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Types of E-functions of Weyl groups and their properties

Typy E-funkcií Weylových grúp a ich vlastnosti

by

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Michal Juránek

Declaration

I declare that I wrote my bachelor's thesis independently and that I used only the sources (literature, papers etc.) acknowledged in the Bibliography.

Prehlásenie

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V Prahe dňa 25.7.2013 ....

Michal Juránek

Názov práce:
Typy E-funkcií Weylových grúp a ich vlastnosti

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Abstrakt: Predložená bakalárska práca sa venuje vlastnostiam E-funkcií Weylových grúp. V prvej časti práce sumarizujeme základné vlastnosti Weylových grúp a afinnych Weylových grúp. V druhej časti sa venujeme špecifickým vlastnostiam párnych podgrúp, ktoré nám umožňujú definovať E-funkcie. Následne uvádzame explicitný tvar E-funkcií v ranku dva a vlastnosti, ktoré spĺňajú. V kľúčovej časti práce je uvedený nami zostrojený explicitný tvar E-funkcií v ranku tri a dôkaz, že spĺňajú diskrétnu ortogonalitu.

Záverom, v tejto práci sa nám podarilo overiť užitočné vlastnosti E-funkcií v ranku tri, najmä diskrétnu ortogonalitu, ktorá umoňuje využiť E-funkcie v spracovaní dát. Táto práca tiež otvára nové otázky, napr. či vlastnosti E-funkcií majú rovnaký tvar vo všetkých rankoch, hlavne v prípade špeciálnej Weylovej grupy typu F_4

Kľúčové slová: Weylova grupa, afinna Weylova grupa, koreňový systém, fundamentálna oblasť, E-funkcia, diskrétna ortogonalita

Title:
Types of E-functions of Weyl groups and their properties

Abstract: The present bachelor thesis focuses on the properties of E-functions of Weyl groups. In the first part of the thesis, we summarize basic properties of Weyl groups and affine Weyl groups. In the second part, we study the specific properties of even subgroups, which allow us to define E-functions. Subsequently, we state the explicit form of E-functions in rank two and their respective properties. In the key part of the thesis, we construct the explicit form of E-functions of rank three and verify their discrete orthogonality relations.

Concluding, the present thesis succeeded in the verification of the useful properties of E-functions of rank three, especially discrete orthogonality with its utilization in processing data. Based on this main conclusion, new questions are raised, e. g., whether the properties of E-functions are valid for any rank in the same form, particularly taking the case of the special Weyl group of type F_4.

Key words: Weyl group, affine Weyl group, root system, fundamental domain, E-function, discrete orthogonality

Contents

Introduction

One type of special functions corresponding to Weyl groups and their root systems, called E-functions, are introduced firstly in [8]. The results of the paper [2], which contains detailed explanation of discrete Fourier calculus of orbit functions, are extended to this type of E-functions in [3]. Additional five types of E-functions for root systems with two different lengths of roots are introduced and studied in detail for simple Lie algebras of rank two in [1]. In the present thesis we study these six types of E-functions for simple Lie algebras of rank three and verify if the properties that are valid in [1] are also valid for simple Lie algebras of rank three. Especially, the discrete orthogonality relations of these functions are of interest.

With further analysis of E-functions, made in [1, 3, 6], the following important properties have been described. E-functions depend on n variables, with n being the rank of the corresponding Lie algebra, and are periodic in various ways in the Euclidean space $\mathbb{R}^n$. Moreover, each pair of the same type of E-functions is orthogonal when integrated over their (bounded) fundamental region in $\mathbb{R}^n$. This property represents continuous orthogonality of E-functions. Similarly, discrete orthogonality also holds — each pair of the same type of E-functions is orthogonal when their values are summed over a discrete lattice of any density in the fundamental region. The symmetry of the lattice is defined by the given Weyl group.

One of the motivations for constructing E-functions is their utilization in processing digitally given data. Each type of E-functions is orthogonal in a region of different shape and therefore might be more suitable for processing specific type of data. This should, in return, increase the processing speed for that specific type of data.

Chapters one, two and three review already known results, except for the last part of chapter two, where for the purpose of discrete calculus of E-functions the description of the fundamental domains is reformulated. Chapter four contains original results concerning the form and orthogonality of E-functions of rank three, except for the basic properties of simple Lie algebras B_3 and C_3 taken from [10].

The thesis is organized as follows. In the first chapter, Weyl groups in general are studied. We define their root systems and classify crystallographic irreducible root systems. From Weyl groups we create affine Weyl groups and discuss their fundamental domains.

In the second chapter, we find three types of subgroups of Weyl groups of index two called even subgroups. We discuss the fundamental domains of these subgroups and define E-functions based on the orbits of these subgroups. We also discuss the properties of these E-functions: their symmetries, orthogonality and product decomposition.

In the third chapter, we summarize the general properties of Weyl groups of type C_2 and G_2 and we review the results of [1] by explicitly stating the E-functions corresponding to the Weyl groups of type C_2 and G_2.

In the fourth chapter, we summarize the general properties of Weyl groups of type B_3, C_3 and explicitly state their E-functions. Also, we verify if they have the same properties as E-functions of Weyl groups of rank two.

Chapter 1

Weyl groups

In this chapter we study Weyl groups in general. We review nomenclature from [1, 2, 4] needed for further definition of E-functions. For a more detailed study of Weyl groups and Coxeter groups see [4].

1.1 Finite reflection groups and root systems

Consider an Euclidian space $\mathbb{R}^n$ with an inner product $\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$. The reflection r_α over the hyperplane H_α with α being the normal vector of the hyperplane is defined as

$$r_\alpha \lambda = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

A finite group generated by reflections is a **finite reflection group**.

Let Φ be a finite set of nonzero vectors in $\mathbb{R}^n$ satisfying these conditions:

- $(\forall \alpha \in \Phi)(\Phi \cap \mathbb{R} \cdot \alpha = \{\alpha, -\alpha\})$
- $(\forall \alpha \in \Phi)(r_\alpha \Phi = \Phi)$

and let W be the group generated by $\{r_\alpha \mid \alpha \in \Phi\}$. Then the set Φ is a **root system with the associated reflection group** W and elements $\alpha \in \Phi$ are called **roots**.

Let Φ be a root system of a finite reflection group W. Then a subset $\Delta \subset \Phi$ with the properties:

- Δ is the basis of $span_{\mathbb{R}}(\Phi)$
- for each $\alpha \in \Phi$ in Δ basis the coefficients all have the same sign (all nonnegative or all nonpositive)

is a **simple system of group** W. From now on, the notation for the simple system $\Delta = \{\alpha_1, \ldots \alpha_n\}$ is used, where n is the dimension of the space $\mathbb{R}^n$. The vectors $\alpha_i \in \Delta$ are called **simple roots**. The reflections corresponding to the simple roots are called **simple reflections** and are denoted

$$r_{\alpha_i} \equiv r_i, \quad \alpha_i \in \Delta.$$

Example 1.1.1. The reflection group of type B_n $(n \geq 3)$, denoted by W_{B_n}, is defined as follows. Let $\{e_i \mid i \in \{1, \ldots, n\}\}$ be the standard basis in $\mathbb{R}^n$. From now on, this notation of the standard basis is used. Then the group W_{B_n} consists of all permutations and all sign changes of the coordinates in the standard basis. This group can be generated by transpositions of the coordinates and one sign change — a sign change on only one coordinate. These generators are clearly orthogonal transformations from $O(n)$. All sign changes are isomorphic to the quotient group $(\mathbb{Z}/2\mathbb{Z})^n$ and all permutations are isomorphic to the permutation group S_n; therefore, the number of elements of W_{B_n} is

$$\left|W_{B_n}\right| = 2^n n!.$$

Considering all combinations of signs, the root system of W_{B_n} is

$$\Phi_{B_n} = \{\pm e_i \pm e_j \mid i, j \in \{1, \dots, n\}, i \neq j\} \cup \{\pm e_i \mid i \in \{1, \dots, n\}\}.$$

The simple system of W_{B_n} is

$$\Delta_{B_n} = \{e_i + e_{i+1} \mid i \in \{1, \dots, n-1\}\} \cup \{e_n\}.$$

We denote the simple roots $\alpha_i = e_i + e_{i+1}$, for $i \in \{1, \dots, n-1\}$, and $\alpha_n = e_n$.

Let Φ be a root system and Δ its simple system then the set

$$\Phi^\vee = \left\{\alpha^\vee \mid \alpha \in \Phi\right\}$$

is the **dual root system to the root system** Φ and roots $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ are called **dual roots** (coroots). The set

$$\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\},$$

where $\alpha_i^\vee = \frac{2\alpha_i}{\langle\alpha_i,\alpha_i\rangle}$, is the **dual simple system to the simple system** Δ and $\alpha_i^\vee$ are called **dual simple roots** (simple coroots).

A root system Φ is **crystallographic** if it satisfies

$$(\forall \alpha, \beta \in \Phi)\left(\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}\right).$$

The integers $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$ are called **Cartan integers**. The elements $C_{ij} = \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}$, where $\alpha_i, \alpha_j \in \Delta$, form the **Cartan matrix**.

Let Φ be a crystallographic root system then the group W generated by $r_\alpha(\alpha \in \Phi)$ is called the **Weyl group** of Φ.

The **root lattice** and the **dual root lattice** are defined as follows:

$$Q = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n, \quad Q^\vee = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee.$$

We also use the following notations for the subsets of Q and $Q^\vee$,

$$\begin{aligned} Q^+ &= \mathbb{Z}_0^+\alpha_1 + \dots + \mathbb{Z}_0^+\alpha_n, & Q^{++} &= \mathbb{N}\alpha_1 + \dots + \mathbb{N}\alpha_n \\ Q^{\vee+} &= \mathbb{Z}_0^+\alpha_1^\vee + \dots + \mathbb{Z}_0^+\alpha_n^\vee, & Q^{\vee++} &= \mathbb{N}\alpha_1^\vee + \dots + \mathbb{N}\alpha_1^\vee. \end{aligned}$$

1.2 Fundamental weights

Let $\{\omega_1, \dots, \omega_n\}$ be a basis of $\mathbb{R}^n$ such that

$$(\forall i, j \in \{1, \dots, n\})(\langle\omega_i, \alpha_j^\vee\rangle = \delta_{ij}).$$

Then $\{\omega_1, \dots, \omega_n\}$ is called the **basis of fundamental weights** and the vectors ω_i are called **weights**.

Let $\omega_i^\vee = \frac{2\omega_i}{\langle\alpha_i,\alpha_i\rangle}$, then trivially it holds that

$$(\forall i, j \in \{1,...,n\})(\langle\omega_i^\vee, \alpha_j\rangle = \delta_{ij}).$$

The basis $\{\omega_1^\vee,...,\omega_n^\vee\}$ is called the **basis of dual weights** and $\omega_i^\vee$ are called **dual weights** (coweights). The notations for these bases are:

$$\Delta = \{\alpha_1,\ldots,\alpha_n\}, \quad \Delta^\vee = \{\alpha_1^\vee,\ldots,\alpha_n^\vee\}$$
$$\Omega = \{\omega_1,\ldots,\omega_n\}, \quad \Omega^\vee = \{\omega_1^\vee,\ldots,\omega_n^\vee\}.$$

Representations of a vector in Δ, $\Delta^\vee$, Ω, $\Omega^\vee$ bases are denoted as

$$(a_1,\ldots,a_n)_\Delta = \sum_{i=1}^n a_i\alpha_i,$$
$$(a_1,\ldots,a_n)_{\Delta^\vee} = \sum_{i=1}^n a_i\alpha_i^\vee,$$
$$(a_1,\ldots,a_n)_\Omega = \sum_{i=1}^n a_i\omega_i,$$
$$(a_1,\ldots,a_n)_{\Omega^\vee} = \sum_{i=1}^n a_i\omega_i^\vee.$$

The **weight lattice** and the **dual weight lattice** are defined as follows

$$P = \mathbb{Z}\omega_1 + ... + \mathbb{Z}\omega_n, \quad P^\vee = \mathbb{Z}\omega_1^\vee + ... + \mathbb{Z}\omega_n^\vee.$$

We also use the following notations for the subsets of P and $P^\vee$,

$$P^+ = \mathbb{Z}_0^+\omega_1 + +... + \mathbb{Z}_0^+\omega_n, \quad P^{++} = \mathbb{N}\omega_1 + ... + \mathbb{N}\omega_n$$
$$P^{\vee+} = \mathbb{Z}_0^+\omega_1^\vee + ... + \mathbb{Z}_0^+\omega_n^\vee, \quad P^{\vee++} = \mathbb{N}\omega_1^\vee + ... + \mathbb{N}\omega_1^\vee.$$

1.3 Classification of irreducible crystallographic reflection groups

Let Φ be a crystallographic root system and let $\alpha, \beta \in \Phi$. Then the order of $r_\alpha r_\beta$ is denoted $m(\alpha,\beta)$, therefore

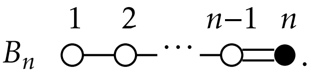
$$(\forall\alpha,\beta\in\Phi)((r_\alpha r_\beta)^{m(\alpha,\beta)} = 1).$$

The **Coxeter–Dynkin diagram** of the Weyl group W with the simple system $\Delta = \{\alpha_1,\ldots,\alpha_n\}$ is constructed as follows:

- the vertices are indexed 1 to n for each simple root,
- a vertix is blank for long roots and full for short roots (if there are two root lengths),
- when $m(\alpha_i,\alpha_j) \geq 3$ then there is an edge connecting vertices i and j,

- the value of the vertix is $m(\alpha_i, \alpha_j)$, in the diagram the value $m(\alpha_i, \alpha_j) = 3$ is denoted by one line, the value $m(\alpha_i, \alpha_j) = 4$ is denoted by two lines and the value $m(\alpha_i, \alpha_j) = 6$ is denoted by three lines.

Example 1.3.1. The Coxeter–Dynkin diagram of W_{B_n} is

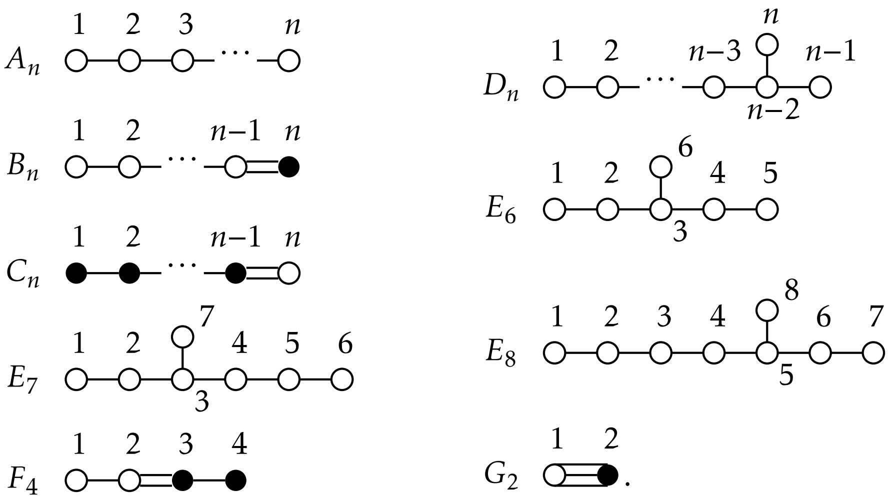


Theorem 1.3.2. Let W, $\widetilde{W}$ be Weyl groups with the same Coxeter–Dynkin diagram then W, $\widetilde{W}$ are conjugate in $O(n)$.
For proof see [4].

A Weyl group is **irreducible** if its Coxeter–Dynkin diagram is connected. Irreducible Weyl groups can be classified by their Coxeter–Dynkin diagrams; these Coxeter–Dynkin diagrams also classify all possible complex simple Lie algebras. The standard notation for the types of all possible Coxeter–Dynkin diagrams of irreducible Weyl groups is

$$A_n\ (n \geq 1),\ \ B_n\ (n \geq 3),\ \ C_n\ (n \geq 2),\ \ D_n\ (n \geq 4),\ \ E_6,\ \ E_7,\ \ E_8,\ \ F_4,\ \ G_2.$$

The corresponding diagrams, taken from [5], are of the following form



1.4 Affine Weyl Groups

The **affine Weyl group** W^{aff} is defined as the semidirect product of the Abelian group of translations of the coroot lattice $Q^\vee$ and of the Weyl group W,

$$W^{\mathrm{aff}} = Q^\vee \rtimes W,$$

and the **dual affine Weyl group** is defined as the semidirect product of the Abelian group of translations of the coroot lattice Q and of the Weyl group W,

$$\widehat{W}^{\mathrm{aff}} = Q \rtimes W.$$

The **highest root** ξ of the affine Weyl group W^{aff} can be expressed, see e. g. [5], as follows

$$\xi = \sum_{i=1}^{n} m_i \alpha_i.$$

The coefficients $m_i \in \mathbb{N}$ can be viewed as attached to the i-th vertix of the Coxeter–Dynkin diagram of W — they are called **marks**. The form of the highest root ξ is for all cases the following:

$$\begin{aligned}
B_n &: \quad \xi = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n\,,\\
C_n &: \quad \xi = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n\,,\\
F_4 &: \quad \xi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,\\
G_2 &: \quad \xi = 2\alpha_1 + 3\alpha_2\,,\\
A_n &: \quad \xi = \alpha_1 + \alpha_2 + \cdots + \alpha_n\,,\\
D_n &: \quad \xi = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n\,,\\
E_6 &: \quad \xi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6\,,\\
E_7 &: \quad \xi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7\,,\\
E_8 &: \quad \xi = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8\,.
\end{aligned}$$

The **highest dual root** η of the dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ can be expressed, see e. g. [2], as follows

$$\eta = \sum_{i=1}^{n} m_i^\vee \alpha_i^\vee$$

and the coefficients $m_i^\vee \in \mathbb{N}$ are called **dual marks.** The form of the highest dual root η is for all cases the following:

$$\begin{aligned}
B_n &: \quad \eta = 2\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee\,,\\
C_n &: \quad \eta = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + \cdots + 2\alpha_n^\vee\,,\\
F_4 &: \quad \eta = 2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee,\\
G_2 &: \quad \eta = 3\alpha_1^\vee + 2\alpha_2^\vee,\\
A_n &: \quad \eta = \alpha_1^\vee + \alpha_2^\vee + \cdots + \alpha_n^\vee,\\
D_n &: \quad \eta = \alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee,\\
E_6 &: \quad \eta = \alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee + 2\alpha_6^\vee\,,\\
E_7 &: \quad \eta = 2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 3\alpha_4^\vee + 2\alpha_5^\vee + \alpha_6^\vee + 2\alpha_7^\vee,\\
E_8 &: \quad \eta = 2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 5\alpha_4^\vee + 6\alpha_5^\vee + 4\alpha_6^\vee + 2\alpha_7^\vee + 3\alpha_8^\vee.
\end{aligned}$$

The **reflection** r_0 is an affine reflection over the hyperplane $\{\chi \in \mathbb{R}^n \mid \langle \chi, \xi \rangle = 1\}$, which is defined as

$$(\forall \chi \in \mathbb{R}^n)\left(r_0 \chi = r_\xi \chi + \frac{2\xi}{\langle \xi, \xi \rangle} \right).$$

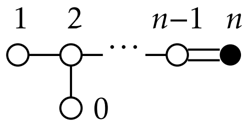
The **reflection** $r_0^\vee$ is an affine reflection over the hyperplane $\{\chi \in \mathbb{R}^n \mid \langle \chi, \eta \rangle = 1\}$, which is defined as

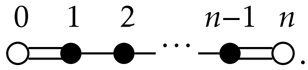
$$(\forall \chi \in \mathbb{R}^n)\left(r_0^\vee \chi = r_\eta \chi + \frac{2\eta}{\langle \eta, \eta \rangle} \right).$$

Theorem 1.4.1. The affine Weyl group W^{aff} is generated by $\{r_i \mid i \in 1,\dots,n\} \cup \{r_0\}$.
For proof see [4].

Let W be a Weyl group and Δ its simple system then the **extended Coxeter–Dynkin diagram** of the affine Weyl group W^{aff} is the Coxeter–Dynkin diagram of W with the root $\alpha_0 \equiv -\xi$ added to the diagram. The vertex representing the root α_0 is indexed with the value 0. Analogously, the extended Coxeter–Dynkin diagram of the dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ is the Coxeter–Dynkin diagram of W with the dual root $\alpha_0^\vee \equiv -\eta$ added to the diagram. The vertex representing the dual root $\alpha_0^\vee$ is indexed with the value 0.

Example 1.4.2. The extended Coxeter–Dynkin diagrams of the Weyl groups of types B_n and C_n are

B_n 

C_n 

The extended Coxeter–Dynkin diagram of $W^{\text{aff}}_{B_n}$ is the same as the extended Coxeter–Dynkin diagram of $\widehat{W}^{\text{aff}}_{C_n}$. The extended Coxeter–Dynkin diagram of $W^{\text{aff}}_{C_n}$ is the same as the extended Coxeter–Dynkin diagram of $\widehat{W}^{\text{aff}}_{B_n}$. These diagrams are dual to each other.

1.5 Fundamental domain

The fundamental domain of a group G is the smallest, in terms of inclusion, connected set $D \subset \mathbb{R}^n$ such that the action of G on D is the whole space $\mathbb{R}^n$.

Theorem 1.5.1. The fundamental domain D of the Weyl Group W is

$$D = \{\lambda \in \mathbb{R}^n | (\forall \alpha \in \Delta)(\langle \lambda, \alpha \rangle \geq 0)\}.$$

For proof see [4].

Theorem 1.5.2. The fundamental domain F of the affine Weyl Group W^{aff} is

$$F = D \cap \{\lambda \in \mathbb{R}^n \mid \langle \lambda, \xi \rangle \leq 1\}.$$

For proof see [4].

The fundamental domain F is equivavently the convex hull of the points $\left\{0, \frac{\omega_1^\vee}{m_1}, \cdots, \frac{\omega_n^\vee}{m_n}\right\}$ and the fundamental domain $F^\vee$ of $\widehat{W}^{\text{aff}}$ is the convex hull of the points $\left\{0, \frac{\omega_1}{m_1^\vee}, \ldots, \frac{\omega_n}{m_n^\vee}\right\}$.

Let W be a Weyl group and Δ its simple system and let $\{X_i \mid i \in \{1, \ldots, n\}\}$ denote the 'mirrors' of reflections of the simple reflections r_i,

$$X_i = \{\chi \in \mathbb{R}^n \mid \langle \chi, \alpha_i \rangle = 0\}.$$

The 'mirror' X_0 of the affine reflection r_0 is

$$X_0 = \{\chi \in \mathbb{R}^n \mid \langle \chi, \xi \rangle = 1\}$$

and its dual counterpart $X_0^\vee$ is

$$X_0^\vee = \{\chi \in \mathbb{R}^n \mid \langle \chi, \eta \rangle = 1\}.$$

The intersections of the 'mirrors' with the fundamental domain F are denoted Y_i and Y_0

$$Y_i = X_i \cap F, \quad Y_0 = X_0 \cap F.$$

The intersections of the 'mirrors' with the fundamental domain $F^\vee$ are analogously denoted $Y_i^\vee$ and $Y_0^\vee$

$$Y_i^\vee = X_i \cap F^\vee, \quad Y_0^\vee = X_0^\vee \cap F^\vee.$$

Chapter 2

E-functions

In this chapter we find three types of normal subgroups of Weyl groups using sign homomorphisms [7]. Using orbits of these subgroups, we define even orbit functions (abbreviated E-functions), and discuss their properties. For a more detailed study see [1, 3, 6].

2.1 Even subgroups

A **sign homomorphism** is a homomorphism $\sigma : W \longmapsto \{1,-1\}$, where $\{1,-1\}$ is the multiplicative group containing elements 1 and -1. One obvious choice of a sign homomorphism is σ_e defined as

$$(\forall w \in W)(\sigma_e(w) = \det(w)).$$

Even subgroups W^σ of a Weyl group W are the kernels of non-trivial sign homomorphisms

$$W^\sigma = \ker\sigma = \{w \in W \mid \sigma(w) = 1\}.$$

The corresponding affine groups are defined as

$$W^{\mathrm{aff}}_\sigma = Q^\vee \rtimes W^\sigma.$$

The maps σ_l and σ_s are sign homomorphisms of a Weyl group W with a simple system Δ. Since simple reflections generate W, it is sufficient to define σ_s and σ_l on simple reflections. They are defined as follows [7]

$$\begin{aligned} \sigma_s(r_s) &= -1\,, \quad \sigma_s(r_l) = 1, \\ \sigma_l(r_s) &= 1\,, \qquad \sigma_l(r_l) = -1, \end{aligned}$$

where r_s are simple reflections over short roots and r_l are simple reflections over long roots. Even subgroups W^e, W^s, W^l are the kernels of sign homomorphisms σ_e, σ_s, σ_l, respectively,

$$W^e = \ker\sigma_e\,, \quad W^s = \ker\sigma_s\,, \quad W^l = \ker\sigma_l.$$

Their corresponding affine groups are

$$W^{\mathrm{aff}}_e = Q^\vee \rtimes W^e, \quad W^{\mathrm{aff}}_s = Q^\vee \rtimes W^s, \quad W^{\mathrm{aff}}_l = Q^\vee \rtimes W^l.$$

Let $\widetilde{Y}_s$ denote the union of all 'mirrors' of short roots intersected with the fundamental domain and $\widetilde{Y}_l$ is the union of all 'mirrors' of long roots intersected with the fundamental domain

$$\widetilde{Y}_s = \widetilde{X}_s \cap F = \bigcup X_i^{(s)} \cap F, \tag{2.1}$$

$$\widetilde{Y}_l = \widetilde{X}_l \cap F = \bigcup X_i^{(l)} \cap F, \tag{2.2}$$

where $X_i^{(s)}$ represent 'mirrors' of short roots and $X_i^{(l)}$ represent 'mirrors' of long roots. The notation $\widetilde{Y}_l^{\vee} = \widetilde{X}_l \cap F^{\vee}$ and $\widetilde{Y}_s^{\vee} = \widetilde{X}_s \cap F^{\vee}$ is also used.

The fundamental domains of the affine groups are denoted F^{e+}, F^{s+} F^{l+} (+ sign notation will be clearer later) and their explicit form is

$$F^{e+} = F \cup rF^{\circ}, \quad F^{s+} = F \cup r_s(F \setminus \widetilde{Y}_s), \quad F^{l+} = F \cup r_l(F \setminus (\widetilde{Y}_l \cup Y_0)).$$

The subsets of weight lattice corresponding to the even subgroups are

$$P_{e+} = P^+ \cup r(P^{++}), \quad P_{s+} = P^+ \cup r_s(P^+ \setminus \widetilde{X}_s), \quad P_{l+} = P^+ \cup r_l(P^+ \setminus \widetilde{X}_l),$$

where r, r_s, r_l are each an arbitrary, but fixed, simple, short simple, long simple reflections respectively.

The stabilizer of $\lambda \in \mathbb{R}^n$ of a group G is defined as

$$\mathrm{Stab}_G(\lambda) = \{g \in G \mid g\lambda = \lambda\}.$$

The coefficients d_λ^e, d_λ^s, d_λ^l denote the orders of stabilizers of W^e, W^s, W^l, respectively,

$$d_\lambda^e = |\mathrm{Stab}_{W^e}(\lambda)|, \quad d_\lambda^s = |\mathrm{Stab}_{W^s}(\lambda)|, \quad d_\lambda^l = |\mathrm{Stab}_{W^l}(\lambda)|.$$

2.2 Maximal torus and its orbits

In this section we define notions needed for the discrete calculus of E-functions. One arbitrary natural number M, which controls the density of the grids appearing in this calculus, is chosen.

Let $\cong$ be an equivalence defined as

$$(\forall \chi, \gamma \in \mathbb{R}^n)(\chi \cong \gamma) \Leftrightarrow (\exists q^\vee \in Q^\vee)(\chi = \gamma + q^\vee),$$

the quotient space $\mathbb{R}^n/\cong$ is the **maximal torus**, denoted $\mathbb{R}^n/Q^\vee$. The quotient space $\mathbb{R}^n/Q$ is defined analogously. For $M \in \mathbb{N}$ the sets $\frac{1}{M}P^\vee/Q^\vee$ and P/MQ are finite grids. Their intersections with fundamental domain F and the magnified dual fundamental domain $MF^\vee$, respectively, are denoted as

$$F_M = \frac{1}{M}P^\vee/Q^\vee \cap F, \quad \Lambda_M = P/MQ \cap MF^\vee.$$

The grid F_M is explicitly given in [3] as

$$F_M = \left\{ \frac{u_1}{M}\omega_1^\vee + \cdots + \frac{u_n}{M}\omega_n^\vee \mid u_0, u_1, \ldots, u_n \in \mathbb{Z}_0^+,\, u_0 + \sum_{i=1}^n u_i m_i = M \right\}. \tag{2.3}$$

The grid Λ_M is explicitly given in [3] as

$$\Lambda_M = \left\{ t_1\omega_1 + \cdots + t_n\omega_n \mid t_0, t_1, \ldots, t_n \in \mathbb{Z}_0^+,\, t_0 + \sum_{i=1}^n t_i m_i^\vee = M \right\}. \tag{2.4}$$

For $\chi \in F_M$ the orders of the stabilizers in the space $\mathbb{R}^n/Q^\vee$ for each type of even subgroup are denoted as

$$h_\chi^e = |\mathrm{Stab}^e \chi|, \quad h_\chi^s = |\mathrm{Stab}^s \chi|, \quad h_\chi^l = |\mathrm{Stab}^l \chi|.$$

and the orders of the orbits are denoted as

$$\epsilon^e(\chi) = |W^e\chi|, \quad \epsilon^s(\chi) = |W^s\chi|, \quad \epsilon^l(\chi) = |W^l\chi|.$$

Of course, the orbit–stabilizer theorem states that

$$\epsilon^e(\chi) = \frac{|W^e|}{h^e_\chi}, \quad \epsilon^s(\chi) = \frac{|W^s|}{h^s_\chi}, \quad \epsilon^l(\chi) = \frac{|W^l|}{h^l_\chi}.$$

For $\lambda \in \Lambda_M$ the orders of the stabilizers in the space $\mathbb{R}^n/Q$ for each type of even subgroup are denoted as

$$h^{\vee e}_\lambda = |\mathrm{Stab}^{\vee e}\lambda|, \quad h^{\vee s}_\lambda = |\mathrm{Stab}^{\vee s}\lambda|, \quad h^{\vee l}_\lambda = |\mathrm{Stab}^{\vee l}(\lambda)|.$$

Finally, using the notation from 2.1, the grids in $\mathbb{R}^n/Q^\vee$ for each type of even subgroup are,

$$F^{e+}_M = F_M \cup rF^\circ_M, \quad F^{s+}_M = F_M \cup r_s(F_M \setminus \widetilde{Y}_s), \quad F^{l+}_M = F_M \cup r_l(F_M \setminus (\widetilde{Y}_l \cup Y_0)),$$

where F°_M denotes $\frac{1}{M}P^\vee/Q^\vee \cap F^\circ$. Their dual counterparts are grids in $\mathbb{R}^n/Q$ defined as

$$\Lambda^{e+}_M = \Lambda_M \cup r\Lambda^\circ_M, \quad \Lambda^{s+}_M = \Lambda_M \cup r_s(\Lambda_M \setminus (M\widetilde{Y}^\vee_s \cup MY^\vee_0)), \quad \Lambda^{l+}_M = \Lambda_M \cup r_l(\Lambda_M \setminus M\widetilde{Y}^\vee_l),$$

where Λ°_M denotes $P/MQ \cap MF^{\vee\circ}$.

2.2.1 Algorithm for calculating orders of stabilizers

Calculating the orders of stabilizers from their definition is not an easy task, but there exists a simple algorithm for calculating the orders of stabilizers using extended Coxeter-Dynkin diagrams [2, 3].

The calculation procedure of $h_\chi \equiv |\mathrm{Stab}_{W^{\mathrm{aff}}}(\chi)|$ and also $\epsilon(\chi) = \frac{|W|}{h_\chi}$ for any $\chi \in F_M$ uses the extended Coxeter-Dynkin diagrams of W^{aff}. The calculation procedure of $h^\vee_\lambda \equiv |\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(\lambda)|$ for any $\lambda \in \Lambda_M$ uses the dual extended Coxeter-Dynkin diagrams of $\widehat{W}^{\mathrm{aff}}$.

Consider a point $\chi \in F_M$.

1. Let $[u_0,\dots,u_n]$ be the corresponding coordinates of $\chi \in F_M$ (from 2.3). If $u_0,\dots,u_n$ are all non-zero, then $h_\chi = 1$.

2. If some of the coordinates $[u_0,\dots,u_n]$ are zero then consider such a subgraph $\widetilde{U}$ of the extended Coxeter–Dynkin diagram of the affine Weyl group W^{aff} consisting only of those nodes i for which $u_i = 0$, $i = 0,\dots,n$. The subgraph $\widetilde{U}$ represents a Weyl group U (non-extended). Take the order of this Weyl group $|U|$. Then it holds that

$$h_\chi = |U|.$$

We proceed similarly to determine $h^\vee_\lambda$ when considering a point $\lambda \in \Lambda_M$.

1. Let $[t_0,\dots,t_n]$ be the corresponding coordinates of $\lambda \in \Lambda_M$ (from 2.4). If $t_0,\dots,t_n$ are all non-zero then $h^\vee_\lambda = 1$.

2. If some of the coordinates $[t_0,\dots,t_n]$ are zero then consider such a subgraph $\widehat{U}$ of the extended Coxeter–Dynkin diagram of the dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ consisting only of those nodes i for which $t_i = 0$, $i = 0,\dots,n$. The subgraph $\widehat{U}$ represents a Weyl group U' (non-extended). Take the order of the Weyl group $|U'|$. Then it holds that

$$h^\vee_\lambda = |U'|.$$

To calculate h^e_χ, h^s_χ, h^l_χ the following relations are used.
For $\chi \in F_M$

$$h^e_\chi = \begin{cases} h_\chi & \text{if } h_\chi = 1 \\ \frac{1}{2} h_\chi & \text{otherwise} \end{cases}$$
$$h^s_\chi = \begin{cases} \frac{1}{2} h_\chi & \text{if } \chi \in \widetilde{Y}_s \\ h_\chi & \text{otherwise} \end{cases}$$
$$h^l_\chi = \begin{cases} \frac{1}{2} h_\chi & \text{if } \chi \in \widetilde{Y}_l \cup Y_0 \\ h_\chi & \text{otherwise.} \end{cases}$$

Similar relations are used to calculate $h^{\vee e}_\lambda$, $h^{\vee s}_\lambda$, $h^{\vee l}_\lambda$.
For $\lambda \in \Lambda_M$

$$h^{\vee e}_\lambda = \begin{cases} h^\vee_\lambda & \text{if } h^\vee_\lambda = 1 \\ \frac{1}{2} h^\vee_\lambda & \text{otherwise} \end{cases}$$
$$h^{\vee s}_\lambda = \begin{cases} \frac{1}{2} h^\vee_\lambda & \text{if } \chi \in \widetilde{Y}_s^\vee \cup Y_0^\vee \\ h^\vee_\lambda & \text{otherwise} \end{cases}$$
$$h^{\vee l}_\lambda = \begin{cases} \frac{1}{2} h^\vee_\lambda & \text{if } \chi \in \widetilde{Y}_l^\vee \\ h^\vee_\lambda & \text{otherwise} \end{cases}$$

The relations for $\chi \in F_M$ allow the calculation of the remaining $\chi \in F^{e+}_M \backslash F_M$, $F^{s+}_M \backslash F_M$, $F^{l+}_M \backslash F_M$, respectively, as

$$h^e_{r\chi} = h^e_\chi, \quad h^s_{r_s\chi} = h^s_\chi, \quad h^l_{r_l\chi} = h^l_\chi,$$

and the relations for $\lambda \in \Lambda_M$ allow the calculation of the remaining $\lambda \in \Lambda^{e+}_M \setminus \Lambda_M$, $\Lambda^{s+}_M \setminus \Lambda_M$, $\Lambda^{l+}_M \setminus \Lambda_M$, respectively, as

$$h^{\vee e}_{r\lambda} = h^{\vee e}_\lambda, \quad h^{\vee s}_{r_s\lambda} = h^{\vee s}_\lambda, \quad h^{\vee l}_{r_l\lambda} = h^{\vee l}_\lambda.$$

2.3 Even orbit functions

In this section we discuss the properties of even orbit functions. The continuous orthogonality relations stated are valid for Weyl groups of rank 2. The discrete orthogonality relations stated are valid for Weyl groups of rank 2 and rank 3.

Let σ be a sign homomorphism then the **even orbit function** of even subgroup W^σ and $\lambda \in P$, $\Psi^\sigma_\lambda : \mathbb{R}^n \longmapsto \mathbb{C}$ is defined as

$$(\forall \chi \in R^n)\left(\Psi^\sigma_\lambda(\chi) = \sum_{w \in W^\sigma} e^{2\pi i \langle w\lambda, \chi\rangle}\right).$$

These functions are invariant with respect to the action of $w \in W^\sigma$

$$\Psi^\sigma_{w\lambda}(\chi) = \Psi^\sigma_\lambda(\chi)$$
$$\Psi^\sigma_\lambda(w\chi) = \Psi^\sigma_\lambda(\chi)$$

and invariant with respect to the action of the Abelian group of translations $Q^\vee$,

$$(\forall q^\vee \in Q^\vee)(\Psi^\sigma_\lambda(\chi + q^\vee) = \Psi^\sigma_\lambda(\chi)).$$

2.3.1 Ξ^{e+} – functions

For $W^{\sigma} = W^{e}$, the even orbit functions are denoted Ξ^{e+} and their explicit form, parametrized by $\lambda \in P_{e+}$, is

$$\Xi_{\lambda}^{e+}(\chi) = \Psi_{\lambda}^{\sigma_e}(\chi) = \sum_{w \in W^e} \mathrm{e}^{2\pi \mathrm{i}\langle w\lambda, \chi\rangle}, \quad \chi \in F^{e+}, \lambda \in P_{e+}.$$

Continuous orthogonality and Ξ^{e+}– transforms

For all $\lambda, \lambda' \in P_{e+}$ it holds that

$$\int_{F^{e+}} \Xi_{\lambda}^{e+}(\chi)\overline{\Xi_{\lambda'}^{e+}(\chi)}\, d\chi = |W^e||F^{e+}|\, d_{\lambda}^{e}\, \delta_{\lambda\lambda'}, \tag{2.5}$$

where $|W^e|$ is the order of the even subgroup W^e and $|F^{e+}|$ is the volume of the fundamental domain. Therefore the set of functions

$$\left\{\Xi_{\lambda}^{e+} \mid \lambda \in P_{e+}\right\}$$

is orthogonal. Let $f \in \mathrm{C}^1(\mathbb{R}^n)$ be invariant to the affine even subgroup W_e^{aff}

$$(\forall \chi \in \mathbb{R}^n)(\forall w \in W^e)(\forall q^{\vee} \in Q^{\vee})(f(w\chi + q^{\vee}) = f(\chi)). \tag{2.6}$$

Then the Ξ^{e+} – transform of f converges to f and in point $\chi \in \mathbb{R}^n$ is defined as

$$\sum_{\lambda \in P_{e+}} c_{\lambda}^{e+}\Xi_{\lambda}^{e+}(\chi), \quad \text{where } c_{\lambda}^{e+} = \frac{1}{|W^e||F^{e+}|\, d_{\lambda}^{e}} \int_{F^{e+}} f(\chi)\overline{\Xi_{\lambda}^{e+}(\chi)} d\chi.$$

Discrete orthogonality and Ξ^{e+} – interpolation

For all $\lambda, \lambda' \in \Lambda_M^{e+}$ it holds that

$$\sum_{\chi \in F_M^{e+}} \epsilon^{e}(\chi)\Xi_{\lambda}^{e+}(\chi)\overline{\Xi_{\lambda}^{e+}(\chi)} = |W^e|M^n \det(C)\, h_{\lambda}^{\vee e}\delta_{\lambda\lambda'}, \tag{2.7}$$

where $\det(C)$ is the determinant of the Cartan matrix and n is the dimension of the space $\mathbb{R}^n$.

Let $g : \mathbb{R}^n \mapsto \mathbb{C}$ be invariant to the affine even subgroup W_e^{aff} (as in 2.6). Then the Ξ^{e+} – interpolation of g, denoted g_M (depends on the density of the grid), in point $\chi \in \mathbb{R}^n$ is

$$g_M(\chi) = \sum_{\lambda \in \Lambda_M^{e+}} k_{\lambda}^{e+}\Xi_{\lambda}^{e+}(\chi), \quad \text{where } k_{\lambda}^{e+} = \frac{1}{|W^e|M^n \det(C)\, h_{\lambda}^{\vee e}} \sum_{\chi \in F_M^{e+}} \epsilon^{e}(\chi) f(\chi)\overline{\Xi_{\lambda'}^{e+}(\chi)}.$$

For $\chi \in F_M^{e+}$ the interpolation function g_M and the function g have the same value

$$(\forall \chi \in F_M^{e+})(g_M(\chi) = g(\chi)).$$

2.3.2 Ξ^{s+} – functions

For $W^{\sigma} = W^{s}$, the even orbit functions are denoted Ξ^{s+} and their explicit form, parametrized by $\lambda \in P_{s+}$, is

$$\Xi_{\lambda}^{s+}(\chi) = \Psi_{\lambda}^{\sigma_s}(\chi) = \sum_{w \in W^s} \mathrm{e}^{2\pi \mathrm{i}\langle w\lambda, \chi\rangle}, \quad \chi \in F^{s+}, \lambda \in P_{s+}.$$

Continuous orthogonality and Ξ^{s+}– transforms

For all $\lambda, \lambda' \in P_{s+}$ it holds that

$$\int_{F^{s+}} \Xi_\lambda^{s+}(\chi)\overline{\Xi_{\lambda'}^{s+}(\chi)}\, d\chi = |W^s|\,|F^{s+}|\, d_\lambda^s\, \delta_{\lambda\lambda'}, \tag{2.8}$$

where $|W^s|$ is the order of the even subgroup W^s and $|F^{s+}|$ is the volume of the fundamental domain. Therefore the set of functions

$$\left\{\Xi_\lambda^{s+} \mid \lambda \in P_{s+}\right\}$$

is orthogonal. Let $f \in C^1(\mathbb{R}^n)$ be invariant to the affine even subgroup W_s^{aff} (as in 2.6). Then the Ξ^{s+} – transform of f converges to f and in point $\chi \in \mathbb{R}^n$ is defined as

$$\sum_{\lambda\in P_{s+}} c_\lambda^{s+}\Xi_\lambda^{s+}(\chi), \quad \text{where } c_\lambda^{s+} = \frac{1}{|W^s|\,|F^{s+}|\, d_\lambda^s}\int_{F^{s+}} f(\chi)\overline{\Xi_\lambda^{s+}(\chi)}d\chi.$$

Discrete orthogonality and Ξ^{s+}– interpolation

For all $\lambda, \lambda' \in \Lambda_M^{s+}$ it holds that

$$\sum_{\chi\in F_M^{s+}} \epsilon^s(\chi)\Xi_\lambda^{s+}(\chi)\overline{\Xi_{\lambda'}^{s+}(\chi)} = |W^s| M^n \det(C)\, h_\lambda^{\vee s}\delta_{\lambda\lambda'}, \tag{2.9}$$

where $\det(C)$ is the determinant of the Cartan matrix and n is the dimension of the space $\mathbb{R}^n$.

Let $g : \mathbb{R}^n \mapsto \mathbb{C}$ be invariant to the affine even subgroup W_s^{aff} (as in 2.6). Then the Ξ^{s+} – interpolation of g, denoted g_M (depends on the density of the grid), in point $\chi \in \mathbb{R}^n$ is

$$g_M(\chi) = \sum_{\lambda\in\Lambda_M^{s+}} k_\lambda^{s+}\Xi_\lambda^{s+}(\chi), \quad \text{where } k_\lambda^{s+} = \frac{1}{|W^s| M^n \det(C)\, h_\lambda^{\vee s}}\sum_{\chi\in F_M^{s+}} \epsilon^s(\chi) f(\chi)\overline{\Xi_\lambda^{s+}(\chi)}.$$

For $\chi \in F_M^{s+}$ the interpolation function g_M and the function g have the same value

$$(\forall\chi \in F_M^{s+})(g_M(\chi) = g(\chi)).$$

2.3.3 Ξ^{l+} – functions

For $W^\sigma = W^l$, the even orbit functions are denoted Ξ^{l+} and their explicit form, parametrized by $\lambda \in P_{l+}$, is

$$\Xi_\lambda^{l+}(\chi) = \Psi_\lambda^{\sigma_l}(\chi) = \sum_{w\in W^l} e^{2\pi i\langle w\lambda,\chi\rangle}, \quad \chi \in F^{l+}, \lambda \in P_{l+}.$$

Continuous orthogonality and Ξ^{l+}– transforms

For all $\lambda, \lambda' \in P_{l+}$ it holds that

$$\int_{F^{l+}} \Xi_\lambda^{l+}(\chi)\overline{\Xi_{\lambda'}^{l+}(\chi)}\, d\chi = |W^l|\,|F^{l+}|\, d_\lambda^l\, \delta_{\lambda\lambda'}, \tag{2.10}$$

where $|W^l|$ is the order of the even subgroup W^l and $|F^{l+}|$ is the volume of the fundamental domain. Therefore the set of functions

$$\left\{\Xi_\lambda^{l+} \mid \lambda \in P_{l+}\right\}$$

is orthogonal. Let $f \in C^1(\mathbb{R}^n)$ be invariant to the affine even subgroup W_l^{aff} (as in 2.6). Then the Ξ^{l+} – transform of f converges to f and in point $\chi \in \mathbb{R}^n$ is defined as

$$\sum_{\lambda \in P_{l+}} c_\lambda^{l+} \Xi_\lambda^{l+}(\chi), \quad \text{where } c_\lambda^{l+} = \frac{1}{|W^l|\,|F^{l+}|\,d_\lambda^l} \int_{F^{l+}} f(\chi)\overline{\Xi_\lambda^{l+}(\chi)} d\chi.$$

Discrete orthogonality and Ξ^{l+}– interpolation

For all $\lambda, \lambda' \in \Lambda_M^{l+}$ it holds that

$$\sum_{\chi \in F_M^{l+}} \epsilon^l(\chi) \Xi_\lambda^{l+}(\chi) \overline{\Xi_{\lambda'}^{l+}(\chi)} = |W^l| M^n \det(C)\, h_\lambda^{\vee l} \delta_{\lambda\lambda'}, \tag{2.11}$$

where $\det(C)$ is the determinant of the Cartan matrix and n is the dimension of the space $\mathbb{R}^n$.

Let $g : \mathbb{R}^n \mapsto \mathbb{C}$ be invariant to the affine even subgroup W_l^{aff} (as in 2.6). Then the Ξ^{l+} – interpolation of g, denoted g_M (depends on the density of the grid), in point $\chi \in \mathbb{R}^n$ is

$$g_M(\chi) = \sum_{\lambda \in \Lambda_M^{l+}} k_\lambda^{l+} \Xi_\lambda^{l+}(\chi), \quad \text{where } k_\lambda^{l+} = \frac{1}{|W^l| M^n \det(C)\, h_\lambda^{\vee l}} \sum_{\chi \in F_M^{l+}} \epsilon^l(\chi) f(\chi) \overline{\Xi_\lambda^{l+}(\chi)}$$

For $\chi \in F_M^{l+}$ the interpolation function g_M and the function g have the same value

$$(\forall \chi \in F_M^{l+})(g_M(\chi) = g(\chi)).$$

2.4 Mixed even orbit functions

In this section we discuss the properties of mixed even orbit functions. The continuous orthogonality relations stated are valid for Weyl groups of rank 2. The discrete orthogonality relations stated are valid for Weyl groups of rank 2 and rank 3.

Let $\sigma, \widetilde{\sigma}$ be two different sign homomorphisms then the **mixed orbit function of even subgroup W^σ and of homomorphism $\widetilde{\sigma}$**, $\Psi_\lambda^{\sigma,\widetilde{\sigma}} : \mathbb{R}^n \longmapsto \mathbb{C}$ is

$$(\forall \chi \in R^n)\left(\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi) = \sum_{w \in \ker \sigma} \widetilde{\sigma}(w) \mathrm{e}^{2\pi \mathrm{i} \langle w\lambda, \chi \rangle}\right).$$

These functions are invariant or anti-invariant with respect to the action of $w \in W^\sigma$

$$\Psi_\lambda^{\sigma,\widetilde{\sigma}}(w\chi) = \widetilde{\sigma}(w) \Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi),$$
$$\Psi_{w\lambda}^{\sigma,\widetilde{\sigma}}(\chi) = \widetilde{\sigma}(w) \Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi).$$

This property is called **$\widetilde{\sigma}$-invariance.** Mixed even orbit functions are invariant with respect to the action of the Abelian group of translations $Q^\vee$

$$(\forall q^\vee \in Q^\vee)(\Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi + q^\vee) = \Psi_\lambda^{\sigma,\widetilde{\sigma}}(\chi)).$$

2.4.1 Ξ^{e-} – functions

The mixed even orbit functions Ξ^{e-} are given by the relation $\Xi^{e-}_\lambda = \Psi^{\sigma_e,\sigma_s}_\lambda = \Psi^{\sigma_e,\sigma_l}_\lambda$, where $\lambda \in P$. The σ_s-invariance implies the following zero points

$$\Xi^{e-}_\lambda(\chi) = 0, \quad \chi \in (Y_0 \cup \widetilde{Y_l}) \cap \widetilde{Y_s}.$$

Some of these functions are also zero

$$\Xi^{e-}_\lambda \equiv 0, \quad \lambda \in P \cap ((X_0 \cup \widetilde{X_l}) \cap \widetilde{X_s}).$$

Therefore the domain and the weight lattice have to be changed to

$$\begin{aligned} F^{e-} &= (F \setminus ((Y_0 \cup \widetilde{Y_l}) \cap \widetilde{Y_s})) \cup r_s F^\circ, \\ P_{e-} &= (P^+ \setminus ((X_0 \cup \widetilde{X_l}) \cap \widetilde{X_s})) \cup r_s P^{++}. \end{aligned}$$

Therefore Ξ^{e-} functions, parametrized by $\lambda \in P_{e-}$, in explict form are

$$\Xi^{e-}_\lambda(\chi) = \sum_{w \in W^e} \sigma_s(w) \mathrm{e}^{2\pi \mathrm{i} \langle w\lambda, \chi \rangle}, \quad \chi \in F^{e-},\, \lambda \in P_{e-}.$$

Continuous orthogonality and Ξ^{e-}– transforms

For all $\lambda, \lambda' \in P_{e-}$ it holds that

$$\int_{F^{e-}} \Xi^{e-}_\lambda(\chi) \overline{\Xi^{e-}_{\lambda'}(\chi)}\, d\chi = |W^e||F^{e-}|\, d^e_\lambda\, \delta_{\lambda\lambda'}, \tag{2.12}$$

where $|W^e|$ is the order of the even subgroup $W^e$and $|F^{e-}|$ is the volume of the fundamental domain. Therefore the set of functions

$$\left\{ \Xi^{e-}_\lambda \mid \lambda \in P_{e-} \right\}$$

is orthogonal. Let $f \in C^1(\mathbb{R}^n)$ be σ_s-invariant to the affine even subgroup W^{aff}_e

$$(\forall \chi \in \mathbb{R}^n)(\forall w \in W^e)(\forall q^\vee \in Q^\vee)(f(w\chi + q^\vee) = \sigma_s(w) f(\chi)). \tag{2.13}$$

Then the Ξ^{e-} – transform of f converges to f and in point $\chi \in \mathbb{R}^n$ is defined as

$$\sum_{\lambda \in P_{e-}} c^{e-}_\lambda \Xi^{e-}_\lambda(\chi), \quad \text{where } c^{e-}_\lambda = \frac{1}{|W^e||F^{e-}|\, d^e_\lambda} \int_{F^{e-}} f(\chi) \overline{\Xi^{e-}_\lambda(\chi)} d\chi.$$

Discrete orthogonality and Ξ^{e-}– interpolation

Due to zero points and zero functions, the grids are changed in the following way

$$\begin{aligned} F^{e-}_M &= (F_M \setminus ((Y_0 \cup \widetilde{Y_l}) \cap \widetilde{Y_s})) \cup r_s F^\circ_M, \\ \Lambda^{e-}_M &= (\Lambda_M \setminus ((M Y_0^\vee \cup M \widetilde{Y_l}^\vee) \cap M \widetilde{Y_s}^\vee)) \cup r_s \Lambda^\circ_M. \end{aligned}$$

For all $\lambda, \lambda' \in \Lambda^{e-}_M$ it holds that

$$\sum_{\chi \in F^{e-}_M} \epsilon^e(\chi) \Xi^{e-}_\lambda(\chi) \overline{\Xi^{e-}_\lambda(\chi)} = |W^e| M^n \det(C)\, h^{\vee e}_\lambda \delta_{\lambda\lambda'}, \tag{2.14}$$

where $\det(C)$ is the determinant of the Cartan matrix and n is the dimension of the space $\mathbb{R}^n$.

Let $g : \mathbb{R}^n \mapsto \mathbb{C}$ be σ_s-invariant to the affine even subgroup W_e^{aff} (as in 2.13). Then the Ξ^{e-} – interpolation of g, denoted g_M (depends on the density of the grid), in point $\chi \in \mathbb{R}^n$ is

$$g_M(\chi) = \sum_{\lambda \in \Lambda_M^{e-}} k_\lambda^{e-} \Xi_\lambda^{e-}(\chi), \quad \text{where } k_\lambda^{e-} = \frac{1}{|W^e| M^n \det(C)\, h_\lambda^{\vee e}} \sum_{\chi \in F_M^{e-}} \epsilon^e(\chi) f(\chi) \overline{\Xi_{\lambda'}^{e-}(\chi)}.$$

For $\chi \in F_M^{e-}$ the interpolation function g_M and the function g have the same value

$$(\forall \chi \in F_M^{e-})(g_M(\chi) = g(\chi)).$$

2.4.2 Ξ^{s-} – functions

The mixed even orbit functions Ξ^{s-} are given by the relation $\Xi_\lambda^{s-} = \Psi_\lambda^{\sigma_s,\sigma_l} = \Psi_\lambda^{\sigma_s,\sigma_e}$, where $\lambda \in P$. The σ_l-invariance implies the following zero points

$$\Xi_\lambda^{s-}(\chi) = 0, \quad \chi \in (Y_0 \cup \widetilde{Y}_l).$$

Some of these functions are also zero

$$\Xi_\lambda^{s-} \equiv 0, \quad \lambda \in P \cap (X_0 \cup \widetilde{X}_l).$$

Therefore the domain and the weight lattice has to be changed to

$$\begin{aligned} F^{s-} &= (F \setminus (Y_0 \cup \widetilde{Y}_l)) \cup r_s F^\circ, \\ P_{s-} &= (P^+ \setminus \widetilde{X}_l) \cup r_s P^{++}. \end{aligned}$$

Therefore Ξ^{s-} functions, parametrized by $\lambda \in P_{s-}$, in explict form are

$$\Xi_\lambda^{s-}(\chi) = \sum_{w \in W^s} \sigma_l(w) e^{2\pi \mathrm{i} \langle w\lambda, \chi \rangle}, \quad \chi \in F^{s-},\ \lambda \in P_{s-}.$$

Continuous orthogonality and Ξ^{s-}– transforms

For all $\lambda, \lambda' \in P_{s-}$ it holds that

$$\int_{F^{s-}} \Xi_\lambda^{s-}(\chi) \overline{\Xi_{\lambda'}^{s-}(\chi)}\, d\chi = |W^s|\, |F^{s-}|\, d_\lambda^s\, \delta_{\lambda\lambda'}, \tag{2.15}$$

where $|W^s|$ is the order of the even subgroup W^s and $|F^{s-}|$ is the volume of the fundamental domain. Therefore the set of functions

$$\left\{ \Xi_\lambda^{e-} \mid \lambda \in P_{s-} \right\}$$

is orthogonal. Let $f \in C^1(\mathbb{R}^n)$ be σ_l-invariant to the affine even subgroup W_s^{aff} (as in 2.13). Then the Ξ^{s-} – transform of f converges to f and in point $\chi \in \mathbb{R}^n$ is defined as

$$\sum_{\lambda \in P_{s-}} c_\lambda^{s-} \Xi_\lambda^{s-}(\chi), \quad \text{where } c_\lambda^{s-} = \frac{1}{|W^s|\, |F^{s-}|\, d_\lambda^s} \int_{F^{s-}} f(\chi) \overline{\Xi_\lambda^{e-}(\chi)} d\chi.$$

Discrete orthogonality and Ξ^{s-}– interpolation

Due to zero points and zero functions the grids are changed in the following way

$$F_M^{s-} = (F_M \setminus (Y_0 \cup \widetilde{Y_l})) \cup r_s F_M^{\circ},$$
$$\Lambda_M^{s-} = (\Lambda_M \setminus M\widetilde{Y_l}^{\vee}) \cup r_s \Lambda_M^{\circ}.$$

For all $\lambda, \lambda' \in \Lambda_M^{s-}$ it holds that

$$\sum_{\chi \in F_M^{s-}} \epsilon^s(\chi) \Xi_\lambda^{s-}(\chi) \overline{\Xi_\lambda^{s-}(\chi)} = |W^s| M^n \det(C)\, h_\lambda^{\vee s} \delta_{\lambda\lambda'}, \tag{2.16}$$

where $\det(C)$ is the determinant of the Cartan matrix and n is the dimension of the space $\mathbb{R}^n$. Let $g : \mathbb{R}^n \mapsto \mathbb{C}$ be σ_l-invariant to the affine even subgroup W_s^{aff} (as in 2.13). Then the Ξ^{s-} – interpolation of g, denoted g_M (depends on the density of the grid), in point $\chi \in \mathbb{R}^n$ is

$$g_M(\chi) = \sum_{\lambda \in \Lambda_M^{s-}} k_\lambda^{s-} \Xi_\lambda^{s-}(\chi), \quad \text{where } k_\lambda^{s-} = \frac{1}{|W^s| M^n \det(C)\, h_\lambda^{\vee s}} \sum_{\chi \in F_M^{s-}} \epsilon^s(\chi) f(\chi) \overline{\Xi_{\lambda'}^{s-}(\chi)}.$$

For $\chi \in F_M^{s-}$ the interpolation function g_M and the function g have the same value

$$(\forall \chi \in F_M^{s-})(g_M(\chi) = g(\chi)).$$

2.4.3 Ξ^{l-} – functions

The mixed even orbit functions Ξ^{l-} are given by the relation $\Xi_\lambda^{l-} = \Psi_\lambda^{\sigma_l,\sigma_s} = \Psi_\lambda^{\sigma_l,\sigma_e}$, where $\lambda \in P$. The σ_s-invariance implies the following zero points

$$\Xi_\lambda^{l-}(\chi) = 0, \quad \chi \in \widetilde{Y_s}.$$

Some of these functions are also zero

$$\Xi_\lambda^{l-} \equiv 0, \quad \lambda \in P \cap \widetilde{X_s}.$$

Therefore the domain and the weight lattice has to be changed to

$$F^{l-} = (F \setminus \widetilde{Y_s}) \cup r_l F^{\circ},$$
$$P_{l-} = (P^+ \setminus \widetilde{X_s}) \cup r_l P^{++}.$$

Therefore Ξ^{l-} functions, parametrized by $\lambda \in P_{l-}$, in explict form are

$$\Xi_\lambda^{l-}(\chi) = \sum_{w \in W^l} \sigma_s(w) \mathrm{e}^{2\pi \mathrm{i} \langle w\lambda, \chi \rangle}, \quad \chi \in F^{l-},\ \lambda \in P_{l-}.$$

Continuous orthogonality and Ξ^{l-}– transforms

For all $\lambda, \lambda' \in P_{l-}$ it holds that

$$\int_{F^{l-}} \Xi_\lambda^{l-}(\chi) \overline{\Xi_{\lambda'}^{l-}(\chi)}\, d\chi = |W^l| |F^{l-}|\, d_\lambda^s\, \delta_{\lambda\lambda'}, \tag{2.17}$$

where $|W^l|$ is the order of the even subgroup W^l and $|F^{l-}|$ is the volume of the fundamental domain. Therefore the set of functions

$$\left\{ \Xi_\lambda^{l-} \mid \lambda \in P_{l-} \right\}$$

is orthogonal. Let $f \in C^1(\mathbb{R}^n)$ be σ_s-invariant to the affine even subgroup W_l^{aff} (as in 2.13). Then the Ξ^{l-} – transform of f converges to f and in point $\chi \in \mathbb{R}^n$ is defined as

$$\sum_{\lambda \in P_{l-}} c_\lambda^{l-} \Xi_\lambda^{l-}(\chi), \quad \text{where } c_\lambda^{l-} = \frac{1}{|W^l| |F^{l-}|\, d_\lambda^l} \int_{F^{l-}} f(\chi) \overline{\Xi_\lambda^{l-}(\chi)} d\chi.$$

Discrete orthogonality and discrete Ξ^{l-}– transforms

Due to zero points and zero functions the grids are changed in the following way

$$F_M^{l-} = (F_M \setminus \widetilde{Y}_s) \cup r_l F_M^\circ,$$
$$\Lambda_M^{l-} = (\Lambda_M \setminus (M\widetilde{Y}_s^{\vee} \cup MY_0^{\vee})) \cup r_l \Lambda_M^\circ.$$

For all $\lambda, \lambda' \in \Lambda_M^{l-}$ it holds that

$$\sum_{\chi \in F_M^{l-}} \epsilon^l(\chi) \Xi_\lambda^{l-}(\chi) \overline{\Xi_\lambda^{l-}(\chi)} = |W^l| M^n \det(C)\, h_\lambda^{\vee l} \delta_{\lambda\lambda'}, \tag{2.18}$$

where $\det(C)$ is the determinant of the Cartan matrix and n is the dimension of the space $\mathbb{R}^n$. Let $g : \mathbb{R}^n \mapsto \mathbb{C}$ be σ_s-invariant to the affine even subgroup W_l^{aff} (as in 2.13). Then the Ξ^{l-} – interpolation of g, denoted g_M (depends on the density of the grid), in point $\chi \in \mathbb{R}^n$ is

$$g_M(\chi) = \sum_{\lambda \in \Lambda_M^{l-}} k_\lambda^{l-} \Xi_\lambda^{l-}(\chi), \quad \text{where } k_\lambda^{l-} = \frac{1}{|W^l| M^n \det(C)\, h_\lambda^{\vee l}} \sum_{\chi \in F_M^{l-}} \epsilon^l(\chi) f(\chi) \overline{\Xi_{\lambda'}^{l-}(\chi)}.$$

For $\chi \in F_M^{s-}$ the interpolation function g_M and the function g have the same value

$$(\forall \chi \in F_M^{s-})(g_M(\chi) = g(\chi)).$$

2.5 Product decomposition of even orbit functions

Different products of E-functions can be decomposed into the sum of E-functions. These relations are valid for Weyl groups of rank 2. For more details see [1].

Decomposition of $\Xi^{e\pm} \cdot \Xi^{e\pm}$

The product of two Ξ^{e+}– functions decomposes into the sum of Ξ^{e+}– functions with the signs of the summands all positive. Two Ξ^{e-}– functions decompose into the sum of Ξ^{e+}– functions with the signs of the summands relative to the sign homomorphism σ_s. The general formula of these decompositions, which hold for any $\lambda, \lambda' \in P$ and $\chi \in \mathbb{R}^2$, is

$$\Xi_\lambda^{e+}(\chi) \cdot \Xi_{\lambda'}^{e+}(\chi) = \sum_{w \in W^e} \Xi_{\lambda + w\lambda'}^{e+}(\chi), \quad \Xi_\lambda^{e-}(\chi) \cdot \Xi_{\lambda'}^{e-}(\chi) = \sum_{w \in W^e} \sigma_s(w) \Xi_{\lambda + w\lambda'}^{e+}(\chi). \tag{2.19}$$

The mixed product of Ξ^{e+}– and Ξ^{e-}– functions decomposes into the sum of Ξ^{e-}– functions with the signs of the summands relative to the sign homomorphism σ_s,

$$\Xi_\lambda^{e+}(\chi) \cdot \Xi_{\lambda'}^{e-}(\chi) = \sum_{w \in W^e} \sigma_s(w) \Xi_{\lambda + w\lambda'}^{e-}(\chi). \tag{2.20}$$

Decomposition of $\Xi^{s\pm} \cdot \Xi^{s\pm}$

The product of two Ξ^{s+}– functions decomposes into the sum of Ξ^{s+}– functions with the signs of the summands all positive. Two Ξ^{s-}– functions decompose into the sum of Ξ^{s-}– functions with the signs of the summands relative to the sign homomorphism σ_l. The general formula of these decompositions, which hold for any $\lambda, \lambda' \in P$ and $\chi \in \mathbb{R}^2$, is

$$\Xi_\lambda^{s+}(\chi) \cdot \Xi_{\lambda'}^{s+}(\chi) = \sum_{w \in W^s} \Xi_{\lambda + w\lambda'}^{s+}(\chi), \quad \Xi_\lambda^{s-}(\chi) \cdot \Xi_{\lambda'}^{s-}(\chi) = \sum_{w \in W^s} \sigma_l(w) \Xi_{\lambda + w\lambda'}^{s+}(\chi). \tag{2.21}$$

The mixed product of $\Xi^{s+}-$ and $\Xi^{s-}-$ functions decomposes into the sum of $\Xi^{s-}-$ functions with the signs of the summands relative to the sign homomorphism σ_l,

$$\Xi^{s+}_{\lambda}(\chi)\cdot\Xi^{s-}_{\lambda'}(\chi)=\sum_{w\in W^s}\sigma_l(w)\Xi^{s-}_{\lambda+w\lambda'}(\chi). \tag{2.22}$$

Decomposition of $\Xi^{l\pm}\cdot\Xi^{l\pm}$

The product of two $\Xi^{l+}-$ functions decomposes into the sum of $\Xi^{l+}-$ functions with the signs of the summands all positive. Two $\Xi^{l-}-$ functions decompose into the sum of $\Xi^{l-}-$ functions with the signs of the summands relative to the sign homomorphism σ_s. The general formula of these decompositions, which hold for any $\lambda,\lambda'\in P$ and $\chi\in\mathbb{R}^2$, is

$$\Xi^{l+}_{\lambda}(\chi)\cdot\Xi^{l+}_{\lambda'}(\chi)=\sum_{w\in W^l}\Xi^{l+}_{\lambda+w\lambda'}(\chi),\quad \Xi^{l-}_{\lambda}(\chi)\cdot\Xi^{l-}_{\lambda'}(\chi)=\sum_{w\in W^l}\sigma_s(w)\Xi^{l+}_{\lambda+w\lambda'}(\chi). \tag{2.23}$$

The mixed product of $\Xi^{l+}-$ and $\Xi^{l-}-$functions decomposes into the sum of $\Xi^{l-}-$functions with the signs of the summands relative to the sign homomorphism σ_s,

$$\Xi^{l+}_{\lambda}(\chi)\cdot\Xi^{l-}_{\lambda'}(\chi)=\sum_{w\in W^l}\sigma_s(w)\Xi^{l-}_{\lambda+w\lambda'}(\chi). \tag{2.24}$$

Chapter 3

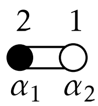
E-functions of rank 2

In this chapter we review explicitly Weyl groups of rank two with two different root lengths – of type C_2 and G_2 – and their E-functions [1, 3, 10].

3.1 Weyl group and Affine Weyl group of type C_2

3.1.1 Basic properties of the Weyl group of type C_2

The Coxeter–Dynkin diagram (numbers are marks m_i) and Cartan matrix with its inverse are the following,



$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \qquad C^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}.$$

Hence α_1 is the short root, α_2 is the long root. The bases of simple roots and fundamental weights are thus related by

$$\begin{aligned} &\alpha_1 = 2\omega_1 - \omega_2, && \omega_1 = \alpha_1 + \tfrac{1}{2}\alpha_2, && \alpha_1^\vee = 2\alpha_1, && \omega_1^\vee = 2\omega_1, \\ &\alpha_2 = -2\omega_1 + 2\omega_2, && \omega_2 = \alpha_1 + \alpha_2, && \alpha_2^\vee = \alpha_2, && \omega_2^\vee = \omega_2. \end{aligned}$$

The four bases are

$$\begin{aligned} \Delta &= \{\alpha_1, \alpha_2\}, \quad \Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee\} \\ \Omega &= \{\omega_1, \omega_2\}, \quad \Omega^\vee = \{\omega_1^\vee, \omega_2^\vee\}. \end{aligned}$$

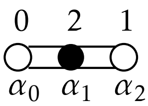
In the orthonormal basis these vectors have the form

$$\begin{aligned} &\alpha_1 = (0,1) = e_2, && \omega_1 = (\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2}e_1 + \tfrac{1}{2}e_2, && \omega_1^\vee = (1,1), \\ &\alpha_2 = (1,-1) = e_1 - e_2, && \omega_2 = (1,0) = e_1, && \omega_2^\vee = (1,0). \end{aligned}$$

The highest root of $W^{\mathrm{aff}}_{C_2}$ and the highest dual root of $\widehat{W}^{\mathrm{aff}}_{C_2}$ are given by these formulas

$$\xi = 2\alpha_1 + \alpha_2 = (1,1), \quad \eta = \alpha_1^\vee + 2\alpha_2^\vee = (2,0).$$

The extended Coxeter–Dynkin diagrams (numbers above vertices are the marks m_i respectively marks $m_i^\vee$) of $W^{\mathrm{aff}}_{C_2}$ and $\widehat{W}^{\mathrm{aff}}_{C_2}$ are



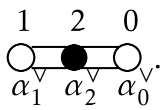


Table 3.1: *The coefficients for discrete calculus of C_2. Positive values of $u_0, u_1, u_2 > 0$ and $t_0, t_1, t_2 > 0$ are assumed.*

$\chi \in F_M(C_2)$	$\epsilon(\chi)$	$\epsilon^e(\chi)$	$\epsilon^s(\chi)$	$\epsilon^l(\chi)$
$[u_0, u_1, u_2]$	8	4	4	4
$[0, u_1, u_2]$	4	4	2	4
$[u_0, 0, u_2]$	4	4	4	2
$[u_0, u_1, 0]$	4	4	2	4
$[0, 0, u_2]$	1	1	1	1
$[0, u_1, 0]$	2	2	1	2
$[u_0, 0, 0]$	1	1	1	1

$\lambda \in \Lambda_M(C_2)$	$h_\lambda^\vee$	$h_\lambda^{e\vee}$	$h_\lambda^{s\vee}$	$h_\lambda^{l\vee}$
$[t_0, t_1, t_2]$	1	1	1	1
$[0, t_1, t_2]$	2	1	1	2
$[t_0, 0, t_2]$	2	1	1	2
$[t_0, t_1, 0]$	2	1	2	1
$[0, 0, t_2]$	4	2	2	4
$[0, t_1, 0]$	8	4	4	4
$[t_0, 0, 0]$	8	4	4	4

3.1.2 E-functions of C_2

In the explicit form of E-functions, the vector from the weight lattice will be in Ω basis and the variable will be in $\Delta^\vee$ basis, denoted $(a,b)_\Omega$ and $(x,y)_{\Delta^\vee}$. The coefficients for discrete calculus of C_2 – assuming $u_0, u_1, u_2 > 0$, $t_0, t_1, t_2 > 0$ – are shown in Table (3.1).

Ξ^{e+} – function of C_2

The explicit form of Ξ^{e+} of C_2 is

$$\Xi^{e+}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} = 2\cos(2\pi b(x-y)) + 2\cos(2\pi(ax - ay + bx - by)),$$

where $(a,b)_\Omega \in P_{e+}$ and $(x,y)_{\Delta^\vee} \in F^{e+}$. The continuous (2.5) and discrete (2.7) orthogonal relations hold. The product decomposition relation (2.19) holds.

Ξ^{s+} – function of C_2

The explicit form of Ξ^{s+} of C_2 is

$$\Xi^{s+}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} = 2[\cos(2\pi(ax+by)) + \cos(2\pi((a+2b)x - by))],$$

where $(a,b)_\Omega \in P_{s+}$ and $(x,y)_{\Delta^\vee} \in F^{s+}$. The continuous (2.8) and discrete (2.9) othogonal relations hold. The product decomposition relation (2.21) holds.

Ξ^{l+} – function of C_2

The explicit form of Ξ^{l+} of C_2 is

$$\Xi^{l+}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} = 2\{\cos(2\pi(ax+by)) + \cos(2\pi(ax - (a+b)y))\},$$

where $(a,b)_\Omega \in P_{l+}$ and $(x,y)_{\Delta^\vee} \in F^{l+}$. The continuous (2.10) and discrete (2.11) othogonal relations hold. The product decomposition relation (2.23) holds.

Ξ^{e-} – function of C_2

The explicit form of Ξ^{e-} of C_2 is

$$\Xi^{e-}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} = 2\{\cos(2\pi(ax+by)) - \cos(2\pi((a+2b)x - (a+b)y))\},$$

where $(a,b)_\Omega \in P_{e-}$ and $(x,y)_{\Delta^\vee} \in F^{e-}$. The continuous (2.12) and discrete (2.14) othogonal relations hold. The product decomposition relations (2.19), (2.20) hold.

Ξ^{s-} **– function of** C_2

The explicit form of Ξ^{s-} of C_2 is

$$\Xi^{s-}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} = 2\{\cos(2\pi(ax+by)) - \cos(2\pi((a+2b)x - by))\},$$

where $(a,b)_\Omega \in P_{s-}$ and $(x,y)_{\Delta^\vee} \in F^{s-}$. The continuous (2.15) and discrete (2.16) othogonal relations hold. The product decomposition relations (2.21), (2.22) hold.

Ξ^{l-} **– function of** C_2

The explicit form of Ξ^{l-} of C_2 is

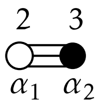
$$\Xi^{l-}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} = 2\{\cos(2\pi(ax+by)) - \cos(2\pi(ax - (a+b)y))\},$$

where $(a,b)_\Omega \in P_{l-}$ and $(x,y)_{\Delta^\vee} \in F^{l-}$. The continuous (2.17) and discrete (2.18) othogonal relations hold. The product decomposition relations (2.23), (2.24) hold.

3.2 Weyl group and Affine Weyl group of type G_2

3.2.1 Basic properties of the Weyl group of type G_2

The Coxeter–Dynkin diagrams (numbers are marks m_i) and Cartan matrix with its inverse are the following,



$$C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

Hence α_2 is the short root and α_1 the long root. The relative lengths of the simple roots are set as $\langle\alpha_2,\alpha_2\rangle = \frac{2}{3}$ and $\langle\alpha_1,\alpha_1\rangle = 2$. The basis of simple roots and fundamental weights are thus related by

$$\begin{aligned} \alpha_1 &= 2\omega_1 - 3\omega_2, & \omega_1 &= 2\alpha_1 + 3\alpha_2, & \alpha_1^\vee &= \alpha_1, & \omega_1^\vee &= \omega_1, \\ \alpha_2 &= -\omega_1 + 2\omega_2, & \omega_2 &= \alpha_1 + 2\alpha_2, & \alpha_2^\vee &= 3\alpha_2, & \omega_2^\vee &= 3\omega_2. \end{aligned}$$

The four bases are

$$\Delta = \{\alpha_1,\alpha_2\}, \quad \Delta^\vee = \{\alpha_1^\vee,\alpha_2^\vee\}$$
$$\Omega = \{\omega_1,\omega_2\}, \quad \Omega^\vee = \{\omega_1^\vee,\omega_2^\vee\}.$$

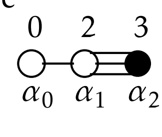
Relative to the orthonormal basis these vectors have the form

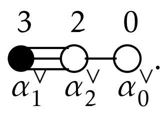
$$\begin{aligned} \alpha_1 &= (\sqrt{2},0) = \sqrt{2}e_1, & \omega_1 &= (\tfrac{1}{\sqrt{2}},\tfrac{\sqrt{3}}{\sqrt{2}}) = \tfrac{1}{\sqrt{2}}e_1 + \tfrac{\sqrt{3}}{\sqrt{2}}e_2, & \omega_1^\vee &= (\tfrac{1}{\sqrt{2}},\tfrac{\sqrt{3}}{\sqrt{2}}), \\ \alpha_2 &= (-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{6}}) = -\tfrac{1}{\sqrt{2}}e_1 + \tfrac{1}{\sqrt{6}}e_2, & \omega_2 &= (0,\tfrac{\sqrt{2}}{\sqrt{3}}) = \tfrac{\sqrt{2}}{\sqrt{3}}e_2, & \omega_2^\vee &= (0,\sqrt{6}). \end{aligned}$$

The highest root of $W^{\mathrm{aff}}_{G_2}$ and the highest dual root of $\widehat{W}^{\mathrm{aff}}_{G_2}$ are given by these formulas

$$\xi = 2\alpha_1 + 3\alpha_2 = \frac{\sqrt{2}}{2}(1,\sqrt{3}), \quad \eta = 3\alpha_1^\vee + 2\alpha_2^\vee = (0,\sqrt{6}).$$

The extended Coxeter–Dynkin diagrams (numbers above vertices are the marks m_i respectively marks $m_i^\vee$) of $W^{\mathrm{aff}}_{G_2}$ and $\widehat{W}^{\mathrm{aff}}_{G_2}$ are





3.2.2 E-functions of G_2

In the explicit form of E-functions, the vector from the weight lattice will be in Ω basis and the variable will be in $\Delta^\vee$ basis, denoted $(a,b)_\Omega$ and $(x,y)_{\Delta^\vee}$. The coefficients for discrete calculus of G_2 –assuming $u_0,u_1,u_2>0$, $t_0,t_1,t_2>0$ – are shown in Table (3.2).

Table 3.2: *The coefficients for discrete calculus of G_2. Positive values of $u_0,u_1,u_2>0$ and $t_0,t_1,t_2>0$ are assumed.*

$\chi \in F_M(G_2)$	$\epsilon(\chi)$	$\epsilon^e(\chi)$	$\epsilon^s(\chi)$	$\epsilon^l(\chi)$
$[u_0,u_1,u_2]$	12	6	6	6
$[0,u_1,u_2]$	6	6	3	6
$[u_0,0,u_2]$	6	6	6	3
$[u_0,u_1,0]$	6	6	3	6
$[0,0,u_2]$	1	1	1	1
$[0,u_1,0]$	2	2	1	2
$[u_0,0,0]$	3	3	3	3

$\lambda \in \Lambda_M(G_2)$	$h^\vee_\lambda$	$h^{e\vee}_\lambda$	$h^{s\vee}_\lambda$	$h^{l\vee}_\lambda$
$[t_0,t_1,t_2]$	1	1	1	1
$[0,t_1,t_2]$	2	1	1	2
$[t_0,0,t_2]$	2	1	2	1
$[t_0,t_1,0]$	2	1	1	2
$[0,0,t_2]$	4	2	2	2
$[0,t_1,0]$	6	3	3	6
$[t_0,0,0]$	12	6	6	6

Ξ^{e+}– function of G_2

The explicit form of Ξ^{e+} of G_2 is

$$\Xi^{e+}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} = 2\cos(\pi\,(ax+by)),$$

where $(a,b)_\Omega \in P_{e+}$ and $(x,y)_{\Delta^\vee} \in F^{e+}$. The continuous (2.5) and discrete (2.7) othogonal relations hold. The product decomposition relation (2.19) holds.

Ξ^{s+} – function of G_2

The explicit form of Ξ^{s+} of G_2 is

$$\begin{aligned}\Xi^{s+}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} =& e^{2\pi\mathrm{i}(ax+by)} + e^{2\pi\mathrm{i}(-ax+(3a+b)y)} + e^{2\pi\mathrm{i}((2a+b)x-(3a+2b)y)}\\ &+ e^{2\pi\mathrm{i}((a+b)x-(3a+2b)y)} + e^{2\pi\mathrm{i}(-(2a+b)x+(3a+b)y)} + e^{2\pi\mathrm{i}(-(a+b)x+by)},\end{aligned}$$

where $(a,b)_\Omega \in P_{s+}$ and $(x,y)_{\Delta^\vee} \in F^{s+}$. The continuous (2.8) and discrete (2.9) othogonal relations hold. The product decomposition relation (2.21) holds.

Ξ^{l+} – function of G_2

The explicit form of Ξ^{l+} of G_2 is

$$\begin{aligned}\Xi^{l+}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} =& e^{2\pi\mathrm{i}(ax+by)} + e^{2\pi\mathrm{i}((a+b)x-by)} + e^{2\pi\mathrm{i}(-(2a+b)x+(3a+2b)y)}\\ &+ e^{2\pi\mathrm{i}((a+b)x-(3a+2b)y)} + e^{2\pi\mathrm{i}(-(2a+b)x+(3a+b)y)} + e^{2\pi\mathrm{i}(ax-(3a+b)y)},\end{aligned}$$

where $(a,b)_\Omega \in P_{l+}$ and $(x,y)_{\Delta^\vee} \in F^{l+}$. The continuous (2.10) and discrete (2.11) othogonal relations hold. The product decomposition relation (2.23) holds.

Ξ^{e-} **– function of** G_2

The explicit form of Ξ^{e-} of G_2 is

$$\begin{aligned}\Xi^{e-}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} =& 2\mathrm{i}\{\sin(2\pi(ax+by)) + \sin(2\pi((3a+b)y) - (2a+b)x)\\ &+ \sin(2\pi((a+b)x - (3a+2b)x))\},\end{aligned}$$

where $(a,b)_\Omega \in P_{e-}$ and $(x,y)_{\Delta^\vee} \in F^{e-}$. The continuous (2.12) and discrete (2.14) othogonal relations hold. The product decomposition relations (2.19), (2.20) hold.

Ξ^{s-} **– function of** G_2

The explicit form of Ξ^{s-} of G_2 is

$$\begin{aligned}\Xi^{s-}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} =& e^{2\pi\mathrm{i}(ax+by)} - e^{2\pi\mathrm{i}(-ax+(3a+b)y)} - e^{2\pi\mathrm{i}((2a+b)x-(3a+2b)y)}\\ &+ e^{2\pi\mathrm{i}((a+b)x-(3a+2b)y)} + e^{2\pi\mathrm{i}(-(2a+b)x+(3a+b)y)} - e^{2\pi\mathrm{i}(-(a+b)x+by)},\end{aligned}$$

where $(a,b)_\Omega \in P_{s-}$ and $(x,y)_{\Delta^\vee} \in F^{s-}$. The continuous (2.15) and discrete (2.16) othogonal relations hold. The product decomposition relations (2.21), (2.22) hold.

Ξ^{l-} **– function of** G_2

The explicit form of Ξ^{s-} of G_2 is

$$\begin{aligned}\Xi^{l-}_{(a,b)_\Omega}(x,y)_{\Delta^\vee} =& e^{2\pi\mathrm{i}(ax+by)} - e^{2\pi\mathrm{i}((a+b)x-by)} - e^{2\pi\mathrm{i}(-(2a+b)x+(3a+2b)y)}\\ &+ e^{2\pi\mathrm{i}((a+b)x-(3a+2b)y)} + e^{2\pi\mathrm{i}(-(2a+b)x+(3a+b)y)} - e^{2\pi\mathrm{i}(ax-(3a+b)y)},\end{aligned}$$

where $(a,b)_\Omega \in P_{l-}$ and $(x,y)_{\Delta^\vee} \in F^{l-}$. The continuous (2.17) and discrete (2.18) othogonal relations hold. The product decomposition relations (2.23), (2.24) hold.

Chapter 4

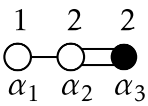
E-functions of rank 3

In this chapter we review explicitly Weyl groups of rank three with two different roots lengths – of type B_3 and C_3 – and their E-functions. Basic properties of these groups are from [10].

4.1 Weyl group and Affine Weyl group of type B_3

4.1.1 Basic properties of the Weyl group of type B_3

The Coxeter–Dynkin diagram (numbers are marks m_i) and Cartan matrix with its inverse are the following,



$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad C^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$

Hence α_3 is a short root and α_2, α_1 are long roots. The relative lengths of the simple roots are set as $\langle \alpha_3, \alpha_3 \rangle = 1$ and $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 2$.

The bases of simple roots and fundamental weights are thus related by

$$\begin{aligned}
&\alpha_1 = 2\omega_1 - \omega_2, && \omega_1 = \alpha_1 + \alpha_2 + \alpha_3, && \alpha_1^\vee = \alpha_1, && \omega_1^\vee = \omega_1,\\
&\alpha_2 = -\omega_1 + 2\omega_2 - 2\omega_3, && \omega_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3, && \alpha_2^\vee = \alpha_2, && \omega_2^\vee = \omega_2,\\
&\alpha_3 = -\omega_2 + 2\omega_3, && \omega_3 = \tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{3}{2}\alpha_3, && \alpha_3^\vee = 2\alpha_3, && \omega_3^\vee = 2\omega_3.
\end{aligned}$$

The four bases are

$$\Delta = \{\alpha_1, \alpha_2, \alpha_3\}, \quad \Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$$
$$\Omega = \{\omega_1, \omega_2, \omega_3\}, \quad \Omega^\vee = \{\omega_1^\vee, \omega_2^\vee, \omega_3^\vee\}.$$

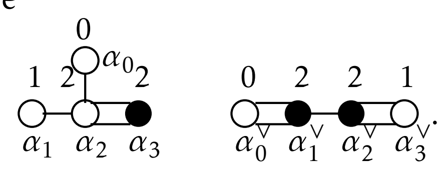
Relative to the orthonormal basis the vectors are of the form

$$\begin{aligned}
&\alpha_1 = (1,-1,0) = e_1 - e_2, && \omega_1 = (1,0,0) = e_1, && \omega_1^\vee = (1,0,0) = e_1,\\
&\alpha_2 = (0,1,-1) = e_2 - e_3, && \omega_2 = (1,1,0) = e_1 + e_2, && \omega_2^\vee = (1,1,0) = e_1 + e_2,\\
&\alpha_3 = (0,0,1) = e_3, && \omega_3 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2}(e_1 + e_2 + e_3), && \omega_3^\vee = (1,1,1) = e_1 + e_2 + e_3.
\end{aligned}$$

The highest root of $W_{B_3}^{\mathrm{aff}}$ and the highest dual root of $\widehat{W}_{B_3}^{\mathrm{aff}}$ are given by these formulas

$$\xi = \alpha_1 + 2\alpha_2 + 2\alpha_3 = (1,1,0), \quad \eta = 2\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee = (2,0,0).$$

The extended Coxeter–Dynkin diagrams (numbers above vertices are the marks m_i respectively marks $m_i^\vee$) of $W_{B_3}^{\mathrm{aff}}$ and $\widehat{W}_{B_3}^{\mathrm{aff}}$ are



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4.1.2 E-functions of B_3

In the explicit form of E-functions, the vector from the weight lattice will be in Ω basis and the variable will be in $\Delta^\vee$ basis, denoted $(a,b,c)_\Omega$ and $(x,y,z)_{\Delta^\vee}$.

For $M=5$, the grid F_5 has 20 elements and its explicit form is

$$\begin{aligned}F_5 =\Big\{&\tfrac{1}{5}(0,0,0,5),\tfrac{1}{5}(0,0,1,3),\tfrac{1}{5}(0,0,2,1),\tfrac{1}{5}(0,1,0,3),\tfrac{1}{5}(0,1,1,1),\tfrac{1}{5}(0,2,0,1),\tfrac{1}{5}(1,0,0,4),\\ &\tfrac{1}{5}(1,0,1,2),\tfrac{1}{5}(1,0,2,0),\tfrac{1}{5}(1,1,0,2),\tfrac{1}{5}(1,1,1,0),\tfrac{1}{5}(1,2,0,0),\tfrac{1}{5}(2,0,0,3),\tfrac{1}{5}(2,0,1,1),\\ &\tfrac{1}{5}(2,1,0,1),\tfrac{1}{5}(3,0,0,2),\tfrac{1}{5}(3,0,1,0),\tfrac{1}{5}(3,1,0,0),\tfrac{1}{5}(4,0,0,1),\tfrac{1}{5}(5,0,0,0)\Big\}.\end{aligned}$$

For $M=5$, the grid Λ_5 has 20 elements and its explicit form is

$$\begin{aligned}\Lambda_5 =\{&(0,0,0,5),(0,0,1,4),(0,0,2,3),(0,0,3,2),(0,0,4,1),(0,0,5,0),(0,1,0,3),\\ &(0,1,1,2),(0,1,2,1),(0,1,3,0),(0,2,0,1),(0,2,1,0),(1,0,0,3),(1,0,1,2),\\ &(1,0,2,1),(1,0,3,0),(1,1,0,1),(1,1,1,0),(2,0,0,1),(2,0,1,0)\}.\end{aligned}$$

Table 4.1: *The coefficients for discrete calculus of B_3. Positive values of $u_0,u_1,u_2,u_3>0$ and $t_0,t_1,t_2,t_3>0$ are assumed.*

$\chi \in F_M(B_3)$	$\epsilon(\chi)$	$\epsilon^e(\chi)$	$\epsilon^s(\chi)$	$\epsilon^l(\chi)$
$[u_0,u_1,u_2,u_3]$	48	24	24	24
$[u_0,u_1,u_2,0]$	24	24	24	12
$[u_0,u_1,0,u_3]$	24	24	12	24
$[u_0,0,u_2,u_3]$	24	24	12	24
$[0,u_1,u_2,u_3]$	24	24	12	24
$[u_0,u_1,0,0]$	6	6	6	6
$[u_0,0,u_2,0]$	12	12	12	12
$[0,u_1,u_2,0]$	12	12	12	12
$[u_0,0,0,u_3]$	8	8	4	8
$[0,u_1,0,u_3]$	8	8	4	8
$[0,0,u_2,u_3]$	12	12	6	12
$[u_0,0,0,0]$	1	1	1	1
$[0,0,u_2,0]$	6	6	6	6
$[0,u_1,0,0]$	1	1	1	1
$[0,0,0,u_3]$	2	2	1	2

$\lambda \in \Lambda_M(B_3)$	$h_\lambda^\vee$	$h_\lambda^{e\vee}$	$h_\lambda^{s\vee}$	$h_\lambda^{l\vee}$
$[t_0,t_1,t_2,t_3]$	1	1	1	1
$[t_0,t_1,t_2,0]$	2	1	1	2
$[t_0,t_1,0,t_3]$	2	1	2	1
$[t_0,0,t_2,t_3]$	2	1	2	1
$[0,t_1,t_2,t_3]$	2	1	1	2
$[t_0,t_1,0,0]$	8	4	4	4
$[t_0,0,t_2,0]$	4	2	4	2
$[0,t_1,t_2,0]$	4	2	2	4
$[t_0,0,0,t_3]$	6	3	6	3
$[0,t_1,0,t_3]$	4	2	2	2
$[0,0,t_2,t_3]$	8	4	4	4
$[t_0,0,0,0]$	48	24	24	24
$[0,0,t_2,0]$	16	8	8	8
$[0,t_1,0,0]$	16	8	8	8
$[0,0,0,t_3]$	48	24	24	24

Ξ^{e+} – function of B_3

The explicit form of Ξ^{e+} of B_3 is

$$\begin{aligned}\Xi^{e+}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {} & e^{-2i\pi(-ax-2bx-cx+by+cz)} + e^{2i\pi(bx-2az-2bz-cz+ay)} + e^{2i\pi(ay+by+cy-ax-cz)} + \\ & + e^{2i\pi(-cx+ay+2by+cy-ax-bx-2bz-cz)} + e^{-2i\pi(-bx-2az-2bz-cz+ay+by)} + \\ & + e^{-2i\pi(ay+2by+cy-ax-bx-2bz-cz)} + e^{2i\pi(-ax-bx-2bz-cz+by)} + \\ & + e^{2i\pi(2by+cy-bx-2az-2bz-cz+ay)} + e^{2i\pi(ax+bx+cy-2bz-cz+by)} + \\ & + e^{-2i\pi(-ax-bx-cx+cy-2bz-cz+by)} + e^{-2i\pi(ax+by+cy-cz)} + e^{-2i\pi(ay+by-ax+cz)} + \\ & + e^{-2i\pi(-cx+ay-ax-bx+2bz+cz)} + e^{2i\pi(ax+by+cz)} + e^{-2i\pi(cx+bx-2az-2bz-cz+ay)} + \\ & + e^{2i\pi(-bx-cx+cy-2az-2bz-cz+ay+by)} + e^{2i\pi(ay-ax-bx+2bz+cz)} + \\ & + e^{2i\pi(-ax-2bx-cx+by+cy-cz)} + e^{2i\pi(-2bx-cx+ay+by-ax+cz)} + \\ & + e^{-2i\pi(-cx+2by+cy-bx-2az-2bz-cz+ay)} + e^{-2i\pi(ax+bx+cx-2bz-cz+by)} + \\ & + e^{-2i\pi(-2bx-cx+ay+by+cy-ax-cz)} + e^{2i\pi(bx+cx-2az-2bz-cz+ay+by)} + \\ & + e^{-2i\pi(bx+cy-2az-2bz-cz+ay+by)},\end{aligned}$$

where $(a,b,c)_\Omega \in P_{e+}$ and $(x,y,z)_{\Delta^\vee} \in F^{e+}$.

The explicit form of the grid F^{e+}_M for any natural M is

$$\begin{aligned} F^{e+}_M = {} & \left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}_0^+,\, d+a+2b+2c = M \right\} \\ & \cup \left\{ \frac{a}{M}\omega_1^\vee + \frac{b+2c}{M}\omega_2^\vee - \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M \right\}. \end{aligned}$$

The explicit form of the grid Λ^{e+}_M for any natural M is

$$\begin{aligned} \Lambda^{e+}_M = {} & \{ a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}_0^+,\, d+2a+2b+c = M \} \\ & \cup \{ a\omega_1 + (b+c)\omega_2 - c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+2a+2b+c = M \}. \end{aligned}$$

The discrete (2.7) othogonal relations hold.

Ξ^{s+} – function of B_3

The explicit form of Ξ^{s+} of B_3 is

$$\begin{aligned}\Xi^{s+}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = 2[& \cos(\pi\,(-bx-cx+az+bz+cz-ay-by)) + \\ & + \cos(\pi\,(ax+bx+cx-by-cy+cz)) + \\ & + \cos(\pi\,(cx-ay-by-cy+ax+bx+bz+cz)) + \\ & + \cos(\pi\,(-ax-by-cy+cz)) + + \cos(\pi\,(bx+az+bz+cz-ay-by)) + \\ & + \cos(\pi\,(ay-ax-bx+bz+cz)) + \cos(\pi\,(ax+by+cz)) + \\ & + \cos(\pi\,(-bx-cy+az+bz+cz-ay-by)) + \\ & + \cos(\pi\,(-ay-by-cy+ax+bx+bz+cz)) + \\ & + \cos(\pi\,(-ax-bx-cx+by+cz)) + \cos(\pi\,(-cx+ay-ax-bx+bz+cz)) + \\ & + \cos(\pi\,(bx+cx-cy+az+bz+cz-ay-by))],\end{aligned}$$

where $(a,b,c)_\Omega \in P_{s+}$ and $(x,y,z)_{\Delta^\vee} \in F^{s+}$.

The explicit form of the grid F^{s+}_M for any natural M is

$$F^{s+}_M = \left\{ \frac{a}{M}\omega^\vee_1 + \frac{b}{M}\omega^\vee_2 + \frac{c}{M}\omega^\vee_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\, d+a+2b+2c = M \right\}$$
$$\cup \left\{ \frac{a}{M}\omega^\vee_1 + \frac{b+2c}{M}\omega^\vee_2 - \frac{c}{M}\omega^\vee_3 \mid a,b,d \in \mathbb{Z}^+_0,\, c \in \mathbb{N},\, d+a+2b+2c = M \right\}.$$

The explicit form of the grid Λ^{s+}_M for any natural M is

$$\Lambda^{s+}_M = \{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\, d+2a+2b+c = M\}$$
$$\cup \{a\omega_1 + (b+c)\omega_2 - c\omega_3 \mid a,b \in \mathbb{Z}^+_0,\, c,d \in \mathbb{N},\, d+2a+2b+c = M\}.$$

The discrete (2.9)) othogonal relations hold.

Ξ^{l+} – function of B_3

The explicit form of Ξ^{l+} of B_3 is

$$\begin{aligned}
\Xi^{l+}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = 2\,[&\cos(\pi\,(-bx-cx+az+bz+cz-ay-by))+\\
&+\cos(\pi\,(ax+bx+cx-by-cy+cz))+\\
&+\cos(\pi\,(cx-ay-by-cy+ax+bx+bz+cz))+\\
&+\cos(\pi\,(-ax-by-cy+cz))+\cos(\pi\,(bx+az+bz+cz-ay-by))+\\
&+\cos(\pi\,(ay-ax-bx+bz+cz))+\\
&+\cos(\pi\,(bx+cx-cy+az+bz+cz-ay-by))+\\
&+\cos(\pi\,(ax+by+cz))+\cos(\pi\,(-bx-cy+az+bz+cz-ay-by))+\\
&+\cos(\pi\,(-ay-by-cy+ax+bx+bz+cz))+\\
&+\cos(\pi\,(-ax-bx-cx+by+cz))+\cos(\pi\,(-cx+ay-ax-bx+bz+cz))],
\end{aligned}$$

where $(a,b,c)_\Omega \in P_{l+}$ and $(x,y,z)_{\Delta^\vee} \in F^{l+}$.

The explicit form of the grid F^{l+}_M for any natural M is

$$F^{l+}_M = \left\{ \frac{a}{M}\omega^\vee_1 + \frac{b}{M}\omega^\vee_2 + \frac{c}{M}\omega^\vee_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\, d+a+2b+2c = M \right\}$$
$$\cup \left\{ \frac{-a}{M}\omega^\vee_1 + \frac{a+b}{M}\omega^\vee_2 + \frac{c}{M}\omega^\vee_3 \mid c \in \mathbb{Z}^+_0,\, a,b,d \in \mathbb{N},\, d+a+2b+2c = M \right\}.$$

The explicit form of the grid Λ^{l+}_M for any natural M is

$$\Lambda^{l+}_M = \{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\, d+2a+2b+c = M\}$$
$$\cup \{-a\omega_1 + (a+b)\omega_2 + c\omega_3 \mid c,d \in \mathbb{Z}^+_0,\, a,b \in \mathbb{N},\, d+2a+2b+c = M\}.$$

The discrete (2.11) othogonal relations hold.

Ξ^{e-} – function of B_3

The explicit form of Ξ^{e-} of B_3 is

$$\begin{aligned}\Xi^{e-}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {}& e^{-2i\pi(-ax-2bx-cx+by+cz)} - e^{2i\pi(bx-2az-2bz-cz+ay)} - e^{2i\pi(ay+by+cy-ax-cz)} +\\
&+ e^{2i\pi(-cx+ay+2by+cy-ax-bx-2bz-cz)} + e^{-2i\pi(-bx-2az-2bz-cz+ay+by)} +\\
&+ e^{-2i\pi(ay+2by+cy-ax-bx-2bz-cz)} - e^{2i\pi(-ax-bx-2bz-cz+by)} -\\
&- e^{2i\pi(2by+cy-bx-2az-2bz-cz+ay)} - e^{2i\pi(ax+bx+cy-2bz-cz+by)} -\\
&- e^{-2i\pi(-ax-bx-cx+cy-2bz-cz+by)} + e^{-2i\pi(ax+by+cy-cz)} - e^{-2i\pi(ay+by-ax+cz)} +\\
&+ e^{-2i\pi(-cx+ay-ax-bx+2bz+cz)} + e^{2i\pi(ax+by+cz)} - e^{-2i\pi(cx+bx-2az-2bz-cz+ay)} +\\
&+ e^{2i\pi(-bx-cx+cy-2az-2bz-cz+ay+by)} + e^{2i\pi(ay-ax-bx+2bz+cz)} +\\
&+ e^{2i\pi(-ax-2bx-cx+by+cy-cz)} - e^{2i\pi(-2bx-cx+ay+by-ax+cz)} -\\
&- e^{-2i\pi(-cx+2by+cy-bx-2az-2bz-cz+ay)} - e^{-2i\pi(ax+bx+cx-2bz-cz+by)} -\\
&- e^{-2i\pi(-2bx-cx+ay+by+cy-ax-cz)} + e^{2i\pi(bx+cx-2az-2bz-cz+ay+by)} +\\
&+ e^{-2i\pi(bx+cy-2az-2bz-cz+ay+by)},\end{aligned}$$

where $(a,b,c)_\Omega \in P_{e-}$ and $(x,y,z)_{\Delta^\vee} \in F^{e-}$.

The explicit form of the grid F^{e-}_M for any natural M is

$$\begin{aligned}F^{e-}_M = &\left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}^+_0,\, (a+b+c)\cdot(c+d) \neq 0,\, d+a+2b+2c = M \right\}\\
\cup &\left\{ \frac{a}{M}\omega_1^\vee + \frac{b+2c}{M}\omega_2^\vee - \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M \right\}.\end{aligned}$$

The explicit form of the grid Λ^{e-}_M for any natural M is

$$\begin{aligned}\Lambda^{e-}_M = &\{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\, (a+b+c)\cdot(c+d) \neq 0,\, d+2a+2b+c = M\}\\
\cup &\{a\omega_1 + (b+c)\omega_2 - c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+2a+2b+c = M\}.\end{aligned}$$

The discrete (2.14) othogonal relations hold.

Ξ^{s-} – function of B_3

The explicit form of Ξ^{s-} of B_3 is

$$\begin{aligned}\Xi^{s-}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {}& e^{-2i\pi(-ax-2bx-cx+by+cz)} - e^{2i\pi(ay+by-ax+cz)} - e^{-2i\pi(ay+by+cy-ax-cz)} -\\
&- e^{2i\pi(cx+bx-2az-2bz-cz+ay)} + e^{2i\pi(-cx+ay+2by+cy-ax-bx-2bz-cz)} +\\
&+ e^{-2i\pi(-bx-2az-2bz-cz+ay+by)} + e^{-2i\pi(ay+2by+cy-ax-bx-2bz-cz)} -\\
&- e^{-2i\pi(2by+cy-bx-2az-2bz-cz+ay)} - e^{2i\pi(-2bx-cx+ay+by+cy-ax-cz)} +\\
&+ e^{-2i\pi(ax+by+cy-cz)} - e^{2i\pi(ax+bx+cx-2bz-cz+by)} + e^{-2i\pi(-cx+ay-ax-bx+2bz+cz)} +\\
&+ e^{2i\pi(ax+by+cz)} - e^{-2i\pi(-2bx-cx+ay+by-ax+cz)} - e^{-2i\pi(bx-2az-2bz-cz+ay)} +\\
&+ e^{2i\pi(-bx-cx+cy-2az-2bz-cz+ay+by)} + e^{2i\pi(ay-ax-bx+2bz+cz)} + e^{2i\pi(-ax-2bx-cx+by+cy-cz)} -\\
&- e^{2i\pi(-ax-bx-cx+cy-2bz-cz+by)} - e^{-2i\pi(-ax-bx-2bz-cz+by)} -\\
&- e^{2i\pi(-cx+2by+cy-bx-2az-2bz-cz+ay)} + e^{2i\pi(bx+cx-2az-2bz-cz+ay+by)} +\\
&+ e^{-2i\pi(bx+cy-2az-2bz-cz+ay+by)} - e^{-2i\pi(ax+bx+cy-2bz-cz+by)},\end{aligned}$$

where $(a,b,c)_\Omega \in P_{s-}$ and $(x,y,z)_{\Delta^\vee} \in F^{s-}$.

The explicit form of the grid F^{s-}_M for any natural M is

$$
\begin{aligned}
F^{s-}_M = &\left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}^+_0,\, (a+b)\cdot d \neq 0,\, d+a+2b+2c = M \right\} \\
\cup &\left\{ \frac{a}{M}\omega_1^\vee + \frac{b+2c}{M}\omega_2^\vee - \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M \right\}.
\end{aligned}
$$

The explicit form of the grid Λ^{s-}_M for any natural M is

$$
\begin{aligned}
\Lambda^{s-}_M = &\{ a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\, (a+b) \neq 0,\, d+2a+2b+c = M \} \\
\cup &\{ a\omega_1 + (b+c)\omega_2 - c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+2a+2b+c = M \}.
\end{aligned}
$$

The discrete (2.16) othogonal relations hold.

Ξ^{l-} – function of B_3

The explicit form of Ξ^{l-} of B_3 is

$$
\begin{aligned}
\Xi^{l-}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = 2\mathrm{i}[&-\sin(2\pi\,(ax+2bx+cx-by-cy+cz))- \\
&-\sin(2\pi\,(-bx-2az-2bz-cz+ay+by))+\sin(2\pi\,(ax+2bx+cx-by-cz))- \\
&-\sin(2\pi\,(bx+cy-2az-2bz-cz+ay+by))+\sin(2\pi\,(ax+by+cz))+ \\
&+\sin(2\pi\,(-cx+ay+2by+cy-ax-bx-2bz-cz))+ \\
&+\sin(2\pi\,(-bx-cx+cy-2az-2bz-cz+ay+by))- \\
&-\sin(2\pi\,(-cx+ay-ax-bx+2bz+cz))-\sin(2\pi\,(ax+by+cy-cz))+ \\
&+\sin(2\pi\,(bx+cx-2az-2z,b-cz+ay+by))+\sin(2\pi\,(ay-ax-bx+2bz+cz))- \\
&-\sin(2\pi\,(ay+2by+cy-ax-bx-2bz-cz))],
\end{aligned}
$$

where $(a,b,c)_\Omega \in P_{l-}$ and $(x,y,z)_{\Delta^\vee} \in F^{l-}$.
The explicit form of the grid F^{l-}_M for any natural M is

$$
\begin{aligned}
F^{l-}_M = &\left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}^+_0,\, c \neq 0,\, d+a+2b+2c = M \right\} \\
\cup &\left\{ \frac{-a}{M}\omega_1^\vee + \frac{a+b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M \right\}.
\end{aligned}
$$

The explicit form of the grid Λ^{l-}_M for any natural M is

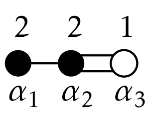
$$
\begin{aligned}
\Lambda^{l-}_M = &\{ a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\, c+d \neq 0,\, d+2a+2b+c = M \} \\
\cup &\{ -a\omega_1 + (a+b)\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+2a+2b+c = M \}.
\end{aligned}
$$

The discrete (2.18) othogonal relations hold.

4.2 Weyl group and Affine Weyl group of type C_3

4.2.1 Basic properties of the Weyl group of type C_3

The Coxeter–Dynkin diagram (numbers are marks m_i) and Cartan matrix with its inverse are the following



$$
C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \qquad C^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}.
$$

Hence α_3 is the long root and α_1 , α_2 are short roots. The relative lengths of the simple roots are set as $\langle\alpha_1,\alpha_1\rangle = \langle\alpha_2,\alpha_2\rangle = 1$ and $\langle\alpha_3,\alpha_3\rangle = 2$. The bases of simple roots and fundamental weights are thus related by

$$\begin{aligned}
&\alpha_1 = 2\omega_1 - \omega_2, && \omega_1 = \alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3, && \alpha_1^\vee = 2\alpha_1, && \omega_1^\vee = 2\omega_1,\\
&\alpha_2 = -\omega_1 + 2\omega_2 - \omega_3, && \omega_2 = \alpha_1 + 2\alpha_2 + \alpha_3, && \alpha_2^\vee = 2\alpha_2, && \omega_2^\vee = 2\omega_2,\\
&\alpha_3 = -2\omega_1 + 2\omega_3, && \omega_3 = \alpha_1 + 2\alpha_2 + \tfrac{3}{2}\alpha_3, && \alpha_3^\vee = \alpha_3, && \omega_3^\vee = \omega_3.
\end{aligned}$$

The four bases are

$$\Delta = \{\alpha_1,\alpha_2,\alpha_3\}, \quad \Delta^\vee = \{\alpha_1^\vee,\alpha_2^\vee,\alpha_3^\vee\}$$
$$\Omega = \{\omega_1,\omega_2,\omega_3\}, \quad \Omega^\vee = \{\omega_1^\vee,\omega_2^\vee,\omega_3^\vee\}.$$

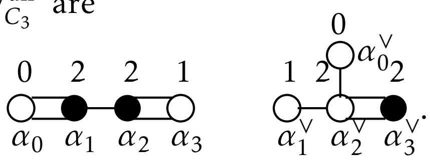
In the orthonormal basis these vectors have the form

$$\begin{aligned}
&\alpha_1 = (\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}},0) = \tfrac{1}{\sqrt{2}}(e_1-e_2), && \omega_1 = (\tfrac{1}{\sqrt{2}},0,0) = \tfrac{1}{\sqrt{2}}e_1, && \omega_1^\vee = (\sqrt{2},0,0),\\
&\alpha_2 = (0,\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}}) = \tfrac{1}{\sqrt{2}}(e_2-e_3), && \omega_2 = (\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}},0) = \tfrac{1}{\sqrt{2}}(e_1+e_2), && \omega_2^\vee = (\sqrt{2},\sqrt{2},0),\\
&\alpha_3 = (0,0,\sqrt{2}) = \sqrt{2}e_3, && \omega_3 = (\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}) = \tfrac{1}{\sqrt{2}}(e_1+e_2+e_3), && \omega_3^\vee = (\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}).
\end{aligned}$$

The highest root of $W_{C_3}^{\mathrm{aff}}$ and the highest dual root $\widehat{W}_{C_3}^{\mathrm{aff}}$ are given by these formulas

$$\xi = 2\alpha_1 + 2\alpha_2 + \alpha_3 = (\sqrt{2},0,0), \quad \eta = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee = (\sqrt{2},\sqrt{2},0).$$

The extended Coxeter–Dynkin diagrams (numbers above vertices are the marks m_i respectively marks $m_i^\vee$) of $W_{C_3}^{\mathrm{aff}}$ and $\widehat{W}_{C_3}^{\mathrm{aff}}$ are



4.2.2 E-functions of C_3

In the explicit form of E-functions, the vector from the weight lattice will be in Ω basis and the variable will be in $\Delta^\vee$ basis, denoted $(a,b,c)_\Omega$ and $(x,y,z)_{\Delta^\vee}$

For $M = 5$, the grid F_5 has 20 elements and its explicit form is

$$\begin{aligned}
F_5 = \Big\{ &\tfrac{1}{5}(0,0,0,5), \tfrac{1}{5}(0,0,1,4), \tfrac{1}{5}(0,0,2,3), \tfrac{1}{5}(0,0,3,2), \tfrac{1}{5}(0,0,4,1), \tfrac{1}{5}(0,0,5,0), \tfrac{1}{5}(0,1,0,3),\\
&\tfrac{1}{5}(0,1,1,2), \tfrac{1}{5}(0,1,2,1), \tfrac{1}{5}(0,1,3,0), \tfrac{1}{5}(0,2,0,1), \tfrac{1}{5}(0,2,1,0), \tfrac{1}{5}(1,0,0,3), \tfrac{1}{5}(1,0,1,2),\\
&\tfrac{1}{5}(1,0,2,1), \tfrac{1}{5}(1,0,3,0), \tfrac{1}{5}(1,1,0,1), \tfrac{1}{5}(1,1,1,0), \tfrac{1}{5}(2,0,0,1), \tfrac{1}{5}(2,0,1,0)\Big\}.
\end{aligned}$$

For $M = 5$, the grid Λ_5 has 20 elements and its explicit form is

$$\begin{aligned}
\Lambda_5 = \{&(0,0,0,5), (0,0,1,3), (0,0,2,1), (0,1,0,3), (0,1,1,1), (0,2,0,1), (1,0,0,4),\\
&(1,0,1,2), (1,0,2,0), (1,1,0,2), (1,1,1,0), (1,2,0,0), (2,0,0,3), (2,0,1,1),\\
&(2,1,0,1), (3,0,0,2), (3,0,1,0), (3,1,0,0), (4,0,0,1), (5,0,0,0)\}.
\end{aligned}$$

Table 4.2: *The coefficients for discrete calculus of C_3. Positive values of $u_0, u_1, u_2, u_3 > 0$ and $t_0, t_1, t_2, t_3 > 0$ are assumed.*

$\chi \in F_M(C_3)$	$\epsilon(\chi)$	$\epsilon^e(\chi)$	$\epsilon^s(\chi)$	$\epsilon^l(\chi)$
$[u_0, u_1, u_2, u_3]$	48	24	24	24
$[u_0, u_1, u_2, 0]$	24	24	12	12
$[u_0, u_1, 0, u_3]$	24	24	24	12
$[u_0, 0, u_2, u_3]$	24	24	24	24
$[0, u_1, u_2, u_3]$	24	24	12	24
$[u_0, u_1, 0, 0]$	6	6	6	3
$[u_0, 0, u_2, 0]$	12	12	12	12
$[0, u_1, u_2, 0]$	12	12	6	12
$[u_0, 0, 0, u_3]$	8	8	8	8
$[0, u_1, 0, u_3]$	12	12	12	8
$[0, 0, u_2, u_3]$	6	6	6	12
$[u_0, 0, 0, 0]$	1	1	1	1
$[0, 0, u_2, 0]$	3	3	3	3
$[0, u_1, 0, 0]$	3	3	3	3
$[0, 0, 0, u_3]$	1	1	1	1

$\lambda \in \Lambda_M(C_3)$	$h_\lambda^\vee$	$h_\lambda^{e\vee}$	$h_\lambda^{s\vee}$	$h_\lambda^{l\vee}$
$[t_0, t_1, t_2, t_3]$	1	1	1	1
$[t_0, t_1, t_2, 0]$	2	1	1	2
$[t_0, t_1, 0, t_3]$	2	1	2	1
$[t_0, 0, t_2, t_3]$	2	1	2	1
$[0, t_1, t_2, t_3]$	2	1	1	2
$[t_0, t_1, 0, 0]$	8	4	4	4
$[t_0, 0, t_2, 0]$	4	2	4	2
$[0, t_1, t_2, 0]$	4	2	2	4
$[t_0, 0, 0, t_3]$	6	3	6	3
$[0, t_1, 0, t_3]$	6	2	2	2
$[0, 0, t_2, t_3]$	4	4	4	4
$[t_0, 0, 0, 0]$	48	24	24	24
$[0, 0, t_2, 0]$	8	8	8	8
$[0, t_1, 0, 0]$	48	8	8	8
$[0, 0, 0, t_3]$	24	24	24	24

Ξ^{e+} – function of C_3

The explicit form of Ξ^{e+} of C_3 is

$$\begin{aligned}\Xi^{e+}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {} & e^{2i\pi(ax+2bx+2cx-by-bz-2cz)} + e^{2i\pi(ay+by+2cy-ax+az+bz)} + e^{-2i\pi(ax+bx+2cx-2cz-bz+by)} + \\ & + e^{-2i\pi(ay+by-ax+az+bz+2cz)} + e^{-2i\pi(bx+2cx-2cy+az+bz-ay-by)} + e^{2i\pi(bx+2cz+az+bz-ay-by)} + \\ & + e^{2i\pi(2by+2cy-bx-az+ay)} + e^{-2i\pi(bx+2cy-az-bz+ay+by)} + e^{2i\pi(ax+bx+2cx-2cy+bz-by)} + \\ & + e^{-2i\pi(2cx-ay-2by-2cy+ax+bx-az)} + e^{-2i\pi(2bx+2cx-ay-by+ax-az-bz-2cz)} + \\ & + e^{2i\pi(bx+2cx-2cz-az-bz+ay+by)} + e^{-2i\pi(ax+by+2cy+bz)} + e^{-2i\pi(ax+bx+2cz+bz-by)} + \\ & + e^{2i\pi(2cx-2by-2cy+bx+az-ay)} + e^{2i\pi(2cx-ay+ax+bx-az-2bz-2cz)} + \\ & + e^{-2i\pi(ay+2by+2cy-ax-bx+az)} + e^{-2i\pi(-bx+2bz+2cz+az-ay)} + e^{2i\pi(ax+by+bz+2cz)} + \\ & + e^{2i\pi(ay-ax-bx+az+2bz+2cz)} + e^{2i\pi(ax+bx+2cy-bz+by)} + e^{-2i\pi(ax+2bx+2cx-by-2cy-bz)} + \\ & + e^{-2i\pi(2cx+bx-2bz-2cz-az+ay)} + e^{2i\pi(2bx+2cx-ay-by-2cy+ax-az-bz)},\end{aligned}$$

where $(a,b,c)_\Omega \in P_{e+}$ and $(x,y,z)_{\Delta^\vee} \in F^{e+}$.

The explicit form of the grid F_M^{e+} for any natural M is

$$\begin{aligned}F_M^{e+} = & \left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}_0^+,\, d+2a+2b+c = M \right\} \\ & \cup \left\{ \frac{a}{M}\omega_1^\vee + \frac{b+c}{M}\omega_2^\vee - \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+2a+2b+c = M \right\}.\end{aligned}$$

The explicit form of the grid Λ_M^{e+} for any natural M is

$$\begin{aligned}\Lambda_M^{e+} = & \{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}_0^+,\, d+a+2b+2c = M\} \\ & \cup \{a\omega_1 + (b+2c)\omega_2 - c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M\}.\end{aligned}$$

The discrete (2.7) othogonal relations hold.

Ξ^{s+} – function of C_3

The explicit form of Ξ^{s+} of C_3 is

$$\begin{aligned}
\Xi^{s+}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {} & e^{2\mathrm{i}\pi(ax+2bx+2cx-by-bz-2cz)} + e^{2\mathrm{i}\pi(ay+by+2cy-ax+az+bz)} + e^{-2\mathrm{i}\pi(ax+bx+2cx-2cz-bz+by)} + \\
& + e^{-2\mathrm{i}\pi(ay+by-ax+az+bz+2cz)} + e^{-2\mathrm{i}\pi(bx+2cx-2cy+az+bz-ay-by)} + \\
& + e^{2\mathrm{i}\pi(bx+2cz+az+bz-ay-by)} + e^{2\mathrm{i}\pi(2by+2cy-bx-az+ay)} + e^{-2\mathrm{i}\pi(bx+2cy-az-bz+ay+by)} + \\
& + e^{2\mathrm{i}\pi(ax+bx+2cx-2cy+bz-by)} + e^{-2\mathrm{i}\pi(2cx-ay-2by-2cy+ax+bx-az)} + \\
& + e^{-2\mathrm{i}\pi(2bx+2cx-ay-by+ax-az-bz-2cz)} + e^{2\mathrm{i}\pi(bx+2cx-2cz-az-bz+ay+by)} + \\
& + e^{-2\mathrm{i}\pi(ax+by+2cy+bz)} + e^{-2\mathrm{i}\pi(ax+bx+2cz+bz-by)} + e^{2\mathrm{i}\pi(2cx-2by-2cy+bx+az-ay)} + \\
& + e^{2\mathrm{i}\pi(2cx-ay+ax+bx-az-2bz-2cz)} + e^{-2\mathrm{i}\pi(ay+2by+2cy-ax-bx+az)} + \\
& + e^{-2\mathrm{i}\pi(-bx+2bz+2cz+az-ay)} + e^{2\mathrm{i}\pi(ax+by+bz+2cz)} + e^{2\mathrm{i}\pi(ay-ax-bx+az+2bz+2cz)} + \\
& + e^{2\mathrm{i}\pi(ax+bx+2cy-bz+by)} + e^{-2\mathrm{i}\pi(ax+2bx+2cx-by-2cy-bz)} + e^{-2\mathrm{i}\pi(2cx+bx-2bz-2cz-az+ay)} + \\
& + e^{2\mathrm{i}\pi(2bx+2cx-ay-by-2cy+ax-az-bz)},
\end{aligned}$$

where $(a,b,c)_\Omega \in P_{s+}$ and $(x,y,z)_{\Delta^\vee} \in F^{s+}$.

The explicit form of the grid F^{s+}_M for any natural M is

$$\begin{aligned}
F^{s+}_M = {} & \left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}^+_0,\ d+a+2b+2c = M \right\} \\
& \cup \left\{ \frac{-a}{M}\omega_1^\vee + \frac{a+b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid c,d \in \mathbb{Z}^+_0,\ a,b \in \mathbb{N},\ d+a+2b+2c = M \right\}.
\end{aligned}$$

The explicit form of the grid Λ^{s+}_M for any natural M is

$$\begin{aligned}
\Lambda^{s+}_M = {} & \{ a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}^+_0,\ d+a+2b+2c = M \} \\
& \cup \{ -a\omega_1 + (a+b)\omega_2 + c\omega_3 \mid c \in \mathbb{Z}^+_0,\ a,b,d \in \mathbb{N},\ d+a+2b+2c = M \}.
\end{aligned}$$

The discrete (2.9)) othogonal relations hold.

Ξ^{l+} – function of C_3

The explicit form of Ξ^{l+} of C_3 is

$$\begin{aligned}
\Xi^{l+}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {} & e^{2\mathrm{i}\pi(ax+2bx+2cx-by-bz-2cz)} + e^{2\mathrm{i}\pi(ay+by+2cy-ax+az+bz)} + e^{-2\mathrm{i}\pi(ax+bx+2cx-2cz-bz+by)} + \\
& + e^{-2\mathrm{i}\pi(ay+by-ax+az+bz+2cz)} + e^{-2\mathrm{i}\pi(bx+2cx-2cy+az+bz-ay-by)} + e^{2\mathrm{i}\pi(bx+2cz+az+bz-ay-by)} + \\
& + e^{2\mathrm{i}\pi(2by+2cy-bx-az+ay)} + e^{-2\mathrm{i}\pi(bx+2cy-az-bz+ay+by)} + e^{2\mathrm{i}\pi(ax+bx+2cx-2cy+bz-by)} + \\
& + e^{-2\mathrm{i}\pi(2cx-ay-2by-2cy+ax+bx-az)} + e^{-2\mathrm{i}\pi(2bx+2cx-ay-by+ax-az-bz-2cz)} + \\
& + e^{2\mathrm{i}\pi(bx+2cx-2cz-az-bz+ay+by)} + e^{-2\mathrm{i}\pi(ax+by+2cy+bz)} + e^{-2\mathrm{i}\pi(ax+bx+2cz+bz-by)} + \\
& + e^{2\mathrm{i}\pi(2cx-2by-2cy+bx+az-ay)} + e^{2\mathrm{i}\pi(2cx-ay+ax+bx-az-2bz-2cz)} + \\
& + e^{-2\mathrm{i}\pi(ay+2by+2cy-ax-bx+az)} + e^{-2\mathrm{i}\pi(-bx+2bz+2cz+az-ay)} + e^{2\mathrm{i}\pi(ax+by+bz+2cz)} + \\
& + e^{2\mathrm{i}\pi(ay-ax-bx+az+2bz+2cz)} + e^{2\mathrm{i}\pi(ax+bx+2cy-bz+by)} + e^{-2\mathrm{i}\pi(ax+2bx+2cx-by-2cy-bz)} + \\
& + e^{-2\mathrm{i}\pi(2cx+bx-2bz-2cz-az+ay)} + e^{2\mathrm{i}\pi(2bx+2cx-ay-by-2cy+ax-az-bz)},
\end{aligned}$$

where $(a,b,c)_\Omega \in P_{l+}$ and $(x,y,z)_{\Delta^\vee} \in F^{l+}$.

The explicit form of the grid F^{l+}_M for any natural M is

$$\begin{aligned}
F^{l+}_M = {} & \left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}^+_0,\ d+2a+2b+c = M \right\} \\
& \cup \left\{ \frac{a}{M}\omega_1^\vee + \frac{b+c}{M}\omega_2^\vee - \frac{c}{M}\omega_3^\vee \mid c,d \in \mathbb{N},\ a,b \in \mathbb{Z}^+_0,\ d+2a+2b+c = M \right\}.
\end{aligned}$$

The explicit form of the grid Λ_M^{l+} for any natural M is

$$\begin{aligned}\Lambda_M^{l+} &= \{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}_0^+,\, d+a+2b+2c = M\}\\ &\cup \{a\omega_1 + (b+2c)\omega_2 - c\omega_3 \mid c \in \mathbb{N},\, a,b,d \in \mathbb{Z}_0^+,\, d+a+2b+2c = M\}.\end{aligned}$$

The discrete (2.11) othogonal relations hold.

Ξ^{e-} – function of C_3

The explicit form of Ξ^{e-} of C_3 is

$$\begin{aligned}\Xi^{e-}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {} & e^{-2i\pi(-ax-2bx-2cx+by+bz+2cz)} - e^{2i\pi(ay+by+2cy-ax+az+bz)} - e^{-2i\pi(ax+bx+2cx-2cz-bz+by)} - \\ & - e^{-2i\pi(ay+by-ax+az+bz+2cz)} + e^{2i\pi(-bx-2cx+2cy-az-bz+ay+by)} + \\ & + e^{-2i\pi(-bx-2cz-az-bz+ay+by)} - e^{2i\pi(2by+2cy-bx-az+ay)} + e^{-2i\pi(bx+2cy-az-bz+ay+by)} - \\ & - e^{-2i\pi(-ax-bx-2cx+2cy-bz+by)} + e^{2i\pi(-2cx+ay+2by+2cy-ax-bx+az)} - \\ & - e^{2i\pi(-2bx-2cx+ay+by-ax+az+bz+2cz)} + e^{2i\pi(bx+2cx-2cz-az-bz+ay+by)} + \\ & + e^{-2i\pi(ax+by+2cy+bz)} - e^{2i\pi(-ax-bx-2cz-bz+by)} - e^{-2i\pi(-2cx+2by+2cy-bx-az+ay)} + \\ & + e^{-2i\pi(-2cx+ay-ax-bx+az+2bz+2cz)} + e^{-2i\pi(ay+2by+2cy-ax-bx+az)} - \\ & - e^{2i\pi(bx-2bz-2cz-az+ay)} + e^{2i\pi(ax+by+bz+2cz)} + e^{2i\pi(ay-ax-bx+az+2bz+2cz)} - \\ & - e^{2i\pi(ax+bx+2cy-bz+by)} + e^{2i\pi(-ax-2bx-2cx+by+2cy+bz)} - \\ & - e^{-2i\pi(2cx+bx-2bz-2cz-az+ay)} - e^{-2i\pi(-2bx-2cx+ay+by+2cy-ax+az+bz)},\end{aligned}$$

where $(a,b,c)_\Omega \in P_{e-}$ and $(x,y,z)_{\Delta^\vee} \in F^{e-}$.

The explicit form of the grid F_M^{e-} for any natural M is

$$\begin{aligned}F_M^{e-} = {} & \left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}_0^+,\, ab+cd \neq 0,\, d+2a+2b+c = M \right\}\\ & \cup \left\{ \frac{a}{M}\omega_1^\vee + \frac{b+c}{M}\omega_2^\vee - \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+2a+2b+c = M \right\}.\end{aligned}$$

The explicit form of the grid Λ_M^{e-} for any natural M is

$$\begin{aligned}\Lambda_M^{e-} &= \{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}_0^+,\, ab+cd \neq 0,\, d+a+2b+2c = M\}\\ &\cup \{a\omega_1 + (b+2c)\omega_2 - c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M\}.\end{aligned}$$

The discrete (2.14) othogonal relations hold.

Ξ^{s-} – function of C_3

The explicit form of Ξ^{s-} of C_3 is

$$\begin{aligned}\Xi^{s-}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = {} & e^{-2i\pi(ax+2bx+2cx-by-bz-2cz)} - e^{-2i\pi(ay+by+2cy-ax+az+bz)} - \\ & - e^{2i\pi(ax+bx+2cx-2cz-bz+by)} - e^{2i\pi(ay+by-ax+az+bz+2cz)} + e^{2i\pi(bx+2cx-2cy+az+bz-ay-by)} + \\ & + e^{-2i\pi(bx+2cz+az+bz-ay-by)} - e^{-2i\pi(2by+2cy-bx-az+ay)} + +e^{2i\pi(bx+2cy-az-bz+ay+by)} - \\ & - e^{-2i\pi(ax+bx+2cx-2cy+bz-by)} + e^{2i\pi(2cx-ay-2by-2cy+ax+bx-az)} - \\ & - e^{2i\pi(2bx+2cx-ay-by+ax-az-bz-2cz)} + e^{-2i\pi(bx+2cx-2cz-az-bz+ay+by)} + e^{2i\pi(ax+by+2cy+bz)} - \\ & - e^{2i\pi(ax+bx+2cz+bz-by)} - e^{-2i\pi(2cx-2by-2cy+bx+az-ay)} + e^{-2i\pi(2cx-ay+ax+bx-az-2bz-2cz)} + \\ & + e^{2i\pi(ay+2by+2cy-ax-bx+az)} - e^{2i\pi(-bx+2bz+2cz+az-ay)} + e^{-2i\pi(ax+by+bz+2cz)} + \\ & + e^{-2i\pi(ay-ax-bx+az+2bz+2cz)} - e^{-2i\pi(ax+bx+2cy-bz+by)} + e^{2i\pi(ax+2bx+2cx-by-2cy-bz)} - \\ & - e^{2i\pi(2cx+bx-2bz-2cz-az+ay)} - e^{-2i\pi(2bx+2cx-ay-by-2cy+ax-az-bz)},\end{aligned}$$

where $(a,b,c)_\Omega \in P_{s-}$ and $(x,y,z)_{\Delta^\vee} \in F^{s-}$.
The explicit form of the grid F_M^{s-} for any natural M is

$$F_M^{s-} = \left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}_0^+,\, cd \neq 0,\, d+a+2b+2c = M \right\}$$
$$\cup \left\{ \frac{-a}{M}\omega_1^\vee + \frac{a+b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M \right\}.$$

The explicit form of the grid Λ_M^{s-} for any natural M is

$$\Lambda_M^{s-} = \{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}_0^+,\, c \neq 0,\, d+a+2b+2c = M\}$$
$$\cup \{-a\omega_1 + (a+b)\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M\}.$$

The discrete (2.16) othogonal relations hold.

Ξ^{l-} – function of C_3

The explicit form of Ξ^{s-} of C_3 is

$$\begin{aligned}
\Xi^{l-}_{(a,b,c)_\Omega}(x,y,z)_{\Delta^\vee} = 2\mathrm{i}[&-\sin(2\pi(-2cx+ay-ax-bx+az+2bz+2cz))-\\
&-\sin(2\pi(bx+2cy-az-bz+ay+by))+\\
&+\sin(2\pi(-2cx+ay+2by+2cy-ax-bx+az))-\\
&-\sin(2\pi(ay+2by+2cy-ax-bx+az))+\sin(2\pi(-ax-2bx-2cx+by+2cy+bz))+\\
&+\sin(2\pi(bx+2cz+az+bz-ay-by))-\sin(2\pi(-ax-2bx-2cx+by+bz+2cz))-\\
&-\sin(2\pi(ax+by+2cy+bz))-\sin(2\pi(-bx-2cx+2cz+az+bz-ay-by))+\\
&+\sin(2\pi(ay-ax-bx+az+2bz+2cz))+\sin(2\pi(ax+by+bz+2cz))+\\
&+\sin(2\pi(-bx-2cx+2cy-az-bz+ay+by))],
\end{aligned}$$

where $(a,b,c)_\Omega \in P_{l-}$ and $(x,y,z)_{\Delta^\vee} \in F^{l-}$.
The explicit form of the grid F_M^{l-} for any natural M is

$$F_M^{l-} = \left\{ \frac{a}{M}\omega_1^\vee + \frac{b}{M}\omega_2^\vee + \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{Z}_0^+,\, ab \neq 0,\, d+2a+2b+c = M \right\}$$
$$\cup \left\{ \frac{a}{M}\omega_1^\vee + \frac{b+c}{M}\omega_2^\vee - \frac{c}{M}\omega_3^\vee \mid a,b,c,d \in \mathbb{N},\, d+2a+2b+c = M \right\}.$$

The explicit form of the grid Λ_M^{l+} for any natural M is

$$\Lambda_M^{l-} = \{a\omega_1 + b\omega_2 + c\omega_3 \mid a,b,c,d \in \mathbb{Z}_0^+,\, abd \neq 0,\, d+a+2b+2c = M\}$$
$$\cup \{a\omega_1 + (b+2c)\omega_2 - c\omega_3 \mid a,b,c,d \in \mathbb{N},\, d+a+2b+2c = M\}.$$

The discrete (2.18) othogonal relations hold.

Conclusion

The properties of E-functions of rank two have been already proven in [1]. In the present thesis, we succeeded in the verification of the useful properties of E-functions of rank three generalized from E-functions of rank two, especially discrete orthogonality with its utilization in data processing.

In addition, the present thesis raises new questions. First of all, it is unknown if the properties of E-functions are valid for any rank in the same form as stated in chapter two, especially taking the case of the special Weyl group of type F_4. Another matter which requires further study are the properties of E-functions indexed by a general point in $\mathbb{R}^n$ (not from the weight lattice). Last but not least, it would be beneficial if we would know under which conditions do the series of functions $\{g_M\}_{M=1}^{\infty}$, for any type of E-function, converge to the functions g.

Potentially, the E-functions can be applied in data processing (image recognition). This may help in data acqusition and data processing in physics (e. g. crystallography), medicine (e. g. magnetic resonance imaging, MRI) and informatics (e. g. data compression and data hiding).

Bibliography

[1] L. Háková, J. Hrivnák, J. Patera *Six types of E-functions of Lie group O(5) and G(2)* J. Phys. A: Math. Theor. **43** 165206 (2010) (arXiv:1202.5031).

[2] J. Hrivnák, J. Patera, *On discretization of tori of compact simple Lie groups*, J. Phys. A: Math. Theor. **42** 385208 (2009).

[3] J. Hrivnák, J. Patera, *On E-discretization of tori of compact simple Lie groups* J. Phys. A: Math. Theor. **43** 165206 (2010).

[4] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, **29** (1990) Cambridge University Press, Cambridge.

[5] A. U. Klimyk, J. Patera, *Antisymmetric Orbit Functions* SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) **3** (2007), 023, 83 pages, math-ph/0702040.

[6] A. U. Klimyk, J. Patera, *E−orbit functions*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) **4** (2008), 002, 57 pages; (arXiv:0801.0822).

[7] R. V. Moody, L. Motlochová, J. Patera, *New families of Weyl group orbit functions*, (arXiv:1202.4415).

[8] J. Patera, *Compact simple Lie goups and their C−,S−,E− transforms* SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) **1** (2005) 025, 6 pages, math-ph/0512029.

[9] R. V. Moody, J. Patera, *Orthogonality within the families of C−, S−, and E−functions of any compact semisimple Lie group*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) **2** (2006) 076, 14 pages, math-ph/0611020.

[10] M. Nesterenko, J. Patera, *Three-dimensional C-, S- and E-transforms* J. Phys. A: Math. Theor. **41** (2008) 475205 (arXiv:0805.3731).