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Kvantové procházky na perkolovaných grafech

Quantum walks on percolated graphs

DIPLOMA THESIS

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Název:
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Abstrakt: Tato práce se zabývá asymptotickým chováním diskrétních kvantových procházek na konečných grafech, jejichž unitární vývoj je narušován procesem dynamické perkolace podkladového grafu. Po zavedení nezbytných konceptů (kvantových procházek, dynamické perkolace a náhodných unitárních operací) je hlavní důraz kladen na nalezení situací, ve kterých asymptotický stav kvantové procházky zůstává stejný jako v případě úplné perkolace. Zjišťujeme, že asymptotické chování perkolované kvantové procházky je z tohoto hlediska velmi robustní efekt a i některé velmi omezené druhy perkolace vedou ke stejným asymptotickým stavům jako perkolace úplná. Důležitým faktem je, že většina výsledků je odvozena pro zcela libovolné grafy nebo grafy splňující pouze určitou obecnou podmínku.

Klíčová slova: perkolace, kvantové procházky, asymptotický stav

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Quantum walks on percolated graphs

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Abstract: This thesis deals with the asymptotic behaviour of coined (discrete time) quantum walks on finite graphs where the unitary evolution is perturbed by a dynamical percolation of the underlying graph. After introducing necessary concepts of dynamical percolation, quantum walks and random unitary operations, the main emphasis is put on finding modified scenarios in which the asymptotic state of the walk is the same as in the cease of the full dynamical percolation. I conclude that the asymptotic behaviour of a percolated quantum walk is very robust from this point of view and even some very restricted percolation schemes give rise to the same asymptotic states as the full percolation. A noticeable fact is, that most of the results are derived for arbitrary graphs, sometimes with some general conditions imposed.

Key words: percolation, quantum walks, asymptotic state

Contents

Foreword

This thesis deals with percolation in quantum walks. The first chapter introduces the basic concept of percolation - a particular example of disordered media. Then we present a modified type - dynamical percolation as a model of fluctuating media. Later we use these models to study discrete quantum walks perturbed by imperfections in the underlying graph. This analysis is the main goal of the thesis.

The second chapter introduces quantum walks. We start with the analogy with a classical random walk and then we define the quantum walk. The definition given here is very general and allows to study quantum walks even in some unusual settings.

In the third chapter we define so called random unitary operations and list some findings about their asymptotic behaviour. Random unitary operations are special examples of systems with open dynamics and have a rather broad range of applications.

In the fourth chapter we introduce percolation into quantum walks and show, that the time evolution of a percolated quantum walk is a random unitary operation. We use results listed in the third chapter and with those we identify some modifications of the percolation process which leave the asymptotic evolution of a percolated quantum walk to be the same as in some already solved cases.

The fifth chapter contains some numerical simulations used to check the analytical results and gain some information about convergence rates and their dependences on the type of percolation.

In the appendix we present a solution of the simplest percolated quantum walk in a great detail, illustrating many concepts used in the main text.

Chapter 1

Percolation

1.1 Static Percolation

The problem of percolation was first stated by Broadbent in 1954 [1]. At that time the question was whether water will percolate (leak) through a block of some porous material if we know the micro structure of this material. This work has also introduced the term percolation.

Today percolation theory is based on graph theory. The basis of the percolation model is a connected, typically infinite graph $G = (V, E)$. Here V is the set of vertices and E the set of edges. We choose a probability p ($0 \leq p \leq 1$) and create a subgraph of G in the following way: we keep all the vertices and then take edges one by one and either leave it present with the probability p or remove it (with the probability $1 - p$) independently of all others. We will denote this subgraph $G_K = (V, K)$ where only edges in the subset $K \subset E$ are present a *percolation graph*.

Through the whole thesis we will be using the term *open edges* for those that remain in the percolation graph and *closed edges* for those that have been removed. Further an *open cluster* is a connected component of edges in the percolation graph.

The version described above is referred to as bond percolation (edges of a graph are often called bonds in the percolation theory). A straightforward analogue is site percolation, where we remove vertices (often called sites in percolation theory) instead of edges.

Percolation theory is basically the study of properties of percolation graphs and their dependence on the original graph G and the probability of retaining edges p. The property of most interest is, whether there exist an infinite cluster of open edges in the percolation graph. The occurrence of such a cluster is what is sometimes called the percolation. Let us denote $P(p)$ the probability for the existence of an infinite open cluster.

1.1.1 The Phase Transition

Suppose that for some values of p there is a non-zero probability, that a vertex is a part of an infinite open cluster. We start at any vertex of our choice v_1 and ask if it is a part of an infinite open cluster. If the answer is no, we take another vertex v_2 that is not in the cluster containing v_1. Such vertex must be present in the infinite original graph because the cluster containing v_1 is finite. We ask the same question about v_2. This procedure yields an infinite sequence of trials each with non-zero probability that there is an infinite open cluster containing some vertex. This means that in this case $P(p) = 1$ - the open cluster almost surely exists. More formally, this is a consequence of the Kolmogorov's zero-one law.

The infinite cluster for a given graph G and given p either almost surely occurs ($P(p) = 1$) or almost surely not ($P(p) = 0$). Because obviously $P(0) = 0$, $P(1) = 1$ and $P(p)$ is certainly a non-decreasing function of p, there must by some value $p_c \in [0, 1]$ such that $P(p) = 0$ for all $p < p_c$ and $P(p) = 1$ for all $p > p_c$. Therefore at $p = p_c$ the property $P(p)$ of the model changes dramatically which would in a physical system be called a *phase transition*.

The question is, what is the value of p_c for a given graph G and also what is $P(p_c)$ (whether there can be an infinite open cluster just at the critical probability).

The most investigated graphs are naturally infinite regular lattices where only closest neighbouring vertices are directly connected by edges. The case of the 1D lattice is not very interesting. As this is just a line of vertices and breaking just one edge cuts of the whole remaining part of graph in one direction, it is clear, that $p_c^{(1)} = 1$. The situation in higher dimensions is much more interesting. It can be shown that $0 < p_c^{(d)} < 1$ for any dimension $d \geq 2$ and actually also $p_c^{(d+1)} < p_c^{(d)}$ [2]. Unfortunately, there are not many more known values of the critical probabilities than $p_c^{(1)} = 1$ and the value $p_c^{(2)} = 1/2$ for a regular 2D square lattice.

There are many open questions in percolation theory but we are not going to discuss them here any further. For an extensive introduction to the field see for example [3]. We will use the model of percolation only to introduce this kind of disorder into quantum walks. In contrast to the percolation theory which is interested in properties of the percolated graph itself we will be examining quantum walks on these graphs. In particular we will be interested in the asymptotic behaviour of the walk which does not even make a good sense on infinite graphs. Therefore we will be using finite graphs and questions about infinite open clusters will not be of our concern. The reason for using percolation is because it is probably the simplest model of perturbed graphs and therefore also a model of perturbed physical systems description of which is based on graphs such as random or quantum walks.

1.2 Dynamic Percolation

The original concept of percolation is now often referred to as *static percolation*. That is due to the existence of another variant - so called *dynamic percolation* that

we will now discus.

In the model of dynamic percolation, we define a discrete time stochastic process based on static percolation. In this process we generate a new percolation graph at every time step. The main question of classical theory in this case is if there are some atypical times, at which an infinite open cluster may emerge.

Dynamical percolation is important for us because it is actually the type of percolation that we will be examining in the context of quantum walks. We will use it as a very simple model of a medium that is not only perturbed but where the perturbation fluctuates in time.

1.3 Application of Percolation in Quantum Walks

In this thesis we are going to use the percolation model in quantum walks. We will not describe it in detail here - we do not even have the definition of a quantum walk yet. What we can say at this point is that the discrete time quantum walk is a discrete time process, that takes place on a graph, and we will be using the dynamic percolation to change the underlying graph of the walk in every step.

Solving analytically the whole dynamics is difficult. We will limit ourselves to the asymptotics as this regime is analytically accessible. This will drive us somewhat far from results of the classical percolation theory. In fact, as we will see later the asymptotic evolution of the walk does not depend on the probability p at all (as long as $0 < p < 1$).

The reason for studying asymptotic states is twofold. The impulse for this are recent publications of new methods suitable for such analysis. The significance of this investigation is related to experimental difficulties with realising quantum walks. Asymptotic states may for example give testable predictions, verification of which does not require precise time control of the experiment.

Asymptotic states have already been found in several cases of percolated quantum walks [9, 10]. In these papers all possible configurations of the percolation graphs have been included in the percolation process. Let us call such scenario a *full percolation*. The main topic of this thesis is the study of what we will call a *partial percolation*. This is motivated by situations that we may encounter in physical systems. The medium may be only slightly perturbed - mostly one or two connections are broken at the time, therefore configurations with many broken edges do not occur. A complementary example is a weakly connected medium - edges are mostly closed and only few get opened. The third situation that we will list here is a system with some additional internal structure. This structure may break the assumption of the independent choice of open/closed edges. We will discuss the example of a site percolation defined in the way that a broken vertex just means broken all edges surrounding it. This creates multi-edge structures that are always open/closed simultaneously. All these situations bring us to the problem of percolation, where only some configurations of edges are possible.

Chapter 2

Quantum Walks

2.1 Random walks

In this chapter we will be dealing with the concept of a quantum walk that has been introduced recently [4, 5, 6]. Since it has it's origin in classical random walks we will introduce random walks to have the basis for the analogy with quantum walks.

The theory of random walks is a large field of study. There are many different types of random walks that may be of interest. Nevertheless, we are going to introduce just the variant that will be useful later in this work.

2.1.1 Random Walk on a Line

Let us start with one of the simplest examples which nicely demonstrates the concept - a random walk on a line. We have a walker with a coin. The walker is flipping the coin and always makes one step to the left or to the right according to the result of the coin flip. An illustrative physical realisation of a random walk may be the Galton board - a board with pins and a ball bouncing of them randomly. A very schematic illustration of a Galton board is shown in figure 2.1. Another (less scientific) example is a drunken sailor walking on a narrow pier.

If we use a fair coin (the probability of going to the left is the same as for going to the right) and ask, what is the probability that the walker will end up at a particular place in a given time step, we obtain a binomial distribution centred at the origin. (We always neglect odd/even positions because those are never occupied at even/odd time steps.) The distribution after 20 steps is shown in figure 2.2.

We may understand the line the walker is moving on as a graph with vertices (places that the walker visits by doing steps) and edges between the closest neighbouring vertices.

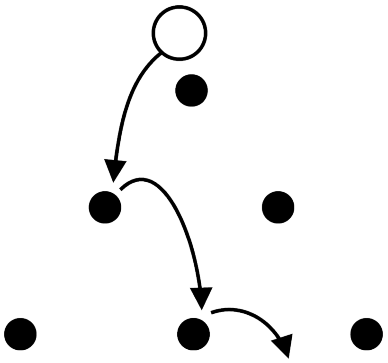

Figure 2.1: Schematic illustration of a Galton board. A ball falling between the pins performs a random walk.

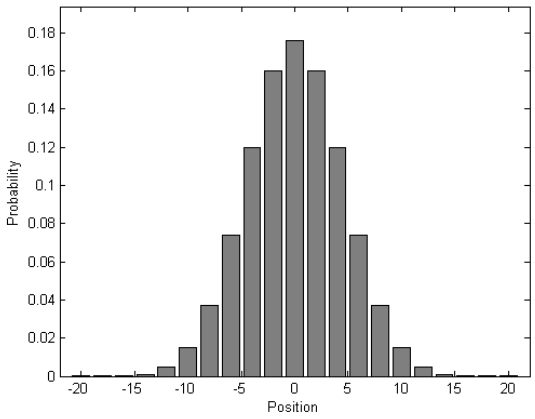

Figure 2.2: The position probability distribution of a classical random walk on a line with a fair coin after 20 steps. If we consider only even positions (odd positions are unoccupied at even steps and vice versa), we get a binomial distribution.

2.1.2 Random Walk on a General Graph

The generalisation of the concept of a random walk to any graph is very straightforward. The only thing that needs to be done is to define transition probabilities. For every edge in a given vertex we define probability, that the walker will leave the vertex through this edge. We may then again ask about the time evolution of the probability distribution at given times or the evolution of any other properties.

2.2 Discrete Time Quantum Walk on a Line

We will use the line graph, but we introduce the concept of quantum walks. The underlying graph is the same, but now the walker is a quantum "particle" and the time evolution is therefore governed by laws of quantum physics.

As for every quantum system, the evolution of the walker is given in a Hilbert space. Our Hilbert space will be of the form $\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_c$, where $\mathcal{H}_x$ is called the position space and $\mathcal{H}_c$ is the coin space. The basis states of $\mathcal{H}$ can be written as $|x\rangle \otimes |L\rangle$ and $|x\rangle \otimes |R\rangle$. Here x denotes a particular vertex of the (line) graph and L/R stands for the direction left/right. In analogy with the classical random walk we will introduce a coin. Here it will be a unitary operator acting on the direction states, so we will call $|L\rangle$ and $|R\rangle$ the coin states. If the line has N vertices, $\mathcal{H}$ is then a $2N$-dimensional Hilbert space. This definition works also for an infinite line, but we will not consider such walks in this thesis.

Typically, the initial state of the walk is of the form $|\psi(0)\rangle = |0\rangle (\alpha |L\rangle + \beta |R\rangle)$. The walker starts localised at some vertex taken as the origin and in a superposition of the coin states.

Now we have to define the time evolution. As we are used to from other quantum systems, the time evolution is described by a unitary operator $U(t)$ as $|\psi(t)\rangle = U(t) |\psi(0)\rangle$. The time variable t is in our case not continuous, but takes only discrete values. The evolution operator has a special form: $U(t) = U^t$, where U is just one given unitary operator. Ultimately a quantum walk evolves in steps and the state in a particular time t is:

$$|\psi(t)\rangle = U^t |\psi(0)\rangle . \tag{2.1}$$

The term *discrete time quantum walk* is often used synonymously to the term *coined quantum walk*. The name coined comes from the fact, that the operator U can be decomposed into a subsequent application of two operators:

$$U = SC, \tag{2.2}$$

where C is the *coin* operator and S is the *step* operator. Further, C is a local operation that acts separately at every vertex, so in the walk on a line it acts only

on two-dimensional subspaces of coin states. The coin operator acting on the whole $\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_c$ can be written as:

$$C = \mathbb{1}_x \otimes C_0, \tag{2.3}$$

where C_0 is some unitary operator on the two-dimensional space.

After applying the coin, the second part of the step is realised - we apply the step operator S[1]. It moves the walker to the neighbouring vertex according to the direction state. Obviously the walker can be in an arbitrary superposition of all states so S actually "moves parts of the walker". The map S can be explicitly written as

$$S = \sum_x |x-1, L\rangle \langle x, L| + |x+1, R\rangle \langle x, R| . \tag{2.4}$$

2.2.1 Hadamard Quantum Walk on a Line

Even in the particular example of a quantum walk on a line, there is still freedom in choosing the coin operator and the initial state. Here we will show probably the most common example - quantum walk with the Hadamard coin. Let us take the two-dimensional Hadamard operator, which can be represented as a matrix:

$$C_0^{(Had)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.5}$$

If we choose a "balanced" initial state - for example $|\psi_0\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|L\rangle + i\,|R\rangle)$ and after several steps sum the probabilities over coin states, we obtain a symmetrical probability distribution over vertices. Nevertheless, due to interference which the walker exhibits, the distribution is very different from the one of a balanced classical random walk. The probability distribution for the Hadamard quantum walk after 100 steps is shown in figure 2.3.

We immediately see that the spreading of the walk is much faster, proportional to t in contrast to the classical random walk where it is proportional to $\sqrt{t}$. This walk has many other interesting properties, but we will now leave the example of a walk on a line and define a quantum walk on a general graph.

2.3 General Coined Quantum Walk

The concept of a general quantum walk can be certainly defined in several ways. We will now give one possible definition that will be used through the rest of the thesis.

[1] Considering the fact, that one step of the walk is actually done by U, it might make more sense to call U the step operator and S the shift operator, but we will keep the name step operator for S.

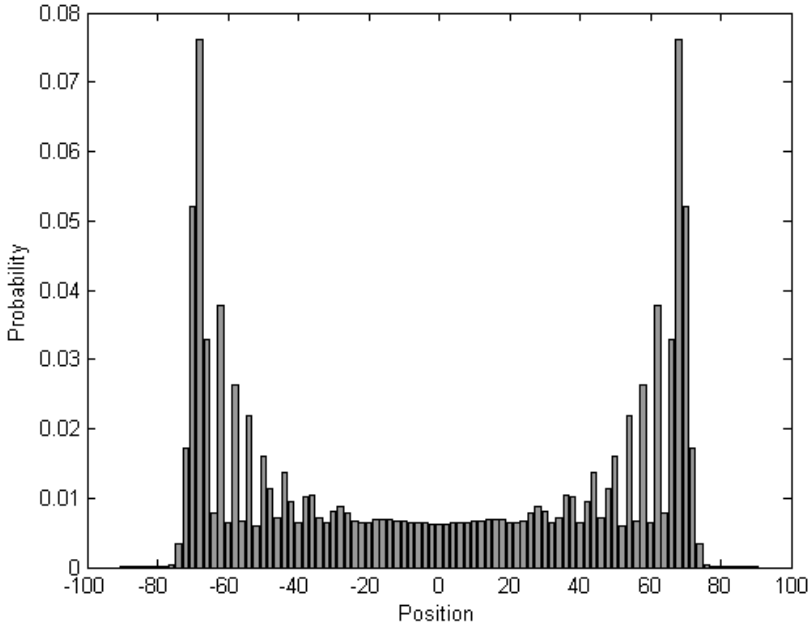

Figure 2.3: Position probability distribution (summed over coin states) for a quantum walk on a line with the Hadamard coin and the initial state $|\psi_0\rangle = |0\rangle \frac{1}{\sqrt{2}}(|L\rangle + i\,|R\rangle)$ after 100 steps. Only even positions are displayed since the probability at odd positions is zero in even time steps. The two-peaked structure is clearly visible.

Let us start with a *directed* graph $G = (V, E_d)$, where V is the set of vertices and E_d the set of directed edges. We require that this graph fulfils one additional condition: every vertex must have the same number of incoming edges as it has outgoing edges. A very convenient and almost exclusively used example of such graph is an arbitrary *undirected* graph - every undirected edge can be interpreted as two directed edges in opposite directions. We have an example of a suitable directed graph in figure 2.4.

We now associate a Hilbert space $\mathcal{H}$ with the set of directed edges - every edge corresponds to one basis state. We may group edges going from a particular vertex $v \in V$ and create a subspace of $\mathcal{H}$ spanned by basis states corresponding to these edges. Then it is possible to write $\mathcal{H}$ as a direct sum of such subspaces over all vertices: $\mathcal{H} = \mathcal{H}_{v_1} \oplus \mathcal{H}_{v_2} \oplus \ldots \oplus \mathcal{H}_{v_n}$, where $v_i \in V$ and $n = |V|$ (number of vertices). It is common to use some regular lattice or its part as the graph for a quantum walk. In such cases the subspaces in all vertices are the same and we may even write $\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_c$. This was also the case of a quantum walk on a line.

A coined quantum walk is a discrete time process and as it was in the case of the walk on a line even in the general definition the time evolution is given by application of a unitary operator U in every time step. This operator consist of subsequent application of the coin operator and the step operator, so $U = SC$. Each of these two operators has its specific properties.

Let us use the structure of the Hilbert space $\mathcal{H}$ and introduce the coin operator C. An important property of this operator is that it does not mix states from

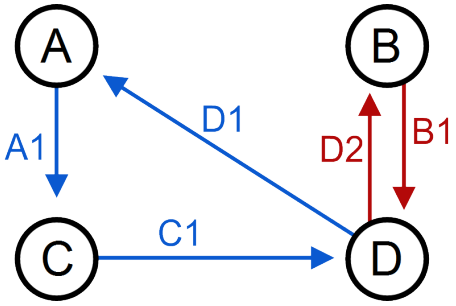

Figure 2.4: Example of a general directed graph that can serve as a basis for a quantum walk.

different vertex subspaces. Therefore a subspace associated with any vertex is an invariant subspace of C. That means that C can be in general decomposed as $C = C_{v_1} \oplus C_{v_2} \oplus \ldots \oplus C_{v_n}$ or specially in the case of a regular lattice $C = \mathbb{1}_x \otimes C_0$ if we choose the coin to be the same in every vertex.

Now we will introduce the step operator S. The main requirement we put on S is, that it has to be a *permutation operator*. This fact will be heavily used and it yields some nice properties of quantum walks and makes the whole argument much more clear. There is one more restriction on S and that is the connection with the graph underlying the quantum walk. If we take a state corresponding to some edge and apply S on this state, the resulting state must lay in the vertex that is the endpoint of the original edge. In simpler words, S must transport the walker along the directed edges.

Ultimately, a general quantum walk is defined by a directed graph where every incoming edge has one outgoing and these directed edges are associated with basis states of a Hilbert space. Then we have to define a coin operation acting locally in every vertex and the last piece of the definition is the step operator. It is partially given by endpoints of the directed edges and we just have do define which state in a given vertex will be chosen by S.

Let us end with the following note. We have defined a quantum walk in rather general way. The aim was to point out the main properties of particular ingredients. In fact, our investigation is one of the cases where solving a general problem will be easier than solving some particular example. Nevertheless, most often quantum walks of our interest are on regular lattices with undirected edges, where we define the coin operator to be the same in every vertex and the step operator is given by some simple rule. We saw such rule in the walk on a line: If the walker was going to the left, he continues to the left and if he was going to the right, he continues to the right.

2.3.1 Example of a General QW

Let us demonstrate the concept of a general quantum walk on one example. If we look at our graph in figure 2.4, we can see four vertices: A, B, C and D. Vertices A, B

and C have only one-dimensional subspaces: $\mathcal{H}_A = \text{span}\{A1\}$, $\mathcal{H}_B = \text{span}\{B1\}$ and $\mathcal{H}_C = \text{span}\{C1\}$. In the vertex D the subspace is two-dimensional: $\mathcal{H}_D = \text{span}\{D1, D2\}$. Overall we have:

$$\mathcal{H}_{example} = \mathcal{H}_A \oplus \mathcal{H}_B \oplus \mathcal{H}_C \oplus \mathcal{H}_D = \text{span}\{A1, B1, C1, D1, D2\}. \tag{2.6}$$

The coin in the vertices A, B and C has to be a multiplication by some complex number with the absolute value 1. That is the only unitary operation on a one-dimensional space. In the vertex D the coin may be for example chosen to be the Hadamard coin from the the Hadamard walk on a line. With one particular choice of the one-dimensional coins and if we have the order of the basis states as it is in the list above, the overall coin operator may be represented as a matrix:

$$C_{example} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \tag{2.7}$$

In figure 2.4, the step operator is depicted by colours. Every state corresponding to an edge is mapped by S onto the state of the same colour in the endpoint vertex. The mappings $S|D1\rangle = |A1\rangle$ and $S|A1\rangle = |C1\rangle$ are the only options in this graph. On the other hand there are two possibilities how to define the remaining action of S. We have chosen that $S|C1\rangle = |D1\rangle$, which now forces us to have $S|D2\rangle = |B1\rangle$ and $S|B1\rangle = |D2\rangle$. The other option would be to have one big cycle in the permutation (not split into two smaller ones) by defining $S|C1\rangle = |D2\rangle$, $S|D2\rangle = |B1\rangle$ and $S|B1\rangle = |D1\rangle$. An image of the situation would not be very helpful here since all edges would have the same colour.

Chapter 3

Asymptotic Evolution of Random Unitary Operations

3.1 Density Matrices and the Jordan Canonical Form

To describe percolated quantum walks we have to use the most general formalism of quantum theory. We have to describe the state of the walker using quantum density matrices and the time evolution using open quantum dynamics language. For these purposes let us start this chapter with some notes about density matrices and about the Jordan canonical form of a matrix.

If we do not have full information about the state of our system (we can not assign a pure state from the Hilbert space to it) or our system is coupled with some other system, which we cannot control or observe, we use the description by a density matrix. We call such state of the system mixed. This may happen when we do not know the exact history of preparation of our state, but there are several possibilities and we are able to assign probabilities to all of them.

If we want to evolve a system in a mixed state $\rho(0)$ by an unitary operator U, the new state is

$$\rho(1) = U^\dagger \rho(0) U. \tag{3.1}$$

Let us now introduce an alternative notation that will be convenient in our later investigation. We will denote entities in this new notation with tildes. In this new notation, density matrices and some other matrices are represented by vectors:

$$\rho = \begin{bmatrix} \rho_{11} & \dots & \rho_{1n} \\ \dots & \dots & \dots \\ \rho_{n1} & \dots & \rho_{nn} \end{bmatrix} \quad \rightarrow \quad \tilde{\rho} = \begin{bmatrix} \rho_{11} \\ \dots \\ \rho_{1n} \\ \rho_{21} \\ \dots \\ \rho_{nn} \end{bmatrix}. \tag{3.2}$$

Now when some linear operator A acts on a matrix as in the equation (3.1), we can write the application of this operator in the new notation as application of one linear operator of the dimension $n^2 \times n^2$. Now we have two ways how to write the evolution:

$$\rho(1) = A^\dagger \rho(0) A \quad \leftrightarrow \quad \tilde{\rho}(1) = \tilde{A}\tilde{\rho}(0). \tag{3.3}$$

There is also a correspondence of scalar products. In particular the standard scalar product of states in the vector form (with a tilde) has the same value as the Hilbert-Schmidt scalar product of states in matrix notation, so:

$$\tilde{\rho}_1 \cdot \tilde{\rho}_2 \quad = \quad \mathrm{Tr}\left(\rho_1^\dagger \rho_2\right). \tag{3.4}$$

If we now have an orthonormal basis $\{\psi\}_{i=1}^{n^2}$ (and $\{\tilde{\psi}\}_{i=1}^{n^2}$), we may decompose every state in this basis as

$$\tilde{\rho} \quad = \quad \sum_{i=1}^{n^2} \left(\tilde{\psi}_i \cdot \tilde{\rho}\right) \tilde{\psi}_i, \tag{3.5}$$

or in matrix notation

$$\rho \quad = \quad \sum_{i=1}^{n^2} \mathrm{Tr}\left(\psi_i^\dagger \rho\right) \psi_i. \tag{3.6}$$

Given an operator $\tilde{A}$, we may find it's eigenvalues and eigenvectors. In general, eigenvectors of $\tilde{A}$ do not form an orthonormal basis and therefore it is sometimes not possible to get $\tilde{A}$ in a diagonal form by a similarity transformation. On the other hand, what is always possible, is to transform $\tilde{A}$ by a similarity transformation into the Jordan canonical form. In this form the matrix is block diagonal where all blocks are of the form:

$$J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, \tag{3.7}$$

where λ is a given eigenvalue of $\tilde{A}$ and blank spaces represent zeros in the matrix. There may be more blocks corresponding to the same eigenvalue λ. If there is an eigenvector corresponding to λ, the block is one-dimensional. Further, if eigenvectors of $\tilde{A}$ form an orthonormal basis, then all blocks in the Jordan form are one-dimensional and the matrix is diagonalised. In the other case the blocks for some eigenvalues are larger, the matrix is not diagonal, and there are generalised eigenvectors corresponding to these eigenvalues. Those are states $\tilde{v}$ fulfilling the equation $\left(\tilde{A} - \lambda\right)^k \tilde{v} = 0$ for some k.

3.2 Unitary Discrete Time Evolution

Let us assume that we have some quantum system, for which the time evolution is given by an unitary operator U. The evolution is discrete in time and every time step is given by one application of the operation U on the state of the system.

Every unitary operator is obviously normal ($UU^\dagger = U^\dagger U = \mathbb{1}$) and therefore diagonalisable. We may use the basis of the Hilbert space $\mathcal{H}$ consisting of eigenvectors of U: $\{|v_i\rangle\}_i$. Therefore $U|v_i\rangle = \lambda_i |v_i\rangle$ for $\forall i$, where λ_i is an eigenvalue corresponding to the eigenvector $|v_i\rangle$. If an eigenvalue has more than one eigenvector - has multiplicity higher than one - we denote it with adequate number of different indices i. For an unitary operator U it must be $|\lambda_i| = 1$ for $\forall i$.

If the initial state is pure, we can decompose it into the eigenstates as

$$|\psi(0)\rangle = \sum_i \langle v_i | \psi(0)\rangle |v_i\rangle . \tag{3.8}$$

Now the application of the operator U gives:

$$U|\psi(0)\rangle = U \sum_i \langle v_i | \psi(0)\rangle |v_i\rangle = \sum_i \langle v_i | \psi(0)\rangle U|v_i\rangle = \sum_i \langle v_i | \psi(0)\rangle \lambda_i |v_i\rangle . \tag{3.9}$$

If we now apply U again, we just get higher powers of eigenvalues in the sum. At the time step t, the state of our system is:

$$|\psi(t)\rangle = U^t |\psi(0)\rangle = \sum_i \langle v_i \,|\, \psi(0)\rangle \, \lambda_i^t \, |v_i\rangle \,. \tag{3.10}$$

We see, that it is rather simple to calculate the state of our system at any given time. Also when the state of the system is described by a density matrix ρ similar decomposition can be done. If we construct $\tilde{U}$ (using the notation from the previous section), the unitarity of U assures, that $\tilde{U}$ has n^2 eigenvectors $\{\tilde{\psi}\}_{i=1}^{n^2}$ and therefore is diagonalisable. Now we again have the time evolution described as:

$$\tilde{\rho}(t) = \tilde{U}^t(\tilde{\rho}(0)) = \sum_i \left(\tilde{\psi}_i \cdot \rho(0)\right) \tilde{U}^t \tilde{\psi}_i = \sum_i \left(\tilde{\psi}_i \cdot \rho(0)\right) \lambda_i^t \tilde{\psi}_i, \tag{3.11}$$

or in the matrix notation

$$\rho(t) = \sum_i \mathrm{Tr}\left(\psi_i^\dagger \rho(0)\right) \lambda_i^t \psi. \tag{3.12}$$

3.3 Definition of RUO and its Basic Properties

Let us imagine a situation, where we have a set of different unitary operators $\{U_i\}_{i=1}^m$ and a system in an initial state described by the density operator $\rho(0)$. The evolution of the system is given by a successive application of different operations U_i, where each one of them has a certain probability p_i ($\sum_i p_i = 1$) to be chosen. Using an inherently statistical description by a density operator, we can interpret such evolution by the application of so called *random unitary operation* acting on the state $\rho(0)$ as:

$$\Phi(\rho(0)) = \sum_{i=1}^{m} p_i U_i \rho(0) U_i^\dagger. \tag{3.13}$$

Obviously, we may use Φ to define a discrete time evolution by defining $\rho(t+1) = \Phi(\rho(t))$.

We can use the same form as for just one unitary operator and rewrite Φ as application of a single matrix of the dimension n^2 on the state ρ in the vector form $\tilde{\rho}$. In contrast to an unitary operator, the eigenvalues of a random unitary operation do not have to have absolute value 1. Nevertheless, there is a restriction that it must be $|\lambda| \leq 1$ [7].

In a general, a RUO is not a normal operator and therefore is not in general diagonalisable. It can be seen if we just evaluate:

$$\Phi^\dagger\left(\Phi(\rho)\right) = \sum_{i,j} p_i p_j U_j^\dagger U_i \rho U_i^\dagger U_j \stackrel{?}{=} \sum_{i,j} p_i p_j U_j U_i^\dagger \rho U_i U_j^\dagger = \Phi\left(\Phi^\dagger(\rho)\right) \tag{3.14}$$

and from the fact, that U_i and $U_j^\dagger$ in general do not commute.

3.4 Attractors for RUO

Even if we cannot in general diagonalize a RUO, we can always find it's eigenvalues and it's Jordan canonical form. The Jordan form of the matrix is associated with a so called Jordan basis. We may again make a decomposition of a mixed state ρ in this basis. Now we will turn our focus on the state of the system after many applications of the RUO. When considering a limit of infinitely many steps, we arrive at an asymptotic behaviour.

From the point of view of the asymptotic behaviour, there are two main classes of eigenvalues of a RUO: those with $|\lambda| = 1$ and those with $|\lambda| < 1$. There are no eigenvalues with $|\lambda| > 1$, as was mentioned earlier. These classes are important because in contrast to the eigenvalues with $|\lambda| = 1$, for $|\lambda| < 1$ the terms λ^t will vanish for t going to infinity and projections to the corresponding subspaces will not be present in the asymptotic evolution.

There are two main results for these two categories of eigenvalues [7]. Firstly, it has been shown that all matrix blocks corresponding to an eigenvalues $|\lambda| < 1$ vanish after sufficiently many steps, so $\lim_{t\to\infty} J^t = 0$. Second, all matrix blocks corresponding to eigenvalues $|\lambda| = 1$ are one-dimensional. Therefore, if we decompose a state ρ in the Jordan basis, parts corresponding to eigenvalues $|\lambda| < 1$ will disappear in the limit of large t and we are left with the dynamics governed by a diagonal matrix. We can write the state in the asymptotic regime as:

$$\rho_{as}(t) = \sum_{\{\lambda_i||\lambda_i|=1\}} \mathrm{Tr}\left(\psi_i^\dagger \rho(0)\right) (\lambda_i)^t \psi. \tag{3.15}$$

We call these matrices (vectors) corresponding to eigenvalues with $|\lambda| = 1$ *attractors*. It is good to note that attractors are in general not valid density operators. Attractors of a RUO can be found [7] as common matrix eigenvectors of all unitary operators that form the random unitary operation Φ, so they must fulfil the equation:

$$U_i X U_i^\dagger = \lambda X \quad \forall U_i. \tag{3.16}$$

The eigenvalue λ must be the same for all U_i for a given attractor. Because we have matrix eigenvectors of unitary operators, the absolute value of λ is always equal to 1.

Chapter 4

Percolation in Quantum Walks

4.1 Definition QWs on Percolation Graphs

Previously, we have defined a general quantum walk on a directed graph. Vertices of this graph define coin invariant subspaces and edges are associated with basis states and determine end-vertices after application of the step operator. This is completed by the definition of a particular coin operation at every vertex and particular states in a given vertex after the step.

Let us start examining perturbations of the quantum walk resulting from a percolation of the graph. In particular, it will be the dynamic percolation, so we will generate a new percolation graph at every step of the walk. At the beginning we have a perfect quantum walk defined on an oriented graph $G = (V, E_d)$, where V is the set of vertices and E_d the set of directed edges. We set some probability p and set every edge of the original graph G to be open with the probability p or closed with the probability $1 - p$. We denote the set of remaining directed edges K_d. This produces a new (percolated) graph $G_{K_d} = (V, K_d)$, where $K_d \subset E_d$. This procedure is done in every step of the walk. Now we need to define a new time evolution on this percolated graph with a given edge configuration K_d, and require this evolution to be unitary.

The coin operation is not affected by the percolation process as well as the dynamics given by the step operator on edges that remain open.

The difference appears where some edge happens to be closed. The behaviour, we would like to get is that instead of passing through the edge the walker will stay in the original vertex. Nevertheless, there may be a serious problem with this approach. We want to keep our step operator to be a permutation operator as in the original definition. It means that the number of states that are mapped by S to states in some given vertex has to be the same as the number of edges going from this vertex. If some edge is closed and we want the walker to stay in the original vertex, we must chose some particular coin state he will be in. The problem is, that we have added one state ending in the vertex and have not removed any. In general all states in this vertex may be "occupied" - they can be resulting states of some other initial states.

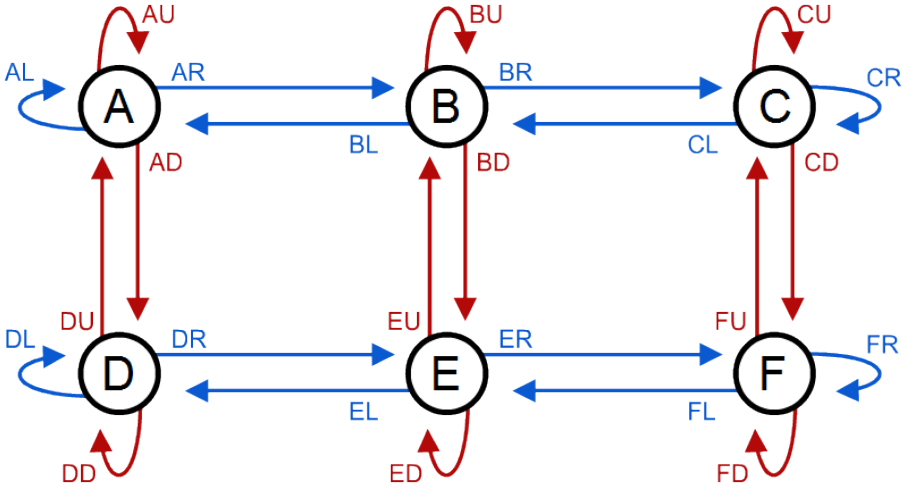

Figure 4.1: Example graph for a percolated quantum walk. The action of the step operator is defined by colouring the edges.

In other words, it may happen that there is not enough "space" for the walker to stay in his original vertex.

We will solve this problem by reducing the generality of our definition of a quantum walk. In percolation processes, we will require that for every edge from A to B, there is also an edge from B to A. There are two ways how to achieve this. Obviously, there can be two edges in opposite directions. A less apparent possibility is just one edge beginning and ending in the same vertex. We will call these "returning" edges.

Let us take a look at the example in figure 4.1. Here we have a cut from the 2D square lattice. There are four edges beginning in every vertex labelled by the name of the vertex and their direction L, D, U or R (left, down, up, right). If there is a vertex in a particular direction, the step operation maps the state of this edge to the state in this neighbouring vertex and keeps the original direction. For example $S|AR\rangle = |BR\rangle$. If we are at the border of the graph and the vertex in desired direction is missing, we turn the edge back to the original vertex - we create a returning edge. The step operator then just flips the direction of the walker and leaves him in the original vertex. For example $S|CR\rangle = |CL\rangle$. This definition is convenient because we have 4 direction states in every vertex, and therefore can have the same coin operation everywhere.

In the percolation process, we will not be removing returning edges. In fact, they already induce the behaviour caused by a broken edge - the walker stays at the vertex he started in, because there is nowhere to go in that direction. In the case of two edges in the opposite directions, we consider them as a pair which can actually be understood as an undirected edge. These pairs or undirected edges are always closed or left open together. In fact, closing an edge means replacing it with returning edges. This assures that if we keep the walker in the original vertex, there is one "unoccupied" state for him - the state that the walker coming through the edge would end up in.

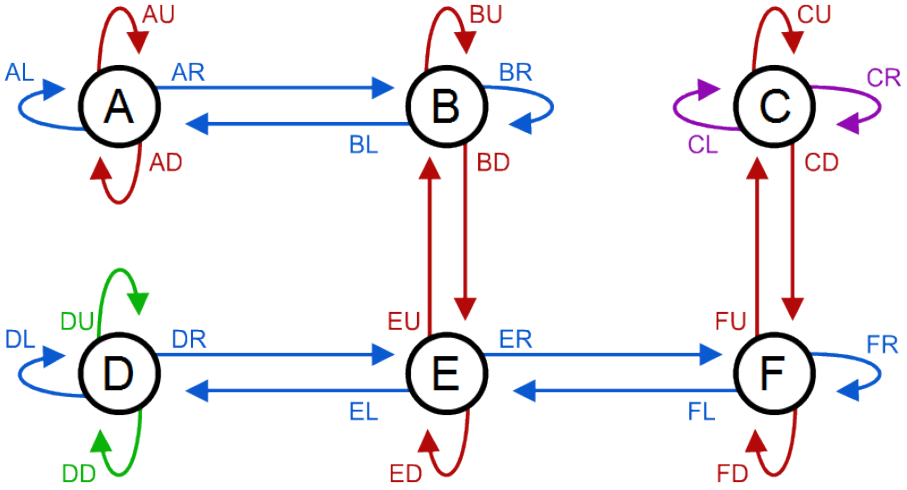

Figure 4.2: Example graph for the percolated quantum walk with two edges removed. The action of the step operator is defined by colouring the edges.

It is definitely helpful to give an example here. In figure 4.2 there is the graph from figure 4.1, where we have removed edges between A and D and between B and C. Now the walker can not pass through the edge AD, so he has to stay in the vertex A. Thanks to the fact that there is also no way of going through the edge DU, we may define $S|AD\rangle = |AU\rangle$. In the original graph it was $|DU\rangle$, that was mapped to $|AU\rangle$.

4.2 Percolation in a Quantum Walk as a Random Unitary Operation

In the previous section we have defined a percolated quantum walk. We have the original directed graph (V, E_d) with the set of vertices V and the set of directed edges E_d. There are two kinds of edges: returning edges and edges in pairs going in the opposite directions. Paired edges always undergo percolation together. From the point of view of percolation, we may work with an undirected graph $G = (V, E)$, where V is the same set of vertices and E is the set of undirected edges representing pairs of edges in opposite directions. The returning edges are not important in creation of the percolated graph.

In every step of the quantum walk we create a new graph by removing (closing) some edges. If we denote the set of edges that remain open K, we then have a quantum walk defined on a new graph $G_K = (V, K)$, where the coin operation is the same as for the original graph, but the step operator is different, say S_K.

Now the evolution for a particular realisation of a percolated walk is $|\psi(t)\rangle = U_{k_t} \dots U_{k_2} U_{k_1} |\psi(0)\rangle$, where $U_{k_i} = S_{k_i} C$ is the evolution operator defined by a particular edge configuration K_i in the i-th step.

Nevertheless, a more adequate description of a percolated quantum walk is using a

density operator instead of a pure state to reflect the classical randomness involved in the percolation process. If we start with some density operator $\rho(t)$, one step of the percolated QW can be described as:

$$\rho(t+1) = \sum_{K \subset E} \pi_K(p) U_K \rho(t) U_K^\dagger, \tag{4.1}$$

where the sum goes over all possible subsets K of all edges E and $\pi_K(p)$ is the probability of occurrence of the configuration K if the percolation probability is p. We immediately see that this time evolution is described by a random unitary operation and therefore we may use the theory prepared earlier.

4.3 Attractors for Quantum Walks

4.3.1 General Attractors

In a percolated quantum walk, the equations for determining attractors of the random unitary operation governing the time evolution of the walk is

$$U_K X_{\lambda,i} U_K^\dagger = \lambda X_{\lambda,i} \text{ for } \forall K. \tag{4.2}$$

The first difference from (3.16) is, that we index different unitary operators by configurations of the underlying graph. Much more important is that we may use the separation of the operator U into the application of the coin operator and the step operator: $U_K = S_K C$. We can therefore rewrite equation for determining attractors of our random unitary operator as:

$$C X_{\lambda,i} C^\dagger = \lambda S_K^\dagger X_{\lambda,i} S_K \text{ for } \forall K. \tag{4.3}$$

We may now note that the left hand side of the equation is independent of a chosen configuration $K \subset E$ and therefore it is some constant matrix for all K. From this we get the condition, that for any two configurations K and K' the terms on the right hand side mus be equal, so:

$$S_K X_{\lambda,i} S_K^\dagger = S_{K'}^\dagger X_{\lambda,i} S_{K'} \text{ for } \forall K, K'. \tag{4.4}$$

This structure allows us to solve the problem in two steps. We can solve the attractor equation for one configuration K:

$$C X_{\lambda,i} C^\dagger = \lambda S_K^\dagger X_{\lambda,i} S_K. \tag{4.5}$$

Here we can avoid a lot of work by choosing the most convenient one. Then we restrict the resulting set of matrices X with what we will call the *step condition* 4.4.

4.3.2 The Step Condition

In the previous section we have introduced the Hilbert space spanned by states consisting of the vertex part and the coin part in each vertex. For the purposes of the following argument we will use the notation without distinguishing vertices, so we will just have states $|x_{v,c}\rangle = |v, c\rangle$ and write the basis as $\{|x_i\rangle\}_i$, where i goes over all possible combinations of vertices and corresponding coin states. Basically i goes over all directed edges.

Here we will present some general findings about the step condition (4.4). We will use the fact, that according to our definition, the step operator S is actually a *permutation* operator acting on the joint position and coin Hilbert space. It means that the step operator just acts as $S_K |x_i\rangle = \left|x_{s_K(i)}\right\rangle$, where s_K denotes the permutation map associated with the edge configuration K.

In these settings and with the basis of a Hilbert space $\mathcal{H}$ denoted as $\{|x_i\rangle\}_i$ the step operator $S \equiv S_K$ can be directly written as

$$S = \sum_i \left|x_{s(i)}\right\rangle \langle x_i| \,, \tag{4.6}$$

$$S^\dagger = \sum_j |x_j\rangle \left\langle x_{s(j)}\right| . \tag{4.7}$$

We can also write an arbitrary operator X in the same manner as:

$$X = \sum_{kl} X_l^k \, |x_k\rangle \langle x_l| \,. \tag{4.8}$$

We use this notation and evaluate the expression of our interest:

$$SXS^\dagger = \sum_{ijkl} X_l^k \left|x_{s(i)}\right\rangle \langle x_i| \, |x_k\rangle \langle x_l| \, |x_j\rangle \left\langle x_{s(j)}\right| = \sum_{ij} X_j^i \left|x_{s(i)}\right\rangle \left\langle x_{s(j)}\right| . \tag{4.9}$$

Let us have two operators S_{K_1} and S_{K_2}. Later we can obviously use this to build a complete step condition by comparing different combinations of operators. Let us denote the permutation map defining S_{K_1} as s and the map defining S_{K_2} as t. Then the step condition in our case is

$$S_{K_1} X S_{K_1}^{\dagger} = S_{K_2} X S_{K_2}^{\dagger}, \tag{4.10}$$

$$\sum_{ij} X_j^i \left|x_{s(i)}\right\rangle \left\langle x_{s(j)}\right| = \sum_{ij} X_j^i \left|x_{t(i)}\right\rangle \left\langle x_{t(j)}\right|, \tag{4.11}$$

$$\sum_{ij} X_{s^{-1}(j)}^{s^{-1}(i)} \left|x_i\right\rangle \left\langle x_j\right| = \sum_{ij} X_{t^{-1}(j)}^{t^{-1}(i)} \left|x_i\right\rangle \left\langle x_j\right|. \tag{4.12}$$

The last step is nothing more than just a reordering the summation indices. We can now compare the corresponding matrix elements and get the step condition in the form:

$$X_{s^{-1}(j)}^{s^{-1}(i)} = X_{t^{-1}(j)}^{t^{-1}(i)} \quad \text{for } \forall i, j. \tag{4.13}$$

The step condition can also be written in a slightly different way:

$$X_j^i = X_{st^{-1}(j)}^{st^{-1}(i)} \quad \text{for } \forall i, j. \tag{4.14}$$

giving us an intuitive interpretation. We can see that every possible pair of step operators that can arise from the percolation process gives the following restriction: every matrix element for a pair of states must be the same as the one obtained by returning by one step according to the first step operator and making one step forward according to the other step operator. The reason for such a simple step condition comes from the fact, that we have defined the step operator as a permutation not allowing it to create superpositions.

Let us investigate a little further what can be the value of $st^{-1}(i)$. We are comparing two step operators, say S_{K_1} and S_{K_2} with corresponding permutation maps s and t respectively. There is a well defined state $t^{-1}(i)$ associated with the edge i. If the edge $t^{-1}(i)$ is present (we mean the undirected edge containing this state) in both K_1 and K_2, then the permutation map s associated with S_1 will send the walker back to i, so st^{-1} is actually an identity. This is due to the fact, that all step operators are derived just from the structure of the percolation graph and therefore the resulting state may not change. The very same result is obtained if the edge $t^{-1}(i)$ is missing in both configuration. See figures 4.3 and 4.4.

The interesting case is when the edge $t^{-1}(i)$ is present only in one configuration and it does not matter in which one. See figures 4.5 and 4.6. Here by applying the map st^{-1} on i, we do not return to i. Instead, the walker ends up in the vertex on the opposite end of the edge - vertex A in our example. The exact state is the one, in which he would end up by passing the edge from B to A.

4.3.3 Construction of the Step Condition

Basically, the step condition is given by the equation (4.14) but it is possible to get further in the general argument and derive a more convenient form of the step

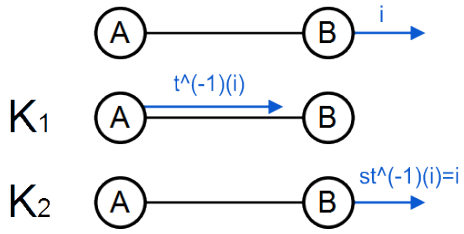

Figure 4.3: Illustration of the step condition. The edge is open in both configurations.

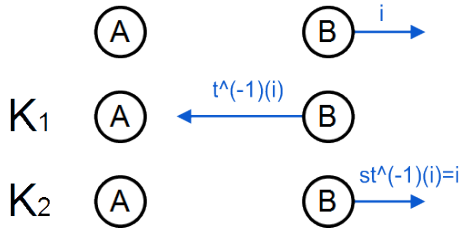

Figure 4.4: Illustration of the step condition. The edge is closed in both configurations.

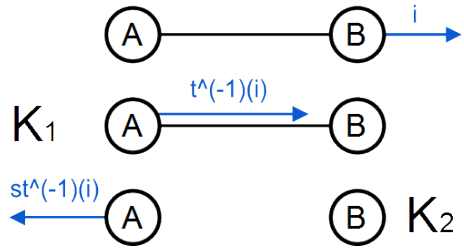

Figure 4.5: Illustration of the step condition. The edge is open only in K_1.

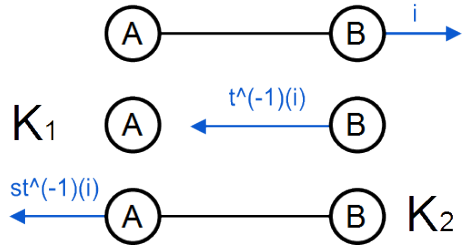

Figure 4.6: Illustration of the step condition. The edge is open only in K_2

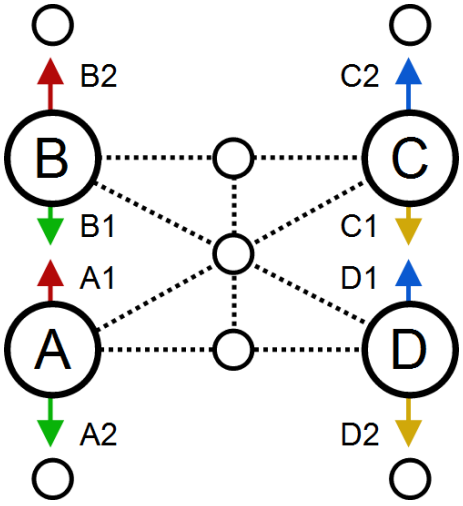

Figure 4.7: Illustration of the notation for the construction of the step condition.

condition. The difference is that now we will list conditions given by percolation of particular edges.

Theoretically, one way how to build the whole step condition is to take various step operators one by one. We would take the first pair and determine all conditions for matrix elements. Than we would take another step operator and by comparing it with one of the previous we would further restrict the attractor matrix elements. Repeating this procedure for all possible step operators would in the end give the full step condition.

Nevertheless, we will use more convenient strategy. Every matrix element is indexed by two states that are parts of two undirected edges or are both on one edge. We will be going through particular pairs of edges and find conditions given by all possible step operators. This is useful because many step operators give the same condition on one matrix element and we just have to care whether there is at least one such pair of step operators.

Let us introduce some useful notation which is shown in figure 4.7. We will be dealing with two edges that will be referred to as X and Y, where X connects vertices A and B and Y connects vertices C and D. Every one of these undirected edges represents two directed edges. The edge X combines $A1$ and $B1$ and the edge Y combines $C1$ and $D1$. The action of the step operator S when both edges X and Y are present is denoted by colours, namely: $S\,|A1\rangle = |B2\rangle$, $S\,|B1\rangle = |A2\rangle$, $S\,|C1\rangle = |D2\rangle$ and $S\,|D1\rangle = |C2\rangle$. We consider a general graph, not necessarily some regular lattice. There are some additional vertices in the figure and also some edges that are drawn by dotted lines indicating that it does not matter if those are open or closed.

Let us go through all relevant possibilities that can occur and determine what step condition we get. If there are two configurations K_1 and K_2 such that in K_1 both X and Y are present and in K_2 none of them is present, than we have the situation shown in figure 4.8. The initial state represents the matrix element X^{B2}_{C2}. The red arrow always denotes the top index of the matrix element and the green arrow

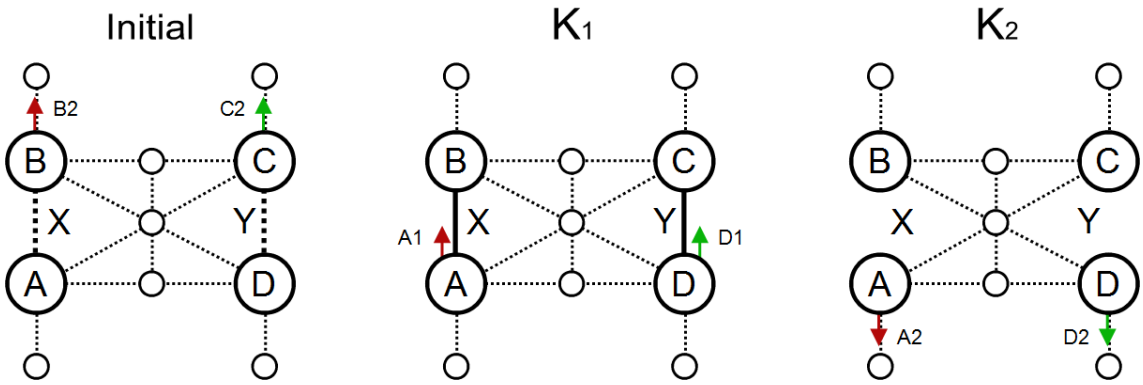

Figure 4.8: Illustration of the situation where we have the initial matrix element X^{B2}_{C2}, in the configuration K_1 both edges are present and in the configuration K_2 both edges are missing.

represents the bottom one. In this situation we therefore obtain the condition:

$$X^{B2}_{C2} = X^{A2}_{D2}. \tag{4.15}$$

It is clear that if we just switched the colours of the arrows, their positions would change in the same way. This means nothing else than that if we have some condition on the matrix elements, we always also have the same condition with top and bottom indices switched. Therefore another part of the step condition is:

$$X^{C2}_{B2} = X^{D2}_{A2}. \tag{4.16}$$

It is important to note that for the same configurations K_1 and K_2, we may choose a different initial matrix element. This is shown in figure 4.9. We see that here we get the conditions:

$$X^{B2}_{D2} = X^{A2}_{C2}, \tag{4.17}$$
$$X^{D2}_{B2} = X^{C2}_{A2}. \tag{4.18}$$

Those are all conditions where both indices of the matrix element differ on the left hand side and on the right hand side of the equality. We can not choose for example $A1$ as one of the indices of the matrix element, because then the result would depend on the presence or absence of different edge than X.

Let us note one more thing. The very same set of conditions as we have derived so far also arises if instead of having one configuration with both edges and one with none of them, we have two configurations K_1 and K_2 such that K_1 contains only X and K_2 contains only Y. This situation is illustrated in figure 4.10. We have shown

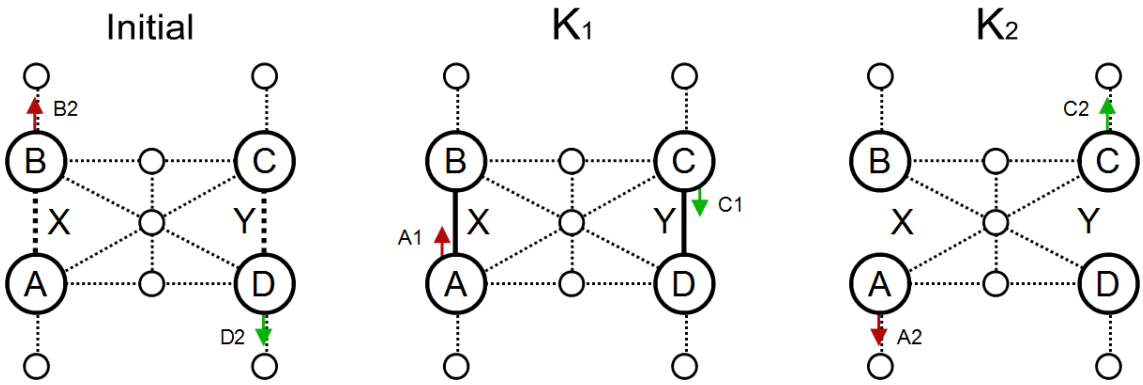

Figure 4.9: Illustration of the situation where again in the configuration K_1 both edges are present and in the configuration K_2 both edges are missing but we have the initial matrix element X_{D2}^{B2}.

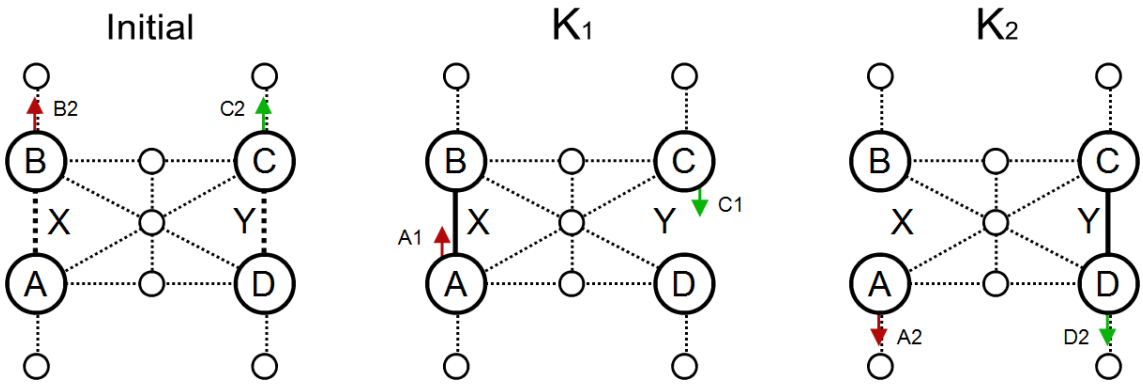

Figure 4.10: Illustration of the situation where we have the initial matrix element X_{C2}^{B2}, in the configuration K_1 only the edge X is present and in the configuration K_2 only the edge Y is present.

here only one choice of the initial matrix element but the equivalence is clear. We are mentioning this because it will be important later in discussion about equivalence of the step condition of a partial percolation scheme with the full percolation.

Let us now have two configurations K_1 and K_2 such that K_1 contains only Y and K_2 contains both X and Y. This is shown in figure 4.11.

The condition obtained here is:

$$X_{C2}^{B2} = X_{C2}^{A2}. \tag{4.19}$$

Obviously, this can be easily generalised for other situations. The index of the matrix element corresponding to the edge that is either open in both configurations or closed in both configurations does not change. The other one changes in the same way as in the previous examples. Therefore in total we have conditions:

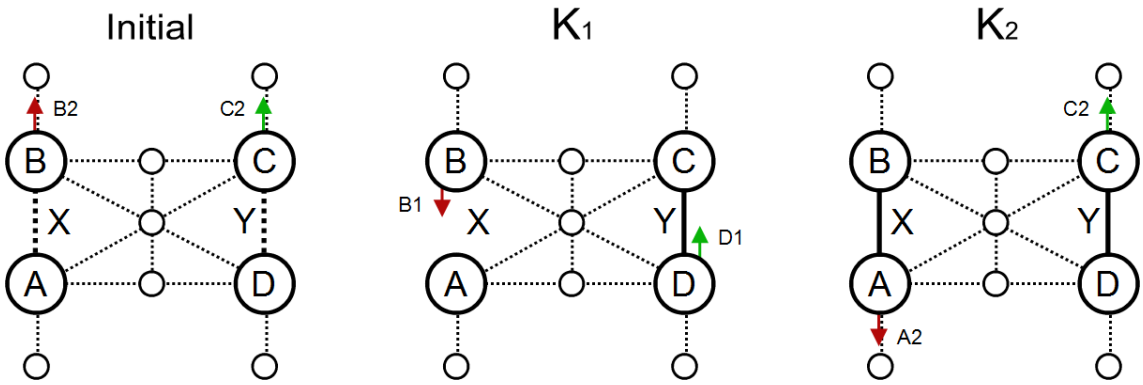

Figure 4.11: Illustration of the situation where the configuration K_1 has only Y and the configuration K_2 has both edges present.

$$X^{B2}_{C2} = X^{A2}_{C2}, \quad X^{B2}_{D2} = X^{A2}_{D2}, \tag{4.20}$$
$$X^{C2}_{B2} = X^{C2}_{A2}, \quad X^{D2}_{B2} = X^{D2}_{A2}, \tag{4.21}$$

$$X^{C2}_{A2} = X^{D2}_{A2}, \quad X^{C2}_{B2} = X^{D2}_{B2}, \tag{4.22}$$
$$X^{A2}_{C2} = X^{A2}_{D2}, \quad X^{B2}_{C2} = X^{B2}_{D2}, \tag{4.23}$$

which can be rewritten into a more compact form:

$$X^{A2}_{C2} = X^{A2}_{D2} = X^{B2}_{D2} = X^{B2}_{C2}, \tag{4.24}$$
$$X^{C2}_{A2} = X^{D2}_{A2} = X^{D2}_{B2} = X^{C2}_{B2}. \tag{4.25}$$

To obtain all these conditions we need at least three configurations. Firstly, we need one containing only X and one containing only Y. Then we need one more and there are two options. It either has to contain both X and Y or none of them, but both these situations are equivalent.

One may note that conditions derived earlier where both indices change may seem useless now. For example, the condition (4.15) can be combined from (4.20) and (4.22) and is directly contained in (4.24). Nevertheless, those conditions are still important. There are situations where conditions (4.20) and (4.22) do not hold but (4.15) does as we will see in the moment.

The last unsolved issue is with returning edges. Those are not associated with some undirected edges, but are still contained in the density matrix and there are restrictions on the corresponding matrix elements. Basically, these states can be treated just as if they were a part of some undirected edge, which is always closed. Therefore we have the conditions, where the index corresponding to the returning edge is fixed, but the other one may be different.

4.3.4 The Full Percolation

By the full percolation we mean the situation where every edge of the original undirected graph $G = (V, E)$ can be open or closed independently of others. Therefore the percolation process may generate any configuration from the power set 2^E. In contrast to the full percolation, we will later discuss the case of a partial percolation where either some edges remain still open, are still closed or have some correlations, like there is a maximum number of open edges in every possible configuration.

Even in this case of the full percolation where we have all configurations that we can have, not all matrix elements must fulfil the strongest condition (4.24).

The first case has already been mentioned - the returning edges. If for example $A2$ is a returning edge, the only condition we can have is:

$$X_{C2}^{A2} = X_{D2}^{A2}, \tag{4.26}$$
$$X_{A2}^{C2} = X_{A2}^{D2}. \tag{4.27}$$

Therefore, there are no connections between two returning edges.

The other case where the step condition is weaker is when the two edges X and Y are chosen to be the same edge. In such a case we obviously can not have configuration containing only X or only Y. Therefore we do not have conditions with only one changing index, but still there are condition where both indices change. Those can be written in this situations as:

$$X_{A2}^{A2} = X_{B2}^{B2}, \tag{4.28}$$
$$X_{B2}^{A2} = X_{A2}^{B2}. \tag{4.29}$$

In all other cases we have the maximal step condition in the case of the full percolation.

4.3.5 Equivalence of Step Conditions

It is obvious that when we allow all possible subsets of E as configuration for a percolated quantum walk, a lot of conditions on matrix operators given by the step condition are given redundantly by many pairs of step operators. A question of quite some interest is, what smaller sets of configurations give the same overall step condition.

The sufficient (but not necessary) condition is to have for every pair of edges X and Y some configurations, where only X is present and only Y is present and either both edges X and Y are present or none of them is present. Symbolic expression might help - we need: $K_X \wedge K_Y \wedge (K_{XY} \vee K_\emptyset)$. The presence of any larger structures has no effect on the step condition. It certainly may be the case, that even if we

take out more configuration, the resulting asymptotic state may be the same due to some symmetry of the coin operation. Nevertheless, we will examine the sufficient condition described above without any concerns about the coin operation.

4.3.6 Adding Empty Configuration

Before turning to examples of subsets of configuration giving the same step condition, let us note one important fact. One of the main simplifications in finding attractors for a quantum walk is using the configuration with all edges closed. This allows us to find the starting solution for one configuration in a localized form - in every vertex separately. In the case when we do not have the all-closed configuration, the situation may be more difficult and probably not solvable analytically. Nevertheless, there may be a trick. If the set of configurations we have at our disposal is such, that adding the all-closed configuration does not change the step condition, we actually may use this configuration for calculation anyway.

If adding the all-closed configuration does not change the step condition, that means that adding the corresponding unitary operation has no effect on the attractor-determining equation 4.3. Therefore the final asymptotic behaviour will be the same in both scenarios.

4.4 Examples of Compatible Sets of Configurations

Let us for now call subsets of possible configurations, which result in the same asymptotic state, as *equivalent sets*. Now we may think of some example sets of configurations fulfilling this requirement.

1-Edge Configurations

From the previous discussion we see, that for example all we need are configurations with exactly one edge. Then we always have configurations with only one edge present plus the configuration with both edges missing. Here we assume that the graph has at least three edges.

This model may represent a very weakly connected fluctuating medium - there are only small "bridges" and most of the edges are closed in every step. We have just derived that this is sufficient to lead to the same asymptotic state as in the case of the full percolation.

Due to the fact that adding more configurations can not change the asymptotic state (it is already the same as for the full percolation), we may also use this model in the situation, where in principle all configurations are possible, but p is so small that chances for configurations with many edges can be neglected. This might be the case in some physical situation and therefore it is good to know that again this set of configurations is compatible with the full percolation.

2-Edge Configurations

Another example is taking only configurations with exactly two edges. Note that this does not mean two "connected" edges (having a common vertex). Nevertheless, even only configurations with pairs of open edges sharing a vertex are enough, if we put a restriction on the underlying graph. This is discussed in the example of site percolation below.

For this example we assume, that our graph has at least 3 edges. Otherwise, the configurations with only one edge would be missing.

n-Edge Configurations

The previous example can be generalised to configurations with exactly n ($n \geq 2$) edges if we assume, that there are at least $n+1$ edges in the graph. This assures that we have configurations with only one edge of any pair plus there are configurations with both edges present.

"Holes"

All previous examples can also be inverted and instead of having only n edges open, we have only n edges closed. (Restrictions on the minimum number of edges in the graph are the same.) This may be of interest in some physical situation where the system undergoes just a small disturbance - even removing just one edge at the time will ultimately lead the system into the same state as the full percolation process.

Site percolation

This example is also quite physically relevant. Until now, we have been concerned only with bond percolation, where edges are being closed. We may actually use this framework to define a kind of site percolation, where vertices are open or closed.

If we define a site percolation in a quantum walk in the way that broken vertex means broken all edges that have this vertex on one side, then the asymptotic states will be again the same as for the full bond percolation if we give one restriction on the underlying graph. There cannot be a pair of vertices that would be connected with more than one edge.

If we take a pair of edges that do not share a vertex, it is trivial to find configurations with both edges closed and also each one open separately with the other one closed.

If our pair of edges share one vertex, there is also no problem. The two configurations where only one edge is open can be reached by leaving the connecting vertex open and breaking the vertex on the other end on of one of the edges.

The problem would be with "double-connected" vertices. If there were two edges connecting one pair of vertices, then for these two edges we would not have configurations with only one of them open. If any of the vertices is broken, also both edges

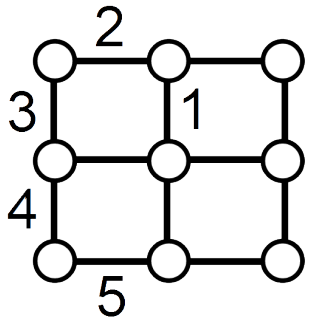

Figure 4.12: The underlying graph for the example with Hamilton paths. Only edges which we will refer to are labelled with numbers.

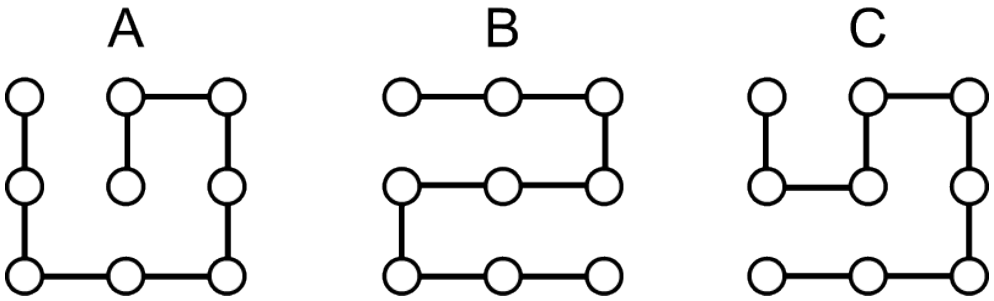

Figure 4.13: Representatives of three classes of Hamilton paths used for showing equivalence of the step condition. The whole class is obtained by rotations and reflections of the representative.

are closed. Therefore we must make this assumption, which is not very restricting - most of the studied cases do not have these double connections between vertices.

Hamilton paths

The last example will demonstrate the procedure of verifying equivalence of the step condition if we are given a particular graph and a structure of allowed configurations. We will have a 3×3 square lattice and the only possible edge configurations in the percolated graph will be Hamilton paths - connected sequences of open edges which go through every vertex exactly once. The underlying graph is shown in figure 4.12.

Our previous examples were chosen because the verification of the equivalence of the step condition was easy and possible for big classes of underlying graphs of arbitrary sizes. In contrast to that here the underlying graph is specified, and the solution will be more tedious.

We will show that for every pair of edges there are Hamilton paths where only one of the edges is open plus Hamilton paths where both these edges are open. For this we will use three classes of paths. One representative of every class is in figure 4.13, and the rest of the class is obtained by rotations and reflections.

Let us start with cases where we want both edges to be open and let us look for suitable choices of paths. We will divide edges into two sets. The edges which do not have the middle vertex as any of their endpoints will be called outer edges and the remaining ones will be called inner edges.

If both edges of interest are outer, we see immediately that the class A provides configurations with both edges open. If we have two inner edges, and they are parallel, we use the class B, and if they are perpendicular, we use C. If one edge is inner and the other one is outer, we again use A.

Now we need configurations with one edge open and one closed. Here if both edges are inner or both are outer, we can use the class A. The class A can also be used if the inner edge is closed and the outer one is open. The situation is more complicated if the inner edge is open and the outer edge is closed. We now use the notation from figure 4.12. The open edge is labelled 1. If the closed edge is 2, we use the class A, if it is 3 or 4, we use the class C, and if it is 4, we use the class B. All other cases are obtained by a reflection.

We have gone through all combinations of two edges and therefore proved that percolation with only Hamilton path configurations has the same asymptotic state as the full percolation on a 3×3 square lattice. It for example shows that in this case covering a 2D graph by 1D structures is from the point of view of asymptotics equivalent to the full 2D percolation.

4.5 Results Summary

In this chapter I presented the main results of the thesis. That is in particular a sufficient condition for a restricted kind of percolation that assures the same asymptotic behaviour as the full percolation. Further, I have listed several examples of such percolation schemes. Most of them are for very general classes of underlying graphs and show the robustness of the asymptotic state. Then I have added one example with a specified graph and large structure configurations - Hamilton paths. This example shows that situation may be rather complicated if the allowed edge configurations have a global structure. Nevertheless, the condition presented here still makes the procedure much more simple than it would be to look for the attractors from scratch.

The aim was not to give some exhaustive list of percolation schemes equivalent with the full percolation. I wanted to show that the condition can be used in a variety of situations and that it cam be rather simple to use it in a particular situation.

Chapter 5

Numerical Results

We will check some of our analytical results by a direct numerical simulation. Previously, I have proposed that only some subsets of all edge configurations are needed in the percolation process to arrive at the same asymptomatic state as in the case of the full percolation.

For this demonstration, we will use the already solved [10] case of a percolated quantum walk on a 2D square lattice. The procedure is straightforward. We have the analytical solution for the asymptotic state, which can be calculated in every step of the walk. We then explicitly construct a step operator for every edge configuration. Using these operators we may construct superoperators Φ_i representing the evolution for some partial percolation[1].

Further, we choose some initial state and evaluate the walk step by step. In every step we compare the analytically calculated asymptotic state with the state evolved using the superoperator Φ_i. The difference between the two density matrices A and B is measured as $||A-B||_{HS}$, where the norm is induced by the Hilbert-Schmidt scalar product, so:

$$||A-B||_{HS} = \sqrt{\langle A-B, A-B\rangle_{HS}} = \sqrt{\mathrm{Tr}\left((A-B)(A-B)^\dagger\right)}. \tag{5.1}$$

5.1 The 2×3 Graph

We will use a very simple example - the 2×3 2D square lattice which is illustrated in figure 5.1.

The vertices are labelled with numbers $\{1,2,3,4,5,6\}$ and edges by letters $\{A,B,C,D,E,F,G\}$. The initial state is chosen to be $|\psi_0\rangle = \frac{1}{2}\left(|2L\rangle + |2D\rangle + |2U\rangle + |2R\rangle\right)$. As the coin operator, we use the four-dimensional Hadamard coin:

[1] If we remove some edge configurations, we have to somehow modify probabilities of particular configurations to make them sum to 1. The asymptotic states do not depend on these probabilities so we use the uniform distribution.

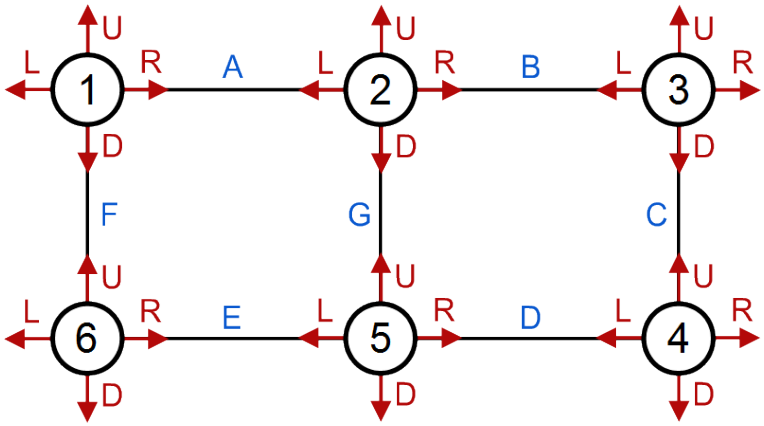

Figure 5.1: Illustration of the 2×3 2D square lattice graph.

$$C_H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

and the four-dimensional Grover coin:

$$C_G = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

The Grover coin is used because the Hadamard coin has a symmetry that might lead to somewhat confusing results. In particular there is a case when even smaller subset of configurations than we would require from our previous investigation is sufficient to lead to the asymptotic state of the fully percolated walk.

We examine five scenarios. Three that should converge to the full-percolation asymptotic state: one-edge configurations, holes and the site percolation. In the case of the site percolation, the subset of configurations is reduced even further. We use only those where just one vertex is open in every configuration, which is enough. The remaining two configurations are in general insufficient to lead to the same asymptotic state as the full percolation. In one case we have only all-closed and all-open configurations. In the other case edges B, C and D are permanently closed.

5.1.1 Grover Coin

The result of the numerical simulation with the Grover coin is shown in figure 5.2.

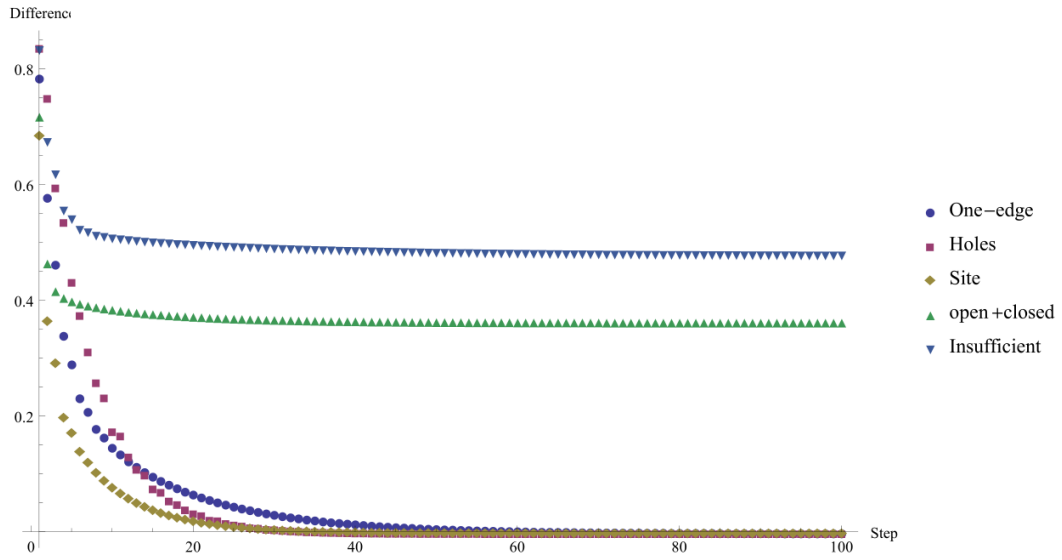

Figure 5.2: Result of the numerical simulation for the 2×3 graph with the Grover coin showing the distance of the state in a particular step of the percolated quantum walk from the calculated asymptotic state of the full percolation for five different scenarios.

We clearly see that all three cases that should be equivalent to the full percolation converge to the predicted asymptotic state. In contrast, sets of configurations that are in general not equivalent stay at a constant distance after some time.

5.1.2 Hadamard Coin

As predicted, also in this scenario all three cases that should be equivalent to the full percolation converge to the predicted calculated state. The result of the numerical simulation with the Hadamard coin is shown in figure 5.3. The difference is that also the set of configurations with only all-closed and all-open configurations converges to the asymptotic state of the fully percolated walk. This is a demonstration of the fact that the condition proposed for equivalence is sufficient but in general not necessary. The configuration with some edges permanently closed still has a different asymptotic state.

5.2 Summary

Numerical simulations presented in this chapter demonstrate several facts. One is that we have verified that in all cases where it was predicted by theory the state of the walk converges to the analytically found asymptotic state. Further the example with the Hadamard coin shows that the condition for equivalence of the step condition is sufficient but not necessary because the scheme with only all-open and all-closed edges does not satisfy our condition but still converges to the same asymptotic state

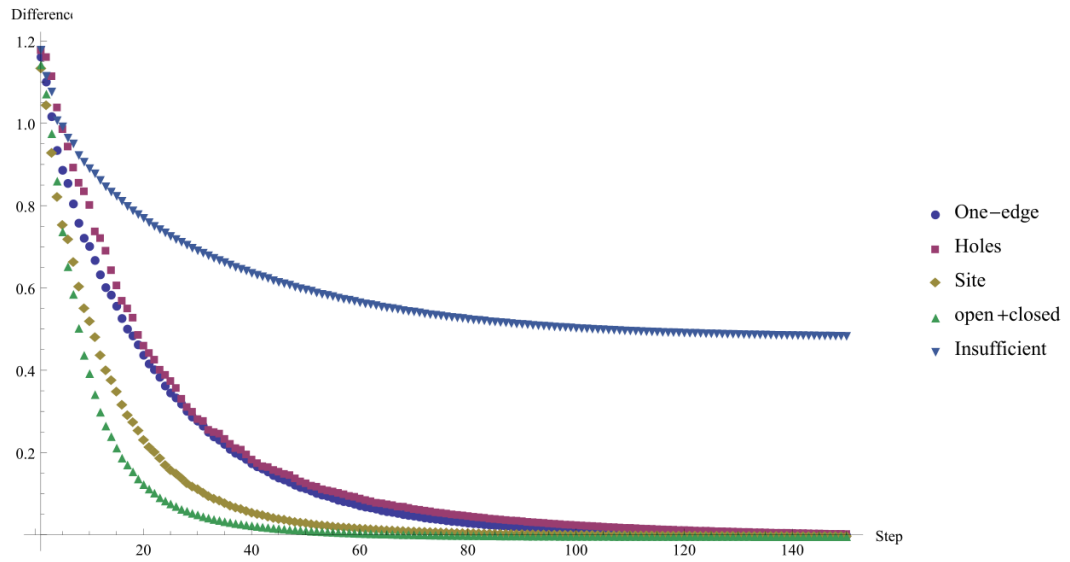

Figure 5.3: Result of the numerical check for the 2×3 graph with the Hadamard coin showing the distance of the state in a particular step of the percolated quantum walk from the calculated asymptotic state of the full percolation for five different scenarios.

as the full percolation.

The last point that I want to emphasise here is a note about convergence rates. Up to now, there is no available analytical method for determining how fast will the system get to the asymptotic regime and therefore we have to rely on numerics. The main result from our simulations is that the rate of convergence depends on the percolation scheme used. In some cases schemes with very restricted set of configurations may converge very fast. This is seen in the Hadamard walk if we use only all-open and all-closed configurations. Here we get rather close to the predicted asymptotic behaviour in order of tens of steps. This suggests that dependence of the rate of convergence on the type of percolation is worth investigating. Particularly, it may be very important for designing experimental settings.

Conclusions

In this thesis we have first introduced the needed concepts of percolation, quantum walks and random unitary operations. Then we combined these concepts to answer some questions about percolated quantum walks.

Throughout the thesis I was trying to work with maximal generality. I consider the requirement posed on the step operator in a quantum walk to be a permutation matrix to be one of the main contributions of this work. On one hand, it still leaves a lot of space for all kinds of settings and on the other hand allows us to find some universal properties of such walks. In particular, the general step condition in finding asymptotic evolution of a percolated quantum walk can be derived for such step operators.

The second and probably more important result of this thesis is the determination of a convenient sufficient (but not necessary) condition on a restricted set of configurations in a percolated graph that gives the same asymptotic behaviour as the fully percolated graph. This condition allows us to use already solved percolated quantum walks and without any additional effort use the solution in other situations. Particularly, it should be pointed out that this allows us to "solve" a site percolation in a quantum walk on a given graph as soon as the bond percolation is solved. With the definition of the site percolation used here the asymptotic states are the same.

Further, I have shown that the class of restricted percolation schemes that have the same asymptotic state as the full percolation is very large. For example, breaking just one edge of the underlying graph at a time is enough, and if we add any other configuration the result is still the same. Let us now discuss several implications of this finding.

It may be a disappointing finding, that by observation of the asymptotic state of some system we get just a little information about what kind of percolation is present in the system. Nevertheless, there is still extremely strong dependence of the asymptotic state on the chosen coin operation. It means that by observing the asymptotic state of some system we may learn something about the local dynamics (the coin operation in the vertex) and we do not need to worry very much whether the percolation is not just partial.

Another situation where it could be possible to use these results is in performing experiments. We want to carry out experiment in the simplest and the most convenient form. It may be very helpful that we only need to implement a very restricted set of configurations. In a graph with n edges, there are 2^n configurations in the

full percolations but only n one-edge configurations. This simplifications may be the difference between feasible and infeasible experiment.

One thing to keep in mind here is the convergence rate. There may be a problem that by restricting the set of configurations used, it may require many more steps to approach the asymptotic regime. Therefore we need to choose the scheme wisely to achieve best results. Further investigation in this area may allow significant optimisation of experiments.

There is one more situation which also has to do with real physical systems. From the results of this thesis we see that the particular asymptotic states of a percolated quantum walk is a robust effect not dependent on some delicate choice of possible edge configurations. Therefore this model has a good chance to describe real systems where we could for example hardly rely on occurrence of all possible edge configurations in reasonable time.

Until now only solutions for walks on 1D line and 2D square lattice have been published. It can be expected that solutions for other graphs like 3D lattices will emerge. These walks may have many interesting and novel properties, but from the point of view of equivalence of the step condition of a restricted percolation with the full percolation, there is no need to modify anything. The results of this thesis are general and in many cases do not depend on the underlying graph or its dimensionality at all. The fact that we are comparing pairs of edges has nothing to do with the dimension of the graph but simply with the fact that matrix elements of a density matrix have two indices.

In conclusion, this thesis presents a convenient condition for equivalence of restricted percolation schemes with the full dynamical percolation in the sense that the asymptotic state of the quantum walk is the same and lists some examples of such schemes which are relevant to experiments and to modelling real physical systems.

Appendix A

The Simplest Percolated QW

A.1 Definition - Percolated QW on 2 Vertices

In this chapter we will examine the simplest example of a percolated quantum walk: QW on two vertices connected by one edge. It is a special case of a percolated quantum walk on a finite line and the asymptotic behaviour has already been solved [9]. We will use this simplest example to complement the chapter on random unitary operations and also the chapter on asymptotic behaviour of quantum walks. The simplicity of this example allows us to show everything in great detail and in particular explicitly list a lot of result, that are very inconvenient to list for any larger graphs (e. g. all $2^{|E|}$ possible step operators).

The definition of our QW is shown in the figure A.1. The Hilbert space $\mathcal{H}$ is only four-dimensional: 2 vertices and in every vertex 2 directions. Therefore the basis states are:

$$|1\rangle \otimes |L\rangle = |1L\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |1\rangle \otimes |R\rangle = |1R\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{A.1}$$

$$|2\rangle \otimes |L\rangle = |2L\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |2\rangle \otimes |R\rangle = |2R\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{A.2}$$

We will again use the 2-dimensional Hadamard matrix as the coin operation in every vertex:

$$C_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{A.3}$$

and therefore the overall coin operator is:

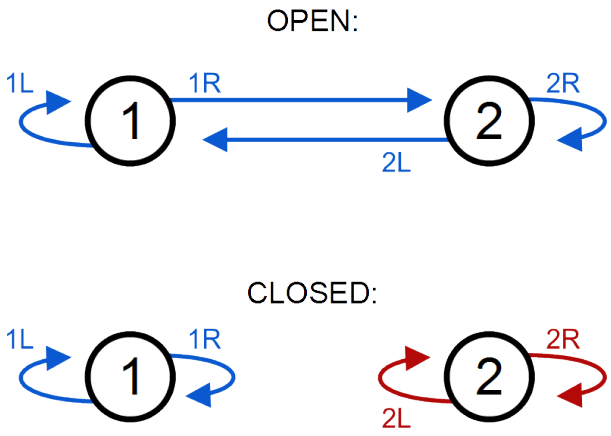

Figure A.1: Percolation in quantum walk on a graph with 2 vertices (1 and 2), only one edge and 4 direction states $1L, 1R, 2L, 2R$. There are only two possible step operators: with the edge closed and open.

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \tag{A.4}$$

There are only two possible step operators and their actions are:

$$S_0 |1L\rangle = |1R\rangle, \quad S_0 |1R\rangle = |1L\rangle, \quad S_0 |2L\rangle = |2R\rangle, \quad S_0 |2R\rangle = |2L\rangle, \tag{A.5}$$
$$S_1 |1L\rangle = |1R\rangle, \quad S_1 |1R\rangle = |2R\rangle, \quad S_1 |2L\rangle = |1L\rangle, \quad S_1 |2R\rangle = |2L\rangle. \tag{A.6}$$

We can represent them by permutation matrices:

$$S_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad S_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{A.7}$$

where S_0 is applied when the edge is closed and S_1 when it is open. These two step operators give rise to two unitary evolution operators:

$$U_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad U_1 = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}. \tag{A.8}$$

Let us keep everything as simple as possible and choose the probability p of the edge being open as $p = \frac{1}{2}$. Now the evolution superoperator Φ governing the time evolution of mixed states is:

$$\Phi(\rho(t+1)) = \frac{1}{2}U_0\rho(t)U_0^\dagger + \frac{1}{2}U_1\rho(t)U_1^\dagger. \tag{A.9}$$

Here we use ρ as a density matrix but Φ can be applied on any 4×4 matrix (not necessarily a valid density operator). We now want to write the action of Φ as application of a matrix. This matrix will be of the dimension 16×16 and will be applied on the density matrix ρ (or other matrix X), that will be written in a vector form, so:

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{13} \\ \rho_{14} \\ \rho_{21} \\ \dots \\ \rho_{44} \end{bmatrix}. \tag{A.10}$$

The matrix form of Φ is already quite nasty looking:

$$\Phi = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{A.11}$$

This operator has 15 eigenvectors and one generalised eigenvector. There are 6 eigenvectors corresponding to eigenvalue 0 (we will present them in the matrix form):

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{A.12}$$

$$X_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.13}$$

three eigenvectors corresponding to eigenvalue 1:

$$X_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad X_8 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad X_9 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{A.14}$$

two eigenvectors corresponding to eigenvalue $\frac{1}{2} - \frac{i}{2}$:

$$X_{10} = \begin{bmatrix} 1 & 0 & 1+i & -i \\ 1-i & -i & 1 & 0 \\ 0 & -1 & i & 1-i \\ i & 1+i & 0 & -1 \end{bmatrix}, \quad X_{11} = \begin{bmatrix} 0 & 1 & -i & -1+i \\ -1 & 0 & -1-i & i \\ i & 1+i & 0 & -1 \\ 1-i & -i & 1 & 0 \end{bmatrix}, \tag{A.15}$$

two eigenvectors corresponding to eigenvalue $\frac{1}{2} + \frac{i}{2}$:

$$X_{12} = \begin{bmatrix} 1 & 0 & 1-i & i \\ 1+i & i & 1 & 0 \\ 0 & -1 & -i & 1+i \\ -i & 1-i & 0 & -1 \end{bmatrix}, \quad X_{13} = \begin{bmatrix} 0 & 1 & i & -1-i \\ -1 & 0 & -1+i & -i \\ -i & 1-i & 0 & -1 \\ 1+i & i & 1 & 0 \end{bmatrix}, \tag{A.16}$$

and two more eigenvectors corresponding to eigenvalues $-i$ and i respectively:

$$X_{14} = \begin{bmatrix} 1 & -i & i & 1 \\ -i & -1 & 1 & -i \\ i & 1 & -1 & i \\ 1 & -i & i & 1 \end{bmatrix}, \quad X_{15} = \begin{bmatrix} 1 & i & -i & 1 \\ i & -1 & 1 & i \\ -i & 1 & -1 & -i \\ 1 & i & -i & 1 \end{bmatrix} \tag{A.17}$$

The generalised eigenvector is of the form:

$$X_{16} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} \\ 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 \\ 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 \\ -\frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}. \tag{A.18}$$

It corresponds to a generalised eigenvalue 0 and in particular fulfils the relation: $(\Phi - \lambda)^2 X = \Phi^2 X = 0X = 0$. Using this orthonormal basis, it is possible to find Jordan normal form of Φ:

$$\tilde{\Phi} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{0}&\mathbf{1}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\mathbf{0}&\mathbf{0}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-i&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&i&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}-\frac{i}{2}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}-\frac{i}{2}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}+\frac{i}{2}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}+\frac{i}{2}
\end{bmatrix}. \tag{A.19}$$

The problematic block, that is not 1-dimensional is highlighted. By an explicit calculation, we see that by second application of $\tilde{\Phi}$ this block disappears:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{A.20}$$

A.1.1 Finding Attractors

We have already found all 5 attractors of the simplest percolated quantum walk. Those are 3 matrices corresponding to the eigenvalue 1 ($X7$, $X8$ and $X9$) and two corresponding to eigenvalues $-i$ ($X14$) and i ($X15$). Nevertheless, we have used a method totally unsuitable for larger graphs. Now we will calculate the same attractors using the method presented in previous chapters, where we do not calculate directly all eigenstates and generalised eigenstates of the superoperator $\tilde{\Phi}$.

Firstly, we have to choose one particular evolution operator to start with. In this example we have only two of them, but even if there are many, the convenient choice is obvious. We choose the one where all edges are closed, because then the problem can be solved separately in every vertex. Let us therefore take the operator U_0:

$$U_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{A.21}$$

Due to the fact that we are dealing with a "regular lattice" where "all" vertices have the same coin operator we can write:

$$U_0 = \mathbb{1}_2 \otimes RC_0, \tag{A.22}$$

where $\mathbb{1}_2$ is a two-dimensional identity matrix, C_0 is the Hadamard coin applied in every vertex and R is a matrix that we will call the reflection operator. It defines the local action of the step operator in a given vertex when all surrounding edges are closed and it can be represented as:

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{A.23}$$

Now we have to solve the equation (4.5) which can be also written as:

$$(S_0C)X(S_0C)^\dagger = \lambda X, \tag{A.24}$$

$$U_0XU_0^\dagger = \lambda X, \tag{A.25}$$

$$(\mathbb{1}_2 \otimes RC_0)X(\mathbb{1}_2 \otimes RC_0)^\dagger = \lambda X. \tag{A.26}$$

Let us write the matrix X in the form:

$$X = \sum_{i=1}^{2}\sum_{j=1}^{2} |i\rangle\langle j| \otimes X^{(i,j)} = \tag{A.27}$$

$$= |1\rangle\langle 1| \otimes X^{(1,1)} + |1\rangle\langle 2| \otimes X^{(1,2)} + |2\rangle\langle 1| \otimes X^{(2,1)} + |2\rangle\langle 2| \otimes X^{(2,2)} = \tag{A.28}$$

$$= \begin{bmatrix} X^{(1,1)} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & X^{(1,2)} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ X^{(2,1)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & X^{(2,2)} \end{bmatrix}, \tag{A.29}$$

where the summing in every index is in general over all vertices and $X^{(i,j)}$ represents a 2×2 block of the matrix X. If we now write the equation (A.24) with X in this

form we can see, that we get four separate equations for different coin blocks. All these equations are of the same form:

$$(RC_0)X^{(i,j)}(C_0R)^\dagger = \lambda X^{(i,j)}. \tag{A.30}$$

This equation can actually be written in the vector form as

$$(RC_0) \otimes (C_0R)^* \tilde{X}^{(i,j)} = \lambda \tilde{X}^{(i,j)}, \tag{A.31}$$

where the star denotes complex conjugate. This is a standard eigenvalue problem for the 4×4 matrix:

$$(RC_0) \otimes (C_0R)^* = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{A.32}$$

Even in this example the importance of these steps is quite obvious. We have moved from a 16×16 matrix to a 4×4 matrix and in general it is from $n^2 \times n^2$ to $n \times n$.

This matrix has two orthogonal eigenvectors corresponding to the eigenvalue 1, which can be written in the form of matrix blocks as:

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{A.33}$$

so the general eigenvector corresponding to the eigenvalue 1 is:

$$x_{(1)} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{A.34}$$

The whole attractor with the eigenvalue 1 must therefore be of the form:

$$X_{(1)} = \begin{bmatrix} \alpha_1 & -\beta_1 & \alpha_2 & -\beta_2 \\ \beta_1 & \alpha_1 & \beta_2 & \alpha_2 \\ \alpha_3 & -\beta_3 & \alpha_4 & -\beta_4 \\ \beta_3 & \alpha_3 & \beta_4 & \alpha_4 \end{bmatrix}. \tag{A.35}$$

Further, there are two eigenvectors corresponding to eigenvalues i and $-i$:

$$x_{(i)} = \alpha \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}, \quad x_{(-i)} = \alpha \begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix} \tag{A.36}$$

with corresponding attractors:

$$X_{(i)} = \begin{bmatrix} -\alpha_1 & i\alpha_1 & -\alpha_2 & i\alpha_2 \\ i\alpha_1 & \alpha_1 & i\alpha_2 & \alpha_2 \\ -\alpha_3 & i\alpha_3 & -\alpha_4 & i\alpha_4 \\ i\alpha_3 & \alpha_3 & i\alpha_4 & \alpha_4 \end{bmatrix}, \quad X_{(-i)} = \begin{bmatrix} -\alpha_1 & -i\alpha_1 & -\alpha_2 & -i\alpha_2 \\ -i\alpha_1 & \alpha_1 & -i\alpha_2 & \alpha_2 \\ -\alpha_3 & -i\alpha_3 & -\alpha_4 & -i\alpha_4 \\ -i\alpha_3 & \alpha_3 & -i\alpha_4 & \alpha_4 \end{bmatrix}. \tag{A.37}$$

The step condition is rather restricted in this case. There is no way how to choose two different edges that could be open. We have only conditions resulting from choosing the only undirected edge for both indices and condition given by choosing this one edge and returning edges. The first option gives us matrix element equalities:

$$X^{1L}_{1L} = X^{2R}_{2R}, \quad X^{1L}_{2R} = X^{2R}_{1L} \tag{A.38}$$

and the other one gives:

$$X^{1L}_{1R} = X^{2R}_{1R}, \quad X^{1L}_{2L} = X^{2R}_{2L}, \tag{A.39}$$

$$X^{1R}_{1L} = X^{1R}_{2R}, \quad X^{2L}_{1L} = X^{2L}_{2R}. \tag{A.40}$$

The step condition can be displayed by the following matrix, where the same letters indicate that the matrix elements must be equal:

$$\begin{bmatrix} a & b & c & d \\ e & & & e \\ f & & & f \\ d & b & c & a \end{bmatrix}. \tag{A.41}$$

If we apply the step condition on the general form of the attractor corresponding to the eigenvalue 1, we get:

$$X_{(1)}(\alpha_1, \beta_1, \beta_2) = \begin{bmatrix} \alpha_1 & -\beta_1 & \beta_1 & -\beta_2 \\ \beta_1 & \alpha_1 & \beta_2 & \beta_1 \\ -\beta_1 & \beta_2 & \alpha_1 & -\beta_1 \\ -\beta_2 & -\beta_1 & \beta_1 & \alpha_1 \end{bmatrix}. \tag{A.42}$$

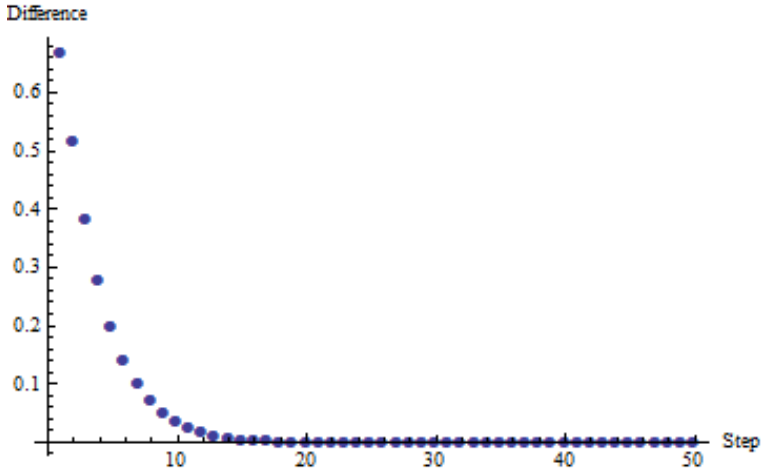

Figure A.2: Difference between the calculated asymptotic state and the state of a percolated quantum walk on a graph with two vertices, Hadamard coin and the initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1L\rangle + |1R\rangle)$.

Now we can easily get attractors found earlier by choosing the three free parameters appropriately. In particular $X7 = X_{(1)}(1,0,0)$, $X8 = X_{(1)}(0,-1,0)$ and $X9 = X_{(1)}(0,0,-1)$. If we take a look at the attractors corresponding to i and $-i$, the step condition reduces the freedom in both cases to only multiplication by a constant and the two original attractors are obtained. This completes our search for attractors using the more scalable method.

Obviously, in this example there is no space for examining some equivalent subsets of configurations. We have only two - with the edge open and with the edge closed. If we would restrict ourselves on either only open or only closed edge, we would get an undisturbed unitary evolution which is certainly different from the non-unitary one given by the superoperator Φ.

Finally, we can check numerically that the walk will really converge to the predicted asymptotic state. Figure A.2 shows the evolution of the difference from the asymptotic state for the initial state chosen to be $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1L\rangle + |1R\rangle)$. The convergence is rather good - 10 steps are enough to bring the system very close to the asymptotic regime.

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