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Thermalization in Quantum Networks

Research Project

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1 Introduction

More than a century ago two immensely successful theories were born, thermodynamics and quantum mechanics. The former focuses on macroscopical phenomena of physical systems expressed in terms like heat or temperature whereas the latter is a microscopical theory par excellence. The goal of quantum mechanics is to precisely describe all the physical phenomena as a result of microscopic principles which the physical bodies and fields follow. Even though both theories should lead to same predictions, as far as thermodynamical issues are concerned, the mutual relations between the two have not been understood in full detail until these days.

Great effort has been put to reveal exact ways how the macroscopic properties of the matter emerge from its underlying microscopic nature and to put their description on a solid mathematical footing [1]. The formation of a thermal equilibrium is one of the long-standing issues whose analysis has given rise to a wealth of various questions [2, 3]. The range of investigated problems is very broad, following recent works [4, 5] it can be divided into several key points.

Asymptotics. Many macroscopic phenomena arise as the asymptotic long-time behaviour of the given quantum system. Determination of such an asymptotic regime for complex quantum systems might be a difficult problem by itself. The central question then is what is the structure and algebraic properties of asymptotic states of the system.

Equilibration. A substantial feature of the thermalization is the occurence of the equilibration. The system should evolve towards some stationary state. However, this situation might not be the case in general. The equilibration in the given system may occur only for some initial states or for large enough systems. In the case the system does equilibrate it is reasonable to investigate the structure of stationary states. If the set of stationary states is sufficiently rich a strong memory effect preserving a lot of information about the initial state is present in the system.

Independency of initial conditions and thermalization. On the contrary, when the set of stationary states is small only few parameters suffice to describe the asymptotic state of the system and the asymptotics of the system is almost independent of the initial conditions. In particular, if the system tends to the Gibbs state $\rho_{\mathrm{th}} \sim e^{-\beta H}$ with β being an inverse temperature and H Hamiltonian of the system we say the given system thermalizes.

As has been already mentioned we are primarily concerned with the above listed issues from the viewpoint of underlying microscopic laws. We intend to study how the equilibrium of a total system establishes via mutual interactions of its numerous subsystems. To be specific, we consider a quantum system composed of many constituents with dynamics analogous to the Boltzmann rare gas. Individual subsystems undergo a free evolution interrupted by short random bipartite collisions. Such systems are referred to as quantum networks. Their nodes correspond to individual quantum subsystems and the set of links captures the geometry of bipartite interactions. An interplay between the free evolution and the random bipartite evolution is responsible for the resulting asymptotic evolution of the system.

In this work we study quantum networks with two properly chosen types of interaction: the partial swap interaction and the controlled-type interaction. The main goal is to explore their asymptotic regime and elucidate the impact of actual collision times, interaction strengths, their probability distribution and the size of the system on the structure of asymptotic states. Especially, we are interested in which details of a quantum network dynamics are irrelevant for its asyptotic evolution and consequently for its equilibration.

Our ambition is to find closed analytical solutions for systems of an arbitrary size. To succesfully accomplish this task the attractor theory developed in [6] for random unitary

operations is employed.

In the first part of this thesis we state precisely conditions under which we explore the quantum network asymptotic behaviour and review indispensable mathematical tools. Two examples of interactions among the network constituents will be taken into account. For either of these we at first examine dynamics governed solely by mutual interactions. Afterwards we consider also the free evolution of the system and examine the dynamics obeyed by the system when both free evolution and mutual interactions are present.

2 Preliminaries

Before we move to investigate the emergence of equilibration in our model system let us recall several fundamental concepts common in quantum theory together with an appropriate notation.

2.1 Mathematical Framework

At the very beginning, let us review the concept of *qudits*. The qudit is a quantum system represented by a d-dimensional Hilbert state space. For $d = 2$ we call such systems *qubits*. In this work we concentrate on systems consisting of finite number of qudits with the same dimension d. Hilbert spaces associated with such systems are constructed as a tensor product of d-dimensional state spaces $\mathscr{H}_d \otimes \mathscr{H}_d \otimes \cdots \otimes \mathscr{H}_d \equiv \mathscr{H}$. When denoting vectors and covectors of the given space, the Dirac notation is used. A vector from $\mathscr{H}$ is denoted as $|\varphi\rangle$ whereas its corresponding covector reads $\langle\varphi|$. The scalar product of two vectors $|\varphi\rangle, |\psi\rangle \in \mathscr{H}$ in this order is thus $\langle\varphi|\psi\rangle$. The set $\mathcal{B}(\mathscr{H})$ of all bounded operators acting on $\mathscr{H}$ forms another Hilbert space. This space is equipped with the Hilbert-Schmidt scalar product—for any two operators $A, B \in \mathcal{B}(\mathscr{H})$ their scalar product is defined as

$$(A, B) \coloneqq \mathrm{Tr}(A^\dagger B). \tag{1}$$

The norm is defined accordingly as $\|A\| \coloneqq \sqrt{(A, A)}$. Mappings taking operators to operators will be referred to as *superoperators*. A very important subset of $\mathcal{B}(\mathscr{H})$ is formed by *density operators*. These are unit-trace positive operators used for a system state description.

Hereafter, all the formulas related to qudits will be expressed in the computational basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ where d stands for qudit dimensionality. For further calculations it is suitable to introduce matrix element notation as follows. Having matrix $X \in \mathbb{C}^{d^N \times d^N}$, its element in the computational basis is denoted as

$$X^{i_1 \ldots i_N}_{j_1 \ldots j_N} \equiv \langle i_1 \ldots i_N | X | j_1 \ldots j_N \rangle, \tag{2}$$

where $i_1, \ldots, i_N \in \{0, \ldots, d-1\}$ label row indices and $j_1, \ldots, j_N \in \{0, \ldots, d-1\}$ label column indices. Henceforth, we will refer to the index of the form (2) as a *multiindex*. In some cases it is helpful to introduce one more symbol. When there are only two pairs of indices (i_l, j_l), (i_m, j_m) relevant for the current calculation the following notation is used

$$X^{i_l, i_m}_{j_l, j_m} \equiv X^{i_1 \ldots i_{l-1}\ i_l\ i_{l+1} \ldots i_{m-1}\ i_m\ i_{m+1} \ldots i_N}_{j_1 \ldots j_{l-1}\ j_l\ j_{l+1} \ldots j_{m-1}\ j_m\ j_{m+1} \ldots j_N}, \tag{3}$$

where inequality $1 \leq l < m \leq N$ is assumed. Similar notation is used also when more than two indices i, j are taken under consideration. In the following we will sometimes call the double (i_l, j_l) or (i_m, j_m) appearing in the index notation introduced in (3) as a *local index* of a matrix element.

2.2 Closed versus Open Dynamics

Taking the quantum system evolution into account two basic classes of systems can be identified—closed quantum systems and open quantum systems. Closed systems are treated as being completely isolated from their environment. Their evolution is well understood and many tools have been developed for their study. From the Schrödinger equation it follows that closed system evolution is fully described by the corresponding evolution operator $U(t)$ whose generator is the system Hamiltonian H. At time t the

system, initially prepared in the state $\rho(0)$, is described by the density operator of the form

$$\rho(t) = U(t)\,\rho(0)\,U^{\dagger}(t). \tag{4}$$

As the Hamiltonian as well as the evolution operator are normal operators they are diagonalizable in some orthonormal basis. Solution of system's dynamics reduces to calculation of corresponding eigenvalues and eigenvectors. Except for Hamiltonian eigenvectors any state of the system undergoes a periodic or quasiperiodic evolution. Consequently, the equilibration in a closed system in the sense defined above is not possible. Provided that oscillations are very quick an alternative weaker definition of equilibration is adopted for closed quantum systems. According to these the system equilibrates whenever its state remains close to an averaged state pertaining to the given evolution [3].

While we abandon the constraint that the system must not interact with its environment we arrive at systems driven by open dynamics. Providing the system evolves from an initial state not correlated with the environment its evolution is captured by *quantum operations* [7]. As a prominent class of these superoperators appear *random unitary operations*. A random unitary operation Φ allows the convex decomposition in the form

$$\Phi(A) = \sum_{\alpha=1}^{M} p_\alpha\, U_\alpha\, A\, U_\alpha^{\dagger} \tag{5}$$

with a set of unitary operators $\{U_\alpha\}_{\alpha=1}^{M}$, probability distribution $\{p_\alpha\}_{\alpha=1}^{M}$ and an arbitrary $A \in \mathcal{B}(\mathscr{H})$. Such an operation can be understood as a weighted average of unitary evolutions (4) generated by different Hamiltonians. Probabilities p_α express our incomplete knowledge about which unitary evolution actually takes place. The operation Φ determines the form of the system state after the time interval Δt. We are concerned with the evolution arising from successive applications of Φ. For our study of equilibration it is thus an important task to find an asymptotic behaviour of the evolution $\lim_{n\to\infty} \Phi^n(\rho)$. Unfortunately, the random unitary operation cannot be diagonalized, in general. Determination of the system evolution is therefore much harder in comparison with closed quantum systems. Nevertheless, as shown in [6] the asymptotic regime ρ_{asymp} of the system, governed by iteration of Φ, lies in the subspace spanned by so called *attractors*. Hereafter we refer to this subspace as the *attractor space*. Attractors are all nontrivial solutions to matrix equations

$$U_\alpha X U_\alpha^{\dagger} = \lambda X, \quad \forall \alpha \in \{1, \ldots, M\}, \tag{6}$$

where λ is an eigenvalue of Φ for which $|\lambda| = 1$. All such eigenvalues form the *attractor spectrum* $\sigma_{|1|}$ for Φ. The equation above has to be satisfied for all unitaries U_α *simultaneously*. Having determined the set of attractors B_λ for each $\lambda \in \sigma_{|1|}$ we can describe the asymptotics of the system as follows. In each eigenspace B_λ we find its suitable orthonormal basis $\{Y_{\lambda,i}\}_{i=1}^{d_\lambda}$, where $d_\lambda = \dim B_\lambda$. It can be proven [6] that two vectors $Y_{\lambda,i}$ and $Y_{\mu,j}$ for different eigenvalues $\mu, \lambda \in \sigma_{|1|}$ are orthogonal, hence the set of all basis vectors $\{Y_{\lambda,i}\}_{\lambda,i}$ represents an orthonormal basis of the attractor space. The state $\Phi^n(\rho)$ approaches the asymptotic regime of the evolution $\rho_{\mathrm{asymp}}(n)$ for increasing number of iterations $n \to \infty$. In the orthonormal basis of the attractor space this asymptotic regime reads

$$\rho_{\mathrm{asymp}}(n) = \sum_{\lambda \in \sigma_{|1|}} \sum_{i=1}^{d_\lambda} (\lambda^n\, \xi_{\lambda,i})\, Y_{\lambda,i}. \tag{7}$$

Coefficients $\xi_{\lambda,i} = (Y_{\lambda,i}, \rho) = \mathrm{Tr}(Y_{\lambda,i}^{\dagger}\, \rho)$ store an information about the initial state ρ. Note the formula above is explicitly dependent on the number of iterations n whenever there is

$\lambda \in \sigma_{|1|}$ such that $\lambda \neq 1$. The asymptotic dynamics may thus exhibit a non-trivial behaviour ranging from a stationary evolution to a quasi-periodic evolution. Another important feature of the evolution asymptotics (7) is that the (nonzero) probabilities $\{p_\alpha\}_\alpha$ play absolutely no role.

The concept of attractors enables us to rephrase and study the presence of equilibration in the system under consideration. Given a physical system we say it *equilibrates* whenever it evolves towards a stationary state.

From the attractor viewpoint the system equilibrates when the asymptotic dynamics (7) is stationary. Inspection of (7) shows this situation happens when the only eigenvalue lying in the attractor spectrum is $\lambda = 1$. Other eivenvalues $\lambda \neq 1$ such that $|\lambda| = 1$ contribute to the asymptotic dynamics by oscillatory terms whose magnitude is directly proportional to the overlap of the initial state ρ with the relevant eigenvectors $Y_{\lambda,i}$.

As can be seen from (6) all the eigenvectors $X_{\lambda=1,i}$ for $\lambda = 1$ represent *fixed points* of the operation Φ, $\Phi(X_{1,i}) = X_{1,i}$. Such attractors are preserved in the time evolution. Hermitian attractors thus play the role of integrals of motion. By an *integral of motion* we mean a hermitian operator whose expectation value in a system state is constant during the time evolution of the system. The overlap of such operators with the initial state ρ of the system tells us how much information carried by ρ will survive after long course of time.

3 Quantum Networks

3.1 Description of the Model

Thermodynamical phenomena result from complex relations within physical systems composed of many mutually interacting constituents. The theoretical treatment of systems with a lot of interacting subsystems almost always poses a challenging problem and some simplifying restrictions are often adopted. In our model system we neglect many-body interactions and assume the two-body interactions are well-separated in time.

The system behaviour is thus governed by binary collisions between individual subsystems. Simultaneously, subsystems themselves also follow their own free evolutions. Suppose we are given two subsystems. During the system evolution there is either no collision between them and then they evolve completely independently, or they do collide and then their evolution is linked together. The structure of these links within the entire system is well embodied by the *network*. The whole system can be understood as a network composed of many nodes, represented by subsystems, which are connected by (in general directed) edges, represented by mutual collisions.

Let collisions be rare enough that no more than one collision can occur at a given time. At each time there is thus at most a single edge active in the whole network. Nevertheless, our ability to determine which subsystems collide at a given instant is limited and we are lead to the introduction of a probability description. Each edge in the network is weighted by the probability with which the associated collision is going to happen at the given moment. We have thence come to the set of subsystems whose free evolution is occasionally interrupted by random mutual collisions. This situation amounts to the concept of the Boltzmann gas.

Our aim is to investigate the equilibration from the first principles stated by quantum mechanics. A quantum description of the system is thus necessary and we arrive at the *quantum networks*. To every quantum network there are attached a Hilbert space $\mathscr{H}$ and a Hamiltonian H governing its evolution. For open systems instead of Hamiltonian another means of an evolution description is present. In our case this means is provided by random unitary operations, as is made precise later.

We use the quantum network comprised of N qudits with the same dimensionality d. Each qudit evolves according to its free evolution Hamiltonian which acts solely on a given qudit and is the same for each qudit. Qudits are also exposed to binary collisions with one another. The form of collisions will be specified below. Throughout the whole thesis we consider only the *complete* quantum networks—every qudit may collide with any other qudit. This ideal setting forms a base upon which generalizations with different topologies can be made. Each collision, say between the a-th and b-th qudits, is assigned a probability p_{ab} expressing how likely is the particular ordered pair (a, b) subjected to an interaction at a given instant. Let the probability distribution $\{p_{ab}\}_{ab}$ be constant in time. For illustration this scheme is depicted in Figure 1.

Each collision lasts a definite amount of time which we will refer to as the *collision time* Δt. We take into account two possibilities. At first, we consider all such times Δt to be equal for each collision. Distinct collisions may be separated by time intervals with purely free evolution of the system. Since we assume Δt to be very small we can think of the purely free evolution time interval as being a multiple of Δt. In other words we study time evolution of the system as a sequence of time intervals Δt. Next, we generalize this setup with collision times being different for distinct pairs (see subsection 3.2).

An interaction of a-th and b-th qudits occuring during the collision-time interval Δt is represented by a unitary operator U_{ab}. This operator acts on the network state and includes also the free evolution of the whole network within the time Δt. With probability

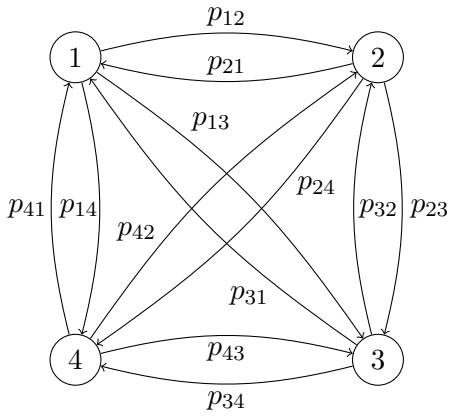

Figure 1: An example of a complete quantum network with $N = 4$. Each node represents a distinct d-dimensional qudit and there is a directed edge between any qudits weighted by corresponding probability of collision p_{ab}.

p_{ab} the system state ρ thus undergoes a path of a closed evolution $U_{ab}\,\rho\,U_{ab}^{\dagger}$. If there is no collision present, the state evolution is driven by operator U^{free}. Recalling everything we have said about our model we see the system evolution within Δt can be rephrased in terms of a random unitary operation (5). That is, operation Φ of the form

$$\Phi(A) = \sum_{\substack{a,b=1\\a\neq b}}^{N} p_{ab}\,U_{ab}\,A\,U_{ab}^{\dagger} + p_0\,U^{\text{free}}\,A\,(U^{\text{free}})^{\dagger}, \tag{8}$$

where $A \in \mathcal{B}(\mathscr{H}^{\otimes N})$ and p_0 stands for a probability of no collision taking place. Two setups may be identified, either $p_0 = 0$ and there is some interaction present *in every interval* Δt of the evolution, or $p_0 \neq 0$. The latter possibility allows for an easier analysis of the system asymptotics.

In further calculations our solution will be parametrized by several quantities (*e.g.* coupling strenght, collision time, attractor eigenvalue). Since we are looking for a generic form of the asymptotic evolution we sometimes intentionally neglect zero-measure set of parameter values to simplify our computations while keeping almost full generality of our results. In an experimental realisation of the quantum network it is effectively impossible to fine-tune parameters to lie in such a zero-measure set and the generic case is thus not affected by these exceptional values.

3.2 Generators of the Evolution

Each of the unitaries U_{ab} in (8) represents a closed evolution of the network within the time interval Δt. Let these evolutions be generated by Hamiltonians H_{ab}. Every H_{ab} encompasses both the free evolution of the whole network and an interaction between qudits a and b in this order. If no collision takes place in a given time step, the governing Hamiltonian is H^{free}, corresponding to the purely free evolution of the whole network. As the *trivial free evolution* we call the free evolution generated by a zero Hamiltonian H^{free}. The Hamiltonian H_{ab} for an actual collision between qudits a and b consists of its free and interaction parts as follows

$$H_{ab} = H^{\text{free}} + H_{ab}^{\text{int}} = \sum_{i\neq a,b} H_i^{\text{free}} + H_a^{\text{free}} + H_b^{\text{free}} + H_{ab}^{\text{int}} = \sum_{i\neq a,b} H_i^{\text{free}} + \tilde{H}_{ab}, \tag{9}$$

where H_i^{free} is the free Hamiltonian of the i-th qudit. All the interaction Hamiltonians H_{ab}^{int} are assumed to describe the same kind of interaction. Symbol $\tilde{H}_{ab}$ denotes the Hamiltonian acting on the subsystem comprised of qudits a and b only. This decomposition of the total Hamiltonian H_{ab} ensures that

$$\left[\tilde{H}_{ab}, \sum_{i\neq a,b} H_i^{\text{free}}\right] = 0. \tag{10}$$

One can make use of this equality to simplify the expression for the evolution operator

$$U_{ab}(\Delta t) = e^{\mathrm{i}\,\Delta t H_{ab}} = e^{\mathrm{i}\,\Delta t \tilde{H}_{ab}}\, e^{\mathrm{i}\,\Delta t \sum_{i\neq a,b} H_i^{\text{free}}} = \tilde{V}_{ab}(\Delta t)\tilde{U}_{ab}(\Delta t), \tag{11}$$

where we defined $\tilde{V}_{ab}(\Delta t) = \exp(\mathrm{i}\,\Delta t\,\tilde{H}_{ab})$ as the part of the unitary operator acting on qudits a and b and $\tilde{U}_{ab}(\Delta t)$ stands for the part of the unitary operator associated with a purely free evolution of remaining qudits.

In the special case when the interaction Hamiltonian H_{ab}^{int} commutes with the free Hamiltonian H^{free} the evolution operator can be rewritten in a more convenient way. We are left with

$$U_{ab}(\Delta t) = e^{\mathrm{i}\,\Delta t H_{ab}} = e^{\mathrm{i}\,\Delta t H_{ab}^{\text{int}}}\, e^{\mathrm{i}\,\Delta t H^{\text{free}}} = V_{ab}(\Delta t)U^{\text{free}}(\Delta t), \tag{12}$$

where U^{free} stands for a free evolution of the *whole* network and V_{ab} is an operator characterizing solely the interaction between qudits a and b.

Let us say few words about the role of time Δt. This quantity is interpreted as a collision time for the interaction emerging between a-th and b-th qudit. In our analysis we explore first the case when collision times are the same for all qudit pairs and equal to one. We refer to this setup as the *unit collision time*. Later on we consider more general case with Δt_{ab} different for each pair of qudits. The purely free evolution will be assigned a time interval Δt_0 in such a setup. On the other hand, we will not come to full generality with Δt_{ab} and Δt_0 being different in each step of the evolution. Similarly to the probability distribution $\{p_{ab}\}_{ab}$ we keep these quantities fixed in time. To sum up, recalling (9) the most general form of the evolution operators U_{ab} and U^{free} we consider reads

$$U_{ab}(\Delta t_{ab}) = e^{\mathrm{i}\,\Delta t_{ab} H^{\text{free}} + \mathrm{i}\,\Delta t_{ab} H_{ab}^{\text{int}}}, \quad U^{\text{free}}(\Delta t_0) = e^{\mathrm{i}\,\Delta t_0 H^{\text{free}}}. \tag{13}$$

So far we have not mentioned the specific form of the Hamiltonians themselves. In what follows we will take into consideration two kinds of free Hamiltonians, which will be specified in appropriate sections, as well as two kinds of interactions. The first interaction is the partial swap operation whereas the second one is the controlled-NOT operation. Both are defined and presented in detail in the following sections.

3.3 Asymptotics of the Quantum Network

In subsection 2.2 we recalled mathematical tools with help of which one determines the asymptotic state of the system evolving according to a random unitary operation. In the same subsection we also revealed connections between the asymptotic evolution and conditions under which the particular system equilibrates. Assume for a while that $p_0 = 0$ in (8). The asymptotic dynamics is obtained via solving of attractor equations (6). The evolution operator decomposition (11) enables us to rewrite these relations into

$$\tilde{U}_{ab}\, X\, \tilde{U}_{ab}^{\dagger} = \lambda \tilde{V}_{ab}^{\dagger}\, X\, \tilde{V}_{ab}, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{14}$$

This expression can be simplified even more. Introducing operator $\overline{V}_{ab} = \tilde{V}_{ab} U_a^\dagger U_b^\dagger$, where $U_i = \exp(\mathrm{i}\,\Delta t H_i^{\mathrm{free}})$, and realizing that $U_a U_b \tilde{U}_{ab} = U^{\mathrm{free}}$ we come to

$$U^{\mathrm{free}}\, X\, (U^{\mathrm{free}})^\dagger = \lambda \overline{V}_{ab}^\dagger\, X\, \overline{V}_{ab}, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{15}$$

If we additionally assume that also no collision might take place during the time interval Δt, *i.e.* $p_0 \neq 0$ (8), attractor equations decouple into

$$U^{\mathrm{free}}\, X\, (U^{\mathrm{free}})^\dagger = \lambda X, \quad \overline{V}_{ab}^\dagger\, X\, \overline{V}_{ab} = X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{16}$$

We have thus separated the purely free evolution part and the interaction part of attractor equations.

Suppose the *interaction Hamiltonian commutes with the free evolution* and unitary operators can thus be rewritten into (12). Importantly, in such a special situation we obtain equations (15) and (16) with $\overline{V}_{ab} = V_{ab}$. We have thus completely separated a purely interaction evolution characterized by V_{ab} and a purely free evolution captured by U^{free}. This situation apparently arises when the free evolution is trivial. One of key consequences is that knowing solution obtained for a trivial free evolution we can easily derive even the general case in (16). These remarkable properties lead us to identify two classes of interactions. One class composed of operations whose Hamiltonian commutes with the free evolution Hamiltonian and the other class composed of the rest of Hamiltonians. In what follows, we choose a representative from either class and study its effects on the system equilibration.

4 Partial Swap-type interaction

As has been demonstrated in the preceding subsection, when the interaction and free Hamiltonians commute with each other the analysis of the system asymptotics simplifies considerably. Surprisingly, there indeed exists an interaction commuting with any free evolution. It is called the *partial swap* operation. In the following we examine the asymptotic evolution of the quantum network when random partial swap interactions are present.

The partial swap (PSW) of qudits a and b is a linear combination of the identity, leaving the qudits unaffected, and a swap operation (SW), interchanging the pair of qudits, *i.e.* $\mathrm{SW}|x\rangle|y\rangle = |y\rangle|x\rangle$. That is

$$\mathrm{PSW}_{ab} = p\,\mathbb{I}_{ab} + q\,\mathrm{SW}_{ab}, \tag{17}$$

where the coefficients p and q appear as parameters. Unitarity of the partial swap restricts values of these parameters as follows

$$p = \cos\left(\frac{\varphi}{2}\right)e^{\mathrm{i}\rho}, \quad q = \pm\mathrm{i}\,\sin\left(\frac{\varphi}{2}\right)e^{\mathrm{i}\rho}, \tag{18}$$

where $\varphi \in (0,\pi)$ and $\rho \in [0,2\pi)$. As the global phase has no observable effects we can put $\rho = 0$. Making use of the involution property $\mathrm{SW}^2 = \mathbb{I}$ one can easily find the Hamiltonian for the partial swap (see Appendix B)

$$H_{\mathrm{PSW}_{ab}} = \pm\frac{\varphi}{2}\mathrm{SW}_{ab}. \tag{19}$$

Thanks to commutation relations (91) derived in Appendix B the evolution operators read $U_{ab}(\Delta t) = V_{ab}(\Delta t)U^{\mathrm{free}}(\Delta t)$, where $V_{ab}(\Delta t) = \exp(\pm\mathrm{i}\,(\Delta t\,\varphi/2)\mathrm{SW}_{ab})$ (12). This form proves useful for future calculations. For unit collision times $\Delta t = 1$ one obtains $V_{ab} = \mathrm{PSW}_{ab}$. Non-unit collision times Δt enter relation (17) only via parameter φ in a way $\varphi \to \varphi\Delta t$. The form of the evolution operator is thus preserved, only parameters p and q modify their values.

Our discussion on the network equilibration is divided into two parts. Firstly, we consider mutual interactions between constituents together with a trivial free evolution $U^{\mathrm{free}} = \mathbb{I}$. We start with the case when interaction times Δt_{ab} for all qudit pairs (a,b) are equal. Later on this constraint is relaxed. Secondly, we incorporate also a non-trivial free evolution.

4.1 Pure Collisions

Before investigating the general setup let us focus on a network evolution when the free part is the trivial one. In other words $U_{ab}(\Delta t) = \overline{V}_{ab}(\Delta t)$. We assume $p_0 = 0$ (8). If we took p_0 nonzero, the attractor equations would directly imply the only nontrivial attractors correspond to $\lambda = 1$. For zero p_0 there appear also other eigenvalues $\lambda \neq 1$. In order to reveal a multi-qudit system evolution generated by the partial swap operation we can plug operators $U_{ab}(\Delta t)$ into attractor equations (15) to obtain

$$p(1-\lambda)\,X + q\,(\mathrm{SW}_{ab}\,X - \lambda X\,\mathrm{SW}_{ab}) = 0, \tag{20}$$

where a and b run through set $\{1,\ldots,N\}$ of all qudits in the system, $a \neq b$. A remark presented above on the effect of different collision times makes it clear that these equations are general enough to encompass all possible settings of Δt_{ab}. For $\lambda = 1$ or $p = 0$ equations (20) reduce to

$$\mathrm{SW}_{ab}X = \lambda X\,\mathrm{SW}_{ab}, \tag{21}$$

i.e. equations (6) for the swap operation only. Let $N \geq 2$ be a number of qudits in the system and $d \geq 2$ their dimension. In the following we provide a solution for the equations (20) or, when the solution has no simple form, we restrict ourselves to count the number of independent solutions. Our findings are collected in the conclusion at the end of this subsection.

4.1.1 Unit Collision Time

Even though equations (20) hold true for all values of collision times, let us treat the case for $\Delta t_{ab} = 1$ first. Generalizations will be straightforward and presented later on.

Attractors for $\lambda = 1$ Setting $\lambda = 1$ simplifies equations (20) into relations (21). Let us begin by recalling that $\mathrm{SW}_{ab} = \mathrm{SW}_{ab}^{-1} = \mathrm{SW}_{ab}^{\dagger}$. The swap operation acts as a swap of row indices while applied to matrix X from the left-hand side. Similarly, it acts as a swap of column indices while applied from the right-hand side. Equations (21) therefore reduce to $\mathrm{SW}_{ab} X \mathrm{SW}_{ab} = X$ which can be expressed in the index notation as

$$X^{i_a,i_b}_{j_a,j_b} = X^{i_b,i_a}_{j_b,j_a}, \quad a,b \in \{1,\ldots,N\}, \quad a \neq b \tag{22}$$

with $i,j \in \{0,\ldots,d-1\}$. That is, all matrix elements which differ by a permutation of their local indices must be equal. These elements form one equivalence class, each class corresponds to an attractor for $\lambda = 1$. Let us look for a number $S_1(N,d)$ of equivalence classes, *i.e.* the dimension of the attractor space. Due to the permutation invariance of the multiindices one can represent each equivalence class merely by a number of local indices $\binom{i}{j}$ appearing in multiindices of relevant matrix elements. These local indices are of any of the following forms

$$\binom{0}{0}, \binom{0}{1}, \ldots, \binom{0}{d-1}, \binom{1}{0}, \binom{1}{1}, \ldots, \binom{1}{d-1}, \ldots, \binom{d-1}{0}, \binom{d-1}{1}, \ldots, \binom{d-1}{d-1}. \tag{23}$$

There are obviously d^2 such local indices which can appear in a given multiindex. The number of equivalence classes is therefore equal to the number of ways one can distribute local indices from above into N-element sets (*i.e.* into multiindices where one does not care about the order of local indices). Since one local index can appear in a given multiindex more than once, we have to use the formula for a number of combinations with repetition. Its general form is $\binom{n+k-1}{k}$ for a subset of size k formed by elements from an n-element set. In our case $k = N$ and $n = d^2$ and we immediately obtain the number of equivalence classes

$$S_1(N,d) = \binom{N+d^2-1}{N}. \tag{24}$$

Recall local indices listed in (23) and focus on some fixed multiindex. In this multiindex there is a certain number of local indices of the form $\binom{0}{0}$. Let this number be denoted by $c(1)$ and similarly for all the remaining local indices from (23). Each equivalence class is thus characterized by d^2-tuple $(c(1),\ldots,c(d^2))$. Making use of this notation we can easily express the general form the attractors associated with $\lambda = 1$ take on. It can be written as a linear combination

$$X = \sum_{c(1),\ldots,c(d^2)} \alpha_{c(1)\ldots c(d^2)} P_{c(1)\ldots c(d^2)}, \tag{25}$$

where α's are coefficents and matrix $P_{c(1)\ldots c(d^2)}$ elements are all zeros except for those elements lying in the equivalence class specified by numbers $c(1),\ldots,c(d^2)$. These special elements are all identical and their values are chosen so as attractors $P_{c(1)\ldots c(d^2)} \equiv P_{\vec{c}}$ are properly normalized according to the Hilbert-Schmidt norm, $\|P_{\vec{c}}\| = 1$. Consequently, matrices $\{P_{\vec{c}}\}_{\vec{c}}$ form an orthonormal basis in the attractor space.

Attractors for $\lambda \neq 1$ Let us focus on the case with $\lambda \neq 1$ in this subsection. Since we assume $p \neq 0$ equation (20) reduces to

$$X + \gamma\left(\mathrm{SW}_{ab}X - \lambda X \mathrm{SW}_{ab}\right) = 0 \tag{26}$$

with $\gamma = \frac{q}{p(1-\lambda)} \neq 0$. By use of the swap operation properties we can express (26) in a local index notation

$$X^{i_a,i_b}_{j_a,j_b} + \gamma\left(X^{i_b,i_a}_{j_a,j_b} - \lambda X^{i_a,i_b}_{j_b,j_a}\right) = 0. \tag{27}$$

In order to find solution to (27) consider the following system of equations

$$\begin{aligned}
X^{a,b}_{c,d} + \gamma\left(X^{b,a}_{c,d} - \lambda X^{a,b}_{d,c}\right) &= 0,\\
X^{b,a}_{c,d} + \gamma\left(X^{a,b}_{c,d} - \lambda X^{b,a}_{d,c}\right) &= 0,\\
X^{a,b}_{d,c} + \gamma\left(X^{b,a}_{d,c} - \lambda X^{a,b}_{c,d}\right) &= 0,\\
X^{b,a}_{d,c} + \gamma\left(X^{a,b}_{d,c} - \lambda X^{b,a}_{c,d}\right) &= 0.
\end{aligned}$$

We can rewrite this system in the matrix form as

$$\begin{pmatrix} 1 & \gamma & -\lambda\gamma & 0\\ \gamma & 1 & 0 & -\lambda\gamma\\ -\lambda\gamma & 0 & 1 & \gamma\\ 0 & -\lambda\gamma & \gamma & 1 \end{pmatrix}\begin{pmatrix} A\\ B\\ C\\ D \end{pmatrix} = 0, \tag{28}$$

with

$$A = X^{a,b}_{c,d},\ B = X^{b,a}_{c,d},\ C = X^{a,b}_{d,c},\ D = X^{b,a}_{d,c}. \tag{29}$$

Unitarity conditions (18) guarantee $p \neq \pm q$ and the determinant of the matrix in (28) is therefore zero iff

$$p^2(\lambda-1)^2 - q^2(1+\lambda)^2 = 0. \tag{30}$$

Expressing the eigenvalue as $\lambda = e^{\mathrm{i}\omega}$ with $\omega \in [0, 2\pi)$ the latter condition can be rewritten as

$$q = \pm \mathrm{i}\, p \tan\left(\frac{\omega}{2}\right),\ \textit{i.e.}\quad \varphi = \pm\omega, \tag{31}$$

where for φ see (18). We have found out that whenever (31) is *not* satisfied, all matrix elements (29) are zero. Since indices a through d were chosen arbitrarily we can conclude while (31) is not satisfied attractor matrix X vanishes. Similarly, for $\lambda = -1$ equality (30) does not hold, associated matrix is thence nonsingular and the only possible solution to (28) is also made of zero elements. That is, for $\lambda = -1$ one obtains a zero attractor matrix.

Let us investigate the case for eigenvalues $\lambda \neq \pm 1$ when (31) *is* satisfied. Then relation (30) is apparently equivalent to $\gamma = \pm\frac{1}{1+\lambda}$ and the matrix in (28) reads

$$\frac{1}{1+\lambda}\begin{pmatrix} 1+\lambda & \pm 1 & \mp\lambda & 0\\ \pm 1 & 1+\lambda & 0 & \mp\lambda\\ \mp\lambda & 0 & 1+\lambda & \pm 1\\ 0 & \mp\lambda & \pm 1 & 1+\lambda \end{pmatrix}.$$

Corresponding kernels of the matrices above are for $\gamma = \frac{1}{1+\lambda}$ and $\gamma = -\frac{1}{1+\lambda}$ of the following form

$$\begin{pmatrix} A\\ B\\ C\\ D \end{pmatrix} = t\begin{pmatrix} -1\\ 1\\ -1\\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A\\ B\\ C\\ D \end{pmatrix} = t\begin{pmatrix} -1\\ -1\\ 1\\ 1 \end{pmatrix} \tag{32}$$

respectively, where $t \in \mathbb{C}$. See (29) and suppose $a = b$ and $\gamma = \frac{1}{1+\lambda}$. From (32) one can easily see that all elements (29) are zero. Similarly for $c = d$ and $\gamma = -\frac{1}{1+\lambda}$. Therefore, if the qudit dimension d is strictly less than a number of qudits N then the entire matrix X of a possible attractor is zero. The reason is following. For $N > d$ there are at least two local row indices and at least two local column indices in a multiindex of any matrix element which are of the same value. For $\gamma = \pm\frac{1}{1+\lambda}$ given we can thus consider either of the two doubles of local indices and apply arguments from above.

For $N \leq d$ we obtain nonzero solutions. Let us focus on the case $\gamma = \frac{1}{1+\lambda}$ first. From (29) and (32) one can see that matrix elements which differ only by permutation of their column indices have the same value. That is, one can gather such elements into an equivalence class. Each class contains elements whose column indices are same up to some permutation. But there is one more constraint. Two elements whose multiindices differ by a transposition of two row indices are inverse to each other. In other words, all elements with a column multiindex fixed have the same value if their row multiindices are identical up to an even permutation. Elements whose row multiindices differ by an odd permutation have opposite values. The same analysis would emerge even for $\gamma = -\frac{1}{1+\lambda}$ with a single exception that the role of row and column multiindices is interchanged.

Let us count the number of independent elements in the attractor matrix X, *i.e.* the number of degrees of freedom present in the solution to (26) for $\gamma = \frac{1}{1+\lambda}$. As we saw in the previous paragraph, having particular matrix element X^r_s fixed, its value determines also the value of all other elements whose multiindices differ from r and s only by permutations. Such elements form an equivalence class. The number of degrees of freedom is thence equal to the number of all equivalence classes. This situation is quite similar to that in subsubsection 4.1.1. Consider the column indices first. We can form an N-element multiindex where each index assumes one of d values, *i.e.* there are $\binom{N+d-1}{N}$ nonequivalent column multiindices for given row multiindex. Regarding the row multiindex we have to take into account only those multiindices with no two indices identical. These multiindices correspond to nonzero matrix elements. Therefore, there is $\binom{d}{N}$ nonequivalent row multiindices. To conclude, the number of degrees of freedom is

$$S_{\neq 1}(N,d) = \binom{N+d-1}{N} \cdot \binom{d}{N}, \tag{33}$$

where N is a number of qudits and d is their dimension. The current degree-of-freedom analysis has been obviously performed for both $\gamma = \pm\frac{1}{1+\lambda}$. Since $\binom{d}{N} = 0$ whenever $d < N$, relation (33) holds for all possible values of d and N.

4.1.2 Conclusion with Unit Collision Time

As we have just seen for the present setting, if the number N of qudits in a network exceeds their dimensionality d, the network equilibrates. Otherwise, the asymptotic evolution is not stationary. Specifically, the attractor form for partial swap operation with $\Delta t_{ab} = 1$ and corresponding number of degrees of freedom read as follows

- $\lambda = 1$ dimension of the attractor space for given N and d is equal to $S_1(N,d)$ (24), the attractor itself satisfies (25),
- $\lambda \neq 1$ for $\varphi \neq \pm\omega$ regardless of $N \geq 2$ matrix X is zero, for $\varphi = \pm\omega$ the dimension of the attractor space for given N and d is equal to $S_{\neq 1}(N,d)$, see (33).

4.1.3 Non-unit Collision Time

So far we have assumed the angle parameter φ be the same for all pairs of qudits. On the other hand, if φ_{ab} vary among different pairs, results do not change much. For $\lambda = 1$ the

only assumption in the above calculations was $q \neq 0$. It is satisfied if and only if $\varphi \neq 2k\pi$ for any $k \in \mathbb{Z}$. (Otherwise the attractor equation (20) trivially holds.) The same result (25) is obtained whenever $\varphi_{ab} \neq 2k\pi$ for all pairs (a, b) and any $k \in \mathbb{Z}$.

Similarly, for $\lambda \neq \pm 1$ we assumed $q \neq 0 \neq p$. In this case the situation is slightly more complicated due to explicit dependence of attractor solutions on the eigenvalue λ. Nevertheless, even for arbitrary angles φ_{ab} there is only a trivial solution $X = 0$ while $N > d$. One could discuss all the remaining possibilities for diverse values of φ_{ab}. In general, the solution would be also zero though. Nontrivial solution is obtained only for a zero-measure set of parameter values.

Still assuming $\Delta t_{ab} = 1$ for all pairs we have come to conclusion that generically the network equilibrates when $N > d$. As a next step towards the general setup let us assume collision times Δt_{ab} be arbitrary. From the form of the Hamiltonian (19) one immediately sees that a value of the collision time effectively modifies a value of a parameter φ_{ab} in (18) as $\varphi_{ab} \to \varphi_{ab}\,\Delta t_{ab}$. The discussion for different collision times is therefore equivalent to the discussion for different angles φ_{ab} done in the preceding paragraph.

4.1.4 Overall Conclusion for Partial Swap

Generically, when the number N of qudits in the network exceeds their dimensionality d, the network equilibrates. The resulting state is then of the form (25). There is only a zero-measure set of parameters φ_{ab} and collision times Δt_{ab} for which the system evolves in a different way. Specifically

- $\lambda = 1$ — if for all pairs $\varphi_{ab}\,\Delta t_{ab} \neq 2k\pi$, $k \in \mathbb{Z}$, then matrix X is of the form (25), dimension of the attractor space for given N and d is equal to $S_1(N, d)$, see (24), otherwise the form of X is more general,
- $\lambda \neq 1$ — if there is (a, b) such that $\varphi_{ab}\,\Delta t_{ab} = 2k\pi$, $k \in \mathbb{Z}$, then X is a zero matrix; if there is (a, b) such that $\varphi_{ab}\,\Delta t_{ab} = (2k+1)\pi$, $k \in \mathbb{Z}$, and $\lambda \neq -1$, then matrix X is zero, for $\lambda = -1$ we obtain more general form; let $\varphi_{ab}\,\Delta t_{ab} \neq k\pi$, $k \in \mathbb{Z}$, for all pairs (a, b), if there is moreover (a, b) such that $\varphi_{ab}\,\Delta t_{ab} \neq \pm\omega \pmod{2\pi}$, then matrix X is zero; finally if for all (a, b) one has $\varphi_{ab}\,\Delta t_{ab} \neq k\pi$, $k \in \mathbb{Z}$ and $\varphi_{ab}\,\Delta t_{ab} = \omega \pmod{2\pi}$ for all pairs or $\varphi_{ab}\,\Delta t_{ab} = -\omega \pmod{2\pi}$ for all pairs, then the dimension of the attractor space for given N and d is equal to $S_{\neq 1}(N, d)$, see (33), if there are two doubles (a, b), (c, d) for which $\varphi_{ab}\,\Delta t_{ab} = \omega \pmod{2\pi}$ and $\varphi_{cd}\,\Delta t_{cd} = -\omega \pmod{2\pi}$, then X is of more general form.

4.2 Composite Evolution

In this subsection we investigate how the nontrivial free evolution affects the asymptotic behaviour governed by partial swap collisions. We have dealt with a trivial free evolution so far, let the free evolution U^{free} be arbitrary in the present case and let $p_0 \neq 0$. As emphasized in subsection 3.3 the interaction Hamiltonians commuting with the free evolution Hamiltonians make the asymptotics determination much easier. Attractor equations decouple into (16)

$$U^{\text{free}}\, X\, (U^{\text{free}})^\dagger = \lambda X, \quad V_{ab}^\dagger\, X\, V_{ab} = X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b, \tag{34}$$

where $V_{ab} = \exp(\pm \mathrm{i}\,(\Delta t\,\varphi_{ab}/2)\mathrm{SW}_{ab})$. The second set of equations above has been already solved. This setup corresponds to the case with a unit eigenvalue as can be seen in subsubsection 4.1.1 and is summarized in subsubsection 4.1.4. To get rid of special unpleasant cases let the collision times satisfy $\varphi_{ab}\,\Delta t \neq 2k\pi$, $k \in \mathbb{Z}$. Then the solution to the second set of equations in (34) takes on the form (25).

At this point we just plug the attractor (25) into attractor equations for purely free evolution. It is important to note that our calculations so far were done with no specification of the computational basis. The permutation invariance (22) of the attractor is thus present in every local basis. For convenience let us treat the first set of equations (34) in the eigenbasis of $U^{\text{free}} \equiv \bigotimes_i U_i$. We do not impose any special requirements on the one-qudit free evolution operator U_i, let $\{\epsilon_l\}_l$ be energy levels of a corresponding free Hamiltonian. Then $U^{\text{free}}(\Delta t) = \sum_i \exp(\mathrm{i}\,\Delta t \sum_k \epsilon_{i_k}) |i_1 \dots i_N\rangle\langle i_1 \dots i_N|$. From this relation it follows

$$U^{\text{free}} |m_1 \dots m_N\rangle\langle n_1 \dots n_N| \, (U^{\text{free}})^\dagger = \exp\left(\mathrm{i}\,\Delta t \sum_k (\epsilon_{m_k} - \epsilon_{n_k})\right) |m_1 \dots m_N\rangle\langle n_1 \dots n_N|.$$

Since the exponent in this equation is the same for all permutations of multiindices $m_1 \dots m_N$ and $n_1 \dots n_N$ we see that (25)

$$U^{\text{free}} \, P_{\vec{c}} \, (U^{\text{free}})^\dagger = \lambda_{\vec{c}} P_{\vec{c}} \tag{35}$$

for suitable eigenvalue $\lambda_{\vec{c}}$. By comparison of (35) with the first set of equations in (34) we conclude that $P_{\vec{c}}$ solves (34) for $\lambda = \lambda_{\vec{c}}$. Put in another way, the free evolution only redistributes attractors $P_{\vec{c}}$ to different eigenvalues $\lambda_{\vec{c}}$. For each of these eigenvalues the associated eigenvector is $P_{\vec{c}}$ and $\{P_{\vec{c}}\}_{\vec{c}}$ can be again used as the orthonormal basis of the attractor space. The present solution is thus identical to (25), only eigenvalues differ.

So far all collision times and the free evolution time were equal to Δt. If we wanted to generalize this setup for different Δt_{ab} and Δt_0 the attractor equations (14) would read

$$\tilde{U}_{ab}(\Delta t_{ab}) \, X \, \tilde{U}_{ab}^\dagger(\Delta t_{ab}) = \lambda \, \tilde{V}_{ab}^\dagger(\Delta t_{ab}) \, X \, \tilde{V}_{ab}(\Delta t_{ab}), \quad \forall a, b \in \{1, \dots, N\}, \quad a \neq b, \tag{36}$$

together with

$$U^{\text{free}}(\Delta t_0) \, X \, (U^{\text{free}})^\dagger(\Delta t_0) = \lambda \, X. \tag{37}$$

This system of equations cannot be simply decomposed as in (16) and further calculations are needed. Since the number of constraints imposed by this general setting is higher than in the case we presented the resulting attractors are likely to be of more restricted form. Nonetheless, we avoid pursuing such a more complicated calculation. Analogous discussion is performed later in the case of CNOT interaction.

4.2.1 Conclusion for Composite Evolution

Without an explicit form of the free evolution we cannot determine whether the system equilibrates or not. On the other hand, we have calculated all the attractors and we also have a formula which can be used to compute associated eigenvalues. From these ingredients it is an easy matter to decide which asymptotic regime the system undergoes in the given case.

5 CNOT-type interaction

Until now we have dealt with the partial swap-type interaction between network constituents. This operation commutes with the free evolution and enables us to find the asymptotic regime of the network evolution relatively easily. In this part we focus on a different kind of interactions related to the *controlled-NOT operation*. These do not commute with the free evolution, in general. Note that decoherence properties of quantum networks with controlled unitary operations were studied in [8]. Analogously to the partial swap, we examine conditions under which the network equilibrates. To this end we solve attractor equations with a trivial free evolution first. As a next step we consider a specific form of the nontrivial free Hamiltonian.

The controlled-NOT (CNOT) is defined for qubits only. Let CNOT_{ab} denote the controlled-NOT operation applied on the a-th and the b-th qubits in this order. The Hamiltonian of this operation is chosen in such a way that for unit interaction time it generates a mapping acting as

$$\mathrm{CNOT}_{ab} = |0\rangle\langle 0| \otimes \mathbb{I}_b + |1\rangle\langle 1| \otimes \sigma_X, \tag{38}$$

where σ_X is a Pauli matrix (Appendix A). Projectors $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ act on the *control* qubit a while the identity and Pauli matrices act on the *target* qubit b. The definition formula may be rewritten to demonstrate explicit action of CNOT. In the computational basis $\{|0\rangle, |1\rangle\}$ we have

$$\mathrm{CNOT}\,|c\rangle|t\rangle = |c\rangle|(c+t) \bmod 2\rangle, \tag{39}$$

where $|c\rangle$ and $|t\rangle$ are control and target qubits, respectively. Calculations in Appendix B show the Hamiltonian for CNOT may be chosen as

$$H_{\mathrm{CNOT}_{ab}} = \frac{\pi}{2}(2l+1)(\mathrm{CNOT}_{ab} - \mathbb{I}_{ab}), \quad l \in \mathbb{Z}. \tag{40}$$

This is not the only possibility though, other forms of the Hamiltonian exist. The CNOT operation satisfies $\mathrm{CNOT}_{ab} = \exp(\mathrm{i}\, H_{\mathrm{CNOT}_{ab}})$ for arbitrary, but distinct, qubits a and b. In the following we will examine not only the "perfect" CNOT operation, but also its $\Delta t_{ab} \neq 1$ versions. These we will denote as $\mathrm{CNOT}_{ab}(\Delta t_{ab}) \equiv \exp(\mathrm{i}\,\Delta t_{ab} H_{\mathrm{CNOT}_{ab}})$. Discussion in Appendix B tells us that

$$\mathrm{CNOT}_{ab}(\Delta t_{ab}) = \frac{1}{2}\left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t_{ab}} + 1\right)\mathbb{I}_{ab} - \frac{1}{2}\left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t_{ab}} - 1\right)\mathrm{CNOT}_{ab}. \tag{41}$$

Unlike the partial swap the controlled-NOT in general does not exhibit nice commutation relations with the free evolution. Therefore we have to confine ourselves to the evolution operator decomposition as demonstrated in (11).

5.1 Pure Collisions

To begin with we consider the system evolves under CNOT-type collisions with only the trivial free evolution of each qudit. Similarly to the partial swap we take $p_0 = 0$ (8). Firstly, we assume all the interaction times are identical and equal to one. Secondly, we relax this assumption and investigate how the asymptotic regime differs from the unit collision time scenario.

5.1.1 Unit Collision Time

Results presented in this part have already been derived in [8]. The present discussion is made in order to generalize them consistently in the upcoming sections. Assuming the

unit collision time for each pair of qubits the hermiticity of the CNOT operation allows us to write the attractor equations (6) in a simplified form

$$\mathrm{CNOT}_{ab}\; X\; \mathrm{CNOT}_{ab} = \lambda X, \quad a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{42}$$

When written in the elementwise fashion this equation transforms into the system of equalities with $i, j \in \{0, 1\}$

$$\begin{array}{lll} X^{0,i}_{0,j} = \lambda X^{0,i}_{0,j}, & & \\ X^{0,i}_{1,0} = \lambda X^{0,i}_{1,1}, & X^{0,i}_{1,1} = \lambda X^{0,i}_{1,0}, & \\ X^{1,0}_{0,j} = \lambda X^{1,1}_{0,j}, & X^{1,1}_{0,j} = \lambda X^{1,0}_{0,j}, & \\ X^{1,0}_{1,0} = \lambda X^{1,1}_{1,1}, & X^{1,1}_{1,1} = \lambda X^{1,0}_{1,0}, & \\ X^{1,0}_{1,1} = \lambda X^{1,1}_{1,0}, & X^{1,1}_{1,0} = \lambda X^{1,0}_{1,1}, & \end{array} \tag{43}$$

where the local indices refer to an ordered pair of qubits (a, b). The rest of indices has to be the same on both sides of equations.

As the CNOT operation is hermitian, the attractor spectrum contains real eigenvalues only, *i.e.* $\sigma_{|1|} \subset \{1, -1\}$. Therefore, in the remainder of this section we focus on solution to equations (42) for parameters $\lambda = \pm 1$. At the very end we present conclusion discussing different forms of attactors based on the eigenvalue λ and number of qubits.

Attractors for $\lambda = 1$ Initially, we find attractors associated with the unit eigenvalue $\lambda = 1$. Combining the system of equations (43) for two ordered pairs of qubits (a, b) and (b, a) one obtains five sets of relations as follows

$$X^{0,0}_{0,0}, \tag{44}$$

$$X^{0,0}_{1,0} = X^{0,0}_{1,1} = X^{0,0}_{0,1}, \tag{45}$$

$$X^{1,0}_{0,0} = X^{1,1}_{0,0} = X^{0,1}_{0,0}, \tag{46}$$

$$X^{1,0}_{1,0} = X^{1,1}_{1,1} = X^{0,1}_{0,1}, \tag{47}$$

$$X^{1,0}_{0,1} = X^{1,1}_{0,1} = X^{0,1}_{1,1} = X^{0,1}_{1,0} = X^{1,1}_{1,0} = X^{1,0}_{1,1}. \tag{48}$$

An expression (44) means there is no condition imposed on matrix element $X^{0,0}_{0,0}$. One can easily verify that relations above cover all possible combinations of row and column indices. These equalities must hold for every choice of qubit pairs and we divide all matrix elements of a possible attractor into five *disjoint* sets.

The first set contains only one element whose indices are all zeros. This element is not subjected to any constraint and its value might be arbitrary. Consider then the set of all matrix elements with zero row multiindices and at least one nonzero column index. These elements lie in the first row of the matrix X. Obviously, only equations (45) apply to such elements. According to these equations all the members of the set have to equal each other. Similar discussion can be done also for a set of all matrix elements whose column multiindex is made of zeros and they have at least one nonzero index in their row multiindex. Such elements represent the first column of matrix X. Equations (46) force these elements to be equal. Moreover, no other equations apply to this set.

The fourth set is comprised of elements lying in the diagonal of the matrix X. Analogously, equations (47) imply that the diagonal terms are identical. Finally, the fifth set of matrix elements collects all the remaining entries. That is, elements *not* lying in the first row or the first column or the diagonal of the matrix. These are subjected to equations (48), which set values of these elements equal.

We have divided all the matrix X elements into five disjoint sets. From discussion in the previous paragraphs it is straightforward to depict solution for equations (42) with $\lambda = 1$

$$X = \begin{pmatrix} a & b & b & b & \dots & b & b \\ c & d & e & e & \dots & e & e \\ c & e & d & e & \dots & e & e \\ c & e & e & d & \dots & e & e \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c & e & e & e & \dots & d & e \\ c & e & e & e & \dots & e & d \end{pmatrix}, \tag{49}$$

where $a, b, c, d, e \in \mathbb{C}$. There is obviously five linearly independent attractors.

Attractors for $\lambda = -1$ From the first equality in (43) one can immediately deduce that the matrix element of attractor X associated with $\lambda = -1$ is zero whenever it has at least one local index of zeros in its multiindex. The remnant matrix elements have to satisfy the following constraints. Similarly to the previous case reprocessing of equations (43) and their twins, where the role of control and target qubits is switched, leads to these relations

$$-X^{0,0}_{1,0} = X^{0,0}_{1,1} = -X^{0,0}_{0,1}, \tag{50}$$
$$-X^{1,0}_{0,0} = X^{1,1}_{0,0} = -X^{0,1}_{0,0}, \tag{51}$$
$$-X^{1,0}_{1,0} = X^{1,1}_{1,1} = -X^{0,1}_{0,1}, \tag{52}$$
$$-X^{1,0}_{0,1} = X^{1,1}_{0,1} = -X^{0,1}_{1,1} = X^{0,1}_{1,0} = -X^{1,1}_{1,0} = X^{1,0}_{1,1}. \tag{53}$$

Recall discussion we pursued for unit eigenvalue $\lambda = 1$. Equations (50), (51) and (52) above assure that the first row of matrix X together with its first column and its diagonal consists of zero entries. It is easily seen when we notice that indices appearing in all these equations have a local index made of zeros. Let us investigate the last system of equations (53). One can rewrite them to two sets as follows

$$X^{1,1}_{0,1} = X^{0,1}_{1,0} = X^{1,0}_{1,1} \equiv a \tag{54}$$
$$X^{1,0}_{0,1} = X^{0,1}_{1,1} = X^{1,1}_{1,0} \equiv -a \tag{55}$$

for some $a \in \mathbb{C}$. First, let the number of qubits be $N \geq 3$ and consider the elements in the form

$$X^{1\ 1\ i}_{0\ 1\ j}, \tag{56}$$

where i and j are indices with an arbitrary value from $\{0, 1\}$. We demonstrate that such elements must be zero and thus a in equations (54) and (55) is zero as well. Hence, the only matrix X satisfying relations in the table above for $N \geq 3$ is a zero matrix only.

To this end, consider all four possible forms the element in the previous formula can take

$$X^{1\ \underline{1}\ \underline{0}}_{0\ \underline{1}\ \underline{0}}, \quad X^{1\ 1\ 0}_{0\ 1\ 1}, \quad X^{\underline{1}\ 1\ \underline{1}}_{\underline{0}\ 1\ \underline{0}}, \quad X^{1\ \underline{1}\ \underline{1}}_{0\ \underline{1}\ \underline{1}}. \tag{57}$$

If we apply equations (51) and (52) to the elements with underlined indices we instantaneously conclude the three asociated elements are zero. For the second element above see (54). This element is equal to

$$X^{1\ 0\ 0}_{1\ 1\ 1}, \tag{58}$$

which is also zero due to (50). We have proven for $N \geq 3$ the corresponding attractor X is a zero matrix. Let us treat the case for $N = 2$. Equations (54) and (55) entirely determine

nonzero matrix elements and the two-qubit attractor reads

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & -a \\ 0 & -a & 0 & a \\ 0 & a & -a & 0 \end{pmatrix} \tag{59}$$

with $a \in \mathbb{C}$. Our investigation reveals the attractor subspace for $\lambda = -1$ is nontrivial only for two qubit systems in which case it is one-dimensional.

5.1.2 Conclusion with Unit Collision Time

We summarize our discussion on possible forms of attractors regarding different values of eigenvalue $\lambda \in \sigma_{|1|} = \{1, -1\}$ and number of qubits $N \geq 2$ with $\Delta t_{ab} = 1$. Up to very special case the asymptotic dynamics is stationary since there is only unit eigenvalue in the attractor spectrum. The detailed discussion reads as follows

- $\lambda = 1$ regardless of N matrix X is of the form (49) and the attractor space is thus five-dimensional,
- $\lambda = -1$ for $N = 2$ matrix X is of the form (59), the attractor space is thus one-dimensional; for $N \geq 3$ there is no nontrivial attractor.

5.1.3 Non-unit Collision Time

In this part we study how various values of collision times Δt_{ab} affect the structure of attractor space pertaining to the quantum network with random CNOT interactions. So far we have taken collision time to be unity, $\Delta t_{ab} = 1$. From now on let $\Delta t_{ab} \neq 1$, but same for all pairs a and b, $\Delta t_{ab} = \Delta t$. Later on we relax even this assumption and take Δt_{ab} different for different qubits. Attractor equations (42) with $\mathrm{CNOT}_{ab}(\Delta t)$ (41) turn into the more complex form

$$(\alpha + \gamma)\, X = \alpha\, \mathrm{CNOT}_{ab}\; X\; \mathrm{CNOT}_{ab} + \mathrm{i}\, s\, (X\; \mathrm{CNOT}_{ab} - \mathrm{CNOT}_{ab}\; X), \tag{60}$$

where $a \neq b$ run through all qubits in the network, $\lambda \in \sigma_{|1|}$, CNOT_{ab} is the ordinary controlled-NOT operation and we defined

$$\begin{aligned} \alpha &= 1 - \cos(\pi \Delta t), \\ s &= \sin(\pi \Delta t), \\ \gamma &= 2(\lambda - 1). \end{aligned}$$

Above we put $l = 0$ in the definition of the Hamiltonian, see (40) and (41). In the following we split our analysis into cases $\lambda = 1$, $\lambda = -1$ and $\lambda \neq \pm 1$. Let us start our examination with the eigenvalue $\lambda = 1$.

Attractors for $\lambda = 1$ For $\lambda = 1$ we have $\gamma = 0$. Let $\alpha = 0$ first, implying $s = 0$. In this setting the CNOT operation reduces to the identity and no collision emerges, see (40). Equations (60) are then satisfied identically and matrix X may be arbitrary. Henceforth let $\alpha \neq 0$. It is convenient to write matrix X as $X = X_R + \mathrm{i}\, X_I$, where X_R and X_I are real matrices. Relations (60) then reduce to the system

$$X_R = \mathrm{CNOT}_{ab}\; X_R\; \mathrm{CNOT}_{ab} + \frac{s}{\alpha}(\mathrm{CNOT}_{ab}\; X_I - X_I\; \mathrm{CNOT}_{ab}), \tag{61}$$

$$X_I = \mathrm{CNOT}_{ab}\; X_I\; \mathrm{CNOT}_{ab} - \frac{s}{\alpha}(\mathrm{CNOT}_{ab}\; X_R - X_R\; \mathrm{CNOT}_{ab}). \tag{62}$$

One can multiply the first equation by CNOT_{ab} from the left and substitute it into the second equation to obtain

$$(X_I - \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab})\left(1 + \frac{s^2}{\alpha^2}\right) = 0 \tag{63}$$

which is satisfied iff $X_I = \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab}$. Substituting this relation back into (61) one sees the same equation holds also for X_R and we can conclude that equations (60) reduce to

$$\mathrm{CNOT}_{ab}\ X\ \mathrm{CNOT}_{ab} = X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{64}$$

We have already dealt with this equation, its solutions is given by (49).

Till now all the collision times were assumed to have the same value. Let us move one step forward and allow the collision times Δt_{ab} to be generically different for each pair (a, b). This general situation is easily solved by noticing that in the above discussion parameters α and s played no role provided $\alpha \neq 0$. That is, the same solution would follow even if the interaction times Δt_{ab} were different for different pairs of qubits and $\alpha_{ab} \neq 0$. If there were (a, b) such that $\alpha_{ab} = 0$ then there is less constraints imposed on solution X and the attractor may have more general form compared to (49).

Attractors for $\lambda = -1$ For $\lambda = -1$ we have $\gamma = -4$. Let $s = 0$ first, implying $\alpha \in \{0, 2\}$. Then equations (60) reduce to $(\alpha - 4)\, X = \alpha\, \mathrm{CNOT}_{ab}\, X\, \mathrm{CNOT}_{ab}$. For $\alpha = 0$ there is only trivial solution $X = 0$. For $\alpha = 2$ (*i.e.* $\Delta t = 2k + 1, k \in \mathbb{Z}$) one obtains the same equation as for $\Delta t_{ab} = 1$

$$\mathrm{CNOT}_{ab}\ X\ \mathrm{CNOT}_{ab} = -X, \quad \forall a, b \in \{1, \ldots, N\}, \quad a \neq b. \tag{65}$$

There is thus a nontrivial attractor (59) only for $N = 2$. Having investigated the case for zero s, let us take $s \neq 0$. One can proceed in the similar way as we did for the unit eigenvalue to obtain

$$\begin{aligned} X_R &= \beta_1\, \mathrm{CNOT}_{ab}\ X_R\ \mathrm{CNOT}_{ab} + \beta_2\, (\mathrm{CNOT}_{ab}\ X_I - X_I\ \mathrm{CNOT}_{ab}), \\ X_I &= \beta_1\, \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab} - \beta_2\, (\mathrm{CNOT}_{ab}\ X_R - X_R\ \mathrm{CNOT}_{ab}), \end{aligned}$$

with $\beta_1 = \frac{\alpha}{\alpha - 4}$ and $\beta_2 = \frac{s}{\alpha - 4}$. Multiplication of the first equation by CNOT_{ab} from the left yields

$$\begin{aligned} &\mathrm{CNOT}_{ab}\ X_R - \beta_1\, X_R\ \mathrm{CNOT}_{ab} = \\ &\qquad - \beta_2(1 - \beta_1)\, \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab} + \beta_2\, (X_I - \beta_1\, \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab}). \end{aligned}$$

We plug the second equation from above into this expression to end up with

$$\begin{aligned} (1 + \beta_2^2)\, \mathrm{CNOT}_{ab}\ X_R - (\beta_1 + \beta_2^2)\, X_R\ \mathrm{CNOT}_{ab} = \\ - \beta_2(1 - \beta_1) \mathrm{CNOT}_{ab}\ X_I\ \mathrm{CNOT}_{ab}. \end{aligned} \tag{66}$$

Along similar lines we obtain also

$$\begin{aligned} (1 + \beta_2^2)\, \mathrm{CNOT}_{ab}\ X_I - (\beta_1 + \beta_2^2)\, X_I\ \mathrm{CNOT}_{ab} = \\ \beta_2(1 - \beta_1) \mathrm{CNOT}_{ab}\ X_R\ \mathrm{CNOT}_{ab}. \end{aligned} \tag{67}$$

Multiplication by CNOT_{ab} from both sides of equation (66) yields the expression for X_I. We can substitute this into (67) and simplify the resulting relation to get

$$(\beta_1^2 + 2\beta_2^2 + 1)\, \mathrm{CNOT}_{ab}\ X_R\ \mathrm{CNOT}_{ab} = 2(\beta_1 + \beta_2^2)\, X_R. \tag{68}$$

Both sides of this equality have to have the same norm (1) and (the Hilbert-Schmidt) norm is preserved under unitary operations. From this we therefore extract the necessary condition in the form

$$(\beta_1^2 + 2\beta_2^2 + 1 - 2|\beta_1 + \beta_2^2|)\|X_R\| = 0. \tag{69}$$

Assuming $X_R \neq 0$ we are left with the constraint imposed on the β's. It is satisfied only for $s = 0$ which is excluded by assumption. That is, only $X_R = 0$ solves (68). When we substitute this solution back into original equations these turn into

$$X_I = \beta_1 \,\mathrm{CNOT}_{ab}\; X_I\; \mathrm{CNOT}_{ab} = \mathrm{CNOT}_{ab}\; X_I\; \mathrm{CNOT}_{ab}. \tag{70}$$

Since $\beta_1 \neq 1$ we can employ the same norm argument to conclude that for $s \neq 0$ inevitably $X = 0$.

It is time to relax the initial constraint and let collision times Δt_{ab} be different for various pairs of qubits. So far we have made use of a single ordered pair of qubits (a, b) to reduce the possible form of the attractor considerably. No other pairs were necessary to calculate $X = 0$ whenever $s_{ab} \neq 0$. Hence in a general setting, if there is a double of qubits such that $s_{ab} \neq 0$, then $X = 0$. Similarly, if for all pairs $s_{ab} = 0$, but there is a double such that $\alpha_{ab} \neq 2$, then the attractor is also trivial, $X = 0$. Otherwise we obtain zero solution for $N \geq 3$ and nonzero solution (59) for $N = 2$.

Attractors for $\lambda \neq \pm 1$ Let us investigate the last case. For $\lambda \neq \pm 1$ we make use of the elementwise representation of equations (60). It reads

$$(\alpha + \gamma)\, X^{i_a,i_b}_{j_a,j_b} = \alpha\, X^{i_a,i}_{j_a,j} + \mathrm{i}\, s \left(X^{i_a,i}_{j_a,j_b} - X^{i_a,i_b}_{j_a,j} \right), \tag{71}$$

where we assumed that qubit a is the control one and b is the target qubit. We denoted $i = (i_a + i_b) \pmod 2$ and $j = (j_a + j_b) \pmod 2$, see (39). When one interchanges roles of a and b as control and target qubits then local indices switch their positions accordingly.

It is easy to see whenever there is a local index of zeros, *i.e.* $(i_a, j_a) = (0, 0)$ or $(i_b, j_b) = (0, 0)$, relation (71) reduces to $\gamma X^{i_a,i_b}_{j_a,j_b} = 0$. Since $\gamma \neq 0$ in the present discussion we can conclude

$$X^{0,i_b}_{0,j_b} = X^{i_a,0}_{j_a,0} = 0, \quad \forall a, b \in \{1, \ldots, N\}. \tag{72}$$

As a next step consider a local index of the form $(i_a, j_a) = (1, 1)$. Corresponding matrix elements are then subjected to constraints

$$\begin{aligned}
(\alpha + \gamma)\, X^{1,0}_{1,0} &= \alpha\, X^{1,1}_{1,1} + \mathrm{i}\, s\, (X^{1,1}_{1,0} - X^{1,0}_{1,1}),\\
(\alpha + \gamma)\, X^{1,0}_{1,1} &= \alpha\, X^{1,1}_{1,0} + \mathrm{i}\, s\, (X^{1,1}_{1,1} - X^{1,0}_{1,0}),\\
(\alpha + \gamma)\, X^{1,1}_{1,0} &= \alpha\, X^{1,0}_{1,1} + \mathrm{i}\, s\, (X^{1,0}_{1,0} - X^{1,1}_{1,1}),\\
(\alpha + \gamma)\, X^{1,1}_{1,1} &= \alpha\, X^{1,0}_{1,0} + \mathrm{i}\, s\, (X^{1,0}_{1,1} - X^{1,1}_{1,0}).
\end{aligned}$$

These can be neatly rewritten into the matrix equation

$$\begin{pmatrix}
-(\alpha+\gamma) & -\mathrm{i}\, s & \mathrm{i}\, s & \alpha \\
-\mathrm{i}\, s & -(\alpha+\gamma) & \alpha & \mathrm{i}\, s \\
\mathrm{i}\, s & \alpha & -(\alpha+\gamma) & -\mathrm{i}\, s \\
\alpha & \mathrm{i}\, s & -\mathrm{i}\, s & -(\alpha+\gamma)
\end{pmatrix}
\begin{pmatrix}
X^{1,0}_{1,0} \\ X^{1,0}_{1,1} \\ X^{1,1}_{1,0} \\ X^{1,1}_{1,1}
\end{pmatrix} = 0. \tag{73}$$

Determinant of the matrix in the equation above is zero iff $\lambda = 1 - \alpha = \pm 1$ which is in contradiction to our assumption. We can perform analogous calculation for a and b interchanged to conclude

$$X^{1,i_b}_{1,j_b} = X^{i_a,1}_{j_a,1} = 0, \quad \forall a, b \in \{1, \ldots, N\}. \tag{74}$$

Let us focus on two last sets of possible local indices of the form either $(i_a, j_a) = (0,1)$ or $(i_a, j_a) = (1,0)$. For the former case equations (71) reduce to

$$(\alpha - \mathrm{i}\, s + \gamma)\, X^{0,i}_{1,1} = (\alpha - \mathrm{i}\, s)\, X^{0,i}_{1,0},$$
$$(\alpha - \mathrm{i}\, s + \gamma)\, X^{0,i}_{1,0} = (\alpha - \mathrm{i}\, s)\, X^{0,i}_{1,1},$$

with $i \in \{0,1\}$. Subtraction of these two equations and the fact that $\gamma \neq 0$ implies equality $X^{0,i}_{1,0} = -X^{0,i}_{1,1}$ for $i \in \{0,1\}$. Taking $i = 0$ the left-hand side element is of the form (72) and both elements in the equation are therefore zero. Similarly, for $i = 1$ the right-hand side element is of the form (74) and both elements are also zero. We could perform analogous investigation also for $(i_b, j_b) = (0,1)$, $(i_a, j_a) = (1,0)$ and $(i_b, j_b) = (1,0)$ with the same result.

Our present investigation has been completely independent of the specific values of the collision time. That is, for $\lambda \neq \pm 1$ one has $X = 0$ regardless of the actual value Δt. Therefore, this result is still valid even in the more general case with generically different values of collision time for each pair Δt_{ab}.

5.1.4 Overall Conclusion for CNOT

Above we have discussed all possible values the collision times of distinct pairs of qubits might assume. By now we have all the information necessary to draw a conclusion on the attractor X form when different eigenvalues, interaction times and number of qubits $N \geq 2$ are considered.

Roughly speaking, the more general case with collision times Δt_{ab} mutually different reproduces results obtained for a special case with $\Delta t_{ab} = 1$. The asymptotic dynamics is stationary, no matter how many qubits constitute the network there are five integrals of motion. Such a behaviour emerges for almost all values of paramaters. Exceptional values form a zero-measure set which is described in detail below

- $\lambda = 1$ — if for all $a, b \in \{1, \ldots, N\}$ it holds that $\Delta t_{ab} \neq 2k$, $k \in \mathbb{Z}$, then the attractor X is of the form (49) regardless of number of qubits N, otherwise matrix X takes a more general form than (49),
- $\lambda = -1$ — for $N \geq 3$ matrix X is zero, for $N = 2$ the attractor matrix is nonzero only when for all doubles it holds that $\Delta t_{ab} = 2k + 1$ for some $k \in \mathbb{Z}$ in which case matrix X is of the form (59),
- $\lambda \neq \pm 1$ — regardless of N and interaction times Δt_{ab} matrix X is zero.

5.2 Composite Evolution

In this subsection we focus on the CNOT operation (38) acting on the multi-qubit system simultaneously with a nontrivial free evolution. The Hamiltonian generating a free evolution is chosen as $H_i^{\mathrm{free}} = s\, \sigma_Z$ (Appendix A).

In the present setup we already consider Δt_{ab} being different and assume $p_0 = 0$ (8). In other words, in every time interval Δt_{ab} there is always some collision taking place. Moreover, we adopt a natural assumption $\Delta t_{ab} = \Delta t_{ba}$ for each pair of qubits to simplify our calculations in a convenient way. One can deal with attractor equations as presented in (14). Due to the form of the Hamiltonian the corresponding evolution operator is diagonal, $U_N = U_N^{\mathrm{diag}}$. It is not hard to see the left-hand side of (14) in the elementwise form reads

$$X^{i_1 \ldots i_N}_{j_1 \ldots j_N} \exp\left(2\mathrm{i}\ \Delta t_{ab}\, s \left(\sum_{k \neq a,b} (j_k - i_k)\right)\right), \tag{75}$$

where Δt_{ab} is the collision time associated with CNOT_{ab}. Operator $\tilde{V}_{ab} = \exp(\mathrm{i}\,\Delta t_{ab}\tilde{H}_{ab})$ (9) acts nontrivialy only on the qubits a and b. Let $\tilde{V}_{ab}|_{(a,b)}$ denote a restriction of this operator to the subsystem of a-th and b-th qubit. Similarly we resctrict also matrix X as follows. Suppose a (row) multiindex $(i_1,\ldots,i_a,\ldots,i_b,\ldots,i_N)$ where all indices i_c pertaining to qubits $c \neq a,b$ are fixed and indices i_a, i_b are left arbitrary. There are four such multiindices for each setting of i_c's. Analogous discussion can be done for column indices. For each setting of i_c's and j_c's we can define a 4×4 matrix $X_{(a,b)}$ composed of elements $X^{i_a,i_b}_{j_a,j_b}$. For every matrix X and every double (a,b) there are thus 2^{N-2} matrices $X_{(a,b)}$. Since the indices i_c and j_c are not directly present in the future calculations, we compute effectively with a single matrix $X_{(a,b)}$. With the new notation in hand equations (14) are reexpressed as

$$X_{(a,b)}\, e^{\mathrm{i}\,\alpha_{ab}} = (\tilde{V}_{ab}|_{(a,b)})^\dagger\, X_{(a,b)}\, (\tilde{V}_{ab}|_{(a,b)}), \quad \forall a,b\in\{1,\ldots,N\}, \quad a\neq b, \tag{76}$$

where $\lambda = e^{\mathrm{i}\omega}$ and we introduced parameter $\alpha_{ab} = 2\Delta t_{ab}\, s\, \sum_{k\neq a,b}(j_k - i_k) - \omega$. The problem is therefore reduced to the solution of matrix equations with effectively 4×4 matrices. As a next step we divide expression (76) into two parts. One part consists of equations determining how the solution $X_{(a,b)}$ depends on a value of α_{ab}. The other part captures the influence of a network structure onto $X_{(a,b)}$.

The first part, *eigenvalue equations*, is obtained from attractor equations (76) when one diagonalizes operators $\tilde{V}_{ab}|_{(a,b)} = S_{ab} D_{ab} S_{ab}^{-1}$ and defines $\tilde{X} \equiv S_{ab}^{-1}\, X_{(a,b)}\, S_{ab}$. One ends up with

$$\tilde{X}\, e^{\mathrm{i}\,\alpha_{ab}} = D_{ab}^\dagger\, \tilde{X}\, D_{ab}, \quad \forall a,b\in\{1,\ldots,N\}, \quad a\neq b. \tag{77}$$

Matrix $\tilde{X}$ is dependent on the order in which eigenvalues are put into D_{ab}. For our choice of D_{ab} eigenvalue equations (77) are equivalent to the following relations when we substitute $\beta = 2s/\pi$ and $b = \sqrt{1+\beta^2}$ and i runs through $\{1,\ldots,4\}$

$$\begin{aligned}
(e^{\mathrm{i}\,\alpha_{ab}} - 1)\tilde{X}_{ii} &= 0, & & \\
(e^{\mathrm{i}\,\alpha_{ab}} - e^{\mathrm{i}\,\pi\beta\Delta t_{ab}})\tilde{X}_{12} &= 0, & (e^{\mathrm{i}\,\alpha_{ab}} - e^{-\mathrm{i}\,\pi\beta\Delta t_{ab}})\tilde{X}_{21} &= 0,\\
(e^{\mathrm{i}\,\alpha_{ab}} - e^{-\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+\beta+b)})\tilde{X}_{13} &= 0, & (e^{\mathrm{i}\,\alpha_{ab}} - e^{\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+\beta+b)})\tilde{X}_{31} &= 0,\\
(e^{\mathrm{i}\,\alpha_{ab}} - e^{-\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+\beta-b)})\tilde{X}_{14} &= 0, & (e^{\mathrm{i}\,\alpha_{ab}} - e^{\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+\beta-b)})\tilde{X}_{41} &= 0,\\
(e^{\mathrm{i}\,\alpha_{ab}} - e^{-\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+3\beta+b)})\tilde{X}_{23} &= 0, & (e^{\mathrm{i}\,\alpha_{ab}} - e^{\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+3\beta+b)})\tilde{X}_{32} &= 0,\\
(e^{\mathrm{i}\,\alpha_{ab}} - e^{-\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+3\beta-b)})\tilde{X}_{24} &= 0, & (e^{\mathrm{i}\,\alpha_{ab}} - e^{\mathrm{i}\,\frac{\pi\Delta t_{ab}}{2}(1+3\beta-b)})\tilde{X}_{42} &= 0,\\
(e^{\mathrm{i}\,\alpha_{ab}} - e^{\mathrm{i}\,\pi\Delta t_{ab}b})\tilde{X}_{34} &= 0, & (e^{\mathrm{i}\,\alpha_{ab}} - e^{-\mathrm{i}\,\pi\Delta t_{ab}b})\tilde{X}_{43} &= 0.
\end{aligned} \tag{78}$$

The other crucial element in the model is the quantum network structure restricting the attractor matrix form immensely. Collision times Δt_{ab} and a free Hamiltonian strength s appear as parameters in our discussion. We intentionally exclude a zero-measure set of values these parameters can take to simplify the network-structure dealing as much as possible. Specifically, instead of taking into account the influence of the whole network onto our ordered pair (a,b) it will be sufficient to consider merely its counterpart (b,a).

Equations (77) have to be satisfied not only for the double (a,b), but also for (b,a). For the latter case one obtains

$$\tilde{Y}\, e^{\mathrm{i}\,\alpha_{ba}} = D_{ab}^\dagger\, \tilde{Y}\, D_{ab}, \quad \forall a,b\in\{1,\ldots,N\}, \quad a\neq b, \tag{79}$$

where now $\tilde{Y} \equiv S_{ab}^{-1}\, P_{ab}\, X_{(a,b)}\, P_{ab}\, S_{ab}$ with $P_{ab} = \mathrm{SW}_{ab}$ being a permutation matrix swapping the a-th and b-th qubit. As we see, both equations (77) and (79) are the same thus having the same set of solutions M. What differs in both is how solutions from this set

M are related to the original matrix $X_{(a,b)}$. From definitions of $\tilde{X}$ and $\tilde{Y}$ it can be seen that $X_{(a,b)}$ must lie in the intersection of two sets as follows

$$X_{(a,b)} \in (S_{ab}\, M\, S_{ab}^{-1}) \cap (P_{ab} S_{ab}\, M\, S_{ab}^{-1} P_{ab}). \tag{80}$$

In other words, for each $X_{(a,b)}$ there must exist $\tilde{X}$ and $\tilde{Y}$ solutions from M such that $X_{(a,b)} = S_{ab}\,\tilde{X} S_{ab}^{-1}$ and $X_{(a,b)} = P_{ab} S_{ab}\,\tilde{Y} S_{ab}^{-1} P_{ab}$. These two expressions relate the form of $\tilde{X}$ and $\tilde{Y}$ matrices. Moreover, this relation is *independent* of the actual forms of solutions to (77) and (79), it reflects the effect of taking into account both (a,b) and (b,a) ordered pairs. We will refer to this relation as *structure equations* in the following. Their explicit form reads

$$\tilde{X} = (S_{ab}^{-1} P_{ab} S_{ab})\,\tilde{Y}\,(S_{ab}^{-1} P_{ab} S_{ab}). \tag{81}$$

Using a vector representation of matrices this condition can be rewritten into $B|\tilde{Y}\rangle = |\tilde{X}\rangle$, where $B = (S_{ab}^{-1} P_{ab} S_{ab}) \otimes (S_{ab}^{-1} P_{ab} S_{ab})^T$, for details see (98) in Appendix C. Since $B^2 = \mathbb{I}$ we have also $|\tilde{Y}\rangle = B|\tilde{X}\rangle$ and roles of $\tilde{X}$ and $\tilde{Y}$ are thus *symmetric*. By suitable permutation of basis vectors within the computational basis the B matrix can be reexpressed in the block-diagonal form

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & B_3 \end{pmatrix}, \tag{82}$$

where the one in the above left corner corresponds to $\tilde{X}_{22}$, B_1 is a 3×3 block associated with elements $\tilde{X}_{24}$, $\tilde{X}_{23}$, $\tilde{X}_{21}$, B_2 is another 3×3 block associated with $\tilde{X}_{42}$, $\tilde{X}_{32}$, $\tilde{X}_{12}$ and B_3 is a 9×9 block corresponding to elements $\tilde{X}_{41}$, $\tilde{X}_{31}$, $\tilde{X}_{11}$, $\tilde{X}_{14}$, $\tilde{X}_{13}$, $\tilde{X}_{43}$, $\tilde{X}_{34}$, $\tilde{X}_{44}$, $\tilde{X}_{33}$ in the respective order.

At this moment, we consider both eigenvalue equations (78) and structure equations represented by matrix B to solve for $X_{(a,b)}$ in a straightforward way. Recalling equations (78) we make distinction between two cases, either $\alpha_{ab} \equiv 0 \pmod{2\pi}$ or $\alpha_{ab} \not\equiv 0 \pmod{2\pi}$. For $\alpha_{ab} \equiv 0 \pmod{2\pi}$ we apparently get $\tilde{X}_{ii}$ being arbitrary. Moreover, if we assume

$$\begin{aligned}
s &\neq \frac{\pi k}{\Delta t_{ab}}, \\
s &\neq \pm\frac{\pi}{4\Delta t_{ab}}\sqrt{4k^2 - \Delta t_{ab}^2}, \\
s &\neq \frac{\pi}{6\Delta t_{ab}}(4k + \Delta t_{ab} \pm \sqrt{64k^2 + 32k\Delta t_{ab} + \Delta t_{ab}^2}), \\
s &\neq \frac{\pi}{10\Delta t_{ab}}(-12k - 3\Delta t_{ab} \pm \sqrt{64k^2 + 32k\Delta t_{ab} + 9\Delta t_{ab}^2}),
\end{aligned}$$

where $k \in \mathbb{Z}$, then all the other elements $\tilde{X}_{ij}$ with $i \neq j$ must be inevitably zero from (78). Thanks to the symmetry also for elements $\tilde{Y}$ we have $\tilde{Y}_{ii}$ being arbitrary and $\tilde{Y}_{ij} = 0$ for $i \neq j$. Under such conditions the matrix B imposes constraints enforcing matrix $X_{(a,b)}$ to be of the form

$$X_{(a,b)} = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \tag{83}$$

with $\gamma, \delta \in \mathbb{C}$. Let us move on to the case $\alpha_{ab} \not\equiv 0 \pmod{2\pi}$. Due to symmetry of $\tilde{X}$ and $\tilde{Y}$ it can be shown that while one puts two variables pertaining to block B_1 equal to zero, the third variable is already zero as well. The same can be shown even for block B_2. From

equations (78) one can read out which values of α_{ab} imply vanishing of the particular matrix element. For instance let us consider α_{ab} such that $\alpha_{ab} \equiv \pi\beta\Delta t_{ab} \pmod{2\pi}$. Then $\tilde{X}_{12}$ might be arbitrary. This variable is associated with block B_2 whose other variables are $\tilde{X}_{42}$ and $\tilde{X}_{32}$. If we now allow only such collision times that

$$s \neq \frac{\pi}{6\Delta t_{ab}}(4k + \Delta t_{ab} \pm \sqrt{64k^2 + 32k\Delta t_{ab} + \Delta t_{ab}^2}), \quad k \in \mathbb{Z}, \tag{84}$$

then our choice $\alpha_{ab} \equiv \pi\beta\Delta t_{ab} \pmod{2\pi}$ assures $\alpha_{ab} \not\equiv \frac{\pi}{2}\Delta t_{ab}(1 + 3\beta \pm b) \pmod{2\pi}$. From (78) it thence follows that $\tilde{X}_{42} = 0 = \tilde{X}_{32}$. We can employ the property of the block B_2 mentioned at the beginning of this paragraph to conclude $\tilde{X}_{42} = \tilde{X}_{32} = \tilde{X}_{12} = 0$ provided that $\alpha_{ab} \equiv \pi\beta\Delta t_{ab} \pmod{2\pi}$ and (84). An analogous discussion can be made for all variables pertaining to blocks B_1 and B_2.

Let us treat the last block B_3. For $\alpha_{ab} \not\equiv 0 \pmod{2\pi}$ we have $\tilde{X}_{ii} = 0 = \tilde{Y}_{ii}$. By plugging these values into equation $B|\tilde{Y}\rangle = |\tilde{X}\rangle$, namely its part corresponding to B_3, we obtain nontrivial constraints on values of some variables $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$, $i \neq j$. These conditions effectively reduce block B_3 into a 3×3 matrix which can be treated in the way we did for blocks B_1 and B_2. Again, when two variables are set to zero the third variable must be also zero. A parameter value discussion similar to that above can be done. As a consequence in the set of collision times we leave only those satisfying each of conditions

$$\begin{array}{llll} s \neq \frac{\pi k}{\Delta t_{ab}}, & & s \neq \pm\frac{\pi}{4\Delta t_{ab}}\sqrt{4k^2 - \Delta t_{ab}^2}, \\ s \neq \frac{\pi}{4\Delta t_{ab}}(2k - \Delta t_{ab}), & & s \neq \frac{k\pi(k-\Delta t_{ab})}{2\Delta t_{ab}(2k-\Delta t_{ab})}, \end{array} \tag{85}$$

$$\begin{aligned} s &\neq \frac{\pi}{6\Delta t_{ab}}\left(2k \pm \sqrt{16k^2 - 3\Delta t_{ab}^2}\right), \\ s &\neq \frac{\pi}{6\Delta t_{ab}}\left(\pm(4k + \Delta t_{ab}) \pm \sqrt{64k^2 + 32k\Delta t_{ab} + \Delta t_{ab}^2}\right), \\ s &\neq \frac{\pi}{10\Delta t_{ab}}\left(-12k - 3\Delta t_{ab} \pm \sqrt{64k^2 + 32k\Delta t_{ab} + 9\Delta t_{ab}^2}\right), \\ s &\neq \frac{\pi}{42\Delta t_{ab}}\left(-20k - 5\Delta t_{ab} \pm \sqrt{64k^2 + 32k\Delta t_{ab} + 25\Delta t_{ab}^2}\right), \\ s &\neq \frac{\pi}{70\Delta t_{ab}}\left(4k + \Delta t_{ab} \pm 3\sqrt{64k^2 + 32k\Delta t_{ab} - 31\Delta t_{ab}^2}\right), \\ s &\neq \frac{\pi}{18\Delta t_{ab}}\left(4k + \Delta t_{ab} \pm \sqrt{64k^2 + 32k\Delta t_{ab} - 23\Delta t_{ab}^2}\right), \end{aligned} \tag{86}$$

where $k \in \mathbb{Z}$. For these suitable parameter values we obtain solution as follows: for $\alpha_{ab} \equiv 0 \pmod{2\pi}$ one has matrix $X_{(a,b)}$ as depicted in (83), for any other value α_{ab} this matrix vanishes. Let conditions (85) and (86) be satisfied for each pair (a, b) of qubits.

Till now we have lead calculations in terms of 4×4 matrices $X_{(a,b)}$. Let us move back to the original problem. Since parameters γ and δ in (83) can also be set to zero, we can without loss of generality assume every solution $X_{(a,b)}$ is of this diagonal form irrespective of the actual value of α_{ab}, see the previous paragraph. A little thought than tells us if the whole matrix X is not zero, then all its diagonal entries are the same except for the very first element which is independent of the others. That is

$$X = \gamma\,|00\ldots0\rangle\langle00\ldots0| + \delta\,\mathbb{I} \tag{87}$$

with some $\gamma, \delta \in \mathbb{C}$. As a final step we translate condition $\alpha_{ab} \equiv 0 \pmod{2\pi}$ into more familiar language consisting of eigenvalue argument ω, collision times etc. From the definition of parameter α_{ab} it follows the condition $\alpha_{ab} \equiv 0 \pmod{2\pi}$ is equivalent to

$$2\Delta t_{ab}\, s \sum_{k\neq a,b} (j_k - i_k) \equiv \omega \pmod{2\pi}. \tag{88}$$

Nevertheless, for diagonal matrices one gets $\sum_{k\neq a,b}(j_k - i_k) = 0$. Consequently, for any nonzero ω the corresponding attractor matrix X vanishes. The only nonzero solution (87) is associated with the only remaining eigenvalue $\lambda = e^{\mathrm{i}0} = 1$.

5.2.1 Conclusion for Composite Evolution

We have considered a composite evolution of CNOT operation with different collision times and a diagonal free Hamiltonian. Apart from a zero-measure set of values s and Δt_{ab}, (85) and (86), the generic asymptotic state is stationary and reads (87). The attractor space is two-dimensional for any number of qubits in the network. In other words

$\lambda \neq 1$	regardless of $N \geq 2$ there is only the trivial attractor $X = 0$,
$\lambda = 1$	regardless of $N \geq 2$ all the attractors X are of the form (87) and the attractor space is two-dimensional.

6 Numerical Simulations

So far we have presented analytical solutions for the asymptotic dynamics of the quantum network evolving under various conditions. Both trivial and non-trivial free evolution together with partial swap and CNOT interactions were investigated and several scenarios were examined regarding interaction times. At first, we considered the simplest case with all collision times being one. Then we changed this value to be different from one and finally we assumed all collision times being of various values. In some cases we simplified our discussion by excluding physically unimportant zero-measure sets of collision times.

In this section we perform numerical simulations of the quantum-network evolution. This approach represents an independent way of calculating the asymptotic regimes of the network which can be subsequently compared with our findings from preceding sections. Numerical simulations also allow us to treat more general settings than we have investigated so far. In particular, unlike in previous sections we can study the quantum network behaviour while collision times Δt_{ab} and probabilities p_{ab} (8) vary during the evolution. Results from these generalized scenarios suggest that solutions to restricted cases, obtained analytically, are also valid in these settings. It should be noted though that the numerical approach suffers from several drawbacks. Apart from round-off errors another disadvantage is that no more than just a few-qudit networks can be examined with the power of the present-day technology.

The numerical approach consists in successive applications of the random unitary operation Φ (8), pertaining to the specific network evolution, to the initial network state. Thanks to the matrix form (99) of the random unitary operation as is shown in Appendix C we can completely get rid of the initial state dependence and focus on the evolution propagator itself. Apart from parameters considered in the analytical solution in the numerical approach we also have to specify the probability distribution in (5) which plays the role of an independent variable. In the following we are interested in the asymptotic form Φ_{asymp} of the propagator obtained after large number of successive applications of Φ. To compare the analytical results with numerical simulations we simply apply the asymptotic propagator to the general matrix of appropriate size and compare the output with the attractor form we had already calculated.

For both CNOT and partial swap interactions three evolution scenarios are taken into account. First two scenarios evaluate the asymptotic propagator Φ_{asymp} as an approximate limit of the iterative sequence $\{\Phi^n\}_{n=1}^{\infty}$, where the random unitary operation Φ is constructed out of the probability distribution and collision times which are kept *constant* during the evolution. Therefore, these cases can be directly compared with their analytical counterparts. The difference between the two scenarios is that the first one takes the uniform probability distribution and unit collision times whereas the second one generates both sets of values randomly before the iterations are triggered. As a consequence, the probability distribution as well as collision times present in the second scenario are in general non-uniform. Since we use random generation merely to produce a particular probabilities and collision times we do not average over their possible values. Averaging would be necessary if the random values of probabilities and collision times resulted from our incomplete knowledge of the network setting. In the plots below the first and second scenarios are depicted by red and blue dots, respectively. We take the quantity $d_n = \|\Phi^{n+1} - \Phi^n\|$ to assess convergence characteristics of the iterative sequence for both scenarios. In general, condition $d_n \to 0$ is merely a necessary condition for the sequence $\{\Phi^n\}_{n=1}^{\infty}$ to converge. Nevertheless, due to intrinsic properties of the random unitary operation Φ [6] this condition is also sufficient. Therefore, if the sequence $\{d_n\}_{n=0}^{\infty}$ is approximately zero from certain step n_0 onward we assume the sequence $\{\Phi^n\}_{n=n_0}^{\infty}$ is

constant and declare the element Φ^{n_0} as the asymptotic propagator Φ_{asymp}. The third scenario represents a generalization of the analytical approach in the sense that the probability distribution of collisions and collision times are no longer kept constant. Instead, they are randomly generated at *each* iteration step and the sequence we study in this setup is thus $\{\prod_{i=1}^{n}\Phi_i\}_{n=1}^{\infty}$, where Φ_i denotes a random unitary operation generated in the i-th step. Purple dots are used to denote this case in the figures below. Analogously to the first two scenarios, as a convergence measure we take the quantity $d_n = \|\prod_{i=1}^{n+1}\Phi_i - \prod_{i=1}^{n}\Phi_i\|$. Again, even in this case we make no averaging over all possibilities.

Each case examined below is performed on a quantum network comprised of three qubits, that is $N = 3$ and $d = 2$. Moreover, in the present discussion we set $p_0 = 0$ (5), but even for non-zero p_0 one obtains very similar results. In other words, the network asymptotics does not care whether there is some collision in every step of its evolution or not, only the rate of convergence is affected. Prior to detailed discussion of the two interactions we can already point out some general properties suggested by plots below. For instance, it turns out the rate of convergence of the second scenario is heavily dependent on the initial probabilities and collision times and varies considerably among different realisations of the network evolution. The third scenario, despite its non-monotonicity, seems to finally converge to the relevant asymptotic state even when the first scenario does not converge. This behaviour might arise due to more restrictions imposed on the attractor space of the third scenario. Contrary to the first case the attractors have to lie in an intersection of attractor spaces attributed to various random unitary operations generated during the evolution.

The partial swap Hamiltonian parameters we used for numerical calculation were chosen as $p = \cos\left(\frac{\pi}{6}\right)$ and $q = \mathrm{i}\sin\left(\frac{\pi}{6}\right)$, see (18). In Figure 2 we consider the partial swap interaction together with the trivial free evolution. Asymptotic states of all scenarios including the third one are identical, up to numerical error, to the analytical solution as is summarized in subsubsection 4.1.4. Since $N > d$, there is only $\lambda = 1$ in the attractor spectrum. In Figure 3 instead of the trivial free evolution we consider a non-trivial free evolution generated by a Pauli matrix Hamiltonian $H_i^{\text{free}} = \sigma_Z$. The first scenario with unit collision times does not converge in this case. Such a behaviour is due to non-unit eigenvalues λ in the attractor spectrum as follows from equation (35). In subsection 2.2 we presented the form of the general asymptotic regime (7) which is explicitly dependent on the number of iterations whenever $\lambda \neq 1$. In such a case the asymptotic regime oscillates and there is not a single propagator capable of capturing the network asymptotics. The first scenario represented by red dots therefore does not converge to zero, it rather approaches the value around four expressing the presence of non-unit eigenvalues. The second scenario is a numerical counterpart of the analytical case represented by equations (36) and (37) we did not solve. Since the second scenario converges we see that mutually different collision times Δt_{ab} and Δt_0 restrict the attractor form so that only the stationary part survives.

Let us move to the CNOT operation. Analogously to the partial swap, the first setup we consider is the CNOT operation acting together with the trivial free evolution. Results for the three scenarios can be seen in Figure 4. Also in this case all scenarios including the third one converge to the analytical solution, see subsubsection 5.1.4. Again, only eigenvalue $\lambda = 1$ appears in the attractor spectrum. In accordance with the analytical treatment of the non-trivial free evolution in subsection 5.2 we choose non-trivial free Hamiltonian as $H_i^{\text{free}} = \sigma_Z$. Results for this choice are shown in Figure 5. In the analytical approach we demanded $\Delta t_{ab} = \Delta t_{ba}$, but as the plot shows such a constraint is redundant since the second scenario with a randomly generated initial collision times converges to the relevant asymptotic evolution very well. The only notable difference between the trivial

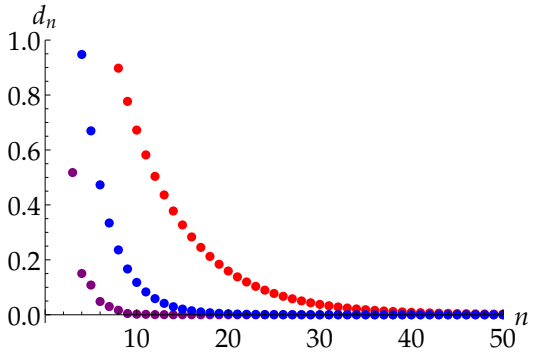

Figure 2: Partial swap-like operation with the trivial free evolution and with $p_0 = 0$— Comparison of rate of convergence for three different scenarios. Red dots correspond to the case with uniform probability distribution and unit collision times. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in each step of the evolution.

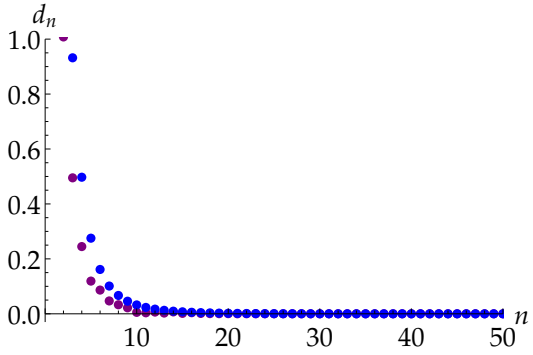

Figure 3: Partial swap-like operation with the non-trivial free evolution and with $p_0 = 0$— Comparison of rate of convergence for three different scenarios. The first scenario with uniform probability distribution and unit collision times does not converge to the stationary analytical solution due to non-unit eigenvalues. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in each step of the evolution.

and non-trivial cases is that in the latter the first scenario converges slower. As opposed to the partial swap case the third scenario fluctuates dramatically before it reaches the asymptotic value.

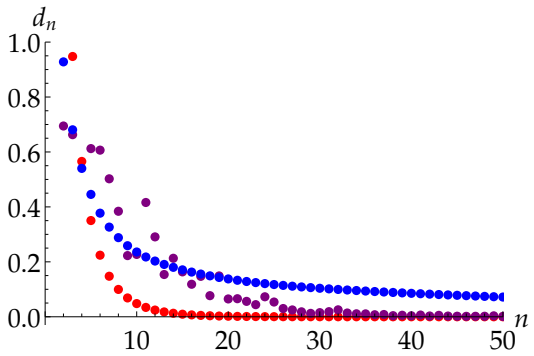

Figure 4: CNOT-like operation with the trivial free evolution and with $p_0 = 0$— Comparison of rate of convergence for three different scenarios. Red dots correspond to the case with uniform probability distribution and unit collision times. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in each step of the evolution.

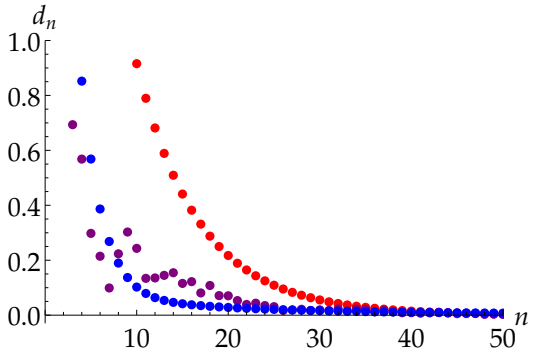

Figure 5: CNOT-like operation with the non-trivial free evolution and with $p_0 = 0$— Comparison of rate of convergence for three different scenarios. Red dots correspond to the case with uniform probability distribution and unit collision times. Blue dots correspond to the case with non-uniform constant probability distribution and collision times. Purple dots correspond to the case with probability distribution and collision times being updated to random values in each step of the evolution.

Examples demonstrated so far suggest analytical results calculated in previous sections are valid and the theoretical framework used for calculations captures the essence of the asymptotic behaviour the quantum network undergoes in different scenarios. Moreover, from the third scenario it seems even in more general setups one obtains attractors already emerging in the restricted cases. From the nature of numerical simulations it is essentially impossible for generated collision times to lie in the zero-measure set of exceptional values for which the solutions are degenerate (see respective conclusions in subsubsections 4.1.4, 5.1.4 and 5.2.1). Hence, we are left with the generic cases only.

7 Conclusion

We have investigated the asymptotic behaviour of the multi-qudit quantum network whose individual constituents undergo free evolution randomly interrupted by swift bipartite interactions. An interplay between the free evolution and mutual interactions possibly leading to the network equilibration has been analyzed. The quantum network was taken to be a complete graph and CNOT and partial swap interactions were considered. We have found closed analytical solutions to these setups for various qudit dimensions, number of qudits and values of interaction times.

For purely partial swap evolution we have found out the generic asymptotic state assumes the form described in (25). The dimension of the attractor space in this case scales with the dimension of qudits and their number according to formula (24). The quantum network with the trivial free evolution and partial swap interactions equilibrates whenever the number of qudits in the network is strictly greater than their dimensionality. When the number of qudits does not exceed their dimensionality the asymptotic behaviour of the quantum network is more complex and equilibration in general does not occur. For purely CNOT evolution the generic asymptotic state is of the stationary form (49) with the dimension of the attractor space being independent of the number of qubits. A quantum network with the trivial free evolution and CNOT interactions between its constituents thence equilibrates regardless of the number of qubits provided that $N \geq 3$.

After investigation of mutual interactions we incorporated also the non-trivial free evolution. For the partial swap we considered general one-qudit free Hamiltonian. The resulting asymptotic state, and hence the equilibration property, is determined by equation (35). Since the proper discussion for CNOT and general one-qubit free Hamiltonian was difficult to complete we considered only a special diagonal form of the Hamiltonian. For this setup we arrived at the asymptotic state as shown in (87). A physically irrelevant zero-measure set of collision-time settings was excluded during the calculations. Nevertheless, numerical simulations show this exclusion is unnecessary. Equilibration in the quantum network evolving under the given non-trivial free Hamiltonian and CNOT interactions is thus very probable.

In future, we would like to consider also quantum networks which are not complete graphs. This setup is more realistic and general than we employed in this work. Other free Hamiltonians and interactions except the partial swap and CNOT can also be taken into account. Especially a composite evolution of CNOT interactions and a free evolution generated by a general two-qubit free Hamiltonian is a challenging problem. Theoretical framework could be augmented to treat evolution having different probability distribution and collision times in each step. Numerical simulations suggest that such a generalized evolution leads to the same network asymptotics and might be analyzed within theoretical framework similar to that we use.

8 References

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A Pauli Matrices

Throughout the thesis we make use of Pauli matrices σ_X, σ_Y and σ_Z defined in the usual way

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

B Hamiltonians

In the main text we presented Hamiltonians for CNOT and PSW operations. This appendix is to show that these Hamiltonians indeed generate appropriate operations as their associated evolution operators. After doing so we also demonstrate explicitly how the evolution operators change their form when the collision time Δt is different from one.

Let us begin with the controlled NOT operation. We claim the corresponding Hamiltonian is of the form

$$H_{\mathrm{CNOT}} = \frac{\pi}{2}(2l+1)(\mathrm{CNOT} - \mathbb{I}), \quad l \in \mathbb{Z}. \tag{89}$$

Since $\mathrm{CNOT}^2 = \mathbb{I}$ we have $(\mathrm{CNOT} - \mathbb{I})^2 = -2(\mathrm{CNOT} - \mathbb{I})$. By mathematical induction one easily proves $(\mathrm{CNOT} - \mathbb{I})^k = (-2)^{k-1}(\mathrm{CNOT} - \mathbb{I})$ for $k \geq 1$. Therefore

$$\begin{aligned} e^{\mathrm{i}H_{\mathrm{CNOT}}} &= \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{i}\frac{\pi}{2}(2l+1))^k(\mathrm{CNOT} - \mathbb{I})^k \\ &= \mathbb{I} + \sum_{k=1}^{\infty} \frac{1}{k!}(\mathrm{i}\frac{\pi}{2}(2l+1))^k(-2)^{k-1}(\mathrm{CNOT} - \mathbb{I}) \\ &= \mathbb{I} - \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{k!}(-\mathrm{i})^k(\pi(2l+1))^k\right)(\mathrm{CNOT} - \mathbb{I}) \\ &= \mathbb{I} - \frac{1}{2}\left(e^{-\mathrm{i}\pi(2l+1)} - 1\right)(\mathrm{CNOT} - \mathbb{I}) \\ &= \mathbb{I} + (\mathrm{CNOT} - \mathbb{I}) = \mathrm{CNOT}. \end{aligned}$$

For partial swap we now prove its corresponding Hamiltonian assumes the form as follows

$$H_{\mathrm{PSW}_{ab}} = \rho\,\mathbb{I} \pm \frac{\varphi}{2}\,\mathrm{SW}_{ab}, \tag{90}$$

where ρ and φ are defined in relations (18). Since a global phase plays no role we can put $\rho = 0$. Similarly to CNOT even for PSW we utilize involution and commutation properties $\mathrm{SW}^2 = \mathbb{I}$ and $[\mathbb{I}, \mathrm{SW}] = 0$ to obtain

$$\begin{aligned} e^{\mathrm{i}H_{\mathrm{PSW}}} &= e^{\mathrm{i}\rho\mathbb{I}}e^{\mathrm{i}\phi\,\mathrm{SW}} = e^{\mathrm{i}\rho}\sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{i}^k(\phi\,\mathrm{SW})^k \\ &= e^{\mathrm{i}\rho}\left(\sum_{k=0}^{\infty}\frac{1}{(2k)!}\mathrm{i}^{2k}(\phi\,\mathrm{SW})^{2k} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\mathrm{i}^{2k+1}(\phi\,\mathrm{SW})^{2k+1}\right) \\ &= e^{\mathrm{i}\rho}\left(\sum_{k=0}^{\infty}\frac{1}{(2k)!}(-1)^k\phi^{2k}\,\mathbb{I} + \mathrm{i}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}(-1)^{2k}\phi^{2k+1}\,\mathrm{SW}\right) \\ &= e^{\mathrm{i}\rho}\cos(\phi)\,\mathbb{I} + \mathrm{i}\,e^{\mathrm{i}\rho}\sin(\phi)\,\mathrm{SW} \\ &= p\,\mathbb{I} + q\,\mathrm{SW} = \mathrm{PSW}, \end{aligned}$$

where we set $\phi = \pm\varphi/2$ for convenience.

Recalling just performed calculations we can modify them by introducing non-unit collision time Δt. It is easy to see that the last but one line of the calculation for CNOT changes to

$$\begin{aligned} e^{\mathrm{i}\,\Delta t H_{\mathrm{CNOT}}} &= \mathbb{I} - \frac{1}{2}\left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t} - 1\right)(\mathrm{CNOT} - \mathbb{I}) \\ &= \frac{1}{2}\left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t} + 1\right)\mathbb{I} - \frac{1}{2}\left(e^{-\mathrm{i}\,\pi(2l+1)\Delta t} - 1\right)\mathrm{CNOT}. \end{aligned}$$

The same procedure for PSW operation yields

$$\begin{aligned} e^{\mathrm{i}\,\Delta t H_{\mathrm{PSW}}} &= e^{\mathrm{i}\,\rho\Delta t}\cos(\phi\Delta t)\,\mathbb{I} + \mathrm{i}\,e^{\mathrm{i}\,\rho\Delta t}\sin(\phi\Delta t)\,\mathrm{SW} \\ &= \tilde{p}\,\mathbb{I} + \tilde{q}\,\mathrm{SW}, \end{aligned}$$

where $\tilde{p}$ and $\tilde{q}$ are p and q parameters, respectively, with angles Δt times bigger than the original ones. That is, the value of collision time only modifies values of parameters and we obtain partial swap again.

Finally, let us calculate the commutator of the partial swap Hamiltonian with the free evolution Hamiltonian pertaining to the subsystem of a-th and b-th qubits. We prove that

$$\left[H_a^{\mathrm{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\mathrm{free}}, H_{\mathrm{PSW}_{ab}}\right] = 0. \tag{91}$$

Analogous discussion could be done for qudits with arbitrary dimensionality. Nevertheless, in here we restrict ourselves to one-qubit Hamiltonians, whose form in a Pauli basis reads

$$H_a^{\mathrm{free}} = \alpha_{\mathbb{I}}^{(a)}\mathbb{I} + \alpha_X^{(a)}\sigma_X + \alpha_Y^{(a)}\sigma_Y + \alpha_Z^{(a)}\sigma_Z. \tag{92}$$

From relation (90) it follows that

$$\left[H_a^{\mathrm{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\mathrm{free}}, H_{\mathrm{PSW}_{ab}}\right] \propto \left[\sum_i (\alpha_i^{(a)}\sigma_i \otimes \mathbb{I}_b + \alpha_i^{(b)}\mathbb{I}_a \otimes \sigma_i), \mathrm{SW}_{ab}\right]. \tag{93}$$

In order to proceed one can derive the swap operation acting on composite subsystem of a-th and b-th qubit takes the form

$$\mathrm{SW}_{ab} = \frac{1}{2}(\mathbb{I}_a \otimes \mathbb{I}_b + \sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z). \tag{94}$$

Substituting this expression into the right-hand side of (93) one gets

$$\frac{1}{2}\sum_{i,j\in\{X,Y,Z\}} \alpha_i^{(a)}\left[\sigma_i \otimes \mathbb{I}_b, \sigma_j \otimes \sigma_j\right] + \frac{1}{2}\sum_{i,j\in\{X,Y,Z\}} \alpha_i^{(b)}\left[\mathbb{I}_a \otimes \sigma_i, \sigma_j \otimes \sigma_j\right]. \tag{95}$$

Employing the commutation relations of Pauli matrices $[\sigma_i, \sigma_j] = 2\mathrm{i}\,\varepsilon_{ijk}\,\sigma_k$ one finds out the commutator reads

$$\left[H_a^{\mathrm{free}} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes H_b^{\mathrm{free}}, H_{\mathrm{PSW}_{ab}}\right] \propto \mathrm{i}\sum_{ijk} \varepsilon_{ijk}\,\sigma_k \otimes \sigma_j\left(\alpha_i^{(a)} - \alpha_i^{(b)}\right). \tag{96}$$

We can conclude that partial swap Hamiltonian commutes with the free evolution if and only if $\alpha_i^{(a)} = \alpha_i^{(b)}$ for all i. In other words the Hamiltonians commute iff the free Hamiltonians for both qubits are the same, which is indeed the present case.

C Superoperators as Matrices

We study the evolution processes of the quantum networks whose generators are random unitary operations, that is superoperators acting on network density matrices $\rho \in \mathbb{C}^{d^N \times d^N}$, see (5). In order to study their properties it is suitable to represent the superoperator itself as a matrix [9]. To that end, one can rewrite an arbitrary matrix $X \in \mathbb{C}^{d^N \times d^N}$ as a ket vector $|X\rangle \in \mathbb{C}^{d^{2N}}$

$$|X\rangle = (X_{1\bullet}, X_{2\bullet}, \ldots, X_{d^N\bullet}), \tag{97}$$

where $X_{i\bullet}$ stands for the i-th row of matrix X. Suppose we are dealing with the superoperator S of the form $S(X) = A\,X\,B \equiv Y$, where $A, B, X, Y \in \mathbb{C}^{d^N \times d^N}$ are four matrices. One can prove the correspondence between different representations of the superoperator S as follows

$$A\,X\,B = Y \quad \sim \quad (A \otimes B^T)|X\rangle = |Y\rangle. \tag{98}$$

The matrix representation of the superoperator S is therefore $S = A \otimes B^T$. Relation (98) is completely general and can be used in its own right. On the other hand, one can make use of this correspondence to rewrite the random unitary operation (5) in the form

$$\Phi|A\rangle = \sum_{a,b=1}^{N} p_{ab} U_{ab} \otimes U_{ab}^{\star} |A\rangle, \tag{99}$$

where star symbol denotes the complex conjugation. After this transformation the mapping composition $\Phi \circ \Phi$ reduces to matrix multiplication $\Phi \cdot \Phi$.