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Faculty of Nuclear Sciences and Physical Engineering

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Bachelor thesis

Operator theoretic approach to the theory of metamaterials

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Supervisor: Mgr. David Krejčiřík, PhD., DSc.

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ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ
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Bakalářská práce

Operátorový přístup k teorii metamateriálů

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V Praze dne

Title:
Operator theoretic approach to the theory of metamaterials

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Abstract: We are dealing here with a mathematical description of metamaterials using operators. From Maxwell's equations we derive at first self-adjoint operator and then using theory of sectorial quadratic forms we find the "invisibility" operator which is non-self-adjoint. We introduce possibilities for application of the metamaterials and describe effect of invisibility which is provided by metamaterials for certain wavelengths. We compute spectrum of both operators and we use these results for mathematical justification of complexification in the "invisibility" operator.

Key words: metamaterials, Maxwell's equations, sectorial quadratic forms, spectral analysis

Název práce:
Operátorový přístup k teorii metamateriálů

Autor: Filip Hložek

Abstrakt: Zabýváme se zde matematickým popisem metamateriálů pomocí operátorů. Z Maxwellových rovnic odvozujeme nejprve samosdružený operátor a potom pomocí teorie sektoriálních kvadratických forem nalezneme operátor "neviditelnosti", který již samosdružený není. Uvádíme možnosti využití metamateriálů a popisujeme efekt neviditelnosti, který pro jisté vlnové délky umožňují. Vypočítáme spektrum obou operátorů a těchto výsledků využijeme k matematickému ospravedlnění komplexifikace v operátoru "neviditelnosti".

Klíčová slova: metamateriály, Maxwellovy rovnice, sektoriální kvadratické formy, spektrální analýza

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Contents

1 Introduction

Metamaterials are artificial materials whose structure is designed so they have properties which may not be found in conventional materials. Although this is a general term, it is used primarily for metamaterials with negative index n where electrical permittivity ε and magnetic permeability μ are simultaneously negative.

The first theoretical study of such materials comes from Russian physicist Viktor Veselago [15]. In this paper he admitted (and concluded) that substances with negative ε and μ have some properties different from ordinary substances with positive values of permittivity and permeability. To understand these properties we shall look at Maxwell's equations where ε and μ appear separately, and not in the form of their product like in squared refractive index $n^2 = \varepsilon\mu$.

$$\mathrm{rot}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.1a}$$

$$\mathrm{rot}\,\vec{H} = \frac{\partial \vec{D}}{\partial t}, \tag{1.1b}$$

and we substitute here from constitutive relations $\vec{B} = \mu\vec{H}, \vec{D} = \varepsilon\vec{E}$. In a plane monochromatic wave all quantities are proportional to $e^{i(\vec{k}\vec{r}-\omega t)}$ where $\vec{k}$ is a wave vector of the monochromatic wave, ω its frequency, $\vec{r}$ is a radius vector and t is time. Because of this fact the expressions above reduce to

$$\vec{k} \times \vec{E} = \omega\mu\vec{H}, \tag{1.2a}$$

$$\vec{k} \times \vec{H} = -\omega\varepsilon\vec{E}. \tag{1.2b}$$

Now we can see that for positive values of ε and μ the vectors $\vec{E}, \vec{H}, \vec{k}$ form a right-handed set of vectors and for negative ε and μ they form a left-handed set. That is why materials with simultaneously negative permittivity and permeability are called “left-handed”. Although the wave vector is in the opposite direction in left-handed substances, the Poynting vector, representing energy flow and defined as $\vec{S} = \vec{E} \times \vec{H}$, forms always a right-handed set with $\vec{E}$ and $\vec{H}$. From this we can see that in the left-handed materials the group velocity (and energy flow) is always opposite to the wave velocity. These results have many interesting consequences, for example reversed Doppler effect and reversed Cerenkov radiation, perfect lens and even metamaterial cloaking.

However the absence of such material in the sixties led to neglect of this subject. After thirty years at the brink of the millennium it was shown how to make materials with negative μ (substance with negative ε were known in plasma physics for relatively long time). Medium with negative permittivity can be created as a system of parallel wires. Medium with negative permeability was designed by John Pendry [9] in 1999. It was created from two concentric conductive rings called split ring resonators (SRR) which were capacitively and inductively coupled. The resonator acts as magnetic dipole with very intense response to the incident wave. These resonators can be built in two-dimensional and three-dimensional bodies and so we get better properties. In 2000 David R. Smith [13] constructed medium with negative refractive index. He combined field of small wires (for negative permittivity) with field of split resonators (for negative permeability) in such way that resonance proceeded at the same frequency.

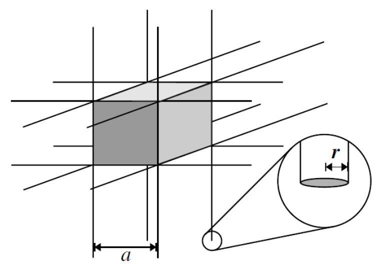

Figure 1: A metamaterial with $\varepsilon < 0$: a periodic structure composed of thin infinite wires arranged in a simple cubic latice, mimics the response of plasma. [10]

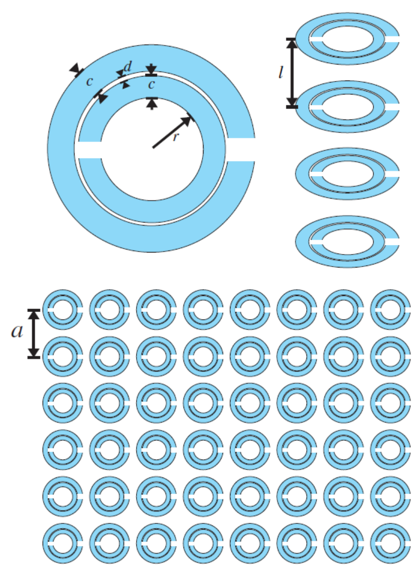

Figure 2: This metamaterial is designed to give a magnetic response to an external magnetic field in the GHz region of the spectrum: rings are manufactured in layers which are then stacked to form an array of resonant columns. [10]

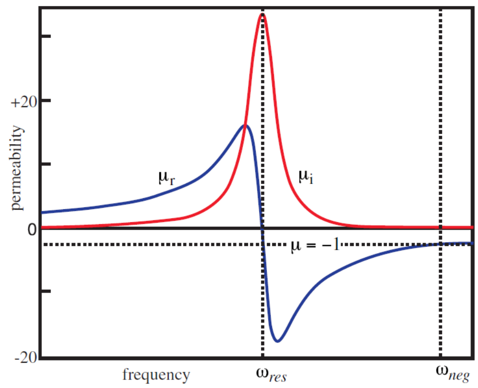

Figure 3: Schematic permeability of the magnetic metamaterial shown in Figure 2 showing the resonant response of the structure at ω_{res}. Note that the frequency at which $\mu_r = -1$ is far removed from the resonant frequency and in this instance is in a region where μ_i is small. [10]

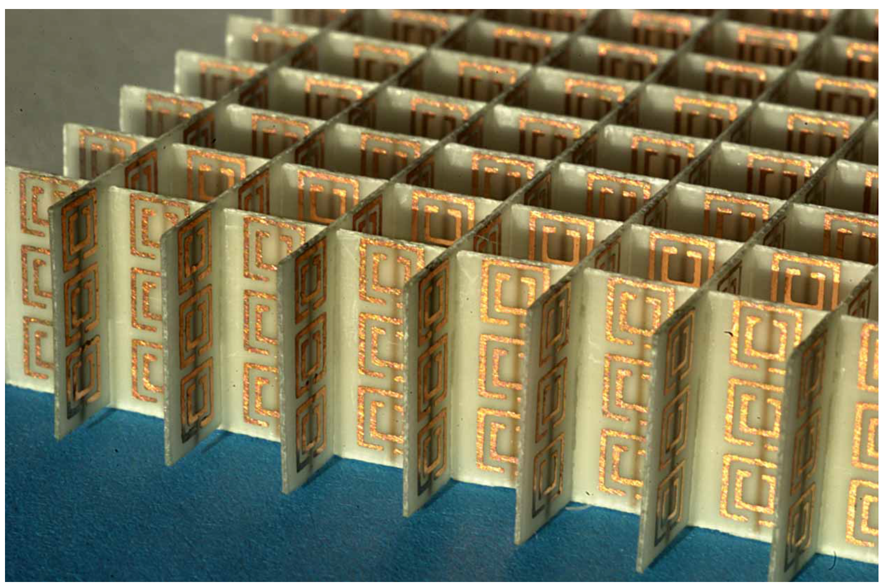

Figure 4: A split ring structure etched into copper circuit board plus copper wires to give negative μ and negative ε. Structure made at USCD by David Smith. [10]

The subject began to develop since those moments. That is because there are many promising applications of metamaterials. For example metamaterial antennas and absorbers, perfect lenses, cloaking devices, terahertz detectors, new high-tech magnetic materials and much more [10].

In this thesis we introduce and examine two operators derived from Maxwell's equations describing discontinuity of permittivity ε in zero. This discontinuity describes how the wave propagates through interface between right-handed and left-handed substances. The first operator is derived immediately from the electric part of Maxwell's equations. The second is a generalization of the first one but there is included the principle of causality which is required by physics. This condition is expressed by adding small positive imaginary parts to ε and μ (in our case only to ε) representing the fact that real systems are always slightly lossy [10].

This thesis is organized as follows. In the next chapter we show how to derive mathematical operator from Maxwell's equations. Also we give here a small physical description of the effect of invisibility. In Chapter 3 we introduce self-ajoint extension of the operator defined in previous chapter. Then in Chapter 4 we point out some important notions from theory of sectorial operators and sesquilinear forms which we use to introduce the "invisibility operator" rigorously. The spectrum of both operators is examined in Chapter 5. There equipped with calculated eigenvalues we can mathematically correctly justify physical complexification. Our results are then summed up in Chapter 6.

2 From Maxwell's equations to the mathematical model

The operators we want to consider are derived from only electric part of Maxwell's equations.

$$\operatorname{div} \vec{D} = \rho, \tag{2.1a}$$

$$\operatorname{rot} \vec{E} = 0. \tag{2.1b}$$

These two equations represent Gauss's and Faraday's law with no magnetic field. Equation (2.1b) expresses that the electrostatic field is potential and therefore we can introduce potential $\vec{E} = \nabla\varphi$. We substitute this equation together with relation $\vec{D} = \varepsilon\vec{E}$ (we assume that the material is homogeneous) to (2.1a) and we get

$$\operatorname{div} \varepsilon(\vec{x}) \operatorname{grad} \varphi = \rho. \tag{2.2}$$

The derived operator in a one-dimensional setting is

$$\begin{aligned} H &= -\frac{d}{dx}\varepsilon(x)\frac{d}{dx} \\ D(H) &= \{\psi \in L^2((-1,1)) | -\frac{d}{dx}\varepsilon(x)\frac{d}{dx}\psi(x) \in L^2((-1,1)), \psi(\pm 1) = 0\} \end{aligned} \tag{2.3}$$

We choose here negative sign for this operator because it is positive in usual case when $\varepsilon > 0$. But because we want to consider metamaterials where permittivity is negative, we define function $\varepsilon(x)$ as

$$\varepsilon(x) = \begin{cases} -1, & x \in \Omega_- = (-1,0) \\ +1, & x \in \Omega_+ = (0,1) \end{cases} \tag{2.4}$$

Now let us have a closer look at the operator H. We understand the derivative in (2.3) in distributional sense. We can see from (2.2) that expression on the left side must be a function. Therefore because of the derivatives in (2.3) we consider $\psi(x)$ and $\varepsilon(x)\frac{d}{dx}\psi(x)$ as the continuous functions in $(-1,1)$. Since the permittivity $\varepsilon(x)$ is defined as (2.4), the only problem with continuity is in zero. All together we get finally an operator

$$\begin{aligned} (H_0\psi)(x) &= \begin{cases} +\psi''(x), & x \in \Omega_- \\ -\psi''(x), & x \in \Omega_+ \end{cases} \\ D(H_0) &= \{\psi \in H^2((-1,0)) \oplus H^2((0,1)) | \psi(-1) = \psi(1) = 0, \\ &\qquad \psi(0^-) = \psi(0^+), \\ &\qquad \psi'(0^-) = -\psi'(0^+)\}, \end{aligned} \tag{2.5}$$

where $H^2((-1,0)) \oplus H^2((0,1))$ is a direct sum of Sobolev spaces [1]. In one-dimensional space we can identify it with $AC^2([-1,0]) \oplus AC^2([0,1])$. Let us note that the boundary conditions $\psi(\pm) = 0$ are as well as in (2.3) only our choice. We examine this operator in Section 3 and prove that it is self-adjoint.

2.1 Effect of invisibility

There are many ways to use metamaterials as was stated in the Introduction. However one of them arise more interest than the others and that is the metamaterial cloaking. In last few years there are exponentially increasing attention to create an electromagnetic cloak of invisibility. It can be based on various schemes [8] for example localized resonance, dipolar scattering cancellation, tunneling light transmittance, sensors and active sources, and transformation optics. Let us summarize some results of the last approach a little bit more [11].

Basics of transformation optics were established in the sixties [3], [12]. However these important studies were neglected and almost forgotten but their usage in metamaterial cloaking brought them back to life. Thus the field of transformation optics has been reestablished [11].

In order to create the electromagnetic cloak we need to have such material that is in design flexible enough to control and direct electromagnetic fields in the way we want. This is the key property of metamaterials which owe it to the subwavelength details of the structure. More informations about design and constructions of metamaterials can be found for example in [14]. In this time we can think of material that can be constructed in a way that permittivity and permeability may vary arbitrarily throughout a material, taking arbitrary sign for its values as desired. It can be shown [11] how electromagnetic quantities: electric displacement field $\vec{D}$, the magnetic field intensity $\vec{B}$ and the Poynting vector $\vec{S}$, can be directed at will. Let us imagine a source which is embedded in some elastic medium that can be pulled and stretched (see Figure 5). We might consider for example a uniform electric field and require that the field lines avoid a given region (which is what we want in the case of invisibility). As we destort the material we find useful to choose Cartesian mesh so it is easy for us to keep track of distortions. We can now understand it as coordinate transformation between the original Cartesian mesh and the distorted mesh. It turns out that Maxwell's equations have the same form in any coordinate system (that is also after this transformation) but the refractive index n, or specifically permittivity ε and permeability μ, are scaled by a common factor. For more details of this transformation see [11].

Now suppose that we want to hide an object contained in a given volume of space. If we imagine this situation, we require that anyone outside the metamaterial cannot see any difference in viewing. They must be completely unaware that something is here concealed. Therefore we need metamaterial to bend the rays around itself and return them to their original trajectory (see Figure 6). An alternative scheme has been recently investigated for the concelment of objects but there is a need for specific knowledge of the shape and the material properties of the object being hidden [11]. Therefore if the object changes the cloak must change as well.

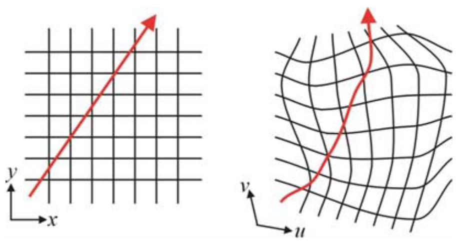

Figure 5: Left: a field line in free space with the background Cartesian coordinate grid shown. Right: the distorted field line with the background coordinates distorted in the same way. [11]

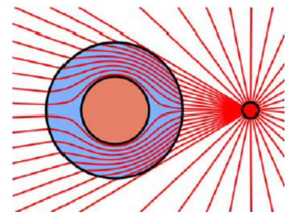

Figure 6: A point charge located near the cloaked sphere. The field is excluded from the cloaked region, but emerges from the cloaking sphere undisturbed. [11]

Although theory implies metamaterial cloaking to be very promising, there are still some issues and complications to deal with [11]. For instance in the example above an arbitrary object may be hidden because it remains untouched by external radiation. There is also an unavoidable singularity in the ray tracing. Look at the Figure 6 and consider a ray heading directly towards the centre of the sphere. Then the ray does not know which way to bend itself. Another problem is whether the cloaking is broad band or specific to a single frequency. In our example above we considered only one frequency. We also wanted the ray to flow around the sphere unchanged but it had to follow longer trajectory than it would have done in free space so in order to have the same phase a phase velocity must be grater than the velocity of light. There is no problem now but one arose when we require also absence of dispersion. For more details see [11].

3 Self-adjoint extension of symmetric operator

3.1 Proof that H_0 is self-adjoint

Now we examine the operator H_0 from Section 2. As was said there we want to find out whether this operator is self-adjoint. Self-adjointness is very useful property because we know that spectrum of such operator is real. Let us recall the definition of self-adjoint operator.

Definition 3.1.1. Operator H is self-adjoint if H is symmetric and $D(H) = D(H^*)$.

It is easy to see that operator (2.5) is symmetric using twice integration by parts. For all $\psi, \varphi \in D(H_0)$ is

$$\begin{aligned}(\psi, H_0\varphi) &= \int_{-1}^{0} \overline{\psi}\varphi'' - \int_{0}^{1} \overline{\psi}\varphi'' = \\ &= -\int_{-1}^{0} \overline{\psi'}\varphi' + \int_{0}^{1} \overline{\psi'}\varphi' + [\overline{\psi}\varphi']_{-1}^{0^-} - [\overline{\psi}\varphi']_{0+}^{1} = \\ &= \int_{-1}^{0} \overline{\psi''}\varphi - \int_{0}^{1} \overline{\psi''}\varphi - [\overline{\psi'}\varphi]_{-1}^{0^-} + [\overline{\psi'}\varphi]_{0+}^{1} + [\overline{\psi}\varphi']_{-1}^{0^-} - [\overline{\psi}\varphi']_{0+}^{1}\end{aligned}$$

The sum of the brackets gives zero (because of the boundary conditions) and so for all ψ, φ from domain of H_0 we get equality $(\psi, H_0\varphi) = (H_0\psi, \varphi)$ which expresses that the operator H_0 is symmetric. To prove equality of the domains we recall definition of adjoint operator.

Definition 3.1.2. Let H be an operator on Hilbert space $\mathscr{H}$. Then adjoint operator H^* is determined by the relation $(\varphi, H\psi) = (H^*\varphi, \psi)$ uniquely for all $\psi \in D(H)$ and $\varphi \in D(H^*)$. The domain of H^* is defined to be the set $D(H^*) = \{\phi \in \mathscr{H} | \exists \eta \in \mathscr{H}, \forall \psi \in D(H), (\phi, H\psi) = (\eta, \psi), \eta = H^*\phi\}$.

We will modify an expression of the form $(\phi, H_0\psi)$ where $\phi \in D(H_0^*)$, $\psi \in D(H_0)$ according to the definition of adjoint operator. Similarly as above we want to use twice integration by parts but first we must find out whether ϕ'' exists. For this we consider a restriction of the operator H_0

$$(\dot{H}_0\psi)(x) = \begin{cases} +\psi''(x), & x \in \Omega_- \\ -\psi''(x), & x \in \Omega_+ \end{cases}$$

$$\begin{aligned} D(\dot{H}_0) = \{\psi \in H^2((-1,0)) \oplus H^2((0,1)) | \psi(-1) = \psi(1) = 0, \\ \psi(0^-) = \psi(0^+) = 0, \\ \psi'(0^-) = \psi'(0^+) = 0\}, \end{aligned} \tag{3.1}$$

and its adjointness [2]

$$\begin{aligned} (\dot{H}_0^*\psi)(x) &= \begin{cases} +\psi''(x), & x \in \Omega_- \\ -\psi''(x), & x \in \Omega_+ \end{cases} \\ D(\dot{H}_0^*) &= \{\psi \in H^2((-1,0)) \oplus H^2((0,1)) | \psi(-1) = \psi(1) = 0\}, \end{aligned} \tag{3.2}$$

Since (3.1) is a restriction of operator (2.5) we know that $\dot{H}_0 \subset H_0$. For their adjointness holds $H_0^* \subset \dot{H}_0^*$ and for operator and its adjointness there is relation $H_0 \subset H_0^*$. All together we have

$$\dot{H}_0 \subset H_0 \subset H_0^* \subset \dot{H}_0^* \tag{3.3}$$

and thus ϕ'' exists. Now we can use integration by parts as we wanted.

$$\begin{aligned}(\phi, H_0\psi) &= \int_{-1}^{0} \overline{\phi}\psi'' - \int_{0}^{1} \overline{\phi}\psi'' = \\ &= \int_{-1}^{0^-} \overline{\phi''}\psi - \int_{0_+}^{1} \overline{\phi''}\psi + [\overline{\phi}\psi']_{-1}^{0^-} - [\overline{\phi}\psi']_{0+}^{1} - [\overline{\phi'}\psi]_{-1}^{0^-} + [\overline{\phi'}\psi]_{0+}^{1} \overset{!}{=} \\ &\overset{!}{=} (\eta, \psi) = (H_0^*\phi, \psi)\end{aligned}$$

Because we want to prove equality between domains of original and adjoint operator, we need the sum of four brackets to be zero.

$$[\overline{\phi}\psi']_{-1}^{0^-} - [\overline{\phi}\psi']_{0+}^{1} - [\overline{\phi'}\psi]_{-1}^{0^-} + [\overline{\phi'}\psi]_{0+}^{1} \overset{!}{=} 0 \tag{3.4}$$

The left side of this equation can be rewritten as follows

$$\begin{aligned}\text{LS} &= \overline{\phi(0^-)}\psi'(0^-) - \overline{\phi(-1)}\psi'(-1) - \overline{\phi(1)}\psi'(1) + \overline{\phi(0^+)}\psi'(0^+) - \\ &- \overline{\phi'(0^-)}\psi(0^-) + \overline{\phi'(-1)}\psi(-1) + \overline{\phi'(1)}\psi(1) - \overline{\phi'(0^+)}\psi(0^+) = \\ &= -\psi(0^+)[\overline{\phi'(0^+)} + \overline{\phi'(0^-)}] + \psi'(0^+)[\overline{\phi(0^+)} - \overline{\phi(0^-)}] - \\ &- \psi'(-1)\overline{\phi(-1)} - \psi'(1)\overline{\phi(1)} \overset{!}{=} 0\end{aligned} \tag{3.5}$$

From the definition of adjoint operator the equality (3.4) must be valid for all $\psi \in D(H_0)$. Therefore we choose some specific functions from the domain to make this equation valid.

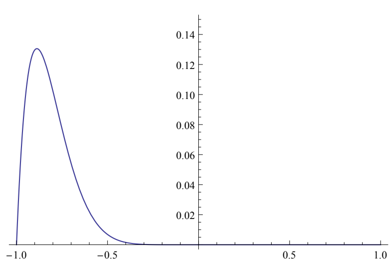

Figure 7: Function vanishing at zero

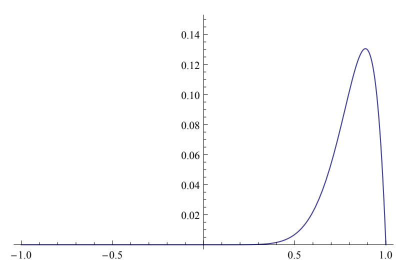

Figure 8: Function vanishing at zero

First we choose function (see Figure 7) that has finite arbitrary nonzero derivative in -1 from which it increases and then continuously decreases and vanishes in a neighbourhood of zero. On interval $(0, 1)$ this function is constantly zero. For such function the expression (3.5) reduces to $-\psi'(-1)\overline{\phi(-1)} = 0$ and therefore we have

$$\phi(-1) = 0 \tag{3.6}$$

The second function (see Figure 8) we choose in reverse to the first one with respect to the y-axis. That means that this function is zero on interval $(-1, 0)$, it vanishes in a neigbourhood of zero from the right and there is finite arbitrary nonzero derivative in 1 to which this function decreases. All together it again reduces the expression (3.5) from which we have

$$\phi(1) = 0 \tag{3.7}$$

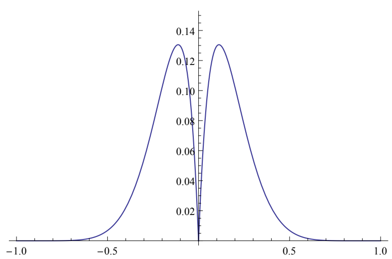

Figure 9: Function vanishing at boundary

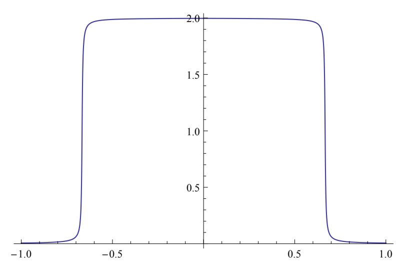

Figure 10: Constant function at zero

In the third and fourth case we want $\psi'(-1) = \psi'(1) = 0$. Thus our third function (see Figure 9) vanishes in a neighbourhood of boundary, there are finite arbitrary nonzero derivatives from the left and right in zero and the value of this function is here zero. Then from (3.5) remains only $\psi'(0^+)[\overline{\phi(0^+)} - \overline{\phi(0^-)}] = 0$ and therefore using the arbitrariness of $\psi'(0)$ we obtain

$$\phi(0^+) = \phi(0^-). \tag{3.8}$$

The last function (see Figure 10) also vanishes in a neighbourhood of boundary but this time we choose it constant and nonzero in zero. Therefore only nonzero part of (3.5) is $-\psi(0^+)[\overline{\phi'(0^+)} + \overline{\phi'(0^-)}] = 0$ and then we obtain the last equation

$$\phi'(0^+) = -\phi'(0^-) \tag{3.9}$$

The equations (3.6) – (3.9) are boundary conditions for the operator H_0^* which are clearly the same as for operator H_0. It means that $D(H_0) = D(H_0^*)$ and by definition the operator H_0 is self-adjoint.

4 Definition of a non-self-adjoint operator

In this section we want to define the "invisibility" operator. The difference between this and the original operator is that we change our earlier defined permittivity (2.4) in a way that we add to both +1 and −1 a positive number ϵ multiplied by imaginary unit. Thus if the parametr ϵ is zero, we have back the operator H_0. We denote such operator H_ϵ and in this chapter we rigorously define it via sesquilinear forms.

But there is a problem with a definition of such operator because the original operator H_0 is not bounded from below so standard methods for introduction such operator cannot be used. Physical explanation can be done by looking at refractive index $n = \sqrt{\varepsilon\mu}$. When both ε and μ are positive, we choose the positive sign for the square root. But in the case when ε and μ are simultaneously negative, causality forces us to choose a negative sign for refractive index. That is because causality requires that both ε and μ have small positive imaginary parts representing the fact that the real systems are always slightly lossy. [10]

4.1 Sectorial operators and sesquilinear forms

In this section we summarize basic facts about sectorial operators and sesquilinear forms. We also state here Representation theorem which we use to introduce operator H_ϵ rigorously.

As was said at the beginning of this chapter the operator H_0 is not bounded from below. However we will show in the next section that the new operator H_ϵ is semibounded (that means bounded either from below or from above) and even sectorial (definition of sectorial operator can be seen further in this section). We have a Representation theorem for such forms [7].

Theorem 4.1.1 (Representation theorem)**.** *Let h be densely defined, closed sectorial form in $\mathscr{H}$. Then the operator*

$$D(H) := \{\psi \in D(h) | \exists \eta \in \mathscr{H}, \forall \phi \in D(h), h(\phi, \psi) = (\psi, \eta)\}, \tag{4.1}$$

$$H\psi := \eta, \tag{4.2}$$

is m-sectorial.

To use this theorem we need first understand the new terms stated there. Let us begin with notion of numerical range.

Definition 4.1.2. Let H be an operator in a Hilbert space $\mathscr{H}$. The numerical range $\Theta(H)$ of H is the set of all complex numbers $(\psi, H\psi)$ where ψ changes over all $\psi \in D(H)$ with $\|\psi\| = 1$.

This notion is important for operators in a Hilbert space $\mathscr{H}$ and we use it in most of our definitions here. Two very important properties of numerical range are that point spectrum $\sigma_p(H)$ of H is subset in $\Theta(H)$ and that numerical range is a convex set. For example we know that spectrum $\sigma(H_0)$ is subset in $\mathbb{R}$ because H_0 is self-adjoint. We show in Section 5 that the point spectrum of H_0 consists of isolated eigenvalues and there are infinitely many of them. Because the numerical range has to be a convex set, it implies that $\Theta(H) = \mathbb{R}$.

For understanding what m-sectorial operator is we need to start with more general accretive operators [7].

Definition 4.1.3. An operator H in $\mathscr{H}$ is said to be accretive if the numerical range $\Theta(H)$ is a subset of the right half-plane, that is, if $\mathrm{Re}(\psi, H\psi) \geq 0$ for all $\psi \in D(H)$.
An operator H which satisfies that $(H+\lambda)^{-1} \in \mathscr{B}(\mathscr{H})$ and $\left\|(H+\lambda)^{-1}\right\| \leq (\mathrm{Re}\,\lambda)^{-1}$ for $\mathrm{Re}\,\lambda > 0$ is said to be m-accretive.

The m-accretive operators have many useful properties. For example an m-accretive operator H is maximal accretive in the sense that H is accretive and there is no proper accretive extension. It can be proved that such operator is equivalent to a closed maximal accretive operator and that H is necessarily densely defined.

Sometimes the numerical range $\Theta(H)$ is not contained in the whole right half-plane or it contains more than that. If such operator still has a structure of accretive operator we have a quasi-accretive operator.

Definition 4.1.4. Operator H is quasi-accretive (quasi-m-accretive) if $H + \alpha$ is accretive (m-accretive) for some scalar α.

This definition expresses that numerical range $\Theta(H)$ is contained in half-plane of the form $\mathrm{Re}\,\zeta \geq \mathrm{const}$. Finally sometimes the numerical range can be a subset of sector which is defined below.

Definition 4.1.5. If numerical range $\Theta(H)$ of H is subset of a sector $|\arg(\zeta - \gamma)| \leq \theta < \pi/2$, then H is said to be sectorial. γ and θ are called a vertex and semi-angle of the sectorial operator H.
Operator H is said to be m-sectorial if it is sectorial and quasi-m-accretive.

So we finally reached definition of m-sectorial operator which we obtain from Representation theorem. Let us just note that spectrum of m-sectorial operator H with a vertex γ and a semi-angle θ is a subset of the sector $|\arg(\zeta - \gamma)| \leq \theta$.

Notion of sectoriality is very similar for forms as for operators. Furthermore it is even easier to handle with quadratic or sesquiliner forms. We will need only two definitions which are almost identical as above.

Definition 4.1.6. The numerical range $\Theta(h)$ of form h is the set of values of $h[\psi]$ for all $\psi \in D(h)$ with $\|\psi\| = 1$.

Definition 4.1.7. The form h is said to be sectorial if its numerical range $\Theta(h)$ is a subset of a sector of the form $|\arg(\zeta - \gamma)| \leq \theta$ where $0 \leq \theta < \frac{\pi}{2}, \gamma \in \mathbb{R}$. We shall call γ a vertex and θ a corresponding semi-angle of the form h.

It is easy to see that the definition above for all $\psi \in D(h)$ means that

$$\mathrm{Re}(h[\psi]) \geq \gamma \tag{4.3}$$
$$|\mathrm{Im}(h[\psi])| \leq (\tan\theta)(\mathrm{Re}(h) - \gamma)[\psi] \tag{4.4}$$

which is useful when we prove that a form is sectorial.

The last two notions of Representation theorem are well known in functional analysis, nevertheless we remind them shortly also because one interesting property for sectorial forms.

Definition 4.1.8. A form h is densely defined if $D(h)$ is dense in $\mathscr{H}$.

Definition 4.1.9. A sectorial form h is closed if $\psi_n \underset{h}{\rightarrow} \psi$ implies that $\psi \in D(h)$ and $h[\psi_n - \psi] \rightarrow 0$.

Symbol $\psi_n \underset{h}{\rightarrow} \psi$ says that a sequence ψ_n is h-convergent to $\psi \in \mathscr{H}$ and that is if $\psi_n \in D(h), \psi_n \rightarrow \psi$ and $h[\psi_n - \psi_m] \rightarrow 0$ for $n, m \rightarrow \infty$. From the definition it follows immediately that h-convergence is equivalent to $h+\alpha$-convergence for any scalar α. In the next section we find very useful following lemma. That is why we also state here a proof of it [7].

Lemma 4.1.10. *Let h be a sectorial form. h-convergence is equivalent to* $\mathrm{Re}(h)$-*convergence.*

Proof. First we denote $\mathrm{Re}\, h = \mathfrak{h}$ and $\mathrm{Im}\, h = \mathfrak{k}$. From the (4.3) and (4.4) it easily follows that

$$|(\mathfrak{h} - \gamma)[\psi, \phi]| \leq (\mathfrak{h} - \gamma)[\psi]^{1/2}(\mathfrak{h} - \gamma)[\phi]^{1/2} \tag{4.5}$$
$$|\mathfrak{k}[\psi, \phi]| \leq (\tan\theta)(\mathfrak{h} - \gamma)[\psi]^{1/2}(\mathfrak{h} - \gamma)[\phi]^{1/2} \tag{4.6}$$

and by adding this two together we have

$$|(h - \gamma)[\psi, \phi]| \leq (1 + \tan\theta)(\mathfrak{h} - \gamma)[\psi]^{1/2}(\mathfrak{h} - \gamma)[\phi]^{1/2} \tag{4.7}$$

It follows further

$$(\mathfrak{h} - \gamma)[\psi] \leq |(h - \gamma)[\psi]| \leq (\sec\theta)(\mathfrak{h} - \gamma)[\psi] \tag{4.8}$$

From this last inequality we get that $(\mathrm{Re}\, h - \gamma)[\psi_n - \psi_m] \rightarrow 0$ if and only if $(h - \gamma)[\psi_n - \psi_m] \rightarrow 0$ □

If we use this lemma in Definition 4.1.9 it follows that h is closed if and only if $\mathrm{Re}\, h$ is closed.

4.2 Operator H_ϵ

Now we are equipped with all basic facts of theory of sectorial forms that we need to introduce the "invisibility" operator H_ϵ. We want it to act as follows

$$(H_\epsilon\psi)(x) = -\frac{d}{dx}\alpha_\epsilon(x)\frac{d}{dx} \tag{4.9}$$

$$\alpha_\epsilon(x) = \begin{cases} -1+i\epsilon & x\in\Omega_- \\ 1+i\epsilon & x\in\Omega_+ \end{cases}, \tag{4.10}$$

Corresponding quadratic form for such operator is

$$h_\epsilon[\psi] = (1+i\epsilon)\left\|\psi'\right\|_+^2 + (-1+i\epsilon)\left\|\psi'\right\|_-^2 \tag{4.11}$$

where $\|\cdot\|_-$ is norm in $L^2((-1,0))$, similary $\|\cdot\|_+$ is norm in $L^2((0,1))$ and $\|\cdot\|$ is norm in $L^2((-1,1))$. For this form we choose domain as $D(h_\epsilon) = H_0^1((-1,1))$.

From the definition of h_ϵ we can see that its numerical range is a subset of complex plane with a positive imaginary part. But according to Definition 4.1.7 we need a subset of half-plane of the form $\operatorname{Re}\zeta \geq \gamma$. To change our quadratic form in such way we multiply it by $e^{i\theta}$. We can easily find that appropriate θ is $-\frac{\pi}{2}$ and so we get "rotated" form

$$\begin{aligned} a_\epsilon[\psi] = -ih_\epsilon[\psi] &= \epsilon(\left\|\psi'\right\|_+^2 + \left\|\psi'\right\|_-^2) + i(\left\|\psi'\right\|_-^2 - \left\|\psi'\right\|_+^2) = \\ &= \epsilon\left\|\psi'\right\|^2 + i(\left\|\psi'\right\|_-^2 - \left\|\psi'\right\|_+^2) \end{aligned} \tag{4.12}$$

Now let's examine whether this form is sectorial using (4.3), (4.4)

$$\operatorname{Re}(a_\epsilon[\psi]) = \epsilon\left\|\psi'\right\|^2 > 0 \tag{4.13}$$

$$|\operatorname{Im}(a_\epsilon[\psi])| = \left|\left\|\psi'\right\|_-^2 - \left\|\psi'\right\|_+^2\right| \leq \left\|\psi'\right\|_+^2 + \left\|\psi'\right\|_-^2 = \frac{1}{\epsilon}\operatorname{Re}(a_\epsilon[\psi]) \tag{4.14}$$

From this we can see that vertex $\gamma = 0$ and for $\theta \in [0, \frac{\pi}{2})$

$$|\operatorname{Im}(a_\epsilon[\psi])| \leq \tan(\theta)(\operatorname{Re}(a_\epsilon) - 0)[\psi], \tag{4.15}$$

Thus $a_\epsilon[\psi]$ is a sectorial form.

Let us note that $\operatorname{Re}(a_\epsilon)$ is a closed form because from (4.13) we can see that it is associated form with operator H_0 (except for the multiplication by ϵ). The operator H_0 is self-adjoint and every self-adjoint operator is also closed, therefore $\operatorname{Re}(a_\epsilon)$ is closed.

Theorem 4.2.1. *There is an m-sectorial operator A_ϵ associated with form* (4.12) *and for all ψ from its domain it acts as*

$$\begin{aligned} (A_\epsilon\psi)(x) &= \begin{cases} (-i-\epsilon)\psi''(x), & x\in\Omega_- = [-1,0] \\ (i-\epsilon)\psi''(x), & x\in\Omega_+ = [0,1] \end{cases} \\ D(A_\epsilon) &= \{\psi\in H_0^1((-1,1)), \psi\in H^2((-1,0))\oplus H^2((0,1))| \\ &\qquad (\epsilon+i)\psi'(0^-) = (\epsilon-i)\psi'(0^+)\}, \end{aligned} \tag{4.16}$$

Proof. It is easy to check that assumptions of Representation theorem 4.1.1 hold. Form a_ϵ is densely defined because space $H^1((-1,1))$ is dense in $\mathscr{H} = L^2((-1,1))$, closed due to corollary of Lemma 4.1.10 and sectorial as was proved above. Therefore we know that there is an m-sectorial operator associated with the form a_ϵ. Now we want to find how it acts and explicit its domain. Following procedure we use is the same as described in Perturbation Theory for Linear Operators by Kato [7].

Let us have sesquilinear form based on definition (4.12)

$$a_\epsilon[\phi,\psi] = \epsilon(\phi',\psi')_+ + \epsilon(\phi',\psi')_- + i(\phi',\psi')_- - i(\phi',\psi')_+ \tag{4.17}$$

Let A_ϵ be the operator associated with the form a_ϵ and for $\psi \in D(A_\epsilon)$ let us have $A_\epsilon\psi = \eta$. The form $a_\epsilon[\phi,\psi]$ where $\phi \in D(a_\epsilon)$ is from Representation theorem equal to (ϕ,η) where $\eta = A_\epsilon\psi$. We write down this equality in the form of integral

$$\int_{-1}^{0}(\epsilon+i)\overline{\phi'}\psi' + \int_{0}^{1}(\epsilon-i)\overline{\phi'}\psi' = \int_{-1}^{0}\overline{\phi}\eta + \int_{0}^{1}\overline{\phi}\eta \tag{4.18}$$

Now we consider z an indefinite integral of η, thus $z' = \eta$. Then

$$\begin{aligned}\int_{-1}^{0}\overline{\phi}\eta + \int_{0}^{1}\overline{\phi}\eta = \int_{-1}^{0}\overline{\phi}z' + \int_{0}^{1}\overline{\phi}z' = -\int_{-1}^{0}\overline{\phi'}z - \int_{0}^{1}\overline{\phi'}z+\\ + \overline{\phi}(0^-)z(0^-) - \overline{\phi}(-1)z(-1) + \overline{\phi}(1)z(1) - \overline{\phi}(0^+)z(0^+)\end{aligned} \tag{4.19}$$

We substitute it into (4.18) and we get

$$\begin{aligned}\int_{-1}^{0}\overline{\phi'}((\epsilon+i)\psi' + z) + \int_{0}^{1}\overline{\phi'}((\epsilon-i)\psi' + z)-\\ - \overline{\phi}(0^-)z(0^-) + \overline{\phi}(-1)z(-1) - \overline{\phi}(1)z(1) + \overline{\phi}(0^+)z(0^+) = 0\end{aligned} \tag{4.20}$$

This equality is true for every $\phi \in D(a_\epsilon)$, that is, every ϕ such that ϕ is absolutely continuous and $\phi' \in L^2(-1,1)$. For any $\phi' \in L^2(-1,1)$ such that $\int_{-1}^{0}\phi' = 0$, $\phi(x) = \int_{-1}^{x}\phi'(x)dx$, $\phi(x) = 0$ for $x \in (0,1]$ satisfies the conditions $\phi \in D(a_\epsilon)$,$\phi(-1) = \phi(0^-) = 0$, so that $[(\epsilon+i)\psi' + z]$ is orthogonal to such ϕ' by (4.20) and thus it must be equal to constant c_-. Similarly for any $\phi' \in L^2(-1,1)$ such that $\int_{0}^{1}\phi' = 0$, $\phi(x) = \int_{x}^{1}\phi'(x)dx$, $\phi(x) = 0$ for $x \in [-1,0)$ satisfies the conditions $\phi \in D(a_\epsilon)$,$\phi(0^+) = \phi(1) = 0$, so that $[(\epsilon-i)\psi' + z]$ is orthogonal to such ϕ' by (4.20) and thus it must be equal to constant c_+. Substituting this into (4.20) we obtain

$$\begin{aligned}\int_{-1}^{0}\overline{\phi}'c_- + \int_{0}^{1}\overline{\phi}'c_+ - \overline{\phi}(0^-)z(0^-) + \overline{\phi}(-1)z(-1) - \overline{\phi}(1)z(1) + \overline{\phi}(0^+)z(0^+) = \\ = (c_- - z(0^-))\overline{\phi}(0^-) + (z(-1) - c_-)\overline{\phi}(-1) + \\ + (c_+ - z(1))\overline{\phi}(1) + (z(0^+) - c_+)\overline{\phi}(0^+) = \\ = (c_- - z(0^-) + z(0^+) - c_+)\overline{\phi}(0) + (z(-1) - c_-)\overline{\phi}(-1) + \\ + (c_+ - z(1))\overline{\phi}(1) = 0\end{aligned} \tag{4.21}$$

Since $\phi(-1)$ and $\phi(1)$ vary over all complex numbers when ϕ varies over $D(a_\epsilon)$, their coefficients in (4.21) must vanish. We write down these equalities and use that the constants c_- and c_+ can be written as $c_\mp = (\epsilon \pm i)\psi'(a) + z(a)$ where a is arbitrary point chosen in a way that z vanishes (thus $a \in -1, 0, 1$).

$$c_- - z(0^-) + z(0^+) - c_+ = (\epsilon + i)\psi'(0^-) - (\epsilon - i)\psi'(0^+) \overset{!}{=} 0 \tag{4.22}$$

$$z(-1) - c_- = -(\epsilon + i)\psi'(-1) \overset{!}{=} 0 \tag{4.23}$$

$$c_+ - z(1) = (\epsilon - i)\psi'(1) \overset{!}{=} 0 \tag{4.24}$$

The equalities (4.23) and (4.24) are satisfied if we choose domain $D(A_\epsilon) \subset H_0^1((-1,1))$. Equation (4.22) is a boundary condition on the derivative at zero which is a generalization of such condition we have for operator H_0. From $(\epsilon + i)\psi' + z = c_-$ and $(\epsilon - i)\psi' + z = c_+$ it follows that $(\epsilon + i)\psi'$ and $(\epsilon - i)\psi'$ are absolutely continuous and $\eta = -((\epsilon + i)\psi')'$ on $[-1, 0)$ and $\eta = -((\epsilon - i)\psi')'$ on $(0, 1]$. In this way we have proved that each $\psi \in D(A_\epsilon)$ has properties that ψ and ψ' are absolutely continuous and $\psi'' \in L^2(-1, 1)$ and that ψ satisfies the boundary conditions (4.22) – (4.24).
In this way we have proved that the operator A_ϵ associated with the form a_ϵ is m-sectorial and acts the same as it is stated in this theorem. □

Now we have the operator A_ϵ and to finally obtain H_ϵ we must multiply A_ϵ by i (rotate the numerical range counter-clockwise by $\pi/2$).

$$\begin{aligned}(H_\epsilon\psi)(x) &= \begin{cases}(1 - i\epsilon)\psi''(x), & x \in \Omega_- = [-1, 0] \\ (-1 - i\epsilon)\psi''(x), & x \in \Omega_+ = [0, 1]\end{cases} \\ D(H_\epsilon) &= \{\psi \in H_0^1((-1,1)), \psi \in H^2((-1,0)) \oplus H^2((0,1)) | \\ &\qquad (\epsilon + i)\psi'(0^-) = (\epsilon - i)\psi'(0^+)\},\end{aligned} \tag{4.25}$$

and that is our "invisibility" operator.

5 Spectral analysis

5.1 Spectrum of the operator H_0

Firstly we take a look at the spectrum of self-adjoint operator H_0 (relation (2.5)). Also we examine a little more general operator which also crosses zero and has a jump discontinuity there but its boundaries does not have to be at $-1, 1$ and this interval $(-1, 1)$ not even need to be symmetric.

5.1.1 Spectrum on symmetric interval $(-1, 1)$

The equation for eigenvalues is $H_0\psi = \lambda\psi$. We have to examine it on two subintervals $\Omega_+ = (0,1), \Omega_- = (-1,0)$ because there the equations differ from each other.

$$\begin{aligned} \Omega_+ &: \psi'' + \lambda\psi = 0 \\ \Omega_- &: \psi'' - \lambda\psi = 0 \end{aligned} \tag{5.1}$$

We first solve the case when $\lambda = 0$. Then there is the same equation for both sets

$$\psi'' = 0 \tag{5.2}$$

and so we can easily write down the solutions

$$\begin{aligned} \Omega_+ &: \psi_1(x) = A_1 x + B_1 \\ \Omega_- &: \psi_2(x) = A_2 x + B_2 \end{aligned} \tag{5.3}$$

From boundary conditions on functions $\psi_{1,2}$ we can determine the constants

$$\psi_1(+1) = A_1 + B_1 = 0 \tag{5.4}$$

$$\psi_2(-1) = -A_2 + B_2 = 0 \tag{5.5}$$

$$\psi_1(0^+) = B_1 = \psi_2(0^-) = B_2 \tag{5.6}$$

$$\psi_1'(0^+) = A_1 = -\psi_2'(0^-) = -A_2 \tag{5.7}$$

thus, we obtain relations: $A_1 = -B_1, A_2 = B_2, B_1 = B_2, A_1 = -A_2$ and so we find the eigenfunctions for eigenvalue $\lambda_0 = 0$ with one arbitrary integration constant:

$$\begin{aligned} \Omega_+ &: \psi_1(x) = A_1 x - A_1 = A_1(x-1) \\ \Omega_- &: \psi_2(x) = -A_1 x - A_1 = -A_1(x+1) \end{aligned} \tag{5.8}$$

For nonzero λ we proceed similarly. When we solve the differential equations (5.1) on each interval, we get the solutions

$$\begin{aligned} \Omega_+ &: \psi_3(x) = C_1 \sin\sqrt{\lambda}x + D_1 \cos\sqrt{\lambda}x \\ \Omega_- &: \psi_4(x) = C_2 \sinh\sqrt{\lambda}x + D_2 \cosh\sqrt{\lambda}x \end{aligned} \tag{5.9}$$

Remark 5.1.1. We should say that although there is a square root in (5.9), we do not have to assume only positive λ. If we consider $\lambda < 0$, we can write it as $\lambda = -\tilde{\lambda}$ where $\tilde{\lambda}$ is opposite number to λ and therefore $\tilde{\lambda}$ is positive. After substitution this negative λ in (5.1) we can easily see that the first equation is changed into the second one and the second equation into the first one. Thus the equations are basically the same but they swapped their intervals Ω_+ and Ω_- with each other. Further we will see that such described sign change of λ does not affect the resulting equation.

Now we use boundary conditions in the same way as above

$$\psi_3(+1) = C_1 \sin\sqrt{\lambda} + D_1 \cos\sqrt{\lambda} = 0 \tag{5.10}$$

$$\psi_4(-1) = -C_2 \sinh\sqrt{\lambda} + D_2 \cosh\sqrt{\lambda} = 0 \tag{5.11}$$

$$\psi_3(0^+) = D_1 = \psi_4(0^-) = D_2 \tag{5.12}$$

$$\psi_3'(0^+) = C_1\sqrt{\lambda} = -\psi_4'(0^-) = -C_2\sqrt{\lambda} \tag{5.13}$$

From the last two equations we get $D_1 = D_2$ and $C_1 = -C_2$, then substitute into the previous two equations and we have

$$\begin{aligned} C_1 \sin\sqrt{\lambda} + D_1 \cos\sqrt{\lambda} = 0 \Rightarrow \frac{D_1}{C_1} = -\tan\sqrt{\lambda} \\ C_1 \sinh\sqrt{\lambda} + D_1 \cosh\sqrt{\lambda} = 0 \Rightarrow \frac{D_1}{C_1} = -\tanh\sqrt{\lambda} \end{aligned} \tag{5.14}$$

$$\tan\sqrt{\lambda} = \tanh\sqrt{\lambda} \tag{5.15}$$

That is a transcendental equation that has infinitely many solutions. Using various computational tools (such as Wolfram Mathematica) we calculate approximate solutions

$$\begin{aligned} \lambda_1 &\approx 15,4182 \\ \lambda_2 &\approx 49,9649 \\ \lambda_3 &\approx 104,248 \\ \lambda_4 &\approx 178,27 \\ \lambda_5 &\approx 272,031 \\ &\dots\dots\dots \end{aligned} \tag{5.16}$$

After substitution these solutions into the equations (5.9) we receive eigenfunctions for appropriate eigenvalues

$$\Omega_+ : \psi_{3,k}(x) = C_1 \sin\sqrt{\lambda_k}x + D_1 \cos\sqrt{\lambda_k}x \tag{5.17}$$

$$\Omega_- : \psi_{4,k}(x) = -C_1 \sinh\sqrt{\lambda_k}x + D_1 \cosh\sqrt{\lambda_k}x \tag{5.18}$$

where $k \in \mathbb{N}$. As was pointed out in Remark 5.1.1 there are also negative eigenvalues identical with (5.16) except for the sign. It is because the boundary conditions are symmertic and if we use the same process as above we obtain again equation (5.15). Some of the eigenfunctions are shown in Figure 11.

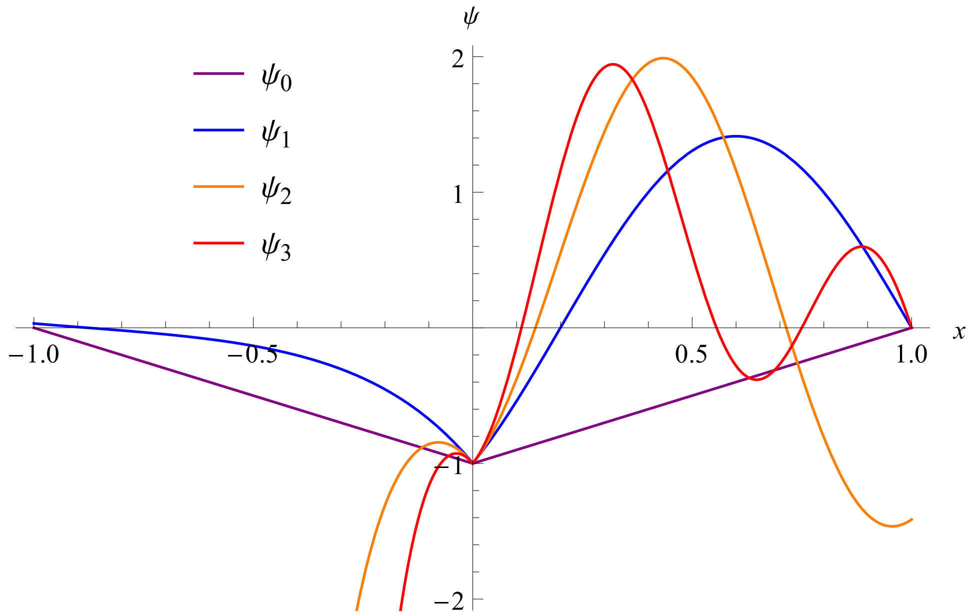

Figure 11: Eigenfunctions of the operator H_0 for $\lambda_{0,1,2,3}$ $(A_1 = C_1 = 1, D_1 = -C_1 \tan(\sqrt{\lambda_k}))$

5.1.2 Spectrum for any interval $(-a_-, a_+)$ containing zero

We change the operator H_0 accordingly to the change of interval:

$$\begin{aligned} D(H_0) = \{\psi \in H^2((-a_-,0)) \oplus H^2((0,a_+)) | \psi(-a_-) = \psi(a_+) = 0, \\ \psi(0^-) = \psi(0^+), \\ \psi'(0^-) = -\psi'(0^+)\} \end{aligned} \tag{5.19}$$

where $a_\pm \in \mathbb{R}^+$. We denote $\alpha := a_+ - a_-$. Let us begin with equations (5.1) and proceed just like in the previous case.

For $\lambda = 0$ we get again equations (5.3) but use the new boundary conditions to determine the constants:

$$\psi_1(a_+) = A_1 a_+ + B_1 = 0 \tag{5.20}$$

$$\psi_2(-a_-) = -A_2 a_- + B_2 = 0 \tag{5.21}$$

$$\psi_1(0^+) = B_1 = \psi_2(0^-) = B_2 \tag{5.22}$$

$$\psi_1'(0^+) = A_1 = -\psi_2'(0^-) = -A_2 \tag{5.23}$$

Thus, we obtain relations: $B_1 = -A_1 a_+, B_2 = A_2 a_-, B_1 = B_2, A_1 = -A_2$ and after the substitutions we find out that

$$a_- = a_+ \tag{5.24}$$

This fact says that $\lambda = 0$ is eigenvalue of the operator H_0 if and only if interval $[-a_-, a_+]$ is symmetric ($\alpha = 0$).

For $\lambda \neq 0$ we get again solutions (5.9) but with new boundary conditions. For this let a_- be fixed and $a_+ = \alpha + a_-$ be a variable.

$$\psi_3(a_+) = C_1 \sin\sqrt{\lambda}(\alpha + a_-) + D_1 \cos\sqrt{\lambda}(\alpha + a_-) = 0 \tag{5.25}$$

$$\psi_4(-a_-) = -C_2 \sinh\sqrt{\lambda}a_- + D_2 \cosh\sqrt{\lambda}a_- = 0 \tag{5.26}$$

$$\psi_3(0^+) = D_1 = \psi_4(0^-) = D_2 \tag{5.27}$$

$$\psi_3'(0^+) = C_1\sqrt{\lambda} = -\psi_4'(0^-) = -C_2\sqrt{\lambda} \tag{5.28}$$

From the last two equations we get again $D_1 = D_2$ and $C_1 = -C_2$ and after substitution we have

$$\begin{aligned} C_1 \sin\sqrt{\lambda}(\alpha + a_-) + D_1 \cos\sqrt{\lambda}(\alpha + a_-) = 0 \Rightarrow \frac{D_1}{C_1} &= -\tan\sqrt{\lambda}(\alpha + a_-) \\ C_1 \sinh\sqrt{\lambda}a_- + D_1 \cosh\sqrt{\lambda}a_- = 0 \Rightarrow \frac{D_1}{C_1} &= -\tanh\sqrt{\lambda}a_- \end{aligned} \tag{5.29}$$

$$\tan\sqrt{\lambda}(\alpha + a_-) = \tanh\sqrt{\lambda}a_- \tag{5.30}$$

The eigenfunctions look the same as (5.17), (5.18) but the eigenvalues λ_k now depend on variable α and therefore the eigenfunctions depend on it as well. This dependence can be seen in Figure 12 where we chose $a_- = 1$. Thanks to this choice we obtain for $\alpha = 0$ same solution (5.16) as before because in this case we have the same interval $[-1, 1]$ where we solve equation (5.30) (or (5.15) in this particular case).

5.2 Spectrum of the operator H_ϵ

Now we examine spectrum of our non-self-adjoint operator H_ϵ (relation (4.25)). Equations for eigenvalues on both intervals Ω_+ and Ω_- are

$$\begin{aligned} \Omega_+ &: (1 + i\epsilon)\psi'' + \lambda\psi = 0 \\ \Omega_- &: (1 - i\epsilon)\psi'' - \lambda\psi = 0 \end{aligned} \tag{5.31}$$

If $\lambda = 0$ we solve again for both intervals the same equation

$$\psi'' = 0 \tag{5.32}$$

which brings once again solutions (5.3). We use boundary conditions

$$\psi_1(1) = A_1 + B_1 = 0 \tag{5.33}$$

$$\psi_2(-1) = -A_2 + B_2 = 0 \tag{5.34}$$

$$\psi_1(0^+) = B_1 = \psi_2(0^-) = B_2 \tag{5.35}$$

$$(\epsilon - i)\psi_1'(0^+) = (\epsilon - i)A_1 = (\epsilon + i)\psi_2'(0^-) = (\epsilon + i)A_2 \tag{5.36}$$

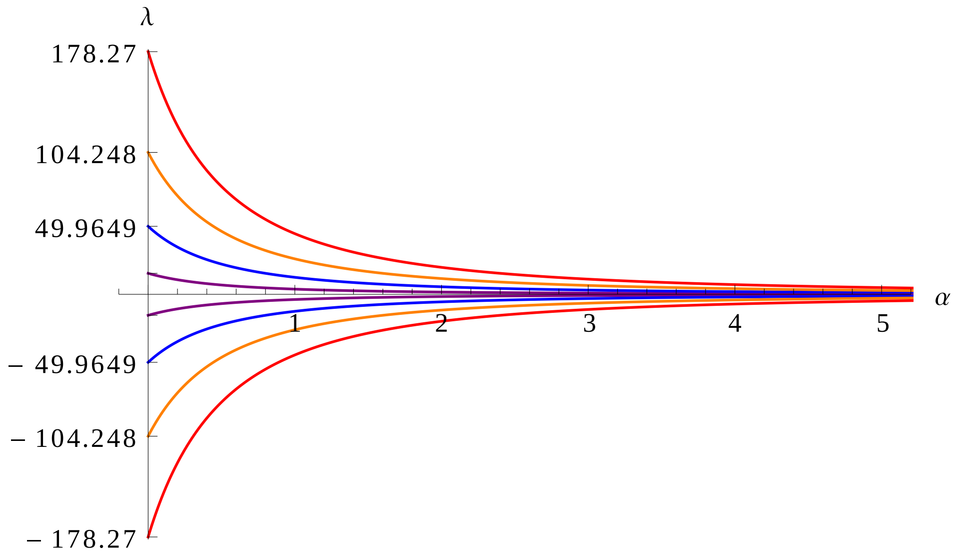

Figure 12: Spectrum of the operator H_0 depending on variable $\alpha \in [0, 5]$

and obtain relations $A_1 = -B_1, A_2 = B_2, B_1 = B_2, (\epsilon - i)A_1 = (\epsilon + i)A_2$. From the first three equalities we see that $A_1 = -A_2$ but the last one says $A_1 = \frac{\epsilon+i}{\epsilon-i}A_2$. This could be satisfied only if $\epsilon = 0$ therefore in general case when ϵ is positive, $\lambda = 0$ can not be eigenvalue of the operator H_ϵ.

More interesting is the case when $\lambda \neq 0$. We rewrite equations (5.31) in better way

$$\begin{aligned} \Omega_+ &: \psi'' + \lambda \frac{(1 - i\epsilon)}{1 + \epsilon^2} \psi = 0 \\ \Omega_- &: \psi'' - \lambda \frac{(1 + i\epsilon)}{1 + \epsilon^2} \psi = 0 \end{aligned} \tag{5.37}$$

and the solutions of these equation are

$$\begin{aligned} \Omega_+ &: \psi_3(x) = C_1 \sin \sqrt{\lambda \frac{1 - i\epsilon}{1 + \epsilon^2}} x + D_1 \cos \sqrt{\lambda \frac{1 - i\epsilon}{1 + \epsilon^2}} x \\ \Omega_- &: \psi_4(x) = C_2 \sinh \sqrt{\lambda \frac{1 + i\epsilon}{1 + \epsilon^2}} x + D_2 \cosh \sqrt{\lambda \frac{1 + i\epsilon}{1 + \epsilon^2}} x. \end{aligned} \tag{5.38}$$

Use of boundary conditions is now a little more difficult

$$\psi_3(1) = C_1 \sin\sqrt{\lambda\frac{1-i\epsilon}{1+\epsilon^2}} + D_1 \cos\sqrt{\lambda\frac{1-i\epsilon}{1+\epsilon^2}} = 0 \tag{5.39}$$

$$\psi_4(-1) = -C_2 \sinh\sqrt{\lambda\frac{1+i\epsilon}{1+\epsilon^2}} + D_2 \cosh\sqrt{\lambda\frac{1+i\epsilon}{1+\epsilon^2}} = 0 \tag{5.40}$$

$$\psi_3(0^+) = D_1 = \psi_4(0^-) = D_2 \tag{5.41}$$

$$(\epsilon - i)\psi_3'(0^+) = (\epsilon - i)C_1\sqrt{\lambda\frac{1-i\epsilon}{1+\epsilon^2}} = (\epsilon + i)\psi_4'(0^-) = (\epsilon + i)C_2\sqrt{\lambda\frac{1+i\epsilon}{1+\epsilon^2}}. \tag{5.42}$$

From the last equality we obtain $C_2 = C_1 \frac{\epsilon - i}{\epsilon + i}\sqrt{\frac{1-i\epsilon}{1+i\epsilon}}$ which we substitute together with (5.41) to the first two equations

$$\begin{aligned} C_1 \sin\sqrt{\lambda\frac{1-i\epsilon}{1+\epsilon^2}} + D_1 \cos\sqrt{\lambda\frac{1-i\epsilon}{1+\epsilon^2}} = 0 \Rightarrow \\ \Rightarrow \frac{D_1}{C_1} = -\tan\sqrt{\lambda\frac{1-i\epsilon}{1+\epsilon^2}} \end{aligned} \tag{5.43}$$

$$\begin{aligned} -C_1\frac{\epsilon - i}{\epsilon + i}\sqrt{\frac{1-i\epsilon}{1+i\epsilon}} \sinh\sqrt{\lambda\frac{1+i\epsilon}{1+\epsilon^2}} + D_1 \cosh\sqrt{\lambda\frac{1+i\epsilon}{1+\epsilon^2}} = 0 \Rightarrow \\ \Rightarrow \frac{D_1}{C_1} = \frac{\epsilon + i}{\epsilon - i}\sqrt{\frac{1+i\epsilon}{1-i\epsilon}} \tanh\sqrt{\lambda\frac{1+i\epsilon}{1+\epsilon^2}} \end{aligned} \tag{5.44}$$

so all together we get final equation

$$\tan\sqrt{\lambda\frac{1-i\epsilon}{1+\epsilon^2}} = \sqrt{\frac{1-i\epsilon}{1+i\epsilon}} \tanh\sqrt{\lambda\frac{1+i\epsilon}{1+\epsilon^2}} \tag{5.45}$$

which can be equivalently written as

$$\tan\sqrt{\frac{\lambda}{1+i\epsilon}} = \sqrt{\frac{1-i\epsilon}{1+i\epsilon}} \tanh\sqrt{\frac{\lambda}{1-i\epsilon}} \tag{5.46}$$

This equation will be solved in the next paragraph.

5.3 Mathematical and physical justification of complexification

It is not difficult to see that if $\epsilon = 0$, we have the same equation as we had for operator H_0 and therefore the same solution as well. But we need to know whether solutions of (5.45) pass continuously in solutions (5.16) of the original equation. To prove this we use Implicit function theorem from [6].

Theorem 5.3.1 (Implicit function theorem)**.** *Let $F(x, y)$ be a function, $[x_0, y_0]$ point in $\mathbb{R}^2$ and n natural number. Let L be an open two-dimensional interval* *which contains point $[x_0, y_0]$ and function $F(x, y)$ is continuously differentiable in L up to order n. Further suppose that $F(x_0, y_0) = 0$, $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$. Then there exist positive numbers Δ_1, Δ_2 such that*

1. $(\forall x \in (x_0 - \Delta_1, x_0 + \Delta_1))(\exists_1 y \in (y_0 - \Delta_2, y_0 + \Delta_2))(F(x, y) = 0)$. *If we denote this y as $f(x)$, the equation $F(x, y) = 0$ is satisfied for all $x \in (x_0 - \Delta_1, x_0 + \Delta_1)$ and $y \in (y_0 - \Delta_2, y_0 + \Delta_2)$ if and only if $y = f(x)$.*
2. *Function $f(x)$ is continuously differentiable up to order n on interval $(x_0 - \Delta_1, x_0 + \Delta_1)$.*

It is easy to see that our function F

$$F(\lambda, \epsilon) = \tan\sqrt{\frac{\lambda}{1+i\epsilon}} - \sqrt{\frac{1-i\epsilon}{1+i\epsilon}}\tanh\sqrt{\frac{\lambda}{1-i\epsilon}} \tag{5.47}$$

is continuously differentiable so we only have to find the point (λ_0, ϵ_0) that satisfies the assumptions of Implicit function theorem.

We already know the point in which function F is zero because when we choose $\epsilon = 0$ we have the same equation as in section 5.1.1 and then we choose λ_0 as a solution of equation (5.15). Now we have to check whether also the derivative of F with respect to x is nonzero in this point.

$$\begin{aligned}
\frac{\partial F}{\partial \epsilon}(\lambda_0, 0) = & \left.\frac{1}{\cos^2\sqrt{\frac{\lambda}{1+i\epsilon}}}\frac{1}{2}\sqrt{\frac{1+i\epsilon}{\lambda}}\frac{-i\lambda}{(1+i\epsilon)^2}\right|_{(\lambda_0,0)} - \\
& - \left.\frac{1}{2}\sqrt{\frac{1+i\epsilon}{1-i\epsilon}}\frac{-i(1+i\epsilon)-i(1-i\epsilon)}{(1+i\epsilon)^2}\tanh\sqrt{\frac{\lambda}{1-i\epsilon}}\right|_{(\lambda_0,0)} - \\
& - \left.\sqrt{\frac{1-i\epsilon}{1+i\epsilon}}\frac{1}{\cosh^2\sqrt{\frac{\lambda}{1-i\epsilon}}}\frac{1}{2}\sqrt{\frac{1-i\epsilon}{\lambda}}\frac{i\lambda}{(1-i\epsilon)^2}\right|_{(\lambda_0,0)} = \\
& = -\frac{i}{2}\sqrt{\lambda_0}\frac{1}{\cos^2\sqrt{\lambda_0}} + i\tanh\sqrt{\lambda_0} - \frac{i}{2}\sqrt{\lambda_0}\frac{1}{\cosh^2\sqrt{\lambda_0}} = \\
& = -i\lambda_0 \left.(\tan\sqrt{\lambda})'\right|_{\lambda=\lambda_0} + i\tanh\sqrt{\lambda_0} - i\lambda_0 \left.(\tanh\sqrt{\lambda})'\right|_{\lambda=\lambda_0}
\end{aligned}$$

Now we use that λ_0 is a solution of (5.15). Because of that we can write $\tan\sqrt{\lambda_0}$ instead of $\tanh\sqrt{\lambda_0}$ and therefore

$$\frac{\partial F}{\partial \epsilon}(\lambda_0, 0) = i\left(\tan\sqrt{\lambda_0} - \frac{\sqrt{\lambda_0}}{\cos^2\sqrt{\lambda_0}}\right) = \frac{i}{\cos^2\sqrt{\lambda_0}}\left(\sin\sqrt{\lambda_0}\cos\sqrt{\lambda_0} - \sqrt{\lambda_0}\right)$$

which is nonzero for all nonzero λ_0. According to Implicit function theorem there exist some open sets containing points $(\lambda_0, 0)$ and thus function F is continuously differentiable around solutions of the origin operator H_0.

Now we can show graphically how the eigenvalues of operator H_ϵ go to the original eigenvalues of operator H_0. It can be seen on Figures 13 and 14 that real part of λ goes with decresaing value of ϵ to the original eigenvalue while the imaginary part goes to zero which is exactly what we wanted.

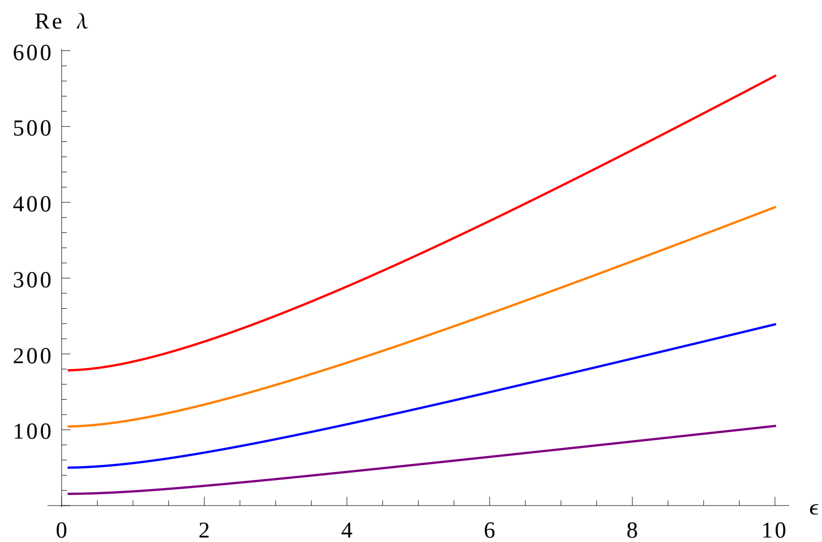

Figure 13: Real part of spectrum of the operator H_ϵ

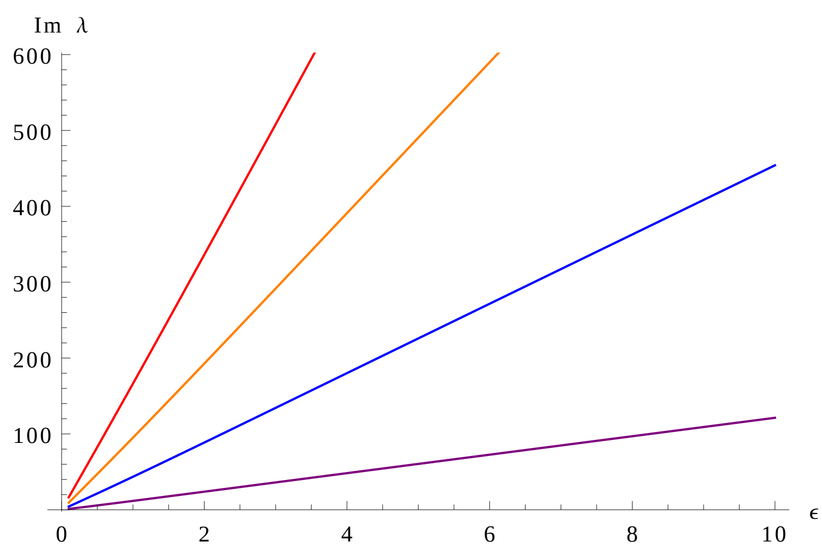

Figure 14: Imaginary part of spectrum of the operator H_ϵ

6 Conclusions

In this thesis we were interested in description of wave propagation from an ordinary medium to the metamaterial using operators. We began with electric part of Maxwell's equation from which we derived formal operator with Dirichlet boundary condition and then specified it with more boundary conditions that were needed. We proved that this operator is self-adjoint and then we constructed more general but non-self-adjoint operator which we called the "invisibility" operator because of such usage in metamaterials. Then we were concerned mainly with this operator, we proved that it is m-sectorial and in the end we found out that in the limit case its spectrum goes continuously into the spectrum of the original self-adjoint operator. Let us here briefly note that an alternative definition of our "invisibility" operator can be done by a new representation theorem for indefinite quadratic forms obtained recently in [5] based on [4].

The motivation for this thesis was to introduce basic facts from the theory of metamaterials and the theory of sectorial quadratic forms as well. The main objective was to define the "invisibility" operator mathematically rigorously and then justify the complexification in it.

There are still many problems to concern and thus many ways how to extend this thesis. For example one could prove that spectrum of the complexed operator goes in the limit case continuously to the spectrum of the original operator generally thus without calculation of the spectra. The alternative definition of our "invisibility" operator should be investigated and used as well. Also one should consider the same problem but on infinite interval instead of the finite one in this case. This thesis also open the way to extend that operator to higher dimension and try to generalize the results stated here.

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