CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

Frame defined by parallel transport for curves in any dimension

Repér definovaný paralelním přenosem pro křivky v libovolné dimenzi

Bachelor Thesis

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- Zadání práce -

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Název práce:

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Abstrakt: Tato práce je věnována zobecnění repéru definovaného paralelním přenosem do vyšších dimenzí. Nejprve jsou prezentovány základní výsledky teorie křivek, diferenciální geometrie a řešení diferenciálních rovnic. Poté je uvedeno porovnání tří různých repérů známých ve třech dimenzích, následované zobecněním repéru definovaného paralelním přenosem a modifikací fundamentálního teorému křivek. Nakonec je diskutována jedna z možných aplikací zobecněného repéru, konstrukce kvantových vlnovodů.

Klíčová slova: křivky, Frenetův repér, repér definovaný paralelním přenosem, vlnovody

Title:

Relatively parallel frame for curves in any dimension

Author: Kateřina Zahradová

Abstract: This thesis is devoted to the generalisation of the relatively parallel adapted frame to higher dimensions. Firstly, the fundamentals of curves, differential geometry and differential equation are provided, followed by an overview and comparison of three different moving frames in three dimensions is given. Then the generalisation of the relatively parallel adapted frame with the modification of the fundamental theorem of curves is presented. Lastly, one of possible applications of the relatively parallel adapted frame, the construction of quantum waveguides, is discussed.

Key words: curves, Frenet frame, relatively parallel frame, waveguides

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Introduction

The history of framing curves can be traced back as far as 1847 [28], when Jean Frenet submitted his doctoral thesis [9] in which he proposed the idea of attaching a frame to each point of a curve, and also included six of the Frenet-Serret formulas. Later, in 1852, this part of his thesis was published as a paper [10]. In 1853, he proposed the aplications in [11]. Although Frenet may not be the first to publish the Frenet formulas, as suggested in [12], the formulas are named after him and Joseph Serret, who independently derived all nine formulas in 1851. According to [31], the modern history of moving frames starts with Élie Cartan who elaborated a completely different theory of looking at curves in [3], namely the method of the repère mobile, by generalising the Darboux kinematical theory. This theory of moving frames formalised previous approaches to curves. For a long time, the use of Frenet frame remained the mostly preferred tool for examining the properties of curves, until Richard L. Bishop pointed out that there are more ways to frame a curve, namely he advocated the usage of the relatively parallel adapted frame in 1975 [1]. From that point, the newly proposed frame was treated in Eucledian space (e.g. [34]), in Minkowski space [16] and in dual space [17]. Until now, the approach taken by Bishop was generalised mostly with respect to the requirements on the curve (e.g. [21]). The generalisation to higher dimension, despite the fact that Bishop himself hinted the requirements in the previously mentioned article, was considered only in the last couple of years. The generalisation to $\mathbb{R}^4$ is presented in [13] and [19]. The proof of existence of the relatively parallel adapted frame in higher dimensions is hinted in [18]. However, to the extent of author's knowledge, no proper proof of the generalisation to an arbitrary dimension for curves of minimal regularity, as presented in this thesis, is known.

Moving frames have a whole range of applications, ranging from theoretical use in the study of (quantum) waveguides (see e.g. [21, 6, 14]) to more practical use in computer graphics, CNC planning [8] or biology and medical science. The practical applications in computer graphics include the generation of ribbons and tubes from 3D space curves, the generation of a new way to control virtual cameras, or rotating the camera orientations relative to stable forward-facing frame [15]. The relatively parallel adapted frame found very interesting applications in biology – such as in a model of protein folding or the DNA (e.g. [25, 27]), in real-time path planning for drug delivery robots [24] or in the study of geometrical risk factors of vascular problems [2]. In many of these applications, the relatively parallel adapted frame is preferred over the Frenet frame, as it has more appropriate behaviour and methods based on it are more robust (cf. [2, 15]).

This thesis is organized as follows. In the first chapter, an overview of curves is given. The second chapter is devoted to a small revision of differential geometry in order to define parallel transport. A recapitulation of solving first order differential equations for an unknown matrix is presented in the third chapter. The fourth chapter explains three different moving frames used in three dimension, two of which are adapted, and provides their comparison. In the fifth chapter, the generalisation of the relatively parallel adapted frame to higher dimension with a modification of the fundamental theorem of curves is given. The last chapter presents one of many possible application of the generalised relatively parallel adapted frame – the construction of waveguides based on the relatively parallel approach in higher dimensions.

Chapter 1

Curves

This chapter is devoted to a small recapitulation of the theory of curves. Most of the results are from [29], while some are also from [20].

Definition 1.0.1. A curve in $\mathbb{R}^n$ is a continuously differentiable mapping $c : I \to \mathbb{R}^n$ from any open interval $I := (a, b)$, i.e. $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, into $\mathbb{R}^n$.

Remark. Note that the only restriction on the interval I is its openness – the interval can be bounded or unbounded.

Definition 1.0.2. A curve $c : I \to \mathbb{R}^n$ is said to be of class $C^k(I)$ if the derivatives $c', \ldots, c^{(k)}$ exist and are continuous on the interval I. The curve c can also be of class $C^\infty(I)$, or smooth, if it has derivatives of all orders on I. Further more, the curve c is of class $C^\omega(I)$, or analytic, if c is smooth and if its Taylor series expansion around any point in its domain converges to the function in some neighbourhood of the point. Lastly, the curve c is of class $C^{k,\alpha}(I)$ if $c \in C^k(I)$ and $c^{(k)}$ satisfies the following condition:

$$(\exists C > 0)\,(\forall x, y \in I)\left(|c^{(k)}(x) - c^{(k)}(y)| \leq C|x - y|^\alpha\right).$$

More specifically, $c \in C^{1,1}(I)$ if c' is continuous and if it satisfies the Lipschitz[1] condition.

Stopping here for a moment, let us introduce two examples of curves which will guide us throughout this thesis – a helix η and the (so-called) Spivak[2] curve σ.

Example. A helix η is defined as

$$\eta(t) := (R\cos(t), R\sin(t), ut), \tag{1.1}$$

where $u, R \in \mathbb{R},\ u \neq 0,\ R > 0$ are constants characterizing the helix. R sets its radius and u defines the rate at which the curve rises (or falls), therefore the helix reduces to a straight line for $R = 0$ and to a circle for $u = 0$. An example of a helix can be found on Fig. 1.1.

[1] Rudolf Lipschitz (1832-1903), German mathematician who contributed to many areas of mathematics, mainly to mathematical analysis and differential geometry. This historical remark, as well as the other ones, were found in [28].

[2] Michal Spivak (*1940), American mathematician specializing in differential geometry. Because of his interest in some properties of the curve σ in [31], we named the curve after him for the purposes of this thesis.

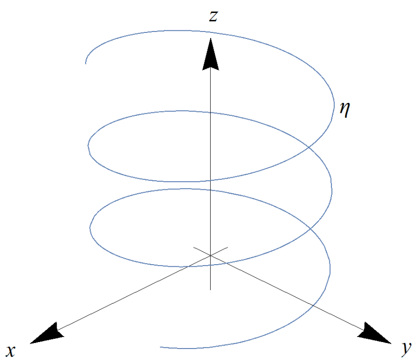

Figure 1.1: A helix η.

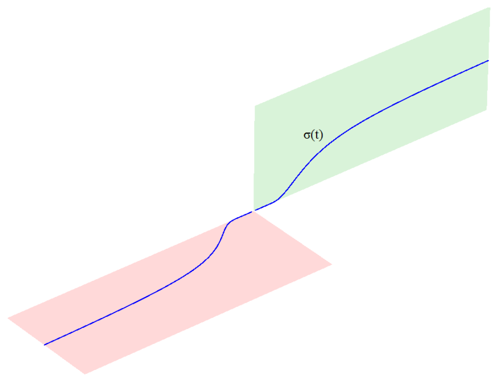

Figure 1.2: The Spivak curve σ.

Example. A Spivak curve σ (Fig. 1.2) is defined as

$$\sigma(t) := \begin{cases} (t, e^{-\frac{1}{t^2}}, 0) & \text{if } t > 0 \\ (t, 0, e^{-\frac{1}{t^2}}) & \text{if } t < 0 \\ 0 & \text{if } t = 0 \end{cases} . \tag{1.2}$$

It is easy to see that $\sigma \in C^\infty(\mathbb{R})$, however, it can be shown that σ is not analytical. It will be clear later that the sudden change of planes in which the curve lives at $t = 0$ makes its analysis harder.

Remark. The variable $t \in I$ is called the parameter of the curve.

Definition 1.0.3. Let $c : I \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$ with its components $c = (c_1, \ldots, c_n)$. For each point $t \in I$, the velocity vector of c at t is a tangent vector

$$c'(t) := \left(\frac{\mathrm{d}c_1}{\mathrm{d}t}(t), \frac{\mathrm{d}c_2}{\mathrm{d}t}(t), \ldots, \frac{\mathrm{d}c_n}{\mathrm{d}t}(t)\right)$$

at the point $c(t)$ in $\mathbb{R}^n$. The speed $|c'|$ of the curve c is defined as

$$|c'| = \left(\left(\frac{\mathrm{d}c_1}{\mathrm{d}t}\right)^2 + \cdots + \left(\frac{\mathrm{d}c_n}{\mathrm{d}t}\right)^2\right)^{1/2} .$$

Definition 1.0.4. Let I and J be open intervals in $\mathbb{R}$. Let $c : I \to \mathbb{R}^n$ be a curve and let $h : J \to I$ be a differentiable (real-valued) function. Then the composite function

$$\tilde{c} := c \circ h : J \to \mathbb{R}^n$$

is a curve and is called the reparametrization of c by h. For $h' \geq 0$ the reparametrization $\tilde{c}$ is called orientation-preserving, resp. orientation-reversing for $h' \leq 0$.

Lemma 1.0.1. If $\tilde{c}$ is a reparametrization of c by h, then

$$\tilde{c}'(s) = c'(h(s))\left(\frac{\mathrm{d}h}{\mathrm{d}s}(s)\right).$$

Proof. Using the chain rule for individual components of $c \circ h$ we obtain

$$c_i(h)'(s) = c_i'(h(s)) \cdot h'(s).$$

Therefore, by the definition of velocity,

$$\tilde{c}'(s) = (c(h(s)))' = (c_1'(h(s)) \cdot h'(s), \ldots, c_n'(h(s)) \cdot h'(s)) = c'(h(s)) \cdot h'(s).$$

□

Definition 1.0.5. Let $c : I \to \mathbb{R}^n$ be a curve. It is called regular if and only if its velocity c' is non-zero on the interval I.

Remark. Sometimes, regular curves are called immersions [31]. Note that the condition of regularity for a curve ensures that the curve will not have any corners or cusps and that it does not "stop".

Definition 1.0.6. The arc-length function s of a regular curve $c \in C^1$, $c : I \to \mathbb{R}^n$ is a real-valued function $s : I \to \mathbb{R}$. The arc-length function is said to be based on $\xi \in (a, b)$, if it is defined as

$$s(t) := \int_\xi^t |c'(u)| \mathrm{d}u.$$

Definition 1.0.7. The curve c is said to be parametrized by its arc-length if $|c'(t)| = 1,\ \forall t \in I$. Such curves are also called unit-speed curves.

Theorem 1.0.2. If c is a regular curve in $\mathbb{R}^n$, then there exists a reparametrization $\tilde{c}$ of c such that $\tilde{c}$ has unit speed.

Proof. Let $c : I \to \mathbb{R}^n$ be any regular curve. Fixing a number $\xi \in I$, we can consider the arc-length function based at $t = \xi$

$$s(t) = \int_\xi^t |c'(u)| \mathrm{d}u.$$

The derivative $\mathrm{d}s/\mathrm{d}t$ is the speed function $|c'|$ of c and since c is regular, $|c'| \neq 0$. Hence $\mathrm{d}s/\mathrm{d}t > 0$ and by the inverse function theorem, there exists s^{-1} which we will denote as $t = t(s)$. Letting $\tilde{c} = c \circ s^{-1} = c \circ t$, we get

$$\tilde{c}'(t(s)) = \frac{\mathrm{d}t}{\mathrm{d}s}(s) c'(t(s)).$$

The speed of $\tilde{c}$ then is

$$|\tilde{c}'(s)| = \frac{\mathrm{d}t}{\mathrm{d}s}(s)|c'(t(s))| = \frac{\mathrm{d}t}{\mathrm{d}s}(s)\frac{\mathrm{d}s}{\mathrm{d}t}(t(s)) = 1.$$

□

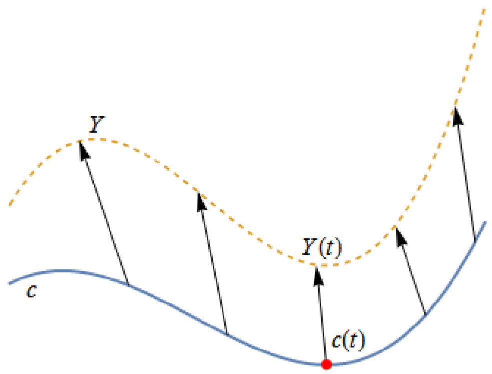

Figure 1.3: Example of a vector field Y on curve c.

Having a unit-speed curve allows us to simplify the mathematical manipulations later on, however, the process of reparametrization can turn a pleasant looking curve into an awful one. Therefore, we will try to give the option for any parametrization when possible.

Example. After the arc-length reparametrization, the helix η will look like

$$\eta(t) = \left(R\cos\left(\frac{t}{\sqrt{u^2+R^2}}\right), R\sin\left(\frac{t}{\sqrt{u^2+R^2}}\right), u\frac{t}{\sqrt{u^2+R^2}}\right). \tag{1.3}$$

The earlier equation (1.1) is much easier to work with even though it is not unit speed.

Now we shift focus towards vectors and finally assign our curve some frame. We will start with the definition of a vector.

Definition 1.0.8. A tangent vector Y_t in $\mathbb{R}^n$ consists of two points of $\mathbb{R}^n$ – its vector part Y and its point of application t.

Remark. Two tangent vectors are equal if and only if they have the same vector part and the same point of application. Tangent vectors Y_t, Y_q with the same vector part but with different points of applications are said to be parallel. Any Y_t, Y_q are different if $t \neq q$.

Definition 1.0.9. A vector field Y on a curve $c : I \to \mathbb{R}^n$ is a function that assign each $t \in I$ a tangent vector $Y(t)$ at the point $c(t)$.

Remark. A velocity of a curve is an example of such a vector field. Note that a tangent vector field does not need to be tangential to c – cf. Fig. 1.3.

However, we can define a vector field tangent to a curve:

Definition 1.0.10. A tangent vector field of $c : I \to \mathbb{R}^n$ is a vector field along c defined by $t \mapsto c'(t)f(t)$, where f is arbitrary scalar function.

Once we have a tangent vector field along a curve, the opportunity for another vector fields along c arises. We would like to assign each curve a collection of such vector fields satisfying special conditions.

Definition 1.0.11. Let $c : I \to \mathbb{R}^n$ be a curve. A moving n-frame along c is a collection of n differentiable mappings

$$e_i : I \to \mathbb{R}^n, \quad i = 1, \ldots, n,$$

such that for all $t \in I$, $e_i(t) \cdot e_j(t) = \delta_{ij}$, where $\cdot$ denotes the scalar product and δ_{ij} is Kronecker[3] delta. Each $e_i(t)$ is a vector field along c. The parameter t can sometimes be called time as it corresponds to the path taken by the curve and the fields.

Definition 1.0.12. A moving n-frame along a curve c is called adapted if all its components are either tangential or normal to the curve.

Remark. Surely, all moving frames incorporating a tangent vector field are adapted and vice versa.

Theorem 1.0.3. Let $c : I \to \mathbb{R}^n$ be a curve and $(e_1, \ldots, e_n)$ its moving n-frame. Then the time development of the frame can be written as

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}' = \mathcal{A} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix},$$

where $\mathcal{A}$ is a skew-symmetrical matrix of dimension n with entries dependent on t and $'$ denotes a differentiation with respect to the parameter.

Proof. Considering the scalar product $e_i \cdot e_j = \delta_{ij}$ and differentiating it with respect to the curve parameter, we obtain the following

$$e_i' \cdot e_j = -e_i \cdot e_j.$$

From that it is clear that $\mathcal{A}_{ij} = -\mathcal{A}_{ji}$ and thus the matrix $\mathcal{A}$ is skew-symmetric. $\square$

Definition 1.0.13. The skew-symmetrical matrix $\mathcal{A}$ from the previous theorem is called a Cartan[4] matrix.

Remark. The reason why skew-symmetrical matrices are called Cartan matrices will be clear in the next chapter.

Knowing what an adapted moving frame is, we can ponder further on some of its possible properties. A natural question is about the spinning of its components, which is resolved by the following definition.

Definition 1.0.14. The moving frame $e_1, \ldots, e_n$ minimizes rotation along e_i, if e_j' can be written as $e_j' = k_j e_i$, $\forall j = 1, \ldots, n$, for some functions k_j, $j = 1, \ldots, n$.

[3]Leopold Kronecker (1823-1891), German mathematician who worked on the algebraic number theory and logic. He believed that mathematics should only deal with integers and finite number of operations.

[4]Élie Joseph Cartan (1869-1951), French mathematician who made significant contributions to the group theory, mathematical physics and differential geometry.

Remark. The rotation minimizing property for some moving frame effectively means that some entries of its Cartan matrix $\mathcal{A}$ are permanently zero.

Focusing only on $\mathbb{R}^3$, one quickly arrives to the conclusion that there are only three different options, due to the skew-symmetry, for rotation minimizing frames with entries a, b:

$$\begin{pmatrix} 0 & a & 0 \\ * & 0 & b \\ 0 & * & 0 \end{pmatrix} \quad \text{as Frenet frame,}$$

$$\begin{pmatrix} 0 & a & b \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \quad \text{as the relatively parallel adapted frame, and}$$

$$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ * & * & 0 \end{pmatrix} \quad \text{as the rotation minimizing osculating frame,}$$

where $*$ denotes entries which could be determined using the skew-symmetry.

Chapter 2

Differential geometry

In this chapter, some essential results from differential geometry are presented, with the focus of those needed later on. This chapter derives from [22] and some results are adopted from [30, 26, 29]. As only the smooth manifolds are usually discussed, we adopted this strategy for our thesis. This, however, does not restrict us as explained later on.

Definition 2.0.15. A k-dimensional differentiable manifold (a k-manifold) is a set M together with a family $(M_i)_{i\in I}$ of subsets such that

1. $M = \cup_{i\in I} M_i$
2. for every $i \in I$ there is an injective map $\varphi_i : M_i \to \mathbb{R}^k$ so that $\varphi_i\,(M_i)$ is open in $\mathbb{R}^k$, and
3. for $M_i \cap M_j \neq \emptyset$, $\varphi_i(M_i \cup M_j)$ is open in $\mathbb{R}^k$ and the composition
$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(M_i \cup M_j) \to \varphi_j(M_i \cup M_j)$$
is differentiable for arbitrary i, j.

Each φ_i is called a chart, φ_i^{-1} is referred to as the parametrization, the set $\varphi_i(M_i)$ is called the parameter domain, and $(M_i, \varphi_i)_{i\in I}$ is called an atlas. The maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(M_i \cup M_j) \to \varphi_j(M_i \cup M_j)$, defined on the intersection of two such charts, are called coordinate transformations or transition functions. We may assume, without loss of generality, that the atlas is the maximal with respect to adding more charts satisfying the conditions 2 and 3 above. A maximal atlas in this sense is then referred to as a differentiable structure.

Definition 2.0.16. Given a k-manifold, we can get additional structure by placing additional requirements on the transformation functions $\varphi_j \circ \varphi_i^{-1}$, which belong to the atlas of the manifold. There are many different possible requirements; however, we will only use the following: if all $\varphi_j \circ \varphi_i^{-1}$ are C^∞-differentiable, the manifold is called a C^∞-manifold. Note that there is a convention that by the term "manifold" we always mean the C^∞-manifold.

Remark. Whitney proved in [33] that every C^1-manifold has a compatible C^∞ atlas, effectively resulting in that every C^1-manifold is a C^∞-manifold. Therefore, our restriction on smooth manifolds is, in fact, not a restriction at all.

Assumption: Our manifold satisfies the Hausdorff separation axiom (T_2-axiom), which states that that for every two distinct points on the manifold p, q, there exist disjoint open neighbourhoods U_p, U_q.

The important thing is that locally the topology of the k-manifold is the same as that of an $\mathbb{R}^k$. In particular, images of open ε-balls in $\mathbb{R}^k$ are open in M. However, as we have yet to define a metric on the manifold, we cannot make sense of a ball in M.

Definition 2.0.17. Let M be an m-manifold, N be an n-manifold and $F : M \to N$ a given map. F is said to be differentiable if for all charts $\varphi : U \to \mathbb{R}^m$, $\psi : V \to \mathbb{R}^n$ with $F(U) \subset V$ the composite function $\psi \circ F \circ \varphi^{-1}$ is also differentiable.

Next, we need to define tangent vectors. There are three equivalent definitions – geometric, algebraic and physical. These are brief versions, for a full definitions see e.g. [22].

Geometric definition: Tangent vectors are tangent to curves lying on the manifold.

Algebraic definition: Tangent vectors are derivations acting on scalar functions.

Physical definition: Tangent vectors are elements of $\mathbb{R}^n$ with a particular transformation behaviour.

Regardless of our definition, every tangent vector at point $p \in M$ can be expressed in a form

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}\bigg|_p,$$

where $X^i = X\varphi^i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi^i \circ \gamma\right)\Big|_{t=0}$ for some curve γ from the geometrical definition and $\frac{\partial}{\partial x^i}|_p$ are basis vectors corresponding to the coordinates x_i. From now on, the Einstein[5] summation is used: sums are formed over indices which occur in formulas both as a subscript and superscripts.

Definition 2.0.18. The tangent space T_pM of M at p is defined as the set of all tangent vectors at the point p. By definition, T_pM and T_qM are disjoint it $p \neq q$. By TM we mean a disjoint union of T_pM, $\forall p \in M$, i.e. $\bigsqcup_{p\in M} T_pM$.

Definition 2.0.19. A differentiable vector field X on a differentiable manifold is a mapping $M \ni p \mapsto X_p \in T_pM$, $(X : M \to TM)$, defined as $X(p) = X^i(p)\frac{\partial}{\partial x^i}$, where X^i are differentiable functions. The set of vector fields on M is denoted by $\mathcal{X}(M)$.

Remark. Note that this definition agrees with the previous one, (1.0.9).

Definition 2.0.20. Let X, Y be vector fields on M. We define a vector field $[X, Y]$ through the relation

$$[X, Y] = X \circ Y - Y \circ X = \left(X(Y^j) - Y(X^j)\right)\frac{\partial}{\partial x^j}.$$

$[X, Y]$ is called the Lie[6] bracket of X, Y and it is also the Lie derivative $\mathcal{L}_X Y$ of Y in the direction X.

[5] Albert Einstein (1879-1955), German-born theoretical physicist who developed both the special and the general theory of relativity and much more.

[6] Marius Sophus Lie (1842-1899), Norwegian mathematician who contributed to the group theory, differential equations and geometry.

Remark. The Lie bracket measures the non-commutativity of the derivatives.

Definition 2.0.21. The dual space T_p^*M to the tangent space T_pM at the point p of a manifold is called the cotangent space. The members of the cotangent space are called 1-forms at point p and usually denoted by small Greek letters, i.e. $\omega : T_pM \to \mathbb{R}$, $\omega(aX + Y) = a\omega(X) + \omega(Y)$. The cotangent space T^*M is defined as $T^*M = \bigsqcup_{p\in M} T_p^*M$.

In order to properly define 1-forms, we need to introduce a whole new structure to our manifold, a fibre bundle.

Definition 2.0.22. A fibre bundle is a manifold E, called the total manifold, equipped with the following additional structures:

1. a manifold M, which is called the base manifold, with a surjective mapping called the projection $\pi : E \to M$ and a open set covering U_α
2. a manifold F called the typical fibre with a diffeomorphisms $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ called local trivializations, fulfilling that for the projection π_1 onto the first component of the set product, $\pi_1 \circ \psi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$.

Definition 2.0.23. A (cross) section of the fibre bundle E is a differentiable map $\sigma : M \to E$, such that $\pi \circ \sigma$ is an identity on M. The set of all sections of E is denoted by $\Gamma(E)$.

Remark. Both the tangent space TM and the cotangent space T^*M can be understood as examples of fibre bundles. Therefore, we may introduce their cross sections – vector fields are cross sections of the tangent space. We used the other definition instead as it is more illustrative.

Definition 2.0.24. A differential 1-form ω defined on a manifold M is a cross section of the cotangent fibre bundle, i.e. $\omega \in \Gamma(T^*M)$.

After defining vector fields and 1-forms, we have something which we can measure, however, we do not have anything to measure it with. Hence, we continue with the definition of a metric.

Definition 2.0.25. A Riemannian[7] metric g on M is an association $p \mapsto g_p \in L^2(T_pM, \mathbb{R}) = \{\alpha : T_pM \times T_pM \to \mathbb{R} \,|\, \alpha \text{ bilinear}\}$ such that the following conditions are satisfied:

1. $g_p(X, Y) = g_p(Y, X)$ for all $X, Y \in T_pM$; (symmetry)
2. $g_p(X, X) > 0$ for all $X \neq 0$; (positive definiteness)
3. the coefficients g_{ij} in every local representation (i.e., in every chart)

$$g_p := g_{ij}(p) \cdot \mathrm{d}x^i\,|_p\, \mathrm{d}x^j\,|_p$$

are differentiable functions. (differentiability)

[7] Georg Friedrich Bernhard Riemann (1826-1866), German mathematician who made revolutionary contributions to analysis, number theory and differential geometry.

The pair (M, g) is called a Riemannian manifold. The Riemannian metric is also called the metric tensor and in local coordinates is given by the matrix (g_{ij}).

Remark. The metric tensor g defines an inner (scalar) product g_p at every point p on the tangent space T_pM. Therefore, the notation $g_p(X, Y) =: \langle X, Y \rangle$ is sometimes used.

Definition 2.0.26. The inverse g^{-1} to the metric g is given by its coefficients g^{il} such that

$$g^{il} g_{lk} = \delta^i_k,$$

where δ^i_k behaves like the normal Kronecker delta.

The question whether a Riemannian metric exists on an arbitrary manifold M is not trivial and it will not be discussed here.

So far we have defined what a manifold is, what a tangent vector is, what a vector field is and we have a scalar product. Nevertheless, we would like to be able to differentiate the vectors as we do with functions in $\mathbb{R}^n$. There are multiple ways how to proceed with this task on a manifold. We can use either the Lie derivative $\mathcal{L}$, the exterior derivative d or the covariant derivative ∇. Our goal is to find the nearest possible way of differentiation to the one known in $\mathbb{R}^n$ and the key is to understand how differentiation actually works. When taking a derivative of a function of one variable, one essentially subtracts values of the function in two different points. This cannot be just mindlessly done for a vector field. We already established that tangent vectors at different points may not be compared in between, so there arises a question how to move vectors on a manifold. It is easy to move vectors in a straight space using a ruler and knowledge from elementary school. However, we are not in a straight space any more – we do not even know how the space looks like or how it behaves, so we cannot just translate the vectors in any arbitrary way. One way to solve that is by means of a connection.

Definition 2.0.27. The affine connection ∇ on a manifold M is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) : (X, Y) \mapsto \nabla(X, Y) =: \nabla_X Y$ that is uniquely defined by the following properties:

1. $\nabla_{fX+Y}(Z) = f\nabla_X(Z) + \nabla_Y(Z)$, (linearity and additivity in the subscript)
2. $\nabla_X(aY + Z) = a\nabla_X(Y) + \nabla_X(Z)$, (additivity and $\mathbb{R}$ linearity in the argument)
3. $\nabla_X(fY) = f\nabla_X(Y) + (Xf)Y$, (Leibniz[8] rule)

$\forall f \in C^\infty$, $\forall a \in \mathbb{R}$, $X, Y, Z \in \mathcal{X}(M)$.

Remark. Note that, due to the third property, the affine connection is not a tensor. However, the difference of any two affine connections is.

[8]Gottfried Wilhelm von Leibniz (1646-1716), German polymath who contributed to philosophy and mathematics. He developed the differential and integral calculus independently of Isaac Newton. His another major mathematical work was on determinants.

When we find ourselves on a Riemannian manifold, then there exists a special affine connection, the Levi-Civita[9] connection, which satisfies the following additional properties:

4. $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, (compatibility with the metric)
5. $\nabla_X Y - \nabla_Y X - [X,Y] = 0$. (symmetry)

Definition 2.0.28. The Levi-Civita connection can also be called a covariant derivative.

We can write out the components of Levi-Civita connection using the Christoffel[10] symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{jl,k} + g_{lk,j} - g_{jk,l} \right),$$

where $g_{ij,k} := \frac{\partial}{\partial x^k} g_{ij}$ as

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

Using the properties listed above we obtain for a general vector $Y = \eta^j \frac{\partial}{\partial x^j}$ and a special choice of $X = \frac{\partial}{\partial x^i}$

$$\nabla_X(Y) = (\frac{\partial \eta^k}{\partial x^i} + \Gamma^k_{ij}\eta^j)\frac{\partial}{\partial x^j}.$$

The Levi-Civita connection is sometimes called the Riemann connection as well as the covariant derivative. Knowing what it is, we may proceed with the following definitions:

Definition 2.0.29.

1. A vector field Y is said to be parallel if $\nabla_X Y = 0$ for every vector field X.
2. A vector field Y along a regular curve c is parallel along the curve c if $\nabla_{c'} Y = 0$.
3. A regular curve c is called a geodesic if $\nabla_{c'} c' = \lambda c'$ for some scalar function λ.

Understanding what a parallel vector field is does not answer the burning question if there exists such field along any curve. The following lemma will answer that.

Lemma 2.0.4.

1. Along an arbitrary regular curve c there is for every $Y_0 \in T_{c(t_0)}M$ a vector field Y along c which is parallel along c and which satisfies $Y(t_0) = Y_0$. This vector field Y is called the parallel displacement of Y_0 along c.
2. Parallel displacement preserves the Riemannian metric, i.e., $\langle Y_1, Y_2 \rangle$ is constant for any two parallel vector fields Y_1, Y_2 along c.

[9]Tullio Levi-Civita (1873-1941), Italian mathematician mostly known for his work on tensor calculus and its applications to the theory of relativity.

[10]Elwin Bruno Christoffel (1829-1900), German mathematician and physicist focused on differential geometry and tensor calculus.

Proof. 1. Let c be a regular curve, denoting the vector field Y as $Y(t) = \eta^j \frac{\partial}{\partial x^j}$, where $\frac{\partial}{\partial x^j}$ are local charts. Using the relations of ∇ mentioned above and using the chain rule, the expression reads

$$\nabla_{c'} Y = \left(c'^i(t) \frac{\partial \eta^k(t)}{\partial x^i} + c'^i(t) \eta^j(t) \Gamma^k_{ij}(c(t)) \right) \frac{\partial}{\partial x^k}.$$

The requirement that Y should be parallel is thus equivalent to the system of ordinary differential equations

$$\eta'^k(t) + c'^i(t) \eta^j(t) \Gamma^k_{ij}(c(t)) = 0$$

for the functions $\eta^k(t)$, $k = 1, \ldots, n$. This system is linear, hence there exists exactly one solution in the given interval for given initial condition $\eta^1(t_0), \ldots, \eta^n(t_0)$ [23].

2. Follows directly from the definition of the Levi-Civita connection, more specifically from

$$\nabla_{c'} \langle Y_1, Y_2 \rangle = \langle \nabla_{c'} Y_1, Y_2 \rangle + \langle Y_1, \nabla_{c'} Y_2 \rangle = 0,$$

for any vector fields Y_1, Y_2 parallel along any regular curve c.

□

Lemma 2.0.5. The parallel displacement of Y_0 along c is unique.

Proof. Let us consider two parallel displacements of Y_0 along c: Y_1, Y_2; $Y_i' = k_i c'$, $i = 1, 2$ with the initial condition $Y_1(t_0) = Y_2(t_0) = Y_0$. Denoting the difference of those two fields as $\Delta Y := Y_1 - Y_2$, we can write $(\Delta Y)' = (k_1 - k_2)c'$. Then we have

$$0 = \langle \Delta Y, (\Delta Y)' \rangle = \frac{1}{2} (|\Delta Y|^2)',$$

from which follows that ΔY is constant along c. However, the initial condition at t_0 yields $\Delta Y(t_0) = 0$, implying that $Y_1 = Y_2$ along the whole curve. □

Now we will look on the Fermi[11]-Walker[12] transport which is a generalisation of the covariant derivative. It is used in the theory of general relativity for the definition of orthonormal frames in such way that the whole curvature of the frame is given only by the presence of energy (or mass) and not by the rotation of the frame.

Definition 2.0.30. Let $(\mathbb{R}^{n+1}, g)$ be a manifold with a metric $g := \mathrm{diag}(-1, +1, \ldots, +1)$. Then g is called the Minkowski[13] metric and is denoted as $\eta := g$. The manifold is called Minkowski space. We can differentiate between vectors of the Minkowski space though their norms as:

$$\begin{aligned} \eta(X, X) &> 0 \quad \text{if } X \text{ is a time-like vector,} \\ &= 0 \quad \text{if } X \text{ is a light-like vector,} \\ &< 0 \quad \text{if } X \text{ is a space-like vector.} \end{aligned}$$

A regular curve $c : I \to \mathbb{R}^{n+1}$ is called space-like (resp. time-like or light-like) if its tangent vector is space-like (resp. time-like or light-like) everywhere.

[11]Enrico Fermi (1901-1954), Italian physicist, creator of the first nuclear reactor, contributor to the development of the quantum theory, nuclear and particle physics, statistical mechanics.

[12]Arthur Geoffrey Walker (1909-2001), British mathematician who made important contributions to physics and cosmology, especially general relativity.

[13]Hermann Minkowski (1864-1909), Polish-German mathematician best known for his geometrical interpretation of special relativity.

Definition 2.0.31. Let A^μ be a vector in the Minkowski space and γ be time-like curve. Then we define the Fermi-Walker derivative of A^μ as

$$\frac{\delta A^\mu}{\delta s} := \frac{DA^\mu}{ds} - (a^\mu u_\nu - u^\mu a_\nu)\, A^\nu,$$

where $\frac{DA^\mu}{ds}$ denotes the parallel transport of the vector A^μ, u^μ is the 4-velocity of γ and $a^\mu = \frac{Du^\mu}{ds}$ is its acceleration. The Fermi-Walker transport of the vector A^μ along a time-like curve γ is then given as the solution of

$$\frac{\delta A^\mu}{\delta s} = 0.$$

The Fermi-Walker derivative of a space-like vector with respect to a time-like vector defines non-inertial but non-rotating frames (under the condition that their Fermi-Walker derivatives vanishes). When applied to inertial frames, the Fermi-Walker behaves just like the normal covariant derivative.

Finally, the reason why the skew-symmetric matrices determining the time evolution of moving frames are called Cartan matrices will be illuminated. Using [31] and [29] as our guides and recalling the moving frames from the previous chapter, we will now study them in more abstract form. Constricting ourselves to a plain Euclid space $\mathbb{R}^n$ with the Euclid metric $g := \mathrm{diag}(1, \ldots, 1)$, we may simplify the notations to the one normally used in $\mathbb{R}^n$. The scalar product, for example, will be denoted as $g(X, Y) := X \cdot Y$.

Definition 2.0.32. Vector field $E_1, \ldots, E_n$ on $\mathbb{R}^n$ constitute a frame field on $\mathbb{R}^n$ provided

$$E_i \cdot E_j = \delta_{ij}, \quad 1 \leq i, j \leq n$$

where δ_{ij} is the Kronecker delta.

The immediate advantage of having a frame field is the possibility to express its covariant derivative in the terms of itself.

Lemma 2.0.6. Let $E_1, \ldots, E_n$ be a frame field on $\mathbb{R}^n$. For each tangent vector V to $\mathbb{R}^n$ at the point p, let

$$\omega_{ij}(V) = \nabla_V E_i \cdot E_j(p), \quad 1 \leq i, j \leq n.$$

Then each ω_{ij} is a 1-form and $\omega_{ij} = -\omega_{ji}$. There 1-forms are called connection forms of the frame field $E_1, \ldots, E_n$.

Proof. For ω_{ij} to be a 1-form, it needs to be a linear real valued function on the tangent vectors. Therefore we must only check the linearity:

$$\begin{aligned}
\omega_{ij}(aV + bW) &= \nabla_{aV+bW} E_i \cdot E_j(p) \\
&= (a\nabla_V E_i + b\nabla_W E_i) \cdot E_j(p) \\
&= a\nabla_V E_i \cdot E_j(p) + b\nabla_W E_i \cdot E_j(p) \\
&= a\omega_{ij}(V) + b\omega_{ij}(W).
\end{aligned}$$

The only remaining thing is to check the skew-symmetry. To prove that, we need to show that $\omega_{ij}(V) = -\omega_{ji}(V)$. Using the definition of the frame field and the constant value of the Kronecker delta, we get

$$0 = V[\delta_{ij}] = V[E_i \cdot E_j] = \nabla_V E_i \cdot E_j(p) + E_i(p) \cdot \nabla_V E_j.$$

Therefore, due to the symmetry of the dot product, $0 = \omega_{ij}(V) + \omega_{ji}(V)$. □

To address the geometrical significance of the connection forms, let us have a closer look at the definition: $\omega_{ij}(V) = \nabla_V E_i \cdot E_j(p)$. It shows that $\omega_{ij}(V)$ is the initial rate at which E_i rotates to E_j as p moves in the direction of V.

Theorem 2.0.7. Let ω_{ij}, $1 \leq i, j \leq n$ be the connection forms of the frame field $E_1, \dots, E_n$ on $\mathbb{R}^n$. Then for any vector field V on $\mathbb{R}^n$,

$$\nabla_V E_i = \omega_{ij} E_j, \quad 1 \leq i, j \leq n.$$

We call these the connection equations of the frame field $E_1, \dots, E_n$.

Proof. For any fixed i, both sides are vector fields. Therefore we must show that at each point p,

$$\nabla_{V(p)} E_i = \omega_{ij}(V(p)) E_j(p).$$

But as the $E_1, \dots, E_n$ is a frame field, we can $\nabla_{V(p)} E_i$ write out in the terms of itself and it is a consequence of the definition of ω_{ij}. □

From studying the connection forms, we find that, thanks to the skew-symmetry, the number of independent ones reduces greatly. Writing them out in a form of a skew-symmetric matrix of 1-forms, we have

$$\omega = \begin{pmatrix} 0 & \omega_{12} & \dots & \omega_{1n} \\ -\omega_{12} & 0 & \dots & \omega_{2n} \\ \vdots & & \ddots & \\ -\omega_{1n} & -\omega_{2n} & \dots & 0 \end{pmatrix}.$$

These connections are sometimes also called Cartan connections and hence the name for the skew-symmetric matrices associated with the frame fields.

Chapter 3

Differential equations

It this chapter, a quick summary of [35] is given on the first order systems of differential equations in matrix form, since these differential equations are not that common. The results are essential later on in Chapter 5 for proving the existence and uniqueness of a relatively parallel adapted moving frame for regular $C^{1,1}$ curve in any dimension.

Notation: Let $J \subset \mathbb{R}$ be any interval, open, closed, half open, bounded or unbounded. Then by $L^1(J;\mathbb{R})$[14] we denote the linear manifold of real valued Lebesgue measurable functions y defined on J for which

$$\int_J |y(t)|\mathrm{d}t =: \int_J |y| < \infty.$$

The linear manifold of locally integrable functions is denoted by $L^1_{loc}(J;\mathbb{R})$. A function y is locally integrable on an interval J if and only if y is integrable on any compact interval $[\alpha,\beta] \subseteq J$. The collection of functions y which are absolutely continuous on all compact intervals $[\alpha,\beta] \subseteq J$ is denoted by $AC_{loc}(J)$.

For any given set S, $M_{n,m}(S)$ denotes the set of $n \times m$ matrices with entries from S. If $n = m$ we write $M_n(S) := M_{n,n}(S)$. A matrix function is absolutely continuous (resp. (locally) integrable) if each of its components are absolutely continuous ((locally) integrable).

Definition 3.0.33. Let J be any interval, $n, m \in \mathbb{N}$. Let $P : J \to M_n(\mathbb{R})$, $F : J \to M_{n,m}(\mathbb{R})$. By a solution of the equation

$$Y' = PY + F \quad \text{on } J$$

we mean a function $Y : J \to M_{n,m}(\mathbb{R})$ which is absolutely continuous on all compact subintervals of J and satisfies the equation above on J.

The theorem about the existence and uniqueness of the solution follows.

Theorem 3.0.8. Let J be any interval, $n, m \in \mathbb{N}$. If

$$P \in M_n(L^1_{loc}(J;\mathbb{R})) \quad \text{and} \quad F \in M_{n,m}(L^1_{loc}(J;\mathbb{R}))$$

then every initial value problem (IVP)

$$\begin{aligned} Y' &= PY + F, \\ Y(u) &= C, \quad u \in J, \quad C \in M_{n,m}(\mathbb{R}) \end{aligned} \tag{3.1}$$

[14]Originally, Zettl uses $\mathbb{C}$ instead of $\mathbb{R}$. However, we work only in real numbers.

has a unique solution defined on all of J.

Proof. There are two possible proofs of this theorem, both included in [35]. Here, the standard successive approximations proof is given. First note that if Y is a solution of IVP (3.1), then an integration yields

$$Y(t) = C + \int_u^t (PY + F), \quad t \in J. \tag{3.2}$$

Conversely, every solution of the integral equation (3.2) is also a solution of the IVP (3.1).

To prove the existence of the solution we construct a solution of the integral equation (3.2) by the method of successive approximations. Define

$$\begin{aligned} Y_0(t) &:= C, \\ Y_{n+1}(t) &:= C + \int_u^t (PY_n + F) \end{aligned}$$

for $n = 0, 1, 2, \dots$ and some $t \in J$. Then Y_n is continuous on J for each $n \in \mathbb{N}_0$. We will show that the sequence $\{Y_n\}_{n=0}^{\infty}$ uniformly converges to a function Y on each compact subinterval of J and that the limit function Y is the unique solution of the integral equation (3.2) and hence also of the IVP (3.1).

Choose $b \in J,\ b > u$ and define

$$p(t) = \int_u^t |P(s)| \mathrm{d}s, \quad t \in J$$

and

$$B_n(t) = \max_{u \le s \le t} |Y_{n+1}(s) - Y_n(s)|, \quad u \le t \le b.$$

Then

$$Y_{n+1}(t) - Y_n(t) = \int_u^t P(s)\,[Y_n(s) - Y_{n-1}(s)]\,\mathrm{d}s, \quad t \in J,\ n \in \mathbb{N}.$$

From this we get, for $u \le t \le b$,

$$\begin{aligned} |Y_2(t) - Y_1(t)| &\le B_0(t) \int_u^t |P(s)| \mathrm{d}s = B_0(t)p(t) \\ &\le B_0(b)p(b), \\ |Y_3(t) - Y_2(t)| &\le \int_u^t |P(s)||Y_2(s) - Y_1(s)| \mathrm{d}s \le \int_u^t |P(s)| B_0(s)p(s) \mathrm{d}s \\ &\le B_0(t) \int_u^t |P(s)|p(s) \mathrm{d}s \le B_0(b)\frac{p^2(t)}{2!} \\ &\le B_0(b)\frac{p^2(b)}{2!}. \end{aligned}$$

Using mathematical induction we get

$$|Y_{n+1}(t) - Y_n(t)| \le B_0(b)\frac{p^n(b)}{n!}, \quad u \le t \le b.$$

Hence for any $k \in \mathbb{N}$

$$
\begin{aligned}
|Y_{n+k+1}(t) - Y_n(t)| &\le |Y_{n+k+1}(t) - Y_{n+k}(t)| + |Y_{n+k}(t) - Y_{n+k-1}(t)| + \cdots + |Y_{n+1}(t) - Y_n(t)| \\
&\le B_0(b)\frac{p^n(b)}{n!}\left[1 + \frac{p(b)}{n+1} + \frac{p^(b)}{(n+2)(n+1)} + \dots\right].
\end{aligned}
$$

Choose m large enough so that for $n > m$, $p(b)/(n+1) \le 1/2$, and so $p^2(b)/[(n+1)(n+2)] \le 1/4$, etc. Then the term $[\dots]$ has the upper bound of 2. Therefore the sequence $\{Y_n\}_{n=0}^{\infty}$ converges uniformly to some Y on $[u, b]$. From this follows that Y satisfies the integral equation (3.2) and hence also the IVP (3.1) on $[u, b]$. There is a similar proof for the case when $b < u$.

To prove the uniqueness of the solution Y, assume another solution Z. Then Z must be also continuous and therefore $|Y - Z| \le M$ for some $M > 0$ on $[u, b]$. Then

$$
|Y(t) - Z(t)| = \left| \int_u^t P(s)\,[Y(s) - Z(s)]\,\mathrm{d}s \right| \le M \int_u^t |P(s)|\mathrm{d}s \le Mp(t), \quad u \le t \le b.
$$

Following the same practice as above,

$$
|Y(t) - Z(t)| \le M\frac{p^n(t)}{n!} \le M\frac{p^n(b)}{n!}, \quad u \le t \le b,\ n \in \mathbb{N}.
$$

Therefore $Y = Z$ on $[u, b]$. It can be proven similarly for $b < u$. Overall, the theorem holds. □

We will later use this theorem to show that there exists a rotation matrix $\mathcal{R} \in M_n(L^1_{loc}(I;\mathbb{R}))$ with some special properties.

Chapter 4

Frames in three dimensions

This chapter provides a summary of the moving frames in three dimensions that are already known. First, the Frenet[15] frame is discussed, followed by the relatively parallel adapted frame and the rotation minimizing osculating frame. Following the approach of Spivak in [31], we will only consider curves $c : (a, b) \to \mathbb{R}^3$ which are immersions, i.e. their first derivative is non-vanishing on the whole interval. This demand is very natural as only for these curves the tangent can be constructed at all points. For such curves the arc-length $s : (a, b) \to \mathbb{R}$ based at $\xi \in (a, b)$, defined as

$$s(t) = \int_{\xi}^{t} |c'(u)| \mathrm{d}u,$$

is a bijection from (a, b) to (L_1, L_2), where $L_1 = s(a), L_2 = s(b)$. As seen in Chapter 1, the arc-length can be used to reparametrize the original curve c so that the final curve $\Gamma = c \circ s^{-1}$ has its derivative equal to one, $|\dot{\Gamma}| \equiv 1$. From now on, any curve with arbitrary parametrization is denoted by small Latin letters (generally by c), the parameter will be usually denoted by t and the derivative will be indicated as c'. On the other hand, for curves parametrized by arc-length, capital Greek letters (Γ), parameter s and derivative $\dot{\Gamma}$ are used. With the conventions established, we can continue with the Frenet frame.

4.1 The Frenet frame

Starting with the Frenet frame is very natural – this frame is the oldest and also the most widely used among the three frames discussed in this thesis. The Frenet-Serret formulas were determined by Frenet in 1847 and independently by Serret[16] in 1851 [29]. As the Frenet frame is adapted, we must start by identifying the tangent vector.

Definition 4.1.1. Let $\Gamma : (L_1, L_2) \to \mathbb{R}^3$ be a C^2 curve. Then the unit tangent vector field on Γ is defined as $T := \dot{\Gamma}$.

Remark. Note that the arc-length parametrization ensures that the tangent has unit length, i.e. $|T(s)| = |\dot{\Gamma}(s)| = 1$.

[15] Jean Frédéric Frenet (1816-1900), French mathematician, astronomer and meteorologist. Best known for discovering the Frenet-Serret formulas.

[16] Joseph Alfred Serret (1819-1885), French mathematician.

Next, the “straightness” of a curve will be inspected. As easy as it is to conclude that a straight line is straight and that the smaller the circle the more it curves, the generalisation is not obvious. By following our intuition that the curvature of a circle is the reciprocal of its radius, as smaller radius means that the cicrle is more curved, we can define the curvature of an arbitrary curve using the same approach as [31]. Nevertheless, our approach inspired by [29] will be less intuitive, but much quicker. Using analogy with mechanics, T represents the speed of a voyager along the curve and hence $\dot{T}$ stands for the curve acceleration. Since we are dealing with unit-speed curves, the derivative of T indicates how much the curve is curving.

Definition 4.1.2. The curvature function $\kappa : (L_1, L_2) \to [0, +\infty)$ of a unit-speed curve $\Gamma \in C^2$ is defined as

$$\kappa(s) := |\dot{T}(s)| = |\ddot{\Gamma}(s)|.$$

Remark. In Chapter 5 we will use the curvature for $C^{1,1}$ curves. In that case we consider κ as the weak derivative of $\dot{\Gamma}$. Also, it is easy to see from the physical analogy that the curvature indicates how much the curve differs from a straight line.

For the rest of this section we have to consider only curves with non-vanishing curvatures (i.e. $\kappa > 0$). This restriction is due to the definition of the Frenet frame and can be very restrictive – any curve which is only partially linear is excluded!

Definition 4.1.3. For any unit-speed curve Γ, we define its principal normal vector field as $N = \frac{1}{\kappa}\dot{T}$.

Remark. Now let us verify that the normal vector field is really normal. Considering the unit length of T, differentiating we conclude that $\dot{T} \cdot T = 0$, therefore $\dot{T}$ is indeed orthogonal to the tangent T. Also, by using the curvature κ to normalize $\dot{T}$, the normal N defined as above has unit length and thus is orthonormal to T.

Definition 4.1.4. The binormal vector field B is defined via vector product: $B := T \times N$.

Remark. The binormal is due to the properties of the vector product orthogonal to both T and N. Also, $|B| = 1$.

Lemma 4.1.1. Let Γ be a unit-speed curve in $\mathbb{R}^3$ with $\kappa > 0$. Then the unit vector fields T, N, B on Γ are mutually orthogonal at each point.

Proof. Can be easily seen from the definitions. □

Definition 4.1.5. A regular C^3 curve in $\mathbb{R}^3$ is called a Frenet curve if and only if $\ddot{\Gamma} \neq 0$ everywhere.

For a Frenet curve we can construct an adapted moving frame, called the Frenet frame, consisting of a tangent T, a principal normal N and a binormal B as defined above. This frame contains full information of the curve Γ and therefore can be used for studying it. Before we can prove that though, we first need to focus on the derivatives of these. We already know that

$$\dot{T} = \kappa N,$$

and we would like to deduce similar equations for $\dot{N}$ and $\dot{B}$. We claim that $\dot{B}$ is a scalar multiple of N.

Lemma 4.1.2. Let Γ be a unit-speed curve in $\mathbb{R}^3$ with Frenet frame T, N, B. Then there exists a real-valued function τ such that $\dot{B} = -\tau N$.

Proof. It is sufficient to prove that $\dot{B} \cdot T = \dot{B} \cdot B = 0$. The latter holds since $|B| = 1$. To prove the former, let us consider the equality $B \cdot T = 0$. Differentiating we get $\dot{B} \cdot T + B \cdot \dot{T} = 0$ and substituting for $\dot{T}$,

$$\dot{B} \cdot T = -B \cdot \dot{T} = -B \cdot \kappa N = 0.$$

□

Definition 4.1.6. The real-valued function τ from the previous lemma is called torsion.

Remark. The minus sign from the definition is a historical convention. Torsion represents how much the curve twists, i.e. how much the curve differs from a plane curve.

Theorem 4.1.3 (Frenet-Serret formulas)**.** Let $\Gamma : (L_1, L_2) \to \mathbb{R}^3$ be a unit-speed curve with curvature $\kappa > 0$, torsion τ and with Frenet frame T, N, B. Then

$$\begin{aligned} \dot{T} &= \kappa N, \\ \dot{N} &= -\kappa T + \tau B, \\ \dot{B} &= -\tau N. \end{aligned}$$

Proof. The first and last formulas are the actual definitions of curvature and torsion. For the proof of the second one, let us consider the following

$$\begin{aligned} T \cdot N &= 0, \\ N \cdot N &= 1, \\ N \cdot B &= 0. \end{aligned}$$

Differentiating we obtain

$$\begin{aligned} &\dot{N} \cdot T = -N \cdot \dot{T} = -N \cdot \kappa N = -\kappa, \\ &\dot{N} \cdot N = 0, \\ &\dot{N} \cdot B = -N \cdot \dot{B} = -N \cdot (-\tau N) = \tau. \end{aligned}$$

□

The Frenet-Serret formulas give us the evolution of the Frenet frame along the curve Γ. They are often written in a matrix form

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

It is then clear that the Frenet frame is a special moving 3-frame with a Cartan matrix (cf. 1.0.13). Further more, as can be seen from the following theorem [32], the Frenet-Serret formulas give us the full information about the curve.

Theorem 4.1.4 (The fundamental theorem of curves)**.** Let κ and τ be continuous functions on some interval (L_1, L_2), with $\kappa(s) > 0$ for all $s \in (L_1, L_2)$. Then there exists a curve parametrised by its arc-length s with its curvature being κ and its torsion being τ. For any two such curves holds that they differ only by a proper Euclidean[17] motion (a translation followed by a rotation).

[17]Euclid (around 300 BC), Greek mathematician known as the "father of geometry", author of the "Elements".

Proof. When confined only to analytic curves, the theorem can very simply be proven by writing our the curve out as its Taylor series and substituting for its derivatives. We will, however, present the proof in the more general way.

Firstly, we write out the differential equations, the Frenet-Serret formulas, which the Frenet frame needs to fulfil:

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \kappa N, \quad \frac{\mathrm{d}N}{\mathrm{d}s} = -\kappa T + \tau B, \quad \frac{\mathrm{d}B}{\mathrm{d}s} = -\tau N.$$

This represents a system of linear homogeneous differential equations and as κ and τ are continuous, there exists a unique set of continuous solutions for a given initial condition [23]. We can put the initial condition at s_0 as

$$T(s_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad N(s_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B(s_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

ensuring that the vectors are orthonormal at s_0. Using the Frenet-Serret formulas we get the following relation between the vectors:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(T_1^2 + N_1^2 + B_1^2) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(1 + 0 + 0) = 0.$$

Similarly, we find

$$T_2^2 + N_2^2 + B_2^2 = 1, \quad T_3^2 + N_3^2 + B_3^2 = 1$$

and three additional relations:

$$\begin{aligned} T_1T_2 + N_1N_2 + B_1B_2 &= 0 \\ T_1T_3 + N_1N_3 + B_1B_3 &= 0 \\ T_2T_3 + N_2N_3 + B_2B_3 &= 0. \end{aligned}$$

Now we will integrate T to obtain the original curve Γ.

$$\Gamma(s) = \int_{s_0}^{s} T(u)\mathrm{d}u$$

gives us a curve not only with a tangent vector field T, but thanks to the Frenet-Serret equation, with the Frenet frame consisting of T, N, B with its curvature κ and torsion τ, parametrised by its arc-length s. Hence we showed the existence part of the theorem.

Moving on to the uniqueness of the curve up to a proper Euclidean motion. We will show that if there are two curves Γ and $\bar{\Gamma}$ with the same curvature and torsion, then they are congruent. Let us move $\bar{\Gamma}(0)$ to $\Gamma(0)$ in such way that the Frenet frames $\bar{T}, \bar{N}, \bar{B}$ and T, N, B coincide. Then as the frames have the same κ and τ,

$$\begin{aligned} \bar{T}\frac{\mathrm{d}T}{\mathrm{d}s} + T\frac{\mathrm{d}\bar{T}}{\mathrm{d}s} + \bar{N}\frac{\mathrm{d}N}{\mathrm{d}s} + N\frac{\mathrm{d}\bar{N}}{\mathrm{d}s} + \bar{B}\frac{\mathrm{d}B}{\mathrm{d}s} + B\frac{\mathrm{d}\bar{B}}{\mathrm{d}s} &= \kappa\bar{T}N + \kappa T\bar{N} - \kappa\bar{N}T + \tau\bar{N}B \\ &\quad - \kappa N\bar{T} + \tau N\bar{B} - \tau\bar{B}N - \tau B\bar{N} = 0, \end{aligned}$$

or

$$T\bar{T} + N\bar{N} + B\bar{B} = \text{const} \overset{s=0}{=} 1.$$

Also for the individual components the following equations hold:

$$T_i\bar{T}_i + N_i\bar{N}_i + B\bar{B}_i = 1, \quad T_i^2 + N_i^2 + B_i^2 = 1, \quad \bar{T}_i^2 + \bar{N}_i^2 + \bar{B}_i^2 = 1.$$

This is equivalent to T, N, B and $\bar{T}, \bar{N}, \bar{B}$ making a zero angle with one another. Hence $T = \bar{T}$, $N = \bar{N}$ and $B = \bar{B}$ for all s. Also, as $\Gamma(0) = \bar{\Gamma}(0)$, the two curves Γ and $\bar{\Gamma}$ must coincide, so the proof is completed. □

4.2 The relatively parallel adapted frame

In the original article [1] from 1975, Bishop introduced a new way of looking on framing a curve – the relatively parallel adapted frame (RPAF). A recapitulation of his original approach and the generalisation onto merely $C^{1,1}$ curves as presented in [21] is given here. Later, in Chapter 5, a generalisation for a regular $C^{1,1}$ curve in any dimension is presented.

The Frenet frame is a standard instrument for examining curves, even though it is very restrictive on the possible curves. Therefore, a search for an alternative approach brought us a new possible way to frame a curve – the relatively parallel adapted frame. While both RPAF and Frenet frame are adapted, meaning that they contain a tangent, their main difference is the choice of the other two normal vector. The normal vectors in Frenet frame are directly dependent on the tangent and therefore on the curve, whereas in RPAF the choice of the normal vectors is more open.

In line with the definition 2.0.29, we say that a normal vector field M along a curve is relatively parallel if its derivative is tangential. Note that this matches the idea that such a vector field turns as little as possible while still being normal. Furthermore, as the field derivative is perpendicular to the field, the length of any relatively parallel normal field remains constant. The idea behind the RPAF is to introduce an adapted frame in which both normals are relatively parallel along the whole curve (and hence they minimize rotation along the tangent vector field). To construct such frame we need some adapted frame (e.g. the Frenet frame, if it exists) to start with. This adapted frame can be (locally) constructed using the Gram[18]-Schmidt[19] process on the tangent T and two parallel fields. Denoting the frame T, M_1, M_2, we can write

$$\dot{T} = p_{01}M_1 + p_{02}M_2, \quad \dot{M}_1 = -p_{01}T + p_{12}M_2, \quad \dot{M}_2 = -p_{02}T - p_{12}M_1.$$

Now we will inspect the condition for a normal field to be relatively parallel. Let θ be a smooth angle function, then we can construct two normal fields N_1, N_2 from the normal fields M_1, M_2 using rotation $\mathcal{R} := \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$:

$$\begin{pmatrix}N_1 \\ N_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}M_1 \\ M_2\end{pmatrix}.$$

[18] Jorgen Pedersen Gram (1850-1916), Danish mathematician who contributed mainly to invariant theory and number theory.

[19] Erhard Schmidt (1876-1959), Baltic German mathematician who greatly contributed to mathematics, e.g. to the theory of integral equations and to functional analysis.

The rotation matrix $\mathcal{R}$ can be arbitrary as long as it is a rotation matrix, i.e. $\mathcal{R}\mathcal{R}^T = 1$, $\det\mathcal{R} = 1$ and $\mathcal{R}$ is differentiable. Differentiating we get

$$\begin{aligned}\dot{N}_1 &= -T(p_{01}\cos\theta - p_{02}\sin\theta) + (p_{12} - \dot{\theta})(M_2\cos\theta - M_1\sin\theta)\\ \dot{N}_2 &= -T(p_{01}\sin\theta + p_{02}\cos\theta) + (p_{12} - \dot{\theta})(M_2\sin\theta - M_1\cos\theta).\end{aligned}$$

Therefore the condition for N_1, N_2 to be relatively parallel to the curve reduces to $\dot{\theta} = p_{12}$ and since there is a solution θ satisfying any initial condition, we can locally construct the relatively parallel adapted frame T, N_1, N_2. For the global existence we connect together the locally defined RPAFs on overlapping intervals. Since we have shown in Chapter 2 that for any given initial vector M_0, the constructed relatively parallel vector field $M(t)$ satisfying that $M(t_0) = M_0$ is uniquely determined, we obtain the overall smoothness of the global RPAF.

To examine the uniqueness of the parallel curvatures k_1, k_2 we can consider a different choice of the initial vectors. Let M_0^1, M_0^2 be the original vectors and $\tilde{M}_0^\mu := \mathcal{R}_{\mu\nu}M_0^\nu$, $\mu, \nu = 1, 2$ be the new initial vectors. Then the parallel curvatures change as $\tilde{k}_\mu = \mathcal{R}_{\mu\nu}k_\nu$. Therefore the parallel curvatures are not unique to the curve. However, when we examine the curvature κ, we have

$$\kappa^2 = |\dot{T}|^2 = |k_1N_1 + k_2N_2|^2 = k_1^2 + k_2^2$$

and so the magnitude of the vector (k_1, k_2) is independent of the choice of the initial vectors and hence the RPAF. Also, if the curve is unit-speed[20] and possesses the Frenet frame, the angle function θ can be written as

$$\theta(s) = \theta_0 + \int_{s_0}^{s} \tau(u)\mathrm{d}u \tag{4.1}$$

and one choice for the parallel curvatures can be

$$(k_1, k_2) = (\kappa\cos\theta, \kappa\sin\theta). \tag{4.2}$$

This nicely illustrates that while the Frenet frame (if it exists) is unique, we can construct a relatively parallel adapted frame from any initial choice of two mutually orthogonal normal vectors and the resulting frames will be considerably different.

4.3 The rotation minimizing osculating frame

Another interesting moving frame, the rotation minimizing osculating frame, will be discussed here to include an example of a non-adapated frame. This frame, which closely follows from the Frenet frame, can be found for example in [7]. The idea behind it enables it to be used in aeronautics where it specifies the so called yaw-free rigid-body motion along a curved path.

The idea behind the rotation minimizing osculating frame (RMOF) is very similar to the one behind RPAF. We start with the Frenet frame on a curve c and instead of rotating N and B along T so that the new frame minimizes rotation along T, we make our reference vector to be the binormal B and we rotate T and N along the B to obtain a new frame F, G, B, where

$$F = \cos\theta\, T - \sin\theta\, N, \quad G = \sin\theta\, T + \cos\theta\, N,$$

[20]If the curve is not parametrised by its arc-length, then we need to integrate $\tau|\dot{c}|$ instead of just τ.

for some function θ. Similarly as above, we find the condition on the angle function θ to be

$$\dot{\theta} = \kappa|\dot{c}|.$$

Then we can write the time evolution of the RMOF as

$$\begin{pmatrix} F \\ G \\ B \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & 0 & t_1 \\ 0 & 0 & t_2 \\ -t_1 & -t_2 & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ B \end{pmatrix},$$

where $t_1 = -\tau \sin\theta$ and $t_2 = \tau\cos\theta$. While the RPAF can be constructed for curves lacking the Frenet frame, from the construction of the RMOF it is clear that it can be applied only to the Frenet curves. This last frame discussed here minimizes rotation about the last missing vector, the binormal B.

4.4 The case study

Here a short comparison of the behaviour of the three different frames will be given for the two curves introduced in the first chapter.

4.4.1 Helix

Firstly, we will start with a helix in $\mathbb{R}^3$, as everything is nice and well defined there. Let us consider the helix η (1.3) from the first chapter,

$$\eta(t) = (R\cos(vt), R\sin(vt), uv), \tag{4.3}$$

where $R, u, v \in \mathbb{R}$, $R > 0$ and $v = \frac{1}{\sqrt{R^2+u^2}}$. It is easy to check that the Frenet frame T, N, B is given by

$$\begin{aligned} T(t) &= (-Rv\sin(vt), Rv\cos(vt), uv), \\ N(t) &= (-\cos(vt), -\sin(vt), 0), \\ B(t) &= (uv\sin(vt), -uv\cos(vt), Rv), \end{aligned}$$

with the (Frenet) curvature and torsion given as

$$\begin{aligned} \kappa = Rv^2 &= \frac{R}{R^2+u^2}, \\ \tau = uv^2 &= \frac{u}{R^2+u^2}. \end{aligned}$$

Using the formulas (4.2), (4.1) for parallel curvatures and the angular function θ, we find that

$$\begin{aligned} \theta(t) &= \theta_0 + v\tau t = \theta_0 + uv^3 t \\ k_1(t) &= \kappa\cos\theta(t) = \frac{R}{R^2+u^2}\cos\theta(t) \\ k_2(t) &= \kappa\sin\theta(t) = \frac{R}{R^2+u^2}\sin\theta(t), \end{aligned}$$

and hence the equations for the RPAF based on the Frenet frame are

$$\begin{aligned}
N_1(t) &= (-\cos\theta\cos(vt) - uv\sin\theta\sin(vt), -\cos\theta\sin(vt) + uv\sin\theta\cos(vt), -Rv\sin\theta),\\
N_2(t) &= (-\sin\theta\cos(vt) + uv\cos\theta\sin(vt), -\sin\theta\sin(vt) - uv\cos\theta\cos(vt), Rv\cos\theta).
\end{aligned}$$

Following similar steps, we can write the RMOF angular function, curvatures and vectors F, G as

$$\begin{aligned}
\theta(t) &= \theta_0 + v\kappa t = \theta_0 + Rv^3 t,\\
t_1(t) &= -\tau\sin\theta = -uv^2\sin\theta,\\
t_2(t) &= \tau\cos\theta = uv^2\sin\theta,\\
F(t) &= (-Rv\cos\theta\sin(vt) + \sin\theta\cos(vt), Rv\cos\theta\cos(vt) + \sin\theta\sin(vt), uv\cos\theta),\\
G(t) &= (-Rv\sin\theta\sin(vt) + \cos\theta\cos(vt), Rv\sin\theta\cos(vt) + \cos\theta\sin(vt), uv\sin\theta).
\end{aligned}$$

On Fig. 4.1 we can see the comparison of Frenet frame vectors N and B with the RPAF vectors U and V on the helix (4.3). Fig. 4.2 compares the Frenet frame T and N with the RMOF F and G on the same helix.

4.4.2 Spivak curve

Considering again the Spivak curve σ, we find that the situation is much more complicated in comparison with the helix. There are three points at which the Frenet frame ceases to exist – at $t = 0$ and $t = \pm\sqrt{\frac{2}{3}}$, and therefore we cannot construct the Frenet frame globally. We can, however, construct it piecewise when we remove the problematic points. To address the problematic points further, we can study the nature of them.

Firstly we will study the points $t = \pm\sqrt{\frac{2}{3}}$. The problem there is due to the curve changing from "convex" to "concave"[21] and vice versa. At these points the curvature κ changes sign and the Frenet frame flips.

Secondly we look at the zero where the situation is slightly different. Recalling the formula for σ (1.2), it is clear that something strange happens there. The Spivak curve σ lies in the xy plane for $t > 0$ and in the xz plane for $t < 0$, effectively switching planes at $t = 0$, while remaining smooth. This switch causes a rotation of the local Frenet frame through $\frac{\pi}{2}$ and is a source of another discontinuity in the frame. This problem is usually discussed when dealing with this curve while the former two problematic points are omitted.

Overall, the final Frenet frame, patched together from the local ones, can be found on left part of Fig. 4.3 with the problematic points emphasized. The result, composed out of 4 different local Frenet frames, has 3 discontinuities at the problematic points where $\kappa = 0$. Here the frame suddenly rotates through either π or $\frac{\pi}{2}$. Therefore, we cannot construct a tube with a non-circular cross section on the curve based on this frame.

Moving on to the relatively parallel adapted frame, we can see on the right part of Fig. 4.3 a much neater result when applied to the Spivak curve. We constructed this frame from the

[21] As the curve σ maps $\mathbb{R}$ to $\mathbb{R}^3$, we cannot talk about convex or concave curves. However, if we restrict ourselves to the plane in which the curve lies, we can apply our notion of convexity or concavity there.

Figure 4.1: Comparison of the Frenet frame principal normal N (green) and binormal B (red) on the left with the RPAF vectors U (green) and V (red) on the right, applied on the helix (4.3) with the parameters set on $R = 2,5$ and $u = 0,7$.

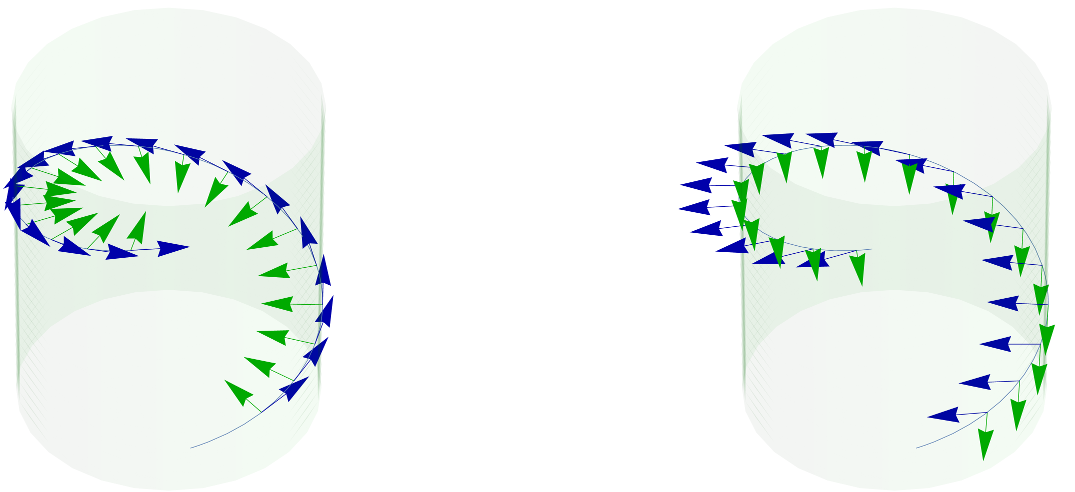

Figure 4.2: Comparison of the Frenet frame tangent T (blue) and principal normal N (green) on the left with the RMOF vectors F (blue) and G (green) on the right, applied on the helix (4.3) with the parameters set on $R = 2,5$ and $u = 0,7$.

Frenet frame on the interval $(-\sqrt{\frac{2}{3}}, 0)$ and then used the parallel transport to extend it on the whole $\mathbb{R}$. Note that this construction is coherent with the one stated earlier as the torsion is 0 for both $t < 0$ and $t > 0$ and therefore the vectors U and V coincide with the principal normal N and the binormal B.

The construction of the rotation minimizing osculating frame is again problematic as it is directly derived from the Frenet frame. The result can be found on the right part of Fig. 4.4 with comparison with the Frenet frame. Only vectors which the two frames do not have in common are shown, i.e. the tangent T and principal normal N from the Frenet frame and the vectors F, G from the RMOF.

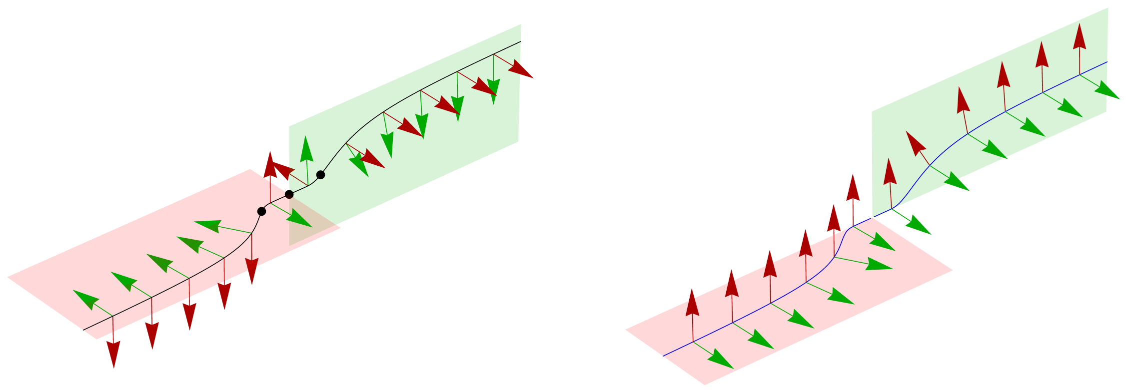

Figure 4.3: Comparison of the Frenet frame and the RPAF on the Spivak curve σ. On the left is the principal normal N (green) and binormal B (red) from the Frenet frame, and points at which the Frenet frame ceases to exist are highlighted to draw attention to the rotation of the frame at these points. On the left figure, the RPAF vectors U (green) and V (red) move smoothly along the curve.

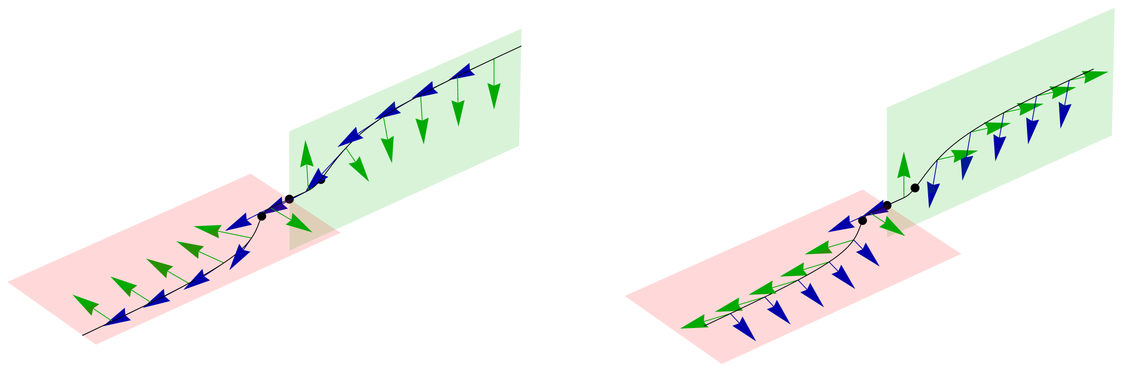

Figure 4.4: Comparison of the Frenet frame and the RMOF on the Spivak curve σ. On the left is the Frenet frame, tangent vector T (blue) and principal normal N (green). The problematic points at which the Frenet frame fails to exist are highlighted. On the right is the RMOF, with the vector F (blue) and G (green).

Chapter 5

Frames in higher dimensions

The first part of this chapter presents the classical results for the Frenet frame in higher dimensions from [20]. In the second part, the main result of the generalisation of the relatively parallel adapted frame to higher dimensions is introduced with the modification of the fundamental theorem of curves.

5.1 The Frenet frame

Definition 5.1.1. A moving n-frame is called the Frenet n-frame, or simply the Frenet frame, if for all k, $1 \leq k \leq n$, the k-th derivative $c^{(k)}(t)$ of $c(t)$ lies in the span of the vectors $e_1(t), \dots, e_k(t)$.

Theorem 5.1.1 (The existence and uniqueness of distinguished Frenet frame)**.** Let $c : I \to \mathbb{R}^n$ be a curve such that for all $t \in I$, the vectors $c'(t), c^{(2)}(t), \dots, c^{(n-1)}(t)$ are linearly independent. Then there exists a unique Frenet frame with the following properties:

1. For k, the vectors $1 \leq k \leq n$, $c'(t), \dots, c^{(k)}(t)$ and $e_1(t), \dots, e_k(t)$ have the same orientation.

2. $e_1(t), \dots, e_n(t)$ have positive orientation.

This frame is called the distinguished Frenet frame.

Proof. The Gram-Schmidt orthogonalization process is used (see e.g. [22]). The assumption of linear independence of $c'(t), c^{(2)}(t), \dots$ implies that $c'(t) \neq 0$. Therefore we may set $e_1(t) := \frac{c'(t)}{|c'(t)|}$. Now we can use the Gram-Schmidt orthogonalization procedure to find vector fields $e_2(t), \dots, e_{n-1}$ by defining

$$\tilde{e}_j(t) := -\sum_{k=1}^{j-1}(c^{(j)}(t) \cdot e_k(t))e_k(t) + c^{(j)}(t)$$

and letting

$$e_j(t) := \frac{\tilde{e}_j(t)}{|\tilde{e}_j(t)|}.$$

It is easy to see that $e_j(t)$, $j < n$, are well defined and satisfy the first assertion. Moreover, we can find $e_n(t)$ so that $e_1(t), \dots, e_n$ have positive orientation. From the definition of a moving frame (1.0.12), we must check the differentiability of the frame. $e_1, \dots, e_{n-1}$ are differentiable by their definition. To justify the differentiability of e_n, one must realize that e_n is continuously dependent on $e_1, \dots, e_{n-1}$ which implies the differentiability of e_n $\square$

5.2 The relatively parallel adapted frame

Before we can proceed with the main theorem about the existence of the relatively parallel adapted frame, we need to prove that we can construct an adapted frame on any regular $C^{1,1}$ curve.

Lemma 5.2.1. Let $\Gamma : (L_1, L_2) \to \mathbb{R}^{n+1}$ be a regular $C^{1,1}$ curve. Then on each compact subinterval $[a, b] \subset (L_1, L_2)$ exists a collection of adapted moving frames on Γ.

Proof. Let $\Gamma = (\Gamma_1, \dots, \Gamma_{n+1})$. Then the tangent vector field T to Γ is defined as

$$T = \left(\dot{\Gamma}_1, \dots, \dot{\Gamma}_{n+1}\right).$$

From

$$|T| = \dot{\Gamma}_1^2 + \dots \dot{\Gamma}_{n+1}^2 = 1$$

follows, that there exists at least one index $i = 1, \dots, n+1$ so that

$$\dot{\Gamma}_i^2(s_0) \geq \frac{1}{n+1}$$

for any fixed $s_0 \in (L_1, L_2)$. Without loss of generality we can assume that this $i = n + 1$, i.e. $|\dot{\Gamma}_{n+1}| \geq 1/\sqrt{n+1}$. As $\dot{\Gamma}$ is continuous, there must exist some $\varepsilon > 0$ such that $|\dot{\Gamma}_{n+1}| > 0$ on $(s_0 - \varepsilon, s_0 + \varepsilon)$. Now, constricting ourselves on the interval $(s_0 - \varepsilon, s_0 + \varepsilon)$, we can construct n normal vectors to the curve as

$$\begin{aligned}
\tilde{N}_1 &= \frac{1}{\sqrt{\dot{\Gamma}_{n+1}^2 + \dot{\Gamma}_1^2}}(\dot{\Gamma}_{n+1}, 0, \dots, 0, -\dot{\Gamma}_1) \\
\tilde{N}_2 &= \frac{1}{\sqrt{\dot{\Gamma}_{n+1}^2 + \dot{\Gamma}_2^2}}(0, \dot{\Gamma}_{n+1}, 0, \dots, 0, -\dot{\Gamma}_2) \\
&\vdots \\
\tilde{N}_n &= \frac{1}{\sqrt{\dot{\Gamma}_{n+1}^2 + \dot{\Gamma}_n^2}}(0, \dots, 0, \dot{\Gamma}_{n+1}, -\dot{\Gamma}_n).
\end{aligned}$$

These vectors are certainly orthogonal to the tangent vector T, they have unit length and they are linearly independent for all $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. However, they do not need to be mutually orthogonal, i.e. $\tilde{N}_i \cdot \tilde{N}_j$ may, or may not be equal to δ_{ij}. To make them orthogonal, we apply the Gram-Schmidt orthogonalization procedure to $T, \tilde{N}_1, \dots, \tilde{N}_n$. The resulting collection of vector fields $T, N_1, \dots, N_n$ will be mutually orthogonal and, if needed, can be normalized. Therefore, we constructed an adapted moving frame $T, N_1, \dots, N_n$ on Γ for the interval $(s_0 - \varepsilon, s_0 + \varepsilon)$.

To construct the collection of frames for the whole compact subinterval $[a, b]$, we realise that $\dot{\Gamma}$ is in fact uniformly continuous on the compact interval $[a, b]$, i.e.

$$(\forall i = 1, \dots, n+1)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in [a, b])(|x - y| < \delta \implies |\dot{\Gamma}_i(x) - \dot{\Gamma}_i(y)| < \varepsilon).$$

As the interest is on the case of $\varepsilon = \frac{1}{\sqrt{n+1}}$, we get a set of $\delta_1, \dots, \delta_{n+1}$ corresponding to $\dot{\Gamma}_1, \dots, \dot{\Gamma}_{n+1}$. Denoting $\delta = \min\{\delta_1, \dots, \delta_{n+1}\}$, we can write

$$(\exists \delta > 0)\,(\forall i = 1, \dots, n+1; x, y \in [a, b])\left(|x - y| < \delta \implies |\dot{\Gamma}_i(x) - \dot{\Gamma}_i(y)| < \frac{1}{\sqrt{n+1}}\right).$$

Starting at $s = a$, we know that there exists at least one component of Γ so that $\dot{\Gamma}_i^2(a) > \frac{1}{n+1}$ and that $\forall s \in [a, \frac{1}{2}\delta]$, $|\dot{\Gamma}_i(s)| > 0$. Therefore we can construct the adapted moving frame as described above on $[a, a + \frac{1}{2}\delta]$. We can continue this process for $[a + \frac{1}{2}\delta, a + \delta]$, $[a + \delta, a + \frac{3}{2}\delta], \dots, [a + \frac{n}{2}\delta, b]$, where $n = \left\lfloor 2\frac{b-a}{\delta} \right\rfloor$. Thus we obtained a collection of adapted moving frames on $[a, b]$ as requested. □

Theorem 5.2.2. Let $c : (a, b) \subset \mathbb{R} \to \mathbb{R}^{n+1}$ be a regular $C^{1,1}$ curve. Then there exists an adapted moving frame $T, N_1, \dots, N_n$ minimazing rotation, i.e.

$$\begin{pmatrix} T \\ N_1 \\ \vdots \\ N_n \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & k_1 & \dots & k_n \\ -k_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -k_n & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ \vdots \\ N_n \end{pmatrix}$$

Proof. We already know from 1.0.2 that every regular curve can be parametrized by its arc-length, ensuring $|\dot{c}| \equiv 1$. Therefore, without loss of generality, let c be a unit-speed. Then the tangent vector field $T := \dot{c}$ is well defined along the whole curve. From Lemma 5.2.1 follows that after dividing the interval (a, b) into any compact subintervals, there exists a finite collection of some adapted moving frames on each subinterval. Using this collection of adapted moving frames we can redivide the interval onto new subinterval so that at each interval we get only one adapted frame from the collection. Initially we will construct the RPAF on one of these subinterval. Then we will justify that the locally constructed RPAFs can be connected in such a way that the overall frame will be smooth.

Firstly, let us consider one of those subintervals, denoting it $[\alpha, \beta]$, and the adapted moving frame on it, denoting the vectors $T, M_1, \dots, M_n$ and the whole frame as U. From theorem 1.0.3 follows that there exists a skew-symmetric matrix $\mathcal{A} \in M_{n+1}\left(L_{loc}\left((a, b)\,; \mathbb{R}\right)\right)$ such that

$$\dot{U} = \mathcal{A}U$$

Also, thanks to the skew-symmetry of $\mathcal{A}$, we can express it as

$$\mathcal{A} = \begin{pmatrix} 0 & \vec{a}^T \\ -\vec{a} & \tilde{\mathcal{A}} \end{pmatrix},$$

where $\vec{a} \in M_{n,1}(L_{loc}((a,b);\mathbb{R}))$ and $\tilde{\mathcal{A}}$ is skew-symmetric matrix from $M_n(L_{loc}((a,b);\mathbb{R}))$.

The construction of the relatively parallel adapted frame consisting of vectors $T, N_1, \ldots, N_n$ and denoted as V will be analogous to the one adopted in the three dimensional case. The desire is to rotate the normal vectors M_i in such way that the rotated vectors will be relatively parallel along the curve on the subinterval, while preserving the tangent vector T. This can be provided by taking a special case of rotation matrix in $\mathbb{R}^{n+1}$ such that

$$\mathcal{R} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathcal{R}} \end{pmatrix},$$

where $\tilde{\mathcal{R}} \in M_n(L_{loc}((a,b);\mathbb{R}))$. Now let us define V as

$$V = \mathcal{R}U. \tag{5.1}$$

This means that we need to seek conditions on $\mathcal{R}$ for V to be a RPAF. On one hand, as V should be minimizing rotation along the tangent, there exists a skew-symmetric matrix $\mathcal{B} \in M_{n+1}(L_{loc}((a,b);\mathbb{R}))$, $\mathcal{B} := \begin{pmatrix} 0 & \vec{k}^T \\ -\vec{k} & 0 \end{pmatrix}$ such that

$$\dot{V} = \mathcal{B}V.$$

On the other hand, differentiating (5.1) and substituting for $\dot{U}$, we get

$$\dot{V} = \dot{\mathcal{R}}U + \mathcal{R}\dot{U} = \dot{\mathcal{R}}U + \mathcal{R}\mathcal{A}U.$$

Comparing these two expressions, we arrive at

$$\left(\dot{\mathcal{R}} + \mathcal{R}\mathcal{A} - \mathcal{B}\mathcal{R}\right)U = 0. \tag{5.2}$$

The matrix U is composed out of orthonormal vectors, meaning that it is orthogonal and there exists an inverse. Therefore after multiplying the equation (5.2) by U^{-1} and writing the matrices out, we get

$$\begin{pmatrix} 0 & 0 \\ 0 & \dot{\tilde{\mathcal{R}}} \end{pmatrix} + \begin{pmatrix} 0 & \vec{a}^T \\ -\tilde{\mathcal{R}}\vec{a} & \tilde{\mathcal{R}}\tilde{\mathcal{A}} \end{pmatrix} - \begin{pmatrix} 0 & \vec{k}^T\tilde{\mathcal{R}} \\ -\vec{k} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vec{a}^T - \vec{k}^T\tilde{\mathcal{R}} \\ \vec{k} - \tilde{\mathcal{R}}\vec{a} & \dot{\tilde{\mathcal{R}}} + \tilde{\mathcal{R}}\tilde{\mathcal{A}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Breaking this matrix equation into its individual components, we get three equations:

$$\begin{aligned} \dot{\tilde{\mathcal{R}}} + \tilde{\mathcal{R}}\tilde{\mathcal{A}} &= 0, \\ \vec{k}^T\tilde{\mathcal{R}} &= \vec{a}^T, \\ \tilde{\mathcal{R}}\vec{a} &= \vec{k}, \end{aligned}$$

with the last two dependent due to $\mathcal{R}$ being a rotation, expressing the same condition just in a different form. Overall, we end up with a first order differential equation for an unknown matrix $\tilde{\mathcal{R}}$ and an equation specifying the vector $\vec{k}$:

$$\begin{aligned} \dot{\tilde{\mathcal{R}}} + \tilde{\mathcal{R}}\tilde{\mathcal{A}} &= 0 \\ \tilde{\mathcal{R}}\vec{a} &= \vec{k}. \end{aligned}$$

From Chapter 3 (resp. [35]) follows that the differential equation has a solution. We will further specify the initial condition as follows: $\tilde{\mathcal{R}}(\alpha) = \mathcal{I}$, where $\mathcal{I}$ denotes the identity matrix and α is the left edge of the interval. With this initial condition the solution is unique and therefore the vector $\vec{k}$, defined as $\vec{k} := \tilde{\mathcal{R}}\vec{a}$ is also uniquely determined. Overall, we constructed a relatively parallel adapted frame on a compact subinterval of (a, b).

Secondly, we need to discuss the construction of the global RPAF. Following the method outlined above, we can construct a collection of relatively parallel adapted frames. To link them together we will use the freedom of choosing the initial condition for the differential equation.

Let us denote the end points of the subintervals for which we can construct the auxiliary adapted frame as $t_i,\ i \in \mathbb{Z}$ and the frame corresponding to $[t_i, t_{i+1}]$ as U_i. Then we can construct the first relatively parallel adapted frame on the interval $[t_{-1}, t_0]$ as described above, with the initial condition $\tilde{\mathcal{R}}(t_0) = \mathcal{I}$. We will, however, change the initial condition for the rest of the intervals as follows:

- $\tilde{\mathcal{R}}(t_i)U_i(t_i) = U_{i-1}(t_i)$ for $i \geq 1$
- $\tilde{\mathcal{R}}(t_{i+1})U_i(t_{i+1}) = U_{i+1}(t_{i+1})$ for $i < 0$.

This choice will ensure that the resulted RPAF will be continuous and its smoothness is a result of uniqueness of the parallel transport 2.0.5. □

Next a modification of the fundamental theorem for curves will be presented. We already know from the previous chapter that the parallel curvatures are not unique for the curve. However, in the next two theorems some correspondence is revealed.

Definition 5.2.1. We say that two curves are congruent if there exist an isometry mapping one curve to another.

Theorem 5.2.3. Let $\alpha,\beta : I \to \mathbb{R}^n$ be arbitrary curves with frame field $E_1, \ldots, E_n$ on α and frame field $F_1, \ldots, F_n$ on β. If

- $\dot{\alpha} \cdot E_i = \dot{\beta} \cdot F_{\pi(i)}, \quad \forall i = 1, \ldots, n$
- $\dot{E}_i \cdot E_j = \dot{F}_{\pi(i)} \cdot F_{\pi(j)}$

for some permutation[22] π. Then α and β are congruent.

Proof. Without loss of generality, we can assume that $0 \in I$. Let G be an isometry such that

$$G(E_i(0)) = F_{\pi(i)}(0).$$

Then for all $t \in I$ we can denote $G(E_i(t)) = \bar{E}_i(t)$.

As G preserves the dot product, $\bar{E}_1, \ldots, \bar{E}_n$ is a frame field on $G(\alpha) = \bar{\alpha}$. Since G preserves velocities and derivatives of vector fields as well, we have

$$\begin{aligned} \bar{\alpha}(0) &= \beta(0), & \dot{\bar{\alpha}} \cdot \bar{E}_i &= \dot{\beta} \cdot F_{\pi(i)}, \\ \dot{\bar{E}}_i \cdot \bar{E}_j &= \dot{F}_{\pi(i)} \cdot F_{\pi(j)}, & \bar{E}_i(0) &= F_{\pi(i)}(0). \end{aligned} \tag{5.3}$$

[22] As the RPAF construction does not specify the order in which the normals can be rearranged, we need to take it into account. Therefore, by permutation π, we mean any possible order of numbers $1, \ldots, n$ such that each number appears only once.

From (5.3) follows that

$$\dot{\bar{E}}_i = \sum_j a_{ij} \bar{E}_j$$

and

$$\dot{F}_{\pi(i)} = \sum_j a_{ij} \bar{F}_{\pi(j)}$$

with the same coefficient functions a_{ij}. We already know from Chapter 1 that $a_{ij} + a_{ji} = 0$[23]. Defining a scalar function $f = \sum_i \bar{E}_i \cdot F_{\pi(i)}$, it follows from the definition of G that $f(0) = n$. Moreover, $f \leq n$ since $\bar{E}_i \cdot F_{\pi(i)} \leq 1$ thanks to the Schwarz inequality. Differentiating f we obtain

$$\dot{f} = \sum_i \dot{\bar{E}}_i \cdot F_{\pi(i)} + \dot{F}_{\pi(i)} \cdot \bar{E}_i = \sum_{i,j} (a_{ij} + a_{ji}) \bar{E}_j \cdot F_{\pi(i)} = 0$$

as a result of the skew-symmetry of (a_{ij}). Thus $f \equiv n$ and from this follows that

$$\bar{E}_i(t) = F_{\pi(i)}(t) \quad \forall t \in I$$

and since

$$\dot{\bar{\alpha}} = \sum_i \dot{\bar{\alpha}} \cdot \bar{E}_i \bar{E}_i \quad \text{and} \quad \dot{\beta} = \sum_i \dot{\beta} \cdot F_{\pi(i)} F_{\pi(i)},$$

$\bar{\alpha}$ and β are parallel, as they can be expressed using vectors which are shown to be are identical (5.3). Therefore, they differ only by a translation in the space. Using that $\bar{\alpha}(0) = \beta(0)$, we will show that $\bar{\alpha} = \beta$. From $\bar{\alpha}, \beta$ being parallel follows that

$$\frac{\mathrm{d}\bar{\alpha}_i}{\mathrm{d}t}(t) = \frac{\mathrm{d}\beta_i}{\mathrm{d}t}(t) \quad \forall i = 1, \ldots, n, \ \forall t \in I,$$

where $\bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. From our knowledge of elementary calculus we deduce that $\beta_i = \bar{\alpha}_i + p_i$ which means $\beta = \alpha + \vec{p}$. However, as $\beta(0) = \bar{\alpha}(0)$, $\vec{p} = 0$. Overall,

$$G(\alpha) = \bar{\alpha} = \beta,$$

as was requested. □

Theorem 5.2.4. Let $\alpha, \beta : I \to \mathbb{R}^n$ be congruent curves, $E_1, \ldots, E_n$ frame field on α with parallel curvatures $k_1, \ldots, k_{n-1}$. Then there exist a frame field on β with the same parallel curvatures $k_1, \ldots, k_{n-1}$.

Proof. As α, β are congruent, there exists an isometry G such that $G(\alpha) = \beta$. Then since G preserves the scalar product, $G(E_i) = F_i$ gives us a frame field $F_1, \ldots, F_n$ on β. From the proof of the previous theorem we know that the coefficient functions a_{ij} are the same for E_i as for F_i. Therefore, we constructed a frame field with the required curvatures.

□

[23] Differentiate $\bar{E}_i \cdot \bar{E}_j = \delta_{ij}$.

Chapter 6

Waveguides

In this chapter the notion of a (quantum) waveguide is introduced throughout an arbitrary deformation of a straight tube, following the approach of [4] or [21] and generalising it for $C^{1,1}$ curves in n-dimensions. Only the relatively parallel adapted frame is used within this chapter as it is the most general frame we can construct so far. Interested readers can find more information about the motivation for (quantum) waveguides in [5].

We begin with a definition of a straight tube.

Definition 6.0.2. Let $\omega \subset \mathbb{R}^n$ be a bounded open connected set. Then a straight tube Ω_0 is defined as $\Omega_0 = \mathbb{R} \times \omega$. We call ω a cross section.

Remark. As ω is bounded, there exists a real number r, $r < \infty$, such that $r = \sup_{t \in \omega} |t|$, cf. Fig. 6.1.

Now we can start the deformation process to achieve an arbitrary tube. Firstly we can bend the straight tube around some curve to produce a curved tube without focusing on what the cross section is doing. Thus let us consider any given regular curve $\Gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ parametrized by its arc-length. It is already known from the previous chapter that for such curve there always exists a relatively parallel adapted frame $(T, N_1, \ldots, N_n)$. Furthermore, the cross section ω can be manipulated along the curve. To achieve this, we will need the following definitions.

Then we can define for a rotation-minimizing adapted moving frame $T, N_1, \ldots, N_n$ a general adapted moving frame $T^{\mathcal{R}}, N_1^{\mathcal{R}}, \ldots, N_n^{\mathcal{R}}$ along Γ as rotation of the original normal $N_1, \ldots, N_n$ by the rotation matrix $\mathcal{R}$, i.e.,

$$\begin{pmatrix} T^{\mathcal{R}} \\ N_1^{\mathcal{R}} \\ \vdots \\ N_n^{\mathcal{R}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R} \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ \vdots \\ N_n \end{pmatrix}.$$

Note that the tangent vector remains unturned. Now we can properly define a curved tube.

Definition 6.0.3. A curved tube Ω of the cross section ω about Γ is defined as the image of Ω_0 for some mapping $\mathcal{L}$ such that

$$\mathcal{L} : \Omega_0 \to \mathbb{R}^{n+1} : (s, t) \mapsto \Gamma(s) + \sum_{i=1}^{n} N_i^{\mathcal{R}}(s) t_i,$$

where $t := (t_1, \ldots, t_n)$. In other words, $\Omega := \mathcal{L}(\Omega_0)$.

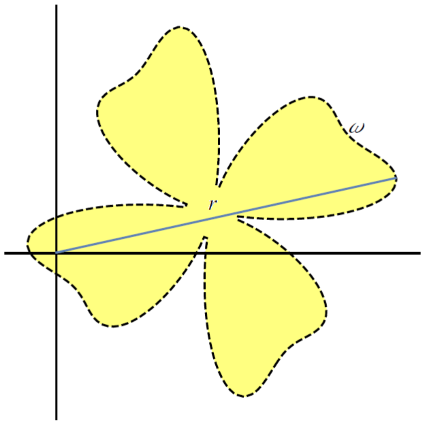

Figure 6.1: An example of a cross-section ω.

However, the aim is to identify Ω with the Riemannian manifold (Ω_0, g_{ij})[24]. To ensure that, further restrictions are needed to impose on $\mathcal{L}$. In the theory of quantum waveguides it is usual to assume that Ω is non-self-intersecting and that therefore $\mathcal{L}$ is injective. This assumption of injectivity yields us the necessary (but not always sufficient) condition of non-vanishing determinant of the metric tensor

$$g_{ij} = \partial_i \mathcal{L} \cdot \partial_j \mathcal{L},$$

where ∂_i is the partial derivative with respect to the i-th variable of $(s, t_1, \ldots, t_n)$. Rewriting $\mathcal{L}$ using the definition of $N_i^{\mathcal{R}}$ and the Einstein summation convention but now summing only over subscripts, we get

$$\mathcal{L}(s,t) = \Gamma(s) + t_i \mathcal{R}_{ij}(s) N_j(s).$$

Using the definition of the relatively parallel adapted frame, we arrive at the following expression for the metric tensor g_{ij}.

$$(g_{ij}) = \begin{pmatrix} h^2 + h_1^2 + \cdots + h_n^2 & h_1 & \ldots & h_n \\ h_1 & 1 & & \\ \vdots & & \ddots & \\ h_n & & & 1 \end{pmatrix}, \tag{6.1}$$

where

$$\begin{aligned} h &= \left(1 - t_i \mathcal{R}_{ij} k_j\right)^2, \\ h_1 &= \mathcal{R}_{1k} \dot{\mathcal{R}}_{mk} t_m, \\ &\vdots \\ h_n &= \mathcal{R}_{nk} \dot{\mathcal{R}}_{mk} t_m. \end{aligned}$$

It is easy to check that

[24]The metric tensor g_{ij} is induced by $\mathcal{L}$, i.e. $g_{ij} := \partial_i \mathcal{L} \cdot \partial_j \mathcal{L}$, where $\cdot$ is the inner product in $\mathbb{R}^{n+1}$.

$$\det g_{ij} = h^2$$

and hence the requirement on the determinant being everywhere non-zero requires that the function h is positive. This can be satisfied only if the parallel curvature functions $k_1, \dots, k_n$ are bounded. Overall we are making the assumption that

1. $\kappa = \sqrt{k_1^2 + \cdots + k_n^2} \in L^\infty(\mathbb{R})$ and $a\|\kappa\|_\infty < 1$;
2. Ω does not overlap itself.

This way $\mathcal{L}$ is a global diffeomorphism and we can identify the curved tube Ω with the Riemannian manifold (Ω_0, g_{ij}).

If the determinant of g is positive, it is possible to find its inverse g^{-1}. The answer to the question how it would look like is answered in the following lemma.

Lemma 6.0.5. Let g be a metric tensor defined as above (6.1). Then its inverse g^{-1} is given as

$$(g_{ij})^{-1} = \frac{1}{h^2}\begin{pmatrix} 1 & -h_1 & -h_2 & -h_3 & \dots & -h_n \\ -h_1 & h^2 + h_1^2 & h_1 h_2 & h_1 h_3 & \dots & h_1 h_n \\ -h_2 & h_2 h_1 & h^2 + h_2^2 & h_2 h_3 & \dots & h_2 h_n \\ -h_3 & h_3 h_1 & h_3 h_2 & h^2 + h_3^2 & \dots & h_3 h_n \\ \vdots & & & & \ddots & \vdots \\ -h_n & h_n h_1 & h_n h_2 & h_n h_3 & \dots & h^2 + h_n^2 \end{pmatrix}.$$

Proof. We will prove it via mathematical induction. Checking it for $n = 1$:

$$(g_{ij})(g_{ij})^{-1} = \frac{1}{h^2}\begin{pmatrix} h^2 + h_1^2 & h_1 \\ h_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -h_1 \\ -h_1 & h^2 + h_1^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now let us consider an n-dimensional metric tensor g, denoting it as g^n to emphasize its size. We can write g^{n+1} as

$$(g_{ij}^{n+1}) = \begin{pmatrix} (g^n) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h_{n+1}^2 & 0 & \dots & 0 & h_{n+1} \\ 0 & & \dots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & 0 \\ h_{n+1} & 0 & \dots & 0 & 1 \end{pmatrix} = G_n + \frac{1}{h^2} H_{n+1}.$$

Similarly, we can divide the supposed inverse metric as

$$\begin{aligned}(g_{ij}^{n+1})^{-1} &= \begin{pmatrix} (g^n)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{h^2}\begin{pmatrix} 0 & & \dots & 0 & -h_{n+1} \\ 0 & & \dots & 0 & h_1 h_{n+1} \\ \vdots & & & & \vdots \\ 0 & & \dots & 0 & h_n h_{n+1} \\ -h_{n+1} & h_{n+1} h_1 & \dots & h_{n+1} h_n & h^2 + h_{n+1}^2 \end{pmatrix} \\ &= G_n^{-1} + \frac{1}{h^2} H_{n+1}^{-1}.\end{aligned}$$

Now let us check that $(g_{ij}^{n+1})(g_{ij}^{n+1})^{-1} = \mathcal{I}$.

$$
\begin{aligned}
(g^{n+1})(g^{n+1})^{-1} &= \left(G_n + \frac{1}{h^2}H_{n+1}\right)\left(G_n^{-1} + \frac{1}{h^2}H_{n+1}^{-1}\right) \\
&= G_nG_n^{-1} + H_{n+1}G_n^{-1} + \frac{1}{h^2}G_nH_{n+1}^{-1} + \frac{1}{h^2}H_{n+1}H_{n+1}^{-1} \\
&= \begin{pmatrix} \mathcal{I} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{h^2}\begin{pmatrix} h_{n+1}^2 & -h_1h_{n+1}^2 & \dots & -h_nh_{n+1}^2 & 0 \\ 0 & & & 0 & 0 \\ \vdots & & & & \vdots \\ h_{n+1} & -h_1h_{n+1} & \dots & -h_nh_{n+1} & 0 \end{pmatrix} \\
&\frac{1}{h^2}\begin{pmatrix} 0 & \dots & 0 & -h^2h_{n+1} \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & 0 \end{pmatrix} + \frac{1}{h^2}\begin{pmatrix} -h_{n+1}^2 & h_1h_{n+1}^2 & \dots & h_nh_{n+1}^2 & h^2h_{n+1} \\ 0 & & & 0 & 0 \\ \vdots & & & \vdots & \\ -h_{n+1} & h_1h_{n+1} & \dots & h_nh_{n+1} & h^2 \end{pmatrix} \\
&= \mathcal{I}.
\end{aligned}
$$

□

Let us end this chapter with an example how the waveguide would look in low-dimensions.

Example. In two dimensional case when $n = 1$, the cross section ω is just some open bounded interval, the curve Γ has just one RPAF curvature $k_1 \equiv k$, and the rotation matrix $\mathcal{R} = (1)$. Then

$$h(s,t) = 1 - k(s)t.$$

In tree dimensions the situation starts to be much more interesting. Let $\omega \subset \mathbb{R}^2$ be a cross section, Γ be a curve possessing relatively parallel adapted frame T, N_1, N_2 with RPAF curvatures k_1, k_2. Making the ansatz

$$\mathcal{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta, \end{pmatrix}$$

for some $\theta : \mathbb{R} \to \mathbb{R}$, $\theta \in C^{0,1}(\mathbb{R})$, we have

$$h(s,t_1,t_2) = 1 - k_1(s)[t_1\cos\theta + t_2\sin\theta] - k_2[-t_1\sin\theta + t_2\cos\theta].$$

You can see examples of waveguides on Fig. 6.2.

We will finish this thesis by assigning a waveguide to the Spivak curve σ (cf. Fig. 6.3) using the relatively parallel adapted frame, which will finally ensure a smooth result even for curves lacking the Frenet frame.

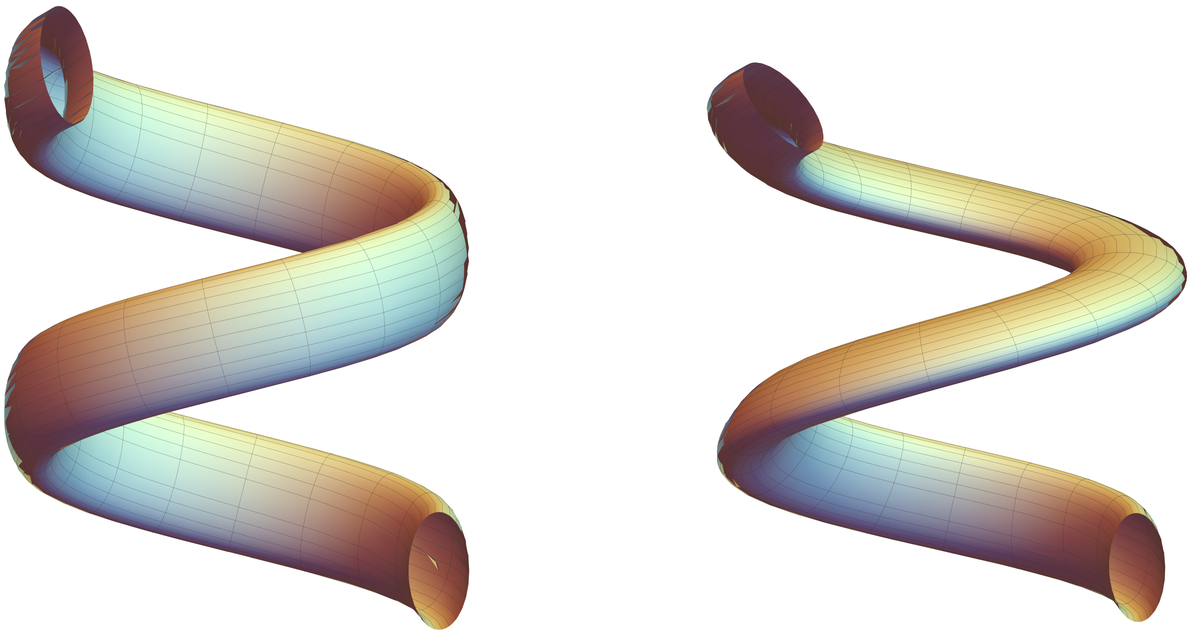

Figure 6.2: The comparison of waveguides constructed from the Frenet frame (left) and the RPAF (right) on the helix (4.3) using an ellipse as the cross section. The major axis corresponds to the binormal B or the normal vector V respectively. At the bottom right both frames coincides.

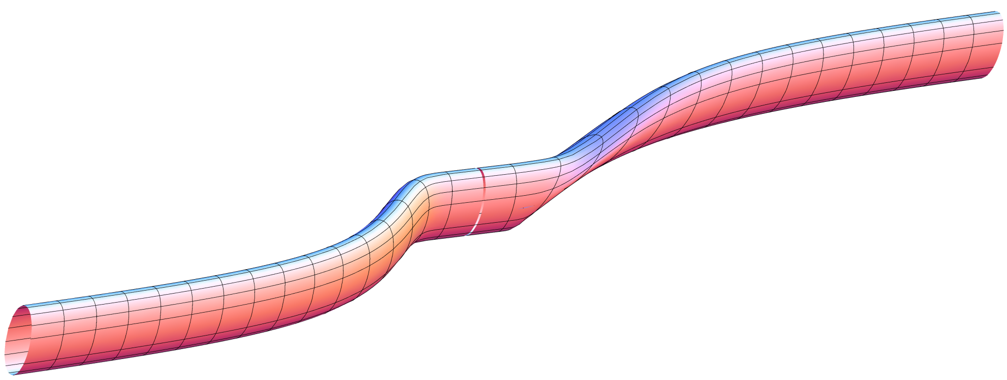

Figure 6.3: A waveguide of ellipse cross section on the Spivak curve σ constructed using the relatively parallel adapted frame.

Conclusion

This thesis was focused on the relatively parallel adapted frame and the moving frames in general. In the first three chapters, the known findings about curves, differential geometry and first order differential equations were recapitulated. Then the three different moving frames were discussed and compared. The theorem about the existence of the relatively parallel adapted frame, generalised to higher dimensions with less restrictions on the curve, was presented and proven. The relatively parallel adapted frame was then used in the construction of wavequides in higher dimensions even for curves not possessing the Frenet frame. Possible continuations of the present work include the strengthening of the modification of the fundamental theorem of curves and further applications. These may include applications of the frame in higher dimensional computer modelling and simulations, in the theory of quantum waveguides or for curves in Minkowski space.

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