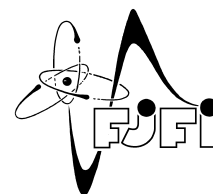


CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering



Quantum walks, search algorithm and state transfer

Kvantové procházky, vyhledávací algoritmus a přenos stavu

Bachelor's Degree Project

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- Zadání práce -

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Prohlášení

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V Praze dne 22. 7. 2015

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Stanislav Skoupý

Název práce:

Kvantové procházky, vyhledávací algoritmus a přenos stavu

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Obor: Matematické inženýrství

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Abstrakt: V této práci se zaměříme na téma kvantových procházek a jejich využití. Představíme základ diskrétní kvantové procházky a zmíníme kontinuální kvantovou procházku. Poté popíšeme Groverův algoritmus a odhadneme jeho složitost. Také stanovíme dolní hranici počtu kroků potřebných k nalezení označeného stavu. Definujeme vyhledávací algoritmus založený na kvantové procházce a opět ukážeme jeho složitost. Provedeme srovnání obou typů algoritmu. Předvedeme jiný způsob odhadu počtu kroků algoritmu založeného na kvantové procházce pomocí vlastností spektra zobecněného evolučního operátoru. Na závěr se zaměříme na uplatnění kvantových procházek při přenosu stavu a komunikaci na mřížce.

Klíčová slova: kvantová procházka, přenos stavu, vyhledávací algoritmus

Title:

Quantum walks, search algorithm and state transfer

Author: Stanislav Skoupý

Abstract: In this text we focus on the topic of quantum walks and their usage. We introduce basic of the discrete time quantum walks and mention the continuous time quantum walk. We describe the Grover's search algorithm and estimate its complexity. We also prove the lower bound of number of steps necessary to find the marked state. We define quantum walk search algorithm and again show its complexity. We present the difference between both kind of algorithm. We show the different method of approximating the number of steps of walk search algorithm using the properties of spectrum of the generalised evolution operator. At the end we focus on use of quantum walks in the state transfer and communication on the regular lattice.

Key words: quantum walks, search algorithms, state transfer

Contents

Introduction	9
1 Mathematical formalism	10
1.1 Quantum Mechanics	10
1.2 Asymptotic notation	10
2 Quantum walk	12
2.1 Discrete time quantum walk	12
2.2 Continuous time quantum walk	13
3 Search algorithm	15
3.1 Grover's algorithm	15
3.2 Quantum walk search algorithm on a hypercube	18
3.3 Quantum walk search algorithm on regular lattice	22
4 Different look on the walk search algorithm	26
4.1 Search on the hypercube	26
4.2 Search on the regular lattice	30
5 State transfer	32
5.1 State transfer as the discrete time quantum walk	32
5.2 An amplitude transfer across regular lattice	34
Conclusion	36

Introduction

In this bachelor's degree project I study the topic of quantum walks and their application in quantum algorithm. The quantum walks are fast developing topic because there are many ways how to physically implement them and test their properties in experiments. I focus on search algorithm and briefly on state transfer.

In the first chapters of this text we introduce the mathematical formalism of the quantum physic that we use and the basic concept of the quantum walks. Then we describe the quantum walk search algorithm and Grover's algorithm, which is not based on quantum walks, and compare their properties. In Chapter 4 we discuss the quantum walk search algorithm again to estimate their complexity from different point of view. Last chapter presents use of discrete time quantum walk for the transfer of the state and communications across the grid thanks to quantum walk search algorithm.

Chapter 1

Mathematical formalism

1.1 Quantum Mechanics

Basic axioms of Quantum mechanic settle that every quantum system can be described with Hilbert space $\mathcal{H}$, which is often called state space, and every state of this system is described by one-dimensional subspace of $\mathcal{H}$. Vector from $\mathcal{H}$ is denoted $|\psi\rangle$. Symbol $\langle\psi|$ represent element from dual space and is assigned to vector $|\psi\rangle$ by Riesz's lemma. Scalar product of $|\psi\rangle$ and $|\varphi\rangle$ is written in form $\langle\varphi|\psi\rangle$. Because one state of the system is represented by vectors from one dimensional subspace, we can choose vector, which norm is equal 1. This will enable us to write simple form of many following expressions.

If Hilbert space has countable orthogonal basis, we can write decomposition of state in the basis

$$|\psi\rangle = \sum_{i=1}^{\infty} \langle i|\psi\rangle |i\rangle. \quad (1.1)$$

In case of finite dimension spaces sum is also finite. One of the most important thing for physics is how we can predict results of measurement. If the state of system is $|\psi\rangle$, then the probability to measure system in the state $|i\rangle$ is equal to $|\langle\psi|i\rangle|^2$.

However quantum system can change trough time, so we have to know how to calculate the time evolution. Evolution of state is determined by the Schödinger equation

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H} |\psi(t)\rangle, \quad (1.2)$$

where $\hat{H}$ is the Hamiltonian operator of the system. Now we take scalar product of solutions from equation (1.2) and execute its time derivative.

$$\begin{aligned} \frac{d}{dt} \langle\varphi(t)|\psi(t)\rangle &= \left(\frac{\partial \langle\varphi(t)|}{\partial t} \right) |\psi(t)\rangle + \langle\varphi(t)| \left(\frac{\partial |\psi(t)\rangle}{\partial t} \right) = \\ &= \frac{i}{\hbar} \langle\varphi(t)|\hat{H}|\psi(t)\rangle - \frac{i}{\hbar} \langle\varphi(t)|\hat{H}|\psi(t)\rangle = 0 \end{aligned} \quad (1.3)$$

From (1.3) follows that the operator of time evolution directed by (1.2) must be unitary to guarantee invariance of scalar product. We want to preserve this property, when we consider discrete time instead of continuous. So every discrete time step will be also performed by unitary operator.

1.2 Asymptotic notation

Another thing we will often use is following standard computer science notation. Let us say $f(n) = O(g(n))$ if there are positive constants c and k such that $0 \leq f(n) \leq cg(n)$ for $n \geq k$.

Similarly we say $f(n) = \Omega(g(n))$ if there are $c, k \geq 0$ and $0 \leq cg(n) \leq f(n)$ for $n \geq k$. If function f comply $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ we say $f(n) = \Theta(g(n))$.

Chapter 2

Quantum walk

In this Chapter we present the basic of quantum walks. First in Section 2.1 we introduce discrete time quantum walk and then in Section 2.2 we briefly describe continuous time quantum walk.

2.1 Discrete time quantum walk

To introduce discrete quantum walks gently we will at first briefly describe classical random walk on discrete line, then we move to the discrete time quantum walk on discrete line and later we introduce quantum walk on general type of graph. Classic random walk on discrete line is a process, where we have one walker on discrete line with grid-length 1. Movement of the walker is random, but we know at each step probabilities for walker to go right and left, let us denote them p_+ for right and p_- for left. Walker starts at position 0 and we are interested in the probability that he is in certain position after the fixed number of steps. Distribution of probability with rising number of steps will approximate to Gauss distribution. Walk is said to be unbiased, if the probabilities are $p_+ = p_- = \frac{1}{2}$.

For introducing quantum walk we have to construct underlying Hilbert space. Let $\{|i\rangle : i \in \mathbb{Z}\}$ be the basis that span position Hilbert space $\mathcal{H}_p$. To use advantages of quantum world we want walker to travel simultaneously in both neighbouring position in each step. But it is easy to see that

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|-1\rangle + \frac{1}{\sqrt{2}}|1\rangle \rightarrow \frac{1}{2}|-2\rangle + |0\rangle + \frac{1}{2}|2\rangle \quad (2.1)$$

is not unitary transformation. Position space need to be augmented with something, which will direct the move of every step. We set another auxiliary space $\mathcal{H}_c$ with basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. This space is often called coin-space. Final state space is $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$. Unitary transition operator, which will shift position depending on the state of coin-space, has the form

$$\hat{S} = |\uparrow\rangle\langle\uparrow| \otimes \sum_i |i+1\rangle\langle i| + |\downarrow\rangle\langle\downarrow| \otimes \sum_i |i-1\rangle\langle i|. \quad (2.2)$$

Time evolution only under (2.2) is quit trivial. State with defined orientation, i.e. $|\uparrow\rangle$ or $|\downarrow\rangle$, will simply shift in one direction. To obtain non-trivial evolution we will first conduct rotation on the coin-space, let us call it coin-flip. Then one step of the walk will consist of coin-flip and application of shift operator $\hat{U} = \hat{S} \cdot (\hat{C} \otimes \hat{I})$. Let us start the walk in the initial state $|\uparrow\rangle \otimes |0\rangle$, take one step and execute measurement on coin-space, after this we would like to get probability distribution of classical unbiased walk. Balanced coin, that is often used to manage this [1], is the so called Hadamard coin

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (2.3)$$

If we measure coin-space after every step, we will get the same probability distribution as for classical unbiased walk. To get proper quantum walk we conduct measurement at the end of the walk. Quantum properties starts to emerge since third step due to interference. As an example of quantum walk we consider 3 step walk with initial state $|\uparrow\rangle \otimes |0\rangle$

$$\begin{aligned}
|\psi_3\rangle &= \hat{U}^3 (|\uparrow\rangle \otimes |0\rangle) = \hat{U}^2 \left(\frac{1}{\sqrt{2}} |\uparrow\rangle \otimes |1\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle \otimes |-1\rangle \right) = \\
&= \hat{U} \left(\frac{1}{2} |\uparrow\rangle \otimes |2\rangle + \frac{1}{2} (|\uparrow\rangle + |\downarrow\rangle) \otimes |0\rangle - \frac{1}{2} |\downarrow\rangle \otimes |-2\rangle \right) = \\
&= \frac{1}{2\sqrt{2}} |\uparrow\rangle \otimes |3\rangle + \frac{1}{2\sqrt{2}} (2|\uparrow\rangle + |\downarrow\rangle) \otimes |1\rangle - \frac{1}{2\sqrt{2}} |\uparrow\rangle \otimes |-1\rangle + \frac{1}{2\sqrt{2}} |\downarrow\rangle \otimes |-3\rangle.
\end{aligned} \tag{2.4}$$

Probability distribution is no longer approaching Gauss distribution. The highest probabilities are not in the centre but on the edges. Walk in this case is also not symmetric, because chosen coin treats states $|\uparrow\rangle$ and $|\downarrow\rangle$ differently [1]. See figure 2.1.

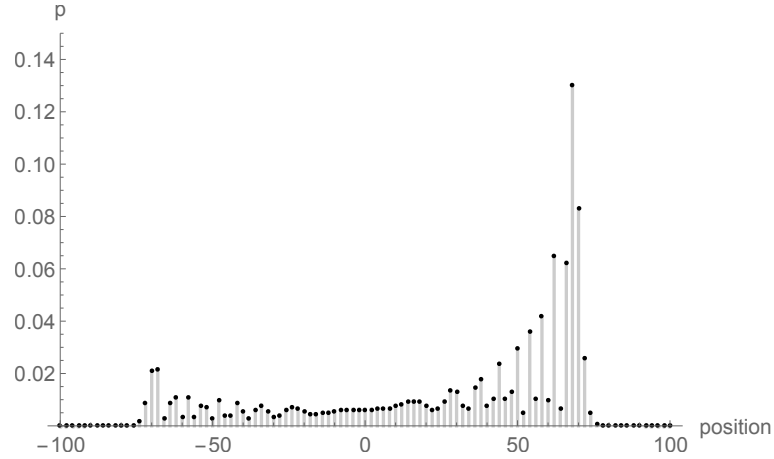


Figure 2.1: The probability distribution of the discrete quantum walk after 100 steps with the initial state $|\uparrow\rangle \otimes |0\rangle$. Only the probability on the even points is plotted. The highest probability is shifted to the right because initial state of the coin is unsymmetrical.

Now we introduce generalization of quantum walks for general graph G . Degree d_v of vertex v of graph G is number of edges outgoing from the vertex. We restrict us on so called regular graph, i.e. all vertices have same degree d . We also label edges outgoing from each vertex from 1 to d . Then underlying Hilbert space of is spanned by orthogonal basis $\{|i\rangle \otimes |v\rangle : v \in G, i \in 1 \dots d\}$. Shift operator is defined as $\hat{S}|i\rangle \otimes |v\rangle = |i\rangle \otimes |w\rangle$, where vertices v and w are connected by edge marked with i on v 's side. Every vertex can have generally different coin. One step of walk now has the form $\hat{U} = \hat{S} \cdot \sum_{v \in G} \hat{C}_v \otimes |v\rangle\langle v|$.

2.2 Continuous time quantum walk

We introduce continuous time quantum walk on regular graph. It take place only on position space $\mathcal{H}_p$ with orthogonal basis $\{|v\rangle : v \in G\}$. Time evolution of the state is determined by (1.2).

We express Hamiltonian of the system in basis

$$\langle v|\hat{H}|w\rangle = \begin{cases} -\alpha & v \neq w, v \text{ and } w \text{ are connected} \\ 0 & v \neq w, v \text{ and } w \text{ are not connected} \\ d\alpha & v = w \end{cases} . \quad (2.5)$$

In classical approximation α can be seen as probability to move from one vertex to another connected vertex [1]. The Hamiltonian (2.5) is time-independent, so from (1.2) we can calculate the time evolution operator

$$\hat{U}(t) = e^{-i\hat{H}t} \quad (2.6)$$

where we set $\hbar = 1$.

Chapter 3

Search algorithm

In this Chapter we introduce the quantum search algorithm. We estimate the number of steps they need to succeed and describe their properties. We start with Grover's algorithm in Section 3.1 and then in Section 3.2 introduce quantum walk algorithm on hypercube. At the end in Section 3.3 we shortly present quantum walk algorithm on regular lattice.

3.1 Grover's algorithm

Grover's algorithm is quantum search algorithm over unsorted data. Before we begin with algorithm, we need to describe the basic quantum unit of information so called qubit and quantum memory register. Classical computer operate with two state systems called bits. Qubits are bits quantum equivalent. Qubit can be in superposition of both states at the same time instead of bit which can be in only one of the two states at the time. For description of qubit state we will use Hilbert space of $\mathbb{C}^2$. Quantum memory register is composed of qubits and underlying Hilbert space is tensor product of particular qubit's spaces. For simplicity of notation we will write $|i\rangle \otimes |j\rangle = |ij\rangle$.

Now we describe the problem for the Grover's algorithm. Let us have quantum register with n qubits. Orthogonal basis $\{|i_1 i_2 \dots i_n\rangle; i_k \in \{0, 1\}\}$ have $N = 2^n$ elements. We label these elements $|S_1\rangle, |S_2\rangle, \dots, |S_N\rangle$. Let there be a unique state from basis, labelled $|S_m\rangle$, that satisfies the condition $C(|S_m\rangle) = 1$, where for all other basis states $C(|S_i\rangle) = 0, i \neq m$. We assume that condition $C(|S_i\rangle)$ can be evaluated in unit time. Now we use Grover's algorithm to find state $|S_m\rangle$.

Algorithm is composed of following steps:

1. Initialize the system in the superposition of basis states

$$\sum_{i=1}^N \frac{1}{\sqrt{N}} |S_i\rangle, \quad (3.1)$$

i.e. there is the same amplitude to be in each of the N states.

2. Repeat following unitary operation $O(\sqrt{N})$ times

- (a) Rotate the phase of basis state $|S\rangle$ if $C(|S\rangle) = 1$ by π . If $C(|S\rangle) = 0$, leave the system unaltered.
- (b) Apply the diffusion transform $\hat{D}$ which is defined by matrix $D = (D_{ij})$ as follows:

$$\langle i | \hat{D} | j \rangle = D_{ij} = \begin{cases} \frac{2}{N} & \text{if } i \neq j \\ -1 + \frac{2}{N} & \text{if } i = j \end{cases} \quad (3.2)$$

3. Measure the the resulting state. This will be state $|S_m\rangle$ with probability of at least 0,5. [2]

Let us show that the algorithm works. It is easy to see from the definition of the matrix D that it can be written in the form

$$D = -I + 2P, \text{ where } P_{ij} = \frac{1}{N} \text{ for all } i, j. \quad (3.3)$$

Amplitudes of vector $\hat{P}|x\rangle$ are equal to the average of amplitudes of vector $|x\rangle$. When we let $\hat{D}$ act on arbitrary vector $|x\rangle$

$$\hat{D}|x\rangle = (-\hat{I} + 2\hat{P})|x\rangle = -|x\rangle + 2\hat{P}|x\rangle \quad (3.4)$$

and label the average of amplitudes as A , we can express i^{th} amplitude of vector $\hat{D}|x\rangle$

$$\langle i|\hat{D}|x\rangle = -\langle i|x\rangle + 2A = (A + (A - \langle i|x\rangle)). \quad (3.5)$$

Expression (3.5) shows that $\hat{D}$ is acting like inversion about the average [2]. Let us apply $\hat{D}$ on vector which have one amplitude equal to $-\sqrt{1-C^2}$ and rest amplitudes equal to $\frac{C}{\sqrt{N}}$, where C is a real number lying between $\frac{1}{2}$ and 1. The average A is approximately equal to $\frac{C}{\sqrt{N}}$ and hence amplitudes $\frac{C}{\sqrt{N}}$ do not change significantly after applying $\hat{D}$. Marked state amplitude becomes positive and increases by $\frac{2C}{\sqrt{N}}$. Thus every iteration in the second step of the algorithm increases amplitude in the marked state by $\frac{2C}{\sqrt{N}}$. As long as the amplitude $\sqrt{1-C^2}$ is less then $\frac{1}{\sqrt{2}}$ the increment is grater then $\frac{1}{\sqrt{2N}}$. This means that there exist a number T less then $\sqrt{N}$ and meeting the condition, that after we repeat T -times second step, we will measure the marked state with probability grater then $\frac{1}{2}$. So the algorithm can find the result with $O(\sqrt{N})$ steps. In figure 3.1 we can see example of the evolution of the probability to measure the marked state.

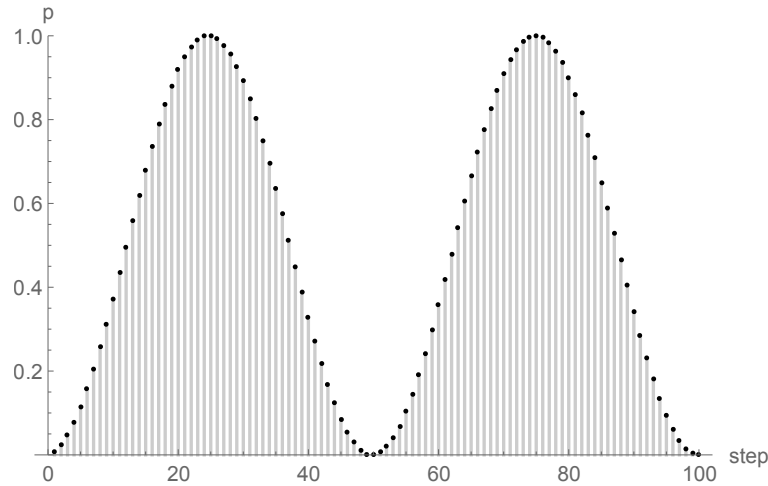


Figure 3.1: The evolution of the probability p to find the marked state during 100 steps. We use the Grover's algorithm and search within 1024 elements.

On the other hand we prove that quantum search algorithm cannot perform better then $\Omega(\sqrt{N})$. In general the quantum search algorithm step is composed from the application of the so called oracle operator $\hat{O}_m = \hat{I} - 2|m\rangle\langle m|$, which changes the phase of the marked state $|m\rangle$ by π ,

and the application of the evolution operator $\hat{U}_n$ in n^{th} step. At beginning of the prove we define the following states

$$\begin{aligned} |\varphi_n\rangle &= \hat{U}_n \dots \hat{U}_2 \hat{U}_1 |\varphi_0\rangle \\ |\varphi_n^m\rangle &= \hat{U}_n \hat{O}_m \hat{U}_{n-1} \dots \hat{U}_2 \hat{O}_m \hat{U}_1 \hat{O}_m |\varphi_0\rangle \end{aligned} \quad (3.6)$$

We show the lower bound of the number of steps by bounding quantity A_n defined as follows

$$A_n = \sum_m \|\varphi_n^m - \varphi_n\|^2 \quad (3.7)$$

For simpler notation we denote the vectors from the Hilbert space $x = |x\rangle$. At first we show that the upper bound of (3.7) is $4n^2$. We prove this by mathematical induction. For $n = 0$ it holds true because $A_0 = 0$. Now we move to the inductive step A_{n+1} . Using $\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ we write

$$\begin{aligned} A_{n+1} &= \sum_m \|\hat{O}_m \varphi_n^m - \varphi_n\|^2 = \sum_m \|\hat{O}_m(\varphi_n^m - \varphi_n) + (\hat{O}_m - \hat{I})\varphi_n\|^2 \leq \\ &\leq \sum_m \left(\|\varphi_n^m - \varphi_n\|^2 + 4\|\varphi_n^m - \varphi_n\| |\langle m|\varphi_n\rangle| + 4|\langle m|\varphi_n\rangle|^2 \right) \end{aligned} \quad (3.8)$$

Now we use Cauchy-Schwarz inequality

$$\left| \sum_{i=1}^N \bar{a}_i b_i \right| \leq \left(\sum_{i=1}^N |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^N |b_i|^2 \right)^{\frac{1}{2}} \quad (3.9)$$

and the fact that

$$\sum_m |\langle m|\varphi_n\rangle|^2 = \|\varphi_n\|^2 = 1 \quad (3.10)$$

to bound A_{n+1} as follows

$$\begin{aligned} A_{n+1} &\leq A_n + 4 \left(\sum_m \|\varphi_n^m - \varphi_n\|^2 \right)^{\frac{1}{2}} \left(\sum_m |\langle m|\varphi_n\rangle|^2 \right)^{\frac{1}{2}} + 4 \leq \\ &\leq A_n + 4\sqrt{A_n} + 4 \leq 4n^2 + 8n + 4 = 4(n+1)^2 \end{aligned} \quad (3.11)$$

which means that $A_n \leq 4n^2$ holds true.

To find the lower bound of A_n we suppose that $|\langle m|\varphi_n^m\rangle|^2 \geq \frac{1}{2}$ which means that we find the marked state with probability higher than $\frac{1}{2}$. We can also suppose without lost of generality that $\langle m|\varphi_n^m\rangle = |\langle m|\varphi_n^m\rangle|$. We can now estimate the following expression

$$\|\varphi_n^m - m\|^2 = 2 - 2|\langle m|\varphi_n^m\rangle| \leq 2 - \sqrt{2}. \quad (3.12)$$

Another useful inequality holds true for any normalized state x

$$\begin{aligned} \sum_m \|x - m\|^2 &\geq \sum_m (2 - 2\Re\langle m|x\rangle) \geq 2N - 2 \sum_m |\langle m|x\rangle| \geq \\ &\geq 2N - 2\sqrt{N} \end{aligned} \quad (3.13)$$

where we use the following estimation

$$\sum_m |\langle m|x \rangle| \leq \left(\sum_m |\langle m|x \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_m 1 \right)^{\frac{1}{2}} \leq \sqrt{N}. \quad (3.14)$$

Using (3.9),(3.12),(3.13) and $\|x - y\|^2 \geq \|x\|^2 - 2\|x\|\|y\| + \|y\|^2$ we can calculate the lower boundary of A_n

$$\begin{aligned} A_n &= \sum_m \|(\varphi_n^m - m) - (\varphi_n - m)\|^2 \geq \\ &\geq \sum_m \|\varphi_n^m - m\|^2 - 2 \sum_m \|\varphi_n^m - m\| \|\varphi_n - m\| + \sum_m \|\varphi_n - m\|^2 \geq \\ &\geq \sum_m \|\varphi_n^m - m\|^2 - 2 \left(\sum_m \|\varphi_n^m - m\|^2 \right)^{\frac{1}{2}} \left(\sum_m \|\varphi_n - m\|^2 \right)^{\frac{1}{2}} + \sum_m \|\varphi_n - m\|^2 \geq. \quad (3.15) \\ &\geq \left[\left(\sum_m \|\varphi_n^m - m\|^2 \right)^{\frac{1}{2}} - \left(\sum_m \|\varphi_n - m\|^2 \right)^{\frac{1}{2}} \right]^2 \\ &\geq \left[(2N - \sqrt{2}N)^{\frac{1}{2}} - (2N - 2\sqrt{N})^{\frac{1}{2}} \right]^2 \end{aligned}$$

From this and $A_n \leq 4n^2$ follows that

$$2n \geq (2N - 2\sqrt{N})^{\frac{1}{2}} - (2N - \sqrt{2}N)^{\frac{1}{2}}. \quad (3.16)$$

So the number of steps of search has lower bound $\frac{c}{2} \sqrt{N}$ where c is approximately equal to $\sqrt{2} - \sqrt{2 - \sqrt{2}}$.

Hence any quantum search algorithm must have at least $\Omega(\sqrt{N})$ steps.

One thing that should be mentioned before we move to another algorithm is that matrix D can be represented as product of three matrices $D = WRW$ [2], where R is the phase rotation matrix and W is so called Walsh-Hadamard Transform Matrix, which are defined as follows

$$\begin{aligned} R_{ij} &= 0 \text{ if } i \neq j, R_{ii} = -1 \text{ if } i \neq 0, R_{00} = 1 \\ W_{ij} &= 2^{-\frac{n}{2}} (-1)^{i \cdot j}. \end{aligned} \quad (3.17)$$

Here i and j are binary representation of numbers i and j , which are lying in $\{0, 1, \dots, N - 1\}$, $i \cdot j$ is a bitwise product [2]. This expression of D is important for the implementation of the Grover's algorithm in quantum computers.

3.2 Quantum walk search algorithm on a hypercube

Quantum walk search algorithm on a hypercube is based on a discrete time quantum walk. Before we introduce the algorithm itself, we at first describe hypercube as a graph, its Hilbert space and unperturbed quantum walk. The hypercube is n -dimensional cube with edge-length 1. It has $N = 2^n$ vertices, which we label with an n -bit binary strings corresponding to their coordinates. Two vertices a and b of hypercube are connected, if their bit strings a and b differ only by a single bit flip. Hence, every vertex have degree equal to n . Hilbert space of a quantum walk on a hypercube is $\mathcal{H} = \mathcal{H}^n \otimes \mathcal{H}^{2^n}$. We denote the states from this Hilbert space as $|d, a\rangle$, where a is a bit string of underlying vertex

and d labels the direction on a hypercube. We define the operator $\hat{S}$ to shift the state $|d, \mathbf{a}\rangle$ to the state $|d, \mathbf{a}_d\rangle$, where $\mathbf{a}_d$ is bit string $\mathbf{a}$ with bit flip on d^{th} place

$$\hat{S} = \sum_{d=1}^n \sum_{\mathbf{a}} |d, \mathbf{a}_d\rangle\langle d, \mathbf{a}|. \quad (3.18)$$

For unperturbed walk we need to denote also the coin operator $\hat{C}_0$ on the coin space. We choose so called Grover diffusion operator

$$\hat{C}_0 = \hat{G} = -\hat{I} + 2|s^C\rangle\langle s^C|, \text{ where } |s^C\rangle = \frac{1}{\sqrt{n}} \sum_{d=1}^n |d\rangle. \quad (3.19)$$

Then the evolution operator of the walk is $\hat{U} = \hat{S}(\hat{C}_0 \otimes \hat{I})$. However, in the quantum walk search algorithm we have some marked state that we need to treat differently than others to produce higher amplitude of probability in the state at the final measurement. At each step we will use different coin on the marked state, denoted by $\hat{C}_m$, and $\hat{C}_0$ will be used on the other states. Coin $\hat{C}_m$ can be chosen simply equal to $-\hat{I}$. Thanks to high symmetry of the hypercube we can assume that the marked state is $|\mathbf{0}\rangle$, where $\mathbf{0}$ is a string consisting only from zeros [4]. Coin operator acting on the whole space have the form

$$\hat{C}' = \hat{C}_0 \otimes \hat{I} + (\hat{C}_m - \hat{C}_0) \otimes |\mathbf{0}\rangle\langle \mathbf{0}| = \hat{C}_0 \otimes \hat{I} - 2|s^C\rangle\langle s^C| \otimes |\mathbf{0}\rangle\langle \mathbf{0}|. \quad (3.20)$$

Shift operator is the same as in the unperturbed walk, so the perturbed operator $\hat{U}'$ is defined as follows

$$\hat{U}' = \hat{S}\hat{C}' = \hat{S}(\hat{C}_0 \otimes \hat{I}) - 2\hat{S}(|s^C\rangle\langle s^C| \otimes |\mathbf{0}\rangle\langle \mathbf{0}|) = \hat{U} - 2\hat{S}(|s^C\rangle\langle s^C| \otimes |\mathbf{0}\rangle\langle \mathbf{0}|). \quad (3.21)$$

Algorithm itself is composed of the following steps:

1. Initialize the system in the superposition of basis states

$$|\varphi_0\rangle = \sum_{d=1}^n \sum_{a=0}^{2^n-1} \frac{1}{\sqrt{n2^n}} |d, \mathbf{a}\rangle, \quad (3.22)$$

where $\mathbf{a}$ is binary representation of a .

2. Apply unitary operator $\hat{U}'$ $t_f = \left\lfloor \frac{\pi}{2} \sqrt{2^n} \right\rfloor$ times.
3. Measure the resulting state [4].

We will show that the marked state have probability $\frac{1}{2} - O\left(\frac{1}{n}\right)$ to be measured at the end of the algorithm. To simplify the problem we first show that the perturbed walk on the hypercube can be collapsed to the walk on the line. Then we introduce two approximate eigenvectors of $\hat{U}'$, initial state $|\varphi_0\rangle$ and $|\varphi_1\rangle$, which both have high overlap with subspace of two eigenvalues of $\hat{U}'$. We denote these eigenvalues by $e^{i\omega}$ and the state $e^{-i\omega}$ and their eigenvectors by $|\omega\rangle$ and $|- \omega\rangle$. We approximate these vectors by linear combination of $|\varphi_0\rangle$ and $|\varphi_1\rangle$. After that we show that application of $\hat{U}'$ is approximation of rotation in subspace spanned by $|\varphi_0\rangle$ and $|\varphi_1\rangle$ with rotation angle approximately equal to $\frac{1}{\sqrt{2^{n-1}}}$. Since we show that the state $|\varphi_1\rangle$ is well approximation of the marked state, the algorithm will finish after $\left\lfloor \frac{\pi}{2} \sqrt{2^n} \right\rfloor$ steps.

Let us define the permutation operator $\hat{P}_{ij}$ as follows

$$\hat{P}_{ij}|d, (a_1, a_2, \dots, a_i, \dots, a_j, \dots, a_n)\rangle = |d_P, (a_1, a_2, \dots, a_j, \dots, a_i, \dots, a_n)\rangle \quad (3.23)$$

where $d_P = d$ for $d \notin \{i, j\}$, $d_P = i$ for $d = j$, $d_P = j$ for $d = i$.

Because the unperturbed evolution operator $\hat{U}$ treats all directions in the same way the operator $\hat{P}_{ij}$ commutes with $\hat{U}$. It is easy to show that $\hat{P}_{ij}$ also commutes with perturbed evolution operator $\hat{U}'$ [4]. From the expression (3.22) of the initial state we see that $|\varphi_0\rangle$ is an eigenvector of $\hat{P}_{ij}$ with eigenvalue 1. From this it follows that $|\varphi_t\rangle = (\hat{U}')^t|\varphi_0\rangle$ is also an eigenvector of $\hat{P}_{ij}$ with eigenvalue 1 for any $t \in \mathbb{N}$ i.e. $(\hat{U}')^t$ preserves the symmetry of initial state with respect to bit swaps. Because of that all intermediate states of algorithm are placed in subspace spanned by eigenvectors of eigenvalue 1 of $\hat{P}_{ij}$. Before we define new basis, we will define function called Hamming weight which assigns to every bit string number of ones in the string. Hamming weight of string $\mathbf{a}$ is denoted by $|\mathbf{a}|$. This function will simplify subsequent notation.

Now we define subspace basis $\{|R, 0\rangle, |L, 1\rangle, |R, 1\rangle, \dots, |R, n-1\rangle, |L, n\rangle\}$ as follows

$$|L, \mathbf{a}\rangle = \sqrt{\frac{1}{a \binom{n}{a}}} \sum_{|\mathbf{a}|=\mathbf{a}} \sum_{a_d=1} |d, \mathbf{a}\rangle \quad (3.24)$$

$$|R, \mathbf{a}\rangle = \sqrt{\frac{1}{(n-a) \binom{n}{a}}} \sum_{|\mathbf{a}|=\mathbf{a}} \sum_{a_d=0} |d, \mathbf{a}\rangle. \quad (3.25)$$

Using these states we can reduce the quantum walk on the hypercube to the quantum walk on the line [4]. Marked state corresponds to $|R, 0\rangle$. Shift operator has the form

$$\hat{S} = \sum_{a=0}^{n-1} |R, \mathbf{a}\rangle \langle L, \mathbf{a}+1| + |L, \mathbf{a}+1\rangle \langle R, \mathbf{a}| \quad (3.26)$$

and the unperturbed coin has the form

$$\hat{C} = \sum_{a=0}^n \begin{pmatrix} \cos(\omega_a) & \sin(\omega_a) \\ \sin(\omega_a) & -\cos(\omega_a) \end{pmatrix} \otimes |\mathbf{a}\rangle \langle \mathbf{a}|, \quad (3.27)$$

where $\cos(\omega_a) = 1 - \frac{2a}{n}$ and $\sin(\omega_a) = \frac{2}{n} \sqrt{a(n-a)}$ and where first part acts on the space spanned by $\{|R\rangle, |L\rangle\}$ and second part acts on position space $\{|0\rangle, |1\rangle, \dots, |n\rangle\}$ [4]. The unperturbed evolution operator $\hat{U}$ on subspace acts as

$$\begin{aligned} \hat{U} = & \sum_{a=0}^{n-1} |R, \mathbf{a}\rangle (-\cos(\omega_{a+1}) \langle L, \mathbf{a}+1| + \sin(\omega_{a+1}) \langle R, \mathbf{a}+1|) + \\ & \sum_{a=1}^n |L, \mathbf{a}\rangle (\sin(\omega_{a-1}) \langle L, \mathbf{a}-1| + \cos(\omega_{a-1}) \langle R, \mathbf{a}-1|) \end{aligned} \quad (3.28)$$

and perturbed operator $\hat{U}'$ acts as

$$\hat{U}' = \hat{U} + \Delta\hat{U} = \hat{U} - 2|L, 1\rangle \langle R, 0|. \quad (3.29)$$

The initial state $|\varphi_0\rangle$ expressed in the new basis have the form

$$|\varphi_0\rangle = \frac{1}{\sqrt{2^n}}|R, 0\rangle + \frac{1}{\sqrt{2^n}}|L, n\rangle + \sum_{a=1}^{n-1} \left(\sqrt{\frac{\binom{n-1}{a}}{2^n}}|R, a\rangle + \sqrt{\frac{\binom{n-1}{a-1}}{2^n}}|L, a\rangle \right). \quad (3.30)$$

Let us defined arc $\mathcal{A}$ of complex numbers with absolute value 1 as follows

$$\mathcal{A} = \left\{ z \mid \operatorname{Re}(z) > 1 - \frac{2}{3n}, |z| = 1 \right\}. \quad (3.31)$$

It can be shown that on this arc lie exactly two eigenvalues of $\hat{U}'$ [4]. They must be a complex conjugate pair. Let us denote them by $e^{i\omega}$ and $e^{-i\omega}$. Corresponding eigenvectors $|\omega\rangle$ and $|- \omega\rangle$ comply $|\omega\rangle = |- \omega\rangle^*$. Let us define the state $|\varphi_1\rangle$ as follows

$$|\varphi_1\rangle = \frac{1}{c} \left(\sum_{a=0}^{\lfloor \frac{n}{2}-1 \rfloor} \frac{1}{\sqrt{2\binom{n-1}{a}}} |R, a\rangle - \frac{1}{\sqrt{2\binom{n-1}{a}}} |L, a+1\rangle \right) \quad (3.32)$$

where c is a normalization constant

$$c = \sqrt{\sum_{a=0}^{\lfloor \frac{n}{2}-1 \rfloor} \frac{1}{\binom{n-1}{a}}}. \quad (3.33)$$

If the system is in the state $|\varphi_1\rangle$ we will measure with high probability p the state $|R, 0\rangle$ which we identify with the marked state. Because for sufficiently large n hold $c \leq 1 + \frac{2}{n}$ we can estimate the probability p

$$p = |\langle R, 0 | \varphi_1 \rangle|^2 = \frac{1}{2c^2} \geq \frac{1}{2\left(1 + \frac{2}{n}\right)^2} = \frac{1}{2} - O\left(\frac{1}{n}\right). \quad (3.34)$$

It can be shown that states $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are approximately eigenvectors of $\hat{U}'$ with eigenvalue 1 [4]. Furthermore, it can be proven that eigenvectors $|\omega\rangle$ and $|- \omega\rangle$ can be written as a linear combination of states $|\varphi_0\rangle$ and $|\varphi_1\rangle$ as follows

$$\begin{aligned} |\omega\rangle &= \sqrt{p_0}|\varphi_0\rangle + \sqrt{p_1}e^{i\eta}|\varphi_1\rangle + \sqrt{1-p_0-p_1}|r_0\rangle \\ |- \omega\rangle &= \sqrt{p_0}|\varphi_0\rangle + \sqrt{p_1}e^{-i\eta}|\varphi_1\rangle + \sqrt{1-p_0-p_1}|r_0\rangle^* \end{aligned} \quad (3.35)$$

where $|r_0\rangle$ is normalized vector orthogonal to $|\varphi_0\rangle$ and $|\varphi_1\rangle$. Also p_0 and p_1 obey $\frac{1}{2} \geq p_0 \geq \frac{1}{2} - \frac{3n}{2^{n+1}}$ and $\frac{1}{2} \geq p_1 \geq \frac{1}{2} - \frac{3n}{8c^2\binom{n-1}{n/2}}$ and $e^{i\eta} = i + \Delta$, where $|\Delta| = O\left(\frac{n}{\binom{n-1}{n/2}}\right)$ [4]. From (3.35) we express initial state as

$$|\varphi_0\rangle = \sqrt{p_0}(|\omega\rangle + |- \omega\rangle) + \delta|r\rangle. \quad (3.36)$$

where $\delta = \sqrt{1-2p_0} = O\left(\sqrt{\frac{n}{2^n}}\right)$ and $|r\rangle$ is a normalized vector orthogonal to $|\omega\rangle$ and $|- \omega\rangle$.

Now we apply t times step operator $\hat{U}'$ on the initial state

$$\begin{aligned} (\hat{U}')^t |\varphi_0\rangle &= \sqrt{p_0}(e^{i\omega t}|\omega\rangle + e^{-i\omega t}|- \omega\rangle) + \delta|r'\rangle = \\ &= 2p_0 \cos \omega t |\varphi_0\rangle - 2\sqrt{p_0 p_1}(\sin \omega t + \operatorname{Re}(e^{-i\omega t} \Delta))|\varphi_1\rangle + \\ &+ \sqrt{1-p_0-p_1}(e^{i\omega t}|r_0\rangle + e^{-i\omega t}|r_0\rangle^*) + \delta|r'\rangle = \\ &= \cos \omega t |\varphi_0\rangle - \sin \omega t |\varphi_1\rangle + O\left(\frac{n^{3/4}}{\sqrt{2^n}}\right)|\tilde{r}\rangle \end{aligned} \quad (3.37)$$

where $|\tilde{r}\rangle$ is some normalized residual vector [4]. From (3.37) it can be seen that after applying operator $\hat{U}'$ $t_f = \frac{\pi}{2|\omega|}$ times the state of the system will approximately rotate from $|\varphi_0\rangle$ to $|\varphi_1\rangle$. Thanks to (3.34) we see that we find the marked state with probability $\frac{1}{2} - O\left(\frac{1}{n}\right)$. To estimate time in relation to dimension of hypercube n we need to estimate ω . It can be shown [4] that

$$\left| \omega + \frac{1}{c\sqrt{2^{n-1}}} \right| \leq O\left(\frac{n^{\frac{2}{3}}}{2^n}\right). \quad (3.38)$$

From (3.38) we can approximate time $t_f = \frac{\pi}{2} \sqrt{2^{n-1}} \left(1 + O\left(\frac{1}{n}\right)\right)$. So after number of steps equal to the closest integer to $t_f = \frac{\pi}{2} \sqrt{2^{n-1}}$ we will measure the marked state with probability $p = \frac{1}{2} - O\left(\frac{1}{n}\right)$. We see the example of evolution of the probability to measure the marked state in figure 3.2.

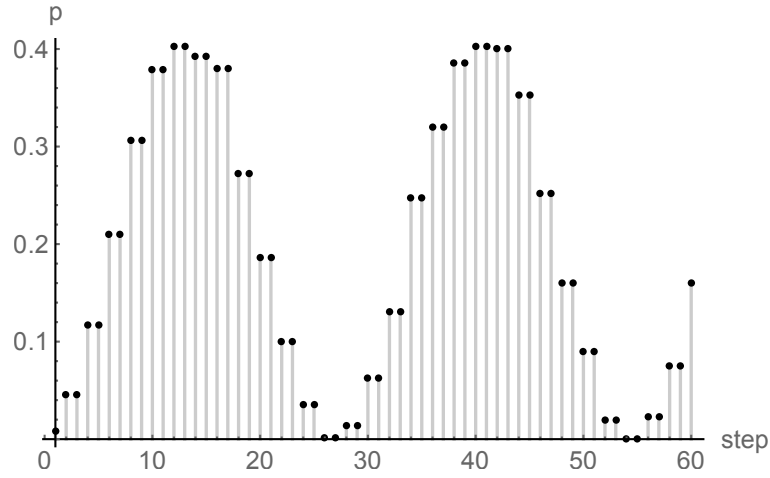


Figure 3.2: The evolution of the probability p to find the marked state during 60 steps. We use the quantum walk algorithm on the hypercube with dimension 7.

Now it is interesting to compare the properties of quantum walk algorithm with the Grover's search algorithm. They have some similarities. They both start with initial state equal to superposition of all states and both change in every step the phase of the marked state by π . Quantum walk algorithm takes $O(\sqrt{2^{n-1}})$ steps and $2^n = N$ is number of vertices, so both algorithm have the same running time of $O(\sqrt{N})$.

There are also some differences. First of them is that the Grover's algorithm can be presented as rotation in two dimensional subspace instead of quantum walk where it is just an approximation of rotation [4]. Also the two dimensional subspace in the Grover's algorithm is spanned by initial state and marked state whereas in the walk algorithm the second state has just a large overlap with marked state. Another difference is in the use of the diffusion operator (coin in the case of the walk algorithm). In the case of the Grover's algorithm it is used on the whole 2^n dimensional space. Quantum walk algorithm uses diffusion operator only on n dimensional space. This property can have some practical use in certain physical implementations of quantum computer [4].

3.3 Quantum walk search algorithm on regular lattice

Quantum walk search algorithm on regular lattice is similar to the walk algorithm on the hypercube. Hence we will just briefly describe the algorithm. We will introduce the regular lattice as the graph and

the similar operators we have introduced in search on the hypercube. Then we define the algorithm and mentioned its properties.

The regular lattice is an n -dimensional grid with dimension n and N vertices arranged as $\sqrt[n]{N} \times \sqrt[n]{N} \times \dots \times \sqrt[n]{N}$. We can identify each vertex with vector $|a_1, a_2, \dots, a_n\rangle$ where $a_i \in \{1, 2, \dots, \sqrt[n]{N}\}$. We add the coin space spanned by $2n$ directions in the regular lattice $\mathcal{H}^{2n}$. Then the final Hilbert space of the walk is $\mathcal{H} = \mathcal{H}^{2n} \otimes \mathcal{H}^N$. We use the same unperturbed coin operator $\hat{C}_0$ as in (3.19). Let us denote the marked vertex by $|m\rangle$. Then coin operator of the perturbed walk has a form similar to (3.20) except there is projector $|m\rangle\langle m|$ instead of $|0\rangle\langle 0|$. Now we define two shift operators $\hat{S}_{ff}$ and $\hat{S}$ in the case of two dimensional lattice.

Let us assume that there is an orthogonal basis of the coin space $\{|\rightarrow\rangle, |\leftarrow\rangle, |\uparrow\rangle, |\downarrow\rangle\}$ which describe the directions. Then $\hat{S}_{ff}$ is defined as follows

$$\begin{aligned}\hat{S}_{ff}|\rightarrow\rangle \otimes |x, y\rangle &= |\leftarrow\rangle \otimes |x+1, y\rangle \\ \hat{S}_{ff}|\leftarrow\rangle \otimes |x, y\rangle &= |\rightarrow\rangle \otimes |x-1, y\rangle \\ \hat{S}_{ff}|\uparrow\rangle \otimes |x, y\rangle &= |\downarrow\rangle \otimes |x, y+1\rangle \\ \hat{S}_{ff}|\downarrow\rangle \otimes |x, y\rangle &= |\uparrow\rangle \otimes |x, y-1\rangle\end{aligned}\tag{3.39}$$

where $x, y \in \{1, \dots, \sqrt[n]{N}\}$. We assume that periodic boundary conditions hold true, i.e. if any x or y is equal to $\sqrt[n]{N}$ and after application of the shift operator it can exceed $\sqrt[n]{N}$, the corresponding coordinate will be shifted to 1 instead, for x or y equal to 1 and shift down it will be shifted to $\sqrt[n]{N}$. We call (3.39) "flip-flop" shift operator, because after the shift the coin changes the direction. The "flip-flop" operator can be defined also in higher dimension obeying the same flipping rule as in two dimension. Operator $\hat{S}$ is defined to not change the directions as follows

$$\begin{aligned}\hat{S}|\rightarrow\rangle \otimes |x, y\rangle &= |\rightarrow\rangle \otimes |x+1, y\rangle \\ \hat{S}|\leftarrow\rangle \otimes |x, y\rangle &= |\leftarrow\rangle \otimes |x-1, y\rangle \\ \hat{S}|\uparrow\rangle \otimes |x, y\rangle &= |\uparrow\rangle \otimes |x, y+1\rangle \\ \hat{S}|\downarrow\rangle \otimes |x, y\rangle &= |\downarrow\rangle \otimes |x, y-1\rangle\end{aligned}\tag{3.40}$$

$\hat{S}$ is a classical shift operator and it can be also expanded to any dimension of regular lattice.

Now we define the unperturbed step operator $\hat{U} = \hat{S}_{ff}(\hat{C}_0 \otimes \hat{I})$ and the perturbed step operator $\hat{U}'$ as follows

$$\hat{U}' = \hat{S}_{ff}\hat{C}' = \hat{U} - 2\hat{S}_{ff}(|s^C\rangle\langle s^C| \otimes |m\rangle\langle m|).\tag{3.41}$$

Then the algorithm is consisting of the following steps:

1. Initialize the system in the superposition of the states

$$|\varphi_0\rangle = \sum_{d=1}^n \sum_{a=1}^N \frac{1}{\sqrt{nN}} |d, a\rangle,\tag{3.42}$$

where a label vertices.

2. Apply unitary operator $\hat{U}'$ T -times.
3. Measure the resulting state.

If the dimension of the lattice is equal or higher then three the number of steps T is required $O(\sqrt{N})$. Then the probability to measure the marked state is $\Theta(1)$, i.e. it is constant [5]. In case of two dimensional lattice after $T = O(\sqrt{N \log(N)})$ probability p to measure the marked state is $p = O\left(\frac{1}{\sqrt{\log(N)}}\right)$ [5]. See example of the search on the two dimensional lattice in figure 3.3. We can use amplitude amplification to increase the probability of success, then the algorithm with complexity $O(\sqrt{N \log(N)})$ will find the marked state with constant probability [5]. It can be also shown that if we use $\hat{S}$ instead of $\hat{S}_{ff}$ algorithm will take at least $\Omega(N)$ steps [5].

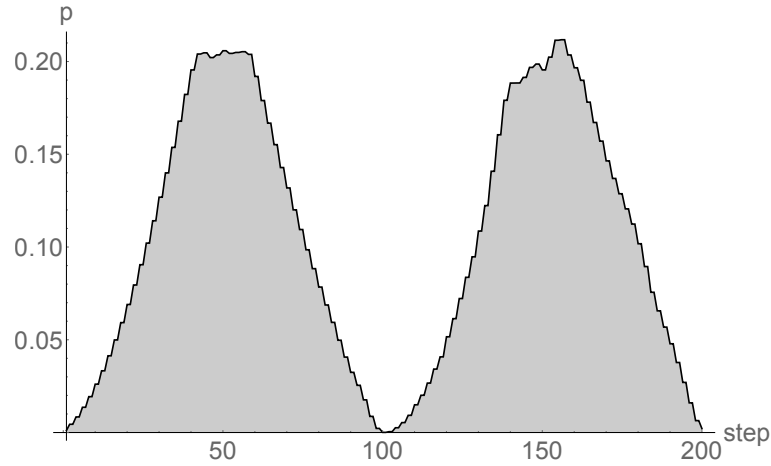


Figure 3.3: The evolution of the probability p to find the marked state during 200 steps. We use the quantum walk algorithm without amplitude amplification on the two dimensional regular lattice with 30 times 30 vertices.

To show that the algorithm works we repeat the same process as in the search on the hypercube. It can be shown that there are two eigenvectors of $\hat{U}'$, denoted by $|\omega\rangle$ and $|- \omega\rangle$, with eigenvalues $e^{i\omega}$ and $e^{-i\omega}$. Then the initial state and the marked state can be approximately written as linear combinations of $|\omega\rangle$ and $|- \omega\rangle$. Application of $\hat{U}'$ is approximately rotation in the subspace spanned by the initial state and the marked state. The number of steps is derived from the angle of rotation which depends on the dimension of the lattice [5].

Before we move to another chapter there is interesting property of regular lattice worth mentioning. In case of the two dimensional lattice there can be used only two dimensional coin instead of four dimensional. Let $\{|0\rangle, |1\rangle\}$ be the standard qubit basis. Then we can choose directions of the coin $|\uparrow\rangle = |0\rangle$, $|\downarrow\rangle = |1\rangle$, $|\leftarrow\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|\rightarrow\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$. If there is no marked state, $\hat{U}$ acts like this

$$\begin{aligned}\hat{U}|\rightarrow\rangle \otimes |x, y\rangle &= |\rightarrow\rangle \otimes |x+1, y\rangle \\ \hat{U}|\leftarrow\rangle \otimes |x, y\rangle &= |\leftarrow\rangle \otimes |x-1, y\rangle \\ \hat{U}|\uparrow\rangle \otimes |x, y\rangle &= |\uparrow\rangle \otimes |x, y+1\rangle \\ \hat{U}|\downarrow\rangle \otimes |x, y\rangle &= |\downarrow\rangle \otimes |x, y-1\rangle\end{aligned}\tag{3.43}$$

If there is a marked state $|m\rangle$, $\hat{U}'$ has form

$$\hat{U}' = \hat{U} (\hat{I} - 2|s, m\rangle\langle s, m|)\tag{3.44}$$

where $|s\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$. It can be shown that the algorithm based on this choice needs the same number of steps $T = O(\sqrt{N} \log(N))$ to succeed [5].

Chapter 4

Different look on the walk search algorithm

In this Chapter we return to the quantum walk search algorithm and estimate the number of steps to succeed from the different perspective. In Section 4.1 we do so for the search algorithm on the hypercube and in Section 4.2 for the search on the regular lattice.

4.1 Search on the hypercube

In this chapter we examine different approach to estimating the number of steps in the search algorithm. We introduce the walk operator $\hat{U}_\lambda$ and show how the number of steps can be derived from the properties of the spectrum of $\hat{U}_\lambda$. We use the definition (3.18) and (3.19). Let us now introduce the relevant eigenvalues of $\hat{U} = \hat{S}(\hat{C}_0 \otimes \hat{I})$. There are following eigenvalues

$$v_k^\pm = e^{\pm i\varphi_k} = 1 - \frac{2k}{n} \pm \frac{2i}{n} \sqrt{k(n-k)} \quad (4.1)$$

in the spectrum of $\hat{U}$ with corresponding eigenvectors

$$\begin{aligned} |v_k^\pm\rangle &= \beta_k \sum_a (-1)^{k \cdot a} \frac{2^{-\frac{n}{2}}}{\sqrt{2}} \sum_{d=1}^n \alpha_{kd}^\pm |d, a\rangle \\ \text{where } \alpha_{kd}^\pm &= \begin{cases} \frac{1}{\sqrt{k}} & \text{if } k_d = 1 \\ \frac{\pm i}{\sqrt{n-k}} & \text{if } k_d = 0 \end{cases} \\ \text{and } \beta_k &= \begin{cases} \sqrt{2} & \text{if } k = 0 \text{ or } k = n \\ 1 & \text{else} \end{cases} . \end{aligned} \quad (4.2)$$

Other eigenvectors of $\hat{U}$ correspond to eigenvalues ± 1 and they are not affected by the perturbation, hence they have no influence on the properties of the algorithm. We denote the marked vertex by $|m\rangle$. From the fact that the coin state $|s^C\rangle$ is eigenvector of $\hat{C}_0$ with eigenvalue 1, i.e. $\hat{C}_0 |s^C\rangle = |s^C\rangle$, we can express $\hat{U}'$ from (3.21) in the form

$$\begin{aligned} \hat{U}' &= \hat{U} - 2\hat{S}(|sm\rangle\langle sm|) = \hat{U} - 2\hat{S}\hat{C}(|sm\rangle\langle sm|) = \\ &= \hat{U}(\hat{I} - 2|sm\rangle\langle sm|). \end{aligned} \quad (4.3)$$

We write $|s^C\rangle \otimes |m\rangle = |sm\rangle$ for simpler notation. Now let us define a family of operators $\hat{U}_\lambda$ as follows

$$\hat{U}_\lambda = \hat{U} \left[\hat{I} + (e^{i\lambda\pi} - 1) |sm\rangle\langle sm| \right]. \quad (4.4)$$

From (4.4) it is easy to see that $\hat{U}_0 = \hat{U}$ and $\hat{U}_1 = \hat{U}'$. Operator $\hat{U}_\lambda$ is also periodic with period 2. If we apply $\hat{U}_\lambda$ on (4.2) we see that the eigenstates orthogonal to $|sm\rangle$ do not change. Hence we can restrict us only to eigenstates not orthogonal to $|sm\rangle$ and from each degenerate eigenspace we choose only one vector. After normalization this vector has the form

$$|\varphi_k^\pm\rangle = \frac{1}{\sqrt{\binom{n}{k}}} \sum_{|j|=k} (-1)^{j \cdot m} |v_j^\pm\rangle. \quad (4.5)$$

Selection of these eigenstate depends on the position of the marked state. Because it can be shown that $\hat{U}_\lambda$ commutes with (3.23)[6], i.e. it preserves high symmetry of the hypercube, we can again choose that the marked state is $|0\rangle$. Then (4.5) has the simpler form

$$|\varphi_k^\pm\rangle = \frac{1}{\sqrt{\binom{n}{k}}} \sum_{|j|=k} |v_j^\pm\rangle. \quad (4.6)$$

These states span the $2n$ dimensional subspace $\mathcal{H}'$. The marked state $|sm\rangle$ is in $\mathcal{H}'$ and the Hilbert space $\mathcal{H}'$ also complies $\hat{U}_\lambda(\mathcal{H}') = \mathcal{H}'$. Again to simplify notation we change index k and $\pm$ for new index l such that $l \in \{-n+1, \dots, n\}$ and $\{k, \pm\} = \{|l|, \text{sgn}(l)\}$. Now we write $e^{\pm i\varphi_k} = e^{i\varphi_l}$ and $|\varphi_k^\pm\rangle = |\varphi_l\rangle$. Thus we can restrict us on Hilbert space $\mathcal{H}'$ with basis $\{|\varphi_l\rangle | l \in \{-n+1, \dots, n\}\}$.

Let us describe the properties of the spectrum of $\hat{U}_\lambda$. The eigenphases φ_l of the unperturbed walk from (4.1) stay largely unchanged with changing λ . Moreover, there are eigenstates $|\rho_\lambda\rangle$ with eigenphase approximately parallel to the line $\frac{\pi\lambda}{2}$ [6]. An avoided crossing can be found at points where eigenphase related to φ_l get near to the eigenphase of $|\rho_\lambda\rangle$. See figure 4.1 for example of eigenphase of $\hat{U}_\lambda$ as a function of λ . It can be shown that the application of $\hat{U}_\lambda$ on $\mathcal{H}'$ near the avoided crossing can be approximately described as evolution in two dimensional subspace, i.e. the other states in $\mathcal{H}'$ are far less influenced. One of the vectors that span the system at l^{th} -crossing is $|\varphi_l\rangle$. It is easy to show that the state $|\varphi_l\rangle$ is approximate eigenvector of (4.4). Let us apply operator (4.4) on the state $|\varphi_l\rangle$ and we get

$$\hat{U}_\lambda |\varphi_l\rangle = e^{i\varphi_l} |\varphi_l\rangle + 2^{-\frac{n+1}{2}} (e^{i\lambda\pi} - 1) \beta_l \sqrt{\binom{n}{|l|}} e^{i\psi_l} \hat{U} |sm\rangle, \quad (4.7)$$

where $e^{i\psi_l} = e^{\pm i\varphi_k} = \frac{\sqrt{k\pm}\sqrt{n-k}}{\sqrt{n}}$ and β_l is given in (4.2). We see that if l is close to 0 or n $|\varphi_l\rangle$ is an approximate eigenstate of $\hat{U}_\lambda$ with exponentially small reminder.

Now let us define function $g(\lambda)$ as a eigenphase of $|\rho_\lambda\rangle$, i.e. $e^{ig(\lambda)}$ is eigenvalue of (4.4). We write $|\rho_\lambda\rangle$ as linear combination of basis states

$$|\rho_\lambda\rangle = 2^{-\frac{n+2}{2}} \sum_{l=-n+1}^n \beta_l \sqrt{\binom{n}{|l|}} e^{i\psi_l} a_l |\varphi_l\rangle \quad (4.8)$$

where coefficients a_l need to be specified yet. We can calculate the scalar product $\langle sm | \rho_\lambda \rangle = 2^{-n-\frac{3}{2}} b$ [6] where b is defined as follows

$$b = \sum_{l=-n+1}^n \beta_l^2 \binom{n}{|l|} a_l. \quad (4.9)$$

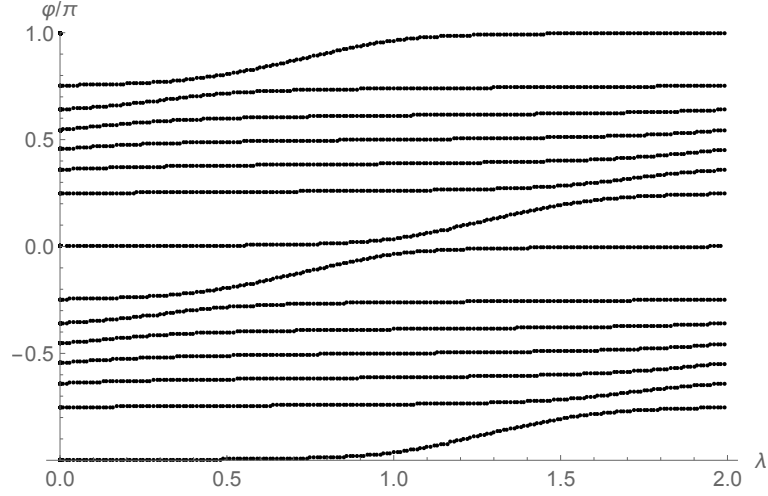


Figure 4.1: The eigenphase of the operator $\hat{U}_\lambda$ as a function of λ for the 7-dimensional hypercube.

Now we choose the coefficients a_l to accomplish $\hat{U}_\lambda|\rho_\lambda\rangle = e^{ig(\lambda)}|\rho_\lambda\rangle$. To do so we apply (4.4) on (4.8) and get

$$\begin{aligned}\hat{U}_\lambda|\rho_\lambda\rangle &= \hat{U}|\rho_\lambda\rangle + (e^{i\lambda\pi} - 1)\hat{U}|sm\rangle\langle sm|\rho_\lambda\rangle \\ &= e^{ig(\lambda)}|\rho_\lambda\rangle + 2^{-\frac{n+1}{2}} \sum_{l=-n+1}^n \beta_l \sqrt{\binom{n}{|l|}} e^{i\psi_l} |\varphi_l\rangle \\ &\quad \left[(-e^{ig(\lambda)} + e^{i\varphi_l}) a_l + (e^{i\lambda\pi} - 1) 2^{-n-1} e^{i\varphi_l} b \right].\end{aligned}\tag{4.10}$$

We see that if we set

$$a_l = -\frac{(e^{i\lambda\pi} - 1) 2^{-n-1} e^{i\varphi_l} b}{-e^{ig(\lambda)} + e^{i\varphi_l}}\tag{4.11}$$

the second term in (4.10) disappears and the state $|\rho_\lambda\rangle$ is eigenstate of $\hat{U}_\lambda$. However constant b also depends on coefficients a_l thus (4.11) and (4.9) represent a set of linear equations. We can express the condition for the equations to have solution by inserting (4.11) in (4.9). We get the expression

$$1 = \sum_{l=-n+1}^n \beta_l^2 \binom{n}{|l|} \frac{(e^{i\lambda\pi} - 1) 2^{-n-1} e^{i\varphi_l}}{e^{i\varphi_l} - e^{ig(\lambda)}}\tag{4.12}$$

which implicitly define $g(\lambda)$.

Now we construct the second vector $|\mu_l\rangle$ in the two dimensional subspace at the l^{th} -crossing by taking from $|\rho_{\lambda_l}\rangle$ only the part which is orthogonal to $|\varphi_l\rangle$. So $|\mu_l\rangle$ complies $|\mu_l\rangle = |\rho_{\lambda_l}\rangle - \langle\varphi_l|\rho_{\lambda_l}\rangle|\varphi_l\rangle$ where we denote λ at l^{th} -crossing by λ_l . From construction of $|\mu_l\rangle$ it follows that $|\mu_l\rangle$ is orthogonal to the state $|\varphi_l\rangle$. The state $|\mu_l\rangle$ expressed in basis $\{|\varphi_l\rangle\}_{l=-n+1}^n$ has the form

$$|\mu_l\rangle = b 2^{-\frac{3n+1}{2}} (1 - e^{i\lambda_l\pi}) \sum_{\substack{k=-n+1 \\ k \neq l}}^n \sqrt{\binom{n}{|k|}} \frac{e^{i(\varphi_k + \psi_k)} \beta_k}{e^{i\varphi_k} - e^{ig(\lambda_l)}} |\varphi_k\rangle\tag{4.13}$$

where b is normalization constant. It can be shown [6] that b is equal to

$$b = \sqrt{\frac{2g'(\lambda)}{\pi}} 2^{n+1}. \quad (4.14)$$

It is easy to see from (4.13) and (4.14) that the term which $|\mu_l\rangle$ differs from $|\rho_{\lambda_l}\rangle$ is exponentially small and thus $|\mu_l\rangle$ is approximately eigenvector of $\hat{U}_{\lambda_l}$. It can be shown that the two dimensional space spanned by the states $|\varphi_l\rangle$ and $|\mu_l\rangle$ has high overlap with true eigenstates $|\omega_l^\pm\rangle$ of $\hat{U}_{\lambda_l}$ at the avoided crossing. Hence we can write $|\varphi_l\rangle \approx \frac{1}{\sqrt{2}}(|\omega_l^+\rangle + |\omega_l^-\rangle)$ and $|\mu_l\rangle \approx \frac{i}{\sqrt{2}}(|\omega_l^+\rangle - |\omega_l^-\rangle)$ [6]. Then we apply $\hat{U}_{\lambda_l}$ on $|\varphi_l\rangle$ for t times and we get the following expression

$$(\hat{U}_{\lambda_l})^t |\varphi_l\rangle \approx \frac{1}{\sqrt{2}} \left(e^{it(\varphi_l + \frac{1}{2}\Delta_l)} |\omega_l^+\rangle + e^{it(\varphi_l - \frac{1}{2}\Delta_l)} |\omega_l^-\rangle \right) \quad (4.15)$$

where Δ_l is gap between two eigenphases at the l^{th} avoided crossing. We see that if we choose the number of steps T_l to comply $e^{iT_l\Delta_l\frac{1}{2}} = i$ we rotate the state $|\varphi_l\rangle$ in the state $|\mu_l\rangle$. In different words after $T_l = \lfloor \frac{\pi}{\Delta_l} \rfloor$ steps we get $(\hat{U}_{\lambda_l})^{T_l} |\varphi_l\rangle \approx ie^{iT_l\varphi_l} |\mu_l\rangle$. The gap between two eigenphases can be calculated for small differences as the gap between two eigenvalues. To estimate the spectral gap we express $\hat{U}_{\lambda_l}$ in the two dimensional subspace as 2×2 matrix W as follows

$$\begin{aligned} W &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ where} \\ A &= \langle \mu_l | U_{\lambda_l} | \mu_l \rangle = e^{ig(\lambda_l)} = e^{i\varphi_l} \\ B &= \langle \varphi_l | U_{\lambda_l} | \mu_l \rangle = 2^{-\frac{n}{2}-1} (e^{i\lambda_l\pi} - 1) e^{i(\varphi_l + \psi_l)} \beta_l \sqrt{\frac{n}{|l|}} \sqrt{\frac{2g'(\lambda_l)}{\pi}} \\ C &= \langle \mu_l | U_{\lambda_l} | \varphi_l \rangle = -2^{-\frac{n}{2}-1} (e^{-i\lambda_l\pi} - 1) e^{i(\varphi_l - \psi_l)} \beta_l \sqrt{\frac{n}{|l|}} \sqrt{\frac{2g'(\lambda_l)}{\pi}} \\ D &= \langle \varphi_l | U_{\lambda_l} | \varphi_l \rangle = e^{i\varphi_l} \left[1 + (e^{i\lambda_l\pi} - 1) 2^{-n-1} \beta_l^2 \binom{n}{|l|} \right] \end{aligned} \quad (4.16)$$

In (4.16) we use the fact that at the l^{th} -crossing we can set $g(\lambda_l) = \varphi_l$ with fulfilling the condition (4.12) up to exponentially small reminder [6]. The spectral gap of W complies

$$(\Delta_l)^2 = (D - A)^2 + 4BC \quad (4.17)$$

and after inserting (4.16) in (4.17) we obtain following approximation of $|\Delta_l|$

$$|\Delta_l| = |e^{i\lambda_l\pi} - 1| 2^{-\frac{n}{2}} \beta_l \binom{n}{|l|} \sqrt{\frac{2g'(\lambda_l)}{\pi}} + O\left(2^{-n} \binom{n}{|l|}\right). \quad (4.18)$$

Entering (4.18) in $T_l = \lfloor \frac{\pi}{\Delta_l} \rfloor$ the number of step that is necessary to rotate the state $|\varphi_l\rangle$ in the state $|\mu_l\rangle$ is

$$T_l = \left\lfloor \frac{\pi 2^{\frac{n}{2}} \sqrt{\frac{\pi}{2g'(\lambda_l)}}}{|e^{i\lambda_l\pi} - 1| \beta_l \sqrt{\frac{n}{|l|}}} \right\rfloor. \quad (4.19)$$

The probability to measure the marked state after T_l steps is

$$|\langle \mu_l | sm \rangle|^2 \approx 2^{-2n-3} b^2 \approx \frac{g'(\lambda_m)}{\pi}. \quad (4.20)$$

The search algorithm on the hypercube constructed in previous chapter corresponds to the avoided crossing at $\lambda = 1$ and $\varphi_0 = 0$, so called 0th-crossing. It can be shown that $g(0) = 1$ and

$$g'(1) = \frac{\pi}{2(1 + \gamma_n - 2^{-n+1})} [6]. \quad (4.21)$$

where $\gamma_n \sim \frac{1}{n}$ for large n

Using this we can estimate the number of steps in the algorithm

$$T = \left\lfloor \frac{\pi 2^{\frac{n}{2}} \sqrt{\frac{\pi}{2g'(1)}}}{2\sqrt{2}} \right\rfloor \approx \left\lfloor \frac{\pi}{4} N \sqrt{2 \left(1 + \frac{1}{n} + 2^{-n+1} \right)} \right\rfloor \approx \left\lfloor \frac{\pi \sqrt{2}}{4} N \right\rfloor \quad (4.22)$$

and the probability to measure the marked state is approximately equal to $\frac{1}{2}$. This matches our results in previous chapter.

From the relation between the number of steps and the eigenphase gap follows that the number of steps declines with the rising gap. So algorithm based on higher avoided crossing will be faster. But it was shown that after crossing particular l the algorithm fails because the probability to measure the marked state falls. Also the initial state of the algorithm at the 0th-crossing $|\varphi_0\rangle$ does not depend on choice of the marked state but on the other crossings the initial state depend on the position of the marked state. So before we start the algorithm based on the other crossings we need to find the initial state and there is no suitable method to do so that still preserve the improvement of the higher crossing.

4.2 Search on the regular lattice

We can perform the same process for estimation of the number of steps on regular lattice search as we did with the search on the hypercube. We define $\hat{U}_\lambda$ by (4.4) and restrict us on the subspace spanned by the spatial eigenvectors of $\hat{U}$ that are not orthogonal to the marked state. Again the avoided crossings in the spectrum of $\hat{U}_\lambda$ is found. The dynamic at the avoided crossing can be describe as rotation in two dimensional subspace spanned by two approximate eigenvectors. The number of steps is inversely proportional to the gap between the eigenphases at the avoided crossing $T = \frac{\pi}{\Delta}$. We focus only on 0th-crossing. It can be shown that the eigenphase gap is equal to

$$\Delta = \frac{4b}{\sqrt{N}} + O\left(\frac{1}{N}\right) \quad (4.23)$$

where b is defined as $\langle sm | \mu_0 \rangle = b$. The constant b is chosen to normalized the state $|\mu_0\rangle$. The value of b depends on the dimension as follows

$$b \approx \begin{cases} \frac{1}{\sqrt{\frac{2}{\pi} \ln(N) + \frac{8}{\pi^2} (2-K) + \frac{2}{\pi} \ln\left(\frac{8}{\pi^2}\right) + O\left(\frac{1}{N}\right)}} = \Theta\left(\frac{1}{\sqrt{\ln(N)}}\right) & \text{for } d = 2 \\ \Theta(1) & \text{for } d \geq 3 \end{cases} \quad (4.24)$$

where $K \approx 0,916$ is Catalan's constant and d is the dimension of the regular lattice. Then after T number of steps

$$T = \begin{cases} \left\lfloor \frac{\pi \sqrt{N}}{4} \sqrt{\frac{2}{\pi} \ln(N) + \frac{8}{\pi^2} (2-K) + \frac{2}{\pi} \ln\left(\frac{8}{\pi^2}\right)} \right\rfloor = O\left(\sqrt{N \ln(N)}\right) & \text{for } d = 2 \\ O(\sqrt{N}) & \text{for } d \geq 3 \end{cases} \quad (4.25)$$

we measure the marked state with probability $p = |\langle sm|\mu_0\rangle|^2 = b^2$

$$p = \begin{cases} \Theta\left(\frac{1}{\sqrt{\ln(N)}}\right) & \text{for } d = 2 \\ \Theta(1) & \text{for } d \geq 3 \end{cases} . \quad (4.26)$$

In the case of the two dimensional lattice it is necessary to use amplitude amplification so the algorithm will find the marked state with probability $\Theta(1)$. By doing so as we discuss in previous chapter algorithm need $O\left(\sqrt{N \ln(N)}\right)$ steps.

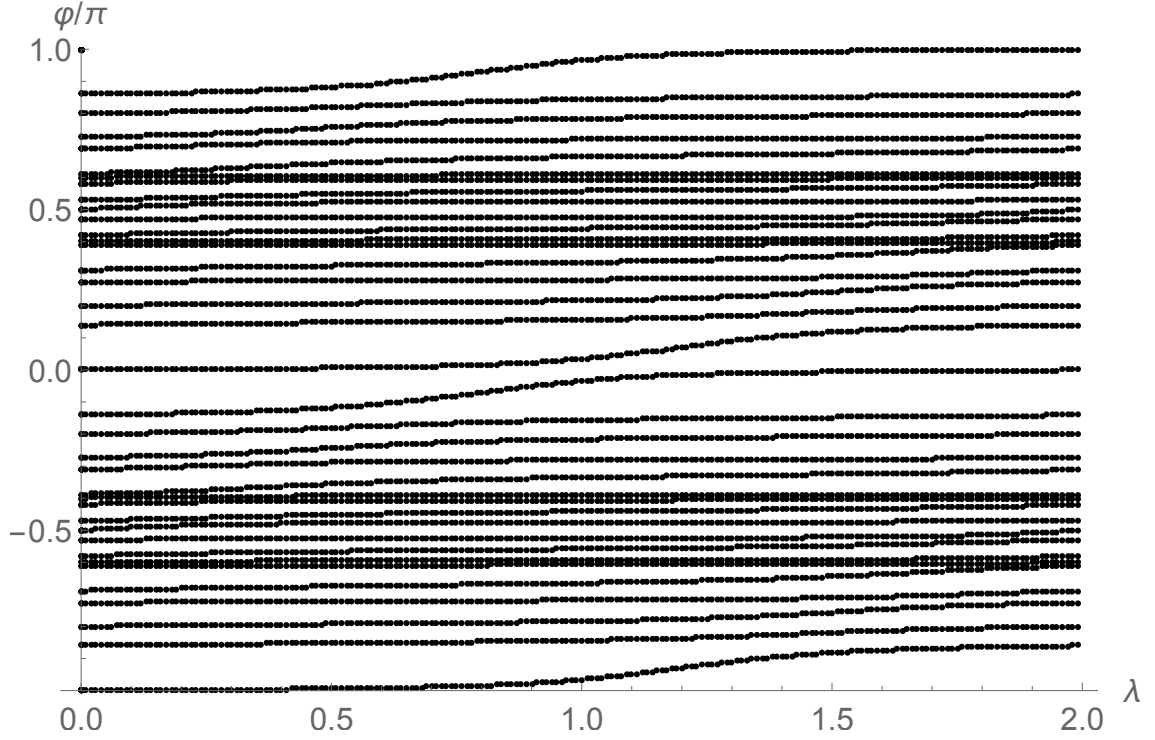


Figure 4.2: The eigenphase of the operator $\hat{U}_\lambda$ as a function of λ for the two dimensional regular lattice with 10 times 10 vertices.

Chapter 5

State transfer

In Section 2.1 we show how quantum discrete walk can be used for the state transfer. In Section 2.2 we use the properties of the quantum search on the regular lattice to communicate across the grid.

5.1 State transfer as the discrete time quantum walk

One of the many use of quantum walks is a state transfer, in particular we focus on the discrete quantum walk realization of the state transfer on a spin chain with position dependent couplings. First we describe in general the system we use and the quantum walk. Then we introduce connection between properties of walk and physical properties of the system. Finally we describe the way the transfer works.

In general the system is a network with every vertex corresponding to spin $\frac{1}{2}$. If there is an interaction between vertex i and j we say that these vertices are connected. We consider an XX model of spin-spin interaction with Hamiltonian of the network

$$\hat{H} = \sum_{\{i,j\} \in E} J_{ij} (\hat{\sigma}_x^i \hat{\sigma}_x^j + \hat{\sigma}_y^i \hat{\sigma}_y^j) \quad (5.1)$$

where E denotes the set of edges, σ_x^i is Pauli X matrix for the i^{th} spin and J_{ij} is strength of spin interaction between i and j , J_{ij} is often called spin coupling parameter [8]. Because (5.1) conserves the total spin number along Z axis $\sum_i \sigma_z^i$ we can restrict us to subspace with fixed number of excitation where by excitation it is meant a spin pointing up along the Z axis. In particular we focus on cases with zero or one excitation. The first case is invariant under the evolution generated by (5.1) and the second case can be consider as walking particle on an underlying network [8]. On the network we marked the two spins as a sender and a receiver. We prepare the sender spin in state $\alpha|\downarrow\rangle + \beta|\uparrow\rangle$ and rest of the network has spin down. The state of the network can be written in the form

$$(\alpha|\downarrow\rangle + \beta|\uparrow\rangle)_s \otimes |\downarrow \dots \downarrow\rangle_{net} |\downarrow\rangle_r. \quad (5.2)$$

We want to use the evolution generated by (5.1) to transfer the state of sender to receiver. So the final state of the system will have the form

$$|\downarrow\rangle_s \otimes |\downarrow \dots \downarrow\rangle_{net} (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)_r. \quad (5.3)$$

We use as the network the spin chain so the interaction occurs only between neighbouring spins.

In the transfer of the state we use the one dimensional discrete time quantum walk. The evolution operator $\hat{U} = \hat{S}\hat{C}$ has in general the form

$$\begin{aligned}\hat{U}|x, \uparrow\rangle &= \cos(\theta)|x+1, \uparrow\rangle - \sin(\theta)|x-1, \downarrow\rangle \\ \hat{U}|x, \downarrow\rangle &= \sin(\theta)|x+1, \uparrow\rangle + \cos(\theta)|x-1, \downarrow\rangle\end{aligned}\quad (5.4)$$

where θ denotes the coin flip rate. The equivalent notation of $\hat{U}$ is

$$\hat{U} = e^{i\hat{P}\hat{\sigma}_z} e^{i\hat{\sigma}_y\theta} \quad (5.5)$$

where $\hat{P}$ is the momentum operator and $\hat{\sigma}_j$ are Pauli matrices acting on the coin space [8]. We can use (5.5) to build the analogy between properties of the walk and physical properties of the system of relativistic particle in one dimension. We write the operator $\hat{U}$ in the form

$$\hat{U} = \hat{U}(t) = e^{i\hat{P}\hat{\sigma}_z vt} e^{i\hat{\sigma}_y \omega t} \quad (5.6)$$

where $vt = 1$ and $\omega t = \theta$. We can identify v with the velocity and ω with the angular frequency. Then if we take the infinitesimal steps and apply the Trotter theorem we get

$$\lim_{n \rightarrow \infty} \left(\hat{U} \left(\frac{t}{n} \right) \right)^n = e^{i(\hat{P}\hat{\sigma}_z v + \hat{\sigma}_y \omega)t} [8]. \quad (5.7)$$

This operator is generated by Hamiltonian, which has the form of one dimensional Dirac Hamiltonian

$$\hat{H} = c\hat{P}\hat{\sigma}_z + mc^2\hat{\sigma}_y \quad (5.8)$$

where we identify v with speed of light and ω with the mass of particle. The Dirac Hamiltonian describe the relativistic particle in one dimension. Because we take $t = 1$ we see that the mass of the particle correspond to the coin flip rate [8]. The operator (5.5) does not differentiate between θ and $\theta + 2\pi$ and in case of $m \rightarrow +\infty$ the flip rate of the coin corresponds to $\theta \rightarrow \frac{\pi}{2}$. Hence we can choose the relation $m = \text{tg}(\theta)$. Another analogy can be constructed using (5.1) to obtain relation between mass and the spin coupling parameter $J = \frac{1}{2m}$ [8].

Now let us introduce the state transfer itself. We work with the finite chains of spins so at the end of the chain we use reversing coin, i.e. coin with flip rate equal to $\theta = \frac{\pi}{2}$. For the transfer we use the engineered couplings protocol. This protocol is based on the properly chosen couplings parameter between each spins. The protocol sets the coupling parameter between spin n and $n+1$ to be $J_n = \frac{\lambda}{2} \sqrt{n(N-n)}$ where λ is real positive parameter common for all couplings [8]. Then the mass for the transfer between n and $n+1$ position is

$$m_n = \frac{1}{2J_n} = \frac{1}{\lambda \sqrt{n(N-n)}}. \quad (5.9)$$

The evolution operator have form

$$U(n) = e^{i\hat{P}\hat{\sigma}_z} e^{i\hat{\sigma}_y \theta_n} \quad (5.10)$$

where θ_n depends on position. In the case of one dimensional discrete quantum walk interference appears only between positions separated by the distance of two, hence we can study two independent walks on even and on odd positions [8]. We can treat even positions as edges and odd as vertices and use for the evolution operator $\hat{U}^2$ instead of $\hat{U}$. So we use N positions to simulated the chain with $\frac{N}{2}$ spins. Coins associated with edges positions correspond to mass and at the vertex position we can set the coin

flip rate $\theta_{2k-1} = 0$, i. e. coin is the identity operator. It can be shown that different choice of the coin on vertex positions produces the similar results [8]. On the edges positions the coin flip rate is derived from its relation with mass and has the form

$$\theta_{2l} = \arctan \left(\frac{1}{2\lambda \sqrt{k(\frac{N}{2} - k)}} \right) \quad (5.11)$$

where $k \in \{1, 2, \dots, \frac{N}{2} - 1\}$.

It was shown in [8] that the transfer strongly depends on the parameter λ . At first the transfer is perfect for small values of λ . This case is called large mass limit. Around $\lambda = \frac{\pi}{N}$ the transfer starts to fall and it is restored for $\lambda = O(1)$ and higher [8], this is called small mass limit. It can be shown that in case of the large mass limit $\hat{U}^2$ is generated by Hermitian operator whose action is the same as the action of the Hamiltonian introduced by Christandl [8]. So it can be shown that the state of the sender is fully transferred to the receiver.

In the case of small mass limit it follows that

$$\cos(\theta_{2k}) \approx 1 \text{ and } \sin(\theta_{2k}) \approx 0. \quad (5.12)$$

Thus the double step operator $\hat{U}^2$ is approximately equal to $e^{2i\hat{P}\hat{\sigma}_z}$. The transfer takes the $\frac{N}{2}$ steps. The initial state of the sender is $\alpha|1, \downarrow\rangle + \beta|1, \uparrow\rangle$. The first part moves left and it is reflected at position 0, then it moves to position $N - 1$, so it results in $\alpha|N - 1, \uparrow\rangle$. $\beta|1, \uparrow\rangle$ first moves to the position N where it is reflected and also its phase is changed by π , then it moves to the position $N - 1$ and results in $-\beta|N - 1, \downarrow\rangle$. After application of operator $\hat{\sigma}_y$ the state of the receiver is $\alpha|N - 1, \downarrow\rangle + \beta|N - 1, \uparrow\rangle$ and so we reach the full state transfer [8].

This shows how discrete time quantum walk can be used to transfer the state across one dimensional chain. The continuous time quantum walk can be also used to transfer the state. However the discrete walk has the additional degree of freedom, the coin, which gives us more possibilities to control the walk [8].

5.2 An amplitude transfer across regular lattice

The process used in the search algorithm on the regular lattice can be used to transfer amplitude across regular lattice. In this section we describe how we can achieve the transfer and its basic properties. First we need to expand the search algorithm for more marked states. Suppose we have l marked states $|m^1\rangle, \dots, |m^l\rangle$ for $l \ll N$. We define a family of operators analogous to (odkaz na U-lambda) with set of parameters $\lambda = (\lambda_1, \dots, \lambda_l)$ as follows

$$\hat{U}_\lambda = \hat{U} \left[1 + \sum_{j=1}^l (e^{i\pi\lambda_j} - 1) |sm^j\rangle\langle sm^j| \right]. \quad (5.13)$$

At $\lambda = (1, \dots, 1)$ around the eigenphase equal to 0 we can find $l - 1$ degenerate eigenvalues and two further values forming an avoided crossing with degenerate subset [9]. The subspace spanned by corresponding vectors has high overlap with space spanned by initial state $|\varphi_0\rangle$ and m approximate eigenvectors $|\rho_{\lambda=1}^1\rangle, \dots, |\rho_{\lambda=1}^l\rangle$. We denote $|\rho_{\lambda=1}^j\rangle = |\rho^j\rangle$ for simpler notation. The states $|\rho^j\rangle$ are orthogonal to $|\varphi_0\rangle$ and they comply $\langle \rho^j | \rho^k \rangle \approx \delta_{jk}$. As in the previous chapter the search algorithm can be constructed on the avoided crossing. It can be shown [9] that the number of steps necessary to find one of the marked

states is inversely proportional to the gap between non-degenerate levels at the avoided crossing. This gap is equal to $\Delta = 2\epsilon\sqrt{l}$ where

$$\epsilon = \frac{2|\langle sm^j|\rho^j\rangle|}{\sqrt{N}} + O\left(\frac{1}{N}\right) \quad (5.14)$$

In (5.14) $|\langle sm^j|\rho^j\rangle|$ does not depend on j . Finally (5.14) and $T \approx \frac{\pi}{\Delta}$ imply that the number of steps of the search algorithm with more marked states is $T = \left\lfloor \frac{\pi}{2} \sqrt{\frac{N}{m}} \right\rfloor$ for dimension higher than 3 and $T = \left\lfloor \frac{\pi}{2} \sqrt{\frac{N}{m \ln(N)}} \right\rfloor$ for dimension equal to 2.

Now we describe the transfer itself. Let us first assume that there exists Hamiltonian $\hat{H}$ which complies $\hat{U}_{\lambda=1} = e^{-i\hat{H}}$. In the basis $\{|\varphi_0\rangle, |\rho^1\rangle, |\rho^2\rangle, \dots, |\rho^l\rangle\}$ the Hamiltonian $\hat{H}$ can be expressed as $(l+1) \times (l+1)$ matrix as follows

$$H^{(l+1) \times (l+1)} = \begin{pmatrix} 0 & -i\epsilon & \dots & -i\epsilon \\ i\epsilon & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ i\epsilon & 0 & \dots & 0 \end{pmatrix} \quad (5.15)$$

where ϵ is defined by (5.14). $H^{(l+1) \times (l+1)}$ has eigenvalues 0 with $l-1$ eigenvectors and $\pm\epsilon\sqrt{l}$ with eigenvectors

$$|v_{\pm}\rangle = \frac{i}{\sqrt{2l}} \left(\mp i\sqrt{l}|\varphi_0\rangle + \sum_{j=1}^l |\rho^j\rangle \right). \quad (5.16)$$

We start the transfer of the amplitude with the system in the state

$$\begin{aligned} |\rho^1\rangle &= \frac{i}{\sqrt{2l}} \left(-|v_+\rangle - |v_-\rangle + i\sqrt{2(l-1)}|v_0\rangle \right) \\ \text{where } |v_0\rangle &= \frac{1}{\sqrt{l(l-1)}} \left[(1-l)|\rho^1\rangle + \sum_{j=2}^l |\rho^j\rangle \right]. \end{aligned} \quad (5.17)$$

We see that $|v_0\rangle$ is orthogonal to $|v_{\pm}\rangle$ so it is eigenstate of (5.15) with eigenvalues 0. Hence it applies that $\hat{U}_1|v_0\rangle = |v_0\rangle$. Now we use the operator $\hat{U}_1$ on the state $|\rho^1\rangle$ T_t times where $T_t = \left\lfloor \frac{\pi}{\epsilon\sqrt{l}} \right\rfloor = 2T$

$$\begin{aligned} \hat{U}_1^{T_t}|\rho^1\rangle &= \frac{i}{\sqrt{2l}} \left(-e^{i\epsilon\sqrt{l}T_t}|v_+\rangle - e^{-i\epsilon\sqrt{l}T_t}|v_-\rangle + i\sqrt{2(l-1)}|v_0\rangle \right) = \\ &= \frac{i}{\sqrt{2l}} \left(|v_+\rangle + |v_-\rangle + i\sqrt{2(l-1)}|v_0\rangle \right) = \\ &= \left[\left(1 - \frac{2}{l} \right) |\rho^1\rangle - \sum_{j=2}^l \frac{2}{l} |\rho^j\rangle \right] \end{aligned} \quad (5.18)$$

So we transfer the amplitude $\frac{2}{m}$ to other states. If we set $l = 2$ we get full transfer. This process can be used for communication between sender and receiver across the regular lattice without the knowledge of their exact position. It can be shown that this process can also track moving receiver though the speed of the receiver is limited by the transfer time $T_t \sim \sqrt{N}$ ($T_t \sim \sqrt{N \ln(N)}$ for dimension equal to 2) [9].

Conclusion

In this text we study the quantum walks and their use in search algorithm and state transfer protocol. We introduce the concept of the discrete time quantum walks and mention difference between quantum walks and classic random walks.

In Chapter 3 we described the Grover's search algorithm and showed that it needs $O(\sqrt{N})$ steps. We also proved the lower bound for quantum search algorithm in general and hence there can not be faster algorithm than $\Omega(\sqrt{N})$. We also presented the quantum search algorithm on the hypercube and on regular lattice and shown that it also takes $O(\sqrt{N})$ steps to succeed except the two dimensional lattice which need $O(\sqrt{N}) \ln(N)$ steps. We compared the properties of the Grover's algorithm and the walk search algorithm and recognised their advantages and disadvantages. In Chapter 4 we again estimated the number of steps of the walk search algorithm from perspective of the avoided crossing of eigenphases of the operator $\hat{U}_\lambda$.

In the last chapter we focused on use of discrete quantum walk to accomplish the transfer of state and communication across the grid thanks to mechanism of the quantum walk search on the regular lattice. We notice that the communication can be achieved without knowledge of position of sender and receiver and that it is possible even with moving receiver.

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