigma) = e^{-r\tau}E[H(S\exp{((r - \frac{\sigma^2}{2}))\tau + \sigma W_\tau})] \tag{1.66}$$

This sum can be calculated numerically with a good proximity now.

To derive the hedging risk, an analogue of Black-Scholes equation and other quantities we will need Ito lemma for Levy processes. Classical Ito lemma does not work for these processes! We will return to this in the next chapter when we have the proper mathematical tools.

Chapter 2

Jump process models

In this chapter we will thoroughly discuss the properties of Levy processes. We will state and explain the famous Levy-Ito decomposition and other important theorems about Levy processes. A special focus will be given to the stochastic calculus. A fundamental result will be a generalization of Ito lemma for jump processes. This will be essential in the last part of this chapter when we derive the option pricing formulas in exp-Levy models.

2.1 Basic mathematical tools

We will state here some basic definitions and results needed for the theory of Levy processes.

Definition 2.1. *Function $f : \langle 0, T\rangle \to \mathbb{R}^d$ is cadlag if it is right continuous with left limits.*

So $f(t+) = f(t)$ and limit $f(t-)$ exists for the cadlag function. We will denote the jump sizes by $\Delta f(t) = f(t) - f(t-)$.

It is well-known that cadlag functions have maximally a countable number of jumps. Furthermore for $\forall \varepsilon > 0$ f has only a finite number of jumps larger than ε.

Remark. *We will also consider left continuous functions with right limits. We will call them caglad functions. We will see that that price processes will be cadlag but our strategy will be a caglad process.*

We will denote the space of all cadlag functions by $D(\langle 0, T\rangle, \mathbb{R}^d)$. Continuous functions form clearly a subspace of $D(\langle 0, T\rangle, \mathbb{R}^d)$. It can also be equipped with a topology. This topology corresponds to "weaker" version of uniform convergence. For details see [34].

The space $D(\langle 0, T\rangle, \mathbb{R}^d)$ equipped with this topology is called Skorokhod space. We will call a process X cadlag if it has cadlag sample paths so

$$X : \Omega \to D(\langle 0, T\rangle, \mathbb{R}^d) \tag{2.1}$$

All Levy processes in particular will be cadlag.

Random times

We will call any positive variable $T(\omega) \geq 0$ the random time. We say that T is adapted with respect to the filtration $\mathcal{F}_t$ if

$$\{T \geq t\} \in \mathcal{F}_t \quad \forall t \geq 0$$

Adapted random time is often called the stopping time. Important class of stopping times are so called hitting times.. For a stochastic process X we define hitting time by

$$T_A = \inf\{t > 0, X_t \in A\}$$

where $A = A^\circ$. Further it can be shown that X is a martingale if for all stopping times τ holds $EX_\tau = EX_0$

Predictable processes

Let us consider a process X as a random variable $X : \langle 0, T\rangle \times \Omega \to \mathbb{R}^d$. We will introduce two important σ-algebras on $\langle 0, T\rangle \times \Omega$.

Definition 2.2. *Optional σ-algebra $\mathcal{O}$ on $\langle 0, T\rangle \times \Omega$ is generated by natural filtrations of all cadlag processes. Process measurable with respect to $\mathcal{O}$ is called optional.*

So cadlag processes generate optional processes in some sense. However an optional process is not generally cadlag. Similarly caglad processes will generate predictable processes.

Definition 2.3. *Predictable σ-algebra $\mathcal{P}$ on $\langle 0, T\rangle \times \Omega$ is generated by natural filtrations of all caglad processes. Process measurable with respect to $\mathcal{P}$ is called predictable.*

This definitions will be very important for us. We already mentioned in the chapter one that a process defining the self-financing strategy must be caglad and predictable. It is so because we control our strategy process so there should not be any “ surprising ” jumps. However the price processes are an opposite case. We assume that the price of an asset can jump unpredictably so the price should be a cadlag process.

Poisson process and random measures

We will briefly summarize properties of Poisson process here. We will introduce a notion of the random measure and Poisson measure will be discussed in particular.

Definition 2.4. *Let us consider iid random variables* $\{\tau_i\}_1^\infty$ *where* $\tau_i \sim exp(\lambda)$. *We will denote* $T_n = \sum_{i=1}^n \tau_i$. *Then we define Poisson process* N_t *with intensity* λ *by*

$$N_t = \sum_{n\geq 1} I_{t\geq T_n} \tag{2.2}$$

We will now review basic properties of this process.

1. $P(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$
2. N_t has independent and stationary increments
3. N_t has cadlag trajectories and is continuous in probability
4. $EN_t = VarN_t = \lambda t$
5. characteristic function of N_t has form

$$\varphi_t(u) = exp(\lambda t(e^{iu} - 1))$$

We will also define compensated Poisson process $\widetilde{N}_t$ by

$$\widetilde{N}_t = N_t - \lambda t \tag{2.3}$$

It can be easily shown that $\widetilde{N}_t$ is a martingale.

We can also consider a more general counting process X_t

$$X_t = \sum_{n\geq 1} I_{t\geq S_n}$$

where S_n is an increasing sequence of random variables and $P(S_n \to \infty) = 1$. It can be shown that the only counting process with stationary and independent increments is Poisson process. The main reason for that is unique property of an exponential distribution, let $V \sim \exp(\lambda)$ then

$$P(V > t + s|V > s) = P(V > t) \tag{2.4}$$

An exponential distribution has no memory. Moreover it is an only probability distribution with this property.

Let us get back to Poisson process, it defines a random measure M by

$$M(\omega, A) = \#\{i \geq 1, T_i(\omega) \in A\} \tag{2.5}$$

In particular $N_t(\omega) = M(\omega, \langle 0, t\rangle)$. We will formalize this notion in the following definition.

Definition 2.5. *Let us consider a probability space* $(\Omega, \mathcal{F}, P)$, *set* $E \subset \mathbb{R}^d$ *and* μ *Radon measure on* $(E, \mathcal{E})$. *Then Poisson measure on* E *with intensity* μ *is* $M : \Omega \times \mathcal{E} \to \mathbb{N}$ *fulfilling*

1. *for bounded* $A \subset E$ *is* $M(\omega, A) < \infty$ *almost surely*
2. $M(\ .\ , A) \sim Poi(\mu(A)) \quad \forall A \in \mathcal{E}$
3. *for disjoint sets* $A_1, .., A_n$ *are* $(M(A_1), .., M(A_n))$ *independent variables*

Further it can be shown that every Poisson measure can be written as

$$M(\omega) = \sum_{n \geq 1} \delta_{X_n(\omega)} \tag{2.6}$$

where X_n is a sequence of random variables. For more details see [15].

We will also define compensated Poisson measure

$$\widetilde{M}(A) = M(A) - \mu(A) \tag{2.7}$$

Clearly $E(\widetilde{M}(A)) = 0$. However $\widetilde{M}(A)$ is not a positive measure.

We will take a brief look on a connection between the random measures and jump processes now. Let us consider $E = \langle 0, T \rangle \times \mathbb{R}^d \setminus \{0\}$ and M Poisson measure on E with intensity μ. We can write now

$$M = \sum_{n \geq 1} \delta_{(T_n, Y_n)} \tag{2.8}$$

where $T_n(\omega) \in \langle 0, T \rangle$ and $Y_n(\omega) \in \mathbb{R}^d$. Intuitively T_n are times of the jumps and Y_n are sizes of jumps. We say that M is adapted to a filtration $\mathcal{F}_t$ if

1. T_n are random times adapted to $\mathcal{F}_t$
2. Y_n are $\mathcal{F}_{T_n}$-measurable.

For any $f : E \to \mathbb{R}$ measurable following relation holds

$$E(M(f)) = \mu(f) = \int_0^T \int_{\mathbb{R}^d} f(s, y) \mu(\mathrm{d}s, \mathrm{d}y) \tag{2.9}$$

If $\mu|f| < \infty$ we can define a $\mathcal{F}_t$-adapted process

$$X_t = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y) M(\mathrm{d}s, \mathrm{d}y) = \sum_{\{n, T_n \in \langle 0, t \rangle\}} f(T_n, Y_n) \tag{2.10}$$

Moreover if we integrate with the respect to $\widetilde{M} = M - \mu$ then the process

$$X_t = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y) \widetilde{M}(\mathrm{d}s, \mathrm{d}y) \tag{2.11}$$

is a martingale. This is an analogical result to martingale property of Ito integral.

Reversely if we consider a cadlag process X_t we can define the measure

$$J_X = \sum_{n \geq 1} \delta_{(T_n, Y_n)} = \sum_{t \in \langle 0,T \rangle}^{\Delta X_t \neq 0} \delta_{(t, \Delta X_t)} \tag{2.12}$$

where set $\{t \mid \Delta X_t \neq 0\}$ is countable. We can interpret $J_X(\langle 0,t\rangle \times A)$ as the number of jumps of X in between time $\langle 0,t \rangle$ with amplitude in A. J_X contains all the information about jumps of the process. We have $J_X = 0$ for continuous processes. J will have for example the following form for Poisson process N

$$J_N = \sum_{n \geq 1} \delta_{(T_n, 1)}$$

The meaning of these notions will become clearer in connection with Levy processes discussed in the next section.

2.2 Levy processes

In this section we will define Levy processes and state basic theorems about them. We will also introduce methods for building new Levy processes with applications to financial models. We will not prove most of the statements stated here, these proofs can be found in [15],[35] or in many other advanced textbooks on stochastic calculus.

2.2.1 Properties of Levy processes

Definition 2.6. *Cadlag stochastic process $X = X_{t \geq 0}$ is called Levy process, if it satisfies following properties*

1. $X_0 = 0$ *a.s*
2. *X has stationary and independent increments*
3. $\lim_{h \to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0 \quad \forall \varepsilon > 0,\ t \geq 0$

Every process fulfilling all these properties except cadlag property is called Levy in law. It can be shown that these processes have cadlag modification.

Property (3.) guarantees that probability of jump at any fixed time t is zero. Mathematically it implies that characteristic function of X is continuous in t.

Obviously Poisson process and Brownian motion with a drift are Levy processes. We will see that every Levy process can be decomposed as linear superposition, possibly uncountable one, of Brownian motion, Poisson process and the drift.

It follows from independence and stationarity of increments

$$\varphi_{t+s}(u) = \varphi_t(u)\varphi_s(u) \tag{2.13}$$

where φ_t is a characteristic function of X_t. With use of the continuity of φ_t we get

$$\varphi_t(u) = (\varphi_1(u))^t \tag{2.14}$$

Levy processes are closely connected with infinitely divisible distributions.

Definition 2.7. *Random variable X is an infinitely divisible if and only if for every $n \in \mathbb{N}$ there exists iid random variables $X_{n,1}, ..., X_{n,n}$ so that*

$$X \stackrel{d}{=} X_{n,1} + ... + X_{n,n}$$

Clearly if X is Levy process then X_t is an infinitely divisible variable. Further the following result holds.

Theorem 2.1. *A random variable Y is infinitely divisible if and only if there exists Levy process X such as*

$$Y \stackrel{d}{=} X_1$$

It also follows from (3.13) and the continuity of φ_t that a characteristic function of Levy process has the form

$$\varphi_t(z) = \exp\left(t\psi(z)\right) \tag{2.15}$$

$\psi(z)$ is a cumulant generating function of X_1. Clearly $\psi_t(z) = t\psi(z)$.

Compound Poisson process

Definition 2.8. *Compound Poisson process (CPP) with intensity λ is defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i \tag{2.16}$$

where N_t is Poisson process with the intensity λ and Y_i are iid random variables.

Clearly CPP is Levy process with piecewise constant trajectories. CPP has the same jump times as underlying Poisson process. It is well-known that cadlag functions can be well approximated by piecewise constant functions. Similarly it is possible to approximate Levy processes by CPP [15].

The following result is also very interesting.

Theorem 2.2. *Every Levy process with piecewise constant trajectories is compound Poisson process.*

We can calculate a characteristic function of CPP X_t. Let F be a cumulative distribution of Y_i then

$$\sum_{n\geq 0} E(e^{iu\sum_{i=1}^{n} Y_i})e^{-\lambda t}\frac{(\lambda t)^n}{n!} = \sum_{n\geq 0}(\int_{\mathbb{R}^d} e^{iux}\mathrm{d}F(x))^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} = e^{-\lambda t\int_{\mathbb{R}^d}(1-e^{iux})\mathrm{d}F(x)}$$

so we have

$$\varphi_t(u) = \exp\,(t\int_{\mathbb{R}^d}(e^{iux}-1)\lambda \mathrm{d}F(x)) \tag{2.17}$$

We will denote $\mathrm{d}\nu(x) = \lambda\mathrm{d}F(x)$. ν is called Levy measure and it corresponds to " activity " of the jumps.

There is a connection between jump measure J_X and Levy measure ν. It si given by the following theorem.

Theorem 2.3. *Let X_t be CPP with Levy measure ν. Then its jump measure J_X is Poisson random measure with the intensity given by*

$$\mu(\mathrm{d}x, \mathrm{d}t) = \mathrm{d}\nu(x)\mathrm{d}t \tag{2.18}$$

This motivates us to define Levy measure for general Levy process as follows.

Definition 2.9. *For every Levy process X and Borel measurable set $A \subset \mathbb{R}^d$ we define Levy measure ν by*

$$\nu(A) = E(\#\{t \in \langle 0,1\rangle,\ \Delta X_t \neq 0,\ \Delta X_t \in A\}) \tag{2.19}$$

Because CPP is a pure jump process it can be written using its jump measure

$$X_t = \sum_{s\in\langle 0,t\rangle} \Delta X_s = \int_{\langle 0,t\rangle\times\mathbb{R}^d} xJ_X(\mathrm{d}s, \mathrm{d}y) \tag{2.20}$$

We should realize that this sum is finite because CPP has almost surely a finite number of jumps in an every finite interval.

Every finite measure ν on $\mathbb{R}^d$ defines Radon measure by (3.18) and this Radon measure defines CPP. It can be shown that there is one to one correspondence between finite measures ν and compound Poisson processes. However there exist Levy processes with infinite Levy measures. We will call them infinite activity processes and discuss them in the next section.

Levy-Ito decomposition

We can consider Levy process X_t given by

$$X_t = \gamma t + W_t + X_t^0 \tag{2.21}$$

Where W is Brownian motion independent on CPP X^0. The question if all Levy processes can be decomposed in this way? Not exactly. The problem is with infinite activity processes. Every Levy measure ν fulfils

$$A \ \ compact, \quad 0 \notin A \implies \nu(A) < \infty$$

this follows directly from the cadlag property. So every Levy measure is Radon measure on $\mathbb{R}^d \setminus \{0\}$. However there is problem when an infinite number of jumps around $\{0\}$ occurs. The following result holds.

Theorem 2.4. *Let X be Levy process in $\mathbb{R}^d$ with Levy measure ν. Then*

1. *ν is Radon measure on $\mathbb{R}^d \setminus \{0\}$ and*

$$\int_{|x|\leq 1} |x|^2 \mathrm{d}\nu(x) < \infty \quad \textit{and} \quad \int_{|x|\geq 1} \mathrm{d}\nu(x) < \infty \tag{2.22}$$

2. *jump measure J_X is Poisson random measure with intensity* $\mathrm{d}\nu(x)\mathrm{d}t$

3. $$X_t = \gamma t + B_t + X_t^1 + \lim_{\varepsilon\to 0} \tilde{X}{}_t^{\varepsilon} \tag{2.23}$$

   where $\gamma \in \mathbb{R}^d$, B_t is Brownian motion with a covariance matrix $\mathbf{A}$ and

$$X_t^1 = \int_0^t \int_{|x|\geq 1} x J_X(\mathrm{d}s, \mathrm{d}x) \quad \textit{and} \quad \tilde{X}{}_t^{\varepsilon} = \int_0^t \int_{\varepsilon\leq|x|\leq 1} x(J_X(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}\nu(x)\mathrm{d}s)$$

   where a convergence is in almost sure sense and uniform on $\langle 0, T\rangle$.

We will also from now on denote

$$\tilde{J}_X(\mathrm{d}s, \mathrm{d}x) = J_X(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}\nu(x)\mathrm{d}s$$

So every Levy Process is uniquely determined by triplet (A, ν, γ). It means that every continuous Levy process has the form $X_t = \gamma t + W_t$. Clearly X_t^1 is a CPP but also

$$X_t^{\varepsilon} = \sum_{s\in\langle 0,t\rangle}^{|\Delta X_t|>\varepsilon}$$

is $\forall\varepsilon > 0$ a compound Poisson process. However the problem at neighbourhood of 0 must be solved by centring of X_t. It can be shown that limit in (3.23) converges thanks to the martingale property of $\tilde{X}{}_t^{\varepsilon}$.

We can easily see now the general form of a characteristic function of Levy process from the previous theorem.

Theorem 2.5. *Let X be Levy process in $\mathbb{R}^d$ with triplet (A, ν, γ) then its characteristic function $\varphi_t(z) = e^{t\psi(z)}$ is given by*

$$\psi(z) = -\frac{1}{2}(z, Az) + i\gamma z + \int_{\mathbb{R}^d} (e^{izx} - 1 - izxI_{|x|\leq 1})\mathrm{d}\nu(x) \tag{2.24}$$

where $(\ ,\ )$ denotes a scalar product in $\mathbb{R}^d$.

The choice of the function $I_{|x|\leq 1}$ in the last term of previous theorems is not obligatory. There are many possible choices. If for example an additional assumption

$$\int_{|x|\geq 1} |x|\mathrm{d}\nu(x) < \infty \tag{2.25}$$

holds then we can rewrite the characteristic function as follows

$$\psi(z) = -\frac{1}{2}(z, Az) + i\gamma_c z + \int_{\mathbb{R}^d} (e^{izx} - 1 - izx)\mathrm{d}\nu(x)$$

where $\gamma_c = \gamma + \int_{|x|\geq 1} x\mathrm{d}\nu(x)$ is called the centre of mass because $EX_t = \gamma_c t$.

It is also important to realize that in the case of infinite activity $\nu(\mathbb{R}^d) = \infty$, has Levy process a countable number of jumps and its jumps are dense in $\langle 0, \infty)$.

Trajectories of Levy processes

Variation of a function f on a finite interval $\langle 0, T\rangle$ is defined by

$$\sup_{\tau} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

where supremum is over every partition of $\langle 0, T\rangle$, $0 = t_0 < ... < t_n = T$.

We say that Levy process has a finite variation if its trajectories have finite variation almost surely. It is well known that Brownian motion has a.s. infinite variation so Levy processes with finite variation do not have a diffusion component. However that is not a sufficient condition.

Theorem 2.6. *Levy process with triplet (A, ν, γ) has a finite variation $\iff$ $A = 0$ and $\int_{|x|\leq 1} |x|\mathrm{d}\nu(x) < \infty$*

Proof. We will only show a main idea of the prove, we need to prove that variation of process $\tilde{X}{}_t^{\varepsilon}$ is finite, this means

$$\lim_{\varepsilon \to 0} \int_0^t \int_{\varepsilon < |x| < 1} |x| J_X(dx.ds) < 0 \quad a.s.$$

which is equivalent to the condition $\int_{|x|\leq 1} |x|\mathrm{d}\nu(x) < \infty$. $\square$

Levy-Ito decomposition can be simplified for Levy processes with finite variation. Let X be a finite variation Levy process with a triplet $(0, \nu, \gamma)$ then

$$X_t = ct + \int_{\langle 0,t\rangle \times \mathbb{R}^d} x J_X(\mathrm{d}s, \mathrm{d}x) \tag{2.26}$$

where

$$c = \gamma - \int_{|x|\leq 1} x\mathrm{d}\nu(x) \tag{2.27}$$

so the characteristic function has the form

$$\varphi_t(u) = \exp\left(t(icu + \int_{\mathbb{R}^d} (e^{iux} - 1)\mathrm{d}\nu(x)\right) \tag{2.28}$$

We will now introduce an important subclass of Levy processes.

Definition 2.10. *Levy process in $\mathbb{R}$ with non-decreasing sample paths is called a subordinator.*

Subordinators will be very useful for the building of new Levy processes. They will be interpreted as new stochastic time. This could be useful in finance where we can interpret this time as the market time. This idea goes back to Mandelbrot [8].

The following theorem is very useful for the recognition of subordinators.

Theorem 2.7. *The following statements are equivalent for every Levy process X with the triplet (A, ν, γ)*

1. *X is subordinator*
2. *$X_t \geq 0$ for some $t > 0$*
3. *$X_t \geq 0$ for every $t > 0$*
4. *$A = 0$, $\nu((-\infty, 0\rangle) = 0$, $\int_0^1 x \mathrm{d}\nu(x) < \infty$ and $c \geq 0$*

So subordinators are processes with positive drift and jumps and also with finite variation, this is not surprising because non-decreasing trajectories imply finite variation.

There is an easy way to define new subordinators. Let $f : \mathbb{R}^d \to \langle 0, \infty)$ be a positive function with the behaviour $f(x) = O(x^2)$ for $x \to 0$. Then for an arbitrary Levy process X_t in $\mathbb{R}^d$ we can define

$$S_t = \sum_{s \in \langle 0,t \rangle}^{\Delta X_t \neq 0} f(\Delta X_s) \tag{2.29}$$

and S_t is subordinator. The condition $f(x) = O(x^2)$ is important because it guarantees convergence of the sum.

We obtain the following process by the choice $f(x) = x^2$

$$S_t = \sum_{s \in \langle 0,t \rangle}^{\Delta X_t \neq 0} (\Delta X_s)^2 \equiv [X, X]^d \tag{2.30}$$

It is called a discontinuous quadratic variation.

Probability density of Levy processes

Clearly not all Levy processes have absolutely continuous distribution. For example the density of Poisson process obviously does not exist.

The following result holds in one dimension.

Theorem 2.8. *Levy process X in $\mathbb{R}$ with a triplet (σ^2, ν, γ) has the continuous density if $\sigma > 0$ or $\nu(\mathbb{R}) = \infty$. Moreover if*

$$\exists \beta \in (0, 2) \liminf_{\varepsilon \to 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} x^2 \mathrm{d}\nu(x) > 0$$

Then $p_t \in C^\infty(\mathbb{R})$ and all its derivatives vanish in $\pm\infty$.

A prove is quite straight forward, see [36].

There is a connection between a density and Levy measure of Levy process. Let us consider CPP X_t with an intensity λ and the jump size given by an absolutely continuous distribution with a density f. We will denote

$$p_t^{ac}(x) = \sum_{n=1}^{\infty} f^{*n}(x) e^{-\lambda t} \frac{(\lambda t)^n}{n!} \tag{2.31}$$

where f^{*n} is n-th convolution. Then we can write

$$P(X_t \in A) = \sum_{n=0}^{\infty} P(X_t \in A | N_t = n) e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-\lambda t} I_{\{0 \in A\}} + \int_A p_t^{ac}(x) \mathrm{d}x \tag{2.32}$$

So the distribution of X is an absolutely continuous on $\mathbb{R} \setminus \{0\}$. We can see that $\lim\limits_{t\to 0} \frac{1}{t} p_t^{ac}(x) = \lambda f(x)$. This implies that for every bounded measurable function f fulfilling $f(0) = 0$ we have

$$\lim_{t\to 0} \frac{1}{t} E(f(X_t)) = \int_{\mathbb{R}^d} f(x) \mathrm{d}\nu(x) \tag{2.33}$$

This result holds even for infinite activity processes if we consider $f(x) = 0$ on neighbourhood of 0. This result implies that Levy measure determines short time behaviour of probability density.

The connection between Levy measure and probability density demonstrated above can be used to prove following useful theorems about existence of the moments.

Theorem 2.9. *Levy process in $\mathbb{R}$ with a triplet (σ^2, ν, γ) fulfils $E|X_t|^n < \infty$ for $\forall t > 0$ or equivalently for arbitrary $t > 0$ if and only if*

$$\int_{|x|\geq 1} |x|^n \mathrm{d}\nu(x) < \infty \tag{2.34}$$

Theorem 2.10. *Levy process in $\mathbb{R}$ with triplet (σ^2, ν, γ) fulfils $E(e^{uX_t}) < \infty$ for $\forall t > 0$ or equivalently for arbitrary $t > 0$ if and only if*

$$\int_{|x|\geq 1} e^{ux} \mathrm{d}\nu(x) < \infty \tag{2.35}$$

we get in that case

$$E(e^{uX_t}) = e^{t\psi(-iu)} \tag{2.36}$$

It is very convenient to calculate moments of Levy processes by differentiating the characteristic or cumulant generating function. Moments $c_n = \frac{1}{i^n} \frac{\partial^n \psi(u)}{\partial u^n}(0)$ are especially interesting because they are linear in t. We have explicitly

$$EX_t = c_1(X_t) = t(\gamma + \int_{|x|\geq 1} x \mathrm{d}\nu(x)) \tag{2.37}$$

$$VarX_t = c_2(X_t) = t(\sigma^2 + \int_{\mathbb{R}} x^2 \mathrm{d}\nu(x)) \tag{2.38}$$

and $c_n(X_t) = t\int_{\mathbb{R}} x^n \mathrm{d}\nu(x)$ for $n \geq 2$. This in particular implies the following scaling relation for a skewness and the kurtosis of Levy processes

$$s(X_t) = \frac{c_3(X_t)}{(c_2(X_t))^{3/2}} = \frac{s(X_1)}{\sqrt{t}},\ \kappa(X_t) = \frac{c_4(X_t)}{(c_2(X_t))^2} = \frac{\kappa(X_1)}{t}$$

These scaling relations can be used as indicators of how well the data can be fitted by Levy processes.

Stable processes

Stable distributions play very important role in the probability theory as attractors of sequences of random variables [3]. See also appendix B for a short review of properties of stable distributions.

So we should assume that stable processes will also play an important role. Their characteristic function has the form

$$\ln \varphi_t(k) = it\gamma k - \sigma t|k|^\alpha (1 + i\beta \frac{|k|}{k}\omega(k,\alpha)) \tag{2.39}$$

where

$$\omega(k,\alpha) = \begin{cases} -tan(\pi\alpha/2) & \text{for } \alpha \neq 1 \\ (2/\pi) ln|k| & \text{for } \alpha = 1 \end{cases}$$

How the characteristic triplet of stable distributions looks is an important question. An answer is given by the following theorem.

Theorem 2.11. *Levy process in $\mathbb{R}$ with generating triplet (σ^2, γ, ν) is α-stable for some $\alpha \in (0, 2\rangle$, if one of these conditions is fulfilled*

1. $\alpha = 2$ *and* $\nu = 0$
2. $\alpha \in (0, 2)$, $\sigma = 0$ *and* $\nu = (c_+ I_{(0,\infty)} + c_- I_{(-\infty,0)})|x|^{-(\alpha+1)}\mathrm{d}x$ *where* $c_\pm \geq 0$

Nice proof can be found in [16]. For the multidimensional version of this theorem see [15].

It holds $\beta = \frac{c_+ - c_-}{c_+ + c_-}$ so $c_\pm = 0$ if and only if $\beta = \mp 1$. The moment generating function $Ee^{\lambda X_t}$ exists in these cases for $\lambda > 0$ respectively $\lambda < 0$. These are only cases when it exists, this follows from the theorem (2.10).

Stable process is Brownian motion or an infinitely active process without a diffusion component. It demonstrates that an infinite activity can in some sense substitute a diffusion component.

Martingales and Levy processes

We will state here two theorems that can help us determine if Levy process or its exponential is a martingale.

Theorem 2.12. *Let X be any process in $\mathbb{R}$ with independent increments then.*

1. *$\forall u \in \mathbb{R}$ is $\frac{e^{iuX_t}}{E(e^{iuX_t})}$ a martingale*
2. *if $E(e^{uX_t}) < \infty,\ \forall t \geq 0$ then $\frac{e^{uX_t}}{E(e^{uX_t})}$ is a martingale*
3. *if $EX_t < \infty,\ \forall t \geq 0$ then $M_t = X_t - EX_t$ is a martingale*
4. *if $VarX_t < \infty,\ \forall t \geq 0$ then $M_t^2 - EM_t^2$ is a martingale*

This theorem follows directly from the independent increments property. The following theorem will also be very useful.

Theorem 2.13. *X Levy process in $\mathbb{R}$ with a triplet (σ^2, ν, γ) is*

1. *martingale $\iff \gamma_c = \gamma + \int_{|x|\geq 1} x\mathrm{d}\nu(x) = 0$ and $\int_{|x|\geq 1} |x|\mathrm{d}\nu(x) < \infty$*
2. *e^{X_t} is martingale $\iff \int_{|x|\geq 1} e^x \mathrm{d}\nu(x) < \infty$ and*
   $\sigma^2 + \frac{\gamma}{2} + \int_{\mathbb{R}}(e^x - 1 - xI_{|x|\leq 1})\mathrm{d}\nu(x) = 0$

The first statement is obvious, we will prove the second statement later with use of Ito lemma for Levy processes.

Levy processes and their generators

It is also worth noting that Levy processes are only Markov processes homogeneous in both the time and space. Mathematically speaking if we define

$$P_{s,t}(x, B) = P(X_t \in B | X_s = x) \tag{2.40}$$

then it holds $P_{s,t}(x, B) = P_{0,t-s}(0, B - x)$.

Generators of Ito diffusions and corresponding Kolmogorov equations are well known [7]. We will now briefly discuss form of generators for Levy processes, we will see that due to jump part of Levy processes its generators will be nonlocal pseudo differential operators.

For every Markov process X and every measurable, bounded function f we can define its time evolution operator by

$$(T_{t,s}f)(x) = E(f(X_t)|X_s = x) \tag{2.41}$$

We will consider only Markov processes for which is $T_{t,s}$ operator on space of bounded functions with supremum norm. These processes are called normal. $T_{t,s}$ is then clearly a linear operator, furthermore it holds $T_{s,r}T_{t,s} = T_{t,r}$ for $r < s < t$ and $\|T_{t,s}\| \leq 1$.

Markov processes with stationary increments are called time-homogeneous. For time homogeneous processes holds

$$T_{t,s} = T_{t-s,0} \qquad T_s T_t = T_{t+s} \quad (T_t f)(x) = E(f(X_t + x)) \tag{2.42}$$

Further we will call time-homogeneous Markov process Feller process if

1. $T_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$
2. $\lim\limits_{t\to 0} \|T_t f - f\| = 0 \quad \forall f \in C_0$

where C_0 denotes continuous function vanishing in infinity. It can be shown that every Levy process is Feller [35].

Theory of semigroups gives us important result that for transition probabilities $T_{t\geq 0}$ of any Feller process there exist a generator of this semi group

$$T_t = e^{tA} \qquad Af = \lim_{t\to 0} \frac{T_t f - f}{t} \tag{2.43}$$

A is called a generator of Feller process and it is in general unbounded, closed operator.

Space homogeneity of Levy processes can be reformulated as follows: Levy processes are only Feller processes with translational invariant transition probabilities.

More rigorously if we define operator $(P_a f)(x) = f(x+a)$ then $[T_t, P_a] = 0$ holds only for Levy processes.

The following theorem gives us form of generator for Levy process.

Theorem 2.14. *Let X be Levy process with cumulant generating function $\psi(z)$ and triplet (B, γ, ν). Let A be generator of X and $f \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes Schwartz space. Then it holds*

1. $(T_t f)(x) = \int_{\mathbb{R}^n} e^{izx} e^{t\psi(z)} (\mathcal{F}^{-1} f)(z) dz$
2. $(Af)(x) = \int_{\mathbb{R}^n} e^{izx} \psi(z) (\mathcal{F}^{-1} f)(z) dz$
3. $(Af)(x) = \gamma^i \partial_i f(x) + \frac{1}{2} B^{ij} \partial_i \partial_j f(x) + \int_{\mathbb{R}^n - \{0\}} (f(x+y) - f(y) - y^i \partial_i f(x) I_{|y|\leq 1}) \nu(dy)$

Proof. Using $(T_t f)(x) = E(f(X_t + x))$, rewriting f as Fourier transform and switching order of integration yields 1.), using this result and a definition of A yields 2.), 3.) follows from elementary properties of Fourier transform. □

Kolmogorov equations for Levy processes

We define $u(t,x) = (T_t f)(x)$. Then from definition of A holds

$$\partial_t u(t,x) = Au(t,x) \qquad u(0,x) = f(x) \tag{2.44}$$

Rewriting this equation for transition density p_t yields

$$\int_{\mathbb{R}^n} f(y) \partial_t p_t(y|x) dy = \int_{\mathbb{R}^n} f(y) A_x p_t(y|x) dy \tag{2.45}$$

using the fact that A and T_t commute yields similarly

$$\int_{\mathbb{R}^n} f(y)\partial_t p_t(y|x)dy = \int_{\mathbb{R}^n} f(y)A_y^\dagger p_t(y|x)dy \tag{2.46}$$

where $A_y^\dagger$ is adjoint operator in variable y. Above equations hold for any function f so we get following equations

$$(A_y^\dagger - \partial_t)p_t(y|x) = 0 \qquad (A_x - \partial_t)p_t(y|x) = 0 \tag{2.47}$$

If we assume no jumps i.e. $\nu = 0$ then we got well-known forward and backward Kolmogorov equation [6]. If we include jumps these equations are incredibly complicated and there is no general expression for adjoint operator .

Under some additional assumptions these equations also hold for Levy-type processes, which are broadly speaking processes we obtain by stochastic integration with respect to Levy process.

2.2.2 Building models from Levy processes

We will introduce few ways to build new Levy processes in this section. We will also introduce few models built in this manner.

There are three main approaches to building new Levy processes - by linear transformation, subordination and by tempering the Levy measure. Proofs of the theorems they can be found in [15].

Linear transformation

A linear transformation of Levy process is of course also Levy. We will state here the following general formula without going into any details.

Theorem 2.15. *Let X be Levy process in $\mathbb{R}^d$ with a triplet (A, ν, γ), $M \in \mathbb{R}^{n,d}$. Then $Y = MX$ is Levy process in $\mathbb{R}^n$ with the characteristic triplet given by*

$$A_Y = MAM^T, \quad \nu_Y(B) = \nu(\{x, Mx \in B\}) \tag{2.48}$$

$$\gamma_Y = M\gamma + \int_{\mathbb{R}^n} y(I_{|y|\le 1} - I_{S_1})\mathrm{d}\nu_Y(y) \tag{2.49}$$

where $S_1 = \{Mx, |x| \le 1\}$

Subordination

We will consider subordinator S_t with a triplet $(0, \rho, c)$. Laplace exponent $l(u)$ given by $E(e^{uS_t}) = e^{tl(u)}$ has in this case the form

$$l(u) = cu + \int_0^\infty (e^{ux} - 1)\mathrm{d}\rho(u) \quad for \quad u \le 0 \tag{2.50}$$

analogically as in (2.28).

We want to interpret S as new stochastic time. This technique can be very useful in finance as well as in physics. A similar technique was for example used in [37] to prove an equivalence between a for dimensional harmonic oscillator and Coulomb potential in the quantum mechanics.

The following result holds.

Theorem 2.16. *Let X be Levy process in $\mathbb{R}^d$ with triplet (A, ν, γ) and the cumulant generating function ψ. Let S_t be a subordinator defined above. Then a process*

$$Y(t, \omega) = X(S(t, \omega), \omega) \tag{2.51}$$

is Levy with a characteristic function and the triplet given by

$$\varphi_{Y_t}(u) = \exp\left(tl(\psi(u))\right) \tag{2.52}$$

$$A_Y = cA, \quad \nu_Y(B) = c\nu(B) + \int_0^\infty p_{X_s}(B)\mathrm{d}\rho(s) \tag{2.53}$$

and

$$\gamma_Y = c\gamma + \int_0^\infty \mathrm{d}\rho(s) \int_{|x|\leq 1} xp_{X_s}(\mathrm{d}x) \tag{2.54}$$

where $p_{X_t} = P \circ X_t^{-1}$

The term (2.52) makes a sense because for every Levy process $\Re(\psi(u)) \leq 0$.

We say that Y is a subordinate to the process X by S. Stable processes are useful subordinators. We will discuss Brownian subordination later in this section. Here we will introduce a α-stable subordinator S_t with $\alpha \in (0, 1)$. Clearly Levy measure of S must fulfil $\nu((-\infty, 0\rangle) = 0$ and we will assume that a drift $c = 0$. Then the characteristic exponent of S has the form

$$l(u) = C_1 \int_0^\infty \frac{e^{ux} - 1}{x^{\alpha+1}}\mathrm{d}x = -\frac{C_1\Gamma(1-\alpha)}{\alpha}(-u)^\alpha \tag{2.55}$$

If we now choose process X to be also a stable symmetric process given by $\psi_X(z) = -C_2|z|^\beta$ then the resulting process Y will be also stable and given by

$$\psi_Y(u) = l(\psi_X(u)) = -C_1 C_2^\alpha \frac{\Gamma(1-\alpha)}{\alpha}|u|^{\alpha\beta}$$

Tempering the Levy measure

We will introduce the last technique for building Levy processes here - a change of Levy measure. Let us consider Levy measure ν such that

$$\int_{|x|\geq 1} e^{\xi x}\mathrm{d}\nu(x) < \infty$$

then $\widetilde{\nu}(\mathrm{d}x) = e^{\xi x}\nu(\mathrm{d}x)$ is also Levy measure. So we can transform process with triplet (A, ν, γ) into the new one with $(A, \widetilde{\nu}, \gamma)$. This transformation is called Esscher transform. We can get a variety of new processes in this manner.

We can also consider asymmetric Esscher transform in the one dimensional case

$$\mathrm{d}\widetilde{\nu}(x) = \mathrm{d}\nu(x)(I_{x>0}e^{-\lambda_+|x|} + I_{x<0}e^{-\lambda_-|x|}) \tag{2.56}$$

where $\lambda_\pm > 0$. This transform will damp large jumps on both tails.

Models based on Levy processes

We have to make some choices when defining Levy model. The first one should be to choose the a jump-diffusion model or an infinity activity model.

Jump-diffusion models are defined as a sum of Brownian motion and compound Poisson process. So there are driven by diffusion and have occasional jumps. The distribution of jump sizes is known in this models but the probability density is not usually available in a closed form. They are also quite easy to simulate.

Infinite activity models usually do not contain a diffusion component and are driven only by jumps. They give realistic predictions of the price process. The probability density in some cases is known in a closed form. However they are harder to simulate and the distribution of jump sizes does not exist for them.

Another choice is how to specify the given model. Usual choices are by Brownian subordination or by by specifying the probability density or Levy measure. All of them have some advantages and disadvantages. For a further discussion see [15].

There is a huge amount of in detail analysed models, we will mention only few of them.

Merton model: we have already encounter this model. Process X, usually interpreted as logarithm of the price, is given by

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \tag{2.57}$$

where N_t is Poison process independent of W_t and with the intensity λ. $Y_i \sim N(m, \delta^2)$ are iid variables. X has a continuous density. This follows from the theorem (2.8). It is given by

$$p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \frac{\exp\left(\frac{-(x-\gamma t-km)^2}{2(\sigma^2 t+k\delta^2)}\right)}{\sqrt{2\pi(\sigma^2 t + k\delta^2)}} \tag{2.58}$$

Levy density has the form

$$\nu(x) = \frac{\lambda}{\sqrt{2\pi\delta^2}} \exp\left(\frac{-(x-m)^2}{2\delta^2}\right) \tag{2.59}$$

by an integration we can get cumulant generating function.

$$\psi(u) = \lambda(\exp\left(\frac{-\sigma^2 u^2}{2} + imu\right) - 1) + icu - \frac{\sigma^2 u^2}{2} \tag{2.60}$$

Where c is defined by (2.27). We can easily calculate all moments now by differentiating ψ.

In conclusion Merton model is jump-diffusion model with four free parameters $\sigma, m, \delta, \lambda$ and the drift. It has heavier tails then normal distribution.

Kou model is also a jump-diffusion model. The Levy density of CPP is given explicitly by

$$\nu(x) = \lambda(p\lambda_+ e^{-\lambda_+|x|} I_{x>0} + (1-p)\lambda_- e^{-\lambda_-|x|} I_{x<0})\mathrm{d}x \tag{2.61}$$

we obtain by an integration

$$\psi(u) = icu - \frac{\sigma^2 u^2}{2} + iu\lambda(\frac{p}{\lambda_+ - iu} - \frac{1-p}{\lambda_- + iu}) \tag{2.62}$$

We can again calculate all moments from ψ. We do not have the probability density in a closed form in Kou model. To conclude Kou model has five free parameters $\lambda, \lambda_\pm, \sigma, p$ and the drift. It has exponential tails $p(x) \sim e^{-\lambda_\pm x}$ in $\pm\infty$.

Brownian subordination has a nice interpretation as Brownian motion in "business" or deformed time. Let S_t be a subordinator with Laplace exponent $l(u)$, Levy density ρ and with a drift c. We consider a model

$$X_t = \sigma W(S_t) + \mu S_t \tag{2.63}$$

using the theorem (2.16) we can write

$$\psi_X(u) = l(i\mu u - \frac{\sigma^2 u^2}{2}) \tag{2.64}$$

We can calculate moments now, if $\mu = 0$ we can write explicitly

$$EX_t = 0, \quad VarX_t = \sigma^2 ES_t, \quad \kappa(X_t) = \frac{3VarS_t}{ES_t^2} \tag{2.65}$$

Levy density of X has from theorem (2.16) the form

$$\nu(x) = \int_0^\infty \exp\left(\frac{-(x-\mu t)^2}{2t\sigma^2}\right)\frac{\mathrm{d}\rho(t)}{\sqrt{2\pi\sigma^2 t}} \tag{2.66}$$

Tempered stable processes: We will mention this model only briefly. They are infinitely activity processes given by Esscher transformation of α-stable processes with $\alpha \in (0,2)$. They do not a have diffusion component and their Levy density is given by

$$\nu(x) = \frac{C_- e^{-\lambda_-|x|}}{|x|^{\alpha+1}} I_{x<0} + \frac{C_+ e^{-\lambda_+|x|}}{|x|^{\alpha+1}} I_{x>0} \tag{2.67}$$

where $c_\pm, \lambda_\pm > 0$. They are much better known as truncated Levy flights in physics [4]. This truncation ensures that tempered stable process has all moments finite. This follows from the theorem (2.9). Physicists usually perform this truncation by making cut-off of approximated probability density. However Esscher transform is a much more elegant approach.

2.3 Stochastic calculus for jump processes

We will introduce an integration with respect to a very general class of processes called semimartingales in this section. We will also state the generalized versions of Ito lemma. We will not rigorously prove most of the statements here but we will explain the meaning and ideas behind them. See [35] for a rigorous approach.

We will start with a brief introduction of this problematic. We want to define an integral in the sense

$$\sum_i \phi_i (S_{T_{i+1}} - S_{T_i}) \equiv \int_0^T \phi_t \mathrm{d}S_t \tag{2.68}$$

We will call $\phi_t = (\phi_t^1, .., \phi_t^n)$ a strategy. $S_t = (S_t^1, .., S_t^n)$ will be a cadlag process. We will consider S even more general than Levy processes later. We will see an interesting fact that the assumptions we imposed on S and ϕ as on an asset price and the strategy will be convenient even in a general theory.

We will start with a definition of the convenient class of integrands.

Definition 2.11. *Process $\phi_{t\in\langle 0,T\rangle}$ is called a simple predictable if it has the form*

$$\phi_t = \phi_0 I_{t=0} + \sum_{i=0}^{n} \phi_i I_{t\in(T_i, T_{i+1}\rangle} \tag{2.69}$$

where $0 = T_0 < ... < T_{n+1} = T$ are adapted random times and ϕ_i are bounded, $\mathcal{F}_{T_i}$-measurable random variables.

An integral with the respect to S has for simple predictable processes the form given by (2.68).

These processes are predictable and caglad so they are viable trading strategies. However they are also important for a theory of the stochastic integration.

We will denote the class of simple predictable processes by $\mathcal{S}(\langle 0,T\rangle)$. Trading strategies will usually be these processes or their limits. Let us note that non-predictable strategies often leads to the arbitrage opportunities so they are inadmissible.

The following result underlines the importance of these processes.

Theorem 2.17. *If S_t is a martingale then for any simple predictable process ϕ is the process*

$$X_t = \int_0^t \phi \mathrm{d}S \tag{2.70}$$

also a martingale.

Where we assume that martingales are cadlag processes from the definition.

To prove this we only need to show $E[X_T|\mathcal{F}_t] = X_t$, which is equivalent to $E[\phi_i(S_{T_{i+1}} - S_{T_i})|\mathcal{F}_t] = \phi_i(S_{T_{i+1}\wedge t} - S_{T_i\wedge t})$. This follows easily from properties of simple processes.

If we further specify the process S then this property will hold for a much larger class of strategies. Note that we used this property in the chapter one where we assumed that ϕ is a "reasonable" process. However we have to consider only the caglad ϕ for this result to hold.

We will encounter a problem that if we start with Levy process X then the integral $\int_0^T \phi \mathrm{d}X$ or the function $f(t, X_t)$ will not be Levy in general. Moreover even Markov processes are not stable under these operations. We will talk about a class of processes that is stable in this sense in the next paragraph.

Semimartingales

The main idea of building the stochastic integral is to define it for simple processes ϕ^n and then for their limits $\phi^n \to \phi$. So we need a convergence $\int_0^t \phi^n \mathrm{d}S \to \int_0^t \phi \mathrm{d}S$ to hold in some sense to define integral for all ϕ. This is the main idea behind the definition of semimartingales.

Definition 2.12. *Adapted cadlag process S is a semimartingale if $\forall \phi^n, \phi \in \mathcal{S}(\langle 0, T\rangle)$ holds*

$$\phi_t^n(\omega) \overset{\Omega\times\langle 0,T\rangle}{\rightrightarrows} \phi_t(\omega) \Longrightarrow \int_0^t \phi^n \mathrm{d}S \overset{P}{\to} \int_0^t \phi \mathrm{d}S \tag{2.71}$$

Intuitively this means that a small change of the price S should not change an integral, i.e. a value of the portfolio, a lot.

Semimartingales are often defined in a different way using a notion of local martingales. However we believe that this definition is more explanatory.

It can be moreover shown that for every semimartingale the convergence on the left side of the implication is uniform.

$$\sup_{t\in\langle 0,T\rangle} |\int_0^t \phi^n \mathrm{d}S - \int_0^t \phi \mathrm{d}S| \overset{P}{\to} 0 \tag{2.72}$$

Every finite variation process X is a semimartingale because if we denote the variation of X by $TV(X)$ we get

$$\sup_{t\in\langle 0,T\rangle} \int_0^t \phi \mathrm{d}X \le TV(X) \sup_{(t,\omega)} |\phi_t(\omega)|$$

It can be also easily shown that every square integrable martingale is a semimartingale. So Brownian motion and Poisson process in particular are semimartingales. Semimartingales form a vector space therefore every Levy process is also a semimartingale. Most of the processes we encounter are semimartingales however for example fractional Brownian motion is not.

We should note that the stochastic integration of simple processes is associative. It means if $S_t = \int_0^t \sigma \mathrm{d}X$ then $\int_0^t \phi \mathrm{d}S = \int_0^t \phi\sigma \mathrm{d}X$.

Semimartingales are stable under the stochastic integration i.e. integral with the respect to the semimartingale is again a semimartingale. This follows from the associativity and definition of semimartingales. Definition of semimartingales also implies that they are the largest set of processes for which the stochastic integral can be defined in a classical Ito or Stratonovich sense.

2.3.1 Stochastic integral for caglad processes

We want to define an integral of a caglad process with the respect to a semimartingale. The following theorem ensures that it is possible in natural way.

Theorem 2.18. *Let us consider ϕ a caglad process, S a semimartingale and $\pi^n = (0 = T_0^n < ... < T_{n+1}^n = T)$ a random partition such that $|\pi^n| = \sup_k |T_k^n - T_{k-1}^n| \overset{a.s.}{\to} 0$. Then the following equation holds*

$$\phi_0 S_0 + \sum_{k=0}^{n} \phi_{T_k}(S_{T_{k+1}\wedge t} - S_{T_k \wedge t}) \overset{P}{\to} \int_0^t \phi_{u-} \mathrm{d}S_u \tag{2.73}$$

Choice of the point T_k in Riemann sum is the same as in Ito integral and it is important for the martingale preserving property to hold. However we do not need to bother with the caglad condition in Ito calculus due to a continuity of Brownian sample paths.

Caglad processes can be written as limits of simple predictable processes. This is important because the crucial martingale preserving property given by theorem (2.17) holds only for them. This implies in particular, that if X is a square integrable martingale and ϕ is caglad and bounded then

$$M_t = \int_0^t \phi \mathrm{d}X \tag{2.74}$$

is also a square integrable martingale.

Integral with respect to Brownian motion

The integration with the respect to BM is given by Ito calculus. We will summarize it in the following theorem, however reader is presumed to be familiar with this theory.

Theorem 2.19. *Let ϕ be a predictable strategy fulfilling $E\int_0^T \phi t^2 \mathrm{d}t < \infty$. Then process $\int_0^t \phi \mathrm{d}W$ is a square integrable martingale and the following properties hold*

1. $E(\int_0^T \phi_t \mathrm{d}W_t) = 0$
2. *isometry property:* $E(\int_0^T \phi_t \mathrm{d}W_t)^2 = E\int_0^T \phi t^2 \mathrm{d}t$

Moreover such strategies can be approximated in the L^2 sense by simple predictable processes ϕ^n

$$E(\int_0^T |\phi_t - \phi_t^n|^2 \mathrm{d}t) \to 0 \tag{2.75}$$

Notice that Ito integral is well defined even for non-caglad processes that are not admissible trading strategies. We already mention that this is due to the continuity of Brownian motion.

Integral with respect to Poisson random measure

We will assume Poisson measure M on $\langle 0,T\rangle \times \mathbb{R}^n$ with an intensity μ. We will as usually denote the compensated measure by $\widetilde{M} = M - \mu$.

Simple predictable processes $\phi : \Omega \times \langle 0,T\rangle \times \mathbb{R}^n \to \mathbb{R}$ are in this case defined by

$$\phi(t,y) = \sum_{i,j=1}^{n,m} \phi_{ij} I_{(T_i,T_{i+1}\rangle}(t) I_{A_j}(y) \tag{2.76}$$

where T_i forms an increasing sequence of random times, ϕ_{ij} are bounded and $\mathcal{F}_{T_i}$-measurable variables and A_i are disjoint sets.

An integral for simple processes is given by

$$\int_0^T \int_{\mathbb{R}^d} \phi(t.y) M(\mathrm{d}t, \mathrm{d}y) = \sum_{i,j=1}^{n,m} \phi_{ij} M((T_i, T_{i+1}\rangle \times A_j) \tag{2.77}$$

Obviously the process

$$\int_0^t \int_{\mathbb{R}^d} \phi(t.y) M(\mathrm{d}t, \mathrm{d}y) = \sum_{i,j=1}^{n,m} \phi_{ij} M((T_i \wedge t, T_{i+1} \wedge t\rangle \times A_j) \tag{2.78}$$

is cadlag and adapted.

We already know that a compensated integral

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(t.y) \widetilde{M}(\mathrm{d}t, \mathrm{d}y) \tag{2.79}$$

is a square integrable martingale. The following isometry property for simple processes can be shown by straightforward calculations

$$EX_t^2 = E(\int_0^t \int_{\mathbb{R}^d} |\phi(t,y)|^2 \mu(\mathrm{d}t, \mathrm{d}y)) \tag{2.80}$$

The crucial point of this construction is to show that for every predictable process fulfilling

$$E(\int_0^t \int_{\mathbb{R}^d} |\phi(t,y)|^2 \mu(\mathrm{d}t, \mathrm{d}y)) < \infty \tag{2.81}$$

there exists a sequence ϕ^n of simple processes such that

$$E(\int_0^t \int_{\mathbb{R}^d} |\phi(t,y) - \phi^n(t,y)|^2 \mu(\mathrm{d}t, \mathrm{d}y)) \to 0 \tag{2.82}$$

therefore

$$\int_0^t \int_{\mathbb{R}^d} \phi^n(t.y)\widetilde{M}(\mathrm{d}t, \mathrm{d}y) \xrightarrow{L^2} \int_0^t \int_{\mathbb{R}^d} \phi(t.y)\widetilde{M}(\mathrm{d}t, \mathrm{d}y) \tag{2.83}$$

So the martingale and isometry property will be preserved for such ϕ. This result is summarized in the following theorem.

Theorem 2.20. *Let ϕ be a predictable process fulfilling $E(\int_0^t \int_{\mathbb{R}^d} |\phi(t, y)|^2 \mu(\mathrm{d}t, \mathrm{d}y)) < \infty$, then the following statements hold*

1. *$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(t.y)\widetilde{M}(\mathrm{d}t, \mathrm{d}y)$ is a square integrable martingale*
2. *isometry property: $EX_t^2 = E(\int_0^t \int_{\mathbb{R}^d} |\phi(t, y)|^2 \mu(\mathrm{d}t, \mathrm{d}y))$*

This is a completely analogical approach to the construction of Ito integral. See [35] for more details.

We will of course work mainly with Poisson measures given as jump measures of Levy processes. For $M = J_S$ a jump measure of Levy process with the intensity $\mathrm{d}\nu(x)\mathrm{d}t$ we have

$$\int_0^T \int_{\mathbb{R}^d} \phi(t.y)M(\mathrm{d}t, \mathrm{d}y) = \sum_{t \in \langle 0,T \rangle}^{\Delta S_t \neq 0} \phi(t, \Delta S_t) \tag{2.84}$$

Integration with respect to martingale measures

Above we presented the heuristic idea of stochastic integration, here we will present more general approach that will cover problems discussed above as particular examples.

We will consider random measure (not necessary positive one) on $\mathbb{R}^+ \times \mathbb{R}^n$. We will denote $M((0, t\rangle \times A) = M(t, A)$ and assume that $M(t, A)$ is $\mathcal{F}_t$-adapted martingale process. We will call M $(2, \rho)$-measure if the following conditions are satisfied

1. $M(\{0\}, A) = 0$ a.s.
2. $M((s, t\rangle, A)$ is independent of $\mathcal{F}_s$
3. $E(M(t, A))^2 = \rho(t, A)$ where ρ is σ-finite measure

L^2 **theory**: We assume integrands $F : \langle 0, T\rangle \times A \times \Omega \to \mathbb{R}$, $A \subset \mathbb{R}^n$ fulfilling

1. $\int_0^T \int_A E(|F(t, x)|^2)\rho(dt, dx) < \infty$
2. F is predictable

These functions form Hilbert space with scalar product given by

$$(F, G) = \int_0^T \int_A E(F(t, x)G(t, x))\rho(dt, dx) \tag{2.85}$$

We denote this space as $\mathcal{H}_2$, simple functions are defined as

$$F = \sum_{j,k} F_k(t_j) I_{(t_j, t_{j+1})} I_{A_k}$$

where $F_k(t_j)$ are bounded and $\mathcal{F}_{t_j}$ measurable, A_k are disjoint sets of finite measure. The standard construction of Lebesgue integral implies that these simple functions are dense in $\mathcal{H}_2$.

Construction now proceeds in the same way as in Poison case, we obtain the following result.

Theorem 2.21. *Let $F \in \mathcal{H}_2$, let M be $(2, \rho)$ measure, then for stochastic integral defined as $I_t(F) = \int_0^t \int_A F(s, x) M(ds, dx)$ holds*

1. *$I_t(F)$ is $\mathcal{F}_t$-adapted square integrable martingale*
2. *$E(I_t(F))^2 = \int_0^t \int_A E(|F(t, x)|^2) \rho(dt, dx)$*

Extended theory: The integration with respect to $(2, \rho)$ measure can be extended for larger class of processes fulfilling

$$P(\int_0^T \int_A E(|F(t, x)|^2) \rho(dt, dx) < \infty) = 1 \tag{2.86}$$

The convergence of an integral will be only in probability sense in this case and the integral itself will be only a local martingale, see [35] for details.

This theory enables to define stochastic integral for large class of processes called Levy type integrals

$$Y_t = Y_0 + \int_0^t G_s ds + \int_0^t F_s dW_s + \int_0^t \int_{|x|<1} H(s.x) \widetilde{J}_X(ds, dx) + \int_0^t \int_{|x| \geq 1} K(s, x) J_X(ds, dx) \tag{2.87}$$

where J_X is jump measure of Levy process X and integrands have to fulfil conditions required in L^2-theory or at least conditions of extended theory.

Quadratic variation and covariation

We will consider partition $\pi = \{0 = t_0 < ... < t_{n+1} = T\}$ then we can define a realized quadratic variation

$$V_X(\pi) = \sum_{t_i \in \pi} (X_{t_{i+1}} - X_{t_i})^2 = X_T^2 - X_0^2 - 2 \sum_{t_i} X_{t_i} (X_{t_{i+1}} - X_{t_i})$$

The last term in the previous equation reminds an integral $-2 \int_0^T X_{u-} \mathrm{d} X_u$ where X_- is a left continuous version of X.

Definition 2.13. *Let us consider a semimartingale X, $X_0 = 0$. We define a cadlag and adapted process*

$$[X,X]_t = X_t^2 - 2\int_0^t X_{u-}\mathrm{d}X_u \tag{2.88}$$

called the quadratic variation

For a sequence of partitions π^n such that $|\pi^n| \to 0$ holds

$$\sum_{t_i \in \pi^n}^{t_i \in \langle 0,t)} (X_{t_{i+1}} - X_{t_i})^2 \xrightarrow{P} [X,X]_t \tag{2.89}$$

We will summarize the properties of quadratic variation:

1. $[X,X]_t \geq [X,X]_s \geq 0$ for $t \geq s$
2. $\Delta[X,X]_t = |\Delta X_t|^2$
3. if X has continuous trajectories and a finite variation then $[X,X] = 0$
4. if X is a martingale and $[X,X] = 0$ then $X_t = X_0$

This implies that the quadratic variation of X is continuous if and only if X is continuous. We can see that the intersection of martingales and continuous processes with a finite variation contains only constants. A decomposition of the process to the square integrable martingale and to the continuous process with a finite variation is unique if exists. Martingales represents a noise and continuous processes with a finite variation the drift in this decomposition.

It holds for every Levy process X with the triplet (σ^2, ν, γ)

$$[X,X]_t = \sigma^2 t + \sum_{s \in \langle 0,t \rangle} |\Delta X_s|^2 \tag{2.90}$$

Moreover a quadratic variation of Levy process is also Levy and a subordinator.

We will briefly mention also the quadratic covariation.

Definition 2.14. *For X, Y semimartingales we define the quadratic covariation by*

$$[X,Y]_t = X_t Y_t - X_0 Y_0 - \int_0^T (X_{s-}\mathrm{d}Y_s + Y_{s-}\mathrm{d}X_s) \tag{2.91}$$

it can be shown similarly as above that if $X_0 = 0$ then

$$\sum_{t_i \in \pi^n}^{t_i \in \langle 0,t)} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \xrightarrow{P} [X,Y]_t$$

2.3.2 Generalized Ito formula

The following relation holds for smooth function f, g in real analysis

$$f(g(t)) - f(g(0)) = \int_0^t f'(g(s))g'(s)\mathrm{d}s \tag{2.92}$$

The stochastic version of this formula is given by Ito lemma for diffusion processes. We will derive a generalization of it for jump processes.

Let us consider a piecewise smooth function $x(t)$ with discontinuities $T_1, .., T_n$

$$x(t) = \int_0^t b(s)\mathrm{d}s + \sum_{T_i \le t} \Delta x_i \tag{2.93}$$

It is a cadlag function. The generalized version of (2.92) for $x(t)$ can be easily derived. It is given by the following theorem.

Theorem 2.22. *Let us consider a function $x(t)$ in the form (2.93) and a real function $f \in C^1$. Then we can write*

$$f(x(T)) - f(x(0)) = \int_0^T b(t)f'(x(t-))\mathrm{d}t + \sum_{i=1}^{n+1}(f(x(T_i-) + \Delta x_i) - f(x(T_i-)) \tag{2.94}$$

This can be directly applied to a process

$$X_t(\omega) = X_0 + \int_0^t b_s(\omega)\mathrm{d}s + \sum_{i=1}^{N_t(\omega)} \Delta X_i(\omega)$$

If we combine this result with classical Ito formula we will obtain Ito lemma for jump-diffusion processes.

Ito lemma for jump-diffusion processes

Theorem 2.23. *Let us consider a process*

$$X_t = X_0 + \int_0^t b_s\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}W_s + \sum_{i=1}^{N_t} \Delta X_i \tag{2.95}$$

where b_t, σ_t are continuous, $E\int_0^T \sigma_t^2\mathrm{d}t < \infty$ and T_i are jump times of N. Then the following holds for any function $f \in C^2$

$$\begin{aligned} f(t, X_t) - f(0, X_0) = \int_0^t (\frac{\partial f}{\partial s} + b_s\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_s^2\frac{\partial^2 f}{\partial x^2})\mathrm{d}s + \int_0^t \sigma_s\frac{\partial f}{\partial x}\mathrm{d}W_s + \\ + \sum_{\{i, T_i \le t\}} (f(s, X_{T_i-} + \Delta X_i) - f(s, X_{T_i-})) \end{aligned} \tag{2.96}$$

we avoided arguments $f = f(s, X_s)$ for simplicity.

The last term can be rewritten as

$$\sum_{\{i, T_i \le t\}} (f(X_{T_i-} + \Delta X_i) - f(X_{T_i-})) = \int_0^t \int_{\mathbb{R}} (f(s, X_{s-} + y) - f(s, X_{s-})) J_X(\mathrm{d}s, \mathrm{d}y) \tag{2.97}$$

The process $f(t, X_t)$ can be decomposed as a sum of a square integrable martingale M_t and a drift V_t. So $f(t, X_t) = M_t + V_t$ where

$$M_t = \int_0^t \int_{\mathbb{R}} (f(s, X_{s-} + y) - f(s, X_{s-})) \widetilde{J}_X(\mathrm{d}s, \mathrm{d}y) + \int_0^t \sigma_s \frac{\partial f}{\partial x} \mathrm{d}W_s \tag{2.98}$$

and

$$V_t = \int_0^t (\frac{\partial f}{\partial s} + b_s \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2}) \mathrm{d}s + \int_0^t \lambda \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}F(y)(f(s, X_{s-} + y) - f(s, X_{s-})) \tag{2.99}$$

Where $\lambda \mathrm{d}F(y)\mathrm{d}t = \mathrm{d}\nu(y)\mathrm{d}t$ is the intensity of J_X.

This decomposition is very convenient for calculation of the expected value.

The term (2.96) can be also rewritten as

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t (\frac{\partial f}{\partial s} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2}) \mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) \mathrm{d}X_s + \\ &+ \sum_{\{i, T_i \le t\}} (f(s, X_{T_i-} + \Delta X_i) - f(s, X_{T_i-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s-})) \end{aligned} \tag{2.100}$$

You should notice that in the integrands with respect to $\mathrm{d}s, \mathrm{d}W_s$ does not matter if we take the left continuous version of X or not. That is why we omit arguments there. However when we integrate with respect to not absolutely continuous measures it is important. We will see that this expression is equivalent to (2.96) in the case of the finite number of jumps, however in an infinite activity case it will be more general.

Ito lemma for Levy processes

The situation is more complicated in the case of infinite activity Levy processes. We cannot distinguish the time evolution given by diffusion component and by the jumps. The approach used above is not sufficient. However with the use of much more sophisticated methods the same result can be obtained.

Theorem 2.24. *Let X be Levy process with a triplet (σ^2, ν, γ) and $f \in C^2$ a real function. Then*

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t (\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}) \mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) \mathrm{d}X_s + \\ &+ \sum_{s \in \langle 0, t \rangle}^{\Delta X_s \neq 0} (f(s, X_{s-} + \Delta X_s) - f(s, X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s-})) \end{aligned} \tag{2.101}$$

If we consider a bounded function $f \in C^2$ with both derivation also bounded then a martingale/drift decomposition $f(t, X_t) = M_t + V_t$ can be obtained. We use Levy-Ito decomposition and we get

$$M_t = \int_0^t \int_{\mathbb{R}} (f(s, X_{s-} + y) - f(s, X_{s-}))\widetilde{J}_X(\mathrm{d}s, \mathrm{d}y) + \int_0^t \sigma \frac{\partial f}{\partial x} \mathrm{d}W_s \tag{2.102}$$

and

$$\begin{aligned} V_t = & \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}\nu(y)(f(s, X_{s-} + y) - f(s, X_{s-}) - yI_{|y|\leq 1} f'(s, X_{s-})) + \\ & + \int_0^t (\frac{\partial f}{\partial s} + \gamma \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2})\mathrm{d}s \end{aligned} \tag{2.103}$$

The conditions imposed on f guarantee that the first integral in (3.83) converges. It can be verified using Lagrange theorem and properties of Levy measures.

We will state for completeness also a multidimensional version of Ito lemma for Levy processes.

Theorem 2.25. *Let $X_t = (X_t^1, ..., X_t^d)$ be Levy process with a triplet (A, ν, γ) and let $f : \langle 0, T \rangle \times \mathbb{R}^d \to \mathbb{R}$ be a function $f \in C^2$ then*

$$\begin{aligned} f(t, X_t) - f(0, X_0) = & \int_0^t \frac{\partial f}{\partial s} \mathrm{d}s + \sum_{i,j=1}^d \int_0^t \frac{1}{2} A_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \mathrm{d}s + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_{s-}) \mathrm{d}X_s^i + \\ & + \sum_{s \in \langle 0, t \rangle}^{\Delta X_s \neq 0} (f(s, X_{s-} + \Delta X_s) - f(s, X_{s-}) - \sum_{i=1}^d \Delta X_s^i \frac{\partial f}{\partial x_i}(s, X_{s-})) \end{aligned} \tag{2.104}$$

Ito lemma for semimartingales

It is worth noticing that Ito formula only depends on trajectories of processes not on their probabilistic structures. So generalizing Ito lemma for semimartingales is not as difficult as it might seem, however we will only state the main result. See [34] for more details.

A quadratic variation of a semimartingale X can be decomposed to a continuous and pure jump part

$$[X, X]_t = [X, X]_t^c + [X, X]_t^{jump}$$

Ito formula for semimartingales has now the following form.

Theorem 2.26. *Let X be a semimartingale and let us consider a function*

$f \in C^2$. *Then*

$$f(t, X_t) - f(0, X_0) = \int_0^t \frac{\partial f}{\partial s} \mathrm{d}s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2} \mathrm{d}[X, X]_s^c + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) \mathrm{d}X_s + \\ + \sum_{s \in \langle 0,t \rangle}^{\Delta X_s \neq 0} \left(f(s, X_{s-} + \Delta X_s) - f(s, X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s-}) \right) \tag{2.105}$$

It follows from this theorem that the class of semimartingales is stable under all sufficiently smooth transformations f because they can be written as a stochastic integral.

Martingale drift decomposition for semimartingales is bit more complicated. We will consider only semimartingales in the form of Levy type integrals

$$Y_t = Y_0 + \int_0^t \gamma_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|x|<1} H(s.x) \widetilde{J}_X(ds, dx) + \int_0^t \int_{|x| \geq 1} K(s, x) J_X(ds, dx) \tag{2.106}$$

Then $d[Y, Y]_t^c = \sigma_t^2 dt$. If we assume that coefficients γ, σ, K, H are "well-behaved", then the drift term has a form

$$V_t = \gamma_t + \int_{|x| \geq 1} K(t, x) \nu(dx) \tag{2.107}$$

where $\nu(dx)dt$ is an intensity of J_X. However if $V_t = 0$ then Y_t is not a martingale but only a local martingale [35], often we will not strictly distinguish martingales and local martingales for purposes of this thesis.

Exponential of Levy processes

In the next section we will build models based on exponentials of Levy processes. It will be convenient for us to make some calculations about them now and have them prepared in advance.

Let X be Levy process with a triplet (σ^2, ν, γ) and a jump measure J_X. Then for $Y_t = \exp(X_t)$ we will using Ito lemma get

$$Y_t = 1 + \int_0^t Y_{s-} \mathrm{d}X_s + \int_0^t \frac{\sigma^2}{2} Y_{s-} \mathrm{d}s + \int_{\langle 0,t \rangle \times \mathbb{R}} Y_{s-}(e^z - 1 - z) J_X(\mathrm{d}s, \mathrm{d}z) \tag{2.108}$$

If $EY_t < \infty$, which is according to the theorem (2.10) equivalent to $\int_{|y| \geq 1} e^y \mathrm{d}\nu(y) < \infty$, then there exists a martingale/drift decomposition in the form

$$M_t = 1 + \int_0^t \sigma Y_{s-} \mathrm{d}W_s + \int_{\langle 0,t \rangle \times \mathbb{R}} Y_{s-}(e^z - 1) \widetilde{J}_X(\mathrm{d}s, \mathrm{d}z) \tag{2.109}$$

and

$$V_t = \int_0^t Y_{s-}(\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^z - 1 - z I_{|z| \leq 1}) \mathrm{d}\nu(z)) \mathrm{d}s \tag{2.110}$$

We can see that the condition $\int_{|y|\geq 1} e^y \mathrm{d}\nu(y) < \infty$ guarantees the existence of integral in (2.110).

So Y_t is a martingale if and only if $V_t = 0$. This is equivalent to the condition given by the theorem (2.13) $\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}}(e^z - 1 - zI_{|z|\leq 1})\mathrm{d}\nu(z) = 0$.

Similarly if we consider X_t to be Levy type integral in the form (2.106) we get with use of Ito lemma for semimartingales the condition for drift term to vanish

$$\gamma_t + \frac{1}{2}\sigma_t^2 + \int_{|x|<1}(e^{H(t,x)} - 1 - H(t,x))\nu(dx) + \int_{|x|\geq 1}(e^{K(t,x)} - 1)\nu(dx) = 0 \tag{2.111}$$

and Y_t is a local martingale if and only if $V_t = 0$. Moreover it can be shown that if $EY_t = 1$ then Y_t is a martingale.

2.4 Option pricing and hedging in exp-Levy models

In this section we will discuss concepts of equivalent martingale measures, hedging and option pricing in exp-Levy models. We have already introduced these notions in the chapter one, here we will discuss them in this special case. The theory of Levy processes we built in this chapter will be essential here.

2.4.1 Equivalence of measures for Levy processes

We demonstrated an importance of equivalent martingale measures in option pricing in the chapter one. We remind that every pricing rule is connected with the martingale measure by the following relation

$$\Pi_t(H) = e^{-r(T-t)}E^Q[H|\mathcal{F}_t]$$

We used Girsanov theorem (1.1) for a recognition of equivalent measures in Black-Scholes case. We will need a similar result for Levy processes. We will state a general result for Levy process, this result is due to Sato [36].

Theorem 2.27. *Let $(X, P), (Y, P')$ be Levy processes in $\mathbb{R}$ with triplets (σ^2, ν, γ) and $(\sigma^{2'}, \nu', \gamma')$ respectively. Then $P\mid_{\mathcal{F}_t} \sim P'\mid_{\mathcal{F}_t}$ for any or equivalently every $t > 0$ if and only if the following conditions hold*

1. $\sigma = \sigma'$
2. $\nu \sim \nu'$ *and* $\int_{\mathbb{R}}(e^{\phi(x)/2} - 1)^2 \mathrm{d}\nu(x) < \infty$
3. *if* $\sigma = 0$ *then* $\gamma' = \gamma + \int_{-1}^{1} x \mathrm{d}(\nu' - \nu)(x)$

where $\phi = \ln \frac{\mathrm{d}\nu'}{\mathrm{d}\nu}$.
Radon-Nikodym derivative has the form then

$$\frac{dP'}{dP}\mid_{\mathcal{F}_t} = e^{M_t} \tag{2.112}$$

where

$$M_t = \alpha X_t^c - \alpha\gamma t - \frac{\alpha^2\sigma^2 t}{2} + \lim_{\varepsilon\to 0}(\int_0^t \int_{|x|>\varepsilon} \phi(x) J_X(ds,dx) - t\int_{|x|>\varepsilon}(e^{\phi(x)}-1)\nu(dx)) \tag{2.113}$$

where X_t^c is continuous part of X and α is given by

$$\alpha\sigma^2 = \gamma' - \gamma - \int_{-1}^{1} x\mathrm{d}(\nu' - \nu)(x) \tag{2.114}$$

if $\sigma > 0$ and $\alpha = 0$ if $\sigma = 0$.

It can be easily shown by some rearranging of terms that M_t is Levy process with characteristic triplet given by

$$\sigma_M^2 = \sigma^2\alpha^2 \tag{2.115}$$

$$\nu_M = \nu\phi^{-1} \tag{2.116}$$

$$\gamma_M = -\frac{\sigma^2\alpha^2}{2} - \int_{\mathbb{R}} (e^y - 1 - yI_{|y|\le 1})(\nu\phi^{-1})(dy) \tag{2.117}$$

We can easily verify that e^{M_t} is a martingale using result (2.110).

A very interesting fact is that we cannot change the drift term in the absence of a diffusion component. However we can temper the Levy measure, which leads to a great variety of models.

This theorem allows us to define new equivalent measure Q by defining Radon-Nykodym derivative. We define $\frac{dQ}{dP}\mid_{\mathcal{F}_t} = e^{M_t}$ where

$$M_t = \gamma_t dt + \sigma_t dW_t + \int_{\mathbb{R}-\{0\}} H(t,x)\widetilde{J}_X(dt,dx) \tag{2.118}$$

e^{M_t} must be a martingale, this determines the coefficient γ_t, using (2.111) yields

$$\gamma_t = -\frac{1}{2}\sigma_t^2 - \int_0^t \int_{\mathbb{R}-\{0\}} (e^{H(s,x)} - 1 - H(s,x))\nu(dx)ds \tag{2.119}$$

This is actually more general form of M_t then given by theorem (2.27), so this change of measure can define models driven by processes with non-stationary increments (case of deterministic, time dependent coefficients) or even by processes without independent increments (random coefficients). The case $\gamma_t, \sigma_t = const.$ and $H(t,x) = \phi(x)$ leads to exp-Levy models.

Under new measure Q is

$$W_Q(t) = W(t) - \int_0^t \sigma_s ds \tag{2.120}$$

standard Brownian motion. This follows from Girsanov theorem. Moreover if we define $\nu_Q(t,A) = \int_0^t \int_A e^{H(s,x)}\nu(dx)ds$ then the random measure

$$\widetilde{J}_Q(t,A) = \widetilde{J}_X(t,A) - \nu_Q(t,A) + \nu(t,A) \quad A \in \mathcal{B}(\mathbb{R}) \tag{2.121}$$

is a martingale under Q. To prove this we need to show that $e^{M_t}\widetilde{J}_Q(t,A)$ is P-martingale. Using Ito lemma yields

$$d(e^{M_t}\widetilde{J}_Q(t,A)) = \widetilde{J}_Q(t,A)de^{M_t} + e^{M_t}\widetilde{J}_Q(dt,A) + de^{M_t}\widetilde{J}_Q(dt,A) \tag{2.122}$$

using $J_X(dt,dx)J_X(dt,dy) = J_X(dt,dx)\delta(x-y)$ yields

$$d(e^{M_t}\widetilde{J}_Q(t,A)) = \widetilde{J}_Q(t,A)de^{M_t} + e^{M_t}\widetilde{J}_Q(dt,A) + \int_A (e^{H(s,x)}-1)J_X(ds,dx) \tag{2.123}$$

so inserting for J_Q yields

$$d(e^{M_t}\widetilde{J}_Q(t,A)) = \widetilde{J}_Q(t,A)de^{M_t} + \int_A e^{M_t}\widetilde{J}_X(dt,dx) + \int_A (e^{H(s,x)}-1)\widetilde{J}_X(dt,dx) \tag{2.124}$$

this proves our statement because both e^{M_t} and $\widetilde{J}_X$ are P-martingales.

Useful way to obtain new equivalent Levy measures is Esscher transform, which we already discussed in section 2.2.2. Let us consider Levy process X with a triplet (σ^2,ν,γ) and let assume $\int_{|x|>1} e^{ux}\nu(dx) < \infty$. Then we can define Radon-Nykodym derivative by

$$\frac{dQ}{dP}\,|_{\mathcal{F}_t} = \frac{e^{uX_t}}{E(e^{uX_t})} = e^{uX_t - t\psi(-iu)} \tag{2.125}$$

where ψ is a cumulant generating function given by (2.24), clearly $e^{uX_t-t\psi(-iu)}$ is a positive martingale so this defines equivalent measures. Moreover this new measure is obtained by tempering of Levy measure $\nu_Q(dx) = e^{ux}\nu(dx)$ and the drift is changed according to $\gamma_Q - \gamma - \int_{-1}^{1} x(e^{ux}-1)\nu(dx) = \sigma^2$. This follows directly from theorem (2.27). We will see that Esscher transform is a very useful tool in the next section.

An important question is if exp-Levy models $S_t = \exp X_t$, X Levy, are arbitrage free. So question is if there always exists equivalent measure Q under which is a discounted price $\hat{S}_t = e^{-rt}S_t$ a martingale. Answer is given by the following theorem.

Theorem 2.28. *Model $S_t = e^{X_t}$ arbitrage free for every Levy process X with neither increasing nor decreasing trajectories.*

The proof of this theorem is quite straightforward but bit technical, it uses the general result (2.27) and Esscher transform. It can be found in [15].

This theorem in particular implies that every Levy process with a diffusion component defines an arbitrage free exp-Levy model.

2.4.2 Hedging in exp-Levy models

We have already discussed hedging in incomplete markets in the chapter one. We know that models with jumps usually generate incomplete markets. We will now revisit general methods described in the chapter one in the framework of exp-Levy models.

We will consider the set of all equivalent martingale measures denoted by $M\{\mathcal{C}\}$, it is important that the incomplete model is not fully defined by a choice of the historical price process but we also need to choose a concrete martingale measure.

Merton approach revisited

We discussed Merton approach thoroughly in the chapter one. We remind that a price of the risky asset is given by

$$S_t = S_0 \exp\left(\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right) = e^{X_t} \tag{2.126}$$

equivalently we can write

$$dS_t = (\mu + \frac{\sigma^2}{2})S_t dt + \sigma S_t dW_t + \int_{\mathbb{R}} S_{t-}(e^y - 1)J_X(dt, dy) \tag{2.127}$$

We chose an equivalent martingale measure Q^M under which only the drift term changed. The theorem (2.27) ensures validity of this choice. We also derived in the chapter one the price of the call option in this model given by (1.64).

We will now make a few more calculations with the use of generalized Ito lemma. We know that under Q^M is a discounted option price $\hat{C}_t^M = C_t^M e^{-rt}$ a martingale. So we can write under Q^M

$$\hat{C}_T^M - \hat{C}_0^M = \int_0^T \frac{\partial C^M}{\partial S} \hat{S}_u \sigma \mathrm{d}W_u^M + \int_0^T \int_{\mathbb{R}} (C^M(u, S_{u-} + z) - C^M(u, S_{u-})) \widetilde{J_X}(\mathrm{d}u, \mathrm{d}z) \tag{2.128}$$

This follows from Ito lemma and the martingale/drift decomposition. We hedged only a diffusion component so we will obtain the same self-financing strategy (ϕ_t^0, ϕ_t) as in Black-Scholes case

$$\phi_t = \frac{\partial C^M}{\partial S}(t, S_{t-}) \tag{2.129}$$

and

$$\phi_t^0 = \frac{C(t, S_{t-}) - \phi_t S_t}{S_t^0} \tag{2.130}$$

The strategy did not change but Black-Scholes equation has a different form here. We will discuss the form of B.-S. equation in exp-Levy models in the next section.

We can also write the discounted hedging error i.e. the risk we did not neutralize as

$$\hat{H}_T - e^{-rT} V_T(\phi) = \hat{C}_T^M - \hat{C}_0^M - \int_0^T \frac{\partial C^M}{\partial S}(u, S_{u-}) \mathrm{d}\hat{S}_u \tag{2.131}$$

Incompleteness of exp-Levy models

We talked about exp-Levy models being incomplete, we will prove this statement in this section.

We will consider model

$$dS_t = S_{t-}dX_t \tag{2.132}$$

where X_t is Levy process with characteristic triplet $(\hat{\sigma}^2, \gamma, \nu)$. Further we will assume $\int_{\mathbb{R}} \max(|x|, x^2) < \infty$, so in particular X_t has finite second moment.

It can be shown that equation (2.132) with initial condition $S_0 = 1$ has unique solution in the form [35]

$$S_t = \varepsilon_X(t) = \exp\left(X_i - \frac{1}{2}[X,X]_t^c\right) \prod_{s \le t} (1 + \Delta X_s) e^{-\Delta X_s} \tag{2.133}$$

Using Ito lemma it can be verified that $\varepsilon_X(t)$ solves (2.132). $\varepsilon_X(t)$ is called stochastic exponential and it can be written as exponential of Levy process when X is Levy.

$\varepsilon_X(t)$ is clearly positive only if jumps $\Delta X_t > -1$, so we will assume $\nu\{(-\infty, -1)\} = 0$. We will consider X_t in the form

$$X_t = mt + \hat{\sigma} W_t + \int_{-1}^{\infty} x \widetilde{J}_X(t, dx), \qquad m = \gamma + \int_1^{\infty} x\nu(dx) \tag{2.134}$$

Using Ito lemma for a discounted price $\hat{S}_t = e^{-rt} S_t$ yields

$$d(\ln \hat{S}_t) = \hat{\sigma} dW_t + (m - r + \frac{\hat{\sigma}^2}{2})dt + \int_{-1}^{\infty} \ln(1+x) \widetilde{J}_X(dt, dx) + \int_{-1}^{\infty} (\ln(1+x) - x)\nu(dx)dt \tag{2.135}$$

We will now consider equivalent measure Q given by $\frac{dQ}{dP} |_{\mathcal{F}_t} = e^{M_t}$ where M_t is given by (2.118). Rewriting (2.135) in terms of $W_Q, \widetilde{J}_Q$ given by (2.120),(2.121) yields

$$\begin{aligned} d(\ln \hat{S}_t) = \hat{\sigma} dW_Q(t) + (m - r + \frac{\hat{\sigma}^2}{2} + \hat{\sigma}\sigma_t + \int_{-1}^{\infty} x(e^{H(t,x)} - 1)\nu(dx))dt + \\ \int_{-1}^{\infty} \ln(1+x) \widetilde{J}_Q(dt, dx) + \int_{-1}^{\infty} (\ln(1+x) - x)\nu_Q(dx)dt \end{aligned} \tag{2.136}$$

We know that under Q are $W_Q, \widetilde{J}_Q$ martingales.

$d(\ln \hat{S}_t)$ can be under Q decomposed as $d(\ln \hat{S}_t) = d(\ln \hat{S}_t^m) + d(\ln \hat{S}_t^d)$, where

$$d(\ln \hat{S}_t^m) = \hat{\sigma} dW_Q(t) + \frac{\hat{\sigma}^2}{2} + \int_{-1}^{\infty} \ln(1+x) \widetilde{J}_Q(dt, dx) + \int_{-1}^{\infty} (\ln(1+x) - x)\nu_Q(dx)dt \tag{2.137}$$

and

$$d(\ln \hat{S}_t^d) = (m - r + \hat{\sigma}\sigma_t + \int_{-1}^{\infty} x(e^{H(t,x)} - 1)\nu(dx))dt \tag{2.138}$$

We can rewrite (2.137) and (2.138) as

$$d\hat{S}_t^m = \hat{\sigma}\hat{S}_t^1 dW_Q(t) + \int_{-1}^{\infty} x\hat{S}_t^1 \widetilde{N}_Q(dt, dx) \tag{2.139}$$

$$d(\hat{S}_t^d) = (m - r + \hat{\sigma}\sigma_t + \int_{-1}^{\infty} x(e^{H(t,x)} - 1)\nu(dx))\hat{S}_t^d dt \tag{2.140}$$

We have $d\hat{S}_t = \hat{S}_t^m d\hat{S}_t^d + \hat{S}_t^d d\hat{S}_t^m$ (because $d\hat{S}_t^d d\hat{S}_t^m = 0$) so for $\hat{S}$ to be a (local) martingale is required $\hat{S}_t^d = 0$. This implies the following condition

$$m - r + \hat{\sigma}\sigma_t + \int_{\mathbb{R}} x(e^{H(t,x)} - 1)\nu(dx) = 0 \tag{2.141}$$

where $m, r, \hat{\sigma}$ are parameters of a model and $\sigma_t, H(t,x)$ are parameters determining the new martingale measure. If we do not want to go beyond exp-Levy models we can consider only $\sigma_t = const., H(t,x) = \phi(x)$.

The equation above has infinitely many solutions because if we assume (σ_t, H) to be a solution then also $(\sigma_t + \int_{\mathbb{R}} f(x)\nu(dx), \ln(e^H - \frac{\hat{\sigma} f}{x}))$ is a solution for any integrable function f.

There are only two cases where this equation has unique solution

1. $\nu = 0$ Black-Scholes model
2. $\hat{\sigma} = 0, \nu(x) = \lambda\delta(x - a)$ Poison process with a drift

all other Levy models are incomplete.

Quadratic hedging in exp-Levy models

We introduced the idea of the quadratic hedging in the chapter one. We will now consider exp-Levy model

$$S_t = \exp(rt + X_t) \tag{2.142}$$

where X is Levy process with a triplet (σ^2, ν, γ). Let us assume that Q is a risk neutral measure under which is $\hat{S}_t$ a martingale. Moreover we want $ES_t^2 < \infty$ to hold. It is equivalent to the condition

$$\int_{|y| \geq 1} e^{2y} \mathrm{d}\nu(y) < \infty$$

We will consider only quadratically integrable strategies (so the L^2-theory applies)

$$E^Q[|\int_0^T \phi_t \mathrm{d}\hat{S}_t|^2] < \infty \tag{2.143}$$

We will denote the set of them by $L^2(S)$.

We will consider self-financing strategy (ϕ_t^0, ϕ_t). The discounted portfolio price $\hat{V}_t(\phi) = \hat{V}_0 + \int_0^t \phi \mathrm{d}\hat{S}$ is clearly a martingale under the new measure.

We want now to find a strategy that minimizes the following term

$$\inf_{\phi \in L^2(S)} |\hat{V}_T(\phi) - \hat{V}_0 - \hat{H}|^2 \tag{2.144}$$

Obviously an expected value of the hedging error is $V_0 - E^Q \hat{H}$. The following theorem states when this problem can be solved explicitly.

Theorem 2.29. *Let us consider exp-Levy model with a risk neutral measure Q*

$$\mathrm{d}\hat{S}_t = \hat{S}_t \mathrm{d}Z_t$$

where Z is Levy process with a diffusion coefficient σ and Levy measure ν.

Let C be an European option with a payoff H fulfilling

$$|H(x) - H(y)| \leq K|x - y|, \quad K > 0$$

so a price of the option can be written as

$$C(t, S) = e^{-r(T-t)} E^Q[H(S_T)|S_t = S]$$

then the optimal strategy given by (3.112) is

$$\phi_t = \theta(t, S_{t-}) \tag{2.145}$$

$$\theta(t, S_t) = \frac{\sigma^2 \frac{\partial C}{\partial S}(t, S) + \frac{1}{S}\int_{\mathbb{R}} z(C(t, S(1+z)) - C(t, S))\mathrm{d}\nu(z)}{\sigma^2 + \int_{\mathbb{R}} z^2 \mathrm{d}\nu(z)} \tag{2.146}$$

Clearly this theorem is applicable for put/call options.

The assumption about pay off function H is important because it implies that the option price will be smooth enough for Ito lemma to apply. When Ito lemma applies then we can prove this statement as follows:

The value of the portfolio is a martingale under Q so we have

$$\hat{V}_T = \int_0^T \phi_t \mathrm{d}\hat{S}_t = \int_0^T \phi_t \sigma \hat{S}_{t-} \mathrm{d}W_t + \int_0^T \int_{\mathbb{R}} \phi_t \hat{S}_{t-} z \widetilde{J}_Z(\mathrm{dt}, \mathrm{z}) \tag{2.147}$$

Notice that if we consider $\hat{S}_t = expX_t$ then the form of $\mathrm{d}\hat{S}_t$ will be given by a martingale drift decomposition (3.97) so

$$\hat{V}_T = \int_0^T \phi_t \mathrm{d}\hat{S}_t = \int_0^T \phi_t \sigma \hat{S}_t \mathrm{d}W_t + \int_0^T \int_{\mathbb{R}} \phi_t \hat{S}_{t-}(e^x - 1) \widetilde{J}_X(\mathrm{dt}, \mathrm{x}) \tag{2.148}$$

We get similarly

$$\hat{C}(t, S_t) - \hat{C}(0, S_0) = \int_0^t \frac{\partial C}{\partial S} \sigma \hat{S}_u \mathrm{d}W_u + \int_0^t \int_{\mathbb{R}} (C(u, S_{u-}(1+z)) - C(u, S_{u-}))\widetilde{J}_Z(\mathrm{d}u, \mathrm{d}z) \tag{2.149}$$

or in the terms of X

$$\hat{C}(t,S_t)-\hat{C}(0,S_0)=\int_0^t \frac{\partial C}{\partial S}\sigma\hat{S}_u \mathrm{d}W_u+\int_0^t\int_{\mathbb{R}}(C(u,S_{u-}e^x)-C(u,S_{u-}))\widetilde{J}_X(\mathrm{d}u,\mathrm{d}x) \tag{2.150}$$

We want to minimize a residual risk now

$$R_T(\phi)=\hat{C}(T,S_T)-\hat{C}(0,S_0)-\hat{V}_T(\phi) \tag{2.151}$$

in the mean square sense. We now obtain the strategy (3.114) by calculating $E(R_T(\phi))^2$ with use of isometry property and minimizing it then with respect to ϕ_t.

Minimal entropy hedging

We mentioned minimal entropy approach already in the chapter 1, its idea is to minimize relative entropy of original measure P and new martingale measure Q. The relative entropy is defined as

$$\varepsilon(Q,P)=E^Q(\ln(\frac{\mathrm{d}Q}{\mathrm{d}P}))=E^P(\frac{\mathrm{d}Q}{\mathrm{d}P}\ln(\frac{\mathrm{d}Q}{\mathrm{d}P})) \tag{2.152}$$

This reminds standard standard definition of Shannon entropy. Function $f(\frac{\mathrm{d}Q}{\mathrm{d}P})=\frac{\mathrm{d}Q}{\mathrm{d}P}\ln(\frac{\mathrm{d}Q}{\mathrm{d}P})$ is convex, therefore it is easy to show using Jensen inequality that $\varepsilon(Q,P)\geq 0$.

It can be shown that in the case of exp-Levy models the entropy $\varepsilon(Q,P)$ only depends on the Levy measures under P and Q [15]. Therefore we should obtain the minimal entropy measure Q by only tempering the Levy measure.

Moreover it turns out that when we consider model $S_t=e^{X_t}$ and X_t is Levy process then it is sufficient to consider Esscher transform $\nu_Q(x)=e^{ux}\nu_P(x)$. It leads to the new measure given by (2.125)

$$e^{M_t(u)}\equiv\frac{dQ_u}{dP}\,|_{\mathcal{F}_t}=\frac{e^{uX_t}}{E(e^{uX_t})} \tag{2.153}$$

Let us assume that X is Levy process with a triplet $(\hat{\sigma}^2,\gamma,\nu)$ given by (2.134). Ito lemma combined with a fact that E^{M_t} is a martingale then gives

$$de^{M_t(u)}=M_{t-}(u)(\hat{\sigma}udW_t+(e^{ux}-1)\widetilde{J}_X(dt,dx)) \tag{2.154}$$

Comparison to the general form of M_t (2.118) gives $H(t,x)=ux$ and $\sigma_t=\hat{\sigma}u$. The equation (2.141) has to be satisfied for Q_u to be a martingale measure, it has the following form

$$m-r+\hat{\sigma}^2u+\int_{\mathbb{R}-\{0\}}x(e^{ux}-1)\nu(dx)=0 \tag{2.155}$$

where m is defined by (2.134) and r is an interest rate. Above equation can be rewritten as

$$z(u)=r-m \tag{2.156}$$

where

$$z(u) = \hat{\sigma}^2 u + \int_{\mathbb{R}-\{0\}} x(e^{ux} - 1)\nu(dx) \tag{2.157}$$

clearly $z'(u) \geq 0$ so $z(u)$ is invertible and the above equation has a solution. The solution yields an equivalent martingale measure Q_u which minimizes the entropy (2.152).

2.4.3 Risk neutral modelling in exp-Levy models

In this section we will assume that the stochastic dynamics under risk-neutral measure is known and we will talk about possible ways to calculate the option prices then.

We will assume an exp-Levy model

$$S_t = S_0 \exp(rt + X_t)$$

where X is Levy process. We assume that a risk neutral dynamics is given by measure Q, so discounted price $\hat{S}_t$ is a martingale under Q.

We assume call options with payoff $H(S_T) = (S_T - K)^+$. We can write the option price under Q as

$$C(t, S, T = \tau + t, K) = e^{-r\tau} E((S_T - K)^+ | S_t = S) \tag{2.158}$$

it can be rewritten as

$$C(t, S, T = \tau + t, K) = Ke^{-r\tau} E(e^{x+X_\tau} - 1)^+ \tag{2.159}$$

where $x = \ln \frac{S}{K} + r\tau$. We used the independence a stationarity of increments to essentially reduce the problem to two degrees of freedom. This is consequence of the time and space homogeneity of Levy processes.

The option price can be also written as convolution

$$C(t, S, T = \tau + t, K) = Ke^{-r\tau} \int_{\mathbb{R}} p_\tau(y) h(x + y) = Ke^{-r\tau}(q_\tau * h)(x) \tag{2.160}$$

where $h(x) = (e^x - 1)^+$ is modified pay off function, p_τ is transition density and $q_\tau(y) = p_\tau(-y)$. This implies that the option price is often continuous even when the pay off function is discontinuous, which is important because Ito lemma can be used then.

Fourier transform methods

We very rarely obtain a closed form solution of the option price in exp-Levy models. We usually do not have a density in a closed form but we have the general form of a characteristic function (2.24). We will assume that risk-neutral dynamic is given i.e. we know the characteristic function of process X_t under martingale measure Q. Methods we introduce here will enable as to find Fourier transform of the option price in this case. We need to perform a numerical inverse transformation then to obtain option price, however numerical approaches to Fourier transformation are very well developed. The typical approach is well-known FFT method [38].

Carr-Madan method: We will assume here that we calculate the option price in the time $t = 0$ and $S_0 = 1$. We also denote $k = \ln K$. We will require the following condition to hold

$$\exists \alpha > 0, \quad \int_{\mathbb{R}} p_T(y) e^{(1+\alpha)y} \mathrm{d}y < \infty \tag{2.161}$$

where p_t is a density of X_t under a risk neutral measure. Equivalently we can write $\int_{\mathbb{R}} e^{(1+\alpha)y} \mathrm{d}\nu(y) < \infty$. We will also denote a characteristic function of X_t under Q by φ_t.

We can write the price of a call option as

$$C(k) = e^{-rT} E(e^{rT+X_T} - e^k)^+ \tag{2.162}$$

The problem is that $C(k)$ is not integrable because $\lim\limits_{k \to -\infty} C(k) > 0$. So we cannot calculate Fourier transform of C directly. We will define

$$z_T(k) = e^{-rT} E(e^{rT+X_T} - e^k)^+ - (1 - e^{k-rT})^+ \tag{2.163}$$

z_T is integrable and we will calculate its Fourier transform $\psi_T(v) = \mathcal{F} z_T(v)$. We can rewrite z_T with use of $E e^{X_t} = 1$

$$z_T(k) = e^{-rT} \int_{\mathbb{R}} p_T(x)(e^{rT+x} - e^k)(I_{k \le x+rT} - I_{k \le rT}) \mathrm{d}x \tag{2.164}$$

We can now calculate Fourier transform of z_T. We obtain after some calculations

$$\psi_T(v) = \mathcal{F} z_T(v) = e^{ivrT} \frac{\varphi_T(v-i) - 1}{iv(1+iv)} \tag{2.165}$$

We used the condition (2.161) that enables us to change the order of integration. We obtain z_T as

$$z_T(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} \psi_T(x) \mathrm{d}x \tag{2.166}$$

and from z_T we get the price of the option. So we only need to perform numerically an inverse Fourier transform to obtain the option price.

Lewis method: We will formulate this method more generally. Let us consider a payoff function h and denote $s = \ln S_0$. We can write the price of the general European option as the function of the initial price

$$C(s) = e^{-rT} \int_{\mathbb{R}} h(e^{s+x+rT}) p_T(x) \mathrm{d}x \tag{2.167}$$

We first need to summarize some results from the theory of complex Fourier transform.

We call the function h Fourier integrable in the strip (a, b) if the following condition holds

$$\int_{\mathbb{R}} (e^{-ay} + e^{-by})|h(y)|\mathrm{d}y < \infty \tag{2.168}$$

Fourier transform is defined for such h and for $z \in \mathbb{C}$, $a < Imz < b$ as

$$\mathcal{F}h(z) = \int_{\mathbb{R}} e^{iyz} h(y)\mathrm{d}y \tag{2.169}$$

It exists and is analytical when above conditions are satisfied. Moreover we can write the inverse transform for such z as

$$h(x) = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} e^{-izx} \mathcal{F}h(z)\mathrm{d}z, \quad a < v < b \tag{2.170}$$

We will require the following conditions for Lewis method

1. $p_T(x)$ is Fourier integrable in the strip S_1
2. $\widetilde{h}(x) = h(e^{x+rT})$ is Fourier integrable in the strip S_2
3. $S = \overline{S}_1 \cap S_2 \neq \emptyset$ where $\overline{S}_1$ denotes a complex conjugate

We want to calculate Fourier transform of the option price now

$$\mathcal{F}C(z) = e^{-rT} \int_{\mathbb{R}} p_T(x)\mathrm{d}x \int_{\mathbb{R}} e^{izs} h(e^{s+x+rT})\mathrm{d}s \tag{2.171}$$

using a substitution $s = y - x$ we obtain a formula

$$\mathcal{F}C(z) = e^{-rT} \varphi_T(-z)\mathcal{F}\widetilde{h}(z), \quad \forall z \in S \tag{2.172}$$

This can be applied to any option. We will show the concrete application for call options now.

The form of the payoff function for the call option is $h(x) = (x - K)^+ = (x - \exp k)^+$. It means that $\widetilde{h}(x)$ is Fourier integrable in the strip $(1, \infty)$. We can calculate Fourier transform of $\widetilde{h}(x)$ and obtain

$$\mathcal{F}\widetilde{h}(z) = \frac{e^{k+iz(k-rT)}}{iz(iz+1)}, \quad Imz > 1 \tag{2.173}$$

We need the density $p_T(x)$ to be Fourier integrable in some strip (a, b) where $a < -1$. The conditions of Lewis method will be satisfied then. This is clearly equivalent to the condition of Carr-Madan method (2.161).

We can use (2.172) when this condition is satisfied and we obtain

$$\mathcal{F}C(z) = \frac{\varphi_T(-z)e^{(1+iz)(k-rT)}}{iz(iz+1)}, \quad 1 < Imz < 1 + \alpha \tag{2.174}$$

We get with the use of the inverse formula (2.170)

$$C(x) = \frac{\exp\left(vx + (1-v)(k - rT)\right)}{2\pi} \int_{\mathbb{R}} \frac{\exp\left(iu(k - x - rT)\right)\varphi_T(-iv - u)}{(iu - v)(iu - v + 1)} \mathrm{d}u \tag{2.175}$$

where $1 < v < 1 + \alpha$.

Lewis method is obviously more complicated than Carr-Madan's. However provided we make a good choice of the parameter v this method is more convenient for numerical purposes.

Black-Scholes equation in exp-Levy models

We will discuss the form of Black-Scholes equation for models with jumps here. However we will first discuss a connection between Black-Scholes equation and a change of measure technique in the local volatility models.

We remind that in the local volatility models has the option pricing equation the form

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2(t,S)S^2\frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{2.176}$$

where $C = C(t, S)$ is the option price. Solving this equation with boundary conditions given by the particular option yields the option price. The case $\sigma = const.$ is just Black-Scholes case.

We also know that under the risk neutral measure we can write the option price with the payoff H as

$$C(t, S) = e^{-r(T-t)}E(H|S = S_t) \tag{2.177}$$

We will demonstrate that a connection between these two approaches is given by the famous Feynman-Kac formula. It states that if we consider parabolic equation for function $u = u(t, x)$ in the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 u}{\partial x^2} + \mu(t,x)\frac{\partial u}{\partial x} - V(t,x)u(t,x) + f(t,x) = 0 \tag{2.178}$$

with the boundary condition $u(x, T) = \psi(x)$ then the solution can be written as the conditional expectation

$$u(t,x) = E^Q\left(\int_t^T e^{-\int_t^s V(\tau,X_\tau)\mathrm{d}\tau} f(s,X_s)\mathrm{d}s + e^{-\int_t^T V(\tau,X_\tau)\mathrm{d}\tau}\psi(X_T)|X_t = x\right) \tag{2.179}$$

where under Q

$$\mathrm{d}X_t = \mu(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t \tag{2.180}$$

Feynman-Kac formula is often written in the terms of path integration due to its connection with the quantum theory and Feynman integral. This formula has an importance in a quantum mechanics in a connection with Schrödinger equation and in the theory of stochastic processes in the connection with parabolic equations and a diffusion. It also has an importance in a theory of the option pricing.

The Black-Scholes equation for local volatility models is clearly parabolic and has the form (2.178) so Feynman-Kac formula can be applied. It also has a boundary condition in the required form. We can write the solution of (2.176) for European options with a payoff $H(S_T)$ using Feynman-Kac formula as

$$C(t,S) = E^Q(e^{-r(T-t)}H(S_T)|S_t = S) \tag{2.181}$$

and under Q

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(t,S_t)S_t\mathrm{d}W_t \tag{2.182}$$

So we can see that the discounted price $\hat{S}_t = e^{-rt}S_t$ is a martingale under Q as required by no-arbitrage argument.

Everything becomes much more complicated when we consider jump models. They are incomplete. So the unique connection between the equation and the measure under which the solution can be written as an expectation does not exist. However Feynman-Kac formula for Levy processes can be found in the literature [35]. It is fairly complicated and the connection between it and integro-differential equations of the form (2.47) is not well understood.

The analogue of Black-Scholes equation for exp-Levy models has the following form

$$\begin{aligned}\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC + \int (C(t,Se^y) - C(t,S)-\\ - S(e^y-1)\frac{\partial C}{\partial S}(t,S))\mathrm{d}\nu(y) = 0\end{aligned} \tag{2.183}$$

It is an integro-differential equation. The integral term makes it non-local equation. Numerical methods for such equations are being extensively developed lately. However they are still extremely difficult to deal with.

We will derive the equation (2.183) now.

Let us consider exp-Levy model $S_t = exp(rt + X_t)$. The discounted price $\hat{S}_t$ has the form given by the martingale decomposition (2.109) under the risk neutral measure. We have for the price itself

$$S_t = S_0 + \int_0^t rS_u\mathrm{d}u + \int_0^t \sigma S_u\mathrm{d}W_u + \int_{\langle 0,t\rangle\times\mathbb{R}} S_{u-}(e^x-1)\tilde{J}_X(\mathrm{d}u,\mathrm{d}x) \tag{2.184}$$

We will now derive the equation (2.183) in a standard way by calculating $\mathrm{d}\hat{C}_t = \mathrm{d}(e^{-rt}C_t)$ and setting the drift term to zero. We need to use Ito lemma for semimartingales. We obtain with a use of $[S,S]^c_t = \int_0^t(\sigma S_u)^2\mathrm{d}u$ the following

$$\begin{aligned}\mathrm{d}\hat{C}_t = e^{-rt}((\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S^2} - rC)\mathrm{d}t + \frac{\partial C}{\partial S}(t,S_{t-})\mathrm{d}S_t)+\\ + e^{-rt}((C(t,S_{t-}e^{\Delta X_t}) - C(t,S_{t-})) - \Delta S_t\frac{\partial C}{\partial S}(t,S_{t-})))\end{aligned} \tag{2.185}$$

Where $\Delta S_t = S_{t-}(e^{\Delta X_t} - 1)$. By taking a drift term now and setting it to zero we obtain the desired equation (2.183).

There is a problem we ignored, the option price C_t might not be smooth enough for Ito lemma. However if we consider the payoff function $H = H(S_T)$ fulfilling

$$|H(x) - H(y)| \leq c|x - y| \qquad c > 0 \tag{2.186}$$

and if process X has a diffusion component $\sigma > 0$ or fulfils condition

$$\liminf_{\varepsilon \to 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} x^2 \mathrm{d}\nu(x) > 0$$

then the above derivation is completely rigorous.

Solving generalized Black-Scholes equation with appropriate boundary conditions is probably the most general approach to the option pricing, however the equation (2.183) is extremely difficult to solve numerically. Overview of numerical methods for this type of equations can be found in [15].

The role of Levy processes in finance became significant in past years. We built a compact mathematical apparat for the option pricing in exp-Levy models. However we did not pay much attention to one of the key steps in the option pricing process - to a calibration of models. We have to calibrate models not only with respect to the historical price process in incomplete markets but also with respect to the option price itself given by a market at time $t = 0$. This is a huge difference compared to complete markets. The calibration theory will certainly be one of the subjects of our future research.

Chapter 3

Anomalous diffusion and fractional processes

In this chapter we will use a more physical approach to stochastic processes. It will be in the contrast with the previous chapter where we approached this problematics with more rigorosity.

In the first section of this chapter we will derive a fractional diffusion equation from a continuous random walk model. We will obtain stable processes as a special solution of these equations. Our approach will be very intuitive however it will well demonstrate an important role of fractional processes. We will also briefly present an anomalous diffusion represented by non-linear Fokker-Planck equation as another approach to systems beyond the central limit theorem and Boltzmann-Gibbs statistics.

In the second part of this chapter we will take a closer look at the double-fractional equation and then introduce possible applications for the option pricing.

Throughout this chapter we will use a notation and facts about the stable distributions and fractional derivatives stated in appendixes B and C.

3.1 Generalized Fokker-Planck equation

A diffusion is one of the most important processes observed in the nature. It occurs in every natural science. It was first observed by biologist Robert Brown, it was further developed by Einstein, Wiener and many others. It also has deep connection with non-relativistic quantum mechanics [19]. A diffusion is mathematically usually described by Fokker-Planck equation or a corresponding Ito stochastic equation.

A classical diffusion is described by Markov processes with Brownian driving noise. This corresponds to the belief that the central limit theorem works. It leads to a linear scaling relation for the variance

$$\langle x^2(t) \rangle \sim Kt \tag{3.1}$$

However in many systems different behaviour can be observed. Fractional processes are typical by a non-linear scaling

$$\langle x^2(t) \rangle \sim K_\alpha t^\alpha \tag{3.2}$$

Fractional behaviour can be found for example in fractal systems, a connection between the fractal geometry and fractional processes is very interesting [22]. Furthermore fractional behaviour can be derived from an ordinary Langevin equation when a trapping or memory effects occur. See [21] for more thorough discussion.

Continuous time random walk

We will briefly introduce a classical diffusion and the basic notions of a continuous time random walk (CTRW) in this section. This formalism will be later useful for a derivation of the fractional diffusion equations.

We will consider random walk on discrete lattice with jumps of sizes Δx and jump times Δt, in classical case master equation has the form

$$W(x, t + \Delta t) = \frac{1}{2} W(x + \Delta x, t) + \frac{1}{2} W(x - \Delta x, t) \tag{3.3}$$

Where $W(x, t)$ is the density of being at the position $x = j\Delta x$ in time $t = k\Delta t$.

Using Taylor expansion

$$W(x, t + \Delta t) = W(x, t) + \Delta t \frac{\partial W}{\partial t} + O(\Delta t)^2 \tag{3.4}$$

and

$$W(x \pm \Delta x, t) = W(x, t) \pm \Delta x \frac{\partial W}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 W}{\partial x^2} + O(\Delta x)^3 \tag{3.5}$$

We obtain the following equation by sending $\Delta x, \Delta t \to 0$ and assuming standard Brownian scaling

$$\frac{\partial W}{\partial t} = K_1 \frac{\partial^2 W}{\partial x^2} \tag{3.6}$$

where

$$K_1 = \lim_{\Delta x, \Delta t \to 0} \frac{(\Delta x)^2}{2\Delta t} \tag{3.7}$$

Th equation (3.6) is the simplest form of Fokker-Planck equation. The density of Brownian motion is the solution of this equation with the initial condition $W(x, 0+) = \delta(x)$

$$W(x, t) = \frac{1}{\sqrt{4\pi K_1 t}} \exp\left(-\frac{x^2}{4K_1 t}\right) \tag{3.8}$$

W can be considered as a diffusion propagator of a free particle. Its Fourier image can be easily obtained

$$\mathcal{F}(W(x,t))(k,t) = \exp(-K_1 k^2 t) \tag{3.9}$$

and the equation (3.6) will have the form

$$\frac{\partial(\mathcal{F}W)}{\partial t} = -K_1 k^2 \mathcal{F}W \tag{3.10}$$

From now on we will denote Fourier transform only by a change of an underlying variable $x \to k$. So $\mathcal{F}(W(x,t))(k,t) \equiv W(k,t)$. Similarly Laplace transform defined as $\mathcal{L}(W(x,t))(x,u) = \int_0^\infty e^{-ut} W(x,t)dt = W(x,u)$ will change a variable $t \to u$.

We will now move to a CTRW. It can be described by a joint density function $\psi(x,t)$ where

$$\lambda(x) = \int_0^\infty \psi(x,t)\mathrm{d}t \tag{3.11}$$

is a density of the jump lengths and

$$w(t) = \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x \tag{3.12}$$

is a density of the waiting time in between jumps. We will usually consider the case $\psi(x,t) = \lambda(x)w(t)$. However waiting times and jump sizes can be also considered correlated for example $\psi(x,t) = \lambda(x|t)w(t)$. An important quantity is a waiting time T

$$T = \int_0^\infty t w(t)\mathrm{d}t \tag{3.13}$$

and the jump variance

$$\Sigma^2 = \int_{\mathbb{R}} x^2 \lambda(x)\mathrm{d}x \tag{3.14}$$

Both of these quantities can be finite or infinite. We will discuss possible cases and their consequences a bit later.

There are many approaches to describing CTRW. We will consider the density η of arriving at x at the time t, then we can write

$$\eta(x,t) = \int_{\mathbb{R}} \mathrm{d}x' \int_0^t \mathrm{d}t' \eta(x',t')\psi(x-x',t-t') + \delta(x)\delta(t) \tag{3.15}$$

We can rewrite this equation applying Fourier and Laplace transform and using properties of a convolution. We have in Fourier/Laplace image

$$\eta(k,u) = \frac{1}{1-\psi(k,u)} \tag{3.16}$$

We can write the density W of being in x at the time t as follows

$$W(x,t) = \int_0^t \eta(x,t')\Psi(t-t')\mathrm{d}t' \tag{3.17}$$

where $\Psi(t)$ is the probability of not jumping up to time t so

$$\Psi(t) = 1 - \int_0^t w(t')\mathrm{d}t' \tag{3.18}$$

we can write in Laplace image

$$\Psi(u) = \frac{1 - w(u)}{u} \tag{3.19}$$

If we now apply Fourier and Laplace transform in the equation (3.17) we obtain

$$W(k,u) = \frac{1 - w(u)}{u} \frac{W_0(k)}{1 - \psi(k,u)} \tag{3.20}$$

where $W_0(k)$ is Fourier image of the initial condition $W_0(x) = W(x, 0+)$.

We will first derive a classical diffusion equation (3.6) using this formalism. Let us consider $\psi(x,t) = \lambda(x)w(t)$ where w has Poisson density

$$w(t) = \frac{\exp(-t/\tau)}{\tau} \tag{3.21}$$

and λ is normally distributed

$$\lambda(x) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp(-\frac{x^2}{4\sigma^2}) \tag{3.22}$$

So clearly both T and Σ^2 are finite. We can calculate Laplace respectively Fourier image of w and λ, up to first order in u respectively k^2. We obtain

$$w(u) = 1 - \tau u + O(u^2) \quad u \to 0 \tag{3.23}$$

and

$$\lambda(k) = 1 - \sigma^2 k^2 + O(k^4) \quad k \to 0 \tag{3.24}$$

It is worth noting that if T and Σ^2 are both finite then every choice of w and λ leads to the same result in lowest orders and to the same behaviour for $t \to \infty$. See [21],[23] for more details.

We will consider the initial condition $W_0(x) = \delta(x)$. We obtain by inserting w and λ to (3.20) and neglecting terms of higher orders the equation

$$W(k,u) = \frac{1}{u + K_1 k^2} \tag{3.25}$$

where $K_1 = \frac{\sigma^2}{\tau}$. We can easily see that this is Fourier/Laplace image of the equation (3.6) so we will obtain the free propagator (3.8) by the inverse transform.

3.1.1 From a continuous time random walk to fractional diffusion

We will now consider the case $T = \infty$ and $\Sigma^2 < \infty$. This is often called fractal time random walk. The concrete form of w, λ is again not very important. We will consider normally distributed λ with Fourier image given by (3.24). We will assume that w has the following behaviour

$$w(t) \sim C_\alpha (\tau/t)^{1+\alpha} \quad \alpha \in (0,1), \quad t \to \infty \tag{3.26}$$

asymptotics in Laplace image is

$$w(u) \sim 1 - (\tau u)^\alpha \qquad u \to 0 \tag{3.27}$$

where the factor 1 ensures proper normalization $\Psi(u=0) = \int_{\mathbb{R}} \Psi(t) = 1$.

We now obtain by using (3.20) and neglecting the terms of higher order

$$W(k,u) = \frac{W_0(k)/u}{1 + K_\alpha u^{-\alpha} k^2} \tag{3.28}$$

where $K_\alpha = \frac{\sigma^2}{\tau^\alpha}$. With the use of a property (C.8) of Laplace transform of Riemann-Liouville derivative we can write

$$W(x,t) - W_0(x) = {}_0D_t^{-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} W(x,t) \tag{3.29}$$

We can now differentiate this equation and with the use of a composition rule (C.4) we get

$$\frac{\partial W}{\partial t} = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} W(x,t) \tag{3.30}$$

This is a fractional diffusion equation. Because Riemann-Liouville derivative is an integral operator this equation is non-local. This implies that the resulting process is not Markov.

We can easily derive a variance. We can see from the equation (3.30) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle x^2(t) \rangle = {}_0D_t^{1-\alpha} 2K_\alpha = 2K_\alpha \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{3.31}$$

so we have

$$\langle x^2(t) \rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha \tag{3.32}$$

This can be equivalently derive by the calculation of $\lim\limits_{k \to 0} -\frac{\mathrm{d}^2}{\mathrm{d}k^2} W(k,u)$ and by performing an inverse Laplace transform. We can get more generally

$$\langle x^{2n}(t) \rangle = (2n)! \frac{2K_\alpha^n}{\Gamma(1+n\alpha)} t^{n\alpha} \tag{3.33}$$

We can rewrite the equation (3.29) into the form

$${}_0D_t^\alpha W(x,t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} W_0(x) = K_\alpha \frac{\partial^2}{\partial x^2} W(x,t) \tag{3.34}$$

So we can see that the decay of the initial condition is proportional to $t^{-\alpha}$. This is the case of sub-diffusion for $\alpha \in (0,1)$, the case $\alpha = 1$ is a standard diffusion.

The solution of a fractional diffusion equation (3.30) can be found in the terms of Fox functions. Further by inverse Fourier transform of (3.28) we can obtain the closed form solution in Laplace image. We will only state the solution in Fourier image. If we perform Fourier transform in (3.30) we can find the solution in the terms of Mittag-Leffler function. We can use (C.23) and obtain

$$W(k,t) = E_\alpha(-K_\alpha k^2 t^\alpha) \tag{3.35}$$

Behaviour of E_α is discussed in appendix C. We know that $W(k,t)$ starts with exponential behaviour and stretches its tails with the time to the heavy tails in a limit. This behaviour was observed for example in polymer systems [21].

We did not include the case of super diffusion $\alpha \in (1,2)$. All calculation above would be in principle also valid in this case, but asymptotic relation (3.27) would not allow proper normalization $\Psi(u=0) = 1$ in this case .

Long jumps - stable case

We discussed the case $T = \infty$ i.e. long rests. We will now take a look at the case $\Sigma^2 = \infty$. We will see that these long jumps lead to stable distributions. We will assume usual $w(t)$ given by (3.21) and λ given by its Fourier image

$$\lambda(k) = \exp(-\sigma^\mu |k|^\mu) \sim 1 - \sigma^\mu |k|^\mu \quad \mu \in (0,2) \quad k \to 0 \tag{3.36}$$

this leads to the well-known asymptotic behaviour of stable distributions

$$\lambda(x) \sim \frac{A_\mu}{|x|^{1+\mu}} \quad x \to \pm\infty \tag{3.37}$$

Inserting into (3.20) and neglecting higher order terms leads to

$$W(k,u) = \frac{W_0(k)}{u + K^\mu |k|^\mu} \tag{3.38}$$

where $K^\mu = \frac{\sigma^\mu}{\tau}$. We obtain a fractional equation

$$\frac{\partial W}{\partial t}(x,t) = {}^0D_x^\mu K^\mu W(x,t) \tag{3.39}$$

where ${}^0D_x^\mu$ is Riezs-Weyl operator with a skewness 0. We can easily see the solution in Fourier image now

$$W(k,t) = \exp(-K^\mu t|k|^\mu) \tag{3.40}$$

So we have obtained a symmetric stable process. This should further justify their importance. It is worth noting that a fractal dimension of trajectories is $max\{1,\mu\}$. It means that in the case $\Sigma^2 = \infty$ trajectories cannot fill the space completely. They create clusters instead.

The last case $T, \Sigma^2 = \infty$ leads to the double fractional equation

$$\frac{\partial W}{\partial t}(x,t) = {}_0D_t^{1-\alpha} K_\alpha^{\mu}{}^0 D_x^\mu W(x,t) \tag{3.41}$$

Where $K_\alpha^\mu = \frac{\sigma^\mu}{\tau^\alpha}$. We will discuss the double fractional equation in a slightly different form later in this chapter. However this should demonstrate its physical importance.

Fractional diffusion-advection equation

Another advantage of a fractional diffusion is a straightforward generalization for a presence of external fields. If we consider a constant external field we obtain in the standard case the following Fokker-Planck equation

$$\frac{\partial W}{\partial t} + v\frac{\partial W}{\partial x} = K_1 \frac{\partial^2 W}{\partial x^2} \tag{3.42}$$

We will assume that in the fractional case will the joint density have Galileo invariant form

$$\phi(x,t) = \psi(x - vt, t) \tag{3.43}$$

where $\psi(x,t)$ is a joint density of a free particle. We can write in Fourier/Laplace image

$$\phi(k,u) = \psi(k, u - ivk) \tag{3.44}$$

We will consider the case $T = \infty$ and $\Sigma^2 < \infty$ now. Once again we use (3.20) and obtain

$$W(k,u) = \frac{W_0(k)}{u(1 + K_\alpha u^{-\alpha} k^2) - ivk} \tag{3.45}$$

this leads to the following equation

$$\frac{\partial W}{\partial t} + v\frac{\partial W}{\partial x} = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} W(x,t) \tag{3.46}$$

This is called a fractional diffusion-advection equation i.e. fractional diffusion in constant external field. We can easily verify that the solution fulfils Galileo invariance

$$W(x,t) = W_{v=0}(x - vt, t) \tag{3.47}$$

where $W_{v=0}$ is a propagator of the free particle with an initial condition $W_{v=0}(x, 0+) = \delta(x)$.

We can calculate moments in the same way as before and we obtain

$$\langle x(t) \rangle = vt \tag{3.48}$$

and

$$\langle x^2(t) \rangle = v^2 t^2 + \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha \tag{3.49}$$

We will also consider the case $T < \infty$ and $\Sigma^2 = \infty$. We obtain in an analogous way the following equation

$$\frac{\partial W}{\partial t} + v\frac{\partial W}{\partial x} = {}^0D_x^\mu K^\mu W(x,t) \tag{3.50}$$

This leads to Markov process. Galileo invariance again holds

$$W(x,t) = W_{v=0}(x - vt, t) \tag{3.51}$$

where in this case $W_{v=0}(x,t)$ is a symmetric stable process.

3.1.2 Fractional Fokker-Planck equation

Physicists usually consider standard Fokker-Planck equation in the form

$$\frac{\partial W}{\partial t} = (\frac{\partial}{\partial x}\frac{V'(x)}{m\rho_1} + K_1\frac{\partial^2}{\partial x^2})W(x,t) \equiv L_{FP}W(x,t) \tag{3.52}$$

where ρ_1 is a friction coefficient and $V(x)$ is a potential.

We will denote eigenvalues of an operator L_{FP} by $-\lambda_n$. We can now write individual modes of this equation as

$$T_n(t) = \exp(-\lambda_n t) \tag{3.53}$$

Further it is well known that stationary solution has the form

$$\lim_{t\to\infty} W(x,t) = W_{st}(x) = N\exp(-\beta V(x)) \tag{3.54}$$

where N is a normalization factor and $\beta = \frac{1}{K_1\rho_1 m}$ reminds Boltzmann factor from equilibrium thermodynamics.

We will now consider a fractional F.-P. equation in the form

$$\frac{\partial W}{\partial t} = {}_0D_t^{1-\alpha}(\frac{\partial}{\partial x}\frac{V'(x)}{m\rho_\alpha} + K_\alpha\frac{\partial^2}{\partial x^2})W(x,t) \equiv {}_0D_t^{1-\alpha}L_{FP}W(x,t) \tag{3.55}$$

We will denote eigenvalues of operator L_{FP} by $-\lambda_{n,\alpha}$. Individual modes can now be written using (C.23) and Mittag-Leffler function

$$T_n(t) = E_\alpha(-\lambda_{n,\alpha}t^\alpha) \tag{3.56}$$

We want to find a stationary solution, it leads to the equation

$${}_0D_t^{1-\alpha}L_{FP}W_{st}(x) = 0 \tag{3.57}$$

A fractional derivative of a constant is not a zero so we have to consider a stationary solution in the form $L_{FP}W_{st}(x) = 0$. So we can write a stationary solution in the same form as in the non-fractional case

$$W_{st}(x) = N\exp(-\frac{V(x)}{mK_\alpha\rho_\alpha}) \tag{3.58}$$

Separation of variables

We will try to find a solution of FFPE in the form

$$W_n(x,t) = T_n(t)\varphi_n(x) \tag{3.59}$$

where φ_n are eigenfunctions of L_{FP}.

The equation (3.55) can now be rewritten into two decoupled equations

$$\frac{\mathrm{d}T_n(t)}{\mathrm{d}t} = -\lambda_{n,\alpha}\ {}_0D_t^{1-\alpha}T_n(t) \tag{3.60}$$

and

$$L_{FP}\varphi_n(x) = -\lambda_{n,\alpha}\varphi_n(x) \tag{3.61}$$

The solution of the first equation is given by (3.56). We can write a general solution by summing over eigenvalues. We will write it in a bit different form with the initial condition $W(x,0|,x'.0) = \delta(x-x')$

$$W(x,t|,x'.0) = e^{\frac{-\Phi(x)-\Phi(x')}{2}}\sum_n \psi_n(x)\psi_n(x')E_\alpha(-\lambda_{n,\alpha}t^\alpha) \tag{3.62}$$

where $\Phi(x) = \frac{V(x)}{mK_\alpha\rho_\alpha}$ and $\psi_n(x) = e^{\Phi(x)/2}\varphi_n(x)$.

It is easy to see that ψ_n are eigenfunctions of a hermitian operator $L = K_\alpha(\frac{\partial^2}{\partial x^2} - (\Phi')^2)$. Moreover L_{FP} and φ_n have the same eigenvalue as L and ψ_n. Because L has negative eigenvalues we can order them $0 \leq \lambda_{0,\alpha} \leq \dots$. The condition $L_{FP}W_{st}(x) = 0$ is equivalent to $\lambda_{0,\alpha} = 0$. So there exists a positive stationary solution $W_{st}(x) = \lim_{t\to\infty} W(x,t)$ if the first eigenvalue is zero.

Smearing kernels

We will derive Laplace image of the solution of the equation (3.55). We will for simplicity consider $\rho_\alpha = \rho_1, K_\alpha = K_1$ We will show that we can write

$$W_\alpha(x,u) = u^{\alpha-1}W_1(x,u^\alpha) \tag{3.63}$$

To verify this we will first rewrite (3.55)

$$W(x,t) - W_0(x) = {}_0D_t^{-\alpha}L_{FP}W(x,t) \tag{3.64}$$

It is not an equivalent form to (3.55). However when we considered the case of long rests we first derived an analogous equation (3.29). Then by differentiation we rewrote it to the form (3.30). So this form can be considered a more fundamental form of fractional F.-P. equation.

We will perform Laplace transform in (3.64) and insert ansatz (3.63). We get

$$u^{\alpha-1}\int_0^\infty e^{-u^\alpha t}W_1(x,t)dt - \frac{W_0(x)}{u} = \frac{1}{u}\int_0^\infty e^{-u^\alpha t}L_{FP}W_1(x,t)dt \tag{3.65}$$

We obtain Laplace image of standard F.-P. equation (3.52) by a substitution $z = u^\alpha$ and W_1 solves this equation. This proves our statement.

We can rewrite (3.63) back to (x, t)

$$W_\alpha(x,t) = \int_0^\infty \mathrm{d}s A(s,t) W_1(x,s) \tag{3.66}$$

where

$$A(s,t) = \mathcal{L}^{-1}\{u^{\alpha-1}\exp(u^\alpha s)\}(s,t) \tag{3.67}$$

$A(s,t)$ is called a smearing kernel. This result demonstrates connection between fractional processes and super-statistics [24].

It can be shown with a use of properties of Fox functions that [21]

$$A(s,t) = \frac{1}{s}\sum_{n=0}^{\infty} \frac{(-1)^n (s/t^\alpha)^{1+n}}{\Gamma(1+n)\Gamma(1-\alpha-n\alpha)} \tag{3.68}$$

This function is closely connected to Wright functions [25] and to and stable distributions [25],[27]. Moreover in some cases the solution can be found in a closed form. For example for $\alpha = 1/2$ Wolfram Mathematica gives

$$A(s,t) = \frac{1}{\sqrt{\pi t}}\exp(-\frac{s^2}{4t}) \tag{3.69}$$

Some remarks on origin of fractional F.-P. equation

We derived fractional F.-P. equation (FFPE) from a continuous time random walk. Naturally there are many alternative approaches. We will mention few of them here. See [21] for more details.

It is possible to derive FFPE directly from the master equation if we consider non-local and non-isotropic jumps. The master equation has the form

$$W_j(t+\Delta t) = \sum_{n=1}^{\infty}(A_{j-n,n}W_{j-n}(t) + B_{j+n,n}W_{j+n}(t)) \tag{3.70}$$

where $A_{j.n}, B_{j,n}$ denote probabilities of jumps of size n from a position j, with a normalization $\sum_{n=1}^{\infty}(A_{j,n} + B_{j,n}) = 1$.

We can derive the FFPE (3.55) from this model. Moreover we obtain a double-fractional F.-P. equation in the case of the infinite jump variance Σ^2

$$\frac{\partial W}{\partial t} = {}_0D_t^{1-\alpha}(\frac{\partial}{\partial x}\frac{V'(x)}{m\rho_\alpha} + K_\alpha^\mu\ {}^0D_x^\mu)W(x,t) \tag{3.71}$$

Similarly if we consider an ordinary master equation

$$W_j(t+\Delta t) = A_{j-1}W_{j-1}(t) + B_{j+1}W_{j+1}(t) \tag{3.72}$$

where jump probabilities A_j, B_j fulfil $A_j + B_j = 1$, we will get standard Fokker-Planck equation. However if we incorporate a trapping, i.e. some random time for which the particle cannot jump, we once again obtain FFPE (2.55).

FFPE equation can be also derived form standard Langevin equation when we consider trapping or memory effects. In general we can say that we obtain fractional equations when we somehow deform the time or space component which leads to non-Brownian scaling.

We derived stable processes as solutions of space fractional diffusion equations. We derived these equations from CTRW with jumps of infinite variance. On the other hand in chapter two we introduced stable processes as processes driven by infinitely many jumps and without diffusion component. Combination of this two views gives us an interesting inside into the origin of these processes. However in chapter 2 we considered only Levy processes i.e. processes with independent increments so we did not encounter a deformation in the time component there.

It is also worth noticing that Kolmogorov equation (2.47) for stable processes is equivalent to the fractional equation (3.39). Comparing these two equation gives us spatial representation of Riezs-Feller derivative, it is same representation as the one given by theorem (2.14) for a characteristic operator of Levy process.

3.1.3 Non-linear Fokker-Planck equation

In this section we briefly introduce a different approach to the anomalous diffusion. We will consider generalized non-linear Fokker-Planck equation in the form

$$\frac{\partial p^\mu}{\partial t} = -\frac{\partial}{\partial x}(K(x,t)p^\mu(x,t)) + Q\frac{\partial^2}{\partial x^2}p^\nu(x,t) \tag{3.73}$$

where $\mu, \nu \in \mathbb{R}$. The case $\mu = \nu = 1$ is the standard case.

We will remind a well-known connection between Ito stochastic differential equation

$$\mathrm{d}X_t = -\mu(X_t,t)\mathrm{d}t + D(X_t,t)\mathrm{d}W_t \tag{3.74}$$

and a dual equation for the density $p(x,t)$ of X_t

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial}{\partial x}(\mu(x,t)p(x,t)) + \frac{1}{2}\frac{\partial}{\partial x^2}(D^2(x,t)p(x,t)) \tag{3.75}$$

If we now consider the case $\mu = 1$ in (3.73) we can write a corresponding stochastic equation

$$\mathrm{d}X_t = K(X_t,t)\mathrm{d}t + \sqrt{Q}(p(X_t,t))^{\frac{\nu-1}{2}}\mathrm{d}W_t \tag{3.76}$$

A generalization for $\mu \neq 1$ is straightforward by a substitution $\widetilde{p} = p^\mu$. So we will consider $\mu = 1$ for now.

We can interpret (3.76) as Langevin equation with a new driving noise $(p(X_t,t))^{\frac{\nu-1}{2}}\mathrm{d}W_t$. If we consider a special case $K = 0$ we will have

$$\mathrm{d}X_t = p(X_t,t)^{\frac{\nu-1}{2}}\mathrm{d}W_t \tag{3.77}$$

Processes in this form are usually called statistical feedback processes. They combine both microscopic level, presented by $\mathrm{d}X_t$, and a macroscopic level presented by $p(x,t)$ in a one equation. This unusual connection leads to non-linear diffusion equations and also to heavy tails as we will see.

Solution of non-linear F.-P. equation

Solving non-linear equations is no easy task however if we consider an equation in the form

$$\frac{\partial p^{\mu}}{\partial t} = -\frac{\partial}{\partial x}(F(x)p^{\mu}(x,t)) + D\frac{\partial^2}{\partial x^2}p^{\nu}(x,t) \tag{3.78}$$

where $F(x) = k_1 - k_2 x,\ k_2 \geq 0$ then explicit solution can be found [30].

The standard case $\mu = \nu = 1$ was solved using ansatz we get by maximizing Shannon entropy

$$S_1[p] = -\int_{\mathbb{R}} p(x)\ln p(x)\mathrm{d}x \tag{3.79}$$

with constraints

$$\int_{\mathbb{R}} p(x)\mathrm{d}x = 1 \tag{3.80}$$

$$\int_{\mathbb{R}} xp(x)\mathrm{d}x = x_M \tag{3.81}$$

$$\int_{\mathbb{R}} (x - x_M)^2 p(x)\mathrm{d}x = \sigma^2 \tag{3.82}$$

we obtain maximizer by straightforward calculations

$$p_1(x) = \frac{e^{-\beta(x-x_M)^2}}{Z_1} \tag{3.83}$$

the normalization factor is

$$Z_1 = \int_{\mathbb{R}} e^{-\beta(x-x_M)^2}\mathrm{d}x = (\pi/\beta)^{1/2} \tag{3.84}$$

Boltzmann factor $\beta = \frac{1}{2\sigma^2}$ in this case. The ansatz for standard F.-P. equation with $\mu = \nu = 1$ is now

$$p_1(x,t) = \frac{e^{-\beta(t)(x-x_M(t))^2}}{Z_1(t)} \tag{3.85}$$

Mean value can be easily calculated as a solution of the equation

$$\frac{\mathrm{d}x_M(t)}{\mathrm{d}t} = k_1 - k_2 x_M(t) \tag{3.86}$$

so we get

$$x_M(t) = \frac{k_1}{k_2} + (x_M(0) - \frac{k_1}{k_2})e^{-k_2 t} \tag{3.87}$$

We can insert our ansatz into (3.78) and compare coefficients of x^2, we get equation

$$\beta'(t) - 2k_2\beta(t) + 4D\beta^2(t) = 0, \qquad \beta(t) = \frac{\beta(0)}{(1 - \frac{2D\beta(0)}{k_2})e^{-2k_2 t} + \frac{2D\beta(0)}{k_2}} \tag{3.88}$$

relation (3.84) implies standard Brownian scaling of time and space

$$Z^2(t)\beta(t) = Z^2(0)\beta(0) = \pi \tag{3.89}$$

We will now consider a general case $\mu, \nu \in \mathbb{R}$. The correct ansatz for non-linear Fokker-Planck equation is given by Tsallis entropy

$$S_q[p] = \frac{1}{q-1}(1 - \int_{\mathbb{R}} (p(x))^q \mathrm{d}x) \quad q \in \mathbb{R} - \{0\} \tag{3.90}$$

We can easily see that in the case $q = 1$ it is just Shannon entropy.

We will maximize S_q with constraints

$$\langle x - x_M \rangle_q = \int_{\mathbb{R}} (x - x_M)(p(x))^q \mathrm{d}x = 0 \tag{3.91}$$

$$\langle (x - x_M)^2 \rangle_q = \int_{\mathbb{R}} (x - x_M)^2 (p(x))^q \mathrm{d}x = \sigma^2 \tag{3.92}$$

The resulting maximizer has the form

$$p_q(x) = \frac{1}{Z_q}(1 - \beta(1-q)(x - x_M)^2)^{\frac{1}{1-q}} \tag{3.93}$$

It is called Tsallis distribution.

Notice that p^q is not normalized so for example $\langle x \rangle_q = \langle x_M \rangle_q \neq x_M$. The idea of these generalized moments lies in the heart of non-extensive thermodynamics [10]. The Tsallis entropy itself is non-extensive. It fulfils for independent systems A, B, i.e. $p_{A*B}(x, y) = p_A(x)p_B(y)$, the following condition

$$S_q(A * B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \tag{3.94}$$

Mathematically this formalism lies in the framework of q-calculus [31]. The q-central limit theorem was formulated in [13] and the limiting distribution there is Tsallis distribution. Recently also generalizes q-stable central limit theorem was introduced [11].

We will get back to solving the equation (3.78) now. The ansatz derived from Tsallis entropy has the form

$$p_q(x, t) = \frac{1}{Z_q(t)}(1 - \beta_q(t)(1-q)(x - x_M(t))^2)^{\frac{1}{1-q}} \tag{3.95}$$

The main result is that if we now choose

$$q = 1 + \mu - \nu \tag{3.96}$$

then we get the following scaling of the time and space

$$Z_q^{2\mu}(t)\beta_q(t) = Z_q^{2\mu}(0)\beta_q(0) = const. \tag{3.97}$$

The case $q > 1$ corresponds to a super-diffusion, $q < 1$ a sub-diffusion.

The equation for the normalization Z_q was also found [30]

$$Z_q(t) = Z_q(0)\left((1-\frac{1}{K_2})e^{-t/\tau} + \frac{1}{K_2})\right)^{1/(\mu+\nu)} \tag{3.98}$$

where

$$K_2 = \frac{k_2}{2\nu D\beta(0)(Z_q(0))^{\mu-\nu}} \tag{3.99}$$

and

$$\tau = \frac{\mu}{k_2(\mu+\nu)} \tag{3.100}$$

This completely solves the equation (3.78). It is remarkable how closely the right choice (3.95) of q connects the classical diffusion and non-linear diffusion.

Application to the option pricing

We will briefly mention how the formalism presented above can be applied to the option pricing problem. We assume the following model of asset returns

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}\Omega_t \tag{3.101}$$

where the process Ω_t is q-Brownian motion, given by

$$\mathrm{d}\Omega_t = P(\Omega_t, t)^{\frac{1-q}{2}} \mathrm{d}W_t \tag{3.102}$$

where $P(\Omega, t)$ is a density of Ω_t. It is given by non-linear Fokker-Planck equation.

$$\frac{\partial}{\partial t} P(\Omega, t|\Omega', t') = \frac{1}{2}\frac{\partial^2}{\partial \Omega^2} P^{2-q}(\Omega, t|\Omega', t') \tag{3.103}$$

We can see that it is equation of the type (3.78) with $F \equiv 0$, $\mu = 1$ and $\nu = 2-q$. So (3.96) implies that Ω_t corresponds to q-dynamics. That is the reason behind this definition.

The solution of this equation has the form (3.94)

$$P_q(\Omega, t|\Omega', t') = \frac{1}{Z(t-t')}(1-\beta(t-t')(1-q)(\Omega-\Omega')^2)^{\frac{1}{1-q}} \tag{3.104}$$

by the choice

$$\beta(t) = c^{\frac{1-q}{3-q}}((2-q)(3-q)t)^{\frac{-2}{3-q}} \tag{3.105}$$

and

$$Z(t) = ((2-q)(3-q)ct)^{\frac{1}{3-q}} \tag{3.106}$$

with constant $c = \beta Z^2$ we ensure the initial condition $P_q(\Omega, t|\Omega_0, 0) = \delta(\Omega - \Omega_0)$ to hold. We can see that time/space scaling (3.97) also holds. An unconditional density $P_q(\Omega, t)$ corresponds to standard q-Brownian motion.

We will consider only $1 < q < 5/3$ for our purposes. The lower bound is given by the fact that for smaller q Tsallis distribution has only a bounded support. The upper bound is given by a requirement of a finite variance

$$E\Omega_t^2 = \frac{1}{(5-3q)\beta(t)} \tag{3.107}$$

This approach is in fact a local volatility model and leads to the following generalized Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 P_q^{1-q} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{3.108}$$

We will now use an equivalent martingale measure approach. It is clear from the theory we presented in the chapter 1 that this model is arbitrage-free and complete. So there exists only one equivalent martingale measure Q under which will S_t have the form

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t P_q^{\frac{1-q}{2}}\mathrm{d}W_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}\Omega_t \tag{3.109}$$

where W_t is standard Brownian motion under Q. Equivalently we can write

$$S_t = S_0 \exp\left(\sigma \int_0^t P_q^{\frac{1-q}{2}}\mathrm{d}W_t + \int_0^t (r - \frac{\sigma^2}{2}P_q^{1-q})\mathrm{d}t\right) \tag{3.110}$$

We have to realize that the following holds to calculate integrals in (3.110)

$$\Omega_t = \sqrt{\frac{\beta(s)}{\beta(t)}}\Omega_s \tag{3.111}$$

so an integration of $\beta(t)\Omega_t^2$ is the same as of a constant. This follows from the fact that under linear transformation of random variable $X \to aX$ density function changes to $p(x) \to \frac{1}{a}p(\frac{x}{a})$.

We obtain by an integration

$$S_t = S_0 \exp\left(\sigma\Omega_t + rt - \frac{\sigma^2}{2}\alpha t^{\frac{2}{3-q}}(1-(1-q)\beta(t)\Omega_t^2)\right) \tag{3.112}$$

where $\int_0^t \frac{1}{Z(s)^{1-q}}\mathrm{d}s = \alpha t^{\frac{2}{3-q}}$ and

$$\alpha = \frac{1}{2}(3-q)\left((2-q)(3-q)c\right)^{(q-1/(3-q))} \tag{3.113}$$

We can write the option price using martingale property

$$C_t = E^Q(e^{-r(T-t)}h(S_T)|\mathcal{F}_t) \tag{3.114}$$

where h is a pay off function, so we can write

$$C_0 = \frac{e^{-rT}}{Z(T)}\int_{\mathbb{R}} h(S_T)(1-\beta(T)(1-q)\Omega_T^2)^{\frac{1}{1-q}}\mathrm{d}\Omega_T \tag{3.115}$$

If we consider European call option with a payoff $H(S_T) = (S_T - K)^+$ then by inserting (3.112) into (3.115) the closed form solution is obtained. It seems to fit empirical data quite well, see [9] for details.

3.2 Fractional processes and option pricing problem

In this section we will first take a closer look at the double fractional diffusion equation. Properties of fractional derivatives and stable distributions stated in appendixes B and C will be frequently used. Applications of fractional processes to the option pricing will be introduced then. In particular we will briefly introduce approach based on fractional Brownian motion. The option pricing based on asymmetric stable processes and double fractional diffusion will be discussed at the end of this section.

3.2.1 Double fractional diffusion

We introduced double fractional diffusion equations in the first part of this chapter. We demonstrated their origin and physical importance. We will now take a closer look at them from a mathematical point of view.

We will consider a double-fractional diffusion equation in the form

$$({}^{*}D_t^{\beta} - {}^{\theta}D_x^{\alpha})p(x,t) = 0 \tag{3.116}$$

where ${}^{*}D_t^{\beta}$ is a Caputo derivative and ${}^{\theta}D_x^{\alpha}$ is Riezs-Feller derivative with the skewness θ.

Ranges of parameters are restricted as follows

$$0 < \alpha \leq 2, \quad 0 < \beta \leq 2, \quad \theta \leq min\{\alpha, 2-\alpha\} \tag{3.117}$$

However not all ranges of parameters allow probabilistic interpretation of $p(t,x)$. It was shown that we have to consider only one of the following cases

1. $0 < \beta \leq 1$
2. $1 < \beta \leq \alpha \leq 2$

We will consider only the first case $0 < \beta \leq 1$. The second case also allow probabilistic representation in the terms of Mellin-Barnes integrals. See [27] for thorough analysis.

Slow diffusion case $\beta \leq 1$

We can easily find Fourier-Laplace image of the equation (3.116). We use a notation where Fourier/Laplace image is denoted only by a change of the variable $x \to k$ respectively $t \to u$. We can write with the use of (C.13) and (C.16)

$$u^{\beta}p(k,u) - u^{\beta-1}p_0(k) = \mathcal{H}_{\alpha,\theta}(k)p(k,u) \tag{3.118}$$

where

$$\mathcal{H}_{\alpha,\theta}(k) = -|k|^{\alpha}e^{isgn(k)\theta\frac{\pi}{2}} \tag{3.119}$$

is a log-characteristic function of a stable variable, see appendix B. We will consider initial condition $p_0(x) = p(x,0) = \delta(x)$ so $p_0(k) = 1$.

We can rewrite the equation (3.118) if $\Re\left(u^\beta - \mathcal{H}_{\alpha,\theta}(k)\right) > 0$ as

$$p(k,u) = \int_0^\infty u^{\beta-1} e^{-lu^\beta} p_0(k) e^{l\mathcal{H}_{\alpha,\theta}(k)} \mathrm{d}l = \int_0^\infty p_1(u,l) p_\alpha(l,k) \mathrm{d}l \tag{3.120}$$

in (x,t) we have

$$p(x,t) = \int_0^\infty p_1(t,l) p_\alpha(l,x) \mathrm{d}l \tag{3.121}$$

Notice the similarity with the equation (3.66). The smearing kernel is in Laplace image given by

$$p_1(u,l) = u^{\beta-1} e^{-lu^\beta} \tag{3.122}$$

The inverse Laplace transform has the form

$$p_1(t,l) = \frac{1}{t^\beta} M_\beta\left(\frac{l}{t^\beta}\right) \tag{3.123}$$

where Wright type function M_β is defined by

$$M_\mu(z) = \sum_{n=0}^\infty \frac{(-z)^n}{\Gamma(n+1)\Gamma(-\mu n + (1-\mu))} \tag{3.124}$$

This function is connected with the stable laws $L_{\alpha,\beta}$ by the following relation

$$\frac{1}{a^{1/\mu}} L_{\mu,1}\left(\frac{z}{a^{1/\mu}}\right) = \frac{a\mu}{z^{\mu+1}} M_\mu\left(\frac{a}{z^\mu}\right) \qquad \mu \in (0,1) \quad a, z > 0 \tag{3.125}$$

The proof and other properties of Wright functions can be found in [25]. We can now write the solution of a double fractional diffusion equation in the form

$$p^\theta_{\alpha,\beta}(x,t) = \int_0^\infty \frac{1}{t^\beta} M_\beta\left(\frac{l}{t^\beta}\right) p_\alpha(l,x) \mathrm{d}l = \int_0^\infty \frac{t}{l\beta} \frac{1}{l^{1/\beta}} L_{\beta,1}\left(\frac{t}{l^{1/\beta}}\right) p_\alpha(l,x) \mathrm{d}l \tag{3.126}$$

Where $p_\alpha(l,x) = \mathcal{F}^{-1}(e^{l\mathcal{H}_{\alpha,\theta}(k)})(x) = (\frac{1}{l^{1/\alpha}}) L^\theta_\alpha(\frac{x}{l^{1/\alpha}})$ is a stable distribution in the parametrization given by (B.2). We considered initial condition $p_0(x) = \delta(x)$.

This will be useful later when we apply this theory to the option pricing problem.

We should note that finding the fundamental solution with initial condition $p(x,0) = \delta(x)$ is sufficient. If we denote it by $G^\theta_{\alpha,\beta}$ we can write an arbitrary solution of the equation (3.116) with the the initial condition $p^\theta_{\alpha,\beta}(x,0) = \varphi(x)$ as

$$p^\theta_{\alpha,\beta}(x,t) = \int_{\mathbb{R}} G^\theta_{\alpha,\beta}(p,t) \varphi(x-p) \mathrm{d}p \tag{3.127}$$

Where we also have to consider a boundary condition $p^\theta_{\alpha,\beta}(\pm\infty,t) = 0$. We have to add another condition $\frac{\partial p}{\partial t}(x,0) = 0$ in the case $1 < \beta \leq 2$.

3.2.2 Black-Scholes pricing in fractional environment

We will introduce some applications of fractional processes into finance now. First we will discuss fractional Brownian motion. We demonstrate its connection with a fractional calculus and outline a possible generalization of Black-Scholes equation.

At the end of this chapter we will generalize Black-Scholes formula with the use of asymmetric stable distributions. This will be done in the framework of double fractional diffusion equations presented above.

Fractional Brownian motion

We have already introduced FBM with regards to its self-similarity in the chapter one, see the definition (1.2). It can be shown that FBM is only H-self similar centred Gaussian process with stationary increments. Its covariance function is given by

$$Cov(B_t^H, B_h^H) = \frac{1}{2}(t^{2H} + h^{2H} - |t-h|^{2H}) \tag{3.128}$$

also following holds

$$EB_t^H(B_{t+h}^H - B_t^H) = \frac{1}{2}(-t^{2H} - h^{2H} + (t+h)^{2H}) \overset{H\neq\frac{1}{2}}{\neq} 0 \tag{3.129}$$

This equation in particular implies that for $H > 1/2$ has FBM positively correlated increments which implies super-diffusive behaviour. On the other hand it is kept oscillating around its mean for $H < 1/2$ which corresponds to a sub-diffusion.

Fractional Brownian motion with Hurst index $H > 1/2$ has a long range dependence. It means that for $r(n) = Cov(B_1^H, B_{n+1}^H - B_n^H)$ holds

$$\sum_{n=1}^{\infty} r(n) = \infty \tag{3.130}$$

The fractional nature of these correlations can be seen from a well-known integral representation of FBM

$$B_t^H = \frac{1}{\Gamma(H+1/2)}\left(\int_{-\infty}^{0}(t-s)^{H-1/2} - (-s)^{H-1/2}\mathrm{d}W_s + \int_0^t (t-s)^{H-1/2}\mathrm{d}W_s\right) \tag{3.131}$$

where W_s is a standard Brownian motion.

We can recognise Riemann-Liouville integral operator it the second term. This clearly demonstrates a connection of the fractional calculus with FBM. The first term is also important because without it FBM would depend significantly on the initial condition [32].

We now want use FBM for a generalization of Black-Scholes theory. A natural approach is to assume that the price of the risky asset is given by

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t^H \tag{3.132}$$

However it is unclear how this equation should be interpreted. We developed stochastic calculus for huge class of stochastic processes called semimartingales in the previous chapter. A problem is that FBM is not a semimartingale so a standard approach cannot be used.

There are few approaches to integration with respect to FBM. The most convenient one for applications seems to be developed in [28]. Wick-Ito stochastic integral developed there for $1/2 < H < 1$ was applied in [29] to the option pricing problem. We will state without going into details that in this framework solution of equation (3.132) has the form

$$S_t = S_0 \exp\left(\sigma B_t^H + \mu t - \frac{\sigma^2 t^{2H}}{2}\right) \tag{3.133}$$

Moreover they have shown that this leads to a complete and arbitrage free market. Generalized Black-Scholes formula for $1/2 < H < 1$ and European call options has the form [29]

$$C_0 = e^{-rT} \int_R \frac{1}{\sqrt{2\pi T^{2H}}} \left(S_0 \exp(\sigma y + rT - \frac{\sigma^2 T^{2H}}{2}) - K\right)^+ \exp\left(-\frac{y^2}{2T^{2H}}\right) \mathrm{d}y \tag{3.134}$$

Clearly the classical model is recovered for $H = 1/2$.

Fractional equations and option pricing problem

We will introduce another generalization of Black-Scholes approach now. We will use probability densities obtained as solutions of fractional equations as driving noises instead of Brownian motion.

We can rewrite a classical Black-Scholes option pricing formula (1.23) for call options into the form

$$C_t = \int_{\mathbb{R}} e^{-r\tau}(S_t e^{r\tau+y} - K)^+ \frac{1}{\sqrt{2\pi\tau\sigma^2}} \exp\left(-\frac{(y+\frac{\sigma^2}{2}\tau)^2}{2\tau\sigma^2}\right) \mathrm{d}y \tag{3.135}$$

where $\tau = T - t$.

We integrate over the density $N(-\frac{\sigma^2\tau}{2}, \sigma^2\tau)$ in (3.135). We now want to replace this normal density by a density of the fractional process.

We are motivated by the generalized central limit theorem to use a stable process. We will assume a model

$$S_t = S_0 \exp\left((r+\lambda)t + \sigma L_{\alpha,\beta}(t)\right) \tag{3.136}$$

where $L_{\alpha,\beta}$ is a stable process.

We want $\hat{S}_t = e^{-rt}S_t$ to be a martingale under new equivalent measure. We will assume $\beta = -1$, $\alpha > 1$. Let the risk neutral dynamic be given by

$$S_t = S_0 \exp\left((r+\mu)t + \sigma L_{\alpha,-1}(t)\right) \tag{3.137}$$

with $\mu = \sigma^\alpha \sec\frac{\pi\alpha}{2}$. It can be verified that $\hat{S}_t$ is a martingale with a use of the relation (B.1).

We made the choice $\beta = -1$ because for $\beta \neq \pm 1$ the moment generating function $Ee^{\sigma L_{\alpha,\beta}(t)}$ does not exist. However stable processes do not have diffusion component. The theorem (2.27) then implies that we cannot change a drift so strictly speaking we cannot obtain model (3.137) from (3.136) by changing measure. To be mathematically rigorous we should assume that risk neutral dynamics is given by (3.137).

We will now assume that risk neutral dynamics is given and we will derive the option pricing formula. We can write the call option price using martingale property as

$$C_t = e^{-r\tau}\int_{\mathbb{R}} \mathrm{d}y\ (S_t e^{r\tau+y} - K)^+ \int_{\mathbb{R}} \mathrm{d}k \frac{e^{-iky}}{2\pi} e^{\tau(ik\mu + \sigma^\alpha \psi_{\alpha,-1}(k))} \tag{3.138}$$

where $\psi_{\alpha,\beta}$ is given by the theorem (B.2). $\psi_{\alpha,-1}(k)$ can be easily calculated with the use of $(ik)^\alpha = i^{\alpha sgn(k)}|k|^\alpha$. We obtain $\psi_{\alpha,-1}(k) = -\sec\frac{\pi\alpha}{2}(ik)^\alpha$. A substitution $y = x + \mu\tau$ now yields

$$C_t = e^{-r\tau}\int_{\mathbb{R}} \mathrm{d}x\ (S_t e^{r\tau+r\mu+x} - K)^+ \int_{\mathbb{R}} \mathrm{d}k \frac{e^{-ikx}}{2\pi} e^{-\tau\mu(ik)^\alpha} \tag{3.139}$$

We will now denote $p_\alpha(x,t) = \int_{\mathbb{R}} \mathrm{d}k \frac{e^{-ikx}}{2\pi} e^{-\tau\mu(ik)^\alpha}$ so we can write the option price as

$$C_t = e^{-r\tau}\int_{\mathbb{R}} \mathrm{d}x\ (S_t e^{r\tau+r\mu+x} - K)^+ p_\alpha(x,\tau) \tag{3.140}$$

Notice that $p_\alpha(x,t)$ is a density of the stable process $\sigma L_{\alpha,-1}(t)$. It fulfils the following fractional equation with a maximal skewness $\theta = 2-\alpha$ corresponding to $\beta = -1$

$$\frac{\partial p_\alpha}{\partial t}(x,t) = -\mu\{{}^\theta D_x^\alpha p_\alpha(x,t)\} \tag{3.141}$$

with the initial condition $p_\alpha(x,0) = \delta(x)$. This can be easily verified. It holds $-\mu\mathcal{H}_{\alpha,2-\alpha}(k) = \sigma^\alpha\psi_{\alpha,-1}(k)$ where $\mathcal{H}_{\alpha,2-\alpha}$ is given by (B.2). So in Fourier image we have

$$\frac{\partial p_\alpha}{\partial t}(k,t) = \sigma^\alpha\psi_{\alpha,-1}(k)p_\alpha(k,t) \tag{3.142}$$

which holds obviously from the definition (C.13). The standard diffusion equation is now recovered in the case of Black-Scholes model.

We want now to generalize this model for both the space and the time fractional diffusion. We have everything we need already prepared. We can write our double fractional diffusion model as

$$C_t = e^{-r\tau}\int_{\mathbb{R}} \mathrm{d}x\ (S_t e^{r\tau+r\mu+x} - K)^+ \frac{1}{\sigma} p_{\alpha,\beta}^\theta(\frac{x}{\sigma},\tau) \tag{3.143}$$

where $p^{\theta}_{\alpha,\beta}(x,t)$ is given by (3.126). It is more convenient for numerical purposes to present $p^{\theta}_{\alpha,\beta}$ in the terms of Mellin-Barnes integrals. This was done in [33] and the calculated option prices seemed to fit the real data quite well.

Chapter 4

Quantum finance

In the first part of this chapter we will reformulate the option pricing problem to the framework of quantum mechanics. Both Hamiltonian and a path integral formulation will be discussed. We will see that various path dependent options can be modelled by adding a potential to Hamiltonian. Special attention will be given to stochastic volatility models, we will see that the path-integral formulation will turn out to be very convenient for these models.

The second part of this chapter will deal with a modelling of interest rates. The standard stochastic approach will be reviewed. We will then introduce a way to model interest rates as a quantum field theory. Both Hamiltonian and Lagrangian formulation will be discussed. The goal of quantum field formulation will be to incorporate non-trivial correlations and to go beyond the standard stochastic formulation.

4.1 Quantum mechanical formulation of the option pricing problem

The Black-Scholes equation (1.7) has a similar form to Schroedinger equation $i\hbar\partial_t\psi = H\psi$ [19]. The difference is in the factor i, also Hamiltonians in QM are self-adjoint while in finance they are not because the drift term is present. This motivates us to formulate option pricing in terms of non-self-adjoint Hamiltonian operators. We will also, in analogy with QM, present the path-integral approach.

We will first take a closer look at stochastic volatility models in the first section. We will derive generalized version of Black-Scholes equation and demonstrate incompleteness of stochastic volatility models.

We will also find corresponding Hamiltonians for both Black-Scholes model and stochastic volatility model. The form of the no-arbitrage condition will be derived in Hamiltonian formulation. We will introduce Hamiltonians with potential terms as a way to model path-dependent options.

In the last section the path integral for the option pricing will be derived and applied to both Black-Scholes and stochastic volatility model.

4.1.1 Stochastic volatility models

The stochastic volatility models were briefly mentioned in the chapter one, here we will discuss them in detail. We will consider a model

$$dS_t = \gamma S_t dt + \sigma_t S_t dW_1(t) \tag{4.1}$$

$$dV_t = (\lambda + \mu V_t)dt + \eta V^{\alpha} dW_2(t), \qquad V_t = \sigma_t^2 \tag{4.2}$$

where we assume correlation between Brownian motions $\frac{E(W_1(t)W_2(t'))}{E(W_1^2(t))E(W_2^2(t))} = \rho \in \langle -1, 1\rangle$ and $\gamma, \lambda, \mu, \eta, \alpha$ are constants.

A change of a measure technique in stochastic volatility models is bit tricky. We can change drifts of both Brownian motions which leads to infinite number of possible martingale measures. There is a lot of mathematical subtleties to deal with when changing measure in stochastic volatility models [42], however we will employ a bit more straight forward approach here.

We will consider a change of a measure under which

$$dS_t = rS_t dt + \sigma_t S_t dW_1(t) \tag{4.3}$$

and V_t is unchanged. Clearly $\hat{S}_t$ is a martingale under this new measure.

We will derive generalized Black-Scholes equation now. We will consider an option $C(S_t, V_t, t)$. The discounted price of this option $\hat{C}_t(S_t, V_t, t) = e^{-rt}C(S_t, V_t, t)$ should be a martingale under the new measure. We can calculate $d\hat{C}_t$ with a use of multidimensional Ito lemma, localize the drift term (a term proportional to dt) and set it to zero. We get the following equation

$$\frac{\partial C}{\partial t} + (\lambda + \mu V)\frac{\partial C}{\partial V} + rS\frac{\partial C}{\partial S} + \frac{VS^2}{2}\frac{\partial^2 C}{\partial S^2} + \rho\eta S V^{\alpha+1/2}\frac{\partial^2 C}{\partial S \partial V} + \frac{\eta^2 V^{2\alpha}}{2}\frac{\partial C^2}{\partial V^2} - rC = 0 \tag{4.4}$$

this is a generalized Black-Scholes equation for our stochastic volatility model. It is often called Merton-Garman equation

We made a concrete choice of the martingale measure in our derivation, however the incompleteness of the market can be presented by introducing a premium risk function $\theta(S.V, r, t)$ and rewriting Merton-Garman equation as

$$\frac{\partial C}{\partial t} + (\lambda + \mu V)\frac{\partial C}{\partial V} + rS\frac{\partial C}{\partial S} + \frac{VS^2}{2}\frac{\partial^2 C}{\partial S^2} + \rho\eta S V^{\alpha+1/2}\frac{\partial^2 C}{\partial S \partial V} + \frac{\eta^2 V^{2\alpha}}{2}\frac{\partial C^2}{\partial V^2} - rC = \theta(S.V, r, t)$$

A problem is that the premium risk is almost impossible to determine from markets.

When we consider $\theta = 0$ and uncorrelated driving noises, i.e. $\rho = 0$, then the solution can be written in an elegant form [41]

$$C_t = \int_0^{\infty} C^{BS}_{\sigma^2 = \widetilde{V}}\, p(\widetilde{V}) d\widetilde{V} \tag{4.5}$$

where $\widetilde{V} = \frac{1}{T-t}\int_t^T V(t')dt'$, $C^{BS}_{\sigma^2=V}$ is Black-Scholes price derived in the first chapter and $p(\widetilde{V})$ is a density function of the random variable $\widetilde{V}$.

4.1.2 Hamiltonian formulation

Black-Scholes equation has a form

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{4.6}$$

after a substitution $S = e^x$ we get an equation for the function $C = C(x, t)$

$$\frac{\partial C}{\partial t} = H_{BS}C \qquad H_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} + r \tag{4.7}$$

The similarity with Schroedinger equation is obvious. We can see that the volatility plays a role of an inverse mass from QM, $\sigma^2 \sim \frac{1}{m}$. The difference is the drift term which makes H_{BS} non-self-adjoint. So the Black-Scholes option pricing is in some sense equivalent to a one dimensional quantum mechanical system.

The Merton-Graman equation (4.4) can be by a substitution $S = e^x, V = e^y$ rewritten to the equation for a function $C(t, x, y)$ in the form

$$\frac{\partial C}{\partial t} = H_{MG}C \tag{4.8}$$

where

$$\begin{aligned} H_{MG} = &-\frac{e^y}{2}\frac{\partial^2}{\partial x^2} - (r - \frac{e^y}{2})\frac{\partial}{\partial x} - (\lambda e^{-y} + \mu - \frac{\eta^2}{2}e^{2y(\alpha-1)})\frac{\partial}{\partial y} - \\ &\rho\eta e^{y(\alpha-1/2)}\frac{\partial^2}{\partial x \partial y} - \frac{\eta^2 e^{2y(\alpha-1)}}{2}\frac{\partial^2}{\partial y^2} + r \end{aligned} \tag{4.9}$$

This means that stochastic volatility models correspond in some sense to a two-dimensional quantum system.

Propagators in option pricing

We want to define pricing kernel in this section - analogue of propagator from QM. We will assume an European path independent option C with a pay off in time T given by a function $f(x, y)$. We define a pricing kernel as a conditional probability density by

$$C(\tau = T - t, x, y) = \int_{\mathbb{R}} p(x, y, \tau, x', y') f(x', y') dx' dy' \tag{4.10}$$

with a boundary condition in time $t = T$

$$p(x, y, 0, x', y') = \delta(x - x')\delta(y - y')$$

A difference from QM is that we have a boundary condition instead of an initial condition.

We will use bra-ket notation as in QM, we consider equation $\partial_t C = HC$. The solution can be written as

$$C(t,x) = e^{tH}C(0,x), \qquad |C,t\rangle = e^{tH}|C,0\rangle \tag{4.11}$$

using the boundary condition $|C,T\rangle = |f\rangle$ yields

$$|C,t\rangle = e^{-(T-t)H}|f\rangle \qquad C(t,x) = \langle x|e^{-\tau H}|f\rangle \tag{4.12}$$

where $\tau = T - t$. The equation above can be rewritten as

$$C(t,x) = \int_{\mathbb{R}} \langle x|e^{-\tau H}|x'\rangle f(x')dx' \tag{4.13}$$

a comparison with (4.10) now yields

$$p(x.\tau,x') = \langle x|e^{-\tau H}|x'\rangle \tag{4.14}$$

This propagator determines the time evolution of the option price, notice that this propagation is backwards in time. This is in a contract with QM.

We will calculate a propagator for Black-Scholes Hamiltonian (4.7) now. We will use a standard notation $\langle x|p\rangle = \frac{e^{ipx}}{\sqrt{2\pi}}$. We can calculate

$$H_{BS}\frac{e^{ipx}}{\sqrt{2\pi}} = \langle x|H_{BS}|p\rangle = (\frac{\sigma^2 p^2}{2} + ip(\frac{\sigma^2}{2} - r) + r)e^{ipx} \tag{4.15}$$

notice that H_{BS} is not self-adjoint so

$$\langle p|H_{BS}|x\rangle = (\langle x|H_{BS}^{\dagger}|p\rangle)^* = (\frac{\sigma^2 p^2}{2} + ip(\frac{\sigma^2}{2} - r) + r)e^{-ipx}$$

The Black-Scholes propagator can be written as

$$p_{BS}(x,\tau,x') = \int_{\mathbb{R}} \langle x|e^{-\tau H_{BS}}|p\rangle\langle p|x'\rangle dx' = e^{-r\tau}\int_{\mathbb{R}} e^{ip(x-x'+\tau(r-\sigma^2/2))}e^{-\tau\sigma^2 p^2/2} \tag{4.16}$$

performing Gaussian integration, see Appendix D, yields

$$p_{BS}(x,\tau,x';\sigma) = e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}}\exp\left(\frac{(x-x'-\tau(r-\sigma^2/2))^2}{2\tau\sigma^2}\right) \tag{4.17}$$

The price of European call option can now be written as

$$C_t = \int_{\mathbb{R}} (e^x - K)^+ p_{BS}(x,\tau,x_t;\sigma)dx \tag{4.18}$$

where asset price $S_t = e^{x_t}$. This result corresponds to the previous result (1.23).

Martingale condition

We derived in the chapter one that a non-arbitrage condition is equivalent to an existence of a martingale measure under which is the discounted price a martingale.

We will now derive under which conditions Hamiltonian generates an arbitrage-free market. We will choose a pay-off function $f(x) = S(x)$ and insert it into (4.13), this yields

$$S(x) = \int_{\mathbb{R}} \langle x|e^{-(T-t)H}|x'\rangle S(x')dx' \tag{4.19}$$

It can be rewritten with use of $S(x) = \langle x|S\rangle$ as

$$e^{-(T-t)H}|S\rangle = |S\rangle \tag{4.20}$$

which is equivalent to

$$H|S\rangle = 0 \tag{4.21}$$

This means that the risk neutral measure exists if and only if $S(x) = e^x$ is an eigenfunction of Hamiltonian with an eigenvalue 0. It is easy to verify that both Black-Scholes and Merton-Graman Hamiltonians satisfy this condition.

Potentials and path dependent options

We will assume Black-Scholes Hamiltonian in this section, we want to describe prices of path dependent options by adding a potential term to H_{BS}

$$H = H_{BS} + V(x) \tag{4.22}$$

Various path dependent options can be modelled in this way [41]. We will demonstrate this method on an example of a double-knock out option.

Example: Double knock-out barrier options - are options similar to European options with one difference - the price of the risky asset $S = e^x$ must stay inside given interval

$$e^a < e^{x_t} < e^b \qquad t \in (0, T) \tag{4.23}$$

otherwise the option becomes worthless.

We will model this option with a potential

$$V(x) = \begin{cases} \infty & x \notin (a, b) \\ 0 & x \in (a, b) \end{cases}$$

Black-Scholes Hamiltonian can be rewritten as

$$H_{BS} = e^{\alpha x}(-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \lambda)e^{-\alpha x} \equiv e^{\alpha x} H_{ef} e^{-\alpha x} \tag{4.24}$$

where

$$\lambda = \frac{1}{2\sigma^2}(r + \frac{\sigma^2}{2})^2 \qquad \alpha = \frac{1}{\sigma^2}(\frac{\sigma^2}{2} - r) \tag{4.25}$$

This is a useful transformation because H_{ef} is self-adjoint.

Solving this problem for $H = H_{ef} + V(x)$ is equivalent to a well-known problem from QM - particle in an infinite potential well. We can write eigenvalues and eigenvectors of this Hamiltonian as

$$(H_{ef} + V(x))\varphi_n(x) = (E_n + \lambda)\varphi_n(x) \tag{4.26}$$

where

$$\varphi_n(x) = \sqrt{\frac{2}{b-a}}\sin\left(p_n(x-a)\right) \qquad E_n = \frac{1}{2}\sigma^2 p_n^2 = \frac{1}{2}\sigma^2(\frac{n\pi}{b-a})^2 \tag{4.27}$$

We can calculate the pricing kernel for this problem, we will denote $H_D = e^{\alpha x}(H_{ef} + V(x))e^{-\alpha x}$. We can use the fact that functions φ_n form an orthonormal basis, which implies $\sum_{n=1}^{\infty} |\varphi_n\rangle\langle\varphi_n| = 1$ to calculate

$$\langle x|e^{-\tau H_D}|x'\rangle = e^{-\lambda\tau+\alpha(x-x')}\sum_{n=1}^{\infty} e^{-\tau E_n}\varphi_n(x)\varphi_n^*(x') \tag{4.28}$$

This kernel can be used to price various options, for European call options the closed form solution can be obtained [41].

4.1.3 Path integral in the option pricing

The idea of path integration in the option pricing is the same as in QM. We can write the propagator as

$$p(x,\tau,x') = \lim_{N\to\infty}\langle x|(e^{-\varepsilon H})^N|x'\rangle \qquad \varepsilon = \frac{\tau}{N} \tag{4.29}$$

It can be further rewritten as

$$p(x,\tau,x') = \prod_{i=1}^{N-1}\int dx_i \left(\prod_{i=1}^{N}\langle x_i|e^{-\varepsilon H}|x_{i-1}\rangle\right) \tag{4.30}$$

where $x_N = x, x_0 = x'$.

We can define a Lagrangian now by

$$\langle x_i|e^{-\varepsilon H}|x_{i-1}\rangle = \mathcal{N}_i(\varepsilon)e^{\epsilon L(x_i,x_{i-1},\varepsilon)} \tag{4.31}$$

We can use that ε is infinitesimal to calculate the above matrix element, typically we can use $e^{-\varepsilon H} = e^{-\varepsilon H_0}e^{-\varepsilon V} + O(\varepsilon^2)$.

We can now write the pricing kernel as

$$p(x,\tau,x') = \int \mathcal{D}_X e^S \tag{4.32}$$

where

$$\int \mathcal{D}_X = \mathcal{N}_N(\varepsilon)\prod_{i=1}^{N-1}\int \mathcal{N}_i(\varepsilon)dx_i \qquad S = \varepsilon\sum_{i=1}^{N} L(x_i,x_{i-1},\varepsilon) \tag{4.33}$$

The normalization constant $\mathcal{N}$ is very often x-independent. Factor S is called action in theoretical physics.

The path integral defined in this manner is called Wiener integral and it is mathematically rigorous. The reason why the path integral in QM does not have a solid mathematical basis is the extra factor i in Schroedinger equation.

Black-Scholes Lagrangian

We have already derived the Black-Scholes propagator (4.17), using this result we can write

$$\langle x_i|e^{-\varepsilon H_{BS}}|x_{i-1}\rangle = e^{-r\varepsilon}\frac{1}{\sqrt{2\pi\varepsilon\sigma^2}}\exp\left(\frac{(\delta x_i-\varepsilon(r-\sigma^2/2))^2}{2\varepsilon\sigma^2}\right) = \mathcal{N}_{BS}(\varepsilon)e^{\varepsilon L_{BS}} \tag{4.34}$$

where $\delta x_i = x_i - x_{i-1}$. We can write the Black-Scholes Lagrangian as

$$L_{BS} = -\frac{1}{2\sigma^2}(\frac{\delta x_i}{\varepsilon}+r-\sigma^2/2)^2 - r \qquad \mathcal{N}_{BS}(\varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon\sigma^2}} \tag{4.35}$$

We can now write the propagator as

$$p_{BS}(x,\tau,x') = \int_{BS}\mathcal{D}_X e^{S_{BS}} \tag{4.36}$$

where

$$\int_{BS}\mathcal{D}_X = \left(\frac{1}{2\pi\varepsilon\sigma^2}\right)^{N/2}\prod_{i=1}^{N-1}\int dx_i \tag{4.37}$$

We implicitly assumed that $x(0) = x'$ and $x(\tau) = x$ and omitted it in our notation of path integration.

The integral (4.36) can be explicitly evaluated in the same manner as a propagator of free particle in QM. Naturally the result (4.17) is again obtained.

Lagrangian for stochastic volatility models

We will consider Merton-Graman Hamiltonian with $\alpha = 1$ and $\lambda = 0$, so it has a form

$$H_{MG} = -\frac{e^y}{2}\frac{\partial^2}{\partial x^2} - (r-\frac{e^y}{2})\frac{\partial}{\partial x} - (\mu-\frac{\eta^2}{2})\frac{\partial}{\partial y} - \rho\eta e^{y/2}\frac{\partial^2}{\partial x\partial y} - \frac{\eta^2}{2}\frac{\partial^2}{\partial y^2} + r \tag{4.38}$$

The Merton-Graman Lagrangian can be easily calculated in momentum space and with use of multidimensional Gaussian integration, see appendix D. After straight forward calculations we get

$$\langle x_i, y_i|e^{-\varepsilon H_{MG}}|x_{i-1}, y_{i-1}\rangle = \frac{1}{2\pi\varepsilon\sqrt{\eta^2 e^y(1-\rho^2)}}e^{\varepsilon L_{MG}(i)} \tag{4.39}$$

where

$$L_{MG} = -\frac{1}{2\varepsilon^2(1-\rho^2)}(e^{-y}\Phi^2 + \frac{1}{\eta^2}\Omega^2 - \frac{2\rho}{\eta}e^{-y/2}\Phi\Omega) + r + O(\varepsilon) \tag{4.40}$$

with

$$\Phi = x_i - x_{i-1} + \varepsilon r - \frac{\varepsilon e^y}{2} \qquad \Omega = y_i - y_{i-1} + \varepsilon\mu - \eta^2\varepsilon/2 \tag{4.41}$$

We define action $S_{MG} = \varepsilon \sum_{i=1}^{N} L_{MG}(i)$. We can write option price now as

$$p(x,y,\tau,x') = \int_{\mathbb{R}} p(x,y,\tau,x',y')dy' = \int \mathcal{D}_X\mathcal{D}_Y e^{S_{MG}} \tag{4.42}$$

where

$$\int \mathcal{D}_X = \frac{e^{-y_N/2}}{\sqrt{2\pi\varepsilon(1-\rho^2)}} \prod_{i=1}^{N-1} \int_{\mathbb{R}} dx_i \frac{e^{-y_i/2}}{\sqrt{2\pi\varepsilon(1-\rho^2)}} \tag{4.43}$$

and

$$\int \mathcal{D}_Y = \int_{\mathbb{R}} dy_0 \prod_{i=1}^{N-1} \int_{\mathbb{R}} \frac{dy_i}{\sqrt{2\pi\varepsilon\eta^2}} \tag{4.44}$$

where in $\mathcal{D}_Y$ is contained an extra integration over y_0.

The Lagrangian can be written as

$$\begin{aligned} L_{MG}(i) = \frac{-e^{-y_i}}{2(1-\rho^2)} \left(\frac{\delta x_i}{2} + r - \frac{e^{y_i}}{2} - \frac{\rho}{\eta} e^{y_i/2}(\mu - \eta^2/2 + \frac{\delta y_i}{\varepsilon}) \right)^2 - \\ \frac{1}{2\eta^2}(\mu - \eta^2/2 + \frac{\delta y_i}{\varepsilon})^2 + r \equiv L_X + L_Y \end{aligned} \tag{4.45}$$

The propagator can be written as

$$p(x,y,\tau,x') = \int \mathcal{D}_Y e^{S_Y} (\int \mathcal{D}_X e^{S_X}) \tag{4.46}$$

The integration over x can be done analytically because L_X is quadratic in x. The calculations are quite straightforward because L_X can be by a substitution rewritten to the form of Lagrangian of free particle in QM. We obtain

$$\int \mathcal{D}_X e^{S_X} = \frac{e^K}{\sqrt{2\pi\varepsilon(1-\rho^2)\sum_{i=1}^{N} e^{y_i}}} \tag{4.47}$$

where

$$K = -\frac{\left(x - x' + \varepsilon \sum_{i=1}^{N} (r - \frac{e^{y_i}}{2} - \frac{\rho}{\eta} e^{y_i/2}(\mu - \eta^2/2 + \frac{\delta y_i}{\varepsilon})) \right)^2}{2\pi\varepsilon(1-\rho^2)\sum_{i=1}^{N} e^{y_i}} \tag{4.48}$$

The integration over y has to be done numerically, these numerical methods are studied in [103] and the results obtained seem to fit the real option prices quite well.

4.2 Quantum field theory of interest rates

We will discuss modelling of interest rates in this section. We considered bonds in previous sections, risk free assets that pay interests

$$P(t,T) = e^{-(T-t)r}P(T,T) \tag{4.49}$$

where r is an interest rate and P is a so called zero coupon bond. We will consider $P(T,T) = 1$ from now on. We considered r to be a constant, but we can also consider $r = r(t,\omega)$ to be time dependent and random. Then we can write

$$P(t,T) = E(e^{-\int_t^T r(t')dt'}) \tag{4.50}$$

This stochastic approach will be discussed in the first part of this chapter. However it is a bit problematic to model bonds in this way, when we consider bonds $P(t,T)$ with different maturities T then it is complicated for function $r = r(t)$ of only one variable to consistently describe the set of bonds with different starting times and maturities.

This is reason to define forward interest rates $f = f(t_1,t_2)$, the price of bonds is then defined as

$$P(t,T) = e^{-\int_t^T f(t,x)dx} \tag{4.51}$$

or equivalently we can define forward rates from observed prices of zero coupons

$$f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T)$$

The forward interest rates are connected with a spot interest rate by a relation $r(t) = f(t,t)$.

The forward rates can be also model stochastically. But when we want both starting times and maturity times of a bond to be independent variables, then we have to choose another approach. We want to model a system $\{f(t,T), t \in (0,t_*), T \in (t, t+T_F)\}$ where every $f(t_1,T_1)$ is a random variable and all these variables are non-trivially correlated. This means that we want to model system with infinitely many degrees of freedom and non-trivial correlations among them. We will see that the framework of quantum field theory is a perfect tool for such a modelling.

4.2.1 Stochastic interest rates

Spot interest rates are modelled by stochastic equations in the form

$$dr(t) = a(r,t) + \sigma(r,t)dW_t \qquad t \in (t_0,T) \tag{4.52}$$

if we assume initial condition $r(t_0) = r_0$ the we have forward Kolmogorov equation for the density of $r(t)$

$$\partial_t P_f(r,t,r_0) = (\frac{1}{2}\partial_r^2\sigma^2(r,t) - \partial_r a(r,t))P_f(r,t,r_0) := H_f P_f(r,t,r_0) \tag{4.53}$$

in the case of a boundary condition $r(T) = R$ we get backward Kolmogorov equation

$$\partial_t P_b(R,t,r) = -H_f^\dagger P_b(R,t,r) := H_b P_b(R,t,r) \tag{4.54}$$

We can also calculate corresponding backward and forward Lagrangians. We will consider only the case $\sigma = \sigma(t)$. We can easily calculate the following matrix element then

$$\langle r|e^{\varepsilon H_f}|r'\rangle = \frac{e^{(-\frac{\varepsilon}{\sigma^2}(r-r'-\varepsilon a)^2-\varepsilon\partial_r a)}}{\sqrt{2\pi\varepsilon\sigma^2}} \tag{4.55}$$

so using (4.31) we can write forward Lagrangian as

$$L_f = -\frac{1}{2\sigma^2(t)}(\frac{dr}{dt} - a(r))^2 - \frac{\partial a(r,t)}{\partial r} \tag{4.56}$$

Similarly we get backward Lagrangian

$$L_b = -\frac{1}{2\sigma^2(t)}(\frac{dr}{dt} - a(r,t)) \tag{4.57}$$

and the action has the form

$$S_b = -\frac{1}{2}\int_{t_0}^{T}\frac{1}{\sigma^2(t)}(\frac{dr}{dt} - a(r,t))dt \tag{4.58}$$

Vasicek model

Probably the most famous model for spot interest rates is Vasicek model. The spot interest rate is given by the following equation

$$dr(t) = a(b-r) + \sigma dW_t \qquad r(t_0) = r_0 \tag{4.59}$$

where a, b, σ are constants.

We want to calculate the bond price

$$P(t_0,T) = E(e^{-\int_{t_0}^T r(t)dt}|r(t_0) = r_0) \tag{4.60}$$

with a boundary condition $P(T,T) = 1$. We have a boundary condition so we have to use a backward action

$$S_b = -\frac{1}{2\sigma^2}\int_{t_0}^{T}(\frac{dr}{dt} - a(b-r))^2 dt \tag{4.61}$$

We will now define a probability density by

$$E_{\langle t_0,T\rangle}(f(r(t))) = \frac{1}{Z}\int \mathcal{D}_r\ e^{S_b} f(r) \tag{4.62}$$

where

$$Z = \int \mathcal{D}_r \; e^{S_b} \qquad \int \mathcal{D}_r = \int_{\mathbb{R}} \prod_{t=t_0}^{T} dr(t) \tag{4.63}$$

where the uncountable product is just a short-hand notation for a countable product on the lattice and sending the lattice distance to zero.

This means that e^{S_b}/Z is a properly normalized probability distribution. We can now write

$$P(t_0, T) = \frac{1}{Z} \int \mathcal{D}_r \; e^{S_b} e^{-\int_{t_0}^{T} r(t)dt} \tag{4.64}$$

where we consider an initial condition $r(t_0) = r_0$ and an open end condition $\frac{dr}{dt}(T) = 0$.

We can use a substitution $r \to r + b$ and rewrite an above equation

$$P(t_0, T) = \frac{1}{Z} \int \mathcal{D}_r \; e^{S} \qquad S = -\frac{1}{2\sigma^2} \int_{t_0}^{T} (\frac{dr}{dt} + ar)^2 dt - \int_{t_0}^{T} (r(t) + b)dt \tag{4.65}$$

We need to perform another substitution now

$$v(t) = \frac{dr}{dt} + ar(t) \implies r(t) = e^{-a(t-t_0)} r_0 + e^{-at} \int_{t_0}^{t} e^{at'} v(t')dt' \tag{4.66}$$

Jacobian $\mathcal{D}_v = det(a + \frac{d}{dt})\mathcal{D}_r$ is a constants so it cancels with the same term from a normalization $1/Z$. We can write now

$$P(t_0, T) = e^{-b(T-t_0)} e^{-f(t_0,T)r_0} \frac{1}{Z} \int \mathcal{D}_v e^{-\frac{1}{2\sigma^2} \int_{t_0}^{T} (v^2(t) + 2\sigma^2 f(t,T) v(t))dt} \tag{4.67}$$

where $f(t, T) = \frac{1 - e^{-a(T-t)}}{a}$.

Performing infinite dimensional Gaussian integration, see appendix D, yields

$$P(t_0, T) = e^{-b(T-t_0)} e^{-f(t_0,T)r_0} e^{\frac{\sigma^2}{2} \int_{t_0}^{T} f^2(t,T)dt} \tag{4.68}$$

We can now also easily derive corresponding forward rates by using the equation (4.51).

HJM model

Heath-Jarrow-Morton model or HJM model describes behaviour of forward rates. We assume the following stochastic dynamic for forward rates $f(t, x)$ where $t < x < T_F$

$$df(t, x) = \alpha(t, x)dt + \sum_{i=1}^{N} \sigma_i(t, x) dW_i(t) \tag{4.69}$$

where Brownian motions W_i are independent. We will analyse this model in detail because the quantum field theory approach will be a direct generalization of this model in a sense. We have N independent driving noises in this model which corresponds to N degrees of freedom. The idea of next sections will be to build a quantum fields theory for forward rates i.e. a theory with infinitely many degrees of freedom.

Martingale condition in HJM model:
The drift coefficient $\alpha(t,x)$ is actually not arbitrary but it is fully determined by the martingale condition

$$P(t_0,T) = E(e^{-\int_{t_0}^{t_1} r(t)dt} P(t_1,T)|P(t_0,T)) \tag{4.70}$$

Moreover this condition in the case of bonds holds under the original measure so we do not need to change to an equivalent martingale measure. This is so because interest rates have to fluctuate around the drift term proportional to e^{rt} otherwise there would be an arbitrage opportunity.

The equation above can be rewritten as

$$P(t_0,T) = \frac{1}{Z}\int \mathcal{D}_W e^{-\int_{t_0}^{t_1} r(t)dt} P(t_1,T) e^{S[W,t_0,t_1]} \tag{4.71}$$

We integrate directly over Brownian sample paths in this case, this means that action has the simplest Gaussian form

$$S[W,t_0,t_1] = -\frac{1}{2}\sum_{i=1}^{N}\int_{t_0}^{t_1} W_i(t)dt \tag{4.72}$$

We can rewrite the integrand as

$$e^{-\int_{t_0}^{t_1} r(t)dt} P(t_1,T) = e^{-\int_{t_0}^{t_1} r(t)dt - \int_{t_1}^{T} f(t_1,x)dx} \tag{4.73}$$

with use of

$$f(t,x) = f(t_0,x) + \int_{t_0}^{t} \alpha(t',x)dt' + \int_{t_0}^{t}\sum_{i=1}^{N}\sigma_i(t',x)W_i(t')dt' \tag{4.74}$$

and equation $r(t) = f(t,t)$ we can rewrite the integrand after some calculations as

$$e^{-\int_{t_0}^{t_1} r(t)dt} P(t_1,T) = e^{-\int_{t_0}^{T} f(t_0,x)dx - \int_{t_0}^{t_1} dt \int_t^T dx\alpha(t,x) - \sum_{i=1}^{N}\int_{t_0}^{t_1} dt\int_t^T dx \sigma_i(t,x)W_i(t)} \tag{4.75}$$

The solution (4.74) of Ito equation (4.69) is written in different form then we are used to. We used there a notation $dW_i(t) = W_i(t)dt$ where $W_i(t)$ is a Gaussian noise, the path integration is then performed with respect to this noise.

We will from now on denote $\int_{t_0}^{t_1} dt \int_t^T dx = \int_A$. The equation (4.71) can now be rewritten as

$$P(t_0,T) = P(t_0,T)e^{-\int_A \alpha(t,x)}\frac{1}{Z}\int \mathcal{D}_W e^{-\sum_{i=1}^{N}\int_A \sigma_i(t,x)W_i(t)} e^{S[W]} \tag{4.76}$$

performing infinite dimensional Gaussian integration and rearranging terms yields a condition

$$e^{\int_A \alpha(t,x)} = e^{\frac{1}{2}\int_{t_0}^{t_1}\sum_{i=1}^{N}(\int_t^T \sigma_i(t,x)dx)^2 dt} \tag{4.77}$$

This condition can be equivalently rewritten as

$$\alpha(t,x) = \sum_{i=1}^{N} \sigma_i(t,x)\int_t^x \sigma_i(t,x')dx' \tag{4.78}$$

This condition is equivalent to a martingale condition (4.70) so this is in fact the martingale condition which uniquely determines the drift term $\alpha(t,x)$.

Bond options in HJM model :

We will consider a call option on a zero bond $P(t_1,T)$ with the expiration time t_1 and a strike price K. The price of this option at time $t_0 < t_1$ is

$$C(t_0, P(t_1,T), K) = E_{(t_0,t_1)}(e^{-\int_{t_0}^{t_1} r(t)dt}(P(t_1,T)-K)_+ | P(t_0,T)) \tag{4.79}$$

Using the definition of zero bonds (4.50) and the formula $\delta(a) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ipa}dp$ allows us to write

$$(P(t_1,T)-K)_+ = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ip(y+\int_{t_1}^T f(t_1,x)dx)}(e^y - K)_+ dp\ dy \tag{4.80}$$

We will now denote

$$\phi(y,t_1,T) = \frac{1}{2\pi}\int_{\mathbb{R}} E(e^{-\int_{t_0}^{t_1} r(t)dt} e^{ip(y+\int_{t_1}^T f(t_1,x)dx)})dp \tag{4.81}$$

so the option price can be written as

$$C(t_0, P(t_1,T), K) = \int_{\mathbb{R}} \phi(y,t_1,T)(e^y - K)_+ dy \tag{4.82}$$

we can rewrite formula for ϕ

$$\phi(y,t_1,T) = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{Z}\int \mathcal{D}_W(e^{-\int_{t_0}^{t_1} r(t)dt} e^{ip(y+\int_{t_1}^T f(t_1,x)dx)} e^{S[W.t_0,t_1]})dp \tag{4.83}$$

We now have to insert (4.74), use $r(t) = f(t,t)$ and perform the path integration. The calculation are lengthy but in principle completely analogical to calculations done above. We get

$$\phi(y,t_1,T) = P(t_0,t_1)\frac{1}{2\pi}\int_{\mathbb{R}} e^{ip(y+\int_{t_1}^T f(t_0,x)dx + \int_{t_0}^{t_1} dt \int_{t_1}^T dx\alpha(t,x))} e^{-(\int_{t_0}^{t_1} dt \int_{t_1}^T dx\alpha(t.x))p^2} dp \tag{4.84}$$

Performing Gaussian integration now yields

$$\phi(y,t_1,T) = P(t_0,t_1)N\left(-\int_{t_1}^T f(t_0,x)dx - \int_{t_0}^{t_1} dt\int_{t_1}^T dx\alpha(t.x), 2\int_{t_0}^{t_1} dt \int_{t_1}^T dx\alpha(t.x)\right) \tag{4.85}$$

Where $N(\mu,\sigma^2)$ denotes a density function of a normal distribution with a mean equal to μ and a variance σ^2. together with (4.82) this determines the option price.

4.2.2 Lagrangian formulation

We want to formulate a quantum field theory of forward interest rates now. The problem with the HJM model is that bonds with different maturities are exactly correlated which is clearly an oversimplification. The idea is to consider set of random variables $\{P(t,T), t_0 < t, t < T < t+T_F\}$, where T_F is maximum time for which we can buy the bond. It is usually 30 years or more so we can typically send $T_F \to \infty$. Variables $P(t,T)$ should be non-trivially correlated. It is beyond standard stochastic calculus to describe such a complicated structure. We need to describe a system with infinite number of degrees of freedom and non-trivial correlations. The quantum field theory was developed exactly for purpose of describing such systems so it is natural to use it also for modelling of forward interest rates.

Action for forward interest rates

We need to define action as a functional of the field $f(t,x)$

$$S[f] = \int_{t_0}^{t_1} dt \int_t^{t+T_F} dx \mathcal{L}[f] := \int_P \mathcal{L}[f] \tag{4.86}$$

We will consider mostly actions quadratic in f because then the path integration can be done analytically. Theories with non-quadratic actions are called non-linear or non-Gaussian. We will briefly introduce such theories later in this chapter.

It can be shown that Lagrangian has to contain the term $\frac{\partial^2 f}{\partial t \partial x}$ otherwise the martingale condition will not be fulfilled. The proposed Lagrangian has the form [41]

$$\mathcal{L}[f] = \mathcal{L}_{kin}[f] + \mathcal{L}_{rigidity}[f] = -\frac{1}{2}\left(\frac{\frac{\partial f}{\partial t} - \alpha(t,x)}{\sigma(t,x)}\right)^2 - \frac{1}{2\mu^2}\left(\frac{\partial}{\partial x}\frac{\frac{\partial f}{\partial t} - \alpha(t,x)}{\sigma(t,x)}\right)^2 \tag{4.87}$$

This is an action of a quantum string - the term α is a drift, $1/\sigma^2$ corresponds to the mass and the term $1/\mu^2$ is a rigidity. The HJM model is recovered in limit of infinite rigidity $\mu \to 0$. The drift term $\alpha(t,x)$ is uniquely determined by the martingale condition so only σ, μ are free parameters of this model.

We also need to consider some boundary conditions. We can assume Dirichlet conditions and fix $f(t,x)$ for $t = t_0$ and $t = t_1$.

We will assume Neumann boundary conditions. They require the surface term to vanish i.e.

$$\frac{\partial}{\partial x}\left(\frac{\partial_t f(t.x) - \alpha(t,x)}{\sigma(t,x)}\right) = 0 \quad for \quad x = t \quad or \quad x = t + T_F \tag{4.88}$$

This condition allows us to rewrite the action

$$S = -\frac{1}{2}\int_P \left(\frac{\partial_t f - \alpha}{\sigma}\right)\left(1 - \frac{1}{\mu^2}\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial_t f - \alpha}{\sigma}\right) \tag{4.89}$$

We will use a standard notation

$$Z = \int \mathcal{D}_f e^{S[f]} \qquad \int \mathcal{D}_f = \prod_{(t,x)\in P} \int_{\mathbb{R}} df(t,x) \tag{4.90}$$

where the uncountable product is just a short-hand notation for a countable product on the lattice and sending the lattice distance to zero.

This definition implies that the term $e^{S[f]}/Z$ is a properly normalized probability distribution.

Autocorrelation functions for velocity field

We will define a drift less velocity field $A(t,x)$ by

$$\frac{\partial f(t,x)}{\partial t} = \alpha(t,x) + \sigma(t,x)A(t,x) \tag{4.91}$$

the Neumann boundary conditions have now the form

$$\frac{\partial A}{\partial x} |_{x=t} = \frac{\partial A}{\partial x} |_{x=t+T_F} = 0 \tag{4.92}$$

The action can be rewritten as

$$S = -\frac{1}{2}\int_P A(t,x)(1 - \frac{1}{\mu^2}\frac{\partial^2}{\partial x^2})A(t,x) \tag{4.93}$$

Jacobian of a transformation $\int \mathcal{D}_f \to \int \mathcal{D}_A$ is just a constant so it cancels with the same term from the normalization $1/Z$.

The velocity field itself cannot be observed, but we can observe autocorrelation functions

$$\langle A(t,x)A(t',x')\rangle = \frac{1}{Z}\int \mathcal{D}_A\ A(t,x)A(t',x')e^{S[A]} \tag{4.94}$$

Autocorrelation functions can be also calculated in a different way, we define a moment generating function

$$Z[J] = \frac{1}{Z}\int \mathcal{D}_A e^{\int_{t_0}^{\infty} dt \int_t^{t+T_F} dx J(t,x)A(t,x)} e^{S[A]} \tag{4.95}$$

where we set $t_1 = \infty$ in the definition (4.86).

We can calculate this integral with use of rules for infinite-dimensional Gaussian integration, see appendix D

$$Z[J] = \exp\left(\frac{1}{2}\int_{t_0}^{\infty} dt \int_t^{t+T_F} dx dx' J(t,x)D(x,x',t,T_F)J(t,x')\right) \tag{4.96}$$

where the propagator $D(x,x',t,T_F)$ is given by the equation

$$(1 - \frac{1}{\mu^2}\frac{\partial^2}{\partial x^2})D(x,x',t,T_F) = \delta(x-x') \tag{4.97}$$

The autocorrelation function can now be obtained by a functional differentiation

$$\langle A(t,x)A(t',x')\rangle = \frac{\delta^2 Z[J]}{\delta J(t,x)\delta J(t',x')}\mid_{J=0} = \delta(t-t')D(x,x',t,T_F) \qquad (4.98)$$

The form of propagators $D(x,x',t,T_F)$ for different boundary conditions is reviewed in appendix E. We obtain the following result from (E.3) when considering Neumann conditions

$$D(x,x',t) = \lim_{T_F\to\infty} D(x,x',t,T_F) = \frac{\mu}{2}(e^{-\mu|x-x'|} + e^{-\mu(x+x'-2t)}) \qquad (4.99)$$

Martingale condition

The drift term $\alpha(t,x)$ is determined by the martingale condition. The derivations are the same as in the case of HJM model. The martingale condition requires

$$P(t_0,T) = E(e^{-\int_{t_0}^{t_1} r(t)dt}P(t_1,T)|P(t_0,T)) \qquad (4.100)$$

or equivalently

$$P(t_0,T) = \frac{1}{Z}\int \mathcal{D}_A e^{-\int_{t_0}^{t_1} r(t)dt}P(t_1,T)e^{S[A]} \qquad (4.101)$$

where $S[A]$ is given by (4.93).

We get an analogical equation to (4.75) in the same way as in HJM model

$$P(t_0,T) = P(t_0,T)e^{-\int_A \alpha(t,x)}\frac{1}{Z}\int \mathcal{D}_A e^{-\sum_{i=1}^N \int_A \sigma_i(t,x)W_i(t)}e^{S[A]} \qquad (4.102)$$

where $\int_A = \int_{t_0}^{t_1} dt \int_t^T dx$.

Performing an infinite dimensional Gaussian integration now yields

$$e^{\int_A \alpha(t,x)} = \exp\left(\frac{1}{2}\int_{t_0}^{t_1} dt \int_t^T dxdx'\sigma(t,x)D(x,x',t,T_F)\sigma(t,x')\right) \qquad (4.103)$$

rewriting this yields the martingale condition

$$\alpha(t,x) = \sigma(t,x)\int_t^x D(x,x',t,T_F)\sigma(t,x')dx' \qquad (4.104)$$

We will introduce a useful technique of discounting the price by a zero bond. We usually consider the discounting factor $e^{-\int_{t_0}^{t_1} r(t)dt}$ so for example $e^{-\int_{t_0}^{t_1} r(t)dt}P(t_1,T)$ is a martingale. However the bond $P(t_0,t_1) = e^{-\int_{t_0}^{t_1} f(t_0,x)dx}$ can be used as a discounting factor also. We want $e^{-\int_{t_0}^{t_1} f(t_0,x)dx}P(t_1,T)$ to be a martingale under the new measure so

$$P(t_0,T) = E_Q(e^{-\int_{t_0}^{t_1} f(t_0,x)dx}P(t_1,T)|P(t_0,T)) \qquad (4.105)$$

or equivalently

$$P(t_0,T) = P(t_0,t_1)E_Q(P(t_1,T)|P(t_0,T)) = P(t_0,t_1)\int \frac{1}{Z}\mathcal{D}_A P(t_1,T)e^{S_Q[A]} \tag{4.106}$$

We can easily obtain the form of $S_Q[A]$ by comparing the above formula with a standard martingale condition (4.70)

$$e^{S_Q} = \frac{e^{-\int_{t_0}^{t_1} r(t)dt}}{P(t_0,t_1)}e^S \tag{4.107}$$

It is easy to verify now that S_Q has the form (4.87) with the same parameter σ and with a drift term

$$\alpha_Q(t,x) = \sigma(t,x)\int_{t_1}^{x} D(x,x',t,T_F)\sigma(t,x')dx' \tag{4.108}$$

So this change of measure is just a change of a drift reminding the Girsanov theorem. This technique will be later used for a pricing of bond options.

Non-Gaussian forward interest rates

We will briefly mention non-Gaussian models here, i.e. models with a non-quadratic action. A problem with these models is that we cannot calculate the path integrals analytically in most of the cases. Moreover this means that it is problematic to determine the drift $\alpha(t,x)$ from the martingale condition. Also other problems known from the quantum field theory appear in these non-linear theories as for example problems with a renormalization. These problems are still not very well understood in the connection with finance [41].

We did not explicitly assumed $f(t,x) > 0$ so far. However this assumption is almost always realistic. We will define a new field ϕ

$$f(t,x) = f_0 e^{\phi(t,x)} > 0 \qquad \frac{\partial f}{\partial t} = f_0\phi\frac{\partial \phi}{\partial t} \simeq f_0\frac{\partial \phi}{\partial t} + O(\phi^2) \tag{4.109}$$

The Lagrangian has now the form

$$\mathcal{L}[\phi] = -\frac{1}{2}\left(\frac{f_0\frac{\partial\phi}{\partial t} - \alpha(t,x)}{\sigma(t,x)}\right)^2 - \frac{1}{2\mu^2}\left(\frac{\partial}{\partial x}\frac{f_0\frac{\partial\phi}{\partial t} - \alpha(t,x)}{\sigma(t,x)}\right)^2 \tag{4.110}$$

We can define the velocity field

$$A(t,x) = \frac{f_0\frac{\partial\phi}{\partial t} - \alpha(t,x)}{\sigma(t,x)} \tag{4.111}$$

The integration over A now leads to fermion path integral and problems with renormalization [41].

If we consider $\sigma(t,x) = \sigma_0 = const.$ then the drift term can be determined uniquely from the martingale condition

$$\alpha(t,x) = -\frac{\sigma_0^2}{2f_0} D(x,x,t,T_F) + \sigma_0^2 \int_t^x D(x,x',t,T_F) e^{\phi(t,x')} dx' \tag{4.112}$$

We will prove this statement in the next section with a use of Hamiltonian formulation of a quantum field theory for the forward interest rates.

Pricing of bond options

We will derive a price of an European call bond option in this paragraph. The derivation follow similar steps as a derivation in HJM model, however calculations become much easier when we use bonds as a discounting factor.

The price of a bond option can be written using (4.106) as

$$C(t_0, P(t_1,T), K) = P(t_0,t_1) E_Q((P(t_1,T) - K)_+ | P(t_0,T)) \tag{4.113}$$

or equivalently

$$C(t_0, P(t_1,T), K) = P(t_0,t_1) \frac{1}{Z} \int \mathcal{D}_A (P(t_1,T) - K)_+ e^{S_Q[A]} \tag{4.114}$$

the integrand can be rewritten as

$$(P(t_1,T) - K)_+ = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ip(y + \int_{t_1}^T f(t_1,x)dx)} (e^y - K)_+ dp\, dy \tag{4.115}$$

The price of a call option can now be written as

$$C(t_0, P(t_1,T), K) = \int_{\mathbb{R}} \phi(y,t_1,T)(e^y - K)_+ dy \tag{4.116}$$

where

$$\phi(y,t_1,T) = P(t_0,t_1) \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{Z} \int \mathcal{D}_A (e^{ip(y + \int_{t_1}^T f(t_1,x)dx)} e^{S_Q[A]}) dp \tag{4.117}$$

inserting for f from the definition (4.91) and calculating a path integral

$$\frac{1}{Z} \int \mathcal{D}_A e^{ip \int_{t_0}^{t_1} dt \int_t^{t_1} dx \sigma(t,x) A(t,x)} e^{S_Q[A]} = \exp\left(-\frac{p^2}{2} \int_{t_0}^{t_1} dt \int_{t_1}^T dx dx' \sigma(t,x) D(x,x',t,T_F) \sigma(t,x') \right) \tag{4.118}$$

we will for brevity denote

$$\Omega = \int_{t_0}^{t_1} dt \int_{t_1}^T dx dx' \sigma(t,x) D(x,x',t,T_F) \sigma(t,x') \tag{4.119}$$

then we can write

$$\phi(y, t_1, T) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ip(y + \frac{\Omega}{2} + \int_{t_1}^{T} f(t_0, x)dx)} e^{-\frac{\Omega p^2}{2}} \tag{4.120}$$

performing Gaussian integration now yields

$$\phi(y, t_1, T) = N\left(-\frac{\Omega}{2} - \int_{t_1}^{T} f(t_0, x)dx,\ \Omega\right) \tag{4.121}$$

where $N(\mu, \sigma^2)$ denotes a density of a normal distribution with a mean μ and a variance σ^2.

Using the martingale condition we can rewrite Ω as

$$\Omega = 2\int_{t_0}^{t_1} dt \int_{t}^{t_1} dx \alpha_Q(t, x) \tag{4.122}$$

after inserting this into (4.121) we can clearly see the similarity with the result (4.85) for HJM model.

4.2.3 Hamiltonian field theory

We will formulate a Hamiltonian field theory of forward interest rates in this section. The Hamiltonian formulation will be equivalent to Lagrangian formulation presented in the previous section. However it is very convenient to have different formulations of the same theory because each of them might be better suited for different problems. We will see that Hamiltonian formulation is better suited to deal with non-linear forward rates where the path integration is not Gaussian so it cannot be performed analytically.

From Lagrangian to Hamiltonian formulation

We need to obtain Hamiltonian - the infinitesimal generator of a time evolution, from the knowledge of Lagrangian. The way to do it is to invert the procedure presented in the section (4.1.3) where we derived Lagrangian from the knowledge of Hamiltonian. However we cannot use the simple relation

$$\langle f_{in} | e^{-(T_{fin} - T_{in})\mathcal{H}} | f_{fin} \rangle = \int \mathcal{D}_f e^{S[f]}$$

because our Hamiltonian is time dependent $\mathcal{H} = \mathcal{H}(t)$.

Hamiltonian has to be time dependent because the space of state vectors

$$|f_t\rangle = \bigotimes_{t \le x \le t + T_F} |f(t, x)\rangle \tag{4.123}$$

changes a form for different times t. This presents some technical difficulties, which we will address in this paragraph.

We will consider a discrete lattice

$$(t,x) \to \varepsilon(n,l) \quad f(t,x) \to f_{n,l} \quad T_F = \varepsilon N_F \tag{4.124}$$

The action has the form

$$S[f] = \sum_n S(n) = \sum_n (\varepsilon \sum_l \mathcal{L}_n[f_{n,l}, f_{n+1,l}]) \tag{4.125}$$

We will denote the state space at time $t = \varepsilon n$ by Π_n, the state vectors have the form

$$|f_n\rangle = \bigotimes_{n \leq l \leq n+N_F} |f_{n,l}\rangle \tag{4.126}$$

We can now define Hamiltonian as an operator $\mathcal{H}_n : \Pi_n \to \Pi_{n+1}$. It is defined by the following relation

$$\rho_n e^{\varepsilon \sum_l \mathcal{L}_n[f_n, f_{n+1}]} = \langle f_n | e^{-\varepsilon \mathcal{H}_n} | f_{n+1} \rangle \tag{4.127}$$

This means that the time evolution operator is acting on the initial state $\langle f_{in}|$. A factor ρ_n can be field dependent, however for linear forward rates it is just a constant.

We get the following formula in the case of the continuous time

$$Z = \int \mathcal{D}_f e^{S[f]} = \langle f_{in} | T \exp{(-\int_{T_{in}}^{T_{fin}} \mathcal{H}(t) dt)} | f_{fin} \rangle \tag{4.128}$$

Where T is a time ordering operator placing the operator with the earliest time in a argument to the left.

The formula above reminds the solution od Schroedinger equation with time dependent Hamiltonian, however the time ordering operator in a quantum mechanics places operators in reverse order. The difference is that the evolution operator in the finance applications is used for discounting the price and hence it propagates backwards in time. This is also the reason why the time evolution operator in finance acts on the initial state.

Hamiltonians for (non)-linear forward rates

In this paragraph we will derive Hamiltonians for linear forward rates defined by Lagrangian (4.87) and for non-linear forward rates defined by (4.110).

The linear forward rates are presented by Lagrangian

$$\mathcal{L}[f] = -\frac{1}{2}\left(\frac{\frac{\partial f}{\partial t} - \alpha(t,x)}{\sigma(t,x)}\right)^2 - \frac{1}{2\mu^2}\left(\frac{\partial}{\partial x}\frac{\frac{\partial f}{\partial t} - \alpha(t,x)}{\sigma(t,x)}\right)^2$$

Using definition (4.125) of $S(n)$ and considering a variable $t = \varepsilon n$ to be discrete and x to be continuous yields

$$S(t = \varepsilon n) = \varepsilon \int_t^{t+T_F} \mathcal{L}_n dx = -\frac{1}{2\varepsilon}\int_t^{t+T_F} A(t,x)(1 - \frac{1}{\mu^2}\frac{\partial^2}{\partial x^2})A(t,x) dx \tag{4.129}$$

where

$$A(t,x)=\frac{f_{t+\varepsilon}(x)-f_t(x)-\varepsilon\alpha(t,x)}{\sigma(t,x)} \tag{4.130}$$

This definition is similar to (4.91) but A is rescaled by ε and a discrete form of differentiation $\frac{\partial f}{\partial t}=\frac{f_{t+\varepsilon}-f_t}{\varepsilon}$ is used.

We can write $e^{S(n)}$ as a path integral now

$$e^{S(n)}=\int \mathcal{D}_p e^{-\frac{\varepsilon}{2}\int_t^{t+T_F} dxdx' p(x)D(x,x',t)p(x')+i\int_t^{t+T_F} dxp(x)A(x)} \tag{4.131}$$

where a propagator $D(x,x',t)$ is given by (4.99), form of $D(x,x',t)$ for different boundary conditions is reviewed in the appendix D.

We can perform a substitution $p(x)\to\sigma(t.x)p(x)$ in the above integral. We get up to the constant, which cancels with the same term from the normalization, the following

$$e^{S(n)}=\int \mathcal{D}_p e^{-\frac{\varepsilon}{2}\int_t^{t+T_F} dxdx' p(x)\sigma(t,x)D(x,x',t)\sigma(t,x')p(x')+i\int_t^{t+T_F} dxp(x)(f_{t+\varepsilon}(x)-f_t(x)-\varepsilon\alpha(t,x))} \tag{4.132}$$

With use of (4.127) we get

$$\rho_n e^{S(n)}=\langle f_t|e^{-\varepsilon\mathcal{H}_n}|f_{t+\varepsilon}\rangle \tag{4.133}$$

$\mathcal{H}_n$ acts on the initial state so we can consider $\mathcal{H}_n=\mathcal{H}_n(\frac{\delta}{\delta f_t})$. Using an identity $\int\mathcal{D}_p|p\rangle\langle p|=1$ yields

$$\langle f_t|e^{-\varepsilon\mathcal{H}_n}|f_{t+\varepsilon}\rangle=\int\mathcal{D}_p\langle f_t|e^{-\varepsilon\mathcal{H}_n}|p\rangle\langle p||f_{t+\varepsilon}\rangle=e^{-\varepsilon\mathcal{H}_n(\frac{\delta}{\delta f_t})}\int\mathcal{D}_p e^{i\int_t^{t+T_F} p(x)(f_t(x)-f_{t+\varepsilon}(x))} \tag{4.134}$$

Comparing this result with (4.132) (and ignoring the multiplication constant) yields the form of Hamiltonian for linear forward rates

$$\mathcal{H}_f(t)=-\frac{1}{2}\int_t^{t+T_F} dxdx'\sigma(t,x)D(x,x',t)\sigma(t,x')\frac{\delta^2}{\delta f(x)\delta f(x')}-\int_t^{t+T_F} dx\alpha(t,x)\frac{\delta}{\delta f(x)} \tag{4.135}$$

We briefly introduced non-linear forward interest rates $f(t,x)=f_0e^{\phi(t,x)}$ in the previous section. However we were unable to make any calculations for them because the non-Gaussian path integration could not be performed analytically. We remind the form of Lagrangian for non-linear forward rates

$$\mathcal{L}[\phi]=-\frac{1}{2}\left(\frac{f_0\frac{\partial\phi}{\partial t}-\alpha(t,x)}{\sigma(t,x)}\right)^2-\frac{1}{2\mu^2}\left(\frac{\partial}{\partial x}\frac{f_0\frac{\partial\phi}{\partial t}-\alpha(t,x)}{\sigma(t,x)}\right)^2 \tag{4.136}$$

It is easy to see that we can derive Hamiltonian for non-linear forward rates in the same way we did in the linear case. We obtain the following result

$$\mathcal{H}_\phi(t)=-\frac{1}{2f_0^2}\int_t^{t+T_F} dxdx'\sigma(t,x)D(x,x',t)\sigma(t,x')\frac{\delta^2}{\delta\phi(x)\delta\phi(x')}-\frac{1}{f_0}\int_t^{t+T_F} dx\alpha(t,x)\frac{\delta}{\delta\phi(x)} \tag{4.137}$$

Martingale condition in Hamiltonian formulation

We will derive a general condition for Hamiltonian to satisfy the martingale condition in this paragraph. This general result will be then used to derive a form of the drift term $\alpha(t,x)$ for both linear and non-linear forward interest rates.

The martingale condition for zero bonds has the form

$$P(t,T) = E_{\langle t,t_1\rangle}(e^{-\int_t^{t_1} r(t)dt} P(t_1,T)|P(t,T)) \tag{4.138}$$

or equivalently

$$P(t,T) = \frac{1}{Z}\int \mathcal{D}_f \rho[f] e^{-\int_t^{t_1} r(t)dt} P(t_1,T) e^{S[f]} \tag{4.139}$$

where a factor $\rho[f]$ is typically field independent, in that case it just cancels with the same term from the normalization $1/Z$.

We obtain be setting $t_1 = t+\varepsilon$, where ε is infinitesimal, the following

$$P(t,T) = \int \mathcal{D}_{f_{t+\varepsilon}} \rho[f_{t+\varepsilon}] e^{-\varepsilon f(t,t)} e^{\varepsilon \mathcal{L}[f_t, f_{t+\varepsilon}]} P[f, t+\varepsilon, T] \tag{4.140}$$

where

$$P[f,t,T] = e^{-\int_t^T f(t,x)dx} = \langle f_t | P(t,T)\rangle$$

We can rewrite the equation (4.140) as

$$\langle f_t | P(t,T)\rangle = \int \mathcal{D}_{f_{t+\varepsilon}} \langle f_t | e^{-\varepsilon f(t,t)} e^{-\varepsilon \mathcal{H}(t)} | f_{t+\varepsilon}\rangle \langle f_{t+\varepsilon} | P(t+\varepsilon, T)\rangle \tag{4.141}$$

or equivalently

$$\langle f_t | P(t,T)\rangle = \langle f_t | e^{-\varepsilon f(t,t)} e^{-\varepsilon \mathcal{H}(t)} | P(t+\varepsilon, T)\rangle \tag{4.142}$$

which is equivalent to

$$e^{-\varepsilon f(t,t)} | P(t,T)\rangle = e^{-\varepsilon \mathcal{H}(t)} | P(t+\varepsilon, T)\rangle \tag{4.143}$$

It holds $e^{-\varepsilon f(t,t)}|P(t,T)\rangle = |P(t+\varepsilon,T)\rangle$ from the definition of zero bonds. This means that the above condition is equivalent to

$$\mathcal{H}(t)|P(t,T)\rangle = 0 \qquad \forall t, T \tag{4.144}$$

It is interesting that we have obtained the same martingale condition for Hamiltonian as the condition (4.21) from the section about a quantum mechanical formulation of the option pricing problem.

We will now use the derived formula (4.144) the calculate the drift $\alpha(t,x)$ for both linear and non-linear forward rates.

The condition (4.144) for linear forward rates has the form

$$\mathcal{H}_f(t)|P(t,T)\rangle = \mathcal{H}_f(t)\exp(-\int_t^T f(t,x)dx) \stackrel{!}{=} 0 \tag{4.145}$$

Using the relation

$$\frac{\delta^n P(t,T)}{\delta f(t,x)^n} = (-1)^n P(t,T)\Theta(T-x) \tag{4.146}$$

and the definition of $\mathcal{H}_f$ (4.135) yields

$$(-\frac{1}{2}\int_t^T dxdx'\sigma(t,x)D(x,x',t)\sigma(t,x') + \int_t^T dx\alpha(t,x))P(t,T) = 0 \tag{4.147}$$

or equivalently

$$\alpha(t,x) = \sigma(t,x)\int_t^x D(x,x',t)\sigma(t,x')dx' \tag{4.148}$$

which is the martingale condition (4.104) we have already derived in Lagrangian formulation.

We will now similarly derive the form of $\alpha(t,x)$ for non-linear forward rates. We were unable to make these derivations in the Lagrangian formulation, however to do so in Hamiltonian formulation is quite easy.

The derivation for non-linear interest rates proceeds in same steps as the previous calculations

$$\mathcal{H}_\phi(t)P(t,T) = \mathcal{H}_\phi(t)e^{-f_0\int_t^T e^{\phi(t,x)}dx} \stackrel{!}{=} 0 \tag{4.149}$$

using the relations

$$\frac{\delta P(t,T)}{\delta\phi(x)} = -f_0 e^{\phi(x,t)}P(t,T)\Theta(T-x)$$

and

$$\frac{\delta^2 P(t,T)}{\delta\phi(x)\delta\phi(x')} = -f_0 e^{\phi(x,t)}P(t,T)\delta(x-x')\Theta(T-x) + f_0^2 e^{\phi(x,t)}e^{\phi(x',t)}P(t,T)\Theta(T-x)\Theta(T-x')$$

and the definition (4.137) of $\mathcal{H}_\phi$ yields after straight forward calculations the martingale condition in the form

$$\alpha(t,x) = -\frac{\sigma^2(x,t)}{2f_0}D(x,x,t) + \sigma(x,t)\int_t^x \sigma(x',t)D(x,x',t)e^{\phi(t,x')} \tag{4.150}$$

The knowledge of Hamiltonian enables in principle to calculate the pricing kernel

$$p[t_0,t_1,f] = \langle f_{t_0}|T\exp(-\int_{t_0}^{t_1}\mathcal{H}_f^Q(t)dt)|f_{t_1}\rangle \tag{4.151}$$

Where the drift term of $\mathcal{H}_f^Q(t)$ is given by equation (4.108), in other words it is adjusted for discounting by a zero bond $P(t_0,T)$.

We can now in principle calculate a price of any path independent bond option. We will consider an option with a pay-off function g, then the option price is given as

$$C(t_0, P(t_1,T), g) = P(t_0,t_1)E_Q(g(P(t_1,T))|P(t_0,t_1)) = P(t_0,t_1)\langle f_{t_0}|T\exp{(-\int_{t_0}^{t_1}\mathcal{H}_f^Q(t)dt)}|g\rangle \tag{4.152}$$

which can be rewritten as

$$C(t_0, P(t_1,T), g) = P(t_0,t_1)\int \mathcal{D}_{f_{t_1}} p[t_0,t_1,f]\langle f_{t_1}|g\rangle \tag{4.153}$$

where $\langle f_{t_1}|g\rangle = g(f_{t_1})$.

Conclusion

This thesis had two main objectives. The first one was to introduce the theory of stochastic processes beyond a traditional framework of Brownian motion and Ito calculus. The other one was to apply this theory to the option pricing problem.

In the first chapter, we focused on a general theory of the option pricing. We introduced standard Black-Scholes theory and talked about its limitations. We derived the two fundamental theorems of asset pricing, defined a completeness of the market, introduced a change of measure technique and discussed incomplete markets.

In chapters 2 and 3, we focused mainly on two approaches to go beyond Brownian motion and a standard diffusion in particular. The first one was physically motivated by an anomalous diffusion observed in many systems nowadays. The main idea was to generalize a standard Fokker-Planck equation to cover the memory effects and other non-standard effects not considered in Brownian models.

We mainly focused on a generalization to fractional Fokker-Planck equation, i.e. a differential equation containing derivatives of a non-integer order. We derived it from a continuous time random walk model and demonstrated its physical significance. We obtained stable processes as a special solution of a space-fractional diffusion. The solutions of both the space and the time fractional diffusion equations yielded non-Markovian processes. The type of memory effects they possess can occur for example in systems with a trapping or in fractal systems. This gave an interesting inside into the origins of fractional processes.

We have also discussed non-linear Fokker-Planck equation. A physical motivation is less clear in this case however we demonstrated a connection between the powers in a non-linear diffusion equation and Tsallis q-entropy.

We also applied both of these approaches to the option pricing problem and derived generalized versions of Black-Scholes formula.

The second class of processes we have discussed were Levy processes, i.e. Markov processes with stationary and independent increments. We showed that they can be interpreted as a superposition of standard Brownian diffusion with Poisson type jumps. We developed a stochastic calculus for these processes including a generalized Ito lemma. We also discussed ways to build Levy models applicable in finance.

An application of Levy processes to the option pricing yielded very interesting results. We calculated option prices in some particular models. However we cannot typically obtain the solutions analytically and we have to use numerical methods. We built a mathematical apparatus to deal with a change of the measure technique in Levy models, we derived generalized Black-Scholes equation and we discussed Fourier transform methods for numerical calculations of option prices.

Levy models bring a lot of new notions into the option pricing theory. Due to their non-diffusion nature Levy analogue of Black-Scholes equation is not a differential but an integro-differential equation. An integral term is due to their jumps. This complicates the situation from a numerical point of view but it also leads to non-uniqueness of an equivalent martingale measure. Such models are called incomplete. We demonstrated a connection of a completeness of the market with Feynman-Kac formula.

While these complications may seem mathematically inconvenient it turns out that the incomplete models describe a reality better than the complete ones do. This is supported mainly by the fact that options are redundant assets in complete markets - they can be perfectly hedged. This does not correspond to the reality, there is a remaining risk that cannot be hedged away in both incomplete models and in a reality .

In the last chapter, we reformulated the option pricing theory in the formalism of the quantum mechanics. We introduced both Hamiltonian and path integral formulation. We derived Hamiltonians and Lagrangians for both Black-Scholes and stochastic volatility model. We also discussed modelling of path-dependent options by adding a potential term. We derived a pricing kernel for a double-knock out barrier option using this method. This problem turned out to be equivalent to a problem with an infinite potential well from the quantum mechanics.

We introduced forward interest rates. A standard way to model them is by stochastic equations. We built a theory to model them as a quantum field then. We chose Lagrangian of the quantum string for this purpose and derived a no-arbitrage condition for linear forward rates and also the price of European bond options. The standard stochastic model was recovered when considering a string with an infinite rigidity. We mainly discussed the Lagrangian formulation of this theory. However, Lagrangian formulation based on a path integration was not convenient for dealing with non-linear forward interest rates. We were unable to perform the path integration for non-linear forward rates analytically, because they have a non-quadratic action. We also introduced the Hamiltonian field formulation and using it we were able to derive the no-arbitrage martingale condition even for non-linear forward rates. The quantum field theory formulation is more general than any model based on stochastic calculus. It turned out that especially the non-linear theory is beyond any standard model.

Appendix A

Ito calculus

We will very briefly summarize some results of Ito calculus. For a thorough review see [7].

We want to define an integral

$$I_t(C) = \int_0^t C_s \mathrm{d}W_s \tag{A.1}$$

for processes fulfilling

1. C is adapted to Brownian motion
2. $\int_0^t EC_s^2 \mathrm{d}s < \infty$

It can be done similarly as in the Riemann case. However we need to choose the left points in each interval of every partition and a convergence must be considered in the L^2 sense. The integral defined in this manner has the following properties

1. $I_t(aC_1 + C_2) = aI_t(C_1) + I_t(C_2) \quad a \in \mathbb{R}$
2. $I_t(C)$ is a martingale with respect to a natural filtration of Brownian motion
3. $EI_t(C) = 0$
4. $E(I_t(C))^2 = \int_0^t EC_s^2 \mathrm{d}s$
5. I_t has continuous trajectories

Another key result of Ito calculus is Ito lemma - an analogue of the chain rule from classical calculus.

Let us consider a process in the form

$$X_t = X_0 + \int_0^t A_s^{(1)} \mathrm{d}s + \int_0^t A_s^{(2)} \mathrm{d}W_s \tag{A.2}$$

it is called Ito process or diffusion. We can state Ito lemma now.

Theorem A.1. *Let us consider Ito process X_t and a function $f(t,x) \in C^2$. Then the following formula holds*

$$f(t, X_t) - f(s, X_s) = \int_s^t \left(\frac{\partial f}{\partial y} + \frac{1}{2}(A_y^{(2)})^2 \frac{\partial^2 f}{\partial x^2} + A_y^{(1)} \frac{\partial f}{\partial x}\right)\mathrm{d}y + \int_s^t A_y^{(2)} \frac{\partial f}{\partial x} \mathrm{d}W_y \tag{A.3}$$

where for the lucidity we omitted arguments $f = f(y, X_y)$

It can be formally proven by Taylor expansion with the use of $(\mathrm{d}W_t)^2 = \mathrm{d}t$ and by neglecting the terms of higher order than $\mathrm{d}t$.

We will also state the multidimensional result.

Theorem A.2. *Let $X_t = (X_t^1, ..., X_t^d)$ be a n-dimensional Brownian motion with a covariance matrix A and a zero mean. Let a function $f : \langle 0, T\rangle \times \mathbb{R}^d \to \mathbb{R}$ be $f \in C^2$. Then the following holds*

$$f(t, X_t) - f(0, X_0) = \int_0^t \frac{\partial f}{\partial s} \mathrm{d}s + \sum_{i,j=1}^{d} \int_0^t \frac{1}{2} A_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \mathrm{d}s + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i} \mathrm{d}X_s^i \tag{A.4}$$

Appendix B

Stable distributions

We will summarize basic properties of stable distributions in this appendix.

Stable distributions have densities invariant under a convolution. It implies that if we consider iid stable variables X_i then it holds

$$\sum_{i=1}^{n} X_i = aX + b$$

where X has the same stable distribution.

An importance of stable distributions lies in the following theorem.

Theorem B.1. *(Levy, Khintchin) A probability density can only be a limiting distribution of a sum of iid random variables, if it is stable.*

Levy and Khintchin have also found the most general class of stable distributions. This class is often called Levy distributions or α - stable distributions.

Theorem B.2. *A probability density $p_{\alpha,\beta}(x)$ is stable $\Leftrightarrow$ a logarithm of its characteristic function has the form :*

$$\psi_{\alpha,\beta}(k) = \ln \varphi_{\alpha,\beta}(k) = i\gamma k - \sigma|k|^{\alpha}(1 + i\beta \tfrac{|k|}{k}\omega(k,\alpha))$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, $\alpha \in (0,2\rangle$ and $\beta \in \langle -1,1\rangle$ and a function ω has the form:

$$\omega(k,\alpha) = \begin{cases} -tan(\pi\alpha/2) & \text{for } \alpha \neq 1 \\ (2/\pi)ln|k| & \text{for } \alpha = 1 \end{cases}$$

The logarithm of the characteristic function $\ln \varphi_{\alpha,\beta}$ is usually called Hamiltonian operator in physics and a cumulant generating function in mathematics. We will denote by $\varphi_{\alpha,\beta}$ the version with $\gamma = 0$ and $\sigma = 1$ from now on.

Meaning of parameters α, β, γ and c can be easily understood. We have a normal distribution with the expected value $\mu = \gamma$ and variance 2σ for $\alpha = 2$

regardless of the parameter β.
The α is a tail exponent or an index of stability for $\alpha \in (0,2)$. The following holds if $\beta \neq \pm 1$

$$p_{\alpha,\beta}(x) \sim \tfrac{1}{|x|^{1+\alpha}} \quad for \quad |x| \to \infty$$

The parameter γ gives a peak position or an expected value, if it exists. β determines how asymmetric a probability density is. The probability density is specially for $\gamma = \beta = 0$ an even function. The distribution is on the other hand for $\beta = \pm 1$ very asymmetric and if at the same time $\alpha \leq 1$ one tail vanishes completely. The support of $p_{\alpha,\beta}$ will be bounded to $\langle \gamma, \infty)$ respectively $(-\infty, \gamma\rangle$ in this case. The parameter σ is a scale factor that determines a width of a distribution.

The moment generating function exists only in the case $\beta = \pm 1$. We have for $\beta = 1$ and $k > 0$

$$\ln E(e^{-kX}) = -\gamma k - \sigma k^{\alpha} \sec \frac{\pi\alpha}{2} \tag{B.1}$$

where $\sec = \frac{1}{\cos}$. The case $\beta = -1$ and $k < 0$ is analogical.

We will also need another parametrization of $\varphi_{\alpha,\beta}$

$$\mathcal{H}_{\alpha,\theta}(k) = \ln \varphi_{\alpha,\theta}(k) = i\gamma k - c|k|^{\alpha} e^{isgn(k)\theta\frac{\pi}{2}} \tag{B.2}$$

Where $|\theta| \leq \min\{\alpha, 2-\alpha\}$. The extreme value od θ always corresponds to $\beta = \pm 1$. For example the case $\theta = 2 - \alpha$ corresponds to $\beta = 1$ for $\alpha > 1$. Analogically $\theta = \alpha - 2$ corresponds to $\beta = -1$. This parametrization will be in particular useful for the fractional differentiation where θ will be interpreted as a skewness.

We will also state the generalized central limit theorem for stable laws.

Theorem B.3. *(generalized CLT) Let X_i be a sequence of iid random variables, with asymptotic behaviour of densities given by*

$$p_{X_i}(x) = C_{\pm}|x|^{-(1+\alpha)} \quad for \quad x \to \pm\infty \ , \ \ \alpha \in (0,2)$$

Let us define the parameter $\beta = \frac{C_+ - C_-}{C_+ + C_-}$ then $\gamma \in \mathbb{R}$ exists such as

$$\tfrac{\sum_{i=1}^{n}(X_i - \gamma)}{n^{1/\alpha}} \xrightarrow{\mathcal{D}} X \sim p_{\alpha,\beta}(x)$$

The proof of this theorem can be found in [3].

Appendix C

Fractional differentiation

We will discuss the concept of fractional calculus i.e. differentiation of non-integer order. There exists many non-equivalent definitions of fractional derivatives. We will mention few of them well applicable to an anomalous diffusion. See for example [39] for more details.

The main idea is to generalize well-known Cauchy integral formula for any $\alpha > 0$

$$ {}_{x_0}\mathcal{I}_x^{\alpha} f(x) \equiv_{x_0} D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-y)^{\alpha-1} f(y) \mathrm{d}y \tag{C.1} $$

We will omit here the discussion about an appropriate domains of these operators. We will consider sufficiently smooth functions only.

It can be easily verified that operators ${}_{x_0}\mathcal{I}_x^{\alpha}$ form an abelian semi-group . We can write for $\alpha, \beta > 0$

$$ {}_{x_0}\mathcal{I}_x^{\alpha} \circ_{x_0} \mathcal{I}_x^{\beta} =_{x_0} \mathcal{I}_x^{\alpha+\beta} \tag{C.2} $$

It can be also easily seen that $\frac{\mathrm{d}}{\mathrm{d}x}({}_{x_0}\mathcal{I}_x^{\alpha+1}) =_{x_0} \mathcal{I}_x^{\alpha}$. The idea now is to define an operator ${}_{x_0}D_x^{\alpha}$ for positive α by

$$ {}_{x_0}D_x^{\alpha} f(x) = \frac{\mathrm{d}^{\lceil \alpha \rceil}}{\mathrm{d}x^{\lceil \alpha \rceil}} (\mathcal{I}_x^{\lceil \alpha \rceil - \alpha} f(x)) \tag{C.3} $$

Where $\lceil \ \rceil$ denotes the ceiling function. We will now discuss properties of these derivatives for certain choices of x_0.

Riemann-Liouville derivative

We obtain fractional Riemann-Liouville derivative by the choice $x_0 = 0$. We will from now on denote ${}_0D_x^{\alpha} \equiv D_0^{\alpha}$.

The composition rule (C.2) does not hold for R.-L. derivative. However it holds in certain cases

$$ D_0^q \circ D_0^Q = D_0^{q+Q} \qquad Q < 0,\ q \in R \tag{C.4} $$

It also holds for $Q < 1$ if we consider bounded functions. It can be formulated even in other cases but it has much more complicated form then [21].

An interesting question is how a derivative of power function x^n looks. We can obtain by straightforward calculation

$$D_0^\alpha x^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} x^{p-\alpha} \tag{C.5}$$

This seems to correspond to a classical derivation of the power function however it implies that a non-integer order differentiation of a constant function does not yield a zero!

R.-L. derivative is very convenient to treat with Laplace transform. We can look at equation (C.1) with $x_0 = 0$ as Laplace convolution. So we can easily obtain for $\alpha > 0$

$$\mathcal{L}(D_0^\alpha f(x))(u) = u^\alpha (\mathcal{L}f)(u) - \sum_{j=0}^{\lfloor \alpha \rfloor} u^j [D_0^{\alpha-1-j} f(x)]_{x=0} \tag{C.6}$$

This can be seen as a direct generalization of the classical rule

$$\mathcal{L}(f^{(n)}(x))(u) = u^n (\mathcal{L}f)(u) - \sum_{j=1}^{n} u^{n-j} f^{j-1}(0) \tag{C.7}$$

Laplace transform has a very nice form for the operator $D_0^{-\alpha} = I_0^\alpha$ where $\alpha > 0$

$$\mathcal{L}(D_0^{-\alpha} f(x))(u) = u^{-\alpha} (\mathcal{L}f)(u) \tag{C.8}$$

This follows directly from properties of a convolution and from $\mathcal{L}(x^p)(u) = \frac{\Gamma(p+1)}{u^{p+1}}$ for $p > -1$.

Riezs-Weyl operator

Another interesting choice $x_0 = -\infty$ yields Riezs-Weyl operator also known as Riezs-Feller derivative. We will denote ${}_{-\infty}D_x^\alpha \equiv D^\alpha$.

One of the desired properties of R.-W. operator is a preservation of the classical differentiation rule for the exponential function

$$D^\alpha \exp{(ax)} = a^\alpha \exp{(ax)} \tag{C.9}$$

This can be easily verified for $a > 0$, however e^{ax} is not in the domain of D^α for $a < 0$. The domain of R.-W. operator differs from the domain of R.-L. derivative for this reason - we have to consider only functions with a sufficient decay in $x \to -\infty$.

Fourier transform is more convenient for treating R.-W. operator then Laplace transform. We can by a straightforward calculation verify that for sufficiently smooth functions holds

$$\mathcal{F}({}_{-\infty}I_x^\alpha f(x))(k) = (-ik)^{-\alpha} \mathcal{F}f(k) \tag{C.10}$$

We can now easily derive a similar relation for D^α

$$\mathcal{F}(D^\alpha f(x))(k) = (-ik)^\alpha \mathcal{F}f(k) \tag{C.11}$$

This is again just a direct generalization of the classical rule for Fourier transform of a derivative.

A very convenient property is the composition rule for R.-W. operator, it holds for any $\alpha, \beta \in \mathbb{R}$ in the form

$$D^\alpha \circ D^\beta = D^{\alpha+\beta} \tag{C.12}$$

This can be easily verified using the composition rule (C.2) and the rule for Fourier transform of a derivative.

Riezs-Weyl operator is often generalized by adding a skewness parameter θ, then it is define by its Fourier image

$$\mathcal{F}(D^\alpha_\theta f(x))(k) = \mathcal{H}_{\alpha,\theta}(k)\mathcal{F}f(k) \tag{C.13}$$

where $\mathcal{H}_{\alpha,\theta}(k) = -|k|^\alpha e^{isgn(k)\theta\frac{\pi}{2}}$ is a log-characteristic function of the stable distribution defined in (B.2). We discussed the possible ranges of (α, θ) parameters in appendix B. The choice $\theta = 0$ corresponds to the stable symmetric distribution. The extreme choice $\theta = min\{\alpha, 2-\alpha\}$ corresponds to an asymmetric stable distribution with $\beta = \pm 1$. Naturally stable distributions are given as the solutions of a space fractional equation

$$\frac{\partial u}{\partial t}(x,t) = {}_xD^\alpha_\theta u(x,t) \tag{C.14}$$

Caputo fractional derivative

The Caputo derivative is defined in a little bit different manner by

$${}^*D^\alpha_0 f(x) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{f^{\lceil\alpha\rceil}(y)}{(x-y)^{1-\lceil\alpha\rceil+\alpha}} \mathrm{d}y \tag{C.15}$$

We can also consider a more general ${}^*D^\alpha_{x_0}$, however we will limit ourselves only to the case $x_0 = 0$.

Laplace image can be easily obtained using properties of convolution

$$\mathcal{L}({}^*D^\alpha_0 f(x))(u) = u^\alpha \mathcal{L}f(u) - \sum_{j=0}^{\lfloor\alpha\rfloor} u^{\alpha-j-1} f^{(j)}(0) \tag{C.16}$$

This is a very convenient form because we can use classical initial conditions.

Caputo and Riemann-Liouville derivatives can be connected via a relation

$${}^*D^\alpha_0 f(x) = D^\alpha_0 (f(x) - \sum_{j=0}^{\lfloor\alpha\rfloor} \frac{x^j}{j!} f^{(j)}(0)) \tag{C.17}$$

Using this relation and relation (C.4) we can easily see that

$$^*D_0^\alpha x^n = D_0^\alpha x^n \qquad n \in \mathbb{N}, \alpha < n \tag{C.18}$$

and

$$^*D_0^\alpha x^n = 0 \qquad n \in \mathbb{N}, \alpha > n \tag{C.19}$$

if we now define Mittag-Leffler function

$$E_{\mu,\nu} = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\mu n + \nu)} \tag{C.20}$$

we can easily see that the function $f(x) = E_{\mu,1}(ax^\mu) \equiv E_\mu(ax^\mu)$ solves the equation

$$^*D_0^\alpha f(x) = af(x) \tag{C.21}$$

Mittag-Leffler function is in the way generalized exponential, clearly $E_1 = \exp$ and $E_2(-x^2) = \cos x$. It also belongs to a more general class of Wright type functions [25].

It is also often defined as inverse Laplace transform [21]. We will only state its asymptotic properties that will be useful in a connection with the fractional diffusion. It has the following behaviour for $t >> \tau$

$$E_\alpha(-(t/\tau)^\alpha) \sim c(t/\tau)^{-\alpha} \tag{C.22}$$

and for $t << \tau$ we get

$$E_\alpha(-(t/\tau)^\alpha) \sim \exp\left(c(t/\tau)^\alpha\right) \tag{C.23}$$

So it starts with exponential behaviour and gets heavier tails in the time.

It is also worth noting that for $\nu > 0$ the equation

$$f(x) - f(0) = -c^\nu D_0^{-\nu} f(x) \tag{C.24}$$

has the solution in the form $f(x) = f(0)E_\nu(-c^\nu t^\nu)$. See [26] for the proof.

Appendix D

Gaussian integration

We will state basic results of multi-dimensional and infinite dimensional Gaussian integration in this appendix. These results are frequently used throughout the fourth chapter of this thesis.

Multidimensional Gaussian integration

We want to calculate Gaussian integral

$$Z[J] = \frac{1}{Z}\int_{\mathbb{R}} e^S dx_1 \dots dx_n \qquad S = -\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} J_i x_i \tag{D.1}$$

where the normalization $1/Z$ is chosen so it would hold $Z[0] = 1$.

Strictly speaking the above integral converges only if the matrix A is positively definite. In that case we can diagonalize matrix A and then by substitution transform the integral into n 1-dimensional Gaussian integrals. We obtain the following result

$$Z[J] = \exp\left(JA^{-1}J\right) \qquad \frac{1}{Z} = \frac{\sqrt{det A}}{(2\pi)^{n/2}} \tag{D.2}$$

Infinite dimensional Gaussian integration

The infinite dimensional Gaussian integration is a path integration with a quadratic action. We consider an action in the form

$$S = -\frac{1}{2}\int_{\mathbb{R}} dt dt' x(t) D^{-1}(t,t') x(t') dt dt' + \int_{\mathbb{R}} J(t) x(t) dt \tag{D.3}$$

We can now calculate the following integral by generalizing the formula (D.2) to infinite dimension

$$Z[J] = \frac{1}{Z}\prod_{t\in\mathbb{R}} dx(t) e^S = \frac{1}{Z}\int \mathcal{D}_x e^{S[x]} = \exp\left(\frac{1}{2}\int_{\mathbb{R}} J(t) D(t,t') J(t') dt dt'\right) \tag{D.4}$$

where $D(t,t')$ is defined by

$$\int_{\mathbb{R}} D^{-1}(t,s)D(s,t')ds = \delta(t-t') \tag{D.5}$$

and the normalization $1/Z$ is chosen so $Z[0] = 1$. This is mathematically far from rigorous because the normalization diverges. However this result is frequently used in the quantum field theory and it leads to right results despite the problems with mathematical rigorosity.

The formula (D.4) is used frequently in the chapter 4. We will demonstrate its use on a typical example from the theoretical physics - on a linear oscillator. Let us consider the action of a harmonic oscillator

$$S = -\frac{m}{2}\int_{\mathbb{R}} (\dot{x}^2(t) + \omega^2 x(t))dt = -\frac{m}{2}\int_{\mathbb{R}} x(t)(-\frac{d^2}{dt^2} + \omega^2)x(t)dt \tag{D.6}$$

This means

$$D^{-1}(t,t') = m(-\frac{d^2}{dt^2} + \omega^2)\delta(t-t') \tag{D.7}$$

This formula can be rewritten with a use of (D.5) as

$$\delta(t-s) = m(-\frac{d^2}{dt^2} + \omega^2)D(t,s) \tag{D.8}$$

This equation has a simple form in Fourier space, the inverse Fourier transform then yields

$$D(t,s) = \frac{1}{2\pi m}\int_{\mathbb{R}} \frac{e^{ik(t-s)}}{k^2+\omega^2}dk = \frac{e^{-\omega|t-s|}}{2m\omega} \tag{D.9}$$

Appendix E

Propagators for linear forward rates

We will state a form of the propagator $D(x, x', t, T_F)$ for different boundary conditions in this appendix. This propagator plays a key role in a quantum field theory of forward interest rates. We will omit any derivation, it can be found in [41].

The propagator for forward interest rates is given by an equation

$$(1 - \frac{1}{\mu^2}\frac{\partial^2}{\partial x^2})D(x, x', t, T_F) = \delta(x - x') \qquad t \leq x \leq t + T_F \tag{E.1}$$

When we consider Neumann boundary conditions

$$\frac{\partial D}{\partial x}\mid_{x=t} = \frac{\partial D}{\partial x}\mid_{x=t+T_F} = 0 \tag{E.2}$$

then the propagator has a form

$$D(x, x', t, T_F) = \mu\frac{\cosh\mu(x-t)\cosh\mu(T_f + t - x)}{\sinh\mu T_F} \qquad x > x' \tag{E.3}$$

When we consider Dirichlet boundary conditions

$$D\mid_{x=t} = D\mid_{x=t+T_F} = 0 \tag{E.4}$$

then the propagator has a form

$$D(x, x', t, T_F) = \mu\frac{\sinh\mu(x-t)\sinh\mu(T_f + t - x)}{\sinh\mu T_F} \qquad x > x' \tag{E.5}$$

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