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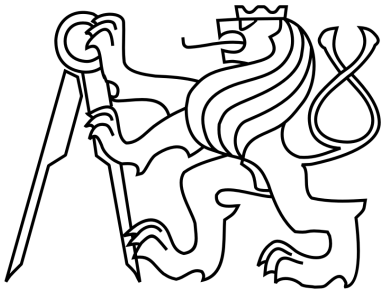

Formation of Equilibrium States in Quantum Networks with Random Interactions

Diploma Thesis

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Jaroslav Kysela

Název práce:	**Formování rovnovážných stavů v kvantových sítích s náhodnou interakcí**
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Abstrakt: Formování tepelné či jakékoliv jiné rovnováhy v různých fyzikálních systémech není dosud zcela plně pochopený jev, chceme-li jeho vznik odvodit ze základních zákonů kvantové mechaniky. Jedním z možných přístupů, jak tento jev popsat, je využití formalizmu otevřených kvantových systémů. V této práci popíšeme vývoj systému náhodnou unitární operací a za model fyzikálního systému vezmeme kvantovou síť, pro níž budeme uvažovat tři třídy interakcí — partial swap, energy exchange a CNOT interakce. Ve všech případech určíme analyticky asymptotický režim kvantové sítě spolu s podmínkami, za kterých síť spěje k rovnováze. Analytická zjištění jsou podpořena numerickými simulacemi pro obecnější režimy vývoje systému. Ukážeme, že kvantová síť spěje k rovnováze pro libovolnou ze zmíněných interakcí, pokud jednotlivé části sítě podstupují triviální volný vývoj. Stejné výsledky obdržíme i pro netriviální volný vývoj a CNOT interakce. Ve všech těchto případech je nastolení rovnováhy obecnou vlastností sítě, nezávislou na konkrétních hodnotách parametrů popisujících daný časový vývoj. Oproti tomu pro partial swap a energy exchange interakce doplněné netriviálním volným vývojem síť již obecně rovnováhy nenabývá.

Klíčová slova: ekvilibrace, rovnovážný stav, kvantová síť, náhodná unitární operace, asymptotická dynamika kvantových systémů

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Abstract: Formation of thermal as well as any other kind of equilibrium in various physical systems is still not a fully understood issue when one wants to derive its emergence from the underlying laws of quantum mechanics. One possible approach to explain this phenomenon is to make use of the open-quantum-system formalism. We utilize a random unitary operation to represent the system evolution and employ a quantum network as our model of a physical system. Three classes of interactions between the network constituents are considered—the partial swap, energy exchange and CNOT interactions. For all setups we determine analytically the asymptotic regime of the network and conclude under which conditions the network tends to equilibrium. Analytical findings are supported by numerical simulations with more general setups. We show that a quantum network tends to equilibrium for any of the three interactions, provided that the free evolution of individual constituents of the network is trivial. The same results are obtained even for a non-trivial free evolution and CNOT interactions. In all these cases the occurrence of equilibrium is a generic property of the network, independent of specific values of parameters present in the setup. On the contrary, for partial swap and energy exchange interactions assisted by non-trivial free evolution the network no longer approaches equilibrium, in general.

Key words: equilibration, equilibrium state, quantum network, random unitary operation, asymptotic dynamics of quantum systems

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Jaroslav Kysela

Contents

1 Introduction

More than a century ago two immensely successful theories were born, statistical physics and quantum theory. The former focuses on macroscopical phenomena of physical systems expressed in terms such as heat or temperature whereas the latter is a microscopical theory par excellence. The goal of quantum theory is to precisely describe all the physical phenomena as a result of microscopic principles which the physical bodies and fields follow. Even though both theories should lead to same predictions, as far as thermodynamical issues are concerned, the mutual relations between the two have not been understood in full detail until these days.

Great effort has been put to reveal exact ways how the macroscopic properties of the matter emerge from its underlying microscopic nature and to put their description on a solid mathematical footing [1]. The formation of a thermal equilibrium is one of the long-standing issues whose analysis has given rise to a wealth of various questions [2, 3]. The range of investigated problems is very broad, following recent works [4, 5] it can be divided into several key points.

Asymptotics. Many macroscopic phenomena arise as the asymptotic long-time behaviour of the given quantum system. Determination of such an asymptotic regime for complex quantum systems might be a difficult problem by itself. The central question then is what is the structure and algebraic properties of asymptotic states of the system.

Equilibration. A substantial feature of the thermalization is the occurence of the equilibration. The system should evolve towards some stationary state. However, this situation might not be the case in general. The equilibration in the given system may occur only for some initial states or for large enough systems. In the case the system does equilibrate it is reasonable to investigate the structure of stationary states. If the set of stationary states is sufficiently rich a strong memory effect preserving a lot of information about the initial state is present in the system.

Independency from initial conditions and thermalization. On the contrary, when the set of stationary states is small only few parameters suffice to describe the asymptotic state of the system and the asymptotics of the system is almost independent of the initial conditions. In particular, if the system tends to the Gibbs state $\rho_{\mathrm{th}} \sim e^{-\beta H}$ with β being an inverse temperature and H Hamiltonian of the system we say the given system thermalizes.

Subsystems. As the physical system consists of many microscopic constituents, one may treat it as being composed of smaller subsystems. Even though the whole system does not have to equilibrate, in general, equilibration of smaller subsystems may still be observed. In such a case one expects the subsystems of the same size approach the same equilibrium state.

As has been already mentioned we are primarily concerned with the above listed issues from the viewpoint of underlying microscopic laws. We intend to study how the equilibrium of a total system establishes via mutual interactions of its numerous subsystems. To be specific, we consider a quantum system composed of many constituents with dynamics analogous to the dilute Boltzmann gas. Individual subsystems undergo a free evolution interrupted by short random bipartite collisions. Such systems are well described by a concept of a quantum network. The network nodes correspond to individual quantum subsystems and the set of links captures the geometry of bipartite interactions. An interplay between the free evolution and the random bipartite interactions is responsible for the resulting asymptotic evolution of the system.

In this work we study quantum networks with three properly chosen types of interactions: the partial swap interaction, the energy exchange interaction and the controlled-NOT-like interaction. The main goal is to explore their asymptotic regime and elucidate

the impact of actual collision times, interaction strengths, their probability distribution and the size of the system on the structure of asymptotic states. Especially, we want to determine which details of a quantum network dynamics are irrelevant for its asymptotic evolution and consequently for its equilibration. Our ambition is to find closed analytical solutions for systems of an arbitrary size. To successfully accomplish this task the attractor theory developed in [6] for random unitary operations is employed.

In the first part of this thesis we state precisely conditions under which we explore the quantum network asymptotic behaviour and review indispensable mathematical tools. Three examples of interactions among the network constituents will be taken into account. For each of these we at first examine dynamics governed solely by mutual interactions. Afterwards we consider also the free evolution of the system and examine the dynamics obeyed by the system when both non-trivial free evolution and mutual interactions are present. The entire discussion is completed by numerical simulations demonstrating the rate of convergence with which the system state evolves towards an equilibrium.

2 Preliminaries

Before we move to investigate the formation of equilibria in our model system let us recall several fundamental concepts common in quantum theory together with appropriate notation.

2.1 Mathematical Framework

At the very beginning, let us review the concept of *qudits*. The qudit is a quantum system represented by a d-dimensional Hilbert state space. For $d = 2$ we call such systems *qubits*. In this work we concentrate on systems consisting of finite number of qudits with the same dimension d. Hilbert spaces associated with such systems are constructed as a tensor product of d-dimensional state spaces $\mathscr{H}_d \otimes \mathscr{H}_d \otimes \cdots \otimes \mathscr{H}_d \equiv \mathscr{H}$. When denoting vectors and covectors of the given space, the Dirac notation is used. A vector from $\mathscr{H}$ is denoted as $|\varphi\rangle$ whereas its corresponding covector reads $\langle\varphi|$. The scalar product of two vectors $|\varphi\rangle, |\psi\rangle \in \mathscr{H}$ in this order is thus $\langle\varphi|\psi\rangle$. The set $\mathcal{B}(\mathscr{H})$ of all bounded operators acting on $\mathscr{H}$ forms another Hilbert space. This space is equipped with the Hilbert-Schmidt scalar product—for any two operators $A, B \in \mathcal{B}(\mathscr{H})$ their scalar product is defined as

$$(A, B) \coloneqq \mathrm{Tr}(A^\dagger B). \tag{1}$$

The norm is defined accordingly as $\|A\| \coloneqq \sqrt{(A, A)}$. Mappings taking operators to operators will be referred to as *superoperators*. A very important subset of $\mathcal{B}(\mathscr{H})$ is formed by *density operators* (also known as *density matrices*). These are unit-trace positive operators which are used to represent the system state.

Suppose we have a quantum system composed of two subsystems. Let this system be characterized by its density operator $\rho_{12} \in \mathcal{B}(\mathscr{H}_1 \otimes \mathscr{H}_2)$. The density operator $\rho_1 \in \mathcal{B}(\mathscr{H}_1)$ of the first subsystem alone then reads $\rho_1 = \mathrm{Tr}_2(\rho_{12})$, where Tr_2 stands for the *partial trace* over the second subsystem. Let $\{|i_1\rangle\}_{i_1=1}^{d_1}$ and $\{|i_2\rangle\}_{i_2=1}^{d_2}$ be orthonormal bases of Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ with dimensions $d_1 = \dim \mathscr{H}_1$ and $d_2 = \dim \mathscr{H}_2$, respectively. The partial trace acts on input operators $A \in \mathcal{B}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ as follows

$$\mathrm{Tr}_2(A) = \sum_{j_1,k_1=1}^{d_1} \Big(\sum_{i_2=1}^{d_2} A_{j_1 i_2, k_1 i_2} \Big) |j_1\rangle\langle k_1|, \tag{2}$$

where $A_{j_1 j_2, k_1 k_2} = \langle j_1 j_2 | A | k_1 k_2 \rangle$ is a matrix element of the operator A in the basis $\{|i_1 i_2\rangle\}_{i_1,i_2}$ of the composite Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. Moreover, suppose that $\mathscr{H}_1 = \mathscr{H}_2 \equiv \mathscr{H}$ with $d = \dim \mathscr{H}$ and we may thus choose the same basis $\{|i\rangle\}_{i=1}^{d}$ for Hilbert spaces of both subsystems. If the form of the operator A is invariant under swapping of the two subsystems, *i.e.* $A_{j_1 j_2, k_1 k_2} = A_{j_2 j_1, k_2 k_1}$, then the partial trace over the first subsystem is equal to the partial trace over the second subsystem

$$\mathrm{Tr}_1(A) = \sum_{j,k=1}^{d} \Big(\sum_{i=1}^{d} A_{ij,ik} \Big) |j\rangle\langle k| = \sum_{j,k=1}^{d} \Big(\sum_{i=1}^{d} A_{ji,ki} \Big) |j\rangle\langle k| = \mathrm{Tr}_2(A). \tag{3}$$

This property can be generalized for many-body composite systems and for partial traces over arbitrarily large subsystems. Specifically, let $A \in \mathcal{B}(\mathscr{H}^{\otimes N})$ be an operator acting on a Hilbert space associated with an N-body system $\mathcal{S}_N$, let Hilbert spaces $\mathscr{H}$ of all bodies be mutually isomorphic, and let us choose the same orthonormal basis $\{|i\rangle\}_{i=1}^{d}$ in each $\mathscr{H}$, $d = \dim \mathscr{H}$. Moreover, let $\mathcal{S}_k$ be a k-body subsystem of $\mathcal{S}_N$ for a fixed $0 < k \leq N$. If the form of A in the basis $\{|i_1 \ldots i_N\rangle\}_{i_1,\ldots,i_N}$ is invariant under permutations of individual bodies, then the value of the partial trace over $\mathcal{S}_k$ is independent of a specific choice of

bodies in $\mathcal{S}_k$. In other words, $\mathrm{Tr}_{\mathcal{S}_k}(A) = \mathrm{Tr}_{\mathcal{S}'_k}(A)$ for any two k-body subsystems $\mathcal{S}_k$ and $\mathcal{S}'_k$ present in $\mathcal{S}_N$.

Hereafter, all the formulas related to qudits will be expressed in the computational basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ where d stands for qudit dimensionality. For further calculations it is suitable to introduce matrix element notation as follows. Having matrix $X \in \mathbb{C}^{d^N \times d^N}$, its element in the computational basis is denoted as

$$X^{i_1 \ldots i_N}_{j_1 \ldots j_N} \equiv \langle i_1 \ldots i_N | X | j_1 \ldots j_N \rangle, \tag{4}$$

where $i_1, \ldots, i_N \in \{0, \ldots, d-1\}$ label row indices and $j_1, \ldots, j_N \in \{0, \ldots, d-1\}$ label column indices. Henceforth, we will refer to the index of the form (4) as a *multiindex*. In some cases it is helpful to introduce one more symbol. When there are only two pairs of indices (i_l, j_l), (i_m, j_m) relevant for the current calculation the following notation is used

$$X^{i_l, i_m}_{j_l, j_m} \equiv X^{i_1 \ldots i_{l-1}\ i_l\ i_{l+1} \ldots i_{m-1}\ i_m\ i_{m+1} \ldots i_N}_{j_1 \ldots j_{l-1}\ j_l\ j_{l+1} \ldots j_{m-1}\ j_m\ j_{m+1} \ldots j_N}, \tag{5}$$

where inequality $1 \leq l < m \leq N$ is assumed. Similar notation is used also when more than two indices i, j are taken into consideration. In the following we will call the double (i_l, j_l) or (i_m, j_m) appearing in the index notation introduced in (5) as a *local index* of a matrix element.

2.2 Closed versus Open Dynamics

Taking the quantum system evolution into account two basic classes of systems can be identified—closed quantum systems and open quantum systems. Closed systems are treated as being completely isolated from their environment. For the study of this class of systems many tools have been developed so far. From the Schrödinger equation it follows that closed system evolution is fully described by the corresponding evolution operator $U(t)$ whose generator is the system Hamiltonian H. At time t the system, initially prepared in the state $\rho(0)$, evolves into the state represented by the density operator of the form

$$\rho(t) = U(t)\, \rho(0)\, U^\dagger(t). \tag{6}$$

As the Hamiltonian as well as the evolution operator are normal operators they are diagonalizable in an orthonormal basis. Determination of the system dynamics reduces to the calculation of corresponding eigenvalues and eigenvectors. Except for Hamiltonian eigenvectors any state of the system undergoes a periodic or quasiperiodic evolution. Consequently, the equilibration in a closed system in the sense defined above is not possible. Provided that oscillations are very quick an alternative weaker definition of equilibration is adopted for closed quantum systems. According to this definition the system equilibrates whenever its state remains close to an averaged state pertaining to the given evolution [3].

When we abandon the constraint that the system must not interact with its environment we arrive at systems driven by open dynamics. Supposing the system evolves from an initial state not correlated with the environment its evolution is captured by *quantum operations* [7]. Among these superoperators the *random unitary operations* form a prominent class. A random unitary operation Φ allows for the convex decomposition in the form

$$\Phi(A) = \sum_{\alpha=1}^{M} p_\alpha\, U_\alpha\, A\, U_\alpha^\dagger \tag{7}$$

with a set of unitary operators $\{U_\alpha\}_{\alpha=1}^M$, probability distribution $\{p_\alpha\}_{\alpha=1}^M$ and an arbitrary $A \in \mathcal{B}(\mathscr{H})$. Such an operation can be understood as a weighted average of unitary evolutions (6) generated by different Hamiltonians. Probabilities p_α express our incomplete

knowledge about which unitary evolution actually takes place. The operation Φ determines the form of the system state after the time interval Δt. We are concerned with the evolution arising from successive applications of Φ. For our study of equilibration it is thus an important task to find the asymptotic behaviour of the evolution $\lim_{n\to\infty} \Phi^n(\rho)$. Unfortunately, the random unitary operation cannot be diagonalized, in general. Determination of the system evolution is therefore much harder in comparison with closed quantum systems. Nevertheless, as shown in [6] the asymptotic regime ρ_{asymp} of the system, governed by iterations of Φ, lies in the subspace spanned by so called *attractors*. Hereafter we refer to this subspace as the *attractor space*. Attractors are all non-trivial solutions to matrix equations

$$U_\alpha X U_\alpha^\dagger = \lambda X, \quad \forall \alpha \in \{1, \ldots, M\}, \tag{8}$$

where λ is an eigenvalue of Φ for which $|\lambda| = 1$. All such eigenvalues form the *attractor spectrum* $\sigma_{|1|}$ of Φ. The equation above has to be satisfied for all unitaries U_α *simultaneously*. Having determined the set of attractors B_λ for each $\lambda \in \sigma_{|1|}$ we can describe the asymptotics of the system as follows. In each eigenspace B_λ we find its suitable orthonormal basis $\{Y_{\lambda,i}\}_{i=1}^{d_\lambda}$, where $d_\lambda = \dim B_\lambda$. It can be proven [6] that two vectors $Y_{\lambda,i}$ and $Y_{\mu,j}$ for different eigenvalues $\mu, \lambda \in \sigma_{|1|}$ are orthogonal, hence the set of all basis vectors $\{Y_{\lambda,i}\}_{\lambda,i}$ represents an orthonormal basis of the attractor space. The state $\Phi^n(\rho)$ approaches the asymptotic regime of the evolution $\rho_{\text{asymp}}(n)$ for increasing number of iterations $n \to \infty$. In the orthonormal basis of the attractor space this asymptotic regime reads

$$\rho_{\text{asymp}}(n) = \sum_{\lambda \in \sigma_{|1|}} \sum_{i=1}^{d_\lambda} (\lambda^n \xi_{\lambda,i})\, Y_{\lambda,i}. \tag{9}$$

Coefficients $\xi_{\lambda,i} = (Y_{\lambda,i}, \rho) = \text{Tr}(Y_{\lambda,i}^\dagger \rho)$ store information about the initial state ρ. Note the formula above is explicitly dependent on the number of iterations n whenever there is $\lambda \in \sigma_{|1|}$ such that $\lambda \neq 1$. The asymptotic dynamics may thus exhibit a non-trivial behaviour ranging from a stationary evolution to a quasi-periodic evolution. Another important feature of the evolution asymptotics (9) is that the (nonzero) probabilities $\{p_\alpha\}_\alpha$ play absolutely no role.

The concept of attractors enables us to study the presence of equilibration in various systems. Given a physical system we say it *equilibrates* whenever it evolves towards a stationary state. From the attractor viewpoint the system equilibrates when the asymptotic dynamics (9) is stationary. Inspection of (9) shows this situation happens when the only eigenvalue lying in the attractor spectrum is $\lambda = 1$. Other eivenvalues $\lambda \neq 1$ such that $|\lambda| = 1$ contribute to the asymptotic dynamics by oscillatory terms whose magnitude is directly proportional to the overlap of the initial state ρ with the relevant eigenvectors $Y_{\lambda,i}$.

As can be seen from (8) all the eigenvectors $X_{1,i}$ for $\lambda = 1$ represent *fixed points* of the operation Φ, $\Phi(X_{1,i}) = X_{1,i}$. Such attractors are preserved during the time evolution. Those operators $X_{1,i}$ that are Hermitian thus play the role of integrals of motion. By an *integral of motion* we mean a Hermitian operator whose expectation value associated with the system state is constant during the system evolution. The overlap of such operators with the initial state ρ of the system tells us how much information carried by ρ will survive after long course of time.

3 Quantum Networks

3.1 Description of Model

Thermodynamic phenomena occurring in physical systems result from collective behaviour of their many mutually interacting constituents. Theoretical treatment of such processes almost always poses a challenging problem and some simplifying restrictions are often adopted. In this work we study a model system with dynamics analogous to a dilute Boltzmann gas. Individual constituents of such systems evolve for most of the time freely, interrupted randomly by short and in time well separated elastic two-body interactions. These interactions will be referred to as *collisions* in the following. We assume that these collisions are rare, so that within a short time interval Δt an occurrence of two collisions is not considered. During this time interval there is thus either no collision between individual subsystems, in which case they all evolve completely independently and freely, or a pair of subsystems a and b collides and then the evolution of both subsystems is linked together. Nevertheless, our ability to determine which subsystems collide within the time interval Δt is limited and we are led to the introduction of a probabilistic description. To each pair (a, b) of system constituents we assign probability p_{ab} with which the given pair undergoes a collision. No-collision case, when the system follows pure free evolution during the interval Δt, arises with probability p_0. These events cover all possibilities that may occur within the time interval Δt, so $\sum_{a\neq b} p_{ab} + p_0 = 1$. Note that $p_{ab} \neq p_{ba}$, in general, as the interaction between subsystems can be asymmetric.

The complex structure of all possible collisions occurring within the interval Δt is naturally captured by a *network*. The nodes in the network are formed by individual subsystems and edges between distinct nodes represent possible collisions. These edges are undirected in case of a symmetric interaction and directed for an asymmetric interaction. Each edge is weighted by an associated probability with which the corresponding pair of subsystems collides. If the probability of a collision between a certain pair is zero, the edge is missing. This network model thus fully describes which subsystems are allowed to interact and how probable these interactions are. Let us emphasize again that this description is associated with a sufficiently short time interval Δt, in which it is not possible for two or more collisions to happen. Throughout the whole thesis we consider only the complete network—each subsystem may collide with any other. This setting forms a base upon which generalizations with different topologies can be made.

So far we have not specified neither the form of individual constituents composing the system nor their free evolution and mutual collisions. In fact, at this level the network model can describe a classical as well as a quantum system. We primarily intent to study the asymptotic dynamics of complex systems that results from microscopic physical laws, which are inevitably of quantum nature. In particular, we assume that our system consists of N qudits. The free evolution of each qudit, undisturbed by collisions, is generated by a free Hamiltonian that has the same form for all qudits. We consider collisions that are short compared to the time interval Δt, but not instantaneous. It allows us to analyze how durations of individual collisions affect the resulting asymptotic dynamics. Each collision lasts a definite amount of time that we will henceforth call the *collision time* or *interaction time*. Let the symbol Δt_{ab} denote a collision time associated with a collision of qudits a and b. Collision times Δt_{ab} may differ for each pair of qudits (a, b), but all collision times must satisfy $\Delta t_{ab} \leq \Delta t$. Even though a more general setting is possible, for mathematical convenience we also demand that all collisions are finished by the end of the time interval Δt. A graphical representation of this setup with different collision times Δt_{ab} and the time interval Δt can be found in Figure 1. Hamiltonians generating mutual collisions are assumed to represent the same type of interaction for all colliding pairs of qudits.

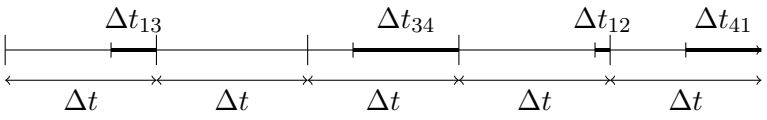

Figure 1: The time evolution of the quantum network may be thought of as a sequence of time intervals Δt. Within each Δt there arises at most a single collision between randomly chosen qudits a and b, which lasts for Δt_{ab} units of time. In our setting the end of any collision coincides with the end of the time interval Δt. In the figure one can see an initial stage of a possible time evolution of the quantum network with four nodes. Interaction times are drawn by thick lines. In the second step of the evolution above no collision emerges.

However, these Hamiltonians may be asymmetric with respect to the swap of the colliding qudits. The form of Hamiltonians and corresponding unitary propagators associated with individual events (collisions or undisturbed free evolution) that may arise during the time interval Δt is presented in greater detail in subsection 3.2.

We have just completed description of a quantum network and its dynamics within the time interval Δt. An example of a four-body quantum network is depicted in Figure 2. In what follows the collision probabilities $\{p_{ab}\}_{ab}$, time interval Δt, interaction times $\{\Delta t_{ab}\}_{ab}$ and free and interaction Hamiltonians will be a priori given and will therefore define the evolution of a whole system during the time interval Δt. The propagator describing an evolution of the system within this time interval incorporates all our lack of knowledge as to which event actually occurs. If the system evolves freely undisturbed by any collision, its evolution is represented by a unitary operation U^{free}. Similarly, with probability p_{ab} there is the collision between qudit a and qudit b. In such a case the network follows a unitary evolution characterized by U_{ab}. Operator U_{ab} encompasses not only the corresponding collision, but also the simultaneous free evolution of all qudits. Since in fact we do not know which pair of qudits collides or whether even any collision arises within Δt, we are forced to sum over all possibilities the evolution may follow to obtain propagator Φ as a linear map

$$\Phi(A) = \sum_{\substack{a,b=1 \\ a\neq b}}^{N} p_{ab}\, U_{ab}\, A\, U_{ab}^{\dagger} + p_0\, U^{\text{free}}\, A\, (U^{\text{free}})^{\dagger}, \tag{10}$$

where $A \in \mathcal{B}(\mathscr{H})$ is an operator acting on the Hilbert space of the whole system. We see the quantum network evolution may be rephrased in terms of a random unitary operation (7) presented in subsection 2.2.

The propagator Φ describes a single step of a quantum-network evolution, which lasts for Δt units of time. The total evolution is defined iteratively as an infinite sequence of these steps. More precisely, the evolution of a network, initially being in the state ρ_0, can be understood as a sequence $\{\Phi^n(\rho_0)\}_{n=0}^{\infty}$ of successive applications of a fixed random unitary operation (10) onto the state ρ_0. Two basic setups may be identified, either $p_0 = 0$ with some bipartite interaction present in every interval Δt of the evolution, or $p_0 \neq 0$ with the possibility of no collision within Δt. The latter setup allows for an easier analysis of the network asymptotics. We have just defined the network evolution by the sequence $\{\Phi^n(\rho_0)\}_{n=0}^{\infty}$ that implicitly assumes all the interaction times $\{\Delta t_{ab}\}_{ab}$, probability distribution $\{p_{ab}\}_{ab}$ as well as the time interval Δt are kept constant during the evolution. In section 7 we present numerical simulations going beyond this setting.

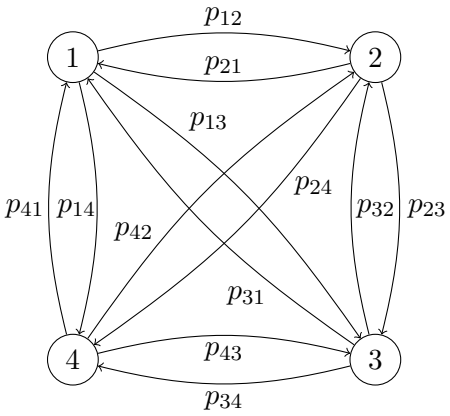

Figure 2: An example of a complete quantum network with $N = 4$ nodes. Each node represents a d-dimensional qudit and there is a directed edge between any qudits weighted by corresponding probability of collision p_{ab}.

3.2 Generators of Evolution

Evolution operator $U^{\text{free}}(\Delta t)$ in (10) captures the free evolution of the network within the time interval Δt. Let this evolution be generated by a free Hamiltonian H^{free}. From now on, as the *trivial free evolution* we call the free evolution generated by a zero Hamiltonian, *i.e.* $H^{\text{free}} = 0$. Since individual qudits are much of the time independent of the others, the total free Hamiltonian H^{free} can be expressed as a sum of local free Hamiltonians H_i^{free}, each of which acts merely on the qudit Hilbert space $\mathscr{H}_d$. That is

$$U^{\text{free}}(\Delta t) = e^{\mathrm{i}\,\Delta t\,H^{\text{free}}}, \quad H^{\text{free}} = \sum_{i=1}^{N} H_i^{\text{free}}. \tag{11}$$

Similarly, each of the unitaries U_{ab} in (10) represents a closed evolution of the network within the time interval Δt, when the interaction of qudit a and qudit b occurs. First $\Delta t - \Delta t_{ab}$ units of time the network evolves freely and the remaining Δt_{ab} units of time both qudits undergo mutual interaction. This process of collision between qudits a and b is explicitly expressed by the unitary operation

$$U_{ab}(\Delta t) = e^{\mathrm{i}\,\Delta t_{ab}\,(H^{\text{free}}+H_{ab}^{\text{int}})}\,e^{\mathrm{i}\,(\Delta t-\Delta t_{ab})\,H^{\text{free}}} = U_{ab}(\Delta t_{ab})\,U^{\text{free}}(\Delta t-\Delta t_{ab}), \tag{12}$$

where H_{ab}^{int} stands for the corresponding interaction Hamiltonian. Interaction Hamiltonians H_{ab}^{int} for all pairs (a,b) are assumed to describe the same kind of interaction. The Hamiltonian $H^{\text{free}} + H_{ab}^{\text{int}}$ governing the total evolution during the collision time of length Δt_{ab} consists of its free and interaction parts as follows

$$H^{\text{free}} + H_{ab}^{\text{int}} = \sum_{i\neq a,b} H_i^{\text{free}} + H_a^{\text{free}} + H_b^{\text{free}} + H_{ab}^{\text{int}} = \sum_{i\neq a,b} H_i^{\text{free}} + \tilde{H}_{ab}, \tag{13}$$

where H_i^{free} is the free Hamiltonian of the i-th qudit. Symbol $\tilde{H}_{ab}$ denotes the Hamiltonian acting on the subsystem comprised of qudits a and b. The decomposition (13) ensures that

$$\left[\tilde{H}_{ab}, \sum_{i\neq a,b} H_i^{\text{free}}\right] = 0. \tag{14}$$

One can make use of this equality to simplify the expression for the evolution operator

$$U_{ab}(\Delta t_{ab}) = e^{\mathrm{i}\,\Delta t_{ab}\,(H^{\text{free}}+H_{ab}^{\text{int}})} = e^{\mathrm{i}\,\Delta t_{ab}\tilde{H}_{ab}}\,e^{\mathrm{i}\,\Delta t_{ab}\,\sum_{i\neq a,b} H_i^{\text{free}}} = \tilde{V}_{ab}(\Delta t_{ab})\,\tilde{U}_{ab}(\Delta t_{ab}), \tag{15}$$

where we defined $\tilde{V}_{ab}(\Delta t_{ab}) = \exp(\mathrm{i}\,\Delta t_{ab}\,\tilde{H}_{ab})$ as the part of the unitary operator acting on qudits a and b and $\tilde{U}_{ab}(\Delta t_{ab})$ stands for the part of the unitary operator associated with a purely free evolution of remaining qudits.

In the special case when the interaction Hamiltonian H^{int}_{ab} commutes with the free Hamiltonian H^{free} the evolution operator (12) can be rewritten in a more convenient way. We are left with

$$U_{ab}(\Delta t) = e^{\mathrm{i}\,\Delta t_{ab} H^{\mathrm{int}}_{ab}}\, e^{\mathrm{i}\,\Delta t H^{\mathrm{free}}} = V_{ab}(\Delta t_{ab})\, U^{\mathrm{free}}(\Delta t), \tag{16}$$

where U^{free} stands for a free evolution of the whole network and V_{ab} is an operator characterizing solely the interaction between qudits a and b.

So far we have not mentioned the specific form of the Hamiltonians themselves. In what follows we will investigate an asymptotic evolution of a complete quantum network with three different types of bipartite interactions as well as three kinds of free Hamiltonians. All of them will be defined and presented in detail in appropriate sections.

3.3 Asymptotics of Quantum Network

In subsection 2.2 we recalled mathematical tools with help of which one determines the asymptotic state of the system evolving according to a random unitary operation. In the same subsection we also revealed connections between the asymptotic evolution and conditions under which the particular system equilibrates. At this moment we finally present attractor equations we will use in the subsequent discussion. In subsection A.1 it is proven that for random unitary operation (10) with $p_0 = 0$ the attractor equations read

$$U^{\mathrm{free}}\, X\, (U^{\mathrm{free}})^{\dagger} = \lambda\, \overline{V}^{\dagger}_{ab}\, X\, \overline{V}_{ab}, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b, \tag{17}$$

where $U^{\mathrm{free}} \equiv U^{\mathrm{free}}(\Delta t)$ and $\overline{V}_{ab} \equiv \tilde{V}_{ab}(\Delta t_{ab})\, U^{\dagger}_{a}(\Delta t_{ab})\, U^{\dagger}_{b}(\Delta t_{ab})$. Symbol $U_i(\Delta t_{ab}) = \exp(\mathrm{i}\,\Delta t_{ab} H^{\mathrm{free}}_i)$ stands for a free evolution operator associated with qudit i. If we additionally assume that also no collision might take place during the time interval Δt, *i.e.* $p_0 \neq 0$ (10), attractor equations decouple into

$$U^{\mathrm{free}}\, X\, (U^{\mathrm{free}})^{\dagger} = \lambda X, \quad \overline{V}^{\dagger}_{ab}\, X\, \overline{V}_{ab} = X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{18}$$

The calculation of attractors is thus accomplished by solving two systems of equations. The first system is associated with solely free evolution of the network and enables us to compute the attractor spectrum $\sigma_{|1|}$, see (8). The second system determines the structure of all possible attractors. Both systems of equations have to be fulfilled simultaneously.

Suppose the *interaction Hamiltonian commutes with the free evolution* and unitary operators can thus be rewritten into (16). Importantly, in such a special case we obtain equations (17) and (18) with $\overline{V}_{ab} = V_{ab}$. We have thus completely separated a pure interaction evolution characterized by V_{ab} and a pure free evolution captured by U^{free}. If we already know, *e.g.* from some previously performed calculations, a solution X for the pure interaction evolution, it is therefore easy to derive even the general case with a non-trivial free evolution. We just plug X into the first system of equations in (18). The structure of the attractor for a composite evolution is thus a special case of the structure of attractors corresponding to the purely interaction evolution. This easier treatment of the network asymptotic regime dependent on the form of Hamiltonians leads us to identify two classes of interactions. One class is composed of unitary operations whose generating interaction Hamiltonian commutes with the free evolution Hamiltonian, and the other class is composed of the rest of Hamiltonians. In what follows, we choose a representative from either class and study its effects on the system equilibration.

In the upcoming discussion we will consider also the case with a trivial free evolution. For this setting the attractor equations simplify to a remarkably simple form

$$V_{ab}(\Delta t_{ab})\, X\, V_{ab}^{\dagger}(\Delta t_{ab}) = \lambda X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{19}$$

Obviously, the above expression is independent of the time interval Δt. Consequently, when all interaction times are identical, we can without loss of generality put $\Delta t_{ab} = \Delta t$. Equation (19) was obtained for $p_0 = 0$. For the decoupled case (18) with $p_0 \neq 0$ (10) the free evolution part reduces to $(1-\lambda)X = 0$ which allows for non-trivial solutions only for $\lambda = 1$. Solving the case with $p_0 = 0$ is thus more general and will be used in future calculations.

3.4 Outline of Analytical Approach

To make our subsequent discussion for each investigated interaction Hamiltonian clear, let us briefly summarize the steps we take to find solutions in each case. First of all, we consider the network evolution driven by mutual interactions accompanied by the trivial free evolution only, *i.e.* $H^{\text{free}} = 0$. Moreover, we keep all interaction strengths appearing in definitions of interaction Hamiltonians identical for all pairs of qudits. The same constraint is imposed also on interaction times Δt_{ab}. In this setting we solve attractor equations for $\lambda = 1$ and $\lambda \neq 1$ independently. After these initial steps we consider more realistic scenario with interaction times and strengths being generically different for each pair.

Having finished our investigation for the trivial free evolution we take into consideration also the non-trivial free Hamiltonian. In this last step we already assume that interaction times and coupling strengths might be different for distinct pairs of qudits. As the analytical treatment of all possible free Hamiltonians is in general impossible, in certain cases we restrict ourselves to investigate only a subclass of these. In further calculations our solution will be parametrized by several quantities such as coupling strengths and collision times. Since we are looking for a generic form of the asymptotic evolution we sometimes intentionally neglect a zero-measure set of parameter values to simplify our computations while keeping almost full generality of our results. In an experimental realisation of the quantum network it is effectively impossible to fine-tune parameters to lie in such a zero-measure set and the generic case is thus not affected by these exceptional values.

4 Partial Swap Interaction

As has been demonstrated in the preceding section, when the interaction and free Hamiltonians commute with each other the analysis of the system asymptotic behaviour simplifies considerably. Surprisingly, there exists an interaction which commutes with any free evolution. This interaction is realized by the *partial swap*. The partial swap operation was introduced for qubits in the context of thermalizing quantum machines and homogenization of a quantum system [8, 9]. We consider a generalization of this operation for an arbitrary dimension of qudits. In the following we examine the asymptotic evolution of the quantum network when random partial swap interactions are present.

The partial swap (PSW) of qudits a and b is a linear combination of the identity, leaving the qudits unaffected, and a swap operation (SW), interchanging the pair of qudits, *i.e.* $\mathrm{SW}\,|x\rangle|y\rangle = |y\rangle|x\rangle$. Explicitly,

$$\mathrm{PSW}_{ab} = p_{ab}\,\mathbb{I}_{ab} + q_{ab}\,\mathrm{SW}_{ab}, \tag{20}$$

where the coefficients p_{ab} and q_{ab} appear as parameters. Unitarity of the partial swap restricts values of these parameters as follows

$$p_{ab} = \cos(\phi_{ab})\, e^{\mathrm{i}\,\rho_{ab}}, \quad q_{ab} = \mathrm{i}\,\sin(\phi_{ab})\, e^{\mathrm{i}\,\rho_{ab}}, \tag{21}$$

where $\phi_{ab} \in (0, 2\pi) \setminus \{\pi/2, \pi, 3\pi/2\}$ and $\rho_{ab} \in [0, 2\pi)$. From the set of values for ϕ_{ab} we exclude those for which either $\mathrm{PSW}_{ab} \propto \mathbb{I}$ or $\mathrm{PSW}_{ab} \propto \mathrm{SW}_{ab}$. As the global phase has no observable effects in the context of attractor equations, we can put $\rho_{ab} = 0$. Making use of the involution property $\mathrm{SW}^2 = \mathbb{I}$ one can easily find, see subsection A.3, the Hamiltonian for the partial swap

$$H_{\mathrm{PSW}_{ab}} = \phi_{ab}\,\mathrm{SW}_{ab}. \tag{22}$$

Thanks to commutation relations (110) derived in subsection A.3 the evolution operators read $U_{ab}(\Delta t) = V_{ab}(\Delta t_{ab}) U^{\mathrm{free}}(\Delta t)$, where $V_{ab}(\Delta t_{ab}) = \exp(\mathrm{i}\,\Delta t_{ab}\,\phi_{ab}\,\mathrm{SW}_{ab})$ (16). This form will be found useful for future calculations. For unit collision times $\Delta t_{ab} = 1$ one obtains $V_{ab} = \mathrm{PSW}_{ab}$. Non-unit collision times Δt_{ab} enter relation (20) only via parameter ϕ_{ab} in a way $\phi_{ab} \to \Delta t_{ab}\,\phi_{ab}$. The form of the evolution operator is thus preserved, only parameters p_{ab} and q_{ab} modify their values. As interaction strength and interaction time always appear together in a product $\Delta t_{ab}\,\phi_{ab}$, from now on we set $\phi_{ab} = 1$ for all pairs of qudits (a, b). This condition does not affect generality of our results and reduces the number of parameters present in our discussion.

The discussion on the network equilibration is divided into two parts. Firstly, we consider mutual interactions between constituents together with a trivial free evolution $U^{\mathrm{free}} = \mathbb{I}$. We start with the case when collision times Δt_{ab} for all qudit pairs (a, b) are equal. Later on this constraint is relaxed. Secondly, we incorporate also a non-trivial free evolution.

4.1 Pure Collisions

Before investigating the general setup let us focus on a network evolution when the free part is the trivial one. For such a setting the attractor equations assume the form (19). In this case $V_{ab}(\Delta t_{ab}) = \mathrm{PSW}_{ab}$ (20) and the attractor equations read

$$p_{ab}(1 - \lambda)\, X + q_{ab}\,(\mathrm{SW}_{ab}\, X - \lambda X\, \mathrm{SW}_{ab}) = 0, \tag{23}$$

where a and b run through the set $\{1, \ldots, N\}$ of all qudits in the quantum network, $a \neq b$. For convenience, we put $p \equiv p_{ab}$ and $q \equiv q_{ab}$. A remark presented above on the effect of

different collision times makes it clear that the form of these equations is general enough to encompass all possible settings of times Δt_{ab} and interaction strengths ϕ_{ab}. For $\lambda = 1$ or $p = 0$ equations (23) reduce to

$$\mathrm{SW}_{ab} X = \lambda X \,\mathrm{SW}_{ab}, \tag{24}$$

i.e. equations (8) for the swap operation only. Let $N \geq 2$ be a number of qudits in the system and $d \geq 2$ their dimension. In the following we provide a solution for the equations (23) and calculate dimensionality of the corresponding attractor space. Our findings are collected in the conclusion at the end of this subsection.

4.1.1 Identical Interaction Times

Even though equations (23) hold true for all values of collision times (and strengths), let us first treat the case with all Δt_{ab} being identical, $\Delta t_{ab} = \Delta t$. Generalizations will be straightforward and presented later on.

Attractors for $\lambda = 1$. Setting $\lambda = 1$ simplifies equations (23) into relations (24). Let us begin by recalling that $\mathrm{SW}_{ab} = \mathrm{SW}_{ab}^{-1} = \mathrm{SW}_{ab}^{\dagger}$. The swap operation acts as a swap of row indices while applied to matrix X from the left-hand side. Similarly, it acts as a swap of column indices while applied from the right-hand side. Equations (24) therefore reduce to $\mathrm{SW}_{ab} X \,\mathrm{SW}_{ab} = X$ which can be expressed in the index notation as

$$X^{i_a,i_b}_{j_a,j_b} = X^{i_b,i_a}_{j_b,j_a}, \quad a,b \in \{1,\ldots,N\}, \quad a \neq b \tag{25}$$

with $i_a, i_b, j_a, j_b \in \{0,\ldots,d-1\}$. That is, all matrix elements which differ by a permutation of their local indices must be equal. These elements form one equivalence class, each class corresponds to a linearly independent attractor associated with $\lambda = 1$. Let us look for a number $S_1(N,d)$ of equivalence classes, *i.e.* the dimension of the attractor space. As the order of local indices $\binom{i_k}{j_k}$ appearing in a matrix-element multiindex $\binom{i_1,\ldots,i_N}{j_1,\ldots,j_N}$ is irrelevant within a given equivalence class, one can represent each equivalence class merely by a number of such local indices. These local indices are of any of the following forms

$$\binom{0}{0},\binom{0}{1},\ldots,\binom{0}{d-1},\binom{1}{0},\binom{1}{1},\ldots,\binom{1}{d-1},\ldots,\binom{d-1}{0},\binom{d-1}{1},\ldots,\binom{d-1}{d-1}. \tag{26}$$

There are obviously d^2 such local indices which can appear in a given multiindex. The number of equivalence classes is therefore equal to the number of ways one can distribute local indices shown above into N-element sets (*i.e.* into multiindices where one does not care about the order of local indices). Since one local index can be present in a given multiindex more than once, we have to use the formula for a number of combinations with repetition. Its general form is $\binom{n+k-1}{k}$ for a subset of size k formed by elements from an n-element set. In our case $k = N$ and $n = d^2$ and we immediately obtain the number of equivalence classes

$$S_1(N,d) = \binom{N+d^2-1}{N}. \tag{27}$$

Recall local indices listed in (26) and focus on some fixed multiindex. In this multiindex there is a certain number of local indices of the form $\binom{0}{0}$. Let this number be denoted by c_0^0 and similarly for all remaining local indices from (26). Each equivalence class is thus characterized by d^2-tuple $(c_0^0, c_1^0, \ldots, c_{d-2}^{d-1}, c_{d-1}^{d-1})$. Making use of this notation we can easily

express the general form the attractors associated with $\lambda = 1$ take on. It can be written as a linear combination

$$X = \sum_{c_0^0,\ldots,c_{d-1}^{d-1}} \alpha_{c_0^0,\ldots,c_{d-1}^{d-1}} P_{c_0^0,\ldots,c_{d-1}^{d-1}}, \tag{28}$$

where α's are coefficients and elements of matrix $P_{c_0^0,\ldots,c_{d-1}^{d-1}}$ are all zeros except for those elements lying in the equivalence class specified by numbers $c_0^0,\ldots,c_{d-1}^{d-1}$. These nonzero elements are all identical and their value is chosen so that attractors $P_{c_0^0,\ldots,c_{d-1}^{d-1}} \equiv P_{\vec{c}}$ are properly normalized according to the Hilbert-Schmidt norm, *i.e.* $\|P_{\vec{c}}\| = 1$. Consequently, matrices $\{P_{\vec{c}}\}_{\vec{c}}$ form an orthonormal basis in the attractor space. Summation in (28) is done over all summation indices $0 \leq c_j^i \leq N$ satisfying additional condition $\sum_{i,j=0}^{d-1} c_j^i = N$. The d^2-tuple $\vec{c} = (c_0^0,\ldots,c_{d-1}^{d-1})$ corresponding to a given row multiindex i and a column multiindex j will be henceforth referred to as the *joint distribution of indices* for i and j. In subsection A.5 we show the explicit form of $P_{\vec{c}}$ and derive the form of its partial trace over a single qudit state space.

Attractors for $\lambda \neq 1$. Let us focus on the case with $\lambda \neq 1$ in this part. Since we assume $p \neq 0 \neq q$ equations (23) reduce to

$$X + \gamma\left(\mathrm{SW}_{ab} X - \lambda X \mathrm{SW}_{ab}\right) = 0 \tag{29}$$

with $\gamma = \frac{q}{p(1-\lambda)} \neq 0$. By use of the swap operation properties we can express (29) in the local index notation as

$$X^{i_a,i_b}_{j_a,j_b} + \gamma\left(X^{i_b,i_a}_{j_a,j_b} - \lambda X^{i_a,i_b}_{j_b,j_a}\right) = 0. \tag{30}$$

In order to find solution to (30) consider the following system of equations

$$\begin{aligned}
X^{a,b}_{c,d} + \gamma\left(X^{b,a}_{c,d} - \lambda X^{a,b}_{d,c}\right) &= 0,\\
X^{b,a}_{c,d} + \gamma\left(X^{a,b}_{c,d} - \lambda X^{b,a}_{d,c}\right) &= 0,\\
X^{a,b}_{d,c} + \gamma\left(X^{b,a}_{d,c} - \lambda X^{a,b}_{c,d}\right) &= 0,\\
X^{b,a}_{d,c} + \gamma\left(X^{a,b}_{d,c} - \lambda X^{b,a}_{c,d}\right) &= 0.
\end{aligned}$$

We can rewrite this system in the matrix form as

$$\begin{pmatrix} 1 & \gamma & -\lambda\gamma & 0 \\ \gamma & 1 & 0 & -\lambda\gamma \\ -\lambda\gamma & 0 & 1 & \gamma \\ 0 & -\lambda\gamma & \gamma & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0, \tag{31}$$

with

$$A = X^{a,b}_{c,d},\ B = X^{b,a}_{c,d},\ C = X^{a,b}_{d,c},\ D = X^{b,a}_{d,c}. \tag{32}$$

Unitarity conditions (21) guarantee $p \neq \pm q$ and the determinant of the matrix in (31) is therefore zero iff

$$p^2(\lambda-1)^2 - q^2(1+\lambda)^2 = 0. \tag{33}$$

Expressing the eigenvalue as $\lambda = e^{\mathrm{i}\omega}$ with $\omega \in [0, 2\pi)$ the latter condition can be rewritten as

$$q = \pm \mathrm{i}\, p \tan\left(\frac{\omega}{2}\right), \quad \textit{i.e.} \quad \pm 2\Delta t \equiv \omega \pmod{2\pi}. \tag{34}$$

We have found out that whenever (34) is *not* satisfied, all matrix elements (32) are zero. Since indices a through d were chosen arbitrarily we can conclude while (34) is not satisfied the attractor matrix X vanishes. Similarly, for $\lambda = -1$ equality (33) does not hold,

an associated matrix is thence nonsingular and the only possible solution to (31) is also made of zero elements. That is, for $\lambda = -1$ one obtains a zero attractor matrix.

Let us investigate the case for eigenvalues $\lambda \neq \pm 1$ when (34) *is* satisfied. Then relation (33) is apparently equivalent to $\gamma = \pm\frac{1}{1+\lambda}$ and the matrix in (31) reads

$$\frac{1}{1+\lambda}\begin{pmatrix} 1+\lambda & \pm 1 & \mp\lambda & 0 \\ \pm 1 & 1+\lambda & 0 & \mp\lambda \\ \mp\lambda & 0 & 1+\lambda & \pm 1 \\ 0 & \mp\lambda & \pm 1 & 1+\lambda \end{pmatrix}.$$

Corresponding kernels of the matrices above are for $\gamma = \frac{1}{1+\lambda}$ and $\gamma = -\frac{1}{1+\lambda}$ of the following form

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = t\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \tag{35}$$

respectively, where $t \in \mathbb{C}$. See (32) and suppose $a = b$ and $\gamma = \frac{1}{1+\lambda}$. From (35) one can easily see that all elements (32) are zero. Similarly for $c = d$ and $\gamma = -\frac{1}{1+\lambda}$. Therefore, if the qudit dimension d is strictly less than the number of qudits N then the entire matrix X of a possible attractor is zero. The reason implying this feature is simple. For $N > d$ there are at least two row indices and at least two column indices in a multiindex of any matrix element which are of the same value. For $\gamma = \pm\frac{1}{1+\lambda}$ given we can thus consider either of the two doubles of indices and apply arguments presented above.

For $N \leq d$ we obtain nonzero solutions. Let us focus on the case $\gamma = \frac{1}{1+\lambda}$ first. From (32) and (35) one can see that matrix elements which differ only by a permutation of their column indices have the same value. As a consequence, one can gather such elements into an equivalence class. Each class contains elements whose column indices are same up to some permutation. But there is one more constraint. Two elements whose multiindices differ by a transposition of two row indices are inverse to each other. In other words, all elements with a column multiindex fixed have the same value if their row multiindices are identical up to an even permutation. Elements whose row multiindices differ by an odd permutation have opposite values. The same analysis would emerge even for $\gamma = -\frac{1}{1+\lambda}$ with a single exception that the role of row and column multiindices is interchanged.

Let us count the number of independent elements in the attractor matrix X, *i.e.* the number of degrees of freedom present in the solution to (29) for $\gamma = \frac{1}{1+\lambda}$. As we saw in the previous paragraph, having particular matrix element X^r_s fixed, its value determines also the value of all other elements whose multiindices differ from r and s only by permutations. Such elements form an equivalence class. The number of degrees of freedom is thence equal to the number of all equivalence classes. This situation is quite similar to that for $\lambda = 1$. Consider the column indices first. We can form an N-element multiindex where each index assumes one of d values, *i.e.* there are $\binom{N+d-1}{N}$ non-equivalent column multiindices for given row multiindex. Regarding the row multiindex we have to take into account only those multiindices with no two indices identical. These multiindices correspond to nonzero matrix elements. Therefore, there is $\binom{d}{N}$ non-equivalent row multiindices. To conclude, the number of degrees of freedom for the non-unit eigenvalues $\lambda = \exp(\pm \mathrm{i}\, 2\Delta t)$, see (34), reads

$$S_{\neq 1}(N,d) = \binom{N+d-1}{N} \cdot \binom{d}{N}, \tag{36}$$

where N is a number of qudits and d is their dimension. The current degree-of-freedom analysis has been obviously performed for both $\gamma = \pm\frac{1}{1+\lambda}$. Since $\binom{d}{N} = 0$ whenever $d < N$, relation (36) holds for all possible values of d and N.

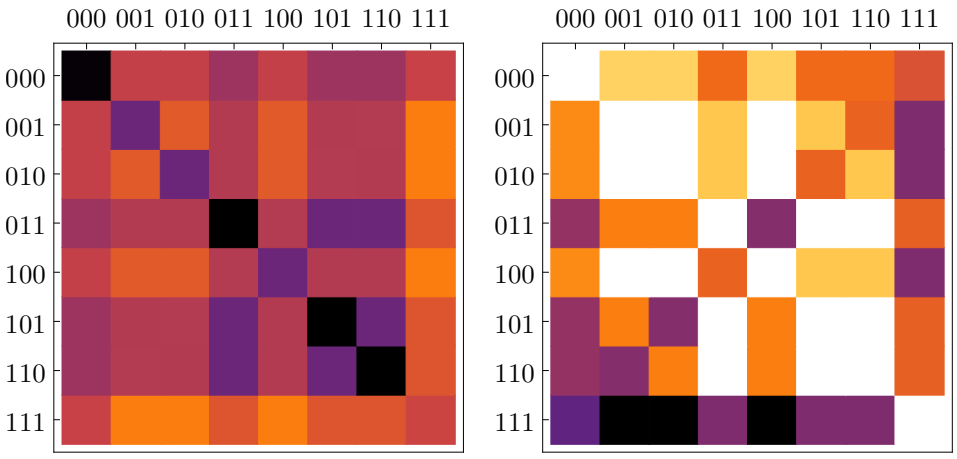

Figure 3: Numerical calculation of the equilibrium state for the quantum network composed of $N = 3$ qubits and governed by partial swap interactions and the trivial free Hamiltonian. The left and right plots depict the real and imaginary parts of the equilibrium state, respectively. Initial state was chosen randomly. Colours represent numerical values of the matrix entries, since values themselves are irrelevant in the present discussion. Apparently, entries whose multiindices belong to the same equivalence class share the same colour. Row and column multiindices are shown explicitly in both plots.

4.1.2 Non-Identical Interaction Times

So far we have assumed the collision times are identical for all pairs of qudits. On the other hand, if Δt_{ab} vary among different pairs, obtained results do not change much. For $\lambda = 1$ the only assumption in the above calculations was $q \neq 0$. It is satisfied if and only if $\Delta t \neq k\pi$ for any $k \in \mathbb{Z}$. (Otherwise the attractor equations (23) trivially hold.) The same result (28) is obtained whenever $\Delta t_{ab} \neq k\pi$ for all pairs (a, b) and any $k \in \mathbb{Z}$.

Similarly, for $\lambda \neq \pm 1$ we assumed $\Delta t \neq k\pi/2$, $k \in \mathbb{Z}$, so that $q \neq 0 \neq p$. In this case the situation is slightly more complicated due to explicit dependence of attractor solutions on the eigenvalue λ. Nevertheless, even for arbitrary times Δt_{ab} there is only a trivial solution $X = 0$ for $N > d$. One could discuss all the remaining possibilities for diverse values of Δt_{ab}. In general, the solution would be also zero though. A non-trivial solution is obtained only for a zero-measure set of parameter values. For a detailed discussion of these solutions see Appendix B.

4.1.3 Conclusion for Pure Collisions

In the last subsection we analyzed the asymptotic regime of a quantum network governed by the trivial free evolution interrupted occasionally by a partial-swap-type interactions between its constituents. Provided that all interaction times Δt_{ab} are identical, $\Delta t_{ab} = \Delta t$, we saw that any sufficiently large quantum network tends towards an equilibrium. More precisely, the network equilibrates whenever the number of qudits N exceeds their dimensionality d. This result holds for an arbitrarily large or small values of the nonzero interaction times Δt. In terms of attractor theory, in such a case there remain only attractors for $\lambda = 1$ generating $\binom{N+d^2-1}{N}$-dimensional space (27) whose orthonormal basis can be chosen as $\{P_{\vec{c}}\}_{\vec{c}}$, see (28). In Figure 3 one can see a numerically calculated example of a three-qubit equilibrium state (in this case $N > d$, so there is only a stationary part of the

asymptotic evolution). Colours are chosen instead of numerical values for better visibility of the state structure. Elements with the same colour belong to the same equivalence class as is discussed in subsubsection 4.1.1.

Due to high symmetry of the partial swap Hamiltonian there are many integrals of motion present in the quantum network evolution and the equilibrium state retains significant amount of information about the initial conditions. Equilibrium states are invariant under permutations of individual qudits. This important observation implies that all subsystems of the same size are described in equilibrium by the same density matrix, as follows from partial trace properties summarized in subsection 2.1. All such subsystems therefore share the same information about the initial state.

For small quantum networks one discovers also non-trivial attractors associated with eigenvalues $\lambda = e^{\pm 2\mathrm{i}\,\Delta t}$. For initial states having an overlap with these attractors there thus arise a non-stationary part of the asymptotic evolution preventing the network from formation of equilibria. Dimension of the corresponding attractor subspace is equal to $S_{\neq 1}(N,d)$ (36). Attractors themselves are antisymmetric under permutations of individual qudits, as follows from discussion preceding the derivation of (36).

Let us sum up our findings when we relax original constraint and enable distinct pairs (a,b) to have different values of interaction times Δt_{ab}. In such a case all the attractors for $\lambda = 1$ remain in the attractor space and the non-stationary part of the asymptotic evolution vanishes. This conclusion is valid for almost all values of interaction times Δt_{ab}. The set of exceptional parameter values excluded from our discussion has a zero measure and thence does not affect a typical behaviour of the network. However, it turns out that attractors (28) associated with $\lambda = 1$ are present in attractor spaces even for regimes when parameters take on such exceptional values. In this sense, equilibrium states of the form (28) constitute a very generic set of asymptotic regimes. Indeed, in our calculations the exceptional values of parameters were peculiar due to the fact, that fewer constraints were imposed on the asymptotic state of the network. Attractors for non-exceptional values of parameters thus emerge also in the attractor spaces for exceptional values of parameters.

As follows from the attractor theory, the resulting network state is completely independent of the probabilities with which each qudit collides with any other qudit, supposing that these probabilities do not vanish. We augment our analytical treatment of the quantum network evolution under the influence of partial-swap interactions by numerical simulations in section 7. It will allow us to study even more general settings with interaction times and collision probabilities for each pair being different in each step of the evolution.

4.2 Composite Evolution

In this subsection we investigate how the non-trivial free evolution affects the asymptotic behaviour governed by partial swap collisions. We have dealt with a trivial free evolution so far, let the free evolution $U^{\mathrm{free}} = U^{\mathrm{free}}(\Delta t)$ be arbitrary in the present case and let collision times Δt_{ab} be already different for each pair, in general. As emphasized in subsection 3.3 the interaction Hamiltonians commuting with the free evolution Hamiltonians make the determination of the asymptotic behaviour much easier. Assuming there might be also no collision present in a time interval Δt, *i.e.* $p_0 \neq 0$ (10), the attractor equations decouple into (18)

$$U^{\mathrm{free}}\, X\, (U^{\mathrm{free}})^{\dagger} = \lambda X, \quad V_{ab}^{\dagger}\, X\, V_{ab} = X, \quad \forall a,b \in \{1,\ldots,N\}, \quad a \neq b, \tag{37}$$

where $V_{ab} = \exp(\pm\mathrm{i}\,\Delta t_{ab}\,\mathrm{SW}_{ab})$ and we still keep $\phi_{ab} = 1$ without loss of generality, see the introductory part of section 4. The second set of equations above has been already treated in subsubsection 4.1.1 and obtained solutions are summarized in subsubsection 4.1.3. To

get rid of special non-interacting cases when $\mathrm{PSW}_{ab} \propto \mathbb{I}$, let the collision times satisfy $\Delta t_{ab} \neq k\pi$, $k \in \mathbb{Z}$. It is important to note that so far our calculations have been done with no specification of the computational basis. The invariance under permutations (25) of the attractor multiindices is thus preserved in every local basis. As a consequence, in every local basis the solution to the second system of equations in (37) may be expressed in the form (28).

At this point we just plug the attractor (28) into attractor equations for purely free evolution, *i.e.* into the first system of equations in (37). For convenience, let us treat this system of equations in the eigenbasis of $U^{\mathrm{free}} \equiv \bigotimes_i U_i$. We do not impose any special requirements on the one-qudit free evolution operator U_i, let $\{\epsilon_l\}_l$ be energy levels of a corresponding free Hamiltonian. Then $U^{\mathrm{free}}(\Delta t) = \sum_m \exp(\mathrm{i}\,\Delta t \sum_k \epsilon_{m_k}) |m_1 \dots m_N\rangle\langle m_1 \dots m_N|$. From this relation it follows that

$$U^{\mathrm{free}} |i_1 \dots i_N\rangle\langle j_1 \dots j_N| \, (U^{\mathrm{free}})^\dagger = \exp\left(\mathrm{i}\,\Delta t \sum_k (\epsilon_{i_k} - \epsilon_{j_k})\right) |i_1 \dots i_N\rangle\langle j_1 \dots j_N|$$

for any fixed multiindices $i = (i_1, \dots, i_N)$ and $j = (j_1, \dots, j_N)$. Let a joint distribution of indices for i and j be equal to $\vec{c}$, as defined in (28). Since the exponent in the equation above is identical for all permutations of multiindices we see that

$$U^{\mathrm{free}} P_{\vec{c}} \, (U^{\mathrm{free}})^\dagger = \lambda_{\vec{c}} P_{\vec{c}}, \quad \text{where} \quad \lambda_{\vec{c}} \equiv \exp\left(\mathrm{i}\,\Delta t \sum_k (\epsilon_{i_k} - \epsilon_{j_k})\right). \tag{38}$$

By comparison of (38) with the first set of equations in (37) we conclude that $P_{\vec{c}}$ solves (37) for $\lambda = \lambda_{\vec{c}}$. Put another way, the free evolution only redistributes attractors $P_{\vec{c}}$ to different eigenvalues $\lambda_{\vec{c}}$. For each of these eigenvalues the associated eigenvector is $P_{\vec{c}}$ and $\{P_{\vec{c}}\}_{\vec{c}}$ can be again used as the orthonormal basis of the attractor space. Except for $P_{\vec{c}}$ matrices there are no other linearly independent attractors. If there were such attractors, they would have to solve the second set of equations in (37). However, as we saw, solutions to these equations can be written in terms of $P_{\vec{c}}$ matrices.

Although the exact asymptotic regime of the network is directly dependent on the spectrum of the free evolution Hamiltonian, some generally valid features may be found. From the expression (38) for $\lambda_{\vec{c}}$ it is clear that eigenvalue $\lambda = 1$ is always present in the attractor spectrum. Indeed, whenever multiindex i and multiindex j differ only by a permutation, the sum in (38) vanishes. Such multiindices share the same distribution of individual indices. This observation can be rephrased in terms of formula (28) as $\sum_k c_m^k = \sum_l c_l^m$ for all $0 \leq m \leq d-1$. The attractor space for $\lambda = 1$ thus always contains attractors of the form

$$X = \sum_{(c_0^0, \dots, c_{d-1}^{d-1}) \in \mathcal{C}} \alpha_{c_0^0, \dots, c_{d-1}^{d-1}} P_{c_0^0, \dots, c_{d-1}^{d-1}}, \tag{39}$$

where $\mathcal{C} = \{(c_0^0, \dots, c_{d-1}^{d-1}) \in \{0, \dots, N\}^{d^2} \,|\, \sum_{k,l} c_l^k = N \wedge \sum_k c_m^k = \sum_l c_l^m, \forall m \in \{0, \dots, d-1\}\}$, α's are coefficients and for P's see (28). Attractors (39) lie in the attractor space associated with $\lambda = 1$ for an arbitrary choice of the energy spectrum $\{\epsilon_l\}_l$ of the free evolution Hamiltonian. In this sense we may say that these attractors form a minimal attractor space for $\lambda = 1$. The number and values of other eigenvalues $\lambda \neq 1$ are determined by a particular free evolution Hamiltonian via equality (38).

Let us find out when the quantum network in the present setup equilibrates. We restrict ourselves to finding only those cases when the equilibration is guaranteed for an arbitrary initial state of the network. Such a situation happens whenever the sum in (38) vanishes for all multiindices i and j. This condition can be expressed as $\sum_{k=0}^{d-1} \epsilon_k (g_k^i - g_k^j) = 0$, where g_k^i is defined as the number of those indices in a multiindex i, which take on value

k (and analogously for multiindex j). Suppose the multiindices i and j are of the forms $(N,0,0,\ldots,0)$ and $(0,N,0,\ldots,0)$, respectively. Then condition $\sum_{k=0}^{d-1}\epsilon_k(g_k^i-g_k^j)=0$ implies $\epsilon_0=\epsilon_1$. By analogous calculations we see the only free evolution Hamiltonians, for which the network definitely equilibrates, are proportional to the identity matrix. Only when there is a single energy level for the free evolution, the network tends toward equilibrium for any initial state.

4.2.1 Conclusion for Composite Evolution

At this point we complete the general discussion on the quantum network evolution generated by an arbitrary free Hamiltonian and partial-swap interactions with various collision times. Many features valid for pure collisions remain valid also in this case. The dimension of the attractor space is equal to $\binom{N+d^2-1}{N}$ (27), as in the trivial-free-evolution setup. The number of degrees of freedom surviving the long-time evolution thus scales with the size of the network and significant amount of information about the initial state is present even in the asymptotic regime. Asymptotic states are still invariant under permutations of individual qudits in the network. Consequently, the subsystems with the same number of qudits have the same density operators and these are also invariant under permutations of individual qudits, see subsection 2.1. As in the trivial-free-evolution regime, the structure of asymptotic states is captured by $P_{\vec{c}}$ matrices (28) and can be therefore seen in Figure 3. For a non-trivial free evolution the matrices $P_{\vec{c}}$ no longer correspond to the unit eigenvalue, in general. Nevertheless, the unit eigenvalue is always present in the attractor spectrum and the corresponding attractor space always contains attractors (39), regardless of the specific energy spectrum of the free Hamiltonian.

Unlike the previous case with the trivial free evolution, in the present setting the quantum network (with a general initial state) does not equilibrate for any number of qudits $N\geq 2$ with arbitrary dimensionality $d\geq 2$. The non-trivial free evolution thus modifies the previous case with the trivial free Hamiltonian tremendously, as far as the form of the asymptotic regime is taken into account. Even though the interactions between qudits suppress the non-stationary behaviour of the network, the free evolution Hamiltonians of individual qudits do not allow the network to attain the equilibrium.

Despite the fact that the entire system does not equilibrate, one might ask a question, whether the equilibration emerges at least in some subsystems. Nevertheless, numerical simulations suggest that for partial swap collisions and randomly chosen free Hamiltonian and initial state, not only the whole network, but also its subsystems do not tend to equilibrium. This property can be also seen from our analytical findings. A general form of the asymptotic state in the present setup reads (9), where roles of $Y_{\lambda,i}$ are played by $P_{\vec{c}}$ matrices associated with different eigenvalues λ. In order for the network subsystem to equilibrate, all the numbers c_l^l in the joint distribution $\vec{c}$ must be equal to zero for all $\lambda\neq 1$, as follows from the partial trace formula (121). However, this condition is in general never satisfied. Consider a row multiindex $i=(0,0,\ldots,0,0)$ and a column multiindex $j=(0,0,\ldots,0,1)$, so $c_0^0=N-1$. Multiindex j cannot be transformed into multiindex i by any permutation and their joint distribution $\vec{c}$ thus not lies in set $\mathcal{C}$ defined in (39). Attractor $P_{\vec{c}}$ is therefore associated with a generally non-unit eigenvalue λ (38). Since the partial trace over a single-qudit subsystem decreases the value of c_0^0 by one we see that we must trace over $(N-1)$-qudit subsystem in order to obtain $c_0^0=0$. However, a tracing-out of the $(N-1)$-qudit subsystem leaves us only a single-qudit state. Even this state is in general non-stationary. Although the subsystems do not equilibrate, their evolution gets synchronized after long enough time. All subsystems with the same number of qudits have the identical non-stationary asymptotic behaviour.

In our discussion we excluded a zero-measure set of collision times and the trivial class of free Hamiltonians, which are proportional to the identity matrix. Nonetheless, from the viewpoint of equilibration, exclusion of such collision times was not necessary as even for these values the quantum network still exhibits a non-stationary asymptotic behaviour.

5 Energy Exchange Interaction

As a transition from the partial swap commuting with any free Hamiltonian to the CNOT not commuting with any non-trivial free Hamiltonian (see subsection A.3), in this section we study the *energy exchange interaction*. This interaction is similar in its form to the former, but its commutation relations are very limited, which is the property shared with the latter. Throughout the section we consider qubit networks only. We define the energy exchange interaction Hamiltonian as

$$H_{\mathrm{EX}_{ab}} = \frac{\kappa_{ab}}{4}(\sigma_+^{(a)} \otimes \sigma_-^{(b)} + \sigma_-^{(a)} \otimes \sigma_+^{(b)}), \tag{40}$$

where $\kappa_{ab} \in \mathbb{R}$ is the interaction strength and $\sigma_\pm = \sigma_X \pm \mathrm{i}\,\sigma_Y$, see subsection A.2. The name of this operation is motivated by an observation that it can be thought of as a Hamiltonian responsible for transfer of an excitation from qubit a to qubit b, and vice versa. Such a viewpoint is justified whenever the energy of the system is captured by a diagonal free Hamiltonian, which will be indeed the case in our future discussion. From the mathematical point of view, the energy exchange interaction is to some extent similar to the partial swap interaction discussed in the previous chapter. This statement follows from the fact, that the Hamiltonian (40) can be rewritten in the form

$$H_{\mathrm{EX}_{ab}} = \kappa_{ab}\,(|0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1|) = \kappa_{ab}\,\mathrm{SW}_{ab} - \kappa_{ab}\,D_{ab}, \tag{41}$$

where $D_{ab} = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|$. In a subsequent discussion we take into consideration this form of the interaction Hamiltonian in addition to the trivial free Hamiltonian $H^{\mathrm{free}} = 0$. Having obtained results for such an evolution governed merely by pure collisions, we augment our model by considering a non-trivial diagonal free Hamiltonian later on. As the most general form of a one-qubit diagonal Hamiltonian it suffices to choose $H_i^{\mathrm{free}} \propto \sigma_Z$.

5.1 Pure Collisions

Before considering more complex scenarios, let us focus on the quantum network whose evolution is governed by mutual interactions between its constituents with the trivial individual free evolution. In subsection A.3 one can find derivation of the evolution operator associated with the energy exchange interaction and the trivial free evolution

$$\mathrm{EX}_{ab}(\Delta t_{ab}) = \mathrm{PSW}_{ab}|_{\rho\to 0,\phi\to\Delta t_{ab}\,\kappa_{ab}} + (1 - e^{\mathrm{i}\,\Delta t_{ab}\,\kappa_{ab}})\,D_{ab}. \tag{42}$$

We demand $\Delta t_{ab}\,\kappa_{ab} \not\equiv 0 \pmod{2\pi}$ to exclude cases when $\mathrm{EX}_{ab} = \mathrm{PSW}_{ab}$ covered in section 4. Since quantities Δt_{ab} and κ_{ab} always appear together as a product $\Delta t_{ab}\kappa_{ab}$ in the formula above, without loss of generality we can put $\kappa_{ab} = 1$. Initially, we suppose all interaction times are identical, $\Delta t_{ab} = \Delta t$.

5.1.1 Identical Interaction Times

As was already mentioned, discussion for the energy exchange interaction will turn out to be similar to that for the partial swap. Attractor equations (19) assume the form

$$p(1-\lambda)X + (1-p-q)(D_{ab}X - \lambda X D_{ab}) + q(\mathrm{SW}_{ab}X - \lambda X \mathrm{SW}_{ab}) = 0, \tag{43}$$

where $p \equiv \cos(\Delta t)$ and $q \equiv \mathrm{i}\sin(\Delta t)$. It is convenient to solve these equations in their index-explicit form since mapping D_{ab} affects only such elements of X for which $i_a = i_b$ or $j_a = j_b$

$$\left(p(1-\lambda) + (1-p-q)(\delta_{i_b}^{i_a} - \lambda\delta_{j_b}^{j_a})\right) X_{j_a,j_b}^{i_a,i_b} + q\left(X_{j_a,j_b}^{i_b,i_a} - \lambda X_{j_b,j_a}^{i_a,i_b}\right) = 0. \tag{44}$$

The index-form attractor equations for elements $X^{i_a,i_b}_{j_a,j_b}$ with $i_a \neq i_b$ or $j_a \neq j_b$ reduce to equations for the mere partial swap interaction (23). Our discussion for the energy exchange interaction thus identifies four situations as follows:

- $\boldsymbol{i_a \neq i_b}$, $\boldsymbol{j_a \neq j_b}$ – Equations (44) reduce to (23), for solution see subsubsection 4.1.3. Specifically, for $\lambda = 1$ we obtain $X^{i_a,i_b}_{j_a,j_b} = X^{i_b,i_a}_{j_b,j_a}$, for $\lambda = \exp(\pm \mathrm{i}\, 2\Delta t)$ it holds that $-X^{0,1}_{0,1} = \pm X^{1,0}_{0,1} = \mp X^{0,1}_{1,0} = X^{1,0}_{1,0}$ and for $\lambda \neq \exp(\pm \mathrm{i}\, 2\Delta t) \wedge \lambda \neq 1$ all relevant matrix elements are zero. In analogous manner to subsubsection 4.1.3 here we assumed $\Delta t \neq k\pi/2$, and we keep this assumption in effect from now on. For $\Delta t = k\pi$ the partial swap reduces to the multiple of the identity and for $\Delta t = (2k+1)\pi/2$ it reduces to the ordinary swap operation.

- $\boldsymbol{i_a \neq i_b}$, $\boldsymbol{j_a = j_b}$ – Equations reduce to $(\lambda - p)X^{i_a,i_b}_{j_a,j_a} = qX^{i_b,i_a}_{j_a,j_a}$. This relation must be valid for both situations, $(i_a, i_b) = (0,1)$ and $(i_a, i_b) = (1,0)$. We thus obtain a system of two equations for two variables in the form $\zeta X^{1,0}_{j_a,j_a} = X^{0,1}_{j_a,j_a}$ and $\zeta X^{0,1}_{j_a,j_a} = X^{1,0}_{j_a,j_a}$ with $\zeta \equiv (\lambda - p)/q$. So $\zeta^2 X^{0,1}_{j_a,j_a} = \zeta X^{1,0}_{j_a,j_a} = X^{0,1}_{j_a,j_a}$ and we see that for $\zeta \neq \pm 1$ we get $X^{0,1}_{j_a,j_a} = 0 = X^{1,0}_{j_a,j_a}$ and for $\zeta = \pm 1$, *i.e.* $\lambda = \exp(\pm \mathrm{i}\, \Delta t)$, there is $X^{1,0}_{j_a,j_a} = \pm X^{0,1}_{j_a,j_a}$, respectively.

- $\boldsymbol{i_a = i_b}$, $\boldsymbol{j_a \neq j_b}$ – Analogously to the previous case the equations above simplify into $(\lambda^\star - p)X^{i_a,i_a}_{j_a,j_b} = qX^{i_a,i_a}_{j_b,j_a}$. By following the discussion in the previous bullet we conclude that for $\lambda \neq \exp(\pm \mathrm{i}\, \Delta t)$ we get $X^{i_a,i_a}_{0,1} = 0 = X^{i_a,i_a}_{1,0}$ and for $\lambda = \exp(\pm \mathrm{i}\, \Delta t)$ there is $X^{i_a,i_a}_{1,0} = \mp X^{i_a,i_a}_{0,1}$, respectively.

- $\boldsymbol{i_a = i_b}$, $\boldsymbol{j_a = j_b}$ – Equations (44) assume the form $(1 - \lambda)X^{i_a,i_a}_{j_a,j_a} = 0$, *i.e.* for $\lambda = 1$ the matrix element is arbitrary, otherwise it vanishes.

Let us review our findings and put them in a more compact form. We assume inequality $\Delta t \neq k\pi/2$, $k \in \mathbb{Z}$, holds throughout. This condition ensures that $p \neq 0 \neq q$ and we already used it above.

Attractors for $\boldsymbol{\lambda = 1}$. From the first bullet we get invariance under permutations of individual qubits $X^{i_a,i_b}_{j_a,j_b} = X^{i_b,i_a}_{j_b,j_a}$. All the nonzero elements are thus distributed over equivalence classes gathering elements whose multiindices differ only by a permutation. According to the discussion for the partial swap, attractors associated with $\lambda = 1$ are of the form (28). The fourth bullet tells us that there are no more constraints imposed on elements $X^{i_a,i_a}_{j_a,j_a}$ in accordance with invariance under permutations. The second and third bullets imply

$$X^{i_a,i_a}_{0,1} = X^{i_a,i_a}_{1,0} = X^{0,1}_{j_a,j_a} = X^{1,0}_{j_a,j_a} = 0.$$

To put it another way, if in multiindices for a given matrix element there is at least one local index of the form $\binom{0}{0}$ or $\binom{1}{1}$ and simultaneously at least one local index of the form $\binom{0}{1}$ or $\binom{1}{0}$, then the matrix element vanishes. This restriction allows for two sets of nonzero elements only. The first set consists of diagonal entries with all local indices of the form $\binom{0}{0}$ or $\binom{1}{1}$. The second set comprises all anti-diagonal entries. These are characterized by having all local indices of the form $\binom{0}{1}$ or $\binom{1}{0}$. There are apparently 2^N diagonal elements collected in $N+1$ equivalence classes and 2^N anti-diagonal elements forming another $N+1$ equivalence classes.

Attractors for $\lambda \neq 1$. Apart from a unit eigenvalue there are only four additional eigenvalues $\lambda = \exp(\pm \mathrm{i}\,\Delta t)$ and $\lambda = \exp(\pm \mathrm{i}\,2\Delta t)$; for any other λ the resulting attractor is zero. Assumptions $\Delta t \neq k\pi/2$ and $\Delta t \neq 2k\pi/3$, $k \in \mathbb{Z}$, ensure that these four eigenvalues are mutually exclusive. As a consequence, the determination of the associated attractors is easy. In short, for $\lambda = \exp(\pm \mathrm{i}\,2\Delta t)$ we obtain

$$-X^{0,1}_{0,1} = \pm X^{1,0}_{0,1} = \mp X^{0,1}_{1,0} = X^{1,0}_{1,0}, \tag{45}$$

respectively, see the first bullet. The second and third bullets imply that for $\lambda = \exp(\pm \mathrm{i}\,\Delta t)$ we have

$$X^{1,0}_{0,0} = \pm X^{0,1}_{0,0}, \quad X^{1,0}_{1,1} = \pm X^{0,1}_{1,1}, \quad X^{0,0}_{1,0} = \mp X^{0,0}_{0,1}, \quad X^{1,1}_{1,0} = \mp X^{1,1}_{0,1}. \tag{46}$$

Values not explicitly shown are zero. Even though it may seem there are non-trivial solutions even for $\lambda \neq 1$, it is true only for $N = 2$. As follows from the fourth bullet, for all $\lambda \neq 1$ we have $X^{i_a,i_a}_{j_a,j_a} = 0$, *i.e.* if there are two identical local indices in the multiindex for a given matrix entry, then this entry vanishes. For $N \geq 3$ there are always at least two local indices of the same form so all entries are zero and we may conclude that for $N \geq 3$ there is no attractor associated with eigenvalue $\lambda \neq 1$.

5.1.2 Non-Identical Interaction Times

This subsection is devoted to perform calculations analogous to that above, but for more general setting when interaction times are not identical for all pairs of qubits. It is not difficult to see that there is in fact no significant modification with respect to the preceding situation at all. From discussion above it is enough to assume $\Delta t_{ab} \neq k\pi/2$ and $\Delta t_{ab} \neq 2k\pi/3$, $k \in \mathbb{Z}$, for all pairs (a, b) to obtain completely identical solution to that above, at least for $N \geq 3$. Moreover, if we assume $\Delta t_{ab} \neq \pm\Delta t_{ab} + k\pi$, $k \in \mathbb{Z}$, then there are no non-trivial attractors for $\lambda \neq 1$ even for $N = 2$, which was not the case with $\Delta t_{ab} = \Delta t$.

5.1.3 Conclusion for Pure Collisions

Calculations presented for the energy-exchange-driven quantum network with the trivial free evolution relied heavily on a discussion made for the partial swap scenario. Indeed, some properties have both interactions in common. One of them being that the network equilibrates whenever $N > d = 2$. We came to this conclusion by reasoning similar to that we employed in the partial-swap case to conclude the network equilibrates whenever $N > d$. Another significant feature of equilibrium states resulting from energy exchange interactions is their invariance under permutations of individual qubits. That is, even in this case we obtain the same density matrices for all subsystems containing the same number of qubits, see subsection 2.1. Similarly to the partial swap the equilibrium state may be expressed in the basis of $P_{\vec{c}}$ matrices (28). (Note that for qubit networks the equivalence-class characterization is provided by a four-tuple $(c^0_0, c^0_1, c^1_0, c^1_1)$ and $P_{\vec{c}}$ matrices have thus four parameters.) Explicitly

$$X = \sum_{k=0}^{N} \alpha_{k,0,0,N-k}\, P_{k,0,0,N-k} + \sum_{k=0}^{N} \alpha_{0,k,N-k,0}\, P_{0,k,N-k,0}, \tag{47}$$

where α's are corresponding coordinates of the state X in the orthonormal basis $\{P_{\vec{c}}\}_{\vec{c}}$. Apparently, the energy exchange interaction imposes more constraints on the asymptotic regime of the network evolution. The only nonvanishing matrix entries are those belonging to the diagonal and anti-diagonal of the matrix, for illustration see Figure 4. There are thus $2(N + 1)$ degrees of freedom, much less than we saw in section 4.

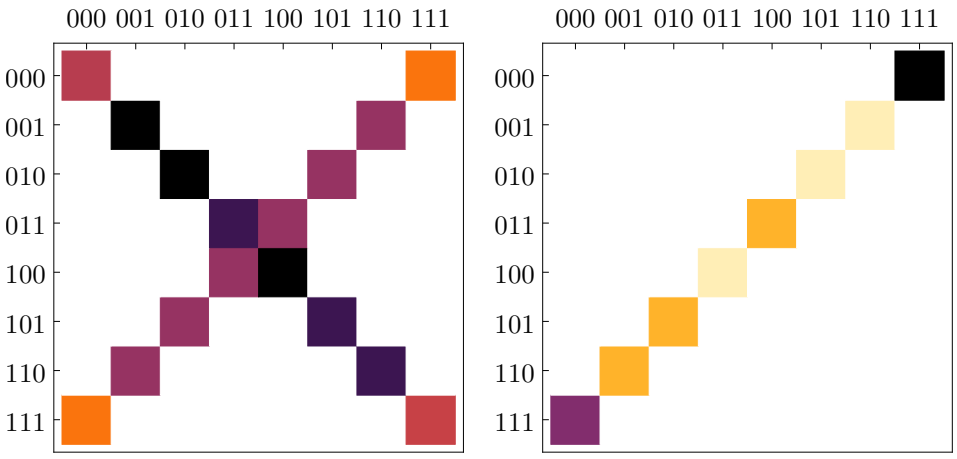

Figure 4: Numerical calculation of the equilibrium state for the quantum network composed of $N = 3$ qubits and governed by energy exchange interactions and the trivial free Hamiltonian. Interaction times Δt_{ab} are different for each pair of qubits and the initial state is chosen randomly. The left and right plots depict the real and imaginary parts of the equilibrium state, respectively. Colours represent numerical values of the matrix entries to emphasize the structure of the equilibrium state. Row and column multiindices are shown explicitly in both plots. Invariance of matrix entries under permutations can be seen by comparing indices associated with elements sharing the same colour.

The above conclusions are valid for both the identical and non-identical collision times. In our discussion we also excluded a zero-measure set of exceptional values of parameters, which enabled us to simplify our computation with effectively no loss of generality. Nonetheless, apart from this zero-measure set the interaction times may have arbitrary value. Even for immensely small values of these parameters the asymptotic behaviour of the network tends to the state (47), after sufficiently long time. In section 7 we demonstrate how the time-scale of such a convergence to the equilibrium state is influenced by small interaction times.

5.2 Composite Evolution

We have investigated asymptotic evolution of the quantum network when only pure collisions generate its dynamics. In this subsection, let us take a non-trivial free evolution of individual qubits into account. Specifically, we study the case when $H_i^{\text{free}} = s\sigma_Z$ for simplicity, since according to subsection A.3 this Hamiltonian commutes with the interaction Hamiltonian and renders calculations easier to handle. Furthermore, we already assume interaction times and interaction strengths are no longer identical, in general.

Suppose for a while that for some time intervals Δt in the network evolution no collision occurs. In other words, $p_0 \neq 0$ in (10). The attractor equations then decouple into (18)

$$U^{\text{free}}(\Delta t)\, X\, (U^{\text{free}})^{\dagger}(\Delta t) = \lambda\, X, \qquad \text{EX}^{\dagger}_{ab}(\Delta t_{ab})\, X\, \text{EX}_{ab}(\Delta t_{ab}) = X, \tag{48}$$

which must hold for all pairs of qubits (a, b). This situation is analogous to the non-trivial free evolution scenario we study in the context of partial swap interactions in section 4. The second system of equations above is already solved. Its solution can be seen in (47). To complete our calculation it suffices to plug this solution into the first system of equations

above, where $U^{\text{free}} = U^{\text{free}}(\Delta t)$ is now a diagonal matrix. Thanks to this fact the left-hand side of the first system of equations in (48) simplifies in an explicit index notation into

$$(U^{\text{free}}\, X\, (U^{\text{free}})^\dagger)^{i_1\ldots i_N}_{j_1\ldots j_N} = r_{ij}\, \lambda\, X^{i_1\ldots i_N}_{j_1\ldots j_N} \tag{49}$$

with $\lambda = \exp(\mathrm{i}\,\omega)$ and

$$r_{ij} = \exp(\mathrm{i}\,\alpha_{ij}), \quad \alpha_{ij} = 2\,s\,\Delta t\,\Sigma_{ij} - \omega, \quad \Sigma_{ij} = \sum_k (j_k - i_k).$$

The free evolution equations in (48) thus reduce in the index notation into $(1-r_{ij})X^{i_a,i_b}_{j_a,j_b} = 0$ which is apparently satisfied whenever $\omega \equiv 2\,s\Delta t\,\Sigma_{ij}$. Let us assume $s\Delta t/\pi \notin \mathbb{Q}$ so that ω is uniquely determined by a single value of s, Δt and Σ_{ij}. Non-vanishing elements of the solution (47) for the trivial free evolution case are collected into two sets—the diagonal and the anti-diagonal. Multiindices of entries from the diagonal apparently satisfy $\Sigma_{ij} = 0$, *i.e.* $\omega = 0$, and there are thus at least $N+1$ linearly independent attractors associated with $\lambda = 1$, see discussion for $\lambda = 1$ in subsubsection 5.1.1. It is not hard to see that for matrices $P_{0,k,N-k,0}$ (47), composing the anti-diagonal, we have $\Sigma_{ij} = 2k - N$, where $0 \le k \le N$. If the number of qubits is even and $k = N/2$, it holds that $\Sigma_{ij} = 0$ and there is thus one more linearly independent attractor associated with $\lambda = 1$. Other values of k lead to emergence of attractors for eigenvalues $\lambda \neq 1$. The first system of equations in (48) therefore only redistributes solutions (47) to different eigenvalues λ. We came to the same conclusion even in the partial-swap case in subsection 4.2.

We have just finished discussion for $p_0 \neq 0$ (10). Let us assume the opposite, *i.e.* there is always a collision occurring in each time interval Δt. Attractor equations (17) in this case read

$$U^{\text{free}}\, X\, (U^{\text{free}})^\dagger = \lambda\, \mathrm{EX}_{ab}{}^\dagger(\Delta t_{ab})\, X\, \mathrm{EX}_{ab}(\Delta t_{ab}), \tag{50}$$

where the left-hand side can be rewritten in the index notation into (49). In subsection 5.1 we put $\kappa_{ab} = 1$ to reduce the number of parameters entering our discussion while keeping the full generality of our results. We keep this simplification in effect even for the subsequent calculations. The right-hand side of (50) is expanded like $\mathrm{PSW}^\dagger X \mathrm{PSW} + (1-p-q)\mathrm{PSW}^\dagger X D + (1-p+q) D^\dagger X \mathrm{PSW} + 2(1-p) D X D$ with $p = \cos(\Delta t_{ab})$, $q = \mathrm{i}\,\sin(\Delta t_{ab})$. When we plug these expressions back into (50), we end up with index-form attractor equations as follows

$$\begin{aligned} 0 = {} & X^{i_a,i_b}_{j_a,j_b}\Big(-r_{ij} + p(p + (1-p+q)\delta^{i_a}_{i_b} + (1-p-q)\delta^{j_a}_{j_b})\Big) + pq\Big(X^{i_a,i_b}_{j_b,j_a} - X^{i_b,i_a}_{j_a,j_b}\Big) \\ & + X^{i_b,i_a}_{j_b,j_a}\Big(2(1-p)\delta^{i_a}_{i_b}\delta^{j_a}_{j_b} - q(q - (1-p+q)\delta^{i_a}_{i_b} + (1-p-q)\delta^{j_a}_{j_b})\Big). \end{aligned} \tag{51}$$

Similarly to subsubsection 5.1.1 we discuss four distinct situations when it comes to values of row and column indices:

- $\boldsymbol{i_a \neq i_b}$, $\boldsymbol{j_a \neq j_b}$ – In (23) we considered attractor equations in the suitable form $\mathrm{PSW}\, X = \lambda X\, \mathrm{PSW}$. Here, on the contrary, we have to keep them in their original form $\mathrm{PSW}\, X\, \mathrm{PSW}^\dagger = \lambda X$, which explicitly reads $(p^2 - \lambda)X + pq(\mathrm{SW}\, X - X\, \mathrm{SW}) - q^2 \mathrm{SW}\, X\, \mathrm{SW} = 0$. In the index notation this formula is identical to equation (51) up to substitution of λ by r_{ij}. The solution to these equations can be therefore found in subsubsection 4.1.3. Namely, for $\alpha_{ij} \equiv 0 \pmod{2\pi}$ we get $X^{i_a,i_b}_{j_a,j_b} = X^{i_b,i_a}_{j_b,j_a}$ for given pair (a,b) and multiindices i and j. For two special values $\alpha_{ij} \equiv \pm 2\Delta t_{ab} \pmod{2\pi}$ the elements satisfy $-X^{0,1}_{0,1} = \pm X^{1,0}_{0,1} = \mp X^{0,1}_{1,0} = X^{1,0}_{1,0}$ and for all other values of α_{ij} these elements go to zero.

- $\boldsymbol{i_a \neq i_b}$, $\boldsymbol{j_a = j_b}$ – For this choice of parameters the equations (51) reduce to simple equality $(p - r_{ij})X^{i_a,i_b}_{j_a,j_a} = qX^{i_b,i_a}_{j_a,j_a}$. By analogous calculation to that in the second bullet in subsubsection 5.1.1 we obtain $X^{0,1}_{j_a,j_a} = \mp X^{1,0}_{j_a,j_a}$ for $\alpha_{ij} \equiv \pm\Delta t_{ab} \pmod{2\pi}$, and $X^{i_a,i_b}_{j_a,j_a} = 0$ otherwise.

- $\boldsymbol{i_a = i_b}$, $\boldsymbol{j_a \neq j_b}$ – Such a choice of parameters reduces equations (51) into another simple formula $(r_{ij} - p)X^{i_a,i_a}_{j_a,j_b} = qX^{i_a,i_a}_{j_b,j_a}$. Again, analogously to the third bullet in subsubsection 5.1.1 we find out that for $\alpha_{ij} \equiv \pm\Delta t_{ab} \pmod{2\pi}$ we have $X^{i_a,i_a}_{0,1} = \pm X^{i_a,i_a}_{1,0}$, otherwise these elements vanish.

- $\boldsymbol{i_a = i_b}$, $\boldsymbol{j_a = j_b}$ – In this special case we end up with equalities $(1 - r_{ij})X^{i_a,i_a}_{j_a,j_a} = 0$. Thence, whenever $\alpha_{ij} \not\equiv 0 \pmod{2\pi}$ all $X^{i_a,i_a}_{j_a,j_a}$ are zero, otherwise they may take on arbitrary values.

Due to quantities r_{ij} and α_{ij} there are non-trivial solutions for matrix elements not only depending on particular choice of a pair of qubits (a, b), but also on specific multiindices the elements have. Concretely, discussion above can be summed up as follows. The attractor matrix element $X^{i_1\ldots i_N}_{j_1\ldots j_N}$ is nonzero for given ω whenever for each pair (a, b) at least one of the following conditions is satisfied

$$\omega \equiv 2\,s\Delta t\,\Sigma_{ij} \pmod{2\pi}, \tag{52}$$
$$\omega \equiv 2\,s\Delta t\,\Sigma_{ij} \pm \Delta t_{ab} \pmod{2\pi}, \tag{53}$$
$$\omega \equiv 2\,s\Delta t\,\Sigma_{ij} \pm 2\Delta t_{ab} \pmod{2\pi}. \tag{54}$$

If there exists (a, b) such that ω is *not* of any of these forms, the resulting matrix element $X^{i_1\ldots i_N}_{j_1\ldots j_N}$ vanishes. We see that condition (52) is independent of any choice of a pair of qubits and is thus satisfied for each (a, b). For such ω's we obtain nonzero matrix elements. Moreover, eigenvalues with ω of the form (53) and (54) are almost never present in the attractor spectrum of the appropriate random unitary operation as these eigenvalues arise only for a zero-measure set of values Δt_{ab}. The typical behaviour of the network is thus unaffected by these exceptional cases and we exclude them from our subsequent consideration, for convenience. In the parameter space of collision times we thus leave only those values which fulfil the following conditions

$$\begin{aligned} \Delta t_{ab} &\not\equiv s\,\Delta t\,m \pmod{\pi}, \\ \Delta t_{ab} - \Delta t_{cd} &\not\equiv s\,\Delta t\,m \pmod{\pi}, \\ \Delta t_{ab} - \Delta t_{cd}/2 &\not\equiv s\,\Delta t\,m \pmod{\pi}, \end{aligned} \tag{55}$$

for any $m \in \{-2N, \ldots, 2N\}$ and all pairs (a, b) and (c, d). The set of excluded values of interaction times apparently has zero measure. Provided that $N \geq 3$, the constraints above ensure there are non-trivial attractors only for ω's satisfying condition (52). This implication can be seen as follows. If we had a fixed pair (a, b), there would be, among others, nonzero matrix elements associated with $\omega \equiv 2\,s\Delta t\,\Sigma_{ij} + \Delta t_{ab} \pmod{2\pi}$. At the same time, for any other pair (c, d) the same ω must be also of one of forms (52), (53) or (54). Nonetheless, conditions (55) ensure that these forms are never attained for allowed values of Δt_{ab} and Δt_{cd}. The matrix elements under consideration are therefore zero and such ω's are not in the attractor spectrum. The same argumentation can be used also for $\omega \equiv 2\,s\Delta t\,\Sigma_{ij} - \Delta t_{ab} \pmod{2\pi}$ and $\omega \equiv 2\,s\Delta t\,\Sigma_{ij} \pm 2\Delta t_{ab} \pmod{2\pi}$.

We have found out that for almost all settings of collision times there are non-trivial attractors associated only with eigenvalues, whose phase satisfies condition $\omega \equiv 2s\Delta t\,\Sigma_{ij} \pmod{2\pi}$ (52). By inspection of bullets above we see that for such ω's the resulting

attractor-matrix elements are invariant under permutations, see the first and fourth bullets. As a consequence, the attractors may be expressed in an orthonormal basis formed by $P_{\vec{c}}$ matrices (28). Furthermore, the second and third bullets imply that the non-vanishing elements lie either in the diagonal or the anti-diagonal of the attractor matrix. All diagonal entries fulfil equality $\Sigma_{ij} = 0$ and thus constitute attractors associated with $\lambda = 1$. Anti-diagonal entries lie in one of equivalence classes characterized by four-tuple $(c_0^0, c_1^0, c_0^1, c_1^1) = (0, k, N-k, 0)$ for some $0 \leq k \leq N$. For these entries we obtain $\Sigma_{ij} = c_1^0 - c_0^1 = 2k - N$ and they thence constitute attractors associated with eigenvalues $\lambda = \exp(\mathrm{i}\, 2s\Delta t(2k - N))$.

To conclude, the non-vanishing elements of the attractor matrix are collected in equivalence classes generated by permutations of their multiindices. All elements in the same class are of the same value. Considering only those with nonzero elements there are 2^N diagonal elements redistributed in $N+1$ equivalence classes and 2^N anti-diagonal elements redistributed in other $N+1$ equivalence classes. Diagonal elements are associated with $\lambda = 1$, anti-diagonal elements are, in general, associated with non-unit eigenvalues, on the other hand.

5.2.1 Conclusion for Composite Evolution

We have completed our analytical study of the quantum network evolution generated by the energy exchange interactions and a free Hamiltonian that is diagonal in the computational basis. From the beginning we assumed the interaction strengths and times are different for different pairs of qubits. Concerning the formation of equilibria, the non-trivial free evolution changes the situation occurring in subsubsection 5.1.3 considerably. The quantum network no longer equilibrates for any $N \geq 3$ and for almost any diagonal free Hamiltonian. The only class of Hamiltonians for which the equilibration emerges irrespective of the form of the initial state is composed of multiples of the identity matrix. This property was already observed in the case of partial swap interactions. The free Hamiltonian does not allow the network to equilibrate, even though mutual interactions between qubits tend to suppress non-stationary parts of the asymptotic evolution. The dimension of the attractor space scales linearly with the network size, as in the trivial free evolution setup (47). Another property that the energy exchange regime has in common with the partial swap regime is that asymptotic states are invariant under permutations of qubits, for both trivial and non-trivial free evolution. These states can be expressed in terms of $P_{\vec{c}}$ matrices (28) corresponding to diagonal and anti-diagonal matrix entries. The structure of the asymptotic states is analogous to that shown in Figure 4, but individual $P_{\vec{c}}$ matrices are associated with different eigenvalues.

Even though the whole system does not tend to equilibrium, the opposite holds true for its subsystems. In contrast to the partial swap case, the energy exchange interactions allow the subsystems of the quantum network to equilibrate. Indeed, there are two sets of nonzero entries in a general attractor matrix. The diagonal entries, which are stationary, and the anti-diagonal entries, whose multiindices are composed of local indices $\binom{0}{1}$ and $\binom{1}{0}$. Joint distributions $\vec{c} = (c_0^0, c_1^0, c_0^1, c_1^1)$ associated with the anti-diagonal entries thus satisfy $c_0^0 = c_1^1 = 0$. Formula (121) derived in subsection A.5 for the partial trace of $P_{\vec{c}}$ immediately implies that all anti-diagonal entries of the system density matrix are traced-out and there remain only stationary diagonal entries in the subsystem density matrix.

In the discussion above we took into account two possibilities of the network evolution. There is either a collision in every time interval Δt, or there are some intervals when no collision occurs. As we saw, the typical asymptotic behaviour of the quantum network governed by energy exchange interactions and a diagonal free Hamiltonian is effectively identical in both situations. Numerical simulations suggest that this feature arises also for

other kinds of interactions. In the analysis above we took into consideration only diagonal free Hamiltonians. Nonetheless, if we consider more general free Hamiltonians and perform numerical simulations, the situation described above changes radically. The network tends to equilibrium and the typical equilibrium state is a maximally mixed state. In this case the loss of information about the initial state is maximal. A general free evolution Hamiltonian has nonzero off-diagonal entries and the interaction Hamiltonian thus no longer commutes with it. This property is apparently responsible for the emergence of stationary asymptotic dynamics. We encounter similar situation in the next section when we investigate CNOT interactions. Similarly to the treatment of partial swap interaction, in our investigation we excluded a zero-measure set of values for collision times. Except for this restriction the collision times may have arbitrary values. In section 7 we will study the role of collision times in more detail.

6 CNOT Interaction

Until now we have dealt with interactions commuting with at least some class of non-trivial free Hamiltonians. Such a feature enables us to find the asymptotic regime of the network evolution relatively easily. In this part we focus on a different kind of interactions related to the *controlled-NOT operation*. These do not commute with any free evolution Hamiltonian different from multiple of the identity, see subsection A.3. Analogously to the previous cases, we examine conditions under which the network equilibrates. To this end we solve attractor equations with a trivial free evolution first. As a next step we consider a specific form of the non-trivial free Hamiltonian. Note that decoherence properties of quantum networks with controlled unitary operations were studied in [10].

The controlled-NOT (CNOT) is defined for qubits only. Let CNOT_{ab} denote the controlled-NOT operation applied on the a-th and the b-th qubits in this order. The Hamiltonian of this operation is chosen in such a way that for unit interaction time it generates a mapping acting as

$$\mathrm{CNOT}_{ab} = |0\rangle\langle 0| \otimes \mathbb{I}_b + |1\rangle\langle 1| \otimes \sigma_X^{(b)}, \tag{56}$$

where σ_X is a Pauli matrix (subsection A.2). Projectors $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ act on the *control* qubit a while the identity and Pauli matrices act on the *target* qubit b. The definition formula may be rewritten to demonstrate explicit action of CNOT. In the computational basis $\{|0\rangle, |1\rangle\}$ we have

$$\mathrm{CNOT}\,|c\rangle|t\rangle = |c\rangle|(c+t) \bmod 2\rangle, \tag{57}$$

where $|c\rangle$ and $|t\rangle$ are control and target qubits, respectively. Calculations in subsection A.3 show the Hamiltonian for CNOT may be chosen as

$$H_{\mathrm{CNOT}_{ab}} = \frac{\pi}{2}(2l+1)(\mathrm{CNOT}_{ab} - \mathbb{I}_{ab}), \quad l \in \mathbb{Z}. \tag{58}$$

Note that other possible forms of Hamiltonian $H_{\mathrm{CNOT}_{ab}}$ exist. The controlled-NOT operation satisfies $\mathrm{CNOT}_{ab} = \exp(\mathrm{i}\, H_{\mathrm{CNOT}_{ab}})$ for arbitrary, but distinct, qubits a and b. This unit-collision time scenario will be investigated first. Later on we consider also asymptotic regimes of the quantum network where collision times Δt_{ab} may assume arbitrary values and the interactions are thus generalized CNOT operations. These we will denote as $\mathrm{CNOT}_{ab}(\Delta t_{ab}) \equiv \exp(\mathrm{i}\,\Delta t_{ab} H_{\mathrm{CNOT}_{ab}})$. Discussion in subsection A.3 tells us that

$$\mathrm{CNOT}_{ab}(\Delta t_{ab}) = \frac{1}{2}\left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t_{ab}} + 1\right)\mathbb{I}_{ab} - \frac{1}{2}\left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t_{ab}} - 1\right)\mathrm{CNOT}_{ab}. \tag{59}$$

Unlike the partial swap the controlled-NOT does not exhibit nice commutation relations with the free evolution. Therefore we have to confine ourselves to the evolution operator decomposition as demonstrated in (15).

6.1 Pure Collisions

To begin with we consider the system evolving under CNOT-type collisions with only the trivial free evolution of each qubit. We also take $p_0 = 0$ (10) in order not to impose too many constraints on the attractor spectrum. If we allowed p_0 to be nonzero, there would remain only attractors for $\lambda = 1$. Firstly, we assume all the interaction times are identical and equal to one. Secondly, we relax this assumption and investigate how the asymptotic regime differs from the unit collision time scenario.

6.1.1 Unit Interaction Time

Results presented in this part have already been derived in [10]. The present discussion is made in order to generalize them consistently in the upcoming sections. Assuming the unit collision time for each pair of qubits the hermiticity of the CNOT operation allows us to write the attractor equations (19) in a simplified form

$$\mathrm{CNOT}_{ab}\; X\; \mathrm{CNOT}_{ab} = \lambda X, \quad a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{60}$$

When written in the elementwise fashion this equation transforms into the system of equalities with $i, j \in \{0, 1\}$

$$\begin{array}{lllllll} X^{0,i}_{0,j} & = & \lambda X^{0,i}_{0,j}, & & & & \\ X^{0,i}_{1,0} & = & \lambda X^{0,i}_{1,1}, & & X^{0,i}_{1,1} & = & \lambda X^{0,i}_{1,0}, \\ X^{1,0}_{0,j} & = & \lambda X^{1,1}_{0,j}, & & X^{1,1}_{0,j} & = & \lambda X^{1,0}_{0,j}, \\ X^{1,0}_{1,0} & = & \lambda X^{1,1}_{1,1}, & & X^{1,1}_{1,1} & = & \lambda X^{1,0}_{1,0}, \\ X^{1,0}_{1,1} & = & \lambda X^{1,1}_{1,0}, & & X^{1,1}_{1,0} & = & \lambda X^{1,0}_{1,1}, \end{array} \tag{61}$$

where the local indices refer to an ordered pair of qubits (a, b). The rest of indices has to be the same on both sides of equations.

As the CNOT operation is Hermitian, the attractor spectrum contains real eigenvalues only, *i.e.* $\sigma_{|1|} \subset \{1, -1\}$. Therefore, in the remainder of this section we focus on a solution to equations (60) for parameters $\lambda = \pm 1$. At the very end we present conclusion discussing different forms of attractors based on the eigenvalue λ and the number of qubits.

Attractors for $\lambda = 1$. Initially, we find attractors associated with the unit eigenvalue $\lambda = 1$. Combining the system of equations (61) for two ordered pairs of qubits (a, b) and (b, a) one obtains five sets of relations as follows

$$X^{0,0}_{0,0}, \tag{62}$$

$$X^{0,0}_{1,0} = X^{0,0}_{1,1} = X^{0,0}_{0,1}, \tag{63}$$

$$X^{1,0}_{0,0} = X^{1,1}_{0,0} = X^{0,1}_{0,0}, \tag{64}$$

$$X^{1,0}_{1,0} = X^{1,1}_{1,1} = X^{0,1}_{0,1}, \tag{65}$$

$$X^{1,0}_{0,1} = X^{1,1}_{0,1} = X^{0,1}_{1,1} = X^{0,1}_{1,0} = X^{1,1}_{1,0} = X^{1,0}_{1,1}. \tag{66}$$

An expression (62) means there is no condition imposed on matrix element $X^{0,0}_{0,0}$. One can easily verify that relations above cover all possible combinations of row and column indices. These equalities must hold for every choice of qubit pairs and we divide all matrix elements of a possible attractor into five *disjoint* sets.

The first set contains only one element whose indices are all zeros. This element is not subjected to any constraint and its value might be arbitrary. Consider then the set of all matrix elements with zero row multiindices and at least one nonzero column index. These elements lie in the first row of the matrix X. Obviously, only equations (63) apply to such elements. According to these equations all the members of the set have to equal each other. Similar discussion can be done also for a set of all matrix elements whose column multiindex is made of zeros and they have at least one nonzero index in their row multiindex. Such elements represent the first column of matrix X. Equations (64) force these elements to be equal. Moreover, no other equations apply to this set.

The fourth set is comprised of elements lying in the diagonal of the matrix X. Analogously, equations (65) imply that the diagonal terms are identical. Finally, the fifth set of matrix elements collects all remaining entries. That is, elements *not* lying in the first row

or the first column or the diagonal of the matrix. These are subjected to equations (66), which set values of these elements equal.

We have divided all the matrix X elements into five disjoint sets. From discussion in the previous paragraphs it is straightforward to write down the solution for equations (60) with $\lambda = 1$

$$X = \begin{pmatrix} a & b & b & b & \dots & b & b \\ c & d & e & e & \dots & e & e \\ c & e & d & e & \dots & e & e \\ c & e & e & d & \dots & e & e \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c & e & e & e & \dots & d & e \\ c & e & e & e & \dots & e & d \end{pmatrix}, \tag{67}$$

where $a, b, c, d, e \in \mathbb{C}$. There is obviously five linearly independent attractors. From discussion above it also follows that the attractors are invariant under permutations of individual qubits. The above expression can therefore be rewritten in terms of $P_{\vec{c}}$ matrices defined in (28). Note that for qubit networks there are only four equivalence classes $(c_0^0, c_1^0, c_0^1, c_1^1)$ and $P_{\vec{c}}$'s thus depend on four indices. Solution (67) can be expressed with help of these matrices as

$$X = a\, P_{N,0,0,0} + b \sum_{k=0}^{N-1} P_{k,N-k,0,0} + c \sum_{k=0}^{N-1} P_{k,0,N-k,0} + d \sum_{k=0}^{N-1} P_{k,0,0,N-k} + e\, S, \tag{68}$$

where $a, b, c, d, e \in \mathbb{C}$ correspond to coefficients in (67), $P_{\vec{c}}$'s are basis matrices from (28), and S is a sum of $P_{\vec{c}}$'s over all remaining equivalence classes. Explicitly,

$$\begin{aligned} S = & \sum_{k=1}^{N-1} \left(P_{0,k,N-k,0} + P_{0,k,0,N-k} + P_{0,0,k,N-k}\right) \\ & + \sum_{k=1}^{N-2} \sum_{l=1}^{N-1-k} \left(P_{k,l,N-k-l,0} + P_{k,l,0,N-k-l} + P_{k,0,l,N-k-l} + P_{0,k,l,N-k-l}\right) \\ & + \sum_{k=1}^{N-3} \sum_{l=1}^{N-2-k} \sum_{m=1}^{N-1-k-l} P_{k,l,m,N-k-l-m}. \end{aligned}$$

Attractors for $\lambda = -1$. From the first equality in (61) one can immediately deduce that the matrix element of attractor X associated with $\lambda = -1$ is zero whenever it has at least one local index of zeros in its multiindex. The remnant matrix elements have to satisfy the following constraints. Similarly to the previous case reprocessing of equations (61) and their twins, where the role of control and target qubits is switched, leads to these relations

$$-X_{1,0}^{0,0} = X_{1,1}^{0,0} = -X_{0,1}^{0,0}, \tag{69}$$

$$-X_{0,0}^{1,0} = X_{0,0}^{1,1} = -X_{0,0}^{0,1}, \tag{70}$$

$$-X_{1,0}^{1,0} = X_{1,1}^{1,1} = -X_{0,1}^{0,1}, \tag{71}$$

$$-X_{0,1}^{1,0} = X_{0,1}^{1,1} = -X_{1,1}^{0,1} = X_{1,0}^{0,1} = -X_{1,0}^{1,1} = X_{1,1}^{1,0}. \tag{72}$$

Recall discussion we pursued for unit eigenvalue $\lambda = 1$. Equations (69), (70) and (71) above ensure that the first row of matrix X together with its first column and its diagonal consists of zero entries. It is easily seen when we notice that indices appearing in all these equations have a local index made of zeros. Let us investigate the last system of equations (72). One can rewrite them to two sets as follows

$$X_{0,1}^{1,1} = X_{1,0}^{0,1} = X_{1,1}^{1,0} \equiv a \tag{73}$$

$$X_{0,1}^{1,0} = X_{1,1}^{0,1} = X_{1,0}^{1,1} \equiv -a \tag{74}$$

for some $a \in \mathbb{C}$. First, let the number of qubits be $N \geq 3$ and consider the elements in the form

$$X^{1\ 1\ i}_{0\ 1\ j},$$

where i and j are indices with an arbitrary value from $\{0, 1\}$. We demonstrate that such elements must be zero and thus a in equations (73) and (74) is zero as well. Hence, the only matrix X satisfying relations in the table above for $N \geq 3$ is a zero matrix only.

To this end, consider all four possible forms the element in the previous formula can take

$$X^{1\ \underline{1}\ \underline{0}}_{0\ \underline{1}\ \underline{0}}, \quad X^{1\ 1\ 0}_{0\ 1\ 1}, \quad X^{\underline{1}\ 1\ \underline{1}}_{\underline{0}\ 1\ \underline{0}}, \quad X^{1\ \underline{1}\ \underline{1}}_{0\ \underline{1}\ \underline{1}}.$$

If we apply equations (70) and (71) to the elements with underlined indices we instantaneously conclude the three asociated elements are zero. For the second element above see (73). This element is equal to

$$X^{1\ 0\ 0}_{1\ 1\ 1},$$

which is also zero due to (69). We have proven for $N \geq 3$ the corresponding attractor X is a zero matrix. Let us treat the case for $N = 2$. Equations (73) and (74) entirely determine nonzero matrix elements and the two-qubit attractor reads

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & -a \\ 0 & -a & 0 & a \\ 0 & a & -a & 0 \end{pmatrix} \tag{75}$$

with $a \in \mathbb{C}$. Our investigation reveals the attractor subspace for $\lambda = -1$ is non-trivial only for two qubit systems in which case it is one-dimensional.

6.1.2 Non-Unit Interaction Time

In this part we study how various values of collision times Δt_{ab} affect the structure of attractor space pertaining to the quantum network with random CNOT interactions. So far we have taken collision times to be unity, $\Delta t_{ab} = 1$. From now on let $\Delta t_{ab} \neq 1$, but the same for all pairs a and b, $\Delta t_{ab} = \Delta t$. Later on we relax even this assumption and take Δt_{ab} different for different qubits. Attractor equations (60) with $\mathrm{CNOT}_{ab}(\Delta t)$ (59) turn into the more complex form

$$(p + r)\, X = p\, \mathrm{CNOT}_{ab}\ X\ \mathrm{CNOT}_{ab} + \mathrm{i}\, q\, (X\ \mathrm{CNOT}_{ab} - \mathrm{CNOT}_{ab}\ X), \tag{76}$$

where $a \neq b$ run through all qubits in the network, $\lambda \in \sigma_{|1|}$, CNOT_{ab} is the ordinary controlled-NOT operation (56) and we defined

$$p = 1 - \cos(\pi \Delta t), \quad q = \sin(\pi \Delta t), \quad r = 2(\lambda - 1).$$

Above we put $l = 0$ in the definition of the Hamiltonian, see (58) and (59). In the following we split our analysis into cases $\lambda = 1$, $\lambda = -1$ and $\lambda \neq \pm 1$. Let us start our examination with the eigenvalue $\lambda = 1$.

Attractors for $\lambda = 1$. For $\lambda = 1$ we have $r = 0$. Let $p = 0$ first, implying $q = 0$. In this setting the CNOT operation reduces to the identity and no collision emerges, see (58). Equations (76) are then satisfied identically and matrix X may be arbitrary. Henceforth

let $p \neq 0$. It is convenient to write matrix X as $X = X_R + \mathrm{i}\, X_I$, where X_R and X_I are real matrices. Relations (76) then reduce to the system

$$X_R = \mathrm{CNOT}_{ab}\ X_R\ \mathrm{CNOT}_{ab} + \frac{q}{p}(\mathrm{CNOT}_{ab}\ X_I - X_I\ \mathrm{CNOT}_{ab}), \tag{77}$$

$$X_I = \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab} - \frac{q}{p}(\mathrm{CNOT}_{ab}\ X_R - X_R\ \mathrm{CNOT}_{ab}). \tag{78}$$

One can multiply the first equation by CNOT_{ab} from the left and substitute it into the second equation to obtain

$$(X_I - \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab})\left(1 + \frac{q^2}{p^2}\right) = 0 \tag{79}$$

which is satisfied iff $X_I = \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab}$. Substituting this relation back into (77) one sees the same equation holds also for X_R and we can conclude that equations (76) reduce to

$$\mathrm{CNOT}_{ab}\ X\ \mathrm{CNOT}_{ab} = X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{80}$$

We have already dealt with these equations, their solution is given by (67).

Till now all the collision times were assumed to have the same value. Let us move one step forward and allow the collision times Δt_{ab} to be generically different for each pair (a, b). This general situation is easily solved by noticing that in the above discussion parameters p and q play no role provided $p \neq 0$. That is, the same solution would follow even if the interaction times Δt_{ab} were different for different pairs of qubits and $p_{ab} \neq 0$. If $p_{ab} = 0$ holds for at least one pair (a, b) then there is less constraints imposed on solution X and thence the attractor associated with $\lambda = 1$ may have more general form compared to (67).

Attractors for $\lambda = -1$. For $\lambda = -1$ we have $r = -4$. Let $q = 0$ first, implying $p \in \{0, 2\}$. Then equations (76) reduce to $(p-4)\, X = p\, \mathrm{CNOT}_{ab}\, X\, \mathrm{CNOT}_{ab}$. For $p = 0$ there is only a trivial solution $X = 0$. For $p = 2$ (*i.e.* $\Delta t_{ab} = 2k + 1, k \in \mathbb{Z}$) one obtains the same equation as for $\Delta t_{ab} = 1$

$$\mathrm{CNOT}_{ab}\ X\ \mathrm{CNOT}_{ab} = -X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{81}$$

There is thus a non-trivial attractor (75) only for $N = 2$. Having investigated the case for zero q, let us take $q \neq 0$. One can proceed in the similar way as we did for the unit eigenvalue to obtain

$$X_R = \beta_1\, \mathrm{CNOT}_{ab}\ X_R\ \mathrm{CNOT}_{ab} + \beta_2\, (\mathrm{CNOT}_{ab}\ X_I - X_I\ \mathrm{CNOT}_{ab}),$$
$$X_I = \beta_1\, \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab} - \beta_2\, (\mathrm{CNOT}_{ab}\ X_R - X_R\ \mathrm{CNOT}_{ab}),$$

with $\beta_1 = \frac{p}{p-4}$ and $\beta_2 = \frac{q}{p-4}$. Multiplication of the first equation by CNOT_{ab} from the left yields

$$\mathrm{CNOT}_{ab}\ X_R - \beta_1\, X_R\ \mathrm{CNOT}_{ab} = \\ - \beta_2(1 - \beta_1)\, \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab} + \beta_2\, (X_I - \beta_1\, \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab}).$$

We plug the second equation appearing above into this expression to end up with

$$(1 + \beta_2^2)\, \mathrm{CNOT}_{ab}\ X_R - (\beta_1 + \beta_2^2)\, X_R\ \mathrm{CNOT}_{ab} = \\ - \beta_2(1 - \beta_1) \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab}. \tag{82}$$

Along similar lines we obtain also

$$(1+\beta_2^2)\,\mathrm{CNOT}_{ab}\;X_I - (\beta_1+\beta_2^2)\,X_I\;\mathrm{CNOT}_{ab} = \\ \beta_2(1-\beta_1)\mathrm{CNOT}_{ab}\;X_R\;\mathrm{CNOT}_{ab}. \quad (83)$$

Multiplication by CNOT_{ab} from both sides of equation (82) yields the expression for X_I. We can substitute this into (83) and simplify the resulting relation to get

$$(\beta_1^2+2\beta_2^2+1)\,\mathrm{CNOT}_{ab}\;X_R\;\mathrm{CNOT}_{ab} = 2(\beta_1+\beta_2^2)\,X_R. \quad (84)$$

Both sides of this equality have to have the same norm (1) and (the Hilbert-Schmidt) norm is preserved under unitary operations. From this we therefore extract the necessary condition in the form

$$(\beta_1^2+2\beta_2^2+1-2|\beta_1+\beta_2^2|)\|X_R\| = 0. \quad (85)$$

Assuming $X_R \neq 0$ we are left with the constraint imposed on the β's. It is satisfied only for $q = 0$ which is excluded by assumption. That is, only $X_R = 0$ solves (84). When we substitute this solution back into original equations these turn into

$$X_I = \beta_1\,\mathrm{CNOT}_{ab}\;X_I\;\mathrm{CNOT}_{ab} = \mathrm{CNOT}_{ab}\;X_I\;\mathrm{CNOT}_{ab}. \quad (86)$$

Since $\beta_1 \neq 1$ we see immediately that for $q \neq 0$ the attractor matrix is inevitably zero.

At this moment we relax the initial constraint and let collision times Δt_{ab} be different for various pairs of qubits. So far we have made use of a single ordered pair of qubits (a,b) to reduce the possible form of the attractor considerably. No other pairs were necessary to calculate $X = 0$ whenever $q_{ab} \neq 0$. Hence in a general setting, if there is a double of qubits such that $q_{ab} \neq 0$, then $X = 0$. Similarly, if for all pairs $q_{ab} = 0$, but there is a double such that $p_{ab} \neq 2$, then the attractor is also trivial, $X = 0$. Otherwise we obtain zero solution for $N \geq 3$ and nonzero solution (75) for $N = 2$.

Attractors for $\lambda \neq \pm 1$. Let us investigate the last case. For $\lambda \neq \pm 1$ we make use of the elementwise representation of equations (76). It reads

$$(p+r)\,X^{i_a,i_b}_{j_a,j_b} = p\,X^{i_a,i}_{j_a,j} + \mathrm{i}\,q\left(X^{i_a,i}_{j_a,j_b} - X^{i_a,i_b}_{j_a,j}\right), \quad (87)$$

where we assumed that qubit a is the control one and b is the target qubit. We denoted $i = (i_a + i_b) \pmod 2$ and $j = (j_a + j_b) \pmod 2$, see (57). When one interchanges roles of a and b as control and target qubits then local indices switch their positions accordingly.

It is easy to see whenever there is a local index of zeros, *i.e.* $(i_a, j_a) = (0,0)$ or $(i_b, j_b) = (0,0)$, relation (87) reduces to $rX^{i_a,i_b}_{j_a,j_b} = 0$. Since $r \neq 0$ in the present discussion we can conclude

$$X^{0,i_b}_{0,j_b} = X^{i_a,0}_{j_a,0} = 0, \quad \forall a,b \in \{1,\ldots,N\}. \quad (88)$$

As a next step consider a local index of the form $(i_a, j_a) = (1,1)$. Corresponding matrix elements are then subjected to constraints

$$\begin{aligned}
(p+r)\,X^{1,0}_{1,0} &= p\,X^{1,1}_{1,1} + \mathrm{i}\,q\,(X^{1,1}_{1,0} - X^{1,0}_{1,1}),\\
(p+r)\,X^{1,0}_{1,1} &= p\,X^{1,1}_{1,0} + \mathrm{i}\,q\,(X^{1,1}_{1,1} - X^{1,0}_{1,0}),\\
(p+r)\,X^{1,1}_{1,0} &= p\,X^{1,0}_{1,1} + \mathrm{i}\,q\,(X^{1,0}_{1,0} - X^{1,1}_{1,1}),\\
(p+r)\,X^{1,1}_{1,1} &= p\,X^{1,0}_{1,0} + \mathrm{i}\,q\,(X^{1,0}_{1,1} - X^{1,1}_{1,0}).
\end{aligned}$$

These can be neatly rewritten into the matrix equation

$$\begin{pmatrix} -(p+r) & -\mathrm{i}\,q & \mathrm{i}\,q & p \\ -\mathrm{i}\,q & -(p+r) & p & \mathrm{i}\,q \\ \mathrm{i}\,q & p & -(p+r) & -\mathrm{i}\,q \\ p & \mathrm{i}\,q & -\mathrm{i}\,q & -(p+r) \end{pmatrix} \begin{pmatrix} X^{1,0}_{1,0} \\ X^{1,0}_{1,1} \\ X^{1,1}_{1,0} \\ X^{1,1}_{1,1} \end{pmatrix} = 0. \tag{89}$$

Determinant of the matrix in the equation above is zero iff $\lambda = 1-p = \pm 1$ which is in contradiction to our assumption. We can perform analogous calculation for a and b interchanged to conclude

$$X^{1,i_b}_{1,j_b} = X^{i_a,1}_{j_a,1} = 0, \quad \forall a, b \in \{1, \ldots, N\}. \tag{90}$$

Let us focus on two last sets of possible local indices of the form either $(i_a, j_a) = (0,1)$ or $(i_a, j_a) = (1,0)$. For the former case equations (87) reduce to

$$\begin{aligned} (p - \mathrm{i}\,q + r)\, X^{0,i}_{1,1} &= (p - \mathrm{i}\,q)\, X^{0,i}_{1,0}, \\ (p - \mathrm{i}\,q + r)\, X^{0,i}_{1,0} &= (p - \mathrm{i}\,q)\, X^{0,i}_{1,1}, \end{aligned}$$

with $i \in \{0,1\}$. Subtraction of these two equations and the fact that $r \neq 0$ implies equality $X^{0,i}_{1,0} = -X^{0,i}_{1,1}$ for $i \in \{0,1\}$. Taking $i = 0$ the left-hand side element is of the form (88) and both elements in the equation are therefore zero. Similarly, for $i = 1$ the right-hand side element is of the form (90) and both elements are also zero. We could perform analogous reasoning also for $(i_b, j_b) = (0,1)$, $(i_a, j_a) = (1,0)$ and $(i_b, j_b) = (1,0)$ with the same result.

Our present investigation has been completely independent of the specific values of the collision times. Hence, for $\lambda \neq \pm 1$ one has $X = 0$ regardless of the actual value Δt. Therefore, this result is still valid even in the more general case with generically different values of collision times Δt_{ab} for each pair of qubits.

6.1.3 Conclusion for Pure Collisions

We have analyzed a scenario when the quantum network is governed by CNOT-like mutual interactions and the trivial free evolution. The situation with different interaction times Δt_{ab} is essentially identical to that with unit collision times. The quantum network with $N \geq 3$ qubits tends to equilibrium. For $N = 2$ and identical interaction times $\Delta t_{12} = \Delta t_{21}$ there emerges a non-stationary asymptotic part (75) of the evolution and the network does not equilibrate. Even though the CNOT operation is not symmetric—there is a clear distinction between the control and the target qubits—the resulting equilibrium state is invariant under permutations of qubits, see (68). This property is due to completeness of the network. An emergence of both CNOT_{ab} and CNOT_{ba} forces the asymptotic state to be symmetric. As a consequence, all subsystems consisting of the same number of qubits are in the same state as is demonstrated in subsection 2.1. As was shown in [10], for not strongly connected networks we may obtain also non-symmetric states.

Invariance under permutations is the property shared with previous types of interactions. Unlike preceding cases though, the dimension of the attractor space does not scale with the size of the network. There are five degrees of freedom for an arbitrary number of qubits composing the quantum network. The CNOT interaction is thus much more restrictive with respect to the two previous scenarios. Vast majority of the information about the initial state is lost during the evolution. These features are in effect for arbitrarily small interaction times Δt_{ab}. Even though the interactions in the system may be arbitrarily weak the asymptotic state is still considerably affected by them. Values of interaction times merely decelerate the convergence of the system evolution towards equilibrium, as will be

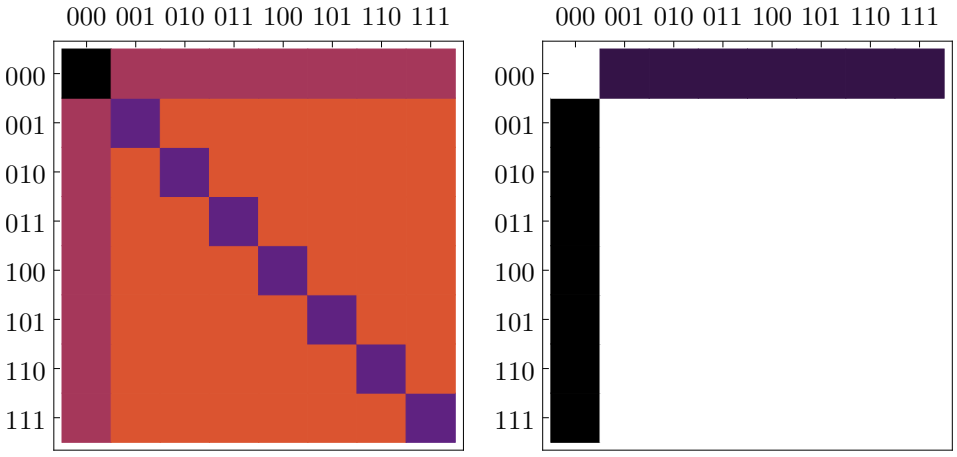

Figure 5: Numerical calculation of the equilibrium state for the quantum network composed of $N = 3$ qubits and governed by CNOT interactions and the trivial free Hamiltonian. Interaction times Δt_{ab} are different for each pair of qubits. The left and right plots depict the real and imaginary parts of the equilibrium state, respectively. The initial state is chosen randomly. These plots confirm validity of the formula (67), which characterizes the structure of attractors. Colours represent numerical values of the matrix entries to emphasize the structure of the equilibrium state. Row and column multiindices are shown explicitly in both plots. Invariance of matrix entries under permutations can be seen by comparing indices associated with elements sharing the same colour.

demonstrated in numerical simulations in section 7. The equilibrium state for $N = 3$ qubits can be seen in Figure 5. Analogously to the partial swap and energy exchange interactions the above conclusion does not hold for special values of interaction times. Nevertheless, these exceptional values form a zero-measure set and as such are in the present discussion negligible.

6.2 Composite Evolution

In this subsection we focus on the CNOT operation (56) acting on the multi-qubit system simultaneously with a non-trivial free evolution. The Hamiltonian generating a free evolution is chosen as $H_i^{\text{free}} = s\,\sigma_Z$ (see subsection A.2). We consider all collision times Δt_{ab} to be identical and equal to the time interval Δt, *i.e.* $\Delta t_{ab} = \Delta t$. We also assume $p_0 = 0$ (10) so there is always a collision taking place during every time interval Δt. Recall attractor equations (17) with operator $\overline{V}_{ab} \equiv \tilde{V}_{ab}(\Delta t)\, U_a^\dagger(\Delta t)\, U_b^\dagger(\Delta t)$. If we move both unitaries $U_i^\dagger(\Delta t)$ to the left-hand side we obtain equations

$$U_a^\dagger U_b^\dagger U^{\text{free}}\, X\, (U^{\text{free}})^\dagger U_b U_a = \lambda\, \tilde{V}_{ab}^\dagger\, X\, \tilde{V}_{ab}, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{91}$$

Operator $U_a^\dagger U_b^\dagger U^{\text{free}} = \exp(\mathrm{i}\,\Delta t \sum_{i\neq a,b} H_i^{\text{free}})$ is diagonal due to the form of the Hamiltonian H_i^{free}. It is not hard to see the left-hand side of the above formula in the elementwise form reads $\lambda\, r_{ab}\, X^{i_1\ldots i_N}_{j_1\ldots j_N}$, where $\lambda = \exp(\mathrm{i}\,\omega)$ and

$$r_{ab} = \exp(\mathrm{i}\,\alpha_{ab}(i,j)), \quad \alpha_{ab}(i,j) = 2s\Delta t \sum_{k\neq a,b} (j_k - i_k) - \omega. \tag{92}$$

Operator $\tilde{V}_{ab} = \exp(\mathrm{i}\,\Delta t\, \tilde{H}_{ab})$ (15) acts non-trivially only on the qubits a and b. Let $\tilde{V}_{ab}|_{(a,b)}$ denote a restriction of this operator to the subsystem of a-th and b-th qubit. Similarly we

resctrict also matrix X as follows. Suppose a (row) multiindex $(i_1, \ldots, i_a, \ldots, i_b, \ldots, i_N)$ where all indices i_c pertaining to qubits $c \neq a, b$ are fixed and indices i_a, i_b are left arbitrary. There are four such multiindices for each setting of i_c's. An analogous discussion can be done for column indices. For each setting of i_c's and j_c's we can define a 4×4 matrix $X_{(a,b)}$ composed of elements $X^{i_a,i_b}_{j_a,j_b}$. For every matrix X and every double (a,b) there are thus 2^{N-2} matrices $X_{(a,b)}$. These matrices satisfy equality

$$X = \sum_{ij} X^{i_1\ldots i_N}_{j_1\ldots j_N} |i_1 \ldots i_N\rangle\langle j_1 \ldots j_N| = \sum_{i',j'} (X_{(a,b)})^{i'}_{j'} \otimes |i'\rangle\langle j'|, \tag{93}$$

where we assume $a < b$ and $i' = (i_1, \ldots, i_{a-1}, i_{a+1}, \ldots, i_{b-1}, i_{b+1}, \ldots, i_N)$ (and similarly for multiindex j'). Since the indices i_c and j_c are not directly present in the future calculations, we work effectively with a single matrix $X_{(a,b)}$. Using the new notation the equations (91) are reexpressed as

$$r_{ab}\, X_{(a,b)} = (\tilde{V}_{ab}|_{(a,b)})^\dagger\, X_{(a,b)}\, (\tilde{V}_{ab}|_{(a,b)}), \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{94}$$

The problem is therefore reduced to the solution of matrix equations with effectively 4×4 matrices. As a next step we divide expression (94) into two parts. One part consists of equations determining how the solution $X_{(a,b)}$ depends on a value of α_{ab}. We will refer to these equations as the *eigenvalue equations*. The other part captures the influence of a network structure onto $X_{(a,b)}$.

The eigenvalue equations are obtained from attractor equations (94) when one diagonalizes operators $\tilde{V}_{ab}|_{(a,b)} = S_{ab} D_{ab} S^{-1}_{ab}$ and defines $\tilde{X} \equiv S^{-1}_{ab}\, X_{(a,b)}\, S_{ab}$. One ends up with

$$r_{ab}\, \tilde{X} = D^\dagger_{ab}\, \tilde{X}\, D_{ab}, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{95}$$

Matrix $\tilde{X}$ is dependent on the order in which eigenvalues are put into D_{ab}. For our choice of D_{ab} the eigenvalue equations (95) are equivalent to the following relations

$$\begin{aligned}
(r_{ab} - 1)\tilde{X}_{ii} &= 0, & & \\
(r_{ab} - e^{\mathrm{i}\,\pi\beta\Delta t})\tilde{X}_{12} &= 0, & (r_{ab} - e^{-\mathrm{i}\,\pi\beta\Delta t})\tilde{X}_{21} &= 0, \\
(r_{ab} - e^{-\mathrm{i}\,\frac{\pi\Delta t}{2}(1+\beta+b)})\tilde{X}_{13} &= 0, & (r_{ab} - e^{\mathrm{i}\,\frac{\pi\Delta t}{2}(1+\beta+b)})\tilde{X}_{31} &= 0, \\
(r_{ab} - e^{-\mathrm{i}\,\frac{\pi\Delta t}{2}(1+\beta-b)})\tilde{X}_{14} &= 0, & (r_{ab} - e^{\mathrm{i}\,\frac{\pi\Delta t}{2}(1+\beta-b)})\tilde{X}_{41} &= 0, \\
(r_{ab} - e^{-\mathrm{i}\,\frac{\pi\Delta t}{2}(1+3\beta+b)})\tilde{X}_{23} &= 0, & (r_{ab} - e^{\mathrm{i}\,\frac{\pi\Delta t}{2}(1+3\beta+b)})\tilde{X}_{32} &= 0, \\
(r_{ab} - e^{-\mathrm{i}\,\frac{\pi\Delta t}{2}(1+3\beta-b)})\tilde{X}_{24} &= 0, & (r_{ab} - e^{\mathrm{i}\,\frac{\pi\Delta t}{2}(1+3\beta-b)})\tilde{X}_{42} &= 0, \\
(r_{ab} - e^{\mathrm{i}\,\pi\Delta tb})\tilde{X}_{34} &= 0, & (r_{ab} - e^{-\mathrm{i}\,\pi\Delta tb})\tilde{X}_{43} &= 0,
\end{aligned} \tag{96}$$

where we substitute $\beta = 2s/\pi$ and $b = \sqrt{1+\beta^2}$ and i runs through the set $\{1, \ldots, 4\}$.

The matrix X is an attractor associated with $\lambda = \exp(\mathrm{i}\,\omega)$ if and only if equations (96) with a fixed ω are satisfied for all pairs of qubits (a,b) simultaneously. Since we consider the quantum network which is represented by a complete graph, the number of these equations is huge. However, we show that it is sufficient when equations (96) are fulfilled simultaneously for pairs (a,b) and (b,a) only. Already for these two sets of equations the possible form of attractors is reduced considerably and the subsequent analysis is relatively easy. When one takes into account equations (95) for the ordered pair of qubits (b,a) one obtains

$$r_{ab}\, \tilde{Y} = D^\dagger_{ab}\, \tilde{Y}\, D_{ab}, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b, \tag{97}$$

where now $\tilde{Y} \equiv S^{-1}_{ab}\, P_{ab}\, X_{(a,b)}\, P_{ab}\, S_{ab}$ with $P_{ab} = \mathrm{SW}_{ab}$ being a permutation matrix swapping the a-th and b-th qubit. As we see, both equations (95) and (97) are the same and therefore

they have the same set of solutions, which we call $\mathcal{M}$. What differs in both is how solutions from this set $\mathcal{M}$ are related to the original matrix $X_{(a,b)}$. From definitions of $\tilde{X}$ and $\tilde{Y}$ it can be seen that $X_{(a,b)}$ must lie in the intersection of the two sets as follows

$$X_{(a,b)} \in (S_{ab}\,\mathcal{M}\,S_{ab}^{-1}) \cap (P_{ab}S_{ab}\,\mathcal{M}\,S_{ab}^{-1}P_{ab}). \tag{98}$$

In other words, for each $X_{(a,b)}$ satisfying (98) there must exist solutions $\tilde{X}$ and $\tilde{Y}$ from $\mathcal{M}$ such that $X_{(a,b)} = S_{ab}\tilde{X}S_{ab}^{-1}$ and $X_{(a,b)} = P_{ab}S_{ab}\tilde{Y}S_{ab}^{-1}P_{ab}$. These two expressions relate the structure of $\tilde{X}$ and $\tilde{Y}$ matrices. Moreover, this relation is *independent* of the actual forms of solutions to (95) and (97), it reflects the effect of taking into account both ordered pairs (a,b) and (b,a). We will refer to this relation as *structure equations* in the following. Their explicit form reads

$$\tilde{X} = (S_{ab}^{-1}P_{ab}S_{ab})\,\tilde{Y}\,(S_{ab}^{-1}P_{ab}S_{ab}). \tag{99}$$

Using a vector representation of matrices this condition can be rewritten into $B|\tilde{Y}\rangle = |\tilde{X}\rangle$, where $B = (S_{ab}^{-1}P_{ab}S_{ab}) \otimes (S_{ab}^{-1}P_{ab}S_{ab})^T$, for details see (117) in subsection A.4. Since $B^2 = \mathbb{I}$ we also have $|\tilde{Y}\rangle = B|\tilde{X}\rangle$ and the roles of $\tilde{X}$ and $\tilde{Y}$ are thus *symmetric*. By a suitable permutation of basis vectors within the computational basis the B matrix can be reexpressed in the block-diagonal form

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_3 \end{pmatrix}, \tag{100}$$

where the one in the above left corner corresponds to $\tilde{X}_{22}$, B_1 is a 3×3 block associated with elements $\tilde{X}_{24}$, $\tilde{X}_{23}$, $\tilde{X}_{21}$, B_2 is another 3×3 block associated with $\tilde{X}_{42}$, $\tilde{X}_{32}$, $\tilde{X}_{12}$ and B_3 is a 9×9 block corresponding to elements $\tilde{X}_{41}$, $\tilde{X}_{31}$, $\tilde{X}_{11}$, $\tilde{X}_{14}$, $\tilde{X}_{13}$, $\tilde{X}_{43}$, $\tilde{X}_{34}$, $\tilde{X}_{44}$, $\tilde{X}_{33}$ in the respective order.

At this moment, we consider both eigenvalue equations (96) and structure equations represented by matrix B to solve for $X_{(a,b)}$ in a straightforward way. Recalling the first line of equations (96) we make distinction between two cases, either $\alpha_{ab} \equiv 0 \pmod{2\pi}$ or $\alpha_{ab} \not\equiv 0 \pmod{2\pi}$. Let us focus on the former case. For $\alpha_{ab} \equiv 0 \pmod{2\pi}$ we obtain $r_{ab} = 1$ and elements $\tilde{X}_{ii}$ are therefore of arbitrary value. Moreover, if we exclude a zero-measure set of parameter values, which is defined by the following equalities

$$\begin{aligned} s &= \frac{\pi k}{\Delta t}, \\ s &= \pm\frac{\pi}{4\Delta t}\sqrt{4k^2 - \Delta t^2}, \\ s &= \frac{\pi}{6\Delta t}(4k + \Delta t \pm \sqrt{64k^2 + 32k\Delta t + \Delta t^2}), \\ s &= \frac{\pi}{10\Delta t}(-12k - 3\Delta t \pm \sqrt{64k^2 + 32k\Delta t + 9\Delta t^2}), \end{aligned}$$

where $k \in \mathbb{Z}$, then all the other elements $\tilde{X}_{ij}$ with $i \neq j$ must be inevitably zero from (96). Thanks to the symmetry also for elements $\tilde{Y}$ we have $\tilde{Y}_{ii}$ being arbitrary and $\tilde{Y}_{ij} = 0$ for $i \neq j$. Under such conditions the matrix B imposes constraints enforcing matrix $X_{(a,b)}$ to be of the form

$$X_{(a,b)} = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \tag{101}$$

with $\gamma, \delta \in \mathbb{C}$. Let us move on to the case $\alpha_{ab} \not\equiv 0 \pmod{2\pi}$. Due to symmetry of $\tilde{X}$ and $\tilde{Y}$ it can be shown that while one puts two variables pertaining to block B_1 equal to zero, the third variable is already zero as well. The same can be shown even for block B_2.

We make use of this property to force all attractors associated with $\lambda \neq 1$ to be zero. The prize for that is an exclusion of a zero-measure set of parameters only. From equations (96) one can read out for which values of α_{ab} the particular matrix element is either zero or have an arbitrary value. For instance let us consider α_{ab} such that $\alpha_{ab} \equiv \pi\beta\Delta t \pmod{2\pi}$. Then $\tilde{X}_{12}$ might be arbitrary according to (96). This variable is associated with block B_2 whose other variables are $\tilde{X}_{42}$ and $\tilde{X}_{32}$. If we now allow only such collision times that

$$s \neq \frac{\pi}{6\Delta t}(4k + \Delta t \pm \sqrt{64k^2 + 32k\Delta t + \Delta t^2}), \quad k \in \mathbb{Z}, \tag{102}$$

then our choice $\alpha_{ab} \equiv \pi\beta\Delta t \pmod{2\pi}$ ensures $\alpha_{ab} \not\equiv \frac{\pi}{2}\Delta t(1+3\beta\pm b) \pmod{2\pi}$. From (96) it thus follows that $\tilde{X}_{42} = 0 = \tilde{X}_{32}$. We can employ the property of the block B_2 mentioned at the beginning of this paragraph to conclude $\tilde{X}_{42} = \tilde{X}_{32} = \tilde{X}_{12} = 0$ provided that $\alpha_{ab} \equiv \pi\beta\Delta t \pmod{2\pi}$ and (102). To conclude, even though for $\alpha_{ab} \equiv \pi\beta\Delta t \pmod{2\pi}$ the element $\tilde{X}_{42}$ might be arbitrary due to eigenvalue equations, by suitable conditions on parameters Δt and s the structure equations force this element to be zero. An analogous discussion can be made for all variables pertaining to blocks B_1 and B_2. As a result, elements $\tilde{X}_{24}$, $\tilde{X}_{23}$, $\tilde{X}_{21}$, $\tilde{X}_{42}$, $\tilde{X}_{32}$ and $\tilde{X}_{12}$ vanish for almost all values of interaction time Δt and the free evolution time scaling s.

Let us treat the last block B_3. For $\alpha_{ab} \not\equiv 0 \pmod{2\pi}$ we have $\tilde{X}_{ii} = 0 = \tilde{Y}_{ii}$. By plugging these values into equation $B|\tilde{Y}\rangle = |\tilde{X}\rangle$, namely its part corresponding to B_3, we obtain non-trivial constraints on values of some variables $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$, $i \neq j$. These conditions effectively reduce block B_3 into a 3×3 matrix which can be treated in the way we did for blocks B_1 and B_2. Again, when two variables are set to zero the third variable must be also zero. A discussion of parameter values similar to that above can be done. As a consequence in the set of collision times we *leave* only those satisfying each of conditions

$$\begin{array}{llll} s \neq \frac{\pi k}{\Delta t}, & & s \neq \pm\frac{\pi}{4\Delta t}\sqrt{4k^2 - \Delta t^2}, \\ s \neq \frac{\pi}{4\Delta t}(2k - \Delta t), & & s \neq \frac{k\pi(k-\Delta t)}{2\Delta t(2k-\Delta t)}, \end{array} \tag{103}$$

$$\begin{array}{rcl} s & \neq & \frac{\pi}{6\Delta t}\left(2k \pm \sqrt{16k^2 - 3\Delta t^2}\right), \\ s & \neq & \frac{\pi}{6\Delta t}\left(\pm(4k + \Delta t) \pm \sqrt{64k^2 + 32k\Delta t + \Delta t^2}\right), \\ s & \neq & \frac{\pi}{10\Delta t}\left(-12k - 3\Delta t \pm \sqrt{64k^2 + 32k\Delta t + 9\Delta t^2}\right), \\ s & \neq & \frac{\pi}{42\Delta t}\left(-20k - 5\Delta t \pm \sqrt{64k^2 + 32k\Delta t + 25\Delta t^2}\right), \\ s & \neq & \frac{\pi}{70\Delta t}\left(4k + \Delta t \pm 3\sqrt{64k^2 + 32k\Delta t - 31\Delta t^2}\right), \\ s & \neq & \frac{\pi}{18\Delta t}\left(4k + \Delta t \pm \sqrt{64k^2 + 32k\Delta t - 23\Delta t^2}\right), \end{array} \tag{104}$$

where $k \in \mathbb{Z}$. For these suitable parameter values we obtain a solution as follows: for $\alpha_{ab} \equiv 0 \pmod{2\pi}$ one has matrix $X_{(a,b)}$ as shown in (101), for any other value α_{ab} this matrix vanishes.

Till now we have lead calculations in terms of 4×4 matrices $X_{(a,b)}$, which express how two qubits, a and b, relate to each other. Let us move back to the original problem. Since parameters γ and δ in (101) can also be set to zero, we can without loss of generality assume every solution $X_{(a,b)}$ is of this diagonal form irrespective of the actual value of α_{ab}, see the previous paragraph. The whole attractor matrix X must be consistent with its each restriction $X_{(a,b)}$. This requirement necessitates the diagonal form of the attractor matrix. It is not hard to see that if the whole matrix X is not zero, then all its diagonal

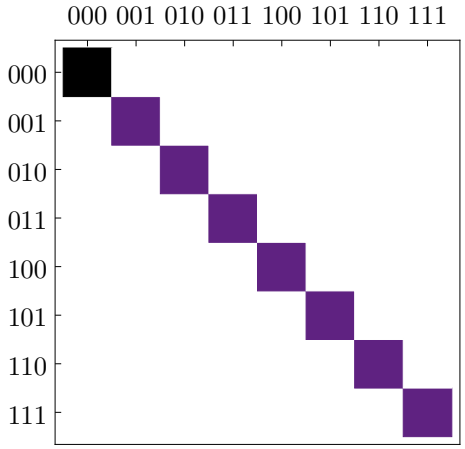

Figure 6: Numerical calculation of the equilibrium state for the quantum network composed of $N = 3$ qubits and governed by CNOT interactions and a diagonal free Hamiltonian. The plot depicts the real part of the equilibrium state, the imaginary part is zero. The initial state is equal to that used in Figure 5. Colours represent numerical values of the matrix entries to emphasize the structure of the equilibrium state. Row and column multiindices are shown explicitly. There are apparently only two degrees of freedom present in the equilibrium state.

entries are identical except for the very first element, which is independent of the others. That is

$$X = \gamma\,|00\ldots0\rangle\langle00\ldots0| + \delta\,\mathbb{I} \tag{105}$$

with some $\gamma, \delta \in \mathbb{C}$. As a final step we analyze condition $\alpha_{ab} \equiv 0 \pmod{2\pi}$ in terms of attractor spectrum, collision time Δt and the time scaling s. From the definition of parameter α_{ab} it follows the condition $\alpha_{ab} \equiv 0 \pmod{2\pi}$ is equivalent to

$$2\Delta t\, s \sum_{k\neq a,b} (j_k - i_k) \equiv \omega \pmod{2\pi}. \tag{106}$$

Nevertheless, we obtained diagonal attractors of the form (105) for which condition (106) reads $\sum_{k\neq a,b}(j_k - i_k) = 0$. Consequently, for any nonzero ω the corresponding attractor matrix X vanishes. The only nonzero solution (105) is associated with the only remaining eigenvalue $\lambda = e^{\mathrm{i}0} = 1$.

6.2.1 Conclusion for Composite Evolution

We have considered a composite evolution consisting of a diagonal free Hamiltonian and CNOT interactions with identical collision times. In contrast to the partial swap and energy exchange interactions the free Hamiltonian does not commute with the interaction Hamiltonian in the present scenario. The entire discussion has been therefore more involved. An asymptotic behaviour of the network is also tremendously different from the other interactions investigated previously. For partial swap and energy exchange interactions the free Hamiltonians redistributed attractors over different eigenvalues and thus precluded formation of equilibria for any number of qudits. In the case of controlled-NOT interactions the equilibration emerges also for non-trivial free Hamiltonians. The quantum network equilibrates regardless of its size and the equilibrium state is obtained as a special case of an attractor associated with the trivial free evolution (67), see (105) and Figure 6.

As a direct consequence the equilibrium state is also invariant under permutations of individual qubits. As follows from the properties of the partial trace, see subsection 2.1, the asymptotic states of subsystems with the same number of qubits are thus identical.

The amount of information surviving the long-time evolution reduces to two degrees of freedom only, irrespective of the number of qubits constituting the network. These results were obtained under assumption that all collision times are equal to the time interval Δt. Numerical simulations show that even if we choose mutually different collision times Δt_{ab}, the asymptotic state is still of the form (105). Numerical simulations also allow us to consider completely general free Hamiltonians, not only the diagonal ones in the computational basis which are employed in the analytical investigation. These simulations suggest that for general free Hamiltonians the only asymptotic (and stationary) state of the network is the maximally mixed state. During the evolution the entire information about the initial state is thus erased. Out of all interactions we studied in this thesis the CNOT interaction imposes the most stringent conditions on the equilibrium state of the system. Since for all three types of interactions we have examined we choose as a free Hamiltonian the same operator $H_i^{\text{free}} = s\,\sigma_Z$, one can easily compare its effect on the equilibrium state for various kinds of the network evolution. When this free evolution Hamiltonian commutes with the interaction Hamiltonian, the network does not tend to equilibrium. On the other hand, as CNOT interaction suggests the opposite may be true for interactions which does not commute with the free Hamiltonian. Similarly to preceding types of interactions we excluded a zero-measure set of values s and Δt, (103) and (104), from our discussion. Expect for this set, the time interval Δt may assume an arbitrary value.

7 Numerical Simulations

So far we have presented analytical solutions for the asymptotic dynamics of the quantum network evolving under various conditions. Both a trivial and a non-trivial free evolution together with three different kinds of interactions were investigated and several scenarios were examined regarding interaction times. At first, we considered the simplest case with all collision times being identical. Afterwards we assumed that all collision times might have mutually different values. In some cases we simplified our discussion by excluding physically unimportant zero-measure sets of parameters.

In this section we perform numerical simulations of the quantum-network evolution. This approach represents an independent way of calculating the asymptotic regimes of the network, which can be subsequently compared with our findings from preceding sections. Numerical simulations also allow us to treat more general settings than we have investigated so far. In particular, unlike in previous sections we can study the quantum network behaviour while collision times Δt_{ab} and probabilities p_{ab} (10) vary during the evolution. Results from these generalized scenarios suggest that solutions to restricted cases, obtained analytically, are also valid in these settings. However, it should be noted that the numerical approach suffers from several drawbacks. Apart from round-off errors another disadvantage is that no more than just a few-qudit networks can be examined with the power of the present-day technology.

The numerical approach consists in successive applications of the random unitary operation Φ (10), associated with the specific network evolution, to the initial network state. Thanks to the matrix form (118) of the random unitary operation, as shown in subsection A.4, we can completely get rid of the initial-state dependence and focus on the evolution propagator itself. Apart from parameters considered in the analytical solution in the numerical approach we also have to specify the probability distribution in (7), which plays the role of an independent variable. In the following we are interested in the asymptotic form Φ_{asymp} of the propagator obtained after large number of successive applications of Φ, *i.e.* $\Phi_{\text{asymp}} = \lim_{n\to\infty} \Phi^n$ (if the limit exists). To compare the analytical results with numerical simulations we simply apply the asymptotic propagator to the general matrix of appropriate size and compare the output with the attractor form we had already calculated.

We focus on two aspects of the network evolution regarding its convergence towards equilibrium. The first aspect is the dependence of the rate of convergence on different distributions of collision times and probabilities. The second aspect is the effect of magnitude of collision times on the rate of convergence. We show differences between the situation when collision times are small and the situation when their magnitudes are comparable with the time interval Δt.

7.1 Dependence on Distributions of Times and Probabilities

Recalling the first aspect, for all three kinds of interactions we have investigated the three evolution scenarios are considered. First two scenarios evaluate the asymptotic propagator Φ_{asymp} as an approximate limit of the iterative sequence $\{\Phi^n\}_{n=1}^{\infty}$, where the random unitary operation Φ is constructed out of the probability distribution and collision times that are kept *constant* during the evolution. Therefore, these cases can be directly compared with their analytical counterparts. The difference between the two scenarios is that the first one takes the uniform probability distribution and unit collision times whereas the second one generates both sets of values randomly before the iterations are triggered. As a consequence, the probability distribution as well as collision times present in the second scenario are in general non-uniform. Since we use random generation merely to produce

a particular probabilities and collision times we do not average over their possible values. Averaging would be necessary if the random values of probabilities and collision times resulted from our incomplete knowledge of the network setting. In the plots below the first and second scenarios are depicted by red and blue dots, respectively. We take the quantity $d_n = \|\Phi^{n+1} - \Phi^n\|$ to assess convergence characteristics of the iterative sequence for both scenarios. In general, condition $d_n \to 0$ is merely a necessary condition for the convergence of the sequence $\{\Phi^n\}_{n=1}^{\infty}$. Nevertheless, due to intrinsic properties of the random unitary operation Φ [6] this condition is also sufficient. Therefore, if the sequence $\{d_n\}_{n=0}^{\infty}$ is approximately zero from certain step n_0 onward we assume the sequence $\{\Phi^n\}_{n=n_0}^{\infty}$ is constant and declare the element Φ^{n_0} to be the asymptotic propagator Φ_{asymp}. The third scenario represents a generalization of the analytical approach in the sense that the probability distribution of collisions and collision times are no longer kept constant. Instead, they are randomly generated at *every* iteration step and the sequence we study in this setup is thus $\{\prod_{i=1}^{n} \Phi_i\}_{n=1}^{\infty}$, where Φ_i denotes a random unitary operation generated in the i-th step. Purple dots are used to denote this case in the figures below. Analogously to the first two scenarios, as a convergence measure we take the quantity $d_n = \| \prod_{i=1}^{n+1} \Phi_i - \prod_{i=1}^{n} \Phi_i \|$ and the asymptotic propagator is defined as $\Phi_{\text{asymp}} = \lim_{n\to\infty} \prod_{i=1}^{n} \Phi_i$ (if the limit exists). Again, even in this case we make no averaging over all possibilities.

Each case examined below is performed on a quantum network comprised of three qubits, that is $N = 3$ and $d = 2$. Moreover, in the present discussion we set the time interval $\Delta t = 1$ and the free-evolution probability $p_0 = 0$ (7), but even for non-zero p_0 one obtains very similar results. The network asymptotic regime thus does not care whether there is a collision in every step of its evolution or not, only the rate of convergence is affected. In the following, by the *non-trivial free Hamiltonian* we always mean the diagonal Hamiltonian $H_i^{\text{free}} = \sigma_Z$. Prior to detailed discussion of the three interactions we can already point out some general properties suggested by plots below. For instance, it turns out that the rate of convergence of the second scenario is heavily dependent on the initial probabilities and collision times and varies greatly among different realisations of the network evolution. The third scenario, despite its non-monotonicity, seems to finally converge to the relevant asymptotic state. We can also observe that its fluctuations are more distinct during the trivial free evolution. The non-trivial free evolution smooths these fluctuations considerably.

Let us begin our analysis with the controlled-NOT interaction as this shows quite different properties compared to the partial swap and the energy exchange interactions. The first setup we consider is the CNOT operation acting together with the trivial free evolution. Results for the three scenarios can be seen in Figure 7. All scenarios including the third one converge to the analytical solution, see subsubsection 6.1.3. The non-trivial free Hamiltonian presented above is chosen in accordance with the analytical treatment of the non-trivial free evolution in subsection 6.2. Results for this choice are shown in Figure 8. We see that the free Hamiltonian decelerates the convergence for each of the scenarios several times. Despite this fact, the network still equilibrates, which is not the case for the two other interactions.

It turns out that behaviour of the quantum network with the partial swap interactions is very similar to the behaviour with the energy exchange interactions. We thus present results only for the partial swap. Unlike the controlled-NOT Hamiltonian the partial swap and energy exchange Hamiltonians are defined with a coupling strengths ϕ_{ab} (22) and κ_{ab} (40), respectively. These numbers represent another set of independent parameters we can modify. However, we set all coupling strengths equal to $\phi_{ab} = \pi/6$ (22) for all pairs of qubits, without loss of generality. For the trivial free evolution the coupling strengths and collision times always appear together in a product and their roles are therefore in-

terchangeable. The second and third scenario thus also correspond to the situation when interaction strengths are diverse for each pair of qubits. Numerical results for the trivial free evolution are shown in Figure 9. Asymptotic states of all scenarios including the third one are identical, up to numerical error, to the analytical solution as is summarized in subsubsection 4.1.3. Since $N > d$, there is only $\lambda = 1$ in the attractor spectrum. The aforementioned smoothing effect of the non-trivial free evolution on the third-scenario fluctuations is well demonstrated by comparison of Figure 9 with Figure 10, where the situation with non-trivial free evolution is depicted. For the non-trivial free evolution and either of the interactions the network does not equilibrate. However, even in these cases one may study the convergence properties of the iterative sequence. We examine these properties in the following subsection.

Examples demonstrated above suggest analytical results calculated in previous sections are valid and the theoretical framework used for calculations captures the essence of the asymptotic behaviour the quantum network undergoes in different scenarios. Moreover, from the third scenario it seems that even in more general setups one obtains attractors already emerging in the restricted cases. From the nature of numerical simulations it is essentially impossible for generated collision times to lie in the zero-measure set of exceptional values for which the solutions are degenerate. Hence, we are left with the generic cases only.

7.2 Dependence on Magnitudes of Collision Times

All the simulations presented so far deal with the rate of convergence in the context of different probability and collision-time distributions. At this moment, we analyze how the magnitude of collision times affects the rate of convergence of the iterative sequence. The network we consider consists of three qubits, as is the case above. We perform simulations similar to the first scenario examined in the previous subsection, but the value of collision times will be modified. In other words, the probability distribution is set to be uniform and collision times for all pairs are identical. For the partial swap and energy exchange interactions we keep all interaction strengths identical for all pairs of qubits. Specifically, we set $\phi_{ab} = \kappa_{ab} = \pi/6$. As a time interval we choose $\Delta t = 1$ and even in this subsection we study three scenarios. All the collision times are equal in the present setting, but we consider three values that these collision times assume: $\Delta t' = \Delta t = 1$, $\Delta t' = \Delta t/5 = 0.2$ and $\Delta t' = \Delta t/10 = 0.1$. We take into consideration all three interactions we investigated analytically. Initially we assume the free evolution is trivial and later on we take $H_i^{\text{free}} = \sigma_Z$ as a free evolution Hamiltonian. The latter is referred to as the *non-trivial free Hamiltonian* in the following.

For the trivial-free-evolution case we obtained qualitatively very similar behaviour for all three interactions. In Figure 11 one can see our results for the CNOT interaction. As can be easily deduced from the plot, the value of collision times truly affects the rate with which the sequence converges. The fastest convergence for our choice of parameters is obtained for the maximum value of collision time $\Delta t' = 1$. In this case the d_n drops below $L = 0.01$ after 14 steps of evolution. For the partial swap interaction the same limit is reached after 143 steps and for the energy exchange interaction 149 steps of evolution are needed to reach this limit. In Figure 11 the first scenario is the fastest and the third scenario is the slowest setting. Nevertheless, for different values of the time interval Δt the order in which individual scenarios converge may differ. Since the choice of collision times and the time interval has a direct impact on the specific form of the propagator Φ, it is not possible to draw simple conclusions about the role of collision time on the rate of convergence. Evolution operators (12) are periodic functions of the collision time $\Delta t'$.

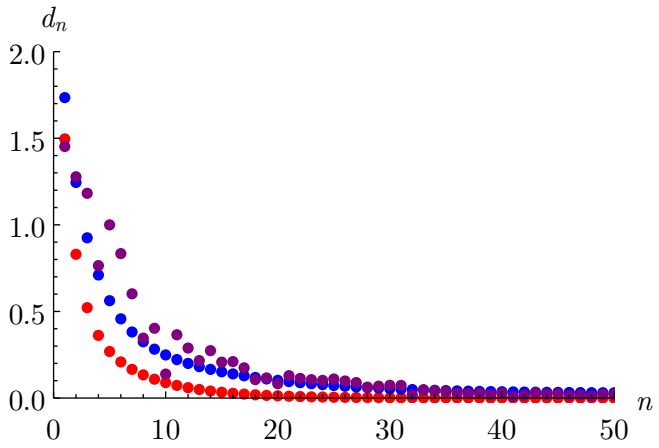

Figure 7: The controlled-NOT interaction with the trivial free evolution and with $p_0 = 0$—Comparison of rate of convergence for three different scenarios. Red dots correspond to the case with uniform probability distribution and unit collision times. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in every step of the evolution.

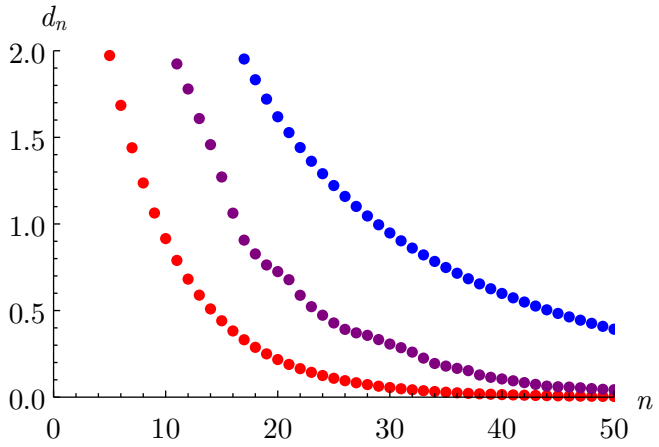

Figure 8: The controlled-NOT interaction with the non-trivial free evolution and with $p_0 = 0$—Comparison of rate of convergence for three different scenarios. Red dots correspond to the case with uniform probability distribution and unit collision times. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in every step of the evolution.

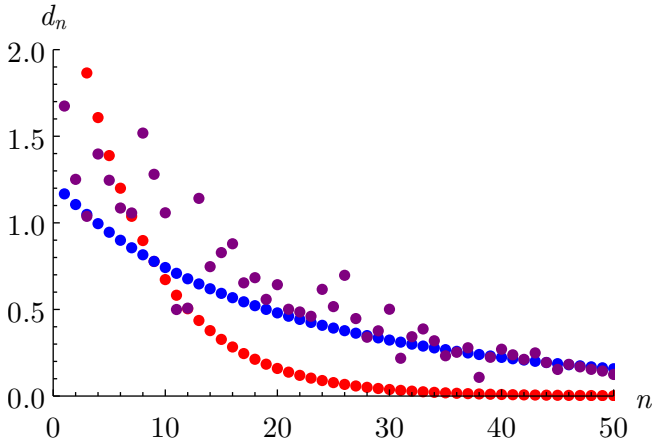

Figure 9: The partial swap interaction with the trivial free evolution and with $p_0 = 0$— Comparison of rate of convergence for three different scenarios. Red dots correspond to the case with uniform probability distribution and unit collision times. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in every step of the evolution.

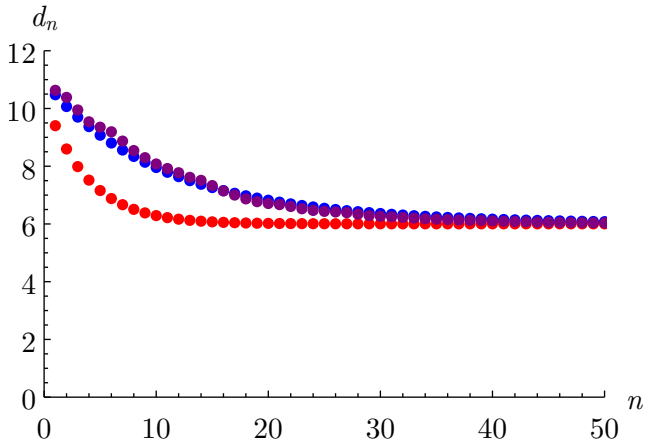

Figure 10: The partial swap interaction with the non-trivial free evolution and with $p_0 = 0$— Comparison of rate of convergence for three different scenarios. Red dots correspond to the case with uniform probability distribution and unit collision times. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in every step of the evolution.

For our choice of parameters the period for partial swap, energy exchange and CNOT interactions is equal to $\Delta t'_p = 12$, $\Delta t'_p = 12$ and $\Delta t'_p = 4$, respectively. The time interval value $\Delta t = 1$ we use in the present setting thus ensures that all three values of collision times are confined to a single period for all three interactions. In this setup the typical behaviour of the network evolution is such that for smaller values of collision times the iterative sequence converges more slowly.

Similar observation holds true even for the non-trivial free evolution, at least for CNOT interactions, see Figure 12. For the partial swap and the energy exchange interactions with the non-trivial free Hamiltonian the network does not equilibrate, but we can still study its evolution towards an asymptotic oscillatory state (9). The situation when the network attains its asymptotic regime can be observed in the plots as a stabilized nonzero value of d_n. Such a behaviour is due to non-unit eigenvalues λ in the attractor spectrum as follows from equation (38). By inspection of Figure 13 and Figure 14 we see that for our choice of parameters the evolution towards asymptotic state gets faster for larger values of collision times, as was the case in the trivial-free-evolution scenario. Nevertheless, for the partial swap we observe different stabilized value of d_n for different collision times. Such a behaviour is not present for energy exchange interactions. Moreover, for a randomly generated non-diagonal free Hamiltonian and energy exchange interactions the sequence converges to zero, but in time-scales that are by several orders of magnitude larger than that for the CNOT.

Complete numerical treatment of the convergence properties of the time evolution sequence $\{\Phi^n\}_{n=1}^{\infty}$ deserves much more attention than we are able to pay it in this section. More detailed discussion of figures presented above is possible, of course. Nonetheless, our main purpose is to demonstrate that in relevant cases the quantum network equilibrates for almost all values of collision times, and that these times affect the rate of its convergence toward equilibrium. Indeed, Figure 11 and Figure 12 demonstrate this property quite well.

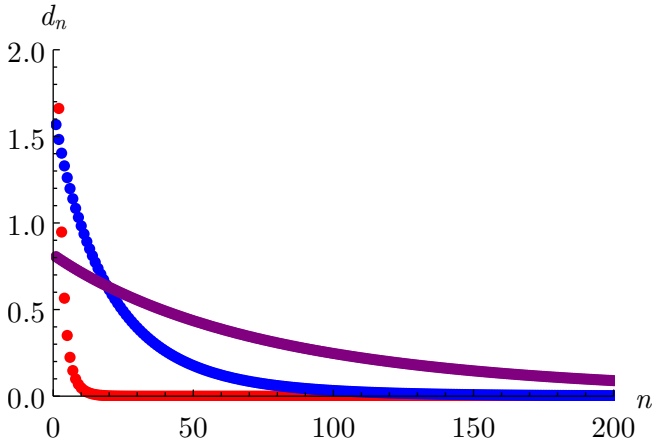

Figure 11: The controlled-NOT interaction with the trivial free evolution and with $p_0 = 0$—Comparison of rate of convergence for three different values of collision times. Red, blue and purple dots correspond to the cases with collision times $\Delta t' = 1$, $\Delta t' = 0.2$ and $\Delta t' = 0.1$, respectively. For the partial swap and energy exchange interactions we would obtain similar trends of rate of convergence for respective values of collision times.

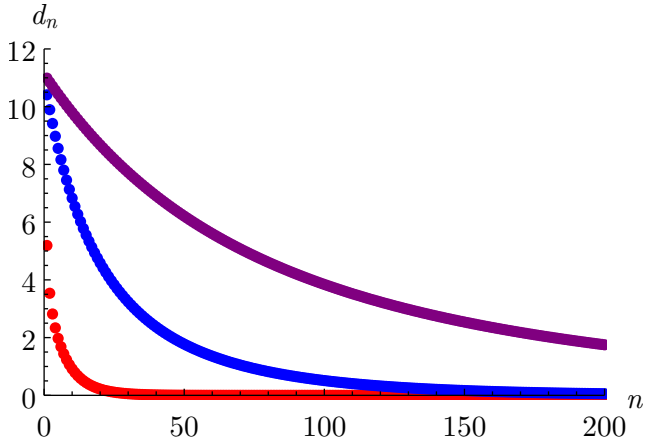

Figure 12: The controlled-NOT interaction with the non-trivial free evolution and with $p_0 = 0$—Comparison of rate of convergence for three different values of collision times. Red, blue and purple dots correspond to the cases with collision times $\Delta t' = 1$, $\Delta t' = 0.2$ and $\Delta t' = 0.1$, respectively.

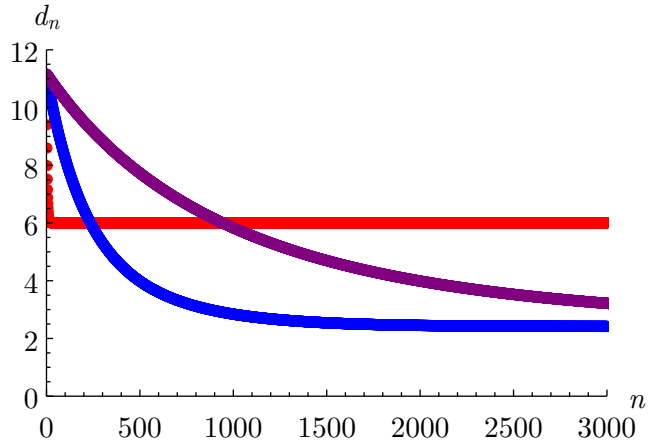

Figure 13: The partial swap interaction with the non-trivial free evolution and with $p_0 = 0$—Comparison of rate of convergence for three different values of collision times. Red, blue and purple dots correspond to the cases with collision times $\Delta t' = 1$, $\Delta t' = 0.2$ and $\Delta t' = 0.1$, respectively.

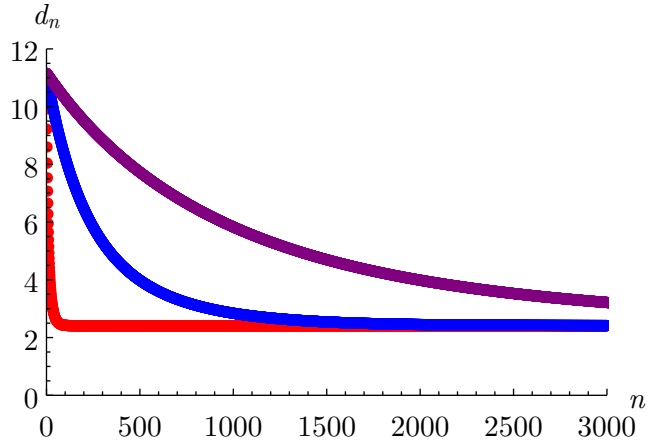

Figure 14: The energy exchange interaction with the non-trivial free evolution and with $p_0 = 0$—Comparison of rate of convergence for three different values of collision times. Red, blue and purple dots correspond to the cases with collision times $\Delta t' = 1$, $\Delta t' = 0.2$ and $\Delta t' = 0.1$, respectively.

8 Conclusion

We investigated the asymptotic behaviour of a multi-qudit quantum network whose individual constituents undergo free evolution randomly interrupted by short bipartite interactions. We were especially focused on formation of equilibria for different quantum-network setups. An interplay between the free evolution and mutual interactions possibly leading to the network equilibration was analyzed. The quantum network was taken to be complete—every qudit could interact with any other qudit—and partial-swap, energy-exchange and CNOT interactions were considered. We found closed analytical solutions for these setups with various qudit dimensions, numbers of qudits and with diverse values of time interval Δt, interaction times and interaction strengths. Finally, we confirmed our analytical results by numerical simulations.

For pure partial-swap evolution we found out the generic asymptotic state assumes the form described in (28). The dimension of the attractor space in this case scales with the dimension of qudits and their number according to formula (27). The quantum network with the trivial free evolution and partial swap interactions tends to equilibrium whenever the number of qudits in the network is strictly greater than their dimensionality. On the contrary, when the number of qudits does not exceed their dimensionality the asymptotic behaviour of the quantum network is more complex and equilibration in general does not occur. Many features present in the partial swap scenario are found also in the case of qubit networks governed by energy exchange interactions. For the trivial free evolution and energy exchange interactions we observe equilibration of the network for any number of qubits $N \geq 3$ and the equilibrium state (47) is a special case of that obtained for the partial swap. The dimension of the attractor space scales linearly with the size of the network. For pure CNOT evolution the generic asymptotic state is of the stationary form (67) with the dimension of the attractor space being independent of the number of qubits. A quantum network with the trivial free evolution and CNOT interactions between its constituents thence equilibrates regardless of the number of qubits provided that $N \geq 3$.

After investigation of mutual interactions we incorporated also the non-trivial free evolution into our model. For the partial swap we considered general one-qudit free Hamiltonian. The resulting asymptotic state, and hence the equilibration property, is determined by equation (38). Nevertheless, for a general non-trivial free evolution and an initial state the network does not tend to equilibrium. The same result we obtained also for networks governed by energy exchange interactions and diagonal free Hamiltonians. However, for a general free evolution Hamiltonian and energy exchange interactions the network tends to equilibrium and numerical simulations suggest the resulting state is a maximally mixed state. Since the proper discussion for CNOT and general one-qubit free Hamiltonian is difficult to complete we considered only diagonal Hamiltonians. For this setup we arrived at the asymptotic state as shown in (105). The quantum network thus equilibrates and the dimension of the attractor space is independent on the size of the network. Numerical simulations suggest that for more general free evolution Hamiltonians the quantum network still tends to equilibrium and the asymptotic state is maximally mixed, similarly to the energy exchange scenario.

Some general features valid for all three interactions were found. Most notably, the asymptotic states of the network are invariant under permutations of individual qudits. This property is preserved even in cases when the network does not equilibrate. As a consequence, all subsystems of the quantum network are in the same reduced state, provided they have the same number of constituents. Another feature valid for all three types of interactions is independence of equilibration on specific values of collision times and other relevant parameters. For the trivial free evolution and any of the three interactions the

network equilibrates for almost any setting of collision times, interaction strengths and time interval Δt.

As a next step in our investigation we can consider also quantum networks that are not complete graphs. This setup is more general than we employed in this work and allows for more regimes of the network evolution. Other free Hamiltonians and interactions except for the partial swap, energy exchange and CNOT can be also taken into account. Especially a composite evolution of CNOT interactions and a free evolution generated by a general one-qubit Hamiltonian is a challenging problem. Theoretical framework could be augmented to treat evolution having different probability distribution and collision times in every iterative step. Numerical simulations suggest that such a generalized evolution leads to the same asymptotic behaviour and might be analyzed within a theoretical framework similar to that we use.

9 References

[1] R. Balian, D. Haar, and J. Gregg, *From Microphysics to Macrophysics: Methods and Applications of Statistical Physics*. Springer, 2006.

[2] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypzyk, "The role of quantum information in thermodynamics — a topical review," *ArXiv e-prints*, May 2015.

[3] C. Gogolin and J. Eisert, "Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems," *ArXiv e-prints*, Mar. 2015.

[4] N. Linden, S. Popescu, A. J. Short, and A. Winter, "Quantum mechanical evolution towards thermal equilibrium," *Phys. Rev. E*, vol. 79, p. 061103, Jun 2009.

[5] P. Reimann, "Canonical thermalization," *New Journal of Physics*, vol. 12, no. 5, p. 055027, 2010.

[6] J. Novotný, G. Alber, and I. Jex, "Asymptotic evolution of random unitary operations," *Central European Journal of Physics*, vol. 8, pp. 1001–1014, Feb. 2010.

[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. New York, NY, USA: Cambridge University Press, 10th ed., 2011.

[8] M. Ziman, P. Štelmachovič, V. Bužek, M. Hillery, V. Scarani, and N. Gisin, "Diluting quantum information: An analysis of information transfer in system-reservoir interactions," *Phys. Rev. A*, vol. 65, p. 042105, Mar 2002.

[9] V. Scarani, M. Ziman, P. Štelmachovič, N. Gisin, and V. Bužek, "Thermalizing quantum machines: Dissipation and entanglement," *Phys. Rev. Lett.*, vol. 88, p. 097905, Feb 2002.

[10] J. Novotný, G. Alber, and I. Jex, "Asymptotic dynamics of qubit networks under randomly applied controlled unitary transformations," *New Journal of Physics*, vol. 13, no. 5, p. 053052, 2011.

[11] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States*. Cambridge University Press, 2008.

A Mathematical Concepts and Proofs

In this appendix we review all necessary mathematical tools which are used extensively in the main text.

A.1 Attractor Equations

As described in section 3 the quantum network evolution is captured by a random unitary operation (10). In order to find out the asymptotic regime of the network evolution we have to solve attractor equations (8), where the role of U_α is played by unitary operators U_{ab} from (10). In this appendix we derive the form of attractor equations that we use throughout the whole thesis.

Assume for a while that $p_0 = 0$ in (10). When we make use of definition (12) the attractor equations (8) for random unitary operation Φ (10) read

$$U^{\text{free}}(\Delta t - \Delta t_{ab})\, X\, (U^{\text{free}})^\dagger(\Delta t - \Delta t_{ab}) = \lambda\, U_{ab}^\dagger(\Delta t_{ab})\, X\, U_{ab}(\Delta t_{ab}),$$

for all pairs (a, b) of qudits. To simplify this expression let us multiply both sides of these equations by $U^{\text{free}}(\Delta t_{ab})$ and $(U^{\text{free}})^\dagger(\Delta t_{ab})$ from left and right, respectively. We obtain

$$U^{\text{free}}(\Delta t)\, X\, (U^{\text{free}})^\dagger(\Delta t) = \lambda\, U^{\text{free}}(\Delta t_{ab})\, U_{ab}^\dagger(\Delta t_{ab})\, X\, U_{ab}(\Delta t_{ab})(U^{\text{free}})^\dagger(\Delta t_{ab}).$$

One can utilize operator decomposition introduced in (15) to reexpress the unitary operator $\overline{V}_{ab}(\Delta t_{ab}) \equiv U_{ab}(\Delta t_{ab})(U^{\text{free}})^\dagger(\Delta t_{ab})$, appearing on the right-hand side, into more convenient form. Specifically, we obtain $\overline{V}_{ab}(\Delta t_{ab}) = \tilde{V}_{ab}(\Delta t_{ab})\, \tilde{U}_{ab}(\Delta t_{ab})\, (U^{\text{free}})^\dagger(\Delta t_{ab}) = \tilde{V}_{ab}(\Delta t_{ab})\, U_a^\dagger(\Delta t_{ab})\, U_b^\dagger(\Delta t_{ab})$, where $U_i(\Delta t_{ab}) = \exp(\mathrm{i}\,\Delta t_{ab} H_i^{\text{free}})$ is a free evolution operator for qudit i. We end up with equations

$$U^{\text{free}}(\Delta t)\, X\, (U^{\text{free}})^\dagger(\Delta t) = \lambda\, \overline{V}_{ab}^\dagger(\Delta t_{ab})\, X\, \overline{V}_{ab}(\Delta t_{ab}), \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b.$$

If we additionally assume that also no collision might take place during the time interval Δt, *i.e.* $p_0 \neq 0$ (10), attractor equations decouple into

$$U^{\text{free}}(\Delta t)\, X\, (U^{\text{free}})^\dagger(\Delta t) = \lambda X,\ \overline{V}_{ab}^\dagger(\Delta t_{ab})\, X\, \overline{V}_{ab}(\Delta t_{ab}) = X,\ \forall a, b \in \{1, \ldots, N\},\ a \neq b.$$

A.2 Pauli Matrices

Throughout the thesis we make use of Pauli matrices σ_X, σ_Y and σ_Z defined in the usual way

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A.3 Hamiltonians

In the main text we presented Hamiltonians for the partial swap, energy exchange and controlled-NOT operations. This appendix is to show that these Hamiltonians indeed generate appropriate operations as their associated evolution operators. After doing so we also demonstrate explicitly how the evolution operators change their form when the collision time Δt is different from one. Finally, we determine which free evolution Hamiltonians commute with the given interaction Hamiltonian.

Let us begin with the controlled-NOT operation. We claim the corresponding Hamiltonian is of the form

$$H_{\text{CNOT}} = \frac{\pi}{2}(2l + 1)(\text{CNOT} - \mathbb{I}), \quad l \in \mathbb{Z}. \tag{107}$$

Since $\mathrm{CNOT}^2 = \mathbb{I}$ we have $(\mathrm{CNOT} - \mathbb{I})^2 = -2(\mathrm{CNOT} - \mathbb{I})$. By mathematical induction one easily proves $(\mathrm{CNOT} - \mathbb{I})^k = (-2)^{k-1}(\mathrm{CNOT} - \mathbb{I})$ for $k \geq 1$. Therefore

$$\begin{aligned} e^{\mathrm{i}\, H_{\mathrm{CNOT}}} &= \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{i}\, \frac{\pi}{2}(2l+1))^k (\mathrm{CNOT} - \mathbb{I})^k \\ &= \mathbb{I} + \sum_{k=1}^{\infty} \frac{1}{k!} (\mathrm{i}\, \frac{\pi}{2}(2l+1))^k (-2)^{k-1} (\mathrm{CNOT} - \mathbb{I}) \\ &= \mathbb{I} - \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{k!} (-\mathrm{i})^k (\pi(2l+1))^k \right) (\mathrm{CNOT} - \mathbb{I}) \\ &= \mathbb{I} - \frac{1}{2} \left(e^{-\mathrm{i}\,\pi(2l+1)} - 1 \right) (\mathrm{CNOT} - \mathbb{I}) \\ &= \mathbb{I} + (\mathrm{CNOT} - \mathbb{I}) = \mathrm{CNOT}. \end{aligned}$$

For partial swap we now prove its corresponding Hamiltonian assumes the form as follows

$$H_{\mathrm{PSW}_{ab}} = \rho\, \mathbb{I} + \phi\, \mathrm{SW}_{ab}, \tag{108}$$

where ρ and ϕ are defined in relations (21). Since a global phase of an evolution operator plays no role in our calculations we can put $\rho = 0$. Similarly to CNOT even for PSW we utilize involution and commutation properties $\mathrm{SW}^2 = \mathbb{I}$ and $[\mathbb{I}, \mathrm{SW}] = 0$ to obtain

$$\begin{aligned} e^{\mathrm{i}\, H_{\mathrm{PSW}}} &= e^{\mathrm{i}\,\rho \mathbb{I}} e^{\mathrm{i}\,\phi\, \mathrm{SW}} = e^{\mathrm{i}\,\rho} \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{i}^k (\phi\, \mathrm{SW})^k \\ &= e^{\mathrm{i}\,\rho} \left(\sum_{k=0}^{\infty} \frac{1}{(2k)!} \mathrm{i}^{2k} (\phi\, \mathrm{SW})^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \mathrm{i}^{2k+1} (\phi\, \mathrm{SW})^{2k+1} \right) \\ &= e^{\mathrm{i}\,\rho} \left(\sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \phi^{2k}\, \mathbb{I} + \mathrm{i} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^{2k} \phi^{2k+1}\, \mathrm{SW} \right) \\ &= e^{\mathrm{i}\,\rho} \cos(\phi)\, \mathbb{I} + \mathrm{i}\, e^{\mathrm{i}\,\rho} \sin(\phi)\, \mathrm{SW} \\ &= p\, \mathbb{I} + q\, \mathrm{SW} = \mathrm{PSW}. \end{aligned}$$

Unlike preceding interactions the energy exchange operation we specified directly by writing down its Hamiltonian, so here we calculate corresponding evolution operator. The energy exchange Hamiltonian is defined as

$$H_{\mathrm{EX}_{ab}} = \kappa\, \mathrm{SW} - \kappa\, D, \tag{109}$$

where $D = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|$ is a diagonal matrix. It is therefore not hard to see its exponential reads $\exp(-\mathrm{i}\,\kappa\, D) = \mathbb{I} + (\exp(-\mathrm{i}\,\kappa) - 1)D$. Moreover, it commutes with the swap operation and $\mathrm{SW} \cdot D = D$, so we have

$$\begin{aligned} e^{\mathrm{i}\, H_{\mathrm{EX}}} &= e^{\mathrm{i}\,\kappa\, \mathrm{SW}} e^{-\mathrm{i}\,\kappa D} = \mathrm{PSW} \cdot (\mathbb{I} + (e^{-\mathrm{i}\,\kappa} - 1)D) \\ &= \mathrm{PSW} + (e^{-\mathrm{i}\,\kappa} - 1)(\cos(\kappa) D + \mathrm{i}\, \sin(\kappa) D) \\ &= \mathrm{PSW} + (e^{-\mathrm{i}\,\kappa} - 1) e^{\mathrm{i}\,\kappa} D = \mathrm{PSW} + (1 - e^{\mathrm{i}\,\kappa}) D \\ &= D + \cos(\kappa)(\mathbb{I} - D) + \mathrm{i}\, \sin(\kappa)(\mathrm{SW} - D), \end{aligned}$$

where we employed the formula (108) with $\rho = 0$ and $\phi = \kappa$.

Recalling just performed calculations we can modify them by introducing non-unit collision time Δt. It is easy to see that the last but one line of the calculation for CNOT changes to

$$\begin{aligned} e^{\mathrm{i}\,\Delta t H_{\mathrm{CNOT}}} &= \mathbb{I} - \frac{1}{2} \left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t} - 1 \right) (\mathrm{CNOT} - \mathbb{I}) \\ &= \frac{1}{2} \left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t} + 1 \right) \mathbb{I} - \frac{1}{2} \left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t} - 1 \right) \mathrm{CNOT}. \end{aligned}$$

The same procedure for the partial swap operation yields

$$\begin{aligned} e^{\mathrm{i}\,\Delta t H_{\mathrm{PSW}}} &= e^{\mathrm{i}\,\rho\Delta t}\cos(\phi\Delta t)\,\mathbb{I} + \mathrm{i}\,e^{\mathrm{i}\,\rho\Delta t}\sin(\phi\Delta t)\,\mathrm{SW} \\ &= \tilde{p}\,\mathbb{I} + \tilde{q}\,\mathrm{SW}, \end{aligned}$$

where $\tilde{p}$ and $\tilde{q}$ are p and q parameters, respectively, with angles Δt times larger than the original ones. That is, the value of collision time only modifies values of parameters and we obtain partial swap again. Analogous result obviously holds even for the energy exchange operation, different collision times only affect value of interaction parameter κ as $\kappa \to \Delta t\,\kappa$.

Finally, let us calculate the commutator of the above Hamiltonians with the free evolution Hamiltonian pertaining to the subsystem of a-th and b-th qudits. For the partial swap we show that

$$\left[H_a^{\mathrm{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\mathrm{free}}, H_{\mathrm{PSW}_{ab}}\right] = 0. \tag{110}$$

This property can be proven for qudits with arbitrary dimensionality. Nevertheless, in here we restrict ourselves to one-qubit Hamiltonians, whose form in a Pauli basis reads

$$H_a^{\mathrm{free}} = \alpha_{\mathbb{I}}^{(a)}\mathbb{I} + \alpha_X^{(a)}\sigma_X + \alpha_Y^{(a)}\sigma_Y + \alpha_Z^{(a)}\sigma_Z. \tag{111}$$

From relation (108) it follows that

$$\left[H_a^{\mathrm{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\mathrm{free}}, H_{\mathrm{PSW}_{ab}}\right] \propto \left[\sum_i (\alpha_i^{(a)}\sigma_i \otimes \mathbb{I}_b + \alpha_i^{(b)}\mathbb{I}_a \otimes \sigma_i), \mathrm{SW}_{ab}\right]. \tag{112}$$

In order to proceed one can demonstrate that the swap operation acting on composite subsystem of a-th and b-th qubit takes the form

$$\mathrm{SW}_{ab} = \frac{1}{2}(\mathbb{I}_a \otimes \mathbb{I}_b + \sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z). \tag{113}$$

By substitution of this expression into the right-hand side of (112) one gets

$$\frac{1}{2}\sum_{i,j\in\{X,Y,Z\}} \alpha_i^{(a)}\left[\sigma_i \otimes \mathbb{I}_b, \sigma_j \otimes \sigma_j\right] + \frac{1}{2}\sum_{i,j\in\{X,Y,Z\}} \alpha_i^{(b)}\left[\mathbb{I}_a \otimes \sigma_i, \sigma_j \otimes \sigma_j\right].$$

Employing the commutation relations of Pauli matrices $[\sigma_i, \sigma_j] = 2\mathrm{i}\,\varepsilon_{ijk}\,\sigma_k$ one finds out the commutator reads

$$\left[H_a^{\mathrm{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\mathrm{free}}, H_{\mathrm{PSW}_{ab}}\right] \propto \mathrm{i}\sum_{ijk} \varepsilon_{ijk}\,\sigma_k \otimes \sigma_j \left(\alpha_i^{(a)} - \alpha_i^{(b)}\right). \tag{114}$$

We can conclude that the partial swap Hamiltonian commutes with the free evolution if and only if $\alpha_i^{(a)} = \alpha_i^{(b)}$ for all i. In other words the Hamiltonians commute iff the free Hamiltonians for both qubits are identical, which is indeed the present case.

Similarly to (113) we can easily prove that matrix D in (109) can be reexpressed in Pauli matrix form as

$$D = \frac{1}{2}(\mathbb{I} \otimes \mathbb{I} + \sigma_Z \otimes \sigma_Z), \tag{115}$$

which is convenient when calculating commutator of H_{EX} with a free Hamiltonian as in (110). From (109) we see

$$\begin{aligned} \left[H_a^{\mathrm{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\mathrm{free}}, H_{\mathrm{EX}_{ab}}\right] \propto &\left[\sum_i (\alpha_i^{(a)}\sigma_i \otimes \mathbb{I}_b + \alpha_i^{(b)}\mathbb{I}_a \otimes \sigma_i), \mathrm{SW}_{ab}\right] \\ &- \left[\sum_i (\alpha_i^{(a)}\sigma_i \otimes \mathbb{I}_b + \alpha_i^{(b)}\mathbb{I}_a \otimes \sigma_i), D\right]. \end{aligned}$$

The first commutator on the right-hand side above is equal to the right hand side of (114) and the second commutator is calculated along similar lines, when instead of SW (113) one uses D (115). The total commutator in the previous formula then vanishes if and only if $\alpha_X^{(a)} = 0 = \alpha_X^{(b)}$, $\alpha_Y^{(a)} = 0 = \alpha_Y^{(b)}$ and $\alpha_Z^{(a)} = \alpha_Z^{(b)} \in \mathbb{R}$. In other words, only diagonal free Hamiltonians commute with the energy exchange interaction Hamiltonian.

In the case of CNOT the resulting conditions imposed by commutators are more stringent in comparison with previous cases as we demonstrate below. Let us calculate for which parameters the following commutator vanishes

$$\left[H_a^{\text{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\text{free}}, H_{\text{CNOT}_{ab}}\right] = 0. \tag{116}$$

Since $|0\rangle\langle 0| = (\mathbb{I} + \sigma_Z)/2$ and $|1\rangle\langle 1| = (\mathbb{I} - \sigma_Z)/2$ one can rewrite (56) into

$$\text{CNOT} = \frac{1}{2}(\mathbb{I}_a \otimes \mathbb{I}_b + \sigma_Z \otimes \mathbb{I}_b + \mathbb{I}_a \otimes \sigma_X - \sigma_Z \otimes \sigma_X).$$

Analogously to preceding cases we plug this expression back into (116) with explicit forms of free Hamiltonians (111). Due to commutation relations for Pauli matrices the commutator reduces to

$$\sum_{i,j\in\{X,Y,Z\}} \mathrm{i}\, \alpha_i^{(a)} \varepsilon_{3ij}\, \sigma_j \otimes (\sigma_X - \mathbb{I}) \;+ \sum_{i,j\in\{X,Y,Z\}} \mathrm{i}\, \alpha_i^{(b)} \varepsilon_{1ij}\, (\sigma_Z - \mathbb{I}) \otimes \sigma_j.$$

When one expands this formula and puts it equal to zero, one obtains simple solution $\alpha_X^{(a)} = \alpha_Y^{(a)} = \alpha_Y^{(b)} = \alpha_Z^{(b)} = 0$. Taking into account the fact the same conditions must apply not only for pair (a, b) but also for (b, a) we see that the only free one-qubit Hamiltonian which commutes with CNOT is a multiple of the identity operator, *i.e.* $H_a^{\text{free}} \propto \mathbb{I}$.

A.4 Superoperators as Matrices

We study the evolution processes of the quantum networks whose generators are random unitary operations, that is superoperators acting on network density matrices $\rho \in \mathbb{C}^{d^N \times d^N}$, see (7). In order to study their properties it is suitable to represent the superoperator itself as a matrix [11]. To that end, one can rewrite an arbitrary matrix $X \in \mathbb{C}^{d^N \times d^N}$ as a ket vector $|X\rangle \in \mathbb{C}^{d^{2N}}$

$$|X\rangle = (X_{1\bullet}, X_{2\bullet}, \ldots, X_{d^N\bullet}),$$

where $X_{i\bullet}$ stands for the i-th row of matrix X. Suppose we are dealing with the superoperator S of the form $S(X) = A\,X\,B \equiv Y$, with matrices $A, B, X, Y \in \mathbb{C}^{d^N \times d^N}$. One can prove the correspondence between different representations of the superoperator S as follows

$$A\,X\,B = Y \quad \sim \quad (A \otimes B^T)|X\rangle = |Y\rangle. \tag{117}$$

The matrix representation of the superoperator S is therefore $S = A \otimes B^T$. Relation (117) is completely general and can be used in its own right. On the other hand, one can make use of this correspondence to rewrite the random unitary operation (7) into the form

$$\Phi\,|A\rangle = \sum_{a,b=1}^{N} p_{ab} U_{ab} \otimes U_{ab}^{\star}\,|A\rangle, \tag{118}$$

where the star symbol denotes the complex conjugation. After this transformation the mapping composition $\Phi \circ \Phi$ reduces to matrix multiplication $\Phi \cdot \Phi$.

A.5 Partial Trace of $P_{\vec{c}}$ Matrices

In section 4, where the network evolution governed by partial swap interactions and a trivial free evolution is studied, we introduce a special class of matrices $\{P_{\vec{c}}\}_{\vec{c}}$ (28). These matrices form an orthonormal basis in an attractor space associated with $\lambda = 1$, where all attractors are invariant under permutations of qudits comprising the network. In this appendix we show their explicit forms and derive some properties these matrices have with respect to the partial trace (2). From description below equality (28) it is not hard to write down the expression for matrix $P_{\vec{c}}$ associated with an index joint distribution $\vec{c}$

$$P_{\vec{c}} = \mathcal{N}_N \sum_{\pi \in \mathcal{S}_N} |\pi(i)\rangle\langle\pi(j)|, \tag{119}$$

where summation is done over the symmetric group $\mathcal{S}_N$ comprised of permutations π that act on N-element multiindices. Row and column multiindices $i = (i_1, \ldots, i_N)$ and $j = (j_1, \ldots, j_N)$, respectively, are any multiindices that have the joint distribution of individual indices equal to $\vec{c}$. For definition of $\vec{c}$ see subsubsection 4.1.1. The prefactor $\mathcal{N}_N = 1/\sqrt{N!(c_0^0)! \ldots (c_{d-1}^{d-1})!}$ is chosen such that the matrix $P_{\vec{c}}$ is normalized according to the Hilbert-Schmidt norm, see (1). In the following we derive a formula with help of which the partial trace of $P_{\vec{c}} \in \mathcal{B}(\mathscr{H}^{\otimes N})$ over a single qudit Hilbert space $\mathscr{H}$ can be expressed as a linear combination of $P_{\vec{z}} \in \mathcal{B}(\mathscr{H}^{\otimes (N-1)})$ matrices defined on an appropriate subspace.

Consider a matrix $P_{\vec{c}} \in \mathcal{B}(\mathscr{H}^{\otimes N})$ for a fixed number of qudits $N > 2$ and some joint distribution $\vec{c} = (c_0^0, c_1^0, \ldots, c_{d-2}^{d-1}, c_{d-1}^{d-1})$. Expression (119) can be rewritten into two sums, where the first sum contains permutations π for which $(\pi(i))_1 = (\pi(j))_1$, and the second sum collects all the remaining permutations. That is, $P_{\vec{c}} = \mathcal{N}_N(\sum_{\pi(i)_1 = \pi(j)_1}(\ldots) + \sum_{\pi(i)_1 \neq \pi(j)_1}(\ldots))$. The first sum reads explicitly

$$\sum_{l=0}^{d-1} |l\rangle\langle l| \otimes c_l^l \left(\sum_{\pi \in \mathcal{S}_{N-1}} |\pi(i^{(l)})\rangle\langle\pi(j^{(l)})| \right), \tag{120}$$

where $i^{(l)}$ and $j^{(l)}$ are such $(N-1)$-element multiindices whose joint distribution of individual indices is equal to $\vec{z}_l = (c_0^0, \ldots, c_{l-1}^l, c_l^l - 1, c_{l+1}^l, \ldots, c_{d-1}^{d-1})$. The most significant difference between the first and second sums is that in the second sum there appears $|k\rangle\langle l|$ with $k \neq l$ instead of $|l\rangle\langle l|$. In the expression for the partial trace the second sum therefore vanishes, as can be seen below. When we plug the expression for $P_{\vec{c}}$ into the formula for the partial trace, we obtain

$$\begin{aligned}
\mathrm{Tr}_1(P_{\vec{c}}) &= \sum_{k=0}^{d-1} \langle k|P_{\vec{c}}|k\rangle = \mathcal{N}_N \sum_{k=0}^{d-1} \langle k| \left(\sum_{\pi(i)_1 = \pi(j)_1} (\ldots) + \sum_{\pi(i)_1 \neq \pi(j)_1} (\ldots) \right) |k\rangle \\
&= \mathcal{N}_N \sum_{k=0}^{d-1} \langle k| \left(\sum_{l=0}^{d-1} c_l^l \, |l\rangle\langle l| \otimes \sum_{\pi \in \mathcal{S}_{N-1}} |\pi(i^{(l)})\rangle\langle\pi(j^{(l)})| \right) |k\rangle \\
&= \mathcal{N}_N \sum_{l=0}^{d-1} c_l^l \sum_{\pi \in \mathcal{S}_{N-1}} |\pi(i^{(l)})\rangle\langle\pi(j^{(l)})| = \mathcal{N}_N \sum_{l=0}^{d-1} c_l^l \, \frac{1}{\mathcal{N}_{N-1}^{(l)}} P_{\vec{z}_l} \\
&= \frac{1}{\sqrt{N}} \sum_{l=0}^{d-1} \sqrt{c_l^l} \, P_{\vec{z}_l},
\end{aligned} \tag{121}$$

where $P_{\vec{z}_l} \in \mathcal{B}(\mathscr{H}^{\otimes (N-1)})$ is a matrix defined analogously to (119) with a normalization factor $\mathcal{N}_{N-1}^{(l)}$. From the last line of equations above we see that whenever the joint distribution $\vec{c}$ satisfy $c_l^l = 0$ for all $0 \leq l \leq d-1$, then the partial trace of matrix $P_{\vec{c}}$ vanishes.

B Discussion for Partial Swap with Pure Collisions

In section 4 we investigated the quantum network evolution governed purely by partial swap-type interactions between individual constituents. For non-identical interaction times we derived the form of the asymptotic state by making use of our results obtained for identical interaction times. We emphasized only the situation when $N > d$ and left more detailed discussion to be summarized in this appendix.

Generically, when the number N of qudits in the network exceeds their dimensionality d, the network equilibrates. The resulting state is then of the form (28). There is only a zero-measure set of parameters ϕ_{ab} and collision times Δt_{ab} for which the system evolves in a different way. (Without loss of generality we again put $\phi_{ab} = 1$.) Specifically

$\lambda = 1$ If for all pairs $\Delta t_{ab} \neq k\pi$, $k \in \mathbb{Z}$, then matrix X is of the form (28); the attractor space dimension for given N and d is equal to $S_1(N, d)$, see (27). Otherwise the form of X is more general.

$\lambda \neq 1$ If there is (a, b) such that $\Delta t_{ab} = k\pi$, $k \in \mathbb{Z}$, then X is a zero matrix. If there is (a, b) such that $\Delta t_{ab} = (2k + 1)\pi/2$, $k \in \mathbb{Z}$, and $\lambda \neq -1$, then matrix X is zero; for $\lambda = -1$ we obtain more general form. Let $\Delta t_{ab} \neq k\pi/2$, $k \in \mathbb{Z}$, for all pairs (a, b). If there is moreover (a, b) such that $2\Delta t_{ab} \not\equiv \pm\omega \pmod{2\pi}$, then matrix X vanishes. Finally, if for all (a, b) one has $\Delta t_{ab} \neq k\pi/2$, $k \in \mathbb{Z}$ and $2\Delta t_{ab} \equiv \omega \pmod{2\pi}$ for all pairs or $2\Delta t_{ab} \equiv -\omega \pmod{2\pi}$ for all pairs, then the dimension of the attractor space for given N and d is equal to $S_{\neq 1}(N, d)$, see (36). If there are two doubles (a, b), (c, d) for which $2\Delta t_{ab} \equiv \omega \pmod{2\pi}$ and $2\Delta t_{cd} \equiv -\omega \pmod{2\pi}$, then X is of more general form.