Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Nodal lines on two dimensional bounded domains

Nodální čáry na dvourozměrných omezených oblastech

Bachelor's Degree Project

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- Zadání práce -

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Candidate's declaration:
I hereby certify that:

- this bachelor's degree project represents my own work;
- the contribution of any supervisors or others to the research or the dissertation itself was consistent with normal supervisory practice;
- external contributions to the research are properly acknowledged and all used sources of information are listed in the bibliography.

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Marie Fialová

Název práce:

Nodální čáry na dvourozměrných omezených oblastech

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Abstrakt: V této práci se zaměřujeme na problematiku tzv. Dirichletovského laplaciánu na omezené oblasti Ω v $\mathbb{R}^2$. Konkrétně se věnujeme kvalitativním vlastnostem množin (známých jako nodální čáry), které jsou vlastní funkcí zobrazeny na nulu. V 60. letech uvedl L.E.Payne doměnku, že tyto čáry druhých vlastních funkcí mají neprázdný průnik s hranicí oblasti Ω. Nejprve se zaměřujeme na oblasti, kde lze řešení najít analyticky. Dále se věnujeme numerickému řešení, které se zakládá na min-max principu a napočítáváme analyticky známá řešení.

Klíčová slova: Dirichletův laplacián, nodální čáry, numerické přiblížení

Title:

Nodal lines on two-dimensional bounded domains

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Abstract: This degree project deals with the so-called Dirichlet Laplacian on a bounded domain Ω in $\mathbb{R}^2$. In particular we are concerned with qualitative properties of the zero sets (that are known as the nodal lines) of its eigenfunctions. It was conjectured by Payne in 1960s that for the second eigenfunction these lines should have non-empty intersection with the boundary of Ω. Firstly we focus on domains that permit explicit analytical solutions. Then we apply numerical methods based on min-max principle to reproduce these results.

Key words: Dirichlet Laplacian, nodal lines, numerical approximation

Contents

Notation

Symbol	Meaning
$\mathbb{C}$	complex numbers
C^∞	smooth functions
$C_0^\infty(\overline{\Omega})$	linear space of smooth functions f on Ω whose partial derivatives can be extended continuously to $\overline{\Omega}$ and $f\|_{\partial\Omega}= 0$
$\mathrm{Dom}(T)$	domain of an operator T
$\partial\Omega$	boundary of region Ω
$\mathscr{H}$	Hilbert space
$\mathrm{Ker}(T)$	kernel of an operator T
$\mathbb{N}$	positive integers
$\mathbb{N}_0$	non-negative integers
$\hat{n}$	integers $1, \ldots, n$
$\overline{\Omega}$	closure of region Ω
$\mathscr{Q}(q)$	domain of a quadratic form q
$\mathbb{R}$	real numbers
$\mathrm{Ran}(T)$	range of an operator T
$W^{1,2}$	Sobolev space
$\langle\cdot,\cdot\rangle$	inner product
$X \subset\subset Y$	X is a subspace of Y

Chapter 1

Introduction

The Laplace operator is one of important operators in physics. It appears in physical equations such as wave, heat and Schrödinger equation. By the separation of variables in these equations we obtain the eigenvalue problem of Laplace operator. For more details see [2].

In this degree project we acquaint with the theory concerning the solution of the eigenproblem for the Dirichlet Laplacian concentrating on the solutions on a bounded region (open, connected and non-empty subset in $\mathbb{R}^N$) Ω with a smooth boundary. The operator is given by its domain and its action. In our case the domain will be restricted by the Dirichlet boundary condition. It means we require for functions from the domain to have zero value on the boundary of the region $\partial\Omega$.

The problem we need to solve is

$$\begin{aligned} -\Delta f &= \lambda f \qquad \text{in } \Omega, \\ f &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where the domain of the operator $-\Delta$ is subspace $C_0^\infty(\overline{\Omega}$ of the Hilbert space $L^2(\Omega)$. This is an elliptic operator that is not self-adjoint. However its closure is self-adjoint (*cf.* Chapter 3).

In Chapter 2 we summarize without proofs the relevant material on unbounded operators, their spectrum and quadratic forms. We introduce the notions of self-adjoint and closed operator and important theorems such as the spectral and representation theorem establishing the relation between quadratic forms and operators with specific properties.

Chapter 3 deals with the Dirichlet boundary condition. We introduce the notion of Sobolev space and show its important properties. Then we look more closely at general and spectral properties of Dirichlet Laplacian.

In particular we are interested in finding the set, where the second eigenfunction of Dirichlet Laplacian vanishes. This set is known as the nodal set. This notion will be precisely introduced in chapter 4. There we also present the results of so-called nodal hypothesis. This hypothesis predicts, that the nodal set for the second eigenfunction of Dirichlet Laplacian on some specific regions is not a closed curve.

Chapter 5 contains the analytical solution of problem (1.1) on a rectangle, a disc and a sector of a disc region. For more complicated regions it is not possible to find the solution analytically. Therefore we used numerical tools for creating a program that could possibly find the eigenfunctions on such a region. We tested this program on regions we solved analytically to find out whether they give some reasonable results. These results are shown and discussed in Chapter 6.

Chapter 2

Basic notions

2.1 Linear operators and spectrum

In this chapter we provide a short introduction to unbounded self-adjoint operators.

Definition 2.1. *A linear operator B is* **bounded** *if there exists a constant $c > 0$ such that $\|B(x)\| \leq c\|x\|$ for all $x \in \mathrm{Dom}(B)$. We define the operator norm by $\|B\| = \sup_{x\in\mathrm{Dom}(B),\|x\|=1} \|Bx\|$.*

However the Laplace operator and many of important operators that appear *e.g.* in quantum mechanics are the unbounded ones. Therefore we present here the theory of the unbounded operators. Many of following terms could be defined on more general spaces, because somewhere the inner product is not needed and the Banach space is sufficient. Since we do not need this generalization we introduce these terms on the Hilbert space $\mathscr{H}$.

Definition 2.2. *Let $\mathrm{Dom}(T)$ be a subspace of $\mathscr{H}$ and T be a linear operator $T : \mathrm{Dom}(T) \to \mathscr{H}$.*
The set $\Gamma(T) := \{(x, Tx) \in \mathscr{H} \oplus \mathscr{H} \mid x \in \mathrm{Dom}(T)\}$ is called the **graph of the operator T**.
We call the operator T **closed** *if its graph is a closed set in $\mathscr{H} \oplus \mathscr{H}$.*

The graph is a Hilbert space with the inner product

$$\langle\langle f_1, g_1\rangle, \langle f_2, g_2\rangle\rangle = \langle f_1, f_2\rangle + \langle g_1, g_2\rangle.$$

(*cf.* [4]-vol1, p. 250).

Closeness is very important property of operators. Of course not all operators are closed. However it is still possible that a closed extension exists. Therefore we need another property called close-ability.

Definition 2.3. *Let T be a linear operator,* $\mathrm{Dom}(T)$ *subspace $\mathscr{H}$ be the domain of T. We say $\tilde{T}$ is an* **extension of *T*** *with domain* $\mathrm{Dom}(\tilde{T})$ *if* $\mathrm{Dom}(\tilde{T})$ *is subspace of $\mathscr{H}$ containing Dom(T) and $\tilde{T}f = Tf$ for all f in* $\mathrm{Dom}(T)$*. An operator is said to be* **closable** *if it has a closed extension.*

Theorem 2.4 (Hellinger-Toeplitz theorem, *cf.* [4]-Section III.5)**.** *Let T be an everywhere-defined linear operator on $\mathscr{H}$ satisfying*

$$\langle x, Ty\rangle = \langle Tx, y\rangle \tag{2.1}$$

for all $x, y \in \mathscr{H}$ then T is bounded.

This theorem is a corollary of the closed graph theorem. It implies that the unbounded operator that obeys (2.1) is not everywhere-defined. Thus we assume that such an operator is only densely defined.

Definition 2.5. *We call operator T to be* **positive** *or* **non-negative** *if it satisfies $\langle Tf, f\rangle > 0$ or $\langle Tf, f\rangle \geq 0$, respectively, for all $f \in \mathrm{Dom}(T) \setminus \{0\}$.*

Definition 2.6. *We call operator T to be* **bounded from bellow** *if it satisfies $\langle Tf, f\rangle \geq M\|f\|^2$, for some $M \in \mathbb{R}$, for all $f \in \mathrm{Dom}(T) \setminus \{0\}$.*

Note 2.7. *Let T be a densely defined linear operator on $\mathscr{H}$. Then there exists at most one y^* such that for all $x \in \mathrm{Dom}(T)$,*

$$\langle y, Tx\rangle = \langle y^*, x\rangle. \tag{2.2}$$

(cf. [1]- 7.1.1.)

This note enables us to define the adjoint operator.

Definition 2.8. *Let T be a linear operator on $\mathscr{H}$ and y^* satisfies (2.2). We define $T^*y := y^*$. We call T^* the* **adjoint operator** *to the operator T with $\mathrm{Dom}(T^*)$ consisting of y satisfying (2.2).*

Theorem 2.9 (*cf.* [4]-VIII.1)**.** *Let T be a densely-defined operator on $\mathscr{H}$. Then*

1. *T^* is closed,*
2. *T is closable if and only if $\mathrm{Dom}(T^*)$ is dense in which case $\overline{T} = T^{**}$,*
3. *if T is closable then $(\overline{T})^* = T^*$.*

Definition 2.10. *The* **spectrum** *$\sigma(T)$ of a closed operator T on a Hilbert space $\mathscr{H}$ is defined by*

$$\sigma(T) := \{\lambda \in \mathbb{C} |\ (T - \lambda) \text{ is not a bijection of } \mathrm{Dom}(T) \text{ onto } \mathscr{H}\}.$$

The situation that $(T - \lambda)$ is not a bijection of $\mathrm{Dom}(T)$ onto $\mathscr{H}$ occurs in two cases excluding each other:

1. *$T - \lambda$ is not an injection, then λ is an* **eigenvalue** *of T,*
2. *$T - \lambda$ is not a surjection.*

We denote $\sigma_p(T)$ the set of all eigenvalues. The set of isolated eigenvalues of a finite multiplicity $\sigma_{disc}(T)$ is called the **discrete spectrum** *of the operator T. The complement of discrete spectrum to the spectrum is called the* **essential spectrum** *of the operator T, $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_{disc}(T)$.*
We call $\rho(T)$ the **resolvent set** *of the operator T defined by*

$$\rho(T) := \mathbb{C} \setminus \sigma(T).$$

By definition, $\lambda \in \rho(T)$ if and only if the operator $T - \lambda$ is a bijection of $\mathrm{Dom}(T)$ onto $\mathscr{H}$. From the closed graph theorem follows that the inverse $(T - \lambda)^{-1}$ is bounded. If $\lambda \in \rho(T)$ we call $R_\lambda := (T - \lambda)^{-1}$ the **resolvent of operator** *T.*

The following lemma explains the importance of the assumption of closeness of the operator in definition of spectrum.

Lemma 2.11. *If a linear operator T is not closed, then $\sigma(T) = \mathbb{C}$. The spectrum $\sigma(T)$ of a linear operator T is always closed set.*

Note 2.12. *If we talk about spectrum of a closable operator we mean the spectrum of its closure.*

Definition 2.13. *A densely-defined linear operator T is called* **symmetric***, if $T \subset T^*$ e.g.* $\mathrm{Dom}(T) \subset$ *Dom*(T^*) *and* $Tx = T^*x, \quad \forall x \in \mathrm{Dom}(T)$. *This means*

$$\langle y, Tx \rangle = \langle Ty, x \rangle, \ \forall x, y \in \mathrm{Dom}(T).$$

We call T a **self-adjoint** *operator, if T is densely defined and satisfies the condition $T = T^*$.*

The following theorem will be needed for the spectral theorem.

Theorem 2.14. *The spectrum of any self-adjoint operator T is real and non-empty. If $z \notin \mathbb{R}$, then*

$$\|(T - z)^{-1}\| \leq |\mathrm{Im} z|^{-1}.$$

Moreover

$$(T - \bar{z})^{-1} = ((T - z)^*)^{-1}.$$

Theorem 2.15 (*cf.* [1]-10.4.3)**.** *A self-adjoint operator is bounded if and only if its spectrum is bounded.*

Definition 2.16. *We define a* **compact operator** *on $\mathscr{H}$ to be an operator T with domain $\mathscr{H}$ such that for any bounded sequence $f_n \in \mathscr{H}$, Tf_n has a norm convergent subsequence.*

Theorem 2.17 (Ries-Schauder, *cf.* [1]- Theorem 6.2.2)**.** *Let T be a compact operator on $\mathscr{H}$, then*

1. $\sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$,
2. *every isolated eigenvalue has finite multiplicity,*
3. $\sigma(T)$ *has at most one limit point $\lambda = 0$,*
4. *The set of eigenvalues is at most countable and we can write the eigenvalues λ_n in decreasing order in such a way that they obey*
$$\lim_{n \to \infty} \lambda_n = 0.$$

Theorem 2.18 (Hilber-Schmidt, *cf.* [1]- Theorem 6.2.4)**.** *Let T be a self-adjoint compact operator on $\mathscr{H}$. Then there exists an orthonormal basis on $\mathscr{H}$ consisting of eigenfunctions of T.*

Note 2.19. *Let T be a linear operator, $z \in \rho(T)$ and $(T - z)^{-1}$ be a self-adjoint compact operator. Further we assume that $(f_n)_{n=1}^{\infty}$ is an orthonormal basis consisting of eigenfunctions of $(T - z)^{-1}$ and the corresponding eigenvalues $\lambda_n \neq 0$ satisfy $\lim_{n\to\infty} \lambda_n = 0$. Then we have*

$$\begin{aligned} (T - z)^{-1} f_n &= \lambda_n f_n \\ \frac{1}{\lambda_n} &= (T - z) f_n \\ T f_n &= \Big(\frac{1}{\lambda_n} + z\Big) f_n. \end{aligned}$$

That means that the eigenvalues $\tilde{\lambda}_n = \left(\frac{1}{\lambda_n} - 1\right)$ of the operator T converge to infinity as $n \to \infty$.

Theorem 2.20 (*cf.* [3]- 4.2.3)**.** *Let T be a self-adjoint operator on $\mathscr{H}$ which is non-negative in the sense that $\sigma(T) \subset [0, \infty)$ then the following conditions are equivalent:*

1. *the resolvent operator $(T + 1)^{-1}$ is compact,*
2. $\sigma_{ess}(T) = \emptyset$,

3. *there exists a complete orthonormal set of eigenvectors $\{f_n\}_{n=1}^{\infty}$ of T with corresponding eigenvalues $\lambda_n \geq 0$ such that* $\lim_{n\to\infty} \lambda_n = \infty$.

Note 2.21 (cf. [3]- 1.1.4.)**.** *Every symmetric operator is closable and its closure is also symmetric.*

Proof. Since T^* is an extension of T the first part of statement follows from 2.9(1)
From 2.9(3) and the first part of this note we get $\overline{T} = T^{**}$. Thus T^{**} is the smallest closed extension of T. Bringing these facts together we get

$$T = T^{**} \subset T^* = (\overline{T})^*.$$

Thus the closure $\overline{T}$ is symmetric.

□

Note 2.22. *A closed operator T is self-adjoint if and only if T^* is symmetric.*

Not all closed symmetric operators are self-adjoint. This is important when we use the spectral theorem which is valid only for self-adjoint operators. Thus we present a criterion for self-adjointness. First we introduce a new useful notion of essential self-adjointness.

Definition 2.23. *A symmetric operator T is* **essentially self-adjoint** *if its closure is self-adjoint. If T is closed we call a subset $D \subset$* Dom(T) *a* **core** *for T if $\overline{T\restriction_D} = T$.*

If T is essentially self-adjoint then it has one and only one self-adjoint extension. The existence follows from the definition and the uniqueness is possible to prove by contradiction (see [4], VIII.2).

We can see the advantage in introducing this notation when getting a non-closed symmetric operator T. If we find out that T is essentially self-adjoint, then the closure is self-adjoint by definition and $\overline{T} = T^{**}$. If we get a self-adjoint operator $\tilde{T}$ we need not to know the exact domain but a knowledge of some core for $\tilde{T}$ is sufficient.

Theorem 2.24 (The basic criterion for self-adjointness, *cf.*[4]- Theorem VIII.3)**.** *Let T be a symmetric operator on $\mathscr{H}$ then the following statements are equivalent:*

1. *T is self-adjoint,*
2. *T is closed and* $\mathrm{Ker}(T^* \pm i) = \{0\}$,
3. $\mathrm{Ran}(T \pm i) = \mathscr{H}$.

The corollary of this theorem is an analogous statement for essentially self-adjoit operators.

Corollary 2.25. *Let T be a symmetric operator on $\mathscr{H}$ then the following statements are equivalent:*

1. *T is essentially self-adjoint,*
2. $\mathrm{Ker}(T^* \pm i) = \{0\}$,
3. $\overline{\mathrm{Ran}(T \pm i)} = \mathscr{H}$.

(cf. [4]- Section VIII.2).

Note 2.26. *A symmetric operator may have many self-adjoint extensions. However it may also happen that there exists no such an extension. (For more details see von Neumann formula in [1]- Theorem 8.3.2).*

Now we introduce the notation for few sets of functions needed in further text.

Definition 2.27. *We denote $C^\infty(\overline{\Omega})$ all the smooth functions on domain $\Omega \subset \mathbb{R}^N$ whose all partial derivatives can be extended continuously on $\overline{\Omega}$.*

Definition 2.28. *We define $C_0^\infty(\overline{\Omega})$ to be the set of all functions $f \in C^\infty(\overline{\Omega})$ satisfying $f \equiv 0$ on the region boundary ($\partial\Omega$).*

Following theorems will enable us to introduce the rational powers of a self-adjoint operator.

Theorem 2.29 (Spectral theorem 1, *cf.* [3]- 2.3.1)**.** *Let T be a self-adjoint operator. Then there exists a unit linear map $f \to f(T)$ from $C_0(\mathbb{R})$ to algebra of all bounded operators $\mathscr{L}(\mathscr{H})$ satisfying*

1. *the map $f \to f(T)$ is multiplicative (which means it is an algebra homomorphism),*
2. *we have $\overline{f}(T) = f(T)^*$ for all $f \in C_0(\mathbb{R})$,*
3. *we have $\|f(T)\| \leq \|f\|_\infty$ for all $f \in C_0(\mathbb{R})$,*
4. *if $w \notin \mathbb{R}$ and $r_w(s) := (s-w)^{-1}$ then $r_w(T) = (T-w)^{-1}$,*
5. *if $f \in C_0(\mathbb{R})$ has support disjoint from $\sigma(T)$ then $f(T) = 0$.*

Functions of self-adjoint operators may be represented by multiplication operators. We can obtain this multiplication operator by a unitary transformation. The spectrum of a self-adjoint operator is an invariant of this transformation. These ideas are summarized in the following theorem.

Theorem 2.30 (Spectral theorem 2, *cf.* [3]- 2.5.1)**.** *Let H be a self-adjoint operator on $\mathscr{H}$ with spectrum S. Then there exists a finite measure μ on $S \times \mathbb{N}$ and a unitary operator*

$$U : \mathscr{H} \to L^2 := L^2(S \times \mathbb{N}, \mathrm{d}\mu)$$

with the following properties: if $h : S \times \mathbb{N} \to \mathbb{R}$ is the function $h(s,n) = s$, then the element $\xi \in \mathscr{H}$ lies in Dom(H) *if and only if $h \cdot U(\xi) \in L^2$. We have*

$$UHU^{-1}\psi = h\psi$$

for all $\psi \in U(\mathrm{Dom}(H))$ and also

$$Uf(H)U^{-1}\psi = f(h)\psi$$

for all $f \in C_0(\mathbb{R})$ and $\psi \in L^2(S \times \mathbb{N}, \mathrm{d}\mu)$.

We can extend Theorem 2.29 originally valid for functions from $C_0^\infty(\mathbb{R})$ to functions from $\mathscr{B}$ denoting the algebra of bounded Borel measurable functions on $\mathbb{R}$.

Definition 2.31. *We say that the sequence $f_n \in \mathscr{B}$* **increases monotonically** *to f if $f_n(x)$ increases pointwise and monotonically to $f(x)$ for every x.*

Note 2.32. *If $f_n \in \mathscr{B}$ increases monotonically to f, then the norms*

$$\|f_n\| := sup\{f_n(x) \mid \ x \in \mathbb{R}\}$$

are uniformly bounded.

Theorem 2.33 (Spectral theorem 3, *cf.* [3]- 2.5.3)**.** *There exists a map $f \to f(H)$ from $\mathscr{B}$ to $\mathscr{L}(\mathscr{H})$ which extends the map from Theorem 2.29 and has the same properties with replacement $C_0(\mathbb{R})$ by $\mathscr{B}$. The extension is unique subject to the further requirement that:*

$$s\text{-}\lim_{n\to\infty} f_n(H) = f(H)$$

whenever $f_n \in \mathscr{B}$ converges monotonically to $f \in \mathscr{B}$.

Note 2.34. *A sequence of bounded operators $(A_n)_{n=1}^\infty$ on a Banach space $\mathscr{X}$ converge strongly to a bounded operator on $\mathscr{X}$ if $\lim_{n\to\infty} \|A_n f - Af\| = 0,\ \forall f \in \mathscr{X}$ and we write $s\text{-}\lim_{n\to\infty} A_n = A$.*

2.2 Quadratic forms

It turns out, that there exists a one-to-one correspondence between quadratic forms with specific properties and self-adjoint operators bounded from below. It is often easier to find the domain of a quadratic form then of an operator. Therefore when we study the operators, it is useful to use the theory of quadratic forms. We summarize the basic definitions and facts from it in this section.

For bounded operator the Riesz lemma implies that there is a one-to-one correspondence between bounded quadratic forms and bounded operators. Thus for a sesquilinear map $q : \mathscr{H} \to \mathscr{H}$ such that $|q(f,g)| < M \|f\| \|g\|$ we can write

$$q(f,g) = \langle f, Ag \rangle,$$

for some bounded operator A. In this section we present an analogues statement for unbounded forms and operators.

Definition 2.35. *We call* **quadratic form** *a map $q : \mathscr{Q}(q) \times \mathscr{Q}(q) \to \mathbb{C}$,*
where $\mathscr{Q}(q)$ is a dense linear subset of Hilbert space $\mathscr{H}$ such that $q(\cdot, f)$ is conjugate linear and $q(g,\cdot)$ is linear for $f, g \in \mathscr{Q}(q)$. We call $\mathscr{Q}(q)$ the **form domain**.

We say that

q is **symmetric** *if*	$q(f,g) = \overline{q(f,g)}$,	$\forall f, g \in \mathscr{Q}$,
q is **positive** *if*	$q(f,f) \geq 0$,	$\forall f \in \mathscr{Q}$,
q is **bounded from below** *if*	$q(f,f) \geq M\|f\|^2$	*for some $M \in \mathbb{R}$.*

Note 2.36. *A positive quadratic form is symmetric if $\mathscr{H}$ is complex. (cf. [1]- Section 1.2).*

Note 2.37. *A quadratic form bounded from below is symmetric if $\mathscr{H}$ is complex. (cf. [4]- Section VIII.6).*

Now we wish to find the connection between self-adjointness and a distinct class of quadratic forms. Since a self-adjoint operator is always closed, we need to introduce the term of closeness for quadratic forms. We do it in accordance with the definition of closed operators. An operator A is closed if its graph is closed. Equivalently we can say that $\mathrm{Dom}(A)$ is complete under the norm $\|f\|_A := \|Af\| + \|f\|$.

Definition 2.38. *A quadratic form q bounded from below, $q(f,f) \geq -M\|f\|^2$, is* **closed** *if $\mathscr{Q}(q)$ is complete under the norm*

$$\|f\|_{+1} = \sqrt{q(f,f) + (1-M)\|f\|^2}.$$

If q is closed and $\mathscr{D} \subset \mathscr{Q}(q)$ is dense in $\mathscr{Q}(q)$ in the norm $\|\cdot\|_{+1}$, then $\mathscr{D}$ is called **form core** *for q. A form q_2 is an* **extension** *of form q_1 if $\mathrm{Dom}(q_1) \subset \mathrm{Dom}(q_2)$ and $q_1(f) = q_2(f), \forall f \in \mathrm{Dom}(q_1)$. A form q is said to be* **closable** *if there exists a closed extension of q. The* **closure** *$\overline{q}$ of the form q is the smallest closed extension.*

Equivalently we can define the core for the closed form q to be the linear subspace $\mathscr{D}$ of domain its $\mathscr{Q}(q)$ satisfying $\overline{q|_{\mathscr{D}}} = \mathscr{Q}(q)$.

Note 2.39. *The norm $\|\cdot\|_{+1}$ comes from the inner product $\langle f, g\rangle_{+1} = q(f,g) + (1-M)\langle f, g\rangle$.*

Note 2.40. *q is closed if and only if for every sequence $(f_n) \subset \mathscr{Q}(q)$ such that*

1. *$f_n \to f$ in $\mathscr{H}$*
2. *$q(f_n - f_m, f_n - f_m) \to 0$ as $m, n \to \infty$,*

$f \in \mathscr{Q}(q)$ and $q(f_n - f, f_n - f) \to 0$.

Theorem 2.41 (Representation theorem, *cf.*[1]- 7.5.8)**.** *Let q be a densely defined closed symmetric and from below bounded quadratic form on $\mathscr{H}$. Then there exists a self-adjoint operator A such that*

1. $\mathrm{Dom}(A) \subset \mathscr{Q}(q)$ *and*
$$q(f,g) = \langle f, Ag\rangle,\ \forall f \in \mathscr{Q}(q),\ g \in \mathrm{Dom}(A),$$
2. $\overline{q\,|_{\mathrm{Dom}(A)}} = q$,
3. *if there exist* $f \in \mathscr{Q}(q)$ *and* $h \in \mathscr{H}$ *such that*
$$q(f,g) = \langle h, g\rangle,\ \forall g \in \mathscr{Q}(q),$$
then $f \in \mathrm{Dom}(A)$ *and* $h = Af$,
4. *if a linear operator T satisfies* $\mathrm{Dom}(T) \subset \mathscr{Q}(q)$ *and* $\langle Tf, g, \rangle = q(f,g)$, $\forall f \in \mathrm{Dom}(T),\ g \in \mathscr{Q}(q)$, *then $T \subset A$. If T is self-adjoint, then $T = A$. Thus the condition (1) defines a unique self-adjoint operator.*

For a symmetric operator bounded from below there always exists at least one closed extension. The problem is that none of these extensions needs to be self-adjoint. For quadratic forms the situation is different. The closed extension does not need to exist but when it does it is automatically self-adjoint.

Since Dirichlet Laplacian is a non-negative operator we will look more closely at this specific case of operators. In following we assume self-adjoint operators. For such a non-negative operator we can introduce the rational powers of the operator using the spectral theorem.

Theorem 2.42 (*cf.* [3]- 4.3.4)**.** *If A is a non-negative self-adjoint operator and* $0 < m < 1$, *then* $f \in \mathrm{Dom}(A)$ *if and only if* $f \in \mathrm{Dom}(A^m)$ *and also* $A^m f \in \mathrm{Dom}(A^{1-m})$. *For such f we have*
$$Af = A^{1-m}(A^m f)$$

Note 2.43. *Using this theorem we can define a quadratic form q for a given self-adjoint operator A by*
$$q(f,g) := \langle A^{1/2} f, A^{1/2} g\rangle$$
where $f, g \in \mathrm{Dom}(A^{1/2})$.

Lemma 2.44 (*cf.* [3]- 4.4.1)**.** *Let A be a non-negative self-adjoint operator on $\mathscr{H}$. Then $f \in \mathscr{H}$ lies in* $\mathrm{Dom}(A)$ *if and only if* $f \in \mathrm{Dom}(A^{1/2})$ *and also there exists $h \in \mathscr{H}$ such that*
$$q(f,g) := \langle h, g\rangle$$
$\forall g \in \mathrm{Dom}(A^{1/2})$. *In this case we have $Af = h$.*

Theorem 2.45 (*cf.* [3]- 4.4.2)**.** *The quadratic form q arises from a non-negative self-adjoint operator if and only if the domain $\mathscr{Q}(q)$ of q is complete under the norm* $\|f\|_q := \sqrt{\|f\|^2 + q(f,f)}$.

Theorem 2.46 (*cf.* [3]- 4.4.5)**.** *Let q be the form defined on the domain $\mathscr{Q}(q)$ of a non-negative symmetric operator A by*
$$q(f,g) := \langle Af, g\rangle.$$
Then the quadratic form q is closable and its closure is associated with a self-adjoint extension of A.

2.3 Variational characterisation of eigenvalues

This section reviews some of the results of the variational method. This method is more closely described in [3]-Chapter 4. We will need the theory of this section for numerical solution of Dirichlet problem in Section 5.2.

Definition 2.47. *Let A be a self-adjoint operator bounded from below on $\mathscr{H}$ and q the associated quadratic form. Then we define*

$$\lambda(L) := \sup\{\langle Af, f\rangle \mid f \in L, \|f\| = 1\} = \sup\{q(f, f) \mid f \in L, \|f\| = 1\},$$

where L is a finite-dimensional subspace of $\mathscr{H}(= Dom(A))$. Further we define

$$\lambda_n := \inf\{\lambda(L) \mid L \subset\subset \mathrm{Dom}(A), \dim(L) = n\}.$$

Theorem 2.48 (Min-max principle, *cf.* [6]- Theorem B.5)**.** *Let A be a self-adjoint operator bounded from below. Let λ_n be a non-decreasing sequence of numbers defined by 2.47. Then*

1. $\lambda_\infty := \lim_{n\to\infty} \lambda_n = \inf \sigma_{ess}(A)$*, with convention that* $\sigma_{ess} = \emptyset$ *if* $\lambda_\infty = +\infty$*,*
2. $\{\lambda_n\}_{n=0}^{\infty} \cap (-\infty, \lambda_\infty) = \sigma_{disc}(A) \cap (-\infty, \lambda_\infty)$ *each* $\lambda_n \in (-\infty, \lambda_\infty)$ *being an eigenvalue repeated according to its multiplicity.*

Corollary 2.49 (*cf.* [6]-Corollary B.2)**.** *If A satisfies the assumptions from 2.48, then*

$$\inf \sigma(A) = \inf_{\substack{f \in \mathrm{Dom}(A)\setminus\{0\} \\ \|f\|=1}} \langle f, Af\rangle = \inf_{\substack{f \in \mathscr{Q}(q)\setminus\{0\} \\ \|f\|=1}} q(f, f).$$

The min-max principle appears also in the form of the so-called max-min principle. The version max-min gives the lower estimates for eigenvalues. This case is the subject of the following theorem.

Theorem 2.50 (Max-min principle, *cf.* [9]- Theorem 12.1)**.** *Let A be a self-adjoint from below bounded operator with associated quadratic form q and let the numbers λ_n be the eigenvalues of A for $n \in \widehat{J-1}$. There $J \in \mathbb{N}$ is the index of first number λ_n, which is not an eigenvalue. If we choose any m linearly independent normalized functions $(g_1, \ldots, g_m) \in \mathrm{Dom}(A) \subset L^2(\mathbb{R}^N)$, then we have for $m \leq J$*

$$\lambda_m = \max_{g_1,\ldots,g_{m-1}} \min\{q(g_m, g_m) \mid \langle g_m, g_i\rangle = 0,\ \forall i \in \widehat{m-1}\}.$$

There we consider the inner products and norms in $L^2(\mathbb{R}^N)$.

Corollary 2.51. *Under the assumptions of 2.50 we have*

$$\lambda_m = \min\{q(g_m, g_m) \mid \langle g_m, f_i\rangle = 0,\ \forall i \in \widehat{m-1}\},$$

where $f_1, \ldots, f_{m-1}$ are the eigenvectors of operator A.

Chapter 3

Dirichlet Laplacian

As mentioned in the introduction when we study the spectrum of a differential operator we need to pay attention to the domain of the operator. The domain may be restricted by a boundary condition. In this project we restrict ourselves on so-called Dirichlet boundary condition. This means we require functions from the domain to have zero value on the boundary of the region $\partial\Omega$. These are the simplest conditions to work with, nevertheless they are useful in many physical problems. For instance they represent ideal conductivity of surface while solving the heat equation. In other words the surroundings have infinitely large thermal capacity. The conditions may be also considered as an infinite potential wall, when we describe the motion of a quantum particle by Schrödinger equation.

Note 3.1. *Dirichlet Laplacian is an elliptic operator.*

Generally we have a second order elliptic operator H on $L^2(\Omega)$ defined by

$$Hf := -b(x)^{-1} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial f}{\partial x_j}\Big), \qquad \forall f \in C_0^\infty(\overline{\Omega}) \tag{3.1}$$

where $a(x) := (a_{ij}(x))$ is a real, symmetric, uniformly positive and bounded matrix satisfying

$$c^{-1}\mathbb{I} \leq a(x) \leq c\mathbb{I}, \qquad \forall x \in \Omega \tag{3.2}$$

for some real constant $c \geq 1$. Further we demand $a_{ij}(x)$ to be defined and differentiable on a region $\Omega \subset \mathbb{R}^n$ and its boundary $(\partial\Omega)$.
The coefficient $b(x)$ is a real, continuous and bounded function defined on the region $\Omega \subset \mathbb{R}^n$ and its boundary satisfying

$$c^{-1} \leq b(x) \leq c, \qquad \forall x \in \Omega \tag{3.3}$$

for some real constant $c \geq 1$.

Lemma 3.2 (*cf.* [3]- Lemma 6.1.1)**.** *By the established conditions for coefficients $a_{ij}(x)$ and $b(x)$ the operator H is symmetric and positive on $\mathscr{H} = L^2(\Omega, b(x)\mathrm{d}x^N)$. Moreover the associated quadratic form*

$$Q(f,g) := \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial \overline{f}}{\partial x_i}\frac{\partial g}{\partial x_j}$$

is closable and its closure on the domain $C_0^\infty(\overline{\Omega})$ is independent of the choice of the particular coefficients $b(x), a(x)$.

Proof. The operator H acts on $L^2(\Omega)$ and especially according to the assumptions of lemma the functions $f, g \in C_0^\infty(\overline{\Omega})$. We use the fact that the Hilbert spaces $L^2(\Omega, \mathrm{d}x^N)$ and $L^2(\Omega, b(x)\mathrm{d}x^N)$ coincide and have only different, but equivalent norms and inner products.

We denote

$$\|f\|_\nabla^2 := \int_\Omega |f|^2 + |\nabla f|^2 \mathrm{d}^N x$$

$$\|f\|_Q^2 := \|f\|^2 + Q(f, f) = \int_\Omega b(x)|f|^2 + \sum_{i,j=1}^N a_{ij}(x) \frac{\partial \overline{f}}{\partial x_i} \frac{\partial f}{\partial x_j},$$

where $\|\cdot\|$ is norm on $\mathscr{H} = L^2(\Omega, b(x)\mathrm{d}x^N)$. Due to the conditions (3.2) and (3.3) we have

$$c^{-1}\|f\|_\nabla^2 \leq \|f\|_Q^2 \leq c\|f\|_\nabla^2.$$

The norm $\|\cdot\|_\nabla$ is independent from the coefficients a_{ij}, b. Thus the norms are comparable and the completion of $C_0^\infty(\overline{\Omega})$ is the same for both of them.

Now we can prove the symmetry of H. That means we need to show the equality

$$\langle f, Hg \rangle = \langle Hf, g \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathscr{H} = L^2(\Omega, b(x)\mathrm{d}x^N)$.
Using integration by parts and Gauss' theorem in connection with the condition that the functions f, g has zero value on $\partial\Omega$ we obtain

$$\langle f, Hg \rangle = \int_\Omega -b(x)^{-1} \sum_{i,j=1}^N \overline{f} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial f}{\partial x_j} \right) b(x)\, \mathrm{d}x^N = \int_\Omega \sum_{i,j=1}^N \frac{\partial \overline{f}}{\partial x_i} \left(a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \mathrm{d}x^N = Q(f, g).$$

In the same manner we show the equality $\langle Hf, g \rangle = Q(f, g)$. Therefore

$$\langle f, Hg \rangle = Q(f, g) = \langle Hf, g \rangle$$

and H is a symmetric operator associated with the quadratic form Q.

For non-negativity we need to verify

$$\langle f, Hf \rangle \geq 0 \qquad \forall f \in C_0^\infty(\overline{\Omega}).$$

We have

$$\langle f, Hf \rangle = Q(f, f) = \int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial \overline{f}}{\partial x_i} \frac{\partial f}{\partial x_j} \qquad \forall f \in C_0^\infty(\overline{\Omega}).$$

Since $a(x)$ is a positive matrix, we have $Q(f, f) \geq 0$. Thus both the form Q and the operator H are non-negative.

The closability is implied by the fact, that Q is a quadratic form defined on a domain of a non-negative symmetric operator by the relation $Q(f, g) = \langle f, Hg \rangle$. (*cf.* Theorem 2.46)

□

Now we introduce the notation for few sets of functions needed in further text.

Definition 3.3. *We call the set* $\mathrm{supp}(f) := \overline{\{x \in \Omega \mid f(x) \neq 0\}}$ *the* **support** *of the function f and define $C_c^\infty(\Omega)$ to be the set of smooth functions with compact supports contained in Ω. The elements of $C_c^\infty(\Omega)$ are called* **test functions**.

Definition 3.4. *We call* **distribution** *a linear function* $\Phi : C_c^\infty(\Omega) \to \mathbb{C}$.

Definition 3.5. *Let $f \in C_c^\infty(\Omega)$. We define* **weak derivative** $D^\alpha \phi$ *of distribution* Φ *by*

$$(D^\alpha \Phi)(f) := (-1)^{|\alpha|}\, \Phi(D^\alpha f)$$

for every multi-index ($\alpha = \alpha_1, \dots, \alpha_N$) consisting of non-negative integers. Here $|\alpha| = \alpha_1 + \dots + \alpha_N$ is called degree of this multi-index.
We define the **multiplication** *by a smooth function g by $(g\,\Phi)f := \Phi(gf)$.*

Note 3.6. *We often use the term derivative in the sense of distributions instead of the notion weak derivative.*

For the next lemma and other work with Dirichlet Laplacian (or elliptic operators in general) we need to introduce the notion of the Sobolev space.

Definition 3.7. *We define the* **Sobolev space** $W^{1,2}(\Omega)$ *for any $\Omega \subset \mathbb{R}^N$ to be the set of all the distributions f satisfying $f \in L^2(\Omega) \wedge \frac{\partial f}{\partial x_i} \in L^2(\Omega),\ \forall i \in \hat{N}$. There the derivatives should be understood in the sense of distributions.*

Note 3.8. *Using this notation the Sobolev space $W^{1,2}(\Omega)$ is the domain of the gradient operator $\nabla : L^2(\Omega) \to L^2(\Omega) \oplus \cdots \oplus L^2(\Omega)$ defined by*

$$\nabla f(x) := \partial_1 f \oplus \cdots \oplus \partial_N f.$$

We define the inner product on $W^{1,2}(\Omega)$ by

$$\langle f, g\rangle_\nabla := \int_\Omega \left(f(x)\overline{g(x)} + \nabla f(x)\overline{\nabla g(x)}\right) \mathrm{d}^N x. \tag{3.4}$$

Lemma 3.9 (*cf.* [3]- Lemma 6.1.2)**.** *With respect to this inner product the Sobolev space $W^{1,2}(\Omega)$ is a Hilbert space for arbitrary bounded region $\Omega \subset \mathbb{R}^N$.*

Proof. According to Note 3.8 it is sufficient to show, that the operator ∇ is a closed linear operator with respect to the norm induced by inner product (3.4).

Let $\{f_n\}_{n=1}^\infty \in W^{1,2}(\Omega)$, $f \in L^2(\Omega)$ and $\{g_n\}_{n=1}^N \in L^2(\Omega)$. We assume, that $\lim_{n\to\infty} \|f_n - f\| = 0$ and $\lim_{n\to\infty} \|\partial_i f_n - g_i\| = 0,\ \forall i \in \hat{N}$, where we denoted $\frac{\partial}{\partial x_i} = \partial_i$. If $\Phi \in C_c^\infty(\Omega)$ is a distribution, then

$$\begin{aligned}
\langle g_i, \Phi\rangle &= \lim_{n\to\infty} \langle \partial_i f_n, \Phi\rangle \\
&= -\lim_{n\to\infty} \langle f_n, \partial_i \Phi\rangle \\
&= -\langle f, \partial_i \Phi\rangle \\
&= \langle \partial_i f, \Phi\rangle.
\end{aligned}$$

We used the continuity of the inner product in the first and third equality and the definition of derivative in the sense of distributions in the second and forth one.

The equality $\langle g_i, \Phi\rangle = \langle \partial_i f, \Phi\rangle$ is valid for all distributions Φ and therefore $g_i = \partial_i f$. Thus $f \in \mathrm{Dom}(\nabla)$ and $\nabla f = (g_1, \dots, g_N)$.

□

Definition 3.10. *We define the* **Sobolev subspace** $W_0^{1,2}(\Omega)$ *by* $W_0^{1,2}(\Omega) := \overline{C_c^\infty(\Omega)}$*, where the closure is taken with respect to the norm* $\|\cdot\|_\nabla$.

Lemma 3.11 (*cf.* [3]- Lemma 6.1.3)**.** *The domain of closure of the quadratic form from Lemma 3.2 satisfies* $Dom(\overline{Q}) = W_0^{1,2}(\Omega)$*. Thus* $W_0^{1,2}(\Omega) = Dom(H^{1/2}) \wedge W_0^{1,2}(\Omega) \supset C_0^\infty(\overline{\Omega})$*. Where H is the self-adjoint closure of the operator from Lemma 3.2 satisfying the Dirichlet boundary condition.*

Proof. We need to show the following inequalities

$$\mathrm{Dom}(\overline{Q}) \subseteq W_0^{1,2}(\Omega),$$
$$\mathrm{Dom}(\overline{Q}) \supseteq W_0^{1,2}(\Omega).$$

From Lemma 3.2 we have $\mathrm{Dom}(\overline{Q}) = C_0^\infty(\overline{\Omega})$. Since $\overline{Q}$ is a closed form, the domain has to be closed with respect to the norm $\|\cdot\|_Q$. Thus $C_0^\infty(\overline{\Omega}) = \overline{C_0^\infty(\overline{\Omega})}$. By definition, it is easily seen that $C_c^\infty(\Omega) \subseteq C_0^\infty(\overline{\Omega})$. Therefore we have $\overline{C_c^\infty(\Omega)} \subseteq \overline{C_0^\infty(\overline{\Omega})}$. These facts together give

$$W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)} \subseteq \overline{C_0^\infty(\overline{\Omega})} = C_0^\infty(\overline{\Omega}) = \mathrm{Dom}(\overline{Q}).$$

This proves the second inequality. To show the first one we introduce for a given $\varepsilon > 0$ a smooth function $F_\varepsilon : \mathbb{R} \to \mathbb{R}$ satisfying

1. $F_\varepsilon(x) = x$, if $|x| > 2\varepsilon$,
2. $F_\varepsilon(x) = 0$, if $|x| \leq \varepsilon$,
3. $|F_\varepsilon(x)| \leq |x|$, if $x \in \mathbb{R}$,
4. $0 \leq F'_\varepsilon(x) \leq 3$, if $x \in \mathbb{R}$.

Let $f \in C_0^\infty(\overline{\Omega})$ be a real function. For a complex function we would do the same steps for the real and imaginary part of the function. We define $f_\varepsilon(x) := F_\varepsilon(f(x))$, $\forall x \in \Omega$. By the definitions of f_ε and F_ε, it is easily seen that f_ε is a smooth function on Ω, which vanishes in a neighbourhood of $\partial\Omega$. Therefore $f_\varepsilon \in C_c^\infty(\Omega)$. The points 3. and 1. give

$$|f_\varepsilon| \leq |f(x)|,$$
$$\lim_{\varepsilon\to 0} f_\varepsilon(x) = f(x),$$

$\forall x \in \Omega$. Using the dominated convergence theorem we get

$$\lim_{\varepsilon\to 0} \|f - f_\varepsilon\| = 0,$$

where $\|\cdot\|$ is the norm on $L^2(\Omega)$. Further the points 4. and 1. give

$$|\nabla f_\varepsilon(x)| \leq 3|\nabla f(x)|,$$
$$\lim_{\varepsilon\to 0} \nabla f_\varepsilon(x) = \nabla f(x),$$

$\forall x \in \Omega$ such that $f(x) \neq 0$. We denote sets $A := \{x \in \Omega \mid f(x) = 0\}$ and $B := \{x \in \Omega \mid f(x) = 0 \wedge \nabla f(x) \neq 0\}$. According to the implicit function theorem the set B is a hypersurface in $\mathbb{R}^N$ of codimension 1. Therefore it is measure-zero set. Then the dominated convergence theorem gives

$$\lim_{\varepsilon\to 0} \|\nabla f - \nabla f_\varepsilon\| = \int_A |\nabla f|^2 \mathrm{d}^N x = \int_B |\nabla f|^2 \mathrm{d}^N x = 0.$$

Since $||f||_\nabla^2 = ||f||^2 + ||\nabla f||^2$, we have

$$\lim_{\varepsilon\to 0} ||f - f_\varepsilon||^2 = \lim_{\varepsilon\to 0} \left(||f - f_\varepsilon||^2 + ||f - f_\varepsilon||_\nabla^2\right) = 0. \tag{3.5}$$

The norms $||\cdot||_Q$ and $||\cdot||_\nabla$ are equivalent. Therefore (3.5) implies that we can approximate a function $f \in C_0^\infty(\overline{\Omega})$ by a function from $C_c^\infty(\Omega)$ with arbitrary precision in the norm $||\cdot||_Q$. In other words $C_c^\infty(\Omega)$ is dense in $C_0^\infty(\overline{\Omega})$. This implies the second inequality.

□

Theorem 3.12. *Under the specified conditions for the coefficients from formula for a general elliptic operator, the quadratic form Q from Lemma 3.2 is closed on the domain $W_0^{1,2}(\Omega)$ in the Hilbert space $L^2(\Omega, b(x)\mathrm{d}^n x)$. There exists a non-negative self-adjoint operator H on $L^2(\Omega, b(x)\mathrm{d}^n x)$ associated to the form in such a way that*

$$\langle H^{1/2} f, H^{1/2} g\rangle = Q(f,g)$$

for all $f, g \in \mathrm{Dom}(H^{1/2}) = W_0^{1,2}(\Omega)$.

Proof. The fact that the quadratic form is closed with respect to the norm on $L^2(\Omega, b(x)\mathrm{d}^n x)$ follows from Lemma 3.2. Theorem 2.45 then proves the second part of this theorem.

□

In following we will look more closely to the self-adjoint extension H of the operator from Lemma 3.2 with the spatial choice of the coefficients $a_{ij}(x) = \delta_{ij}$ and $b(x) = 1$. We call this operator the Dirichlet Laplacian and denote $H = -\Delta$. According to Theorem 2.41 this self-adjoint extension is unique. In particular we will study the spectrum of Dirichlet Laplacian. Thus we solve

$$\begin{aligned} -\Delta f &= \lambda f \qquad \text{in } \Omega, \\ f &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3.6}$$

for $f \in W_0^{1,2}(\Omega)$.

Note 3.13. *According to Theorem 2.14 the Dirichlet Laplacian has real and non-empty spectrum.*

Lemma 3.14 (*cf.* [3]- 6.2.2)**.** *Let Ω be a cube and λ_n the eigenvalues of Dirichlet Laplacian on Ω written in increasing order and repeated by multiplicity. Then there exists a constant $c > 0$ such that*

$$\lim_{n\to\infty} \lambda_n n^{-2/N} = c.$$

Theorem 3.15 (*cf.* [3]- 6.2.3)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded region, then H defined above has empty essential spectrum and compact resolvent.*

Proof. We denote

$$\begin{aligned} \tilde{Q} &:= \int_\Omega |\nabla f(x)|^2 \mathrm{d}^N x, \qquad f \in C_c^\infty(\Omega) \\ \lambda_n(\Omega) &:= \inf\{\lambda(L) \mid L \subset\subset C_c^\infty(\Omega),\ \dim L = n\}, \end{aligned}$$

where $\lambda(L) := \sup\{\tilde{Q}(f) \mid f \in L, ||f|| \le 1\}, \forall L \subset\subset C_c^\infty$.

Let $\Omega \subseteq \Omega'$, then $\lambda_n(\Omega) \ge \lambda_n(\Omega')$ since $\lambda(\Omega)$ is infimum over a smaller set. If we choose Ω' such that it is a cube, then the previous lemma implies $\lim_{n\to\infty} \lambda_n(\Omega) = \infty$. Then Theorem 2.48 implies that $\sigma_{ess}(H) = \emptyset$. According to Theorem 2.20 this is equivalent to the fact that H has compact resolvent.

□

Note 3.16. *This theorem with Note 3.13 give* $\sigma(H) = \sigma_{disc}(H)$.

Note 3.17. *We denote* $W^{2,2}(\Omega)$ *the set of all the distributions* f *satisfying* $f \in L^2(\Omega) \wedge \frac{\partial^2 f}{\partial x_i^2} \in L^2(\Omega)$, $\forall i \in \hat{N}$. *There the derivatives should be understood in the sense of distributions. Then any weak solution of 3.6 satisfies the extra regularity* $f \in W^{2,2}(\Omega)$. *(cf. [6]- Theorem D.3)*

Chapter 4

Nodal hypothesis

In this chapter we introduce the important notion of nodal line more precisely then we did in the introduction. We summarize the up to date results of the so-called nodal hypothesis.

This hypothesis originates from the conjecture of L.E. Payne from 1960s. Payne assumed the nodal hypothesis to be valid for bounded regions in $\mathbb{R}^2$. Later on this was proved with the additional assumptions on convexity and smoothness of the boundary. The hypothesis was reformulated for higher dimensional cases and proved even for some unbounded regions with some specific properties.

Definition 4.1. *We denote λ_n and f_n the nth eigenvalue and eigenfunction of Dirichlet Laplacian on a region Ω and define the* **nodal set** *of f_n to be $\mathcal{N}(f_n) = f_n^{-1}(0) \cap \Omega$.*

One can verify, that $\mathcal{N}(f_1) = \emptyset$ and if $\langle f_1, f_2 \rangle = 0$, then $\mathcal{N}(f_2) \neq \emptyset$. On planar domains this non-empty set is a nodal line. In our case we are interested in the nodal lines of the second eigenfunction. The reason is simple. These are the eigenfunctions the nodal hypothesis concerns about.

The original conjecture of L.E. Payne was formulated (in 1967) for Ω bounded as follows. Let $\Omega \subset \mathbb{R}^2$ be bounded. If

$$\begin{aligned} -\Delta f &= \lambda f \qquad &&\text{in } \Omega \\ f &= 0 \qquad &&\text{on } \partial\Omega \end{aligned}$$

then $\mathcal{N}(f_2)$ cannot be a closed curve. Thus its closure must intersect the boundary.

Hypothesis 4.2 (The nodal set conjecture)**.** *For all bounded and simply-connected domains Ω the second eigenfunction f_2 satisfies*

$$\mathcal{N}(f_2) \cap \partial\Omega \neq \emptyset.$$

The conjecture was proved for bounded convex domains in $\mathbb{R}^2$ with smooth boundary by A.D. Melas in 1992. It was reformulated and proved for some unbounded domains such as long thin convex domains in $\mathbb{R}^N$ by D. Jerison in 1995, thin curved tubes in $\mathbb{R}^N$, possibly non-convex and multi-connected by P.Freitas and D. Krejčiřík in 2007 and diminishing fibre bundle of arbitrary dimension with a compact base space of dimension at most 3 by J. Lampart, S. Taufel in 2014. On the contrary it was disproved for a certain multi-connected domain in $\mathbb{R}^2$ by M.T. Hoffmann-Ostenhof and N. Nadirashvili in 1997 and the same in $\mathbb{R}^N$ by S. Fournais in 2001.

Chapter 5

Examples

5.1 Analytical solution

A way how to obtain the nodal lines of Dirichlet Laplacian is to study the eigenfunctions of this operator. Therefore we solve the following linear differential equation of the second order, which is the equation for the eigenvalue problem

$$-\Delta f = \lambda f, \tag{5.1}$$

where $f \in \mathrm{Dom}(H)$. In fact we can find the analytical solution of this problem only on some very specific domains like disc (or a sector of a disc) and rectangle. When we want the nodal lines on a more difficult region we need to use an approximation and numerical tools.
In this section we show the analytical solutions on the above mentioned regions.

5.1.1 Rectangle

For the rectangle the region is

$$\Omega = \{(x, y)|\, x \in (0, a) \wedge y \in (0, b)\},$$

where constants $a, b > 0$. Due to the Dirichlet boundary condition we have

$$f(0, y) = f(x, 0) = f(a, y) = f(x, a) = 0.$$

Using separation of variables by solving (5.1) we get the eigenvalues $\lambda_{n,m}$ and associated eigenvectors $f_{n,m}$,

$$\lambda_{n,m} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \tag{5.2}$$

$$f_{n,m} = \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right), \tag{5.3}$$

where $n, m \in \mathbb{N}$. The obtained functions are not normalized. It is easy to notice that the eigenvalues are degenerated whenever the condition $a^2(m^2 - \tilde{m}^2) + b^2(n^2 - \tilde{n}^2) = 0$ for some positive integers $n \neq \tilde{n}, m \neq \tilde{m}$ is satisfied. The eigenfunction corresponding to the second eigenvalue on rectangle is shown in Figure 6.1a on p. 31.

5.1.2 Disc

Now the region Ω is a disc of radius $r = 1$. To solve the problem (5.1) we transform Laplacian to the polar coordinates (r, θ). If we denote $\frac{\partial f}{\partial x} = \partial_x f$ and $\frac{\partial^2 f}{\partial x^2} = \partial^2_{xx} f$ then the equation for the eigenvalues is

$$\partial^2_{rr} v + \frac{1}{r}\partial_r v + \frac{1}{r^2}\partial^2_{\theta\theta} v + \lambda v = 0. \tag{5.4}$$

The boundary condition is represented by the equality $v(1, \theta) = 0$.

Putting $v(r, \theta) = f(r)\, h(\theta)$, we solve this problem by separation of variables. The equation (5.4) now takes the form:

$$\frac{f''(r) + \frac{1}{r} f'(r) + \lambda f(r)}{f(r)} = -\frac{h''(\theta)}{h(\theta)}.$$

Since both sides are functions of different variables, they have to be a constant. We denote this constant c. Now we obtain the general solution for the angular part of the equation: $h(\theta) = a\cos(n\theta) + b\sin(n\theta)$, where $n \in \mathbb{N}_0$ and satisfies $n^2 = c$.

For the solution of the radial part we denote $f(r) = y$ and obtain

$$r^2 y'' + r y' + (r^2\lambda - n^2) y = 0.$$

For $\lambda \neq 0$ we substitute: $r\sqrt{\lambda} = \rho(\lambda)$ and obtain the Bessel equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}y}{\mathrm{d}\rho} + \left(1 - \frac{n^2}{\rho^2}\right) y = 0. \tag{5.5}$$

The solutions of (5.5) are the Bessel functions $J_n(\rho)$. (The Bessel functions are more closely described in [8], Chapters 9., 10.) We denote $\sqrt{\lambda} = k \in \mathbb{R}$. The boundary condition gives $v(1, \theta) = J_n(k)\, h(\theta) = J_n(\sqrt{\lambda})\, h(\theta) = 0$. The Bessel function has infinite number of zeros for every $n \in \mathbb{N}_0$. If we denote the mth zero of the nth Bessel function $k_{m,n}$ (for $m, n \in \mathbb{N}_0$), then the corresponding eigenvalue $\lambda_{m,n}$ is the square of this zero.

Then the eigenfunctions take the form

$$v_{n,m}(r, \theta) = J_n(k_{m,n} r)\, (a\cos(n\theta) + b\sin(n\theta)), \tag{5.6}$$

where $k \in \mathbb{R}$, $n \in \mathbb{N}_0$ and a, b are real constants. The eigenfunctions corresponding to the second eigenvalue on a disc are shown in Figures 6.2a and 6.2c on p. 32.

5.1.3 Sector of a disc

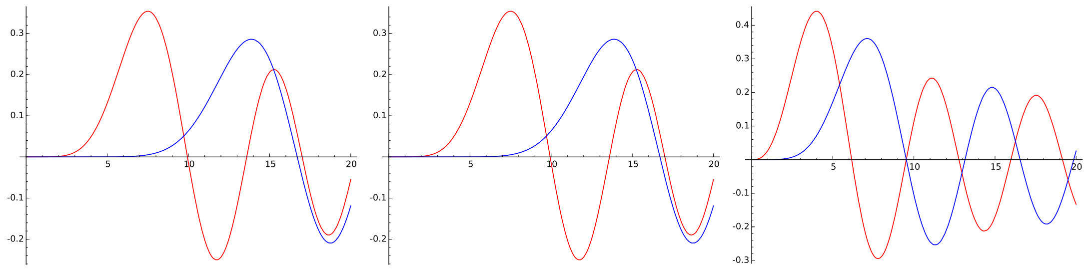
Now we do not assume the whole disc, but just a part of it. It means that the angular variable is restricted by a real number $\alpha \in (0, 2\pi)$. We use very similar approach as we did in the case of Ω being a disc. Thus we get

$$-\frac{h''(\theta)}{h(\theta)} = c$$

where c is a constant and $\theta \in (0, \alpha)$. Since the boundary condition now gives $h(0) = h(\alpha) = 0$, the solution $h(\theta) = A\sin(\frac{m\pi\theta}{\alpha})$, $m \in \mathbb{Z}$, for arbitrary constant $A \in \mathbb{R}$ satisfies this differential equation. We

Figure 5.1: The Bessel functions J_ν for $\nu = \frac{\pi}{\alpha}$ (red colour) and $\nu = \frac{2\pi}{\alpha}$ (blue colour) for angles $\alpha = \frac{\pi}{6}$, $\alpha = \frac{11\pi}{6}$ and $\alpha = 1.1125$

(a) $\alpha = \frac{\pi}{6}$ (b) $\alpha = \frac{11\pi}{6}$ (c) $\alpha = 1.1125$



also get the form of the constant $c = \left(\frac{\pi m}{\alpha}\right)$, $m \in \mathbb{Z}$. Now we make the substitution for $\lambda \neq 0$ in the same way as we did in the previous case: $r\sqrt{\lambda} = \rho(\lambda)$. The form of Bessel equation we obtain now is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}y}{\mathrm{d}\rho} + \left(1 - \frac{\left(\frac{m\pi}{\alpha}\right)^2}{\rho^2}\right) y = 0$$

and Bessel functions $J_\nu(\rho)$, $\nu = \frac{m\pi}{\alpha}$ solve this equation. Since we need to satisfy the boundary condition $f(1) = 0$ we have $y(\sqrt{\lambda}) = J_\nu(\sqrt{\lambda}) = 0$. Thus the eigenvalue λ is a square of the first root of the Bessel function and the corresponding eigenfunctions take form

$$v(r,\theta) = J_\nu(\sqrt{\lambda} r)\, A \sin\left(\frac{m\pi\theta}{\alpha}\right) \qquad m \in \mathbb{Z}. \tag{5.7}$$

In Figures 5.1a and 5.1b the Bessel functions for angles $\alpha = \frac{\pi}{6}$ and $\alpha = \frac{11\pi}{6}$ for $m = 1, 2$ are shown. For the smaller angle $\alpha = \frac{\pi}{6}$ the second eigenvalue is the square of the second zero of the Bessel function corresponding to $m = 1$. However for the bigger angle $\alpha = \frac{11\pi}{6}$ the second eigenvalue is the first square of the first zero of the Bessel function corresponding to $m = 2$. We expect there exists an angle β for which we obtain two different Bessel functions J_ν for $\nu = \frac{m\pi}{\beta}$, $m = 1, 2$ such that they share one same root. Thus we search for β satisfying

$$J_{\frac{\pi}{\beta}}(\sqrt{\lambda}) = J_{\frac{2\pi}{\beta}}(\sqrt{\lambda}).$$

This condition gives $\beta \doteq 1.113$ and correspondent eigenvalue is $\lambda \doteq 9.528$. The Bessel functions for this angle are shown in Figure 5.1c. The second eigenfunctions on sectors of discs for various angles are shown in Figures 6.4 and 6.3 on p. 33 and p. 34.

Note 5.1. *The nodal lines for the degenerated eigenvalues are not given uniquely, since the eigenfunction is an arbitrary linear combination of the linearly independent eigenfunctions corresponding to the degenerated eigenvalue.*

5.2 Numerical approximation

The max-min principle (Theorem 2.50) enable us to make the numerical approximation of eigenvalues and eigenfunctions. It provides the lower estimates for the eigenvalues. We remind that we have $H = -\Delta$ with the associated quadratic form Q and search for the nodal lines of H on various bounded regions Ω.

The script for the numerical approximation of eigenfunctions of Dirichlet Laplacian was created with assistance of consultant Ing. Tomáš Kalvoda,Ph.D. We used the open-source mathematical software SAGE. It is based mainly on Python. It uses already existing open-source libraries. We describe here the libraries and functions we used in the source code of our program for finding the eigenfunctions of H.

The used method is called finite element method. At first we divide the region we are interested in into smaller parts called finite elements. Especially we choose these finite elements to be triangles in the process called triangulation.

According to the max-min principle (Theorem 2.50) we have to choose m linearly independent normalized functions $(g_1, \ldots g_m)$ to obtain the mth eigenvalue. We choose g_i to be a linear combination of functions from a particular set $\mathscr{X}$. We define a function from $\mathscr{X}$ for every point of the triangulation excluding the points of the boundary as follows. If we denote a the chosen point, B the set of all its neighbouring points and C the set of the remaining points, then we define the function in discrete points of triangulation by

$$f_a(x) = \begin{cases} 1 & \text{for } x = a \\ 0 & \text{for } x \in B \\ 0 & \text{for } x \in C. \end{cases}$$

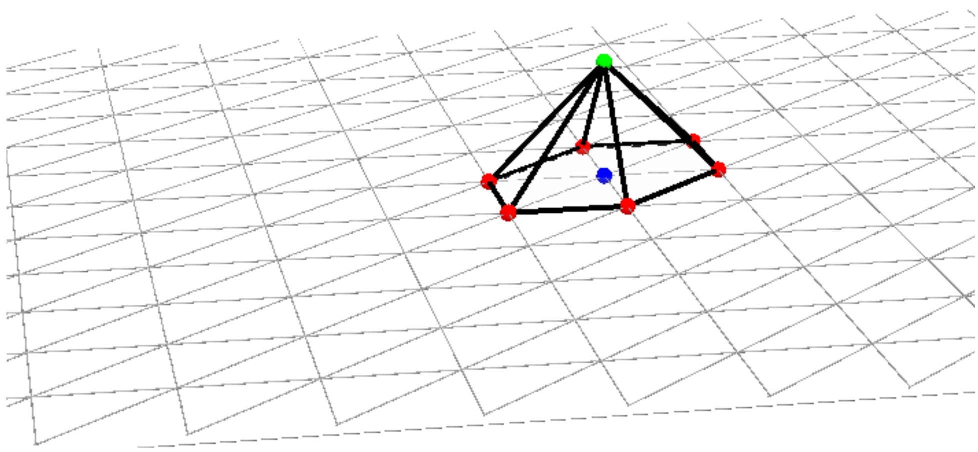
We denote the coordinates of points a and $b, c \in B$ such that $b \neq c$: $a = (a_x, a_y)$, $b = (b_x, b_y)$ and $c = (c_x, c_y)$, then the function on the triangle with vertices a, b, c takes the form

$$f_a(x, y) = \frac{1}{a_x\,(b_y - c_y) + b_x\,(c_y - a_y)}((b_x\,c_y - b_y\,c_x) + (b_y - c_y)\,x + (c_x - b_y)\,y) \qquad \text{if } (x, y) \in \text{triangle } a, b, c.$$

Outside this triangle the function has zero value. If we assign numbers to all points of the triangulation omitting points on the boundary and denote n the total number of these points then $(f_i)_{i=1}^{n}$ form a linearly independent system of piece-wise linear functions. (We will not differentiate the point and the number assigned to it when using the index of the function.)

For illustration the function $f_a \in \mathscr{X}$ is depicted in Figure 5.2.

Figure 5.2: Illustrative figure of a function f_a form set $\mathscr{X}$. The blue point is our chosen point a, the green one represents the function value $f_a(a)$ and the red ones are neighbouring points of a. Intersections of lines are the remaining points of the triangulation.



Since we choose the function g_i to be a linear combination of functions from $\mathscr{X}$ we write

$$g_i = \sum_{k=1}^{n} g_k^i f_k$$

where g_k^i, $\forall k \in \hat{n}$ is a real coefficient and n is the total number of points of the triangulation except the points on the boundary. We denote the vector of coefficients $\tilde{g}_i := (g_1^i, \ldots, g_n^i)^T \in \mathbb{R}^n$.

We define the matrix of the quadratic form $\tilde{Q}$ by

$$\tilde{Q}_{kj} = \langle \nabla f_k, \nabla f_j \rangle \qquad f_i \in \mathscr{X}, \ \forall i \in \hat{n}. \tag{5.8}$$

The inner product on the left-hand side of (5.8) has non-zero value only in the case of $k = j$ and k, j denoting the neighbouring points of triangulation. We denote F_{aa} the diagonal elements and F_{ab} the non-diagonal non-zero elements of $\tilde{Q}$. Then we have

$$\begin{aligned} F_{aa} = \langle \nabla f_a, \nabla f_a \rangle &= \int_\Omega (\nabla f_a)^2 \, \mathrm{d}^2 x = \int_{triangle\, a,b,c} (\nabla f_a)^2 \, \mathrm{d}^2 x \\ &= \frac{1}{2} \frac{(b_x^2 + b_y^2 - 2 b_x\, c_x + c_x^2 - 2 b_y\, c_y + c_y^2)}{|a_y\, b_x - a_x\, b_y - (a_y - b_y)\, c_x + (a_x - b_x)\, c_y|}\,, \end{aligned}$$

$$\begin{aligned} F_{ab} = \langle \nabla f_a, \nabla f_b \rangle &= \int_\Omega \nabla f_a \, \nabla f_b \, \mathrm{d}^2 x = \int_{triangle\, a,b,c} \nabla f_a \, \nabla f_b \, \mathrm{d}^2 x \\ &= -\frac{1}{2} \frac{(a_x\, b_x + a_y\, b_y - (a_x + b_x)\, c_x + c_x^2 - (a_y + b_y)\, c_y + c_y^2)}{|a_y\, b_x - a_x\, b_y - (a_y - b_y)\, c_x + (a_x - b_x)\, c_y|}\,. \end{aligned}$$

We wish to investigate the first and second eigenvalue and eigenfunction of H. According to the max-min principle we search for the minimum over functions $(g_1, \ldots, g_m)$ to find the approximation of the mth eigenvalue. If we denote g_1 the wanted approximation of the first eigenfunction then we have

$$\lambda_1 \approx \min_{\substack{g_i^1 \in \mathbb{R} \\ \|g^1\|=1}} \sum_{k,j=1}^{n} \langle \nabla(g_k^1\, f_k), \nabla(g_j^1\, f_j) \rangle = \min_{\substack{g_i^1 \in \mathbb{R} \\ \|g^1\|=1}} \sum_{k,j=1}^{n} g_k^1\, \tilde{Q}_{kj}\, g_j^1 = \min_{\substack{g_i^1 \in \mathbb{R} \\ \|g^1\|=1}} \langle \tilde{g}_1, \tilde{Q} \tilde{g}_1 \rangle \tag{5.9}$$

There the norm is in $L^2(\Omega)$. Finding this minimum we obtain the vector of coefficients $(g_1^1, \ldots, g_n^1)^T$ which determines the eigenfunction g_1. We denote g_2 the wanted approximation of the second eigenfunction. Then the approximation of the second eigenvalue λ_2 takes form

$$\lambda_2 \approx \min_{\substack{g_i^2 \in \mathbb{R} \\ \|g_2\|=1 \\ \langle g_1, g_2 \rangle = 0}} \sum_{k,j=1}^{n} \langle \nabla(g_k^2\, f_k), \nabla(g_j^2\, f_j) \rangle = \min_{\substack{g_i^2 \in \mathbb{R} \\ \|g_2\|=1 \\ \langle g_1, g_2 \rangle = 0}} \sum_{k,j=1}^{n} g_k^2\, \tilde{Q}_{kj}\, g_j^2 = \min_{\substack{g_i^2 \in \mathbb{R} \\ \|g_2\|=1 \\ \langle g_1, g_2 \rangle = 0}} \langle \tilde{g}_2, \tilde{Q} \tilde{g} \rangle. \tag{5.10}$$

Again the norm is in $L^2(\Omega)$. We obtain the vector of coefficients $(g_1^2, \ldots, g_n^2)^T$ determining the function g_2.

Now we move to the exact functions in SAGE we used to solve the task introduced bellow. When the points (the vertices of triangles) are set, then the algorithm for getting the second eigenvalue and eigenfunction is universal for any bounded region.

- At first we set the points covering given region. We want to assign numbers to these points. For this there exists the function *array* in the library NumPy which is abbreviation for numerical python.

- The function *Delaunay* from library scipy.spatial (ScientificPython) focused on spatial algorithms and data structures makes the triangulation.

- Further we get the inner products in the right-hand sides of the equations (5.9) and (5.10) using the function *np.dot* from library numpy. Thus we obtain a sparse matrix. For work with such a type of matrices we use the the library scipy.sparse. The class *dok_matrix* saves only the non-zero elements of the matrix. This class is suitable in our case because non-zero elements are only the diagonal ones (these containing the inner products of two same functions of the basis) and few others next to the diagonal (inner products of a function from the basis and its neighbour function, in the sense that both have value of 1 in different but neighbouring points).

- The task is now to minimize the expressions given in (5.9) and (5.10), where the matrix $\tilde{H}$ is this sparse matrix. The minimum is found by the function *minimize* from the library scipy.optimize intended for optimization and root finding with method *SLSQP* which is abbreviation for Sequential Least SQuares Programming. This method accepts additional constraints. In our case the additional conditions are the normalisation of the function g, when searching for the first eigenfunction and the normalisation of g with orthogonality of g to the first eigenvector, when searching for the second eigenvector.

- Finishing this process we obtained a piecewise linear function on original region Ω satisfying the given conditions. However this function is smooth. To smooth it we use the function *LinearND-Interpolator* which means Piecewise linear interpolant in N dimensions from scipy.interpolate.

Further information about the used libraries and functions are available in [7]. The script is saved on the enclosed CD.

Chapter 6

Discussion on results

As was mentioned in Section 5.1 computing nodal lines analytically is possible only for some specific regions. In that section we found the second eigenfunction of Dirichlet Laplacian on a rectangle, a disc and a sector of a disc. Thus we tested our program for numerical approximation described in Section 5.2 on these regions. In this chapter the results obtained both analytically and numerically are shown and discussed.

It is noticeable that both methods give results following the nodal hypothesis introduced in Chapter 4. That means that the nodal lines are indeed not closed and their closures intersect the boundary.

The following figures depict the obtained eigenfunctions and their nodal lines. For disc and rectangle with specific choice of lengths of edges we have the degeneracies of the second eigenvalue. And we obtain linearly independent eigenfunctions corresponding to the same eigenvalue. We found also these degeneracies solving this problem numerically. As we can see in Figures 6.2 and 6.1 the numerical approximation gives the eigenfunctions with a sufficient precision for these regions.

The only imprecision occurs near the boundary. The size of gradient of a monotonous function on a certain interval depends on the difference of function values in the edges of this interval. Thus the eigenfunction decreases from maximum to minimum rapidly and the nodal line can be found with sufficient precision. Near the boundary the eigenfunction needs to satisfy the Dirichlet condition. Therefore in intersection of the neighbourhood of the boundary and regions around the zero sets of eigenfunction there appear large areas, where the eigenfunction value is close to zero. As a result the exact zero value (nodal line) cannot be found with sufficient precision.

The numerical imprecision is particularly noticeable in the results of the sector of a disc region. The reason is the same as we mentioned in the previous paragraph. Therefore the problem occurs mainly the disc centre. In the case of angles smaller then some specific angle the nodal line goes in the direction of the angular variable. For angles larger then this specific angle the nodal line goes in the direction of the axis of the angle. In the first case the results are more precise since the area with values close to zero is not very large compared to the area with values close to zero in the second case. In the second case the bigger the angle we choose the more precise are the results (compare Figures 6.3f and 6.3b).

We have also found the specific angle in which the first case changes into the second one. For this angle $\theta \doteq 1.113$ the Bessel functions J_ν for $\nu = \frac{\pi}{\alpha}$ and $\nu = \frac{2\pi}{\alpha}$ have one same root. Therefore the eigenvalue is degenerated. Thus we obtain two eigenfunctions with nodal lines in the angular and in the radial direction respectively. These eigenfunctions are shown in Figures 6.4a- 6.4d.

In Table 6.1 are given the eigenvalues obtained both analytically and numerically for comparison.

We conclude, that the numerical approach is useful for the illustration of eigenfunctions and their nodal lines. However for the numerical imprecision it is not suitable for direct confrontation of the Nodal hypothesis.

Figure 6.1: The second eigenfunction on rectangle region of edges $a = 1, b = 2$ obtained both analytically and numerically.

(a) Analytical solution (b) Numerical solution

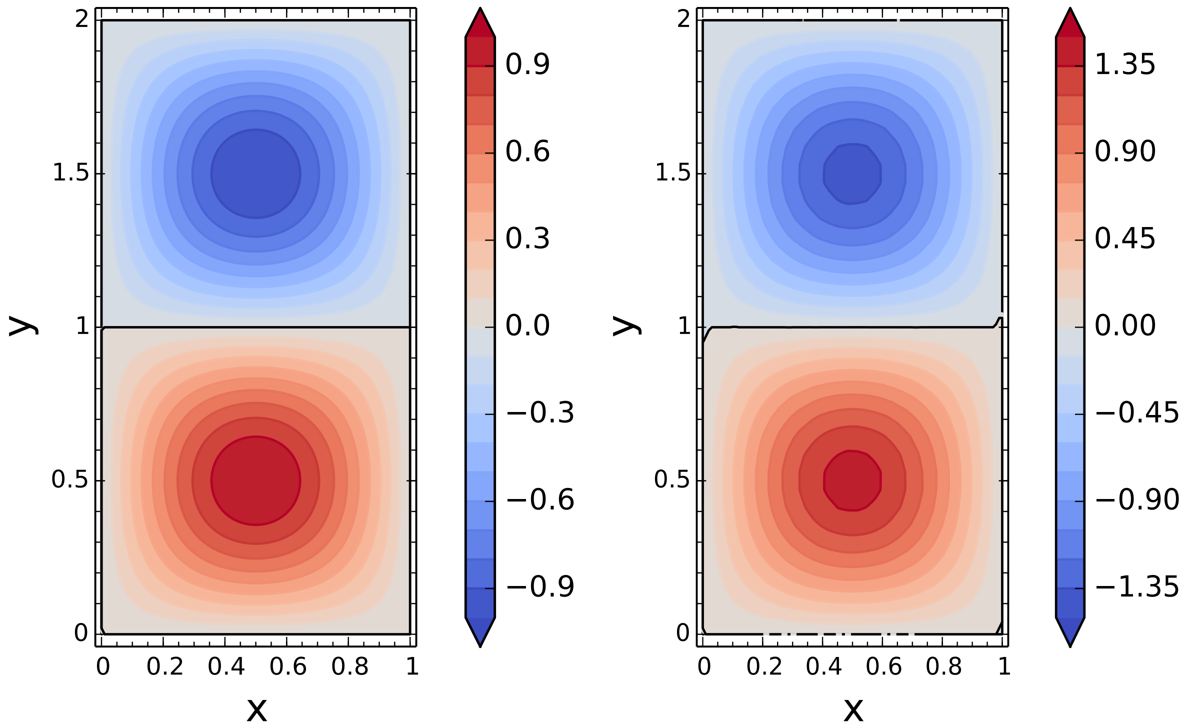


Table 6.1: The eigenvalues λ

Region	Analytical solution		Numerical solution	
	Eigenvalue	Eigenfunction	Eigenvalue	Eigenfunction
rectangle, $a = 1, b = 2$	19.739	Figure 6.1a	19.856	Figure 6.1b
disc, $r = 1$	14.682	Figure 6.2a, 6.2c	14.735	Figure 6.2b, 6.2d
sector, $r = 1, \theta = \frac{\pi}{6}$	184.669	Figure 6.3a	187.193	Figure 6.3b
sector, $r = 1, \theta = 1.1125$	90.774	Figure 6.4a, 6.4c	91.695	Figure 6.4b, 6.4d
sector, $r = 1, \theta = \pi$	26.375	Figure 6.3c	26.545	Figure 6.3d
sector, $r = 1, \theta = \frac{11\pi}{6}$	15.632	Figure 6.3e	15.801	Figure 6.3f

Figure 6.2: The second eigenfunction of the corresponding degenerated eigenvalue on a disc region obtained both analytically and numerically. We denoted f_{1a} and f_{1b} the analytical solution (5.6) from Section 5.1 for the choice of coefficients $a = 1, b = 0$ and $a = 0, b = 1$, respectively.

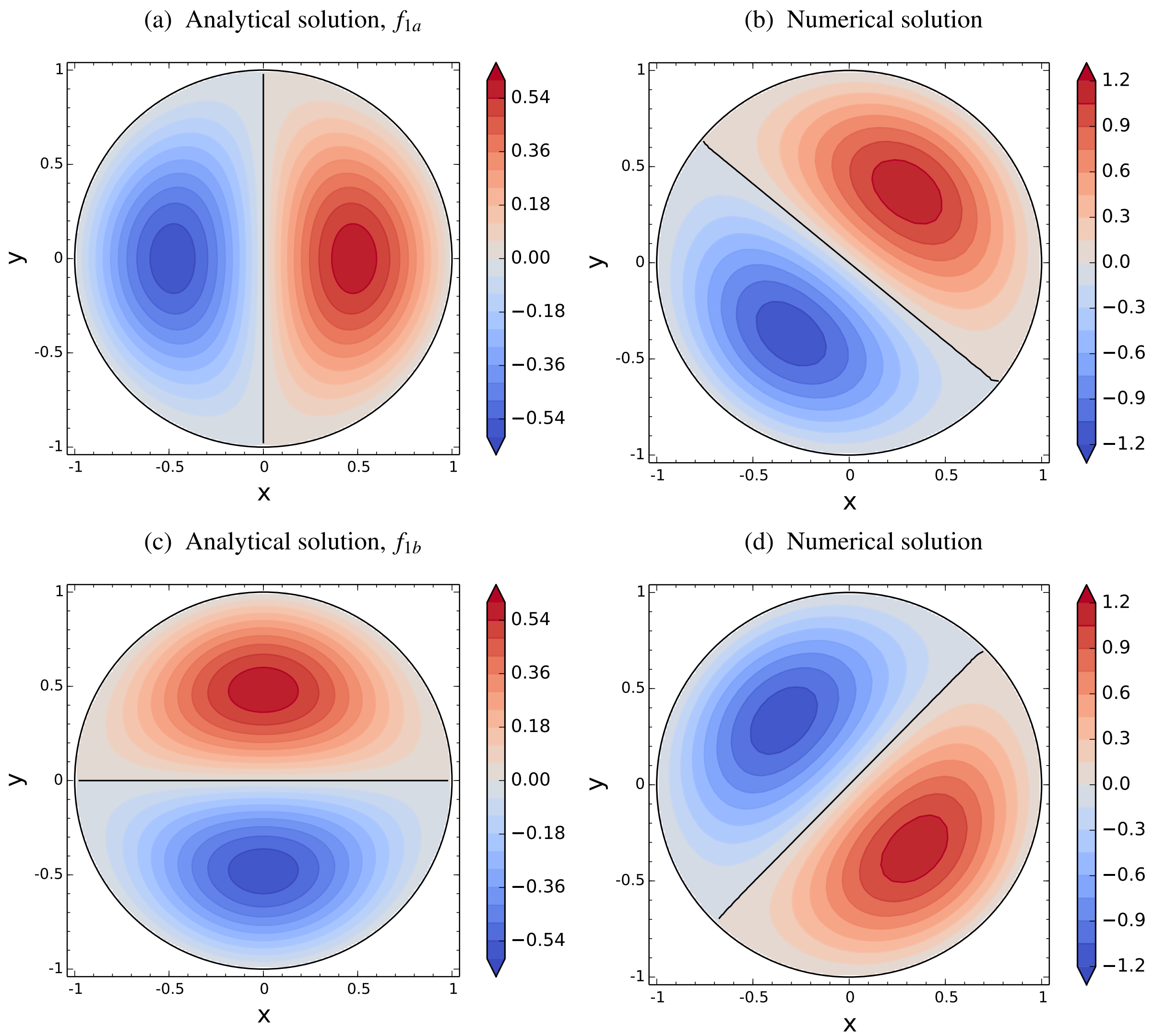
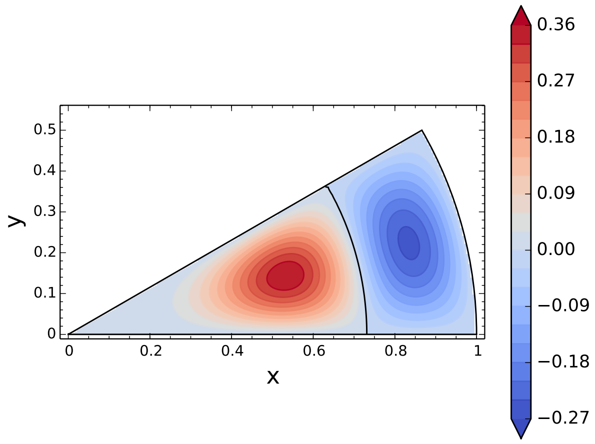
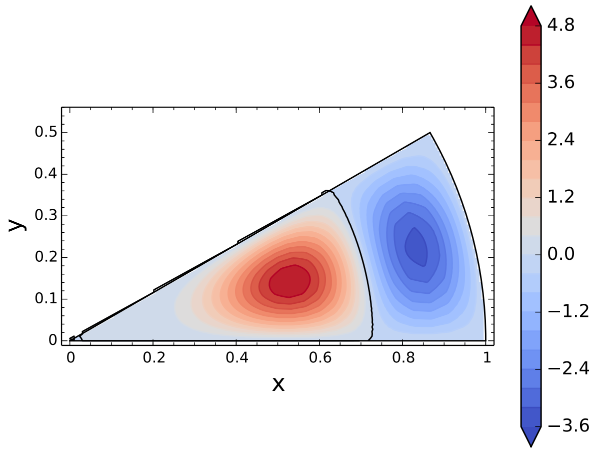


Figure 6.3: The second eigenfunction on sector of a disc region for angle θ obtained both analytically and numerically.

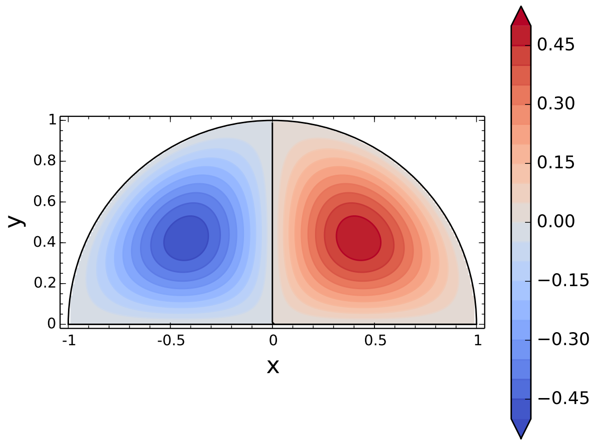
(a) Analytical solution, $\theta = \frac{\pi}{6}$.



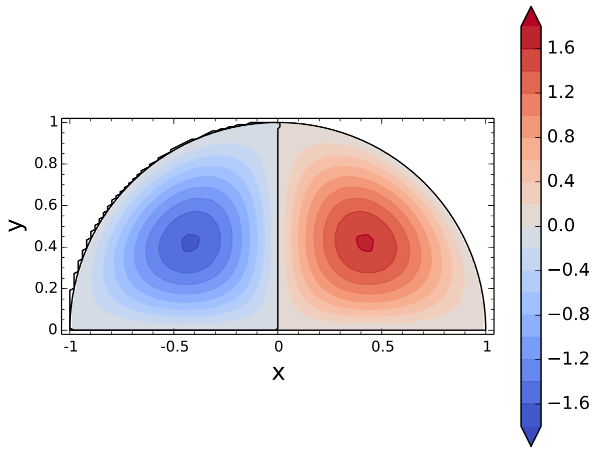
(b) Numerical solution, $\theta = \frac{\pi}{6}$.



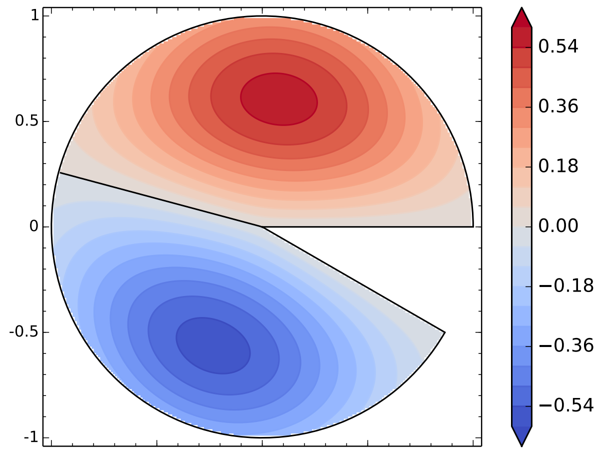
(c) Analytical solution, $\theta = \pi$



(d) Numerical solution, $\theta = \pi$



(e) Analytical solution, $\theta = \frac{11\pi}{6}$



(f) Numerical solution, $\theta = \frac{11\pi}{6}$

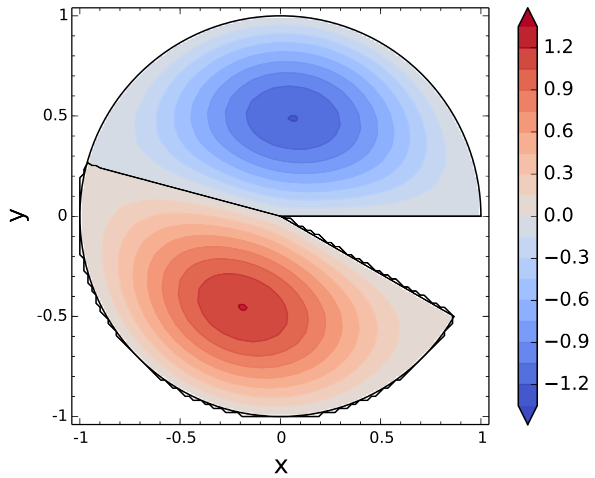
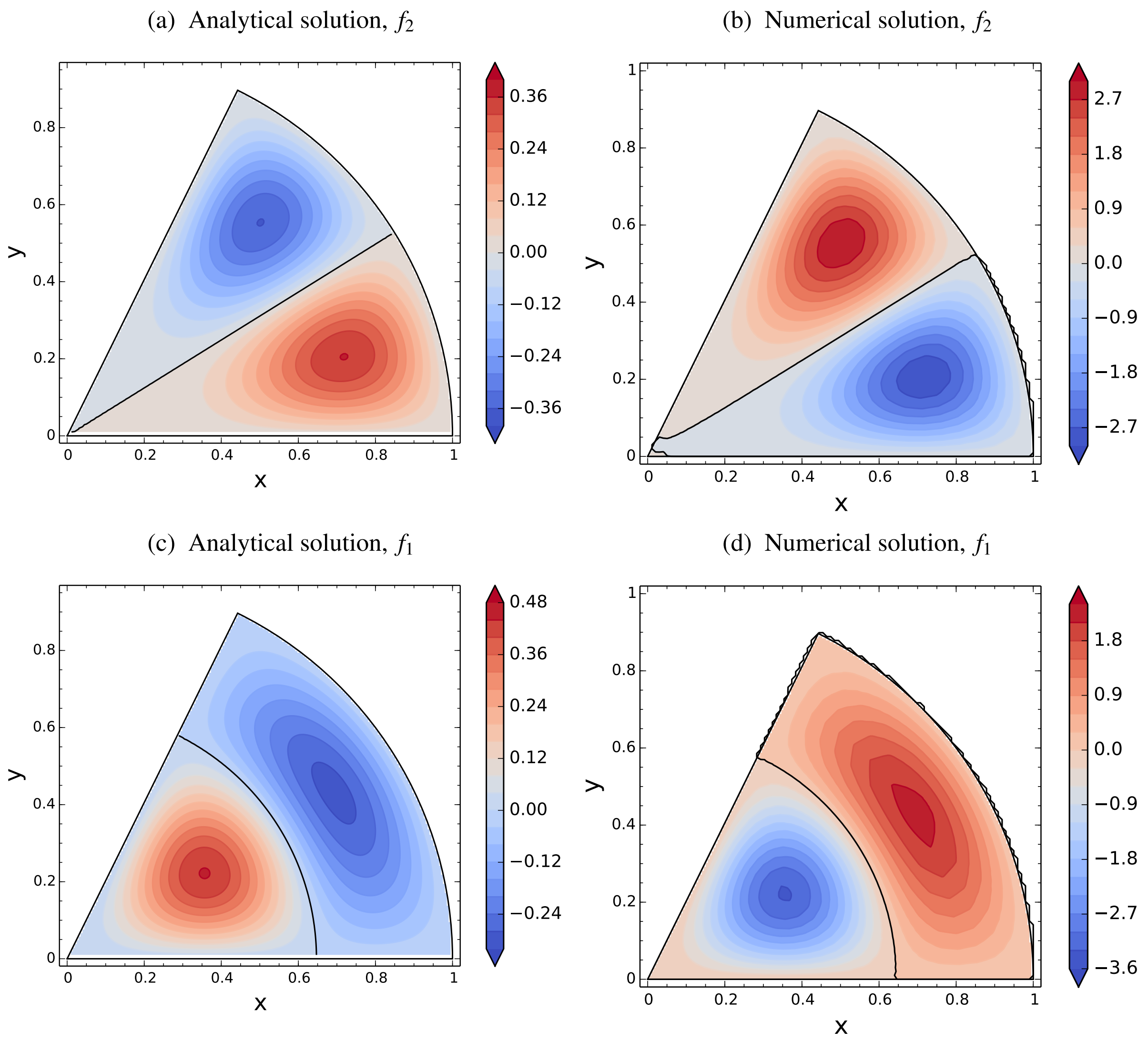


Figure 6.4: The second eigenfunctions of corresponding degenerated eigenvalue on a sector of a disc region for angle $\theta = 1.1125$. We denoted f_1 and f_2 the analytical solution (5.7) from Section 5.1 for the choice of $m = 1$ and $m = 2$, respectively.



Summary

In this degree project we dealt with the Dirichlet Laplacian on bounded regions in $\mathbb{R}^2$. This operator is important in the physics since it occurs in equations such as wave, Schrödinger and heat equation. This is a self-adjoint closed non-negative operator. In Chapter 2 we reviewed some of standard facts on the theory concerning such an operator and its spectrum and summarized without proofs the relevant material on quadratic forms. In Chapter 3 we showed the important general properties of Dirichlet Laplacian resulting in its specific spectral properties.

Section 5.1 contains the analytical solution for the eigenvalue problem of Dirichlet Laplacian on specific regions, that provides such a solution. We found the eigenfunctions and eigenvalues on a rectangle, a disc and a sector of a disc. We found possible degeneracies of second eigenvalues on these regions.

Then we created a script for numerical approximation of solution of the eigenvalue problem of Dirichlet Laplacian. This program provides the solution of this problem on regions solved analytically. The software and the algorithms we used are described in Section 5.2. The script could be also used for some other bounded regions. However the first step of the algorithm (the triangulation) needs to be done separately for every particular region, which is not always very easy task.

We discussed the results of analytical and numerical solutions in Chapter 6. Comparing the results of both methods we found out that the program gives reasonable results. We also checked, that the solutions are in agreement with the nodal hypothesis, whose results are summarized in Chapter 4.

It would be challenging to find the nodal lines using the numerical tools for more complicated regions, on which we cannot find the analytical solution.

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