CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering
Department of Physics

DIPLOMA THESIS

Věrný přenos informace s poruchami

Perfect state transfer in the presence of perturbations

Author: Bc. Antonín Hoskovec
Supervisor: Prof. Ing. Igor Jex, DrSc.
Consultant: Aurél Gábris, Ph.D.
Academic year: 2011/2012

Prohlášení

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

Nemám závažný důvod proti užití tohoto školního díla ve smyslu § 60 Zákona č.121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

Declaration

I declare that I wrote my diploma thesis independently, and that I have not made use of any aid (literature, projects, SW etc.) other than those acknowledged.

I agree with the usage of this thesis in the sense of the § 60 Act 121/2000 (Copyright Act of Czech Republic).

V Praze dne 4. května 2012 .....

Bc. Antonín Hoskovec

Acknowledgments

I would like to gratefully acknowledge the support from Prof. Ing. Igor Jex, DrSc. and Aurél Gábris, Ph.D. over the past years, the kind understanding for silly mistakes I made, the never ending opportunities for consultations and the immense patience. I truly feel lucky that I was ever able to work with them on the topics investigated, an opportunity not available to many. It was them who introduced me into the fundamental research of quantum computation, they supervised my bachelor's thesis too. I also wish to express my gratitude towards Dr. Georgios M. Nikolopoulos, whom I worked together with on some of the research presented here (namely part IV).

Název práce: Věrný přenos informace s poruchami

Autor: Bc. Antonín Hoskovec

Obor: Matematické inženýrství / Matematická fyzika

Druh práce: Diplomová práce

Vedoucí práce: Prof. Ing. Igor Jex, DrSc., Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze

Konzultant: Aurél Gábris, Ph.D., Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze

Abstrakt: Teoretický úvod obsahuje základní přehled kvantových výpočtů, od definování základních požadavků, přes metody na jejich splnění až po praktické návrhy na jejich uskutečnění. Vlastní výzkum se pak týká přenosu informace, kdy jsou nejprve diskutovány možnosti uplatnění známých metod na přenos informace na uvězněných iontech a později jsou v detailu rozebrány poruchy způsobené ohyby lineárních řetízků qubitů. Ohyby jsou očekávatelné poruchy při manipulaci s informací ve více dimenzích. V závěru je pak popsána navrhovaná metoda kompenzace za ztráty způsobené ohyby. Poruchy byly prozkoumány numerickými metodami.

Klíčová slova: kvantové počítače, kvantová informace, přenos stavu, kvantové sítě s poruchami.

Title: Perfect state transfer in the presence of perturbations

Author: Bc. Antonín Hoskovec

Abstract: The theoretical introduction part contains basic overview of quantum computation, from defining the fundamental criteria, over the methods of fulfilling them, to practical designs of quantum computers. The original research is about the state transfer, where first application of known methods for state transfer on trapped ions is discussed and then follows the detailed investigation of perturbations caused by bendings of linear qubit chains. The bendings are expected defects when manipulating with the information in more than one dimension. The final sections are devoted to description of proposed method of compensation for the losses from bendings. The perturbations were treated numerically.

Key words: quantum computation, quantum information, state transfer, perturbed quantum networks.

Contents

Part I
Basics of quantum computation and information

1 Introduction

Quantum computation is a term for many growing branches of science, born in 1982 in publications of Richard Feynman [1–3]. The main motivation why quantum computers are so thoroughly investigated, both theoretically and experimentally, is the impact building a quantum computer is expected to have. Impact on perhaps every field of science. In 1994 Peter Shor discovered the famous Shor's algorithm for factorization of integers in polynomial time on quantum computers. It is exponentially more efficient algorithm then the best known classical algorithm - the general number field sieve. Shor's protocol is a beautiful example of efficient quantum computation that would forever change cryptography. The possible benefits do not stop here. Let us mention the quantum encryption algorithms, originating 1984 in articles by Charles H. Bennett and Gilles Brassard [4]. More to the point we should call these the quantum key distribution algorithms for secure production and sharing of a secret key that can be used for encryption. From the nature of quantum mechanics it is impossible to eavesdrop such a communication, the communicating parties should be able to detect it. These algorithms are being brought to real life experiments in the present. Quantum key distribution has been presented to work at 1Mbit/s over the distance of 20km of optical fiber. And even much longer distances up to approximately 150km have been achieved by some research organizations. The progress in this field went as far as to commercial companies getting involved in the development.

In 1996 Seth Lloyd has shown that any quantum computer could be used for efficient simulation of any quantum system, even for example DNA. The methods used nowadays for decrypting DNA sequences and their effects require large computational power and many times rely on databases of known sequences for comparison of the chemical constitution [5]. The quantum simulations would uncover the effects much faster. The simulations might help with further miniaturization of microprocessors. Today's companies are capable of producing microprocessors fabricated with characteristic dimensions as low as 45nm. On this level of manipulation with matter quantum behavior is usually considered disturbing, because it allows electrons to jump unpredictably.

The idea of quantum computation however beautiful would not suffice without experimental implementation. This thesis gives several examples of quantum computer candidates, which are being investigated by different teams. Let us mention the recent development in the trapped ions field, which seems to have the biggest potential. Last year a 14 qubit system was presented [6] together with open-system quantum simulator [7] and universal digital quantum simulation [8] all working with trapped ions. An article about simulation of Ising model on hundreds of qubits under the label of trapped ions has recently been published [9].

By one of the DiVincenzo criteria [10] an efficient way of performing transfer of a quantum information is needed for building a functioning quantum computer. It is a natural consequence of the need to read, write and manipulate the information before, during and after the computation as well as the need of different parts of the quantum computer to communicate. For example we could design a quantum processor that would be driven by states of registers similarly to how some operations are performed on classical computers. Then the perfect state transfer (PST) would be needed to bring the information from the register to the computational unit. Quantum networks can be considered a means of communication even between different quantum computers. The problem of PST has been first addressed in 2003 by Sougato Bose [11] and in 2004 by Georgios M. Nikolopoulos, David Petrosyan and Peter Lambropoulos [12]. These two papers stimulated quite much activity in the field of perfect state transfer [13–17].

We aim to present here the basics of quantum computation and in some detail describe the necessary conditions on quantum systems to be considered suitable for quantum computation, then portray some

of the quantum computer candidates to really give understanding of why PST is needed and how it is incorporated into the process of quantum computation. Then we would like to present recent ideas about engineering systems capable of PST and give a description of our original results concerning PST on trapped ions and two dimensional arrangements of quantum networks, which have not been investigated in the literature before.

1.1 Notation

We will use the standard Dirac notation and formulations of quantum mechanics introduced by Paul Dirac and John von Neumann [18]. When we will be working with a quantum system, the Hilbert space of all possible states will be denoted by $\mathscr{H}$ with vectors denoted by *kets* $|\psi\rangle \in \mathscr{H}$ and corresponding dual vectors denoted by *bras* $\langle\psi| \in \mathscr{H}^*$. A concrete Hilbert space will be always specified in corresponding section, it can be for example the space $\mathbb{C}^2$ for spin states or the L^2 for the space of wave functions.

One important fact to mention is the correspondence of a vector from $\mathscr{H}$ to an actual physical state of the system in this notation. The vectors $|\psi\rangle$ and

$$\alpha \cdot |\psi\rangle\,,\ \alpha \in \mathbb{C},\ \alpha \neq 0, \tag{1.1}$$

are considered to describe the same physical state. Hence it is sufficient to consider only vectors normalized to 1. We will always be using normalized vectors to represent a state, which simplifies notations for values of observables, mean values, different probabilities etc.

If not mentioned otherwise, observables will always be associated with Hermitian operators on $\mathscr{H}$ and denoted by capital Latin letters only. The only exception being the creation and annihilation operators $a^\dagger$, a.

Throughout this work units with $\hbar = 1$ are used for the sake of simplicity. That also means that instead of using τ in time units we will rather be using a dimensionless quantity $t \equiv \tau/\mathrm{J}$, where J is our time units.

1.2 Qubit

Quantum computation and information work with *qubits*, physical systems that serve as information carriers in an analogue to classical bits in classical computation. In this subsection we will discuss some similarities and differences between the two. In this work we refer to qubits as to abstract mathematical concepts even though some physical realizations will be presented and discussed.

A qubit is a system that can likewise the classical bit be in two possible states (the Hilbert (sub)space $\mathscr{H} = \mathbb{C}^2$). The usual notation of the two states of bit is 0 and 1, we will denote the corresponding qubit states by $|0\rangle$ and $|1\rangle$. The main and the most important difference between the two is that qubits can be found not only in the two states, but in any linear combination or, in other words, the *superposition* of the two

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \tag{1.2}$$

where $\alpha, \beta \in \mathbb{C}$. The set of the states $|0\rangle$ and $|1\rangle$ is called a *computational basis* and it has been chosen so the vectors always form an orthonormal basis in a two-dimensional subspace of $\mathscr{H}$. Because we require the representatives of a state to be normalized, the following condition must be kept in mind

$$|\alpha|^2 + |\beta|^2 = 1, \tag{1.3}$$

which allows for some very nice visualizations of the qubit state and its global phase such as the Bloch or Poincaré spheres [19].

Although there are phenomena from the world around us directly connected to it like the Brewster's angle is connected to polarization of light, the superposition property is rather difficult to grasp with common human intuition only as it has some interesting consequences. For example if we repeatedly measure a qubit in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,, \tag{1.4}$$

which is a *coherent superposition*, in half of the measurements it will be found in state $|0\rangle$ and the the other half in state $|1\rangle$. This is sometimes described as qubit being simultaneously in both states. In general the coefficients do not have to be the same or real, then the absolute values squared give probabilities of finding the qubit in the respective state, which is part of the definitions of quantum physics that we are using. Linear combinations of basis states of qubits and their properties could be described in entire sections devoted to it, interested reader might want to consult for example [19, 20] for detailed description of these properties and their impacts on quantum computation. It is the linearity of quantum physics that allows quantum physicists to construct algorithms for quantum computers that could be so efficient.

Despite this strange behavior there is a wide variety of physical systems that can be considered as qubit representatives, because they follow the two dimensional state space requirement and can be manipulated in ways that allow for quantum computation. In the following sections we shall present some examples of these i.e. photon states, nuclear spin states of trapped atoms, electronic spin states, electronic states in quantum dots [6, 12, 19, 21–29].

Sometimes quantum computation can be performed on more dimensional objects than qubits, three level systems are then called qutrits and general d dimensional objects qudits. We will, however, not be using them either for computation or further discuss their properties.

2 Quantum computation

A quantum computer is a device that uses the coherence phenomena in quantum physics world to perform calculations. The quantum computer designs harvest powers of qubits, entanglement and other aspects of quantum world. It has been shown by Scott Aaronson [30] that the class of problems solvable in polynomial time on quantum *Turing machine*, that is a quantum computer that can implement any unitary transformation on its qubits, is larger to the class of problems solvable in exponential time on classical computers. Larger in the sense that the class is equivalent to probabilistically polynomial time class (PP) of problems, which contains the non-deterministic polynomial time (NP) class of problems.

The heart of quantum computation are quantum algorithms like the famous Shor's algorithm, which are defined as sequences of unitary operations on qubits. This is why we will aim at presenting results from the areas of unitary transformations and their actual realization on various physical systems when giving an overview of quantum computer candidates.

David DiVincenzo from IBM has defined requirements on a physical system to be considered quantum computer candidate [10]. According to his criteria the system *must* have these properties:

1. A scalable physical system with well characterized qubits
2. The ability to initialize the state of the qubits to a simple fiducial state, such as $|000\ldots\rangle$
3. Long relevant decoherence times, much longer than the gate operation time
4. A *universal* set of quantum gates
5. A qubit-specific measurement capability
6. The ability to interconvert stationary and flying qubits
7. The ability faithfully to transmit flying qubits between specified locations

We shall discuss these in more detail in sections about the quantum computer candidates, both generally and then for each system individually. The universal set of quantum gates will have an entire section devoted to it to really see which set of gates can be considered universal and why. In some more recent publications these requirements are relaxed in some aspects, but the core remains the same.

The research presented in this thesis is focused on the sixth and seventh points of the DiVincenzo requirements, namely the perfect state transfer. Nevertheless, we will present past results related to almost all the DiVincenzo criteria as well to show how the state transfer fits into the quantum computation frame.

2.1 Quantum gates

Let us compare the building block of classical logic operations with quantum ones. In this work we will be using the quantum circuit model of a quantum computer [19]. The two main elements of the model are quantum wires and quantum gates. The quantum wires and quantum gates are counterparts of wires and logical circuits in classical computers. The main results in this thesis are related to quantum wires and state transfer, but a quantum computer is more complex device and to understand its operation we give a short overview of the computation methods as well.

The simplest logic we can talk about in a classical computer would be the logic involving only one bit. The only non-trivial logical operation acting only on one bit is the NOT operation. It is given by its truth table that does the $0 \longrightarrow 1$, $1 \longrightarrow 0$ transformation on bits. To perform a similar quantum operation acting on states of qubit we need to define how this operation acts on a superposition of states as well (we only know how it should act on the basis states). Because we allow only unitary operations, it must act *linearly*

$$\mathrm{NOT}\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right) \equiv \alpha\mathrm{NOT}\left|0\right\rangle + \beta\mathrm{NOT}\left|1\right\rangle, \tag{2.1}$$

It is common to denote the quantum analogue of the NOT gate with the X letter for historical reasons and it is common to write it in a form of the following matrix in the chosen basis

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

which is acting on the state

$$\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle, \tag{2.3}$$

in vector notation written as

$$\left|\psi\right\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{2.4}$$

If you act with this matrix on this vector, it is easy to see that you get exactly the desired result, namely

$$X\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \tag{2.5}$$

This is an example of a quantum gate operation and we see that such quantum gates can be represented by 2×2 matrices. We immediately see that compared to the classical case, where the only non trivial logical operation that we need to consider is the NOT operation, here we can consider all the 2×2 unitary matrices. The constraint of unitarity can be seen to be equivalent to the normalization condition, because even the resulting state of a transformation has to follow the condition (1.3) on the unit norm of the vector. Giving us general condition

$$\left|\left|A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right|\right|^2 = 1, \text{ where} \tag{2.6}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{C}, \tag{2.7}$$

A is the most general matrix 2×2. This condition can be shown to be equivalent [19] to

$$A^\dagger A = I, \tag{2.8}$$

where I is the identity matrix and $A^\dagger$ is the adjoint matrix of A,

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.9}$$

The unitarity is the only constraint on the quantum gate operations we will ever require, when talking about quantum gate, we will always refer to some unitary operation. The most often used one qubit gates except the X gate are the Z gate

$$Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.10}$$

the Hadamard gate H

$$H \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.11}$$

the phase gate

$$S \equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \tag{2.12}$$

and the $\frac{\pi}{8}$ gate

$$T \equiv \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}. \tag{2.13}$$

One of the most important facts to mention about the unitary 2×2 matrices for computation purposes is the following theorem [19]

Theorem 2.1. *Arbitrary 2×2 unitary matrix may be decomposed as*

$$U = e^{i\alpha} \begin{pmatrix} e^{-\frac{i\beta}{2}} & 0 \\ 0 & e^{\frac{i\beta}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\delta}{2}} & 0 \\ 0 & e^{\frac{i\delta}{2}} \end{pmatrix}, \tag{2.14}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Proof. Lets denote the Pauli matrices as

$$X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.15}$$

$$Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2.16}$$

$$Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.17}$$

these matrices generate the $SO(3)$ Lie group of rotations in a 3 dimensional real Euclidean space. Let us denote the rotations generated through the exponentials of the Pauli matrices

$$\begin{aligned} R_x(\theta) \equiv e^{-i\theta\frac{X}{2}} &= \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \\ R_y(\theta) \equiv e^{-i\theta\frac{Y}{2}} &= \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \\ R_z(\theta) \equiv e^{-i\theta\frac{Z}{2}} &= \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) Z = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}. \end{aligned} \tag{2.18}$$

these are easily calculated using the relation

$$\exp(iAx) = \cos(x) I + i\sin(x) A, \tag{2.19}$$

which is valid for x being an arbitrary real number and A a matrix such that

$$A^2 = I, \tag{2.20}$$

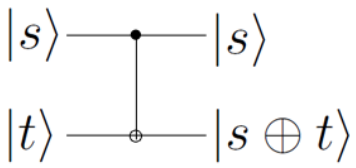


Figure 2.1: $\mathrm{C}-\mathrm{NOT}$ gate

which is satisfied by all the Pauli matrices [19].

Since U is a unitary matrix, both the rows and columns have to be mutually orthonormal, from this it follows that there exist

$$\alpha, \beta, \gamma, \delta \in \mathbb{R}, \tag{2.21}$$

such that

$$U = \begin{pmatrix} e^{i\left(\alpha-\frac{\beta}{2}-\frac{\delta}{2}\right)}\cos\left(\frac{\gamma}{2}\right) & -e^{i\left(\alpha-\frac{\beta}{2}+\frac{\delta}{2}\right)}\sin\left(\frac{\gamma}{2}\right) \\ e^{i\left(\alpha+\frac{\beta}{2}-\frac{\delta}{2}\right)}\sin\left(\frac{\gamma}{2}\right) & e^{i\left(\alpha+\frac{\beta}{2}+\frac{\delta}{2}\right)}\cos\left(\frac{\gamma}{2}\right) \end{pmatrix}. \tag{2.22}$$

Using elementary matrix multiplication it is easy to see that this can be written as

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta), \tag{2.23}$$

which is the needed form. □

The importance of this theorem is in showing how to perform only certain operations to be able to perform any single qubit unitary operation on physical systems that we would like to use as a quantum computer. It can also be shown that we do not need to perform these operations for arbitrary $\alpha, \beta, \gamma, \delta$, but only a finite set of operations that can be repeated is needed to have these with arbitrary precision in all the parameters. We will comment this in the following section. The universality lets anyone building a quantum computer know exactly which operations he needs to perform any transformation.

We would like to continue in this spirit even for more general cases, for example for two qubit gates and see what is needed to perform any unitary operation on them (find a universal set of gates). For example on classical bits one can perform AND, OR, XOR, NAND or NOR operations on two or more bits and it is well known result [19] that you only need the NAND operation to construct arbitrary function on bits. It is the so called *universal gate*, it is not a trivial property, for example XOR alone or even XOR with NOT are not sufficient.

A very similar result for qubits in the sense of universality underlies the $\mathrm{C}-\mathrm{NOT}$ operation - the controlled NOT operation. Usually it is considered for a two qubit gate, but it could be easily generalized to more than one qubit. For the sake of simplicity we will now focus on two qubits only (we shall later see that there is no loss of generality here). One of these qubits is called the *control/source* qubit while the other one the *target* qubit. The $\mathrm{C}-\mathrm{NOT}$ operation quantum circuit is represented in the figure 2.1 (for an introduction to reading and writing quantum circuits consult the section 2.3). The $\mathrm{C}-\mathrm{NOT}$ is defined as follows: if the control qubit is in the state $|0\rangle$, the target qubit is left untouched and if the control qubit is in the state $|1\rangle$, the X operation is applied on the second qubit. In the basis of vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ the $\mathrm{C}-\mathrm{NOT}$ matrix reads

$$U_{CN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2.24}$$

which is easily seen to be unitary. The decomposition of any unitary matrix into U_{CN} and the one qubit unitaries will be discussed in the following section.

2.2 Universality of Hadamard, $\mathrm{C}-\mathrm{NOT}$ and $\frac{\pi}{8}$ gates

Let us focus on the possibility to decompose any unitary operation into the gates from the title, namely the one qubit gates: Hadamard and $\frac{\pi}{8}$ gates and one two qubit gate, the $\mathrm{C}-\mathrm{NOT}$. We want to work with any unitary matrix, not just 2×2 or 4×4 matrices. We will proceed in three steps. First we discuss the relation of arbitrary unitary operators with domain D to operators acting non trivially only on two dimensional subspaces of the domain D, then we will show that any unitary operator may be expressed using only single qubit and the $\mathrm{C}-\mathrm{NOT}$ gates. Finally we show how to approximate with an arbitrary given precision any single qubit gate using the mentioned single qubit operators.

Let U be a unitary matrix acting on some d-dimensional Hilbert space. We will show that it is possible to decompose the matrix into a product of unitary matrices that act non trivially only on a two dimensional subspaces of the Hilbert space. We will call these matrices *two-level system matrices* [29]. First let us consider a 3×3 matrix ($d=3$) of the form

$$U=\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix}, \tag{2.25}$$

where all the parameters are complex numbers. We will find U_1, U_2, U_3 unitary two-level system matrices such that they form together the decomposition:

$$U_3U_2U_1U = I, \tag{2.26}$$

$$\Rightarrow U = U_1^\dagger U_2^\dagger U_3^\dagger. \tag{2.27}$$

If $b\neq 0$, we choose

$$U_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I, \tag{2.28}$$

and if $b=0$, we choose

$$U_1=\frac{1}{\sqrt{|a|^2+|b|^2}}\begin{pmatrix} a^\star & b^\star & 0 \\ b & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.29}$$

which in any case is a two level matrix (the $.^\star$ is a complex conjugation). Next multiply the matrices and rename the parameters to get

$$U_1U=\begin{pmatrix} a' & d' & g' \\ 0 & e' & h' \\ c' & f' & j' \end{pmatrix}, \tag{2.30}$$

now let us choose similarly U_2 to get a zero in the bottom left. If $c'=0$, we choose

$$U_2=\begin{pmatrix} a'^\star & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.31}$$

and for $c'\neq 0$

$$U_2=\frac{1}{\sqrt{|a'|^2+|c'|^2}}\begin{pmatrix} a'^\star & 0 & c'^\star \\ 0 & 1 & 0 \\ c' & 0 & -a' \end{pmatrix}, \tag{2.32}$$

after multiplication and re-scaling again we get

$$U_2U_1U=\begin{pmatrix} 1 & d'' & g'' \\ 0 & e'' & h'' \\ 0 & f'' & j'' \end{pmatrix}, \tag{2.33}$$

however U, U_1, U_2 are all unitary so $U_2 U_1 U$ is unitary as well and so

$$d'' = g'' = 0, \tag{2.34}$$

if we now choose U_3 to be

$$U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e''^{\star} & f''^{\star} \\ 0 & h''^{\star} & j''^{\star} \end{pmatrix}, \tag{2.35}$$

we obtain the desired decomposition.

We can now use the process of induction to prove the same for any d; for arbitrary d we can find $U_1, \ldots U_{d-1}$ such that $U_{d-1} \ldots U_1 U$ has 1 in the top left and zeros in the first row and column everywhere else by the process from above and consequently do the same for the $d-1$ dimensional bottom right block of this matrix and so on until we get the decomposition

$$U = V_1 \ldots V_k,$$

for some k, where V_i is a two level matrix for all i.

The procedure just presented gives the recipe for the decomposition of an arbitrary unitary operation on many qubits using only two-level system matrices. This decomposition may not be optimal as the worst case scenario is that we will need $2^{n-1}(2^n - 1)$ two-level system matrices. On the other hand it might happen that we will only need $d-1$ matrices.

Let us proceed to the second step. We want to show single qubit gates and $\mathrm{C-NOT}$ gate are universal as mentioned earlier. Using the previous paragraphs to show that any unitary operation can be decomposed into single qubit gates and $\mathrm{C-NOT}$ gates (universality) we now only need to show how to decompose any two level unitary matrix into these.

In order to do this we first need to introduce the Gray code. Let s and t be binary numbers of the same length n. A Gray code connecting s and t is a sequence $g_1, \ldots, g_k$ of binary numbers from s to t where the two consequent binary numbers differ only in one digit. For example for

$$\begin{aligned} s &= 101001, \\ t &= 110011, \end{aligned} \tag{2.36}$$

the Gray sequence would be

$$\begin{aligned} g_1 &= 101001, \\ g_2 &= 101011, \\ g_3 &= 100011, \\ g_4 &= 110011. \end{aligned} \tag{2.37}$$

Notice that $k \leq n+1$ as that is the case when the two numbers are different in all the digits.

Before we proceed show how to decompose, we need to stop at the Hilbert space we are now working with. As we are working with n qubits, where each computational basis state is either $|0\rangle$ or $|1\rangle$. From now on we will be using two possible notations of the overall state of the qubits. The first one is a mere abbreviation of a long tensor product

$$|\psi_1 \psi_2 ... \psi_n\rangle \equiv |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle, \tag{2.38}$$

for example

$$|101\rangle = |1\rangle |0\rangle |1\rangle. \tag{2.39}$$

The second notation uses only one qubit present in the state $|1\rangle$, lets say j-th qubit is in this state, then the overall state will be denoted as

$$|\psi\rangle = |j\rangle, \tag{2.40}$$

and when all the qubits are in the $|0\rangle$ state, we will write

$$|\psi_0\rangle = |0\rangle\,, \tag{2.41}$$

as an overall state. Notice that this formalism can also be used outside of the n qubit case for our general case with no confusion because our d-dimensional Hilbert space has an orthonormal basis of d vectors and $|i\rangle$ can then be used to denote that the system is in a state of the i-th vector from the aforementioned basis.

Let U be the two level matrix we want to decompose. Let the two vectors it involves non trivially be the $|s\rangle$ and $|t\rangle$ from the computational basis. And let the $g_1, \ldots g_m$ be the Gray sequence of s and t in binary. Also let $\widetilde{U}$ be a contraction of U on the subspace spanned by $|s\rangle$ and $|t\rangle$. We can think of it as of a one qubit operator. We will be using it in the form of the controlled $\widetilde{U}$ operator. It works just as the $\mathrm{C-NOT}$ operator but instead of applying NOT operator on the target qubit when the control qubit is in $|1\rangle$ state, we apply the $\widetilde{U}$ operator on the target.

Implementing the U operator now proceeds in the following steps. The first step is to swap the state $|g_1\rangle$ and $|g_2\rangle$, remember - g's are binary numbers. Let g_1 and g_2 differ at the i-th digit. The swap can then be easily accomplished by a controlled bit flip ($\mathrm{C-NOT}$) using the qubits that are in the same state in the sequences of g_1 and g_2 (they only differ in one digit), this means that the condition for applying the NOT operator is all the remaining qubits being in states corresponding to the common sub sequence of g_1 and g_2. Next we swap $|g_2\rangle$ and $|g_3\rangle$. We continue until we perform $m-2$ operations and get

$$\begin{aligned} |g_1\rangle &\to |g_{m-1}\rangle\,, \\ |g_2\rangle &\to |g_1\rangle\,, \\ |g_3\rangle &\to |g_2\rangle\,, \\ &\cdots \\ |g_{m-1}\rangle &\to |g_{m-2}\rangle\,, \end{aligned} \tag{2.42}$$

all the other states from computational basis are untouched by this transformation. The next step involves the bit that is different between g_{m-1} and g_m (t). Let us say that they differ at j-th position. We perform a controlled $\widetilde{U}$ operation with j-th qubit as the target and the condition being: all the remaining qubits are in the state described by the common sub sequence of g_m and g_{m-1}. The procedure is finished by undoing the swap operations by performing them in reversed order, that is swapping first $|g_{m-1}\rangle$ with $|g_{m-2}\rangle$ and so on.

The final step remaining is to show how to perform arbitrary one qubit operation using only Hadamard and $\frac{\pi}{8}$ gates with arbitrary precision. It is obvious that using only these one cannot perform any unitary operation exactly. However, how do we measure the precision of an operator approximation? A good measure for many reasons [19] is the error function

$$E\left(U,V\right) \equiv \max_{|\psi\rangle} \left(||\left(U-V\right)|\psi\rangle||\right), \tag{2.43}$$

where U is the approximated operator, V is the operator we want to use to approximate U and the maximum is over all the states from computational basis. It can be shown [19] that if this error is small, errors in measurements will also be very small. This measure also has the nice chaining property, it can be shown [19] that if we try to approximate a sequence of unitaries $U_1, \ldots, U_m$ by a sequence $V_1, \ldots, V_m$ then the error can be estimated by errors done by individual approximations

$$E\left(U_m U_{m-1} \ldots U_1, V_m V_{m-1} \ldots V_1\right) \leq \sum_{j=1}^{m} E\left(U_j, V_j\right). \tag{2.44}$$

We will show how to approximate any unitary U by sequences of the H Hadamard gates and the T $\frac{\pi}{8}$ gate. Consider T and HTH operators, the T is up to a global phase rotation by $\frac{\pi}{4}$ around the

$\hat{z}$ axis and the HTH is a rotation by $\frac{\pi}{4}$ around the $\hat{x}$ axis (this can be visualized easily on the Bloch sphere). The composition of these operations gives

$$\exp\left(-i\frac{\pi}{8}Z\right)\exp\left(-i\frac{\pi}{8}X\right) = \tag{2.45}$$

$$= \left(\cos\frac{\pi}{8}I - i\sin\frac{\pi}{8}Z\right)\left(\cos\frac{\pi}{8}I - i\sin\frac{\pi}{8}X\right) \tag{2.46}$$

$$= \cos^2\frac{\pi}{8}I - i\left[\cos\frac{\pi}{8}(X+Z) + \sin\frac{\pi}{8}Y\right]\sin\frac{\pi}{8}. \tag{2.47}$$

This is a rotation around the axis along $\vec{n} = \left(\cos\frac{\pi}{8}, \sin\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$ by an angle θ, where θ is defined by the relation

$$\cos\frac{\theta}{2} = \cos^2\frac{\pi}{8}. \tag{2.48}$$

It can be shown that θ is thus an irrational multiple of 2π [19]. We will denote this rotation by $R_{\vec{n}}(\theta)$. Note, we only needed the T and H gates to obtain it. This rotation can be used to approximate $R_{\vec{n}}(\alpha)$ to an arbitrary precision, lets say $\delta > 0$ being the desired precision, with multiple iterations. Let N be an integer greater than $\frac{2\pi}{\delta}$ and $\theta_k \equiv (k\theta) \mod 2\pi$ ($\theta_k \in [0, 2\pi)$). The pigeonhole principle now tells us that there are distinct i and j integers smaller then N such that

$$|\theta_i - \theta_j| \leq \frac{2\pi}{N} < \delta, \tag{2.49}$$

where we can assume $j > i$. Since $i \neq j$ and θ is an irrational multiple of 2π, the

$$\theta_{j-i} \equiv |\theta_i - \theta_j| \neq 0. \tag{2.50}$$

Therefore the sequence $\theta_{l(j-i)}$ fills up the interval $[0, 2\pi)$ and as we change l, the adjacent members of this sequence are no more than δ apart. This almost proves the following theorem [19]

Theorem 2.2. *For arbitrary $\epsilon > 0$ there exists n such that $E\left(R_{\vec{n}}(\alpha), R_{\vec{n}}(\theta)^n\right) < \frac{\epsilon}{3}$, where all the symbols have been previously defined.*

Proof. The proof follows easily from previous text and the fact that for arbitrary α and β it is true that

$$E\left(R_{\vec{n}}(\alpha), R_{\vec{n}}(\alpha+\beta)\right) = \left|1 - \exp\left(i\frac{\beta}{2}\right)\right|. \tag{2.51}$$

□

How to approximate any unitary matrix? It is easily seen that

$$HR_{\vec{n}}(\alpha)H = R_{\vec{m}}(\alpha), \tag{2.52}$$

where

$$\vec{m} = \left(\cos\frac{\pi}{8}, -\sin\frac{\pi}{8}, \cos\frac{\pi}{8}\right). \tag{2.53}$$

hence

$$E\left(R_{\vec{m}}(\alpha), R_{\vec{m}}(\theta)^n\right) < \frac{\epsilon}{3}, \tag{2.54}$$

as well. From the theorem 2.1 it follows that arbitrary unitary may be written as

$$U = R_{\vec{n}}(\beta)R_{\vec{m}}(\gamma)R_{\vec{n}}(\delta), \tag{2.55}$$

when neglecting irrelevant global phase. We arrive at the final theorem [19]

Theorem 2.3. *For arbitrary $\epsilon > 0$ there exist $n_1, n_2, n_3 \in \mathbb{N}$ such that*

$$E\left(U, R_{\vec{n}}(\theta)^{n_1}HR_{\vec{n}}(\theta)^{n_2}HR_{\vec{n}}(\theta)^{n_3}\right) < \epsilon. \tag{2.56}$$

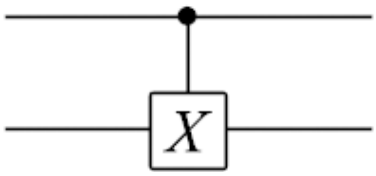


Figure 2.2: $C - NOT$ gate

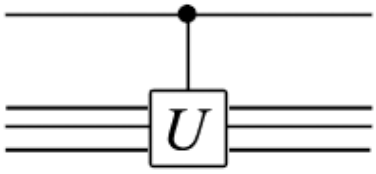


Figure 2.3: Controlled three qubit U gate

Proof. Follows easily from the chaining rule (2.44) if we put into it notes (2.54) and (2.55). □

We see that we can approximate any unitary transformation on any number of qubits using only the $C - NOT$ gate, Hadamard gate and the $\frac{\pi}{8}$ gate. It is a very important result in relation to the point (4) of the DiVincenzo requirements. Sometimes to this selection of operators the phase gate is added for the purposes of error reduction as this sequence is as one can imagine not the best to implement and this gate helps to improve it [19]. We do not want to present error reduction procedures here, so we can neglect it. In any case we now know which operations we need to be able to perform on any system we would like to consider a quantum computer candidate.

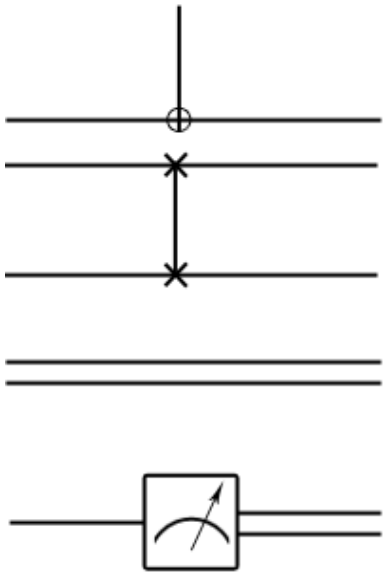
2.3 Quantum circuits

There are various ways how to represent a quantum circuit, we saw an example of this in figure 2.1 where the $C - NOT$ gate was depicted in form of a quantum circuit. The diagrams are supposed to be read from left to right and each line from left to right represents a time evolution of a certain quantum bit. If at the beginning of the line there is no state specified, the qubit is assumed to be in the state $|0\rangle$, which is usually chosen to be the ground state.

The vertical lines depict influences the qubits have on each other. These effects can have various reasons behind them - a multi qubit unitary transformation of some sort. If the vertical lines connect to the horizontal ones with a full black circle, it means "reading" or "usage" of the qubit without influencing its state. The most general notation of a unitary transformation applied to some qubit is a connection of lines into a square box with the transformation written inside. If there is a vertical line connected to this box, it usually means it takes the connected qubit as an input, but does not influence it compared to a situation when horizontal line enters the box. When the box takes the qubit from the left, it is an input qubit but the transformation influences it. The most simple example of this might be the $C - NOT$ gate depicted with X gate, the NOT counterpart operator, as shown in the figure 2.2. This can easily be generalized to performing any controlled U gate even on many qubits as in the figure 2.3.

Some of the common gates are so frequent that symbols for the simplicity of the diagrams have been introduced. The table of the most common symbols is the table 2.1.

The addition modulo 2 takes a state from the top qubit and if it is $|0\rangle$, it does nothing and if it is



addition modulo 2 ($C-NOT$)

swap gate

classical bit

measurement

Table 2.1: Quantum circuit symbols

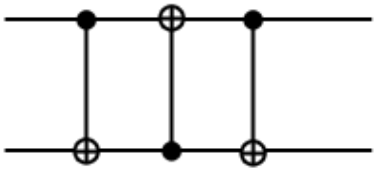


Figure 2.4: swap explicitly

in the state $|1\rangle$, it flips the target qubit (horizontal line), it is the simplest notation for the $C-NOT$ gate.

The swap gate is a quantum circuit abbreviation for the circuit from the figure 2.4. Indeed just by writing down the circuit from this figure one can see that it performs a swap operation

$$\begin{aligned} |a,b\rangle &\to |a, a\oplus b\rangle \\ &\to |a\oplus(a\oplus b), a\oplus b\rangle = |b, a\oplus b\rangle \\ &\to |b, (a\oplus b)\oplus b\rangle = |b,a\rangle\,. \end{aligned} \tag{2.57}$$

The classical bit means that at this time we can measure the qubit only either in the state $|0\rangle$ or in the state $|1\rangle$ but not in their linear combination. As is common this is a result of a measurement performed on the qubit, which is depicted on the symbol used for measurement.

Let us point out some important properties of these circuits resulting from quantum laws of physics. For example two horizontal lines can never be joined together, in electronic circuits joining two horizontal lines means performing OR operation on them (the so called FANIN gate), but this operation is not unitary and so for us it is forbidden and we must never join two horizontal lines together. Another property is that quantum circuits are not allowed to be cyclic, nowhere in any quantum circuit may be feedback, reason for this is that the unitary operations in some physical systems correspond to a rather instantaneous processes and we want to be general enough to have these systems in our thoughts. The last remark we are going to make is that it is forbidden to split a line, in electronic circuits this is called the FANOUT gate and what it does is making an exact copy of input bits to multiple output

bits, but quantum mechanics forbids cloning (theorem 2.4) and so this operation is also forbidden (it is not unitary as we will see in a moment).

Our favorite C − NOT circuit (fig. 2.1) can serve as a nice illustration of the no cloning theorem. Firstly, what would we call "cloning", intuitively for two qubits it would be a process where at the start we would have one qubit in an arbitrary state $|\psi\rangle$ which we want to copy, we do not care about the initial state of the second qubit, and at the end of cloning we want to have both qubits in the state $|\psi\rangle$. Could the C − NOT circuit serve this purpose? Let us assume the target qubit is in its ground state $|0\rangle$, which is not much of a restriction as we want to be able to prepare all the qubits in an arbitrary state.

If we apply the C − NOT operation on two qubits, where the control qubit is in the general state $|\psi\rangle = a|0\rangle + b|1\rangle$, we get

$$a|00\rangle + b|10\rangle \underset{\text{C-NOT}}{\rightarrow} a|00\rangle + b|11\rangle, \tag{2.58}$$

which exactly acts as cloning when $a = 0$ or $b = 0$. But we see that it fails for any other linear combination as

$$|\psi\rangle|\psi\rangle = a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle, \tag{2.59}$$

also note that it in fact is true that the C − NOT would give the correct result for $ab = 0$. Let us now formulate this more precise [29].

Theorem 2.4. *There is no cloning of quantum states in the sense of a unitary transformation U_C acting in the following way*

$$U_C|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle, \tag{2.60}$$

on two quantum systems with arbitrary initial state $|\psi\rangle$.

Proof. There are many ways of proving this theorem [19, 29]. We will show that the existence of such a unitary transformation is in contradiction with the linearity of quantum physics from the reference [19]. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be arbitrary possible states of the first system. When the U_C acts on a linear combination of these states, we get

$$U_C(c|\psi_1\rangle + d|\psi_2\rangle)|0\rangle = c|\psi_1\rangle|\psi_1\rangle + d|\psi_2\rangle|\psi_2\rangle, \tag{2.61}$$
$$\text{where c, d} \in \mathbb{C} \text{ and } |\text{c}|^2 + |\text{d}|^2 = 1,$$

while directly from the definition of U_C

$$\begin{aligned} U_C(c|\psi_1\rangle + d|\psi_2\rangle)|0\rangle &= (c|\psi_1\rangle + d|\psi_2\rangle)(c|\psi_1\rangle + d|\psi_2\rangle) \\ &= c^2|\psi_1\psi_1\rangle + cd|\psi_1\psi_2\rangle + cd|\psi_2\psi_1\rangle + d^2|\psi_2\psi_2\rangle. \end{aligned} \tag{2.62}$$

These two expressions are equal if and only if $cd = 0$, which does not hold for arbitrary $c, d \in \mathbb{C}$ and therefore there is no such unitary transformation. □

Let us now present another rather popular example of a quantum circuit. The quantum teleportation circuit is shown in the figure 2.5. The idea behind teleportation of a state between say Alice and Bob is to use the the *Bell state* of two systems (qubits) to communicate over an arbitrary distance. We want Alice to send an arbitrary state $|\psi\rangle = a|0\rangle + b|1\rangle$ exactly to Bob, classically she might need an infinite amount of information because of the continuity of linear transformation. The coefficients in the linear combination might be irrational. We will need a classical communication to communicate results of measurements.

Alice has two qubits, one in the state we want to communicate and the other one is in the Bell state with the one qubit that Bob possesses, this is the particular order that we choose to use from now on in the tensor products: Alice's secret qubit - 1 in the figure, Alice's part of the Bell pair - 2 in the figure, Bob's part - 3 in the figure. We can write the overall initial state of the circuit

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[a|0\rangle\left(|00\rangle + |11\rangle\right) + b|1\rangle\left(|00\rangle + |11\rangle\right)\right]. \tag{2.63}$$

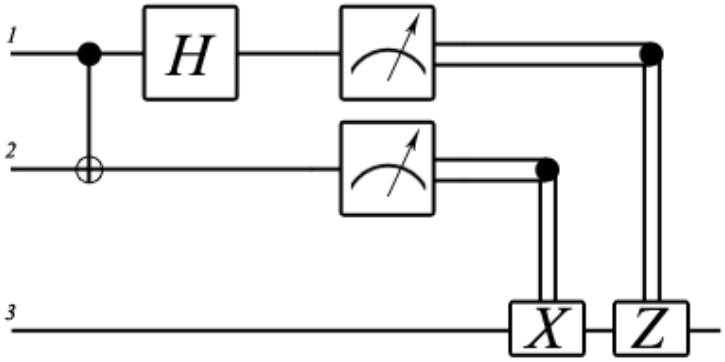


Figure 2.5: Quantum teleportation circuit

00	I
01	X
10	Z
11	ZX

Table 2.2: Bob's transformations

As shown in the figure 2.5, Alice first performs a $\mathrm{C}-\mathrm{NOT}$ gate, where the qubit to be teleported (*teleportee*) is the control qubit and the Alice's Bell qubit is the target. It results in

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|10\rangle + |01\rangle)\right]. \tag{2.64}$$

Then the Hadamard gate is applied on the teleportee and one gets

$$|\psi_2\rangle = \frac{1}{2}\left[a\,(|0\rangle + |1\rangle)\,(|00\rangle + |11\rangle) + b\,(|0\rangle - |1\rangle)\,(|10\rangle + |01\rangle)\right]. \tag{2.65}$$

This is the desired result. We can rewrite the expression to get

$$\begin{aligned}|\psi_2\rangle =& \frac{1}{2}\left[|00\rangle\,(a\,|0\rangle + b\,|1\rangle) + |01\rangle\,(a\,|1\rangle + b\,|0\rangle)\right.\\ &\left. + |10\rangle\,(a\,|0\rangle - b\,|1\rangle) + |11\rangle\,(a\,|1\rangle - b\,|0\rangle)\right].\end{aligned} \tag{2.66}$$

This state is almost what we wanted (we want the Bob's qubit to be in the teleportee's initial state $a\,|0\rangle + b\,|1\rangle$), now Alice only needs to perform measurement on her two qubits and communicate the results to Bob, he will then apply transformations conditioned on the results of Alice. To see what gate Bob needs to apply lets look at possible results of measurements and corresponding resulting states of the Bob's qubit

$$00 \to |\psi_3(00)\rangle = a\,|0\rangle + b\,|1\rangle \tag{2.67}$$
$$01 \to |\psi_3(01)\rangle = a\,|1\rangle + b\,|0\rangle \tag{2.68}$$
$$10 \to |\psi_3(10)\rangle = a\,|0\rangle - b\,|1\rangle \tag{2.69}$$
$$11 \to |\psi_3(11)\rangle = a\,|1\rangle - b\,|0\rangle. \tag{2.70}$$

The transformations Bob needs to apply after receiving the results of Alice's measurements are shown in the table 2.2. Notice how the order of ZX is reversed to the position in the diagram 2.5.

Teleportation does not contradict the finiteness of the speed by which information can propagate. Classical communication is still needed for faithful reconstruction on Bob's side. Note also that it is not in conflict with the no cloning theorem as the teleportee is at the end in one of the basis states. Quantum teleportation has many theoretical applications which are beyond the scope of this work, but interested reader might find these in the references [19, 29].

Part II
Quantum computer candidates

In Part I of this thesis we have introduced the basic ideas about quantum computers, and recollected which are the required basic quantum operations. The present part of the thesis will be dedicated to the discussion of physical systems which have the potential for realization of a quantum computer. Some of these systems have already been experimentally demonstrated to posses much of the required capabilities. We feel that it is important to present the latter results here also because the state transfer is a problem of quantum computation and the formalism that we will present encapsulates many of present ideas of quantum computer implementations.

Although we have presented above an idea of what is necessary for quantum computer to be able to perform any unitary operation, we will pay detailed attention to the DiVincenzo criteria, a collection of formal requirements that a quantum computer must fulfill to be practical for any use. The importance of these criteria is that real world generally places obstacles in the way of easily fulfilling all these criteria at once. The very first criteria we will discuss is the addressability of every qubit (point 5 of DiVincenzo). It must be possible (the design has to give a way to do this) to write on qubits any binary number and we need to be able to read (measure) the state of any one qubit at any time. The reason behind this are algorithms that consider partial reading of the information at some point and we are not allowed to disturb the rest of the system doing so as that would endanger any previous and future computations.

The system under consideration should be well isolated from environmental effects and decoherence for a time necessary to perform a calculation with reading the outputs (DiVincenzo point (3)). There are many kinds of perturbations that influence the system. Usually the condition is given for the number of cycles one can perform

$$N_C = \frac{T_{\text{relax}}}{T_C}, \tag{2.71}$$

where T_{relax} is the characteristic time of decoherence (time after which the errors caused by the environment outweigh the unitary time evolution of system), T_C is the time of a typical calculation we would like to perform. It is usual to consider systems for which $N_C \geq 1$ [29] quantum computer candidates, although DiVincenzo proposes much more strict criteria.

To be able to perform any operation on qubits the qubits must be allowed to interact (we shall see that once we try to present a way of implementing the elementary transformations). And we must be able to control the interaction well. This condition is very restrictive of systems based on photons as even though photons are excellent information carriers even over long times and distances, they tend to interact mutually very weekly. Recently new results concerning quantum walks using optical circuits have gone with N_C as far as to 30 [31]. On the other side of the spectra sit electrons or electronic states of ions, because electrons interact very strongly via the Coulombic interaction, but on the other hand they suffer from the same effect from environmental fields and are perturbed easily.

The last constraint on a real world quantum computer is scalability with respect to the number of qubits (point 1). It might be possible to create a quantum computer consisting of two qubits, but this system would hardly be of any computational use. So the researches are faced with the obstacle of performing the manipulations with larger and larger numbers of qubits, one of the successful experimental presentations of trapped ions went recently as far as to 14 qubits [6].

3 Quantum harmonic oscillator

3.1 Description of the system

Harmonic oscillators are the most common of the tools of a physicist. Many systems can, under certain conditions, be approximated by classical or quantum harmonic oscillators. Multiple Harmonic

oscillators are even used to model more complex systems. Classically we say that a harmonic oscillator is any system under the influence of the potential well

$$V(x) = \frac{1}{2}m\omega^2 x^2. \tag{3.1}$$

This can be for example a mass on a spring, some oscillations of an electric circuit or just a particle in external driving field. The total energy of this system is a sum of the potential energy and of the kinetic term. The energy is passing from one component to other as the system moves.

When the oscillator is either cooled enough to exhibit quantum behavior or the harmonic oscillations are used as a mere approximation, the most important characteristic of the quantum oscillator is that the energy becomes quantized and can only take certain discrete values and in the case of the harmonic oscillator, these values are equidistant, usually the difference between two consequent values is labeled as

$$\Delta E = \omega. \tag{3.2}$$

An example of such a system are electromagnetic cavity field modes where each quantum of energy represents one photon present in the system.

3.2 The Hamiltonian

The Hamiltonian of the classical harmonic oscillator reads (3.1)

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{3.3}$$

which is then used to write the quantum version just by replacing the position and momentum of the particle by corresponding operators of the momentum and the position:

$$x = x\cdot\,, \tag{3.4}$$

$$p = -i\frac{\partial}{\partial x}. \tag{3.5}$$

The time-less Schrödinger equation reads

$$\frac{1}{2m}\frac{d^2\psi_n(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi_n(x) = E_n\psi_n(x)\,, \tag{3.6}$$

with the conditions on the eigenfunctions

$$\psi_n(x) \underset{x\to\pm\infty}{\to} 0, \tag{3.7}$$

$$\int |\psi_n|^2 = 1. \tag{3.8}$$

The explicit formulas for the eigenfunctions are not important now, we are more interested in the algebraic structures of the solutions. The most important is the eigenenergies are equidistant. We can define the creation and annihilation operators as

$$a^\dagger \equiv \frac{1}{\sqrt{2m\omega}}(m\omega x - ip)\,, \tag{3.9}$$

$$a \equiv \frac{1}{\sqrt{2m\omega}}(m\omega x + ip)\,, \tag{3.10}$$

using the position and momentum operators. The Hamiltonian can be expressed using these as

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right), \tag{3.11}$$

the factor $\frac{\omega}{2}$ is often neglected as it only gives a change of global phase for the eigenfunctions which is irrelevant. By direct computation we can show

$$\left[H, a^\dagger\right] = \omega a^\dagger. \tag{3.12}$$

Suppose now that there exists solution of (3.6) (from the theory of differential equations it follows that there has to be one), let us refer to it as $|\psi\rangle$ for some energy E, using the commutation relation one can show how the creation operator acts on the eigenfunction

$$Ha^\dagger |\psi\rangle = \left(\left[H, a^\dagger\right] + a^\dagger H\right) |\psi\rangle = (\omega + E)\, a^\dagger |\psi\rangle . \tag{3.13}$$

This shows that if $|\psi\rangle$ is an eigenstate, then $a^\dagger |\psi\rangle$ is also an eigenstate with energy $E+\omega$. By exactly the same procedure it is possible to show that $a|\psi\rangle$ is an eigenstate with energy $E-\omega$. It is this property that gave the operators alternative names *raising* and *lowering* operators.

By a simple thought process one sees that

$$|\varphi_n\rangle = \left(a^\dagger\right)^n |\psi\rangle , \tag{3.14}$$

are eigenstates with energies

$$E_n = E + n\omega, \tag{3.15}$$

and there is infinitely many of them for $n \in \mathbb{N}$. Also since H is positively definite, there has to exist a state $|\psi_0\rangle$ for which

$$a|\psi_0\rangle = 0, \tag{3.16}$$

this is the *ground state* of the harmonic oscillator. An abbreviation for writing the n-th state obtained from the ground state is often used

$$|n\rangle \equiv \left(a^\dagger\right)^n |\psi_0\rangle . \tag{3.17}$$

Using the property of the ground state it is beneficial to realize following relations

$$H|n\rangle = \left(n + \frac{1}{2}\right) |n\rangle , \tag{3.18}$$

$$N|n\rangle \equiv a^\dagger a |n\rangle = n|n\rangle , \tag{3.19}$$

where we have defined the number of excitations operator N,

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle , \tag{3.20}$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle . \tag{3.21}$$

3.3 Application to quantum computation

We would now like to perform similar operations to those presented on qubits but only using a single harmonic oscillator. A natural way of doing so is to create a $1-1$ mapping between at first two qubit states and states $|n\rangle$ of the harmonic oscillator. We know the most important two qubit operation in sense of universality is the C$-$NOT gate which on two qubits performs the transformation

$$|00\rangle \rightarrow |00\rangle , \tag{3.22}$$

$$|01\rangle \rightarrow |01\rangle , \tag{3.23}$$

$$|10\rangle \rightarrow |11\rangle , \tag{3.24}$$

$$|11\rangle \rightarrow |10\rangle . \tag{3.25}$$

Lets choose the proposed mapping as

$$|00\rangle \rightarrow |0\rangle, \tag{3.26}$$
$$|01\rangle \rightarrow |2\rangle, \tag{3.27}$$
$$|10\rangle \rightarrow \frac{|4\rangle + |1\rangle}{\sqrt{2}}, \tag{3.28}$$
$$|11\rangle \rightarrow \frac{|4\rangle - |1\rangle}{\sqrt{2}}. \tag{3.29}$$

Now let us have the system at time $t = 0$ in a state spanned by these four vectors. And we let the system evolve to a time $t = \frac{\pi}{\omega}$. What happens to basis states $|n\rangle$? The Schrödinger equation

$$\exp\left(-i\pi a^\dagger a\right)|n\rangle = (-1)^n |n\rangle, \tag{3.30}$$

immediately gives us for the states involved in the computational basis that $|0\rangle$,$|2\rangle$ and $|4\rangle$ remain unchanged (this should shed some light on the choice of the mapping) and the state $|1\rangle$ goes to $-|1\rangle$. Directly we see that this in the computational basis is exactly the $\mathrm{C-NOT}$ gate. And we achieved it only by the time evolution of the system!

In a general case we would need to have a mapping of n qubits to the states of harmonic oscillator. The basis of such a state space would have 2^n vectors. And the mapping could then be achieved using the $|0\rangle, |1\rangle, \ldots, |2^n\rangle$ states in a very similar way to the one before. And performing arbitrary unitary transformation would be done by matching the eigenvalues of the desired unitary transformation to the eigenvalues of $a^\dagger a$ operator exactly as before (if possible - the spectrum of arbitrary U does not have to be known). Or one could compose it from the elementary transformations.

The obvious drawbacks of this construction are very high energy levels of the oscillator for even a low number of qubits (two level systems), and we have not even considered any way of writing and reading information from/onto qubits as the Hamiltonian of the Harmonic oscillator is very general.

4 Electromagnetic cavities

4.1 Description of the system

As mentioned before, excellent carriers of quantum information are photons and their internal polarization states. However, it is very difficult to make photons interact directly, which is needed for quantum computation. The qubits need to influence each other. For this purpose matter always has to come in play as a mediator, in case of Cavity quantum electrodynamics (cavity QED) the inter-mediator of the influence are atoms.

One of the obstacles when working with light is creating, maintaining and controlling single photon states. The controlled interaction with matter can also be very difficult to achieve. The situation changes if we introduce the *Fabry-Perot* cavities, which enhance the interaction between the electromagnetic field and the atom.

Usually the Fabry-Perot cavity is created as two parallel metallic plates with a certain reflection coefficient R and the light is introduced in as in the figure 4.1. The result of such a setting is a stationary electromagnetic field inside the cavity [19]. For the sake of simplicity we will consider only one dimensional simplification of the cavity that is an abscissa of the length L and think of the inside of the cavity as of an empty space.

The field inside the cavity has to obey the Maxwell equations

$$\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \tag{4.1}$$
$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}, \tag{4.2}$$

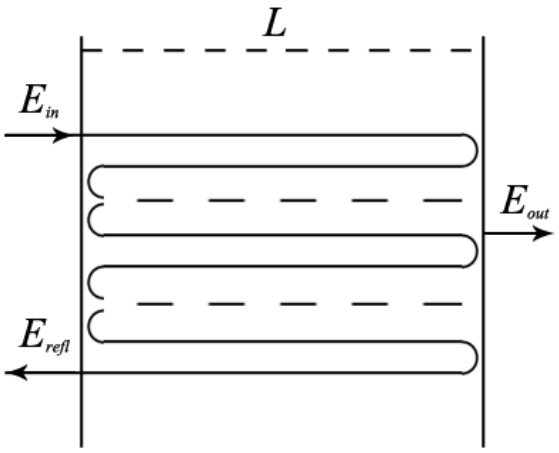


Figure 4.1: Fabry-Perot cavity, the reflections leading to the stationary electromagnetic field

with boundary conditions at the cavity surface

$$\nabla \cdot \vec{E} = 0, \tag{4.3}$$
$$\nabla \cdot \vec{H} = 0. \tag{4.4}$$

The set of equations leads directly to the wave equation

$$\left(\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\right) \vec{E} = 0, \tag{4.5}$$

which is an equation of a wave propagating with the velocity

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \tag{4.6}$$

With the aid of the boundary conditions we get only certain possible modes inside the cavity, each mode can be occupied by photons.

The modes in the cavity have to obey the condition

$$2L = n\lambda, \tag{4.7}$$

where λ is the wavelength and $n \in \mathbb{N}$.

Consequently the wave numbers and frequencies of these modes are respectively

$$k_n = \frac{2\pi}{\lambda} = \frac{\pi}{L} n, \tag{4.8}$$
$$\omega_n = c k_n = \frac{\pi c}{L} n. \tag{4.9}$$

Now we can fill the expressions with real-world numbers to see if we really separated the modes of the light sufficiently. The distance between two consequent modes is

$$\Delta f = \frac{\Delta \omega_n}{2\pi} = \frac{c}{2L}, \tag{4.10}$$

which for $L = 1\text{cm}$ gives frequency separation of 15GHz.

Partial solution of the wave equation corresponding to the mode n is

$$u_n(z) = \sqrt{\frac{2}{L}} \sin k_n z, \tag{4.11}$$

which is already chosen to be normalized, because

$$\int_0^L u_n(z)\, u_m(z)\, dz = \delta_{nm}. \tag{4.12}$$

Because the modes can be separated so well, we are allowed to look always at one mode and drop the indexes. To get a solution for electromagnetic field inside the cavity, we can write for the electric field

$$E_x = E_0 q(t)\, u(z),\ E_y(z) = 0, \tag{4.13}$$

and to have a solution of (4.1) we choose

$$H_y(z) = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \frac{\dot{q}(t)}{\omega} \sqrt{\frac{2}{L}} \cos kz. \tag{4.14}$$

To get the energy, we integrate the expression

$$\begin{aligned} E &= \frac{1}{2} \int_0^L \left(\varepsilon_0 E_x^2(z) + \mu_0 H_y^2(z)\right) dz = \\ &= \frac{\varepsilon_0 E_0^2}{\omega^2} \left(\frac{1}{2} \dot{q}^2(t) + \frac{1}{2} \omega^2 q^2(t)\right), \end{aligned} \tag{4.15}$$

because these are the only non-zero contributions. The energy (4.15) is clearly the energy of a harmonic oscillator from (3) except for the constant prefactor. We can now easily move to operators Q and P and get exactly the harmonic oscillator Hamiltonian and we introduce the creation and annihilation operators in the same way as in (3.9) and (3.10).

If we drop the constant terms in the Hamiltonian of the oscillator, we may write the Hamiltonian for multiple modes as

$$H = \sum_k \omega_k a_k^\dagger a_k, \tag{4.16}$$

where the index k runs through all the modes and we need to add commutation relations for operators of different modes,

$$\left[a_k, a_n^\dagger\right] = \delta_{kn}. \tag{4.17}$$

And the overall quantum state is now

$$|\psi\rangle = |n_1, n_1, \ldots\rangle, \tag{4.18}$$

where n_i is the number of photons in the i-th mode.

4.2 The Hamiltonian

As mentioned before, we would like the light in the cavity to interact with the matter or atoms in the cavity to have an interaction between the photons. There are many models for this interaction and choosing the right one is usually related to the frequency of the incident light as well as characteristics of the matter (heavy or light particles etc.). One of the most common, and for the experiments we have in mind suitable, is the *Jaynes-Cummings model* [19, 20]. For now we will assume in correspondence to

common methods that the matter can be in certain discrete energy states, denoted as $|i\rangle$. For certain frequencies (we show why in section 5.2) it is sufficient to consider only two processes, the photon absorption

$$|j\rangle \to a\,|i\rangle\,, \tag{4.19}$$

where a is the annihilation operator of the electromagnetic field in the cavity and $E_i > E_j$, the corresponding term in the Hamiltonian would be

$$H_a = a\,|i\rangle\,\langle j|\,. \tag{4.20}$$

The second process is the release of a photon to the cavity, which is basically the reverse of the absorption and the related term in the Hamiltonian is therefore

$$H_e = a^\dagger\,|j\rangle\,\langle i| = (a\,|i\rangle\,\langle j|)^\dagger\,. \tag{4.21}$$

The Hamiltonian of the field, atom and the interaction can now be written as

$$H = \sum_k \omega_k a_k^\dagger a_k + \sum_i E_i\,|i\rangle\,\langle i| + \sum_{k,i>j} g_{ij}\left(a_k\,|i\rangle\,\langle j| + a_k^\dagger\,|j\rangle\,\langle i|\right), \tag{4.22}$$

where g_{ij} are usually referred to as coupling constants. In general these can be complex numbers, but usually are considered real (just as we have them real, which can be seen from that the Hamiltonian has to be Hermitian). In any case they describe the strengths of the absorption and emission processes between different energy levels of the atom.

In the proof of the theorem 2.1 we have defined the Pauli matrices. If we consider only two level atom and a single mode of the field, which is a good approximation in many cases [29], we get the famous Jaynes-Cummings Hamiltonian

$$H = \omega a^\dagger a + \frac{\Omega}{2}\,(1+\sigma_3) + g\left(a\sigma^+ + a^\dagger\sigma^-\right), \tag{4.23}$$

where

$$\sigma_3 \equiv Z, \tag{4.24}$$

$$\sigma^\pm \equiv \frac{1}{2}\,(X \pm iY)\,, \tag{4.25}$$

and Ω is just the energy of the excited state of the atom and the Pauli operators are acting on $\{|0\rangle\,, |1\rangle\}$, two states of the atom.

In general we will write the overall state of "the field and the atom" in this particular order, for example we could have a state $|15, 1\rangle$ that would refer to a state with 15 photons and an excited atom. Using this notation we can now guess the solution of the system to be of the form

$$|\psi_n\rangle = \exp\left(-i\omega n t\right)\left(c_0\,|n, 0\rangle + c_1\,|n-1, 1\rangle\right). \tag{4.26}$$

c_0 and c_1 are to be determined from the Schrödinger equation

$$i\frac{d}{dt}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 & g\sqrt{n} \\ g\sqrt{n} & \Delta \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \tag{4.27}$$

where

$$\Delta = \Omega - \omega, \tag{4.28}$$

is called *detuning*. Because the eigenvalues of this matrix are

$$\lambda_\pm = \frac{1}{2}\,(\Delta \pm \Omega_n)\,, \tag{4.29}$$

with

$$\Omega_n^2 = \Delta^2 + 4g^2 n, \tag{4.30}$$

if we consider initial conditions equivalent to the atom being initially in its ground state

$$c_0(0) = 1,\ c_1(0) = 0, \tag{4.31}$$

we can find the solutions of the Schrödinger equation to be

$$c_0(t) = \exp\left(-i\frac{\Delta t}{2}\right)\left(\cos\frac{\Omega_n t}{2} + i\frac{\Delta}{\Omega_n}\sin\frac{\Omega_n t}{2}\right), \tag{4.32}$$

$$c_1(t) = -\frac{2ig\sqrt{n}}{\Omega_n}\exp\left(-i\frac{\Delta t}{2}\right)\sin\frac{\Omega_n t}{2}. \tag{4.33}$$

This enlightens the choice of Ω_n, called the *Rabi frequency* (which is a little different from the Rabi frequency in semi-classical approach as we will see later, which is proportional to the field intensity), which is as an effective frequency of the oscillations between the grounded and the excited state of the atom. It is important that the total state space splits into blocks of different n that are uncoupled.

Lets take a look on some special choices of time evolutions. If we choose time to be

$$\Omega_n t_\pi = \pi, \tag{4.34}$$

which is called the π-pulse, by substituting it into the equations for oscillations we get

$$c_0(t_\pi) \underset{\Delta\to 0}{\longrightarrow} 0, \tag{4.35}$$

$$c_1(t_\pi) \underset{\Delta\to 0}{\longrightarrow} -i. \tag{4.36}$$

The limit $\Delta \to 0$ means $\Omega \sim \omega$, which signifies the relation between the frequency of the electromagnetic field and the energy difference between the excited and the ground state of the atom.

Another pulse that deserves our attention is the 2π-pulse, which refers to the choice of time

$$\Omega_n t_{2\pi} = 2\pi. \tag{4.37}$$

If we do the same limit, we get

$$c_0(t_{2\pi}) \underset{\Delta\to 0}{\to} -1, \tag{4.38}$$

$$c_1(t_{2\pi}) = 0. \tag{4.39}$$

This pulse has some different properties from the π pulse. The first is that it up to phase restores the initial state, if we want the initial state to be restored completely, we need a 4π-pulse. And the second note comes from $c_1(t_{2\pi}) = 0$, independently of Δ, which can be used for manipulation with the phase in c_0.

Note also that the time of the interaction is mode-dependent (it depends on the Rabi frequency), therefore it was beneficial for us to consider single mode only. If more modes are present and they interact with the atoms, very fast dephasing becomes imminent [29].

4.3 Application to quantum computation

So far we have not discussed which states we will choose as the carriers of quantum information nor have we talked about couplings between these states. For example we could couple the photonic states by sending atoms through the cavities and use their electronic transitions. Overall there are two main ideas as how to encode information and perform unitary transformations on them.

Scheme I

The first scheme we will discuss involves a four level system (atom), where a ground state $|0\rangle$ is coupled to a doublet state $|1\pm\rangle$ (state of a two level system), which is successively coupled to another singlet state $|2\rangle$ through two modes of the electromagnetic field. The doublet state can be two Zeeman states of an atom for $l = 1$ when neglecting the state with $m_l = 0$. This can be done if we choose well polarization of photons and the Zeeman states can be achieved with a weak magnetic field that leaves the energy states more or less degenerate.

We make the choices of couplings for two modes in the cavity. We will denote the annihilation operators of the two fields by $a_\pm$ and $b_\pm$, where the indexes apply to two polarizations of each mode. And polarization state of the photons in the fields will be our two qubits. The Hamiltonian that describes this is

$$H = g\left(a_+ |1+\rangle\langle 0| + a_- |1-\rangle\langle 0| + b_+ |2\rangle\langle 1-| + b_- |2\rangle\langle 1+| + \text{h.c.}\right). \tag{4.40}$$

We want to be using polarization state of the two photons in the modes as our two qubits, let the overall state of them be

$$|\psi_{in}\rangle = \left(\alpha_+ a_+^\dagger + \alpha_- a_-^\dagger\right)\left(\beta_+ b_+^\dagger + \beta_- b_-^\dagger\right)|0\rangle. \tag{4.41}$$

And let us choose the encoding to be

$$\begin{aligned} a_+^\dagger |0\rangle &= |0\rangle_a, && (4.42)\\ a_-^\dagger |0\rangle &= |1\rangle_a, && (4.43)\\ b_+^\dagger |0\rangle &= |0\rangle_b, && (4.44)\\ b_-^\dagger |0\rangle &= |1\rangle_b, && (4.45) \end{aligned}$$

where a and b are now used to mark the qubits.

We have to show how to perform a controlled operation on these. We are almost done as we only need to apply a π-pulse (calculated for the lower transition i.e. $\Omega_{10}t = \pi$) to this system while choosing the detunings so that for the lower transition

$$\Delta_{10} \sim 0, \tag{4.46}$$

and for the higher transition the detuning is

$$\Delta_{21} \gg g. \tag{4.47}$$

This in the lower transition causes movement of the population from the a field to states $|1\pm\rangle$ proportionally to the coefficients α_+ and α_- as we discussed earlier. The choice of the detuning for the higher transition causes the coefficients c_0 and c_1 for this transition to be

$$\begin{aligned} c_1(t) &\approx O\left(\frac{g}{\Delta_{21}}\right) \sim 0, && (4.48)\\ c_0(t) &\approx \exp\left(-i\frac{\Delta_{21}t}{2}\right)\left(\cos\frac{\Omega_{21}t}{2} + i\sin\frac{\Omega_{21}t}{2}\right) && (4.49)\\ &= \exp\left(i\frac{\Omega_{21}-\Delta_{21}}{2}t\right). \end{aligned}$$

intuitively we can image this as any population in the states $|1\pm\rangle$ influences immediately the b field in the sense of changing the phases of states $b_\pm^\dagger|0\rangle$. The phase of the lower state of the higher transition is varying with the speed

$$\frac{\Omega_{21}-\Delta_{21}}{2} \sim \frac{g^2}{\Delta_{21}}, \tag{4.50}$$

which allows us to denote

$$\Phi \equiv \frac{g^2}{\Delta_{21}} t, \tag{4.51}$$

the phase. Note that as we mentioned before, by choosing Δ_{21} properly, we can influence the change in the phase. And so what happens after the π-pulse (time defined by the lower transition) with the higher transition? We can illustrate that on the two qubit state.

$$\begin{aligned} &\left(\alpha_+\beta_+ a_+^\dagger b_+^\dagger + \alpha_+\beta_- a_+^\dagger b_-^\dagger + \alpha_-\beta_+ a_-^\dagger b_+^\dagger + \alpha_-\beta_- a_-^\dagger b_-^\dagger\right)|0\rangle \\ \underset{\pi-\text{pulse}}{\rightarrow} &\alpha_+\beta_+ b_+^\dagger |1+\rangle + \alpha_+\beta_- e^{i\Phi} b_-^\dagger |1+\rangle + \alpha_-\beta_+ e^{i\Phi} b_+^\dagger |1-\rangle + \alpha_-\beta_- b_-^\dagger |1-\rangle \\ &= \alpha_+\left(\beta_+ b_+^\dagger + e^{i\Phi}\beta_- b_-^\dagger\right)|1+\rangle + \alpha_-\left(e^{i\Phi}\beta_+ b_+^\dagger + \beta_- b_-^\dagger\right)|1-\rangle . \end{aligned} \tag{4.52}$$

This shows us that the polarization of the photon in the b field is shifted by $\pm\Phi$ depending directly on the state of he photon in the field a. The process can be reversed with another π-pulse as we mentioned before. Practical implementation has to overcome problems with choosing different detunings for the two fields as well as problems with decoherence times, because we need a single photon to trigger the π-pulse as well as reading polarization state of the photon has to be read afterwords. Still this approach is better than encoding information into the presence of the photon in the field, because detecting a single photon's existence is a difficult task.

Scheme II

The second scheme we want to present here is very similar to the first one in that it involves two photons, a and b - in the same mode (with the same detuning). This time we will be using a three level system instead of the four level to couple them. Let us say that photon a couples the transition $|0\rangle \rightarrow |1\rangle$ and photon b the higher transition $|0\rangle \rightarrow |2\rangle$. And we will encode the information into the presence of the photons, which means, that $|1\rangle_a$ is photon is present and $|0\rangle_a$ stands for there is no photon (ground state of the mode). We already mentioned that encoding information this way brings its risks as the detection of a single photon is not a reliable task.

In analogy to the previous scheme the initial state of the two qubits is

$$|\psi\rangle_{in} = \left(\alpha_0 + \alpha_1 a^\dagger\right)\left(\beta_0 + \beta_1 b^\dagger\right)|0\rangle , \tag{4.53}$$

and the described interaction can be written

$$H_I = g\left(a|1\rangle\langle 0| + b|2\rangle\langle 0| + \text{h.c.}\right). \tag{4.54}$$

Now apply a 2π pulse. If there is no photon a present, we know what will happen with the state, we will get

$$\left(\beta_0 + \beta_1 b^\dagger\right)|0\rangle \rightarrow \left(\beta_0 - \beta_1 b^\dagger\right)|0\rangle , \tag{4.55}$$

similarly if the photon b is not present

$$\left(\alpha_0 + \alpha_1 a^\dagger\right)|0\rangle \rightarrow \left(\alpha_0 - \alpha_1 a^\dagger\right)|0\rangle . \tag{4.56}$$

The situation is more complicated if both photons are present, but similarly to the first scheme, a phase shift arises and in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the transformation is

$$U_{2\pi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & e^{i\Phi} \end{pmatrix}, \tag{4.57}$$

which is an elementary logic operation similar to the $\mathrm{C-NOT}$ gate [19, 29]. Another issue of this setup is that a single photon can cause only certain phase changes and not arbitrary ones.

5 Trapped ions

Partly summarized in [32], we have spent time working in this area, which is therefore more familiar to us than the other physical implementations of quantum computers. The theory below is a summary of previous results, some original results concerning state transfer on trapped ions will be presented in the sections about state transfer 8.

5.1 Description of the system

Almost every student when he first hears of the electromagnetic force and its properties thinks to himself that somehow this could be used for levitation and controlling objects invisibly even over some non-zero distances, it reminds him or her of magic. And as it turns out, it is possible to trap charged particles using electromagnetic fields in the *Paul traps* which is a name given to the experiments after their designer, a German physicist Wolfgang Paul.

The first thought that comes to mind is using a static electric field of some sort to create a trap, but we need to keep in mind the Earnshaw theorem (static configurations of charged particles cannot be achieved by electrostatic interactions only), namely for the potential the theorem can be written

$$\nabla^2 V(r) = 0, \tag{5.1}$$

which has to be valid in an empty space. Mathematically this condition forbids any extremes occurring inside the area we are interested in, the only place an extreme might occur is at the edges, which would not help us trap a charged particle in the area. Thus we have to consider time dependent fields.

Let us write down the equation of motion for a charged particle in an oscillating electric field in one dimension

$$m\ddot{x} = \lambda x \cos(\omega t), \tag{5.2}$$

where m is the mass of the particle, x the position, λ is the amplitude of the oscillations and ω is the frequency of the oscillations. Usually this does not make any trapping possible, but if we take $\omega \gg 1$, we can observe a separation of movement into two components, one being a slow motion and the other one quickly varying term (the Kapitza method [33]). To see this, we need to put it as an ansatz into the equation and verify it being a solution. Lets call the quickly varying term *micromotion* and denote it by $\xi(t)$, similarly the slow term will be called *smooth* and denoted $X(t)$. We want to test the ansatz of the combination of these two

$$x(t) = X(t) + \xi(t), \tag{5.3}$$

which put into the equation of motion gives

$$m\ddot{X} + m\ddot{\xi} = \lambda X \cos\omega t + \lambda \xi \cos\omega t. \tag{5.4}$$

Expressing the descriptions of the components mathematically would be

$$\xi \ll X, \tag{5.5}$$

$$\ddot{\xi} \gg \ddot{X}. \tag{5.6}$$

Now we perform a Taylor expansion of the (5.3) and compare the lowest order terms

$$m\ddot{\xi} = \lambda X \cos\omega t. \tag{5.7}$$

The solution

$$\xi(t) = -\frac{\lambda^2 X(t)}{m\omega^2} \cos\omega t, \tag{5.8}$$

can be inserted into the (5.4) to get the $X(t)$ explicitly. The equation for the smooth part

$$m\ddot{X} = \lambda \xi \cos\omega t = -\frac{\lambda^2 X}{m\omega^2} \cos^2\omega t, \tag{5.9}$$

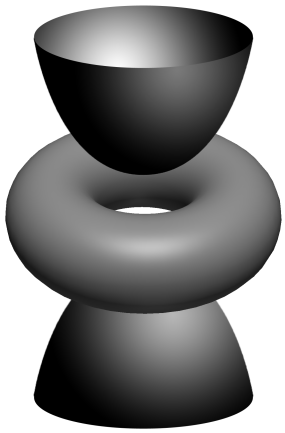


Figure 5.1: Electrode setup for Paul trapping potential

can be further worked with, because we expect the $\omega \gg 1$. We replace the $\cos^2 \omega t$ with its time average $(\frac{1}{2})$ to get the equation of the harmonic oscillator after some relabeling

$$\ddot{X} = -\Omega^2 X, \tag{5.10}$$

$$\Omega \equiv \frac{\lambda}{\sqrt{2}m\omega}. \tag{5.11}$$

Thus we now know how to achieve trapping that results in harmonic oscillations of the particle.

Now that we have shown a simple procedure on how to derive trapping, we can show the most common potential used for trapping in the laboratories, which is

$$V(x, y, z) = V_0 \left(x^2 + y^2 - 2z^2\right) \cos \omega t. \tag{5.12}$$

This potential conveys the trapping in all three directions. The electrode setup capable of creating such a potential is in figure 5.1. The potential barrier created by this setup is in order of couple of electron volts and thus for many purposes quite stable (for cold ions this is a large barrier). The frequencies of the oscillations of the ions in this trap vary from 100kHz to 10MHz and the driving field is usually chosen to be in the order of 100MHz (which illustrates the difference of time scales of the micromotion and the smooth motion).

It should be mentioned that the trapping can only be achieved for cooled ions and the process of their cooling is not trivial, usually a sequence of cooling methods ought to be used. The final steps are usually achieved by sideband cooling or some other very clever process.

When we introduce multiple ions into the trap, the fact that they are not neutral triggers Coulombic interactions between them and strong repulsive forces. The potentials for trapping multiple ions are usually fabricated so that the ions in the trap form a linear chain, that is, the fields are chosen stronger in the $\vec{x}$ and $\vec{y}$ directions while in the $\vec{z}$ direction the ions are allowed to oscillate. These trapping potentials together with the mutual Coulombic repulsion ensure results like in the figure 5.2, where experimental results of ion trapping are presented.

It is not difficult to show that the Coulombic forces result in collective modes of oscillation of the ions [19], it is a similar process to the one used in the so called small oscillations, it is a result of Taylor approximation of the overall potential and transformation into normal coordinates. The images of the modes are very intuitive and are similar to those of particles joined by springs (joined oscillators). The

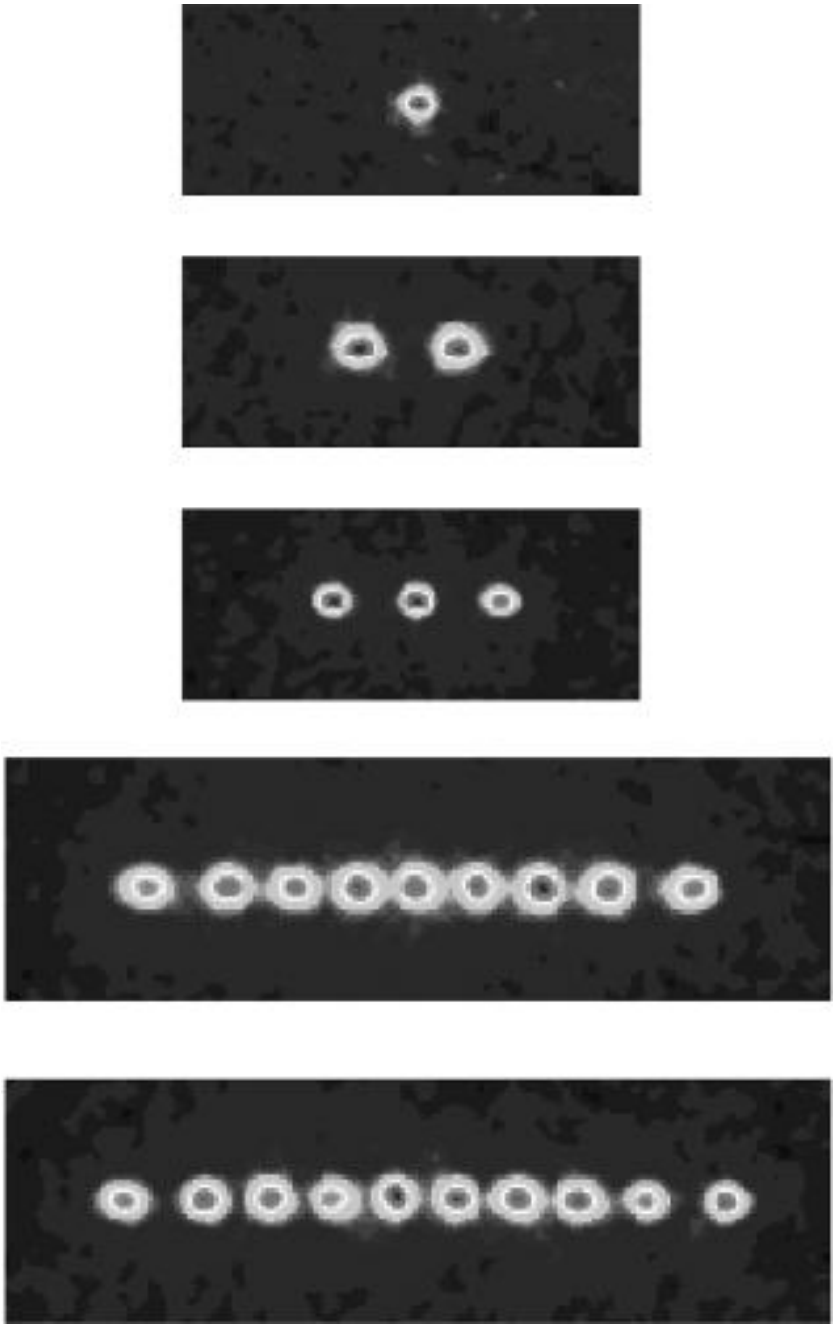


Figure 5.2: Multiple ions in the trap, picture taken from [20, 29]

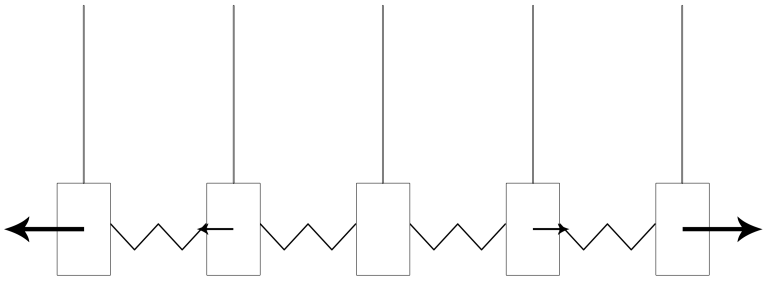


Figure 5.3: Breathing mode

first excited mode that one can think of is the center of mass oscillation mode, where all the particles oscillate with the same frequency Ω from left to right with the center of mass. Another mode of the oscillations is the *breathing* mode, which is very similar to the one of the system of pendulums joined by springs, see figure 5.3. The frequency of this mode is $\Omega' > \Omega$. These two modes are depicted in the figure 5.4. The pictures of the ions are taken from [20, 29], originally from the work of Rainer Blatt's group in Innsbruck.

The collective motion is the basis of any quantum computation performed on trapped ions as it can be coupled to the internal state of each ion and addressed individually using lasers. It is the Coulombic repulsion that separates ions enough so that we are able to focus lasers on a single ion in the chain. This approach is not the most efficient one and there has been a slightly different approach developed [22–24, 27], which uses additional magnetic field applied to the trap which causes the ions to be addressable each on a different frequency of the radiation, which allows us to illuminate all the ions in the trap and influence only one of them. Also the frequency of the lasers gets shifted towards the microwave or the radio-frequency range, instead of optical lasers that need to be used in the setup described above and are the most expensive and experimentally demanding part of the apparatus.

The collective oscillation is called the *information bus*, because if we approximate it by the ground and first excited state of a harmonic oscillator, this can be used to encode one qubit, which can be coupled to the internal states of the ions. The information that is to be used for calculation purposes will be encoded on the internal states of the ions. We will be using ions with $\frac{1}{2}$ spin. Consequently each ion can either have a projection of the spin in the $\hat{z}$ direction $+\frac{1}{2}$ or $-\frac{1}{2}$. Let us call the $+\frac{1}{2}$ state $|e\rangle$ and the other one $|g\rangle$, which will together serve as our computational basis. In the next section we will present past results [19, 20, 29] of how to use lasers and dipole interaction to couple the internal state of the ions to the information bus and then how to use this for computational purposes.

5.2 The Hamiltonian

The description of the illumination of the ions (the ion-field interaction) starts with dipole interaction. For a single ion in the trap the dipole interaction reads

$$H_I = -\vec{\mu} \cdot \vec{B}, \tag{5.13}$$

where $\vec{\mu}$ is the dipole moment proportional to the spin operator

$$\vec{\mu} = \mu_m \vec{S}, \tag{5.14}$$

μ_m is the respective magneton and

$$\vec{S} = \frac{1}{2}\vec{\sigma} \equiv \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{5.15}$$

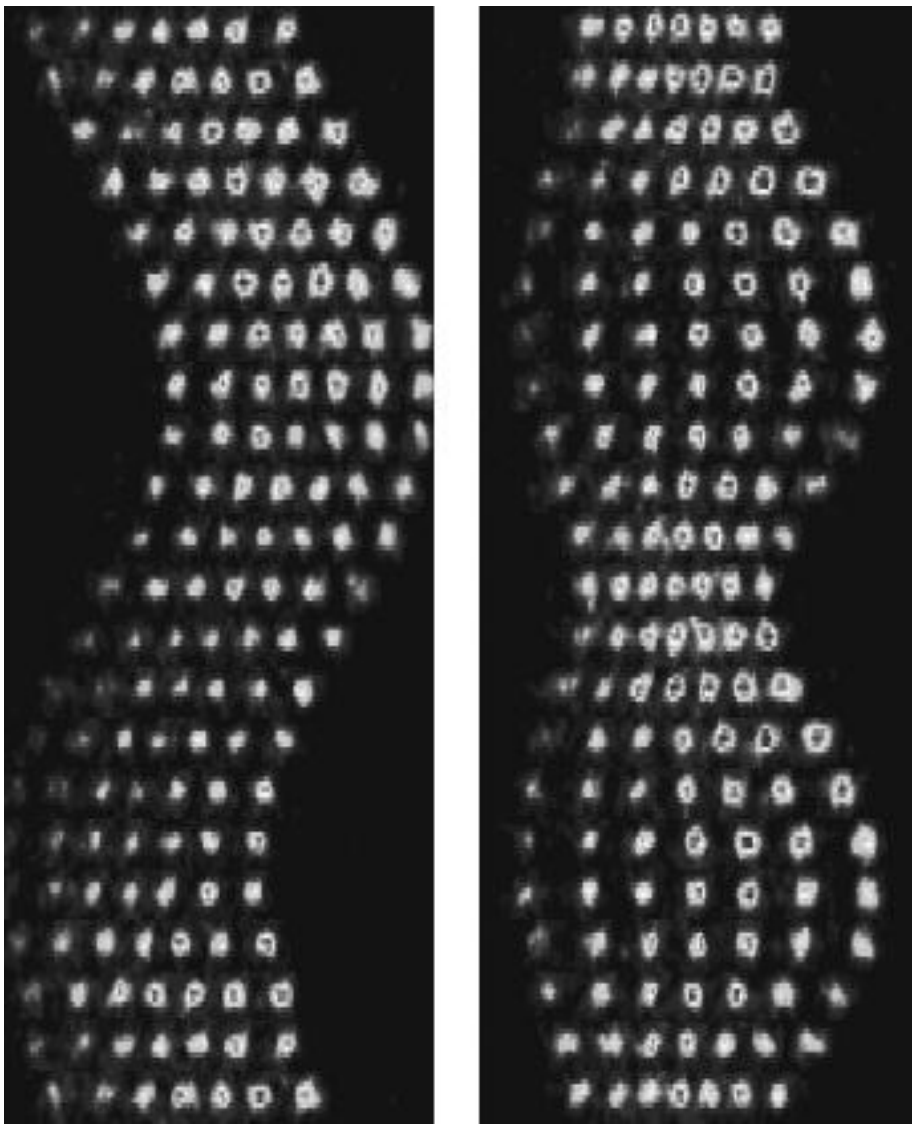


Figure 5.4: Modes of the ion chain, picture taken from [20, 29]

the spin operator is a scaled Pauli vector.

The incident magnetic wave can be described by

$$\vec{B} = B_1 \vec{x} \cos\left(kz - \omega t + \varphi\right), \tag{5.16}$$

B_1 is the field strength, k being the momentum in the $\vec{z}$ direction and ω, φ are the frequency and the phase of the field.

The dipole interaction does not suffice as a description of the wheels behind the experiment, the intuition for this is that the ion in the trap is trapped harmonically and the field that he sees is dependent on its position. Let us denote the energy scale of the harmonic trapping of the ion by ω_z. The ion has been cooled enough so that its position becomes quantized, which allows us to write down its position with the raising and lowering operators of a harmonic oscillator

$$z = z_0 \left(a^\dagger + a\right). \tag{5.17}$$

We expect the ion to be near its ground state so that the width of its oscillation is small compared to the wavelength of the laser, correctly expressing this is saying that the *Lamb-Dicke* parameter

$$\eta \equiv k z_0, \tag{5.18}$$

is small. In this scenario on the contrast to (4.30) we define the Rabi frequency

$$\Omega \equiv \mu_m \frac{B_1}{2}. \tag{5.19}$$

Because [19]

$$S_x = \frac{S_+ + S_-}{2}, \tag{5.20}$$

where $S_\pm$ are atomic raising and lowering operators, we can rewrite the interaction Hamiltonian in the following way

$$\begin{aligned} H_I = -\vec{\mu}\vec{B} \quad \approx \quad & \left[\frac{\Omega}{2}\left(S_+ e^{i(\varphi-\omega t)} + S_- e^{-i(\varphi-\omega t)}\right)\right] \\ & + \left[\frac{i\eta\Omega}{2}\left(S_+ a + S_- a^\dagger + S_+ a^\dagger + S_- a\right)\left(e^{i(\varphi-\omega t)} - e^{-i(\varphi-\omega t)}\right)\right]. \end{aligned} \tag{5.21}$$

Where we see the difference in that the ion is oscillating, the first square bracket is the Jaynes-Cummings interaction model that would suffice if there were no oscillation, and the second bracket is the coupling of the internal state of the ions and the harmonic oscillation state.

We can now use the free particle Hamiltonian

$$H_0 = \omega_0 S_z + \omega_z a^\dagger a, \tag{5.22}$$

to transform into the interaction picture. In the interaction picture we use the operators

$$\begin{array}{cc} S_+(t) = S_+ e^{i\omega_0 t} & S_-(t) = S_- e^{-i\omega_0 t} \\ a^\dagger(t) = a^\dagger e^{i\omega_z t} & a(t) = a e^{-i\omega_z t}. \end{array} \tag{5.23}$$

Based on the frequency of the incident light we can rewrite the Hamiltonian (5.21) as

$$H_I' = \begin{cases} i\frac{\eta\Omega}{2}\left(S_+ a^\dagger e^{i\varphi} - S_- a e^{-i\varphi}\right) & \text{for } \omega = \omega_0 + \omega_z \\ i\frac{\eta\Omega}{2}\left(S_+ a e^{i\varphi} - S_- a^\dagger e^{-i\varphi}\right) & \text{for } \omega = \omega_0 - \omega_z, \end{cases} \tag{5.24}$$

where we have dropped the non-dominant terms (the rotating wave approximation). It is beneficial for us to focus on the

$$\omega = \omega_0 - \omega_z, \tag{5.25}$$

case from now on, which has fixed the laser frequency we want to be using. Now we will transform back into the Schrödinger picture and express the atomic raising and lowering operators in the computational basis

$$H_I = \frac{\Omega}{2}\left(a\left|e\right\rangle\left\langle g\right| + a^\dagger\left|g\right\rangle\left\langle e\right|\right). \tag{5.26}$$

Notice that the approximations yielded a self-adjoint operator, thus we have obtained an effective unitary evolution , this is a non-trivial property as the lasers can easily bring energy into the system.

The complete Hamiltonian for the case $\omega = \omega_0 - \omega_z$ can therefore be written down

$$H = \omega_e\left|e\right\rangle\left\langle e\right| + \omega_g\left|g\right\rangle\left\langle g\right| + \omega_z a^\dagger a + \frac{\Omega}{2}\left(a\left|e\right\rangle\left\langle g\right| + a^\dagger\left|g\right\rangle\left\langle e\right|\right), \tag{5.27}$$

where ω_e and ω_g respond to eigenenergies of $\omega_0 S_z$ and spin projections $+\frac{1}{2}$ and $\frac{-1}{2}$ respectively.

The very last step of this section is to generalize the Hamiltonian to multiple ions in the trap. That is easily achieved by replacing $\Omega \to \frac{\Omega}{\sqrt{N}}$ and replacing the $a^\dagger$ and a operators with operators that act not on the oscillation level of the single ion, but on the information bus (the collective oscillation mode). Note also that this approach does not consider any effect on the incident light and it turns out that this is a very good approximation [19], because the light does not tend to become entangled with the system it interacted with up to a very good degree.

5.3 Application to quantum computation

It can be shown [19] that by choosing appropriately the phase of the incident light and the duration of the interaction it is possible to construct the elementary rotations from the theorem 2.1 for arbitrary angles and in this way apply arbitrary single qubit unitary transformation.

In this section we will be illuminating only a single ion and therefore couple only its internal state to the information bus, not the internal state of any other ion. This is why we need to focus only on the single particle Hamiltonian. Lets call *coupling* the following part of the Hamiltonian

$$\Sigma \equiv a\left|e\right\rangle\left\langle g\right| + a^\dagger\left|g\right\rangle\left\langle e\right|, \tag{5.28}$$

when laser is pointed to our ion, this coupling generates the transformation

$$U_\theta = \exp\left(-i\theta\Sigma\right), \tag{5.29}$$

where θ stands for the phase of the electromagnetic field, the coupling strength (the Rabi frequency) and the time - the length of the interaction.

We will be using a particular ordering of notations, where the last position in the tensor product will always belong to the oscillation level, while the previous ones will be internal spin states of the ions. It is important, the the energy level of the $N+1$-th qubit will therefore be different from the remaining ones, that will come into play later.

From how the time evolution is defined in quantum mechanics, it is not difficult to see

$$U_\theta\left|g\right\rangle\left|g\right\rangle = \left|g\right\rangle\left|g\right\rangle, \tag{5.30}$$

in literature you might see this state referred to as to $|g,0\rangle$ [29] - we do not need this now, in any case it is the ground state both in the internal spin state and the oscillation. We now need to prepare a few tools, one of them being

$$\Sigma^2 = aa^\dagger\left|e\right\rangle\left\langle e\right| + aa^\dagger\left|g\right\rangle\left\langle g\right|, \tag{5.31}$$

which we will let together with the coupling act on some of the the basis states

$$\begin{array}{ll} \Sigma^2\left|e\right\rangle\left|g\right\rangle = \left|e\right\rangle\left|g\right\rangle & \Sigma\left|e\right\rangle\left|g\right\rangle = \left|g\right\rangle\left|e\right\rangle \\ \Sigma^2\left|g\right\rangle\left|e\right\rangle = \left|g\right\rangle\left|e\right\rangle & \Sigma\left|g\right\rangle\left|e\right\rangle = \left|e\right\rangle\left|g\right\rangle, \end{array} \tag{5.32}$$

table that proves to be particularly useful when we expand the time evolution and group certain terms in a following way

$$\begin{aligned} \exp\left(-i\theta\Sigma\right) &= \sum_{k=0}^{\infty} \frac{\left(-i\theta\right)^k}{k!}\Sigma^k \\ &= \sum_{k=0}^{\infty} \frac{\left(-i\theta\right)^{2k}}{(2k)!}\Sigma^{2k} + \sum_{k=0}^{\infty} \frac{\left(-i\theta\right)^{2k+1}}{(2k+1)!}\Sigma^{2k+1}, \end{aligned} \tag{5.33}$$

because we now can simplify all the expressions and just write

$$U_\theta \left|g\right\rangle \left|e\right\rangle = \cos\theta \left|g\right\rangle \left|e\right\rangle - i\sin\theta \left|e\right\rangle \left|g\right\rangle, \tag{5.34}$$

$$U_\theta \left|e\right\rangle \left|g\right\rangle = \cos\theta \left|e\right\rangle \left|g\right\rangle - i\sin\theta \left|g\right\rangle \left|e\right\rangle. \tag{5.35}$$

Special cases of this transformation are a π-pulse ($2\theta = \pi$), which does

$$\left|g\right\rangle \left|e\right\rangle \rightarrow -i \left|e\right\rangle \left|g\right\rangle, \tag{5.36}$$

$$\left|e\right\rangle \left|g\right\rangle \rightarrow -i \left|g\right\rangle \left|e\right\rangle, \tag{5.37}$$

and a 2π-pulse, that acts

$$\left|g\right\rangle \left|e\right\rangle \rightarrow -\left|g\right\rangle \left|e\right\rangle, \tag{5.38}$$

$$\left|e\right\rangle \left|g\right\rangle \rightarrow -\left|e\right\rangle \left|g\right\rangle. \tag{5.39}$$

The generalization to N ions is very straightforward and similar to before, we only change $\Omega \rightarrow \frac{\Omega}{\sqrt{N}}$ and change the meaning of the harmonic raising and lowering operators (or equivalently the last part of the tensor products is now the information bus). It is now possible to show how to perform a $\mathrm{C-NOT}$ gate on two qubits [19, 20, 29] only with these operations and a single qubit phase shift operation, the procedure is exactly the same as in the case of electromagnetic cavities, finding a sequence of pulses that performs it using the action on basis states. We shall be using these elementary operations later in the sections about perfect state transfer 8.

6 Quantum dots

6.1 Description of the system

In this section we would like to present results in the field of quantum dots [19, 20, 29, 34]. Quantum dots belong amongst solid state systems that have the potential to become a quantum computer one day. The solid state systems have the advantage over the others in that in the end if there will exist a commercially available quantum computer, it will have to be driven by today's computers and many of the solid state designs rely on semiconductor physics and are already compatible with them.

It is the revolutionary ideas in the engineering precise up to nanometers that allowed for creation of the quantum dots. It is sometimes possible to manipulate even molecules to the desired shape of the product. The research of quantum dots has attracted attention from various fields of science including biology, chemistry, material science and physics [34]. The quantum dots are solid state systems that restrict movement of charge carriers in all three directions and are of nanometer scale. Simultaneously each dot can be driven by external control, for example by changing the voltage.

Sometimes are the quantum dots called artificial atoms, because the electrons trapped in them exhibit very similar properties to those of atoms in nature. Example of these is discrete energy levels. Further more, properties of quantum dots can be influenced in a wide range by external electromagnetic fields.

We will be focusing on semiconductor dots. These are usually defined to be a structure smaller than 1μm in every dimension. Typically these are fabricated in the size of $10^{-8} \sim 10^{-7}$m. The trapping

potential that limits motion of charge carriers is usually of the order of 0.1eV $\sim$ 1eV, which leads to discrete energy level separations of electrons by meV. And the devices need to be cooled often below 1K temperature to operate.

From semiconductor dots the most promising seem to be the electrostatic quantum dots, which are formed by multiple layers and source and drain electrodes placed as the bottom and top layer respectively. This ensures the control, sometimes more electrodes are placed in the experiment to ensure more precise control. Usually these setups have a cylindrical symmetry and two of the designs are the most explored. The shape of an etched pillar and a layer sequence with a metal cap. The number of dots corresponds to a number of layers present. The trapping potential is created both by conduction band offsets and the electrodes.

The potential is, although it cannot be measured, in some cases (for example the two shapes mentioned) known from theoretical calculations by solving the Poisson equation for the entire dot [34, 35]. And the results are that the potential can be parametrized either by a Gaussian function or generally by an exponential function of the form

$$V = -V_0 \exp\left[-\left(\frac{r}{R}\right)^p - (\frac{|z|}{Z})^p\right], \tag{6.1}$$

where $V_0 > 0$ is the depth of the well, $r = \sqrt{x^2 + y^2}$, $p > 1$ and R and Z are constants characterizing the trapping in the lateral directions $\vec{x}, \vec{y}$ and the vertical direction $\vec{z}$ respectively. For $p = 2$ it describes the Gaussian potential and for $p > 10$ the shape of the well reminds a rectangle.

For quantum computation purposes can serve either spin states of the electrons or excitons inside the quantum dots or charge properties of the dots. Exciton is a bound electron-hole system, which gets created after the absorption of a photon with energy of the magnitude of the forbidden energy of a semiconductor and it has a lifetime in microseconds, after which it recombines and emits a photon.

It seems that the most promising are the calculations based on the spin states of the electrons. Because the spin states have sufficiently long decoherence times ($T_{decoh} \simeq 1\mu$s) as well as relaxation times. Also the spin states are easily manipulated with external magnetic fields.

6.2 The Hamiltonian

Because we would like to present the C $-$ NOT gate, let us now work with only two-electron system in two coupled quantum dots, indexed by $j = 1, 2$. Let us assume that each dot involves different energy levels, that can be ensured for example by adding a magnetic field with non-zero gradient between the two dots, which influences the Zeeman energy states. Let us also assume that the thickness and the material of the layer separating the two dots allows them to be coupled.

We assume the Hamiltonian to be of the form

$$H = H_1 + H_2 + H_{int}, \tag{6.2}$$

where H_j is the one particle Hamiltonian of an electron in magnetic field

$$H_j = \omega_j S_{z,j}, \tag{6.3}$$

$S_{z,j}$ is the spin operator of the j-th particle we used before and ω_j is related to the energy of the Zeeman states.

The interaction part is

$$H_{int} = 4\Omega S_{z,1} S_{z,2}, \tag{6.4}$$

the Ω is the coupling strength between the spins. And we choose our computational basis to be

$$S_{z,1}\left|0,l\right\rangle = \frac{1}{2}\left|0,l\right\rangle, \; S_{z,1}\left|1,l\right\rangle = -\frac{1}{2}\left|1,l\right\rangle, \tag{6.5}$$

where $|k,l\rangle$ are the two-spin states and $k, l \in \{0, 1\}$. And the same choice we make for the $S_{z,2}$ operator. The setups are fabricated so that we can control the Ω, even switch it off.

We mentioned before, $\omega_1 \neq \omega_2$. This allows us to drive the states of the electrons, because each electron responds to interaction with a photon on different frequency, therefore we do not have to focus an electromagnetic field somehow on one electron, but we can enlighten them both and affect only one of them. A photon on the frequency ω_1 causes transitions $|0,l\rangle \leftrightarrow |1,l\rangle$ either by the process of absorption or by stimulated emission. The same is valid for the photon on the frequency ω_2.

6.3 Application to quantum computation

Notice that we can only drive the ions with photons on the energy mentioned for $\Omega = 0$, because after switching the coupling on, the energies shift. To see this, let us first write the eigenstates and the corresponding eigenenergies of the Hamiltonian (6.2) with $\Omega = 0$:

$$\begin{aligned} \epsilon_1 &= -\frac{1}{2}(\omega_1 + \omega_2) \quad : \quad |1,1\rangle \\ \epsilon_2 &= -\frac{1}{2}(\omega_1 - \omega_2) \quad : \quad |1,0\rangle \\ \epsilon_3 &= +\frac{1}{2}(\omega_1 - \omega_2) \quad : \quad |0,1\rangle \\ \epsilon_4 &= +\frac{1}{2}(\omega_1 + \omega_2) \quad : \quad |0,0\rangle , \end{aligned} \tag{6.6}$$

and second the eigenenergies for $\Omega \neq 0$, notice that it so happens that the Hamiltonian has the same eigenvectors:

$$\begin{aligned} \epsilon_1 + \Omega \quad &: \quad |1,1\rangle \\ \epsilon_2 - \Omega \quad &: \quad |1,0\rangle \\ \epsilon_3 - \Omega \quad &: \quad |0,1\rangle \\ \epsilon_4 + \Omega \quad &: \quad |0,0\rangle , \end{aligned} \tag{6.7}$$

and it is this shift by $\pm\Omega$ in the eigenenergies that allows quantum computation to take place.

Because we can apply pulse on frequency that selectively induces change in the state of one electron depending on the state of the other. For example a pulse on the frequency

$$\omega = \omega_2 - 2\Omega, \tag{6.8}$$

switches states $|1,0\rangle \leftrightarrow |1,1\rangle$ and any other state is left untouched. If we write this in the computational basis, we see it is exactly the $\mathrm{C-NOT}$ gate.

Note also that this is only one of the explored ways of implementing the quantum computation on quantum dots. There are other methods based on measuring spin through charge, spontaneous magnetization of dots, electron spin resonance and measurements of singlet-triplet splitting with the help of a Faraday rotation [34].

This model of quantum dots and spin-spin interaction is rather simplified, but from previous sections we should have an intuition as to why it can be used. The description should enlighten basics of quantum dot systems.

Part III
State transfer

State transfer is a means for accomplishing points 6 and 7 of the DiVincenzo criteria for a functioning quantum computer. Since there will always be need to store and share information, there will also be the need to move it, whether it is from a large storage media or just as a means of communication. As we know from the theorem 2.4 the cloning of quantum states is not allowed by the laws of nature, thus we need to be looking for methods of transferring the state in accordance with this principle.

There are several ways of performing the transfer of a quantum state. In the literature, generally two main groups of techniques are identified. One group is based on the use of active, the other is based on passive quantum wires. The lengthy definition can be found in the reference [15], in the following we will limit ourselves to a detailed illustration of the two approaches. The difference is in the amount of control one has to put into the information transfer. Active quantum wires require lots of external control. Example of an active quantum wire can be a photon, on whose polarization state we imprint one qubit, then let the photon travel to where it is needed through optical media and then write the information onto the target qubit. The qubit that travels is usually called the *flying qubit*. In active quantum wires a flying qubit implementation is usually chosen to be very stable and reliable even long distances and long period of time. The weak point of it is the large amount of external control, there is no reason why any physical system that can serve as a qubit would have to be able to interact with the photon for example and transferring the information from the source qubit onto the flying one and back can be extremely difficult to perform without errors. Sometimes active quantum wires are defined as wires that for some time become decoupled from the systems they transfer the information between.

Passive quantum wires are setups that require minimal or none external control, because for the transfer the designs intend to use the quantum computer itself or at least fabricate the wire between computer components from the same physical systems as the computer. This ensures very high compatibility with any quantum computer design. The other side of passive wires is that it is hard to imagine building such a passive wire over very long distances. This approach is more suitable for communication within the computer itself, over short distances, between registers. Passive quantum wires have attracted large amount of attention over the past years [11–17, 21, 25, 26, 36–39]. Articles have been written both about system-specific designs for state transfer as well as many general ideas regarding for example spin networks have been published.

Recently the attention is shifting from theoretical perfect state transfer to problems of a real world. As some protocols for state transfer are known, physicists are working on practical designs that consider errors both from fabrication and running an imperfect quantum computer, that consider needs to route the information to various places, transfer of multiple excitations or even create entangled pairs of qubits using the passive wires.

State transfer is in its essence a unitary operation, because we only want to transfer information, not destroy or create it. To show how to perform arbitrary unitary transformation is the first task designers of quantum computer candidates face. Therefore it should be possible to perform the state transfer on anything that can be considered a quantum computer candidate. Performing the unitary transformation however does not always have to be passive, because typically this is done by series of pulses, voltage modifications etc.

We are looking for ways to achieve the state transfer with the time evolution only, usually this is a task of choosing proper couplings between the qubits. In the following sections we present an overview of past work in the field, which is derivation of a rather general set of tools for state transfer, then we show how to apply these tools on trapped ions quantum computer in section 8, this section contains some original results, and finally we present our recent results on quantum networks with bendings in section IV.

7 Perfect state transfer

7.1 Transfer of single excitation

Let us first present results [11–17, 21, 25, 26, 36–40]. First we need to define what we mean under the term PST. Let us consider a finite network of n sites labeled

$$\mathbb{S}_n = \{1, 2, 3, \dots n\}\,, \tag{7.1}$$

where at each site sits a qubit and the qubits are mutually coupled, we do not specify how yet. Usually the information to be transferred is encoded on the first k qubits,

$$|\psi_{gen}\rangle = |\psi_1, \dots, \psi_k, \text{arbitrary state}\rangle\,, \tag{7.2}$$

state transfer could be the unitary operation that performs

$$|\psi_{gen}\rangle \underset{U_{PST}}{\rightarrow} |\text{arbitrary state}, \psi_1, \dots, \psi_k\rangle\,. \tag{7.3}$$

Would we know this transformation, performing transfer even to some subsets of nodes should be possible, because we expect to have a complete control over the interactions of qubits and we could redesign the network based on what we want to achieve. Of course in reality it does not have to be possible to create the same couplings between nodes 1 and n as it would be between 1 and 2.

Following previous work we shall first focus on a transfer of single excitation and later use the knowledge for multiple excitations. Transfer of a single excitation is in many cases enough for a transfer of multiple excitations, because we can just perform the procedure multiple times (but we are neglecting the possibility of the excitations themselves to interact). We want to focus on a general network without mentioning if we have any physical implementation in mind and see what are the restrictions for the couplings between the qubits.

We shall restrict the condition (7.3) and the state of the network somewhat now. We will restrict our studies to energy (or number of excitations) conserving Hamiltonians. We assume $|0\rangle$ state is chosen the ground state of every qubit. Thus we need to focus only on transfer of the state $|1\rangle$, because if we can faithfully transfer it, we can transfer any linear combination of the two. Moreover at the time $t = 0$ we assume our network to be in the state

$$|\psi_0\rangle = |1\rangle\,|0\rangle \dots |0\rangle = |1\rangle\,, \tag{7.4}$$

where on the right hand side of the expression we used the notation, where $|j\rangle$ stands for the state with excitation present at the j-th site introduced earlier. Because, as mentioned above, we are interested in energy conserving Hamiltonians only, we will from now on work in a single excitation subspace of the general state space, the set

$$\mathbb{S} = \{|j\rangle \mid j = 1, \dots n\}\,, \tag{7.5}$$

can be used to form the computational basis. The difference between the equation (7.3) and this is that we want the remaining sites to be in ground states. This should not be a problem, because we are expecting to have a control over the individual states of qubits. We define state transfer as a unitary transformation expressed in the computational basis as a permutation

$$P = \begin{pmatrix} 0 & & & \\ \vdots & & \widetilde{P} & \\ 0 & & & \\ 1 & 0 & \dots & 0 \end{pmatrix}, \tag{7.6}$$

where $\widetilde{P}$ is a permutation matrix. This is more restrictive than (7.3), because we are explicitly defining how the transformation should act on the qubits that do not contain any information.

The idea of passive wires is to look for Hamiltonians that after a well defined time t lead to the permutation through the time evolution of quantum mechanics. That is with no external control in the process. We want to prepare the network and let it transfer the excitation by itself, the fundamental equation we will attempt to solve is

$$U(t) = e^{iHt} = P, \tag{7.7}$$

where the unknowns are the Hamiltonian and the time t. It has proven useful to divide the process of solving for one-cycle permutations and many-cycle permutations.

One-cycle permutations

From all the one-cycle permutations of the form (7.6) we only need to consider one, namely

$$P = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \tag{7.8}$$

because we can get all the other $(n-2)! - 1$ permutations from this one just by relabeling the sites $\{2, \dots n-1\}$. Let us now present the whole class $\mathfrak{C}_H$ of Hamiltonians that give the result (7.7) [15]. It is straightforward to find the eigenvalues of the permutation (7.8)

$$\sigma = \left\{ \lambda_j \mid \lambda_j = \exp\left(i2\pi \frac{j}{n}\right), j = 0, 1, \dots, n-1 \right\}, \tag{7.9}$$

with the corresponding eigenvectors, expressed in the computational basis,

$$\left|y_{\lambda_j}\right\rangle = \frac{1}{\sqrt{n}} \sum_{\alpha \in \mathbb{S}_n} \lambda_j^{\alpha - 1} |\alpha\rangle = \frac{1}{\sqrt{n}} \left(1, \lambda_j^1, \lambda_j^2, \dots, \lambda_j^{n-1}\right), \tag{7.10}$$

and use these to write the spectral decomposition of the permutation (7.8)

$$P = \sum_{\lambda_j \in \sigma} \lambda_j |\lambda_j\rangle \langle\lambda_j| . \tag{7.11}$$

Because it is true for finite dimensional matrices - from definition of matrix exponential - that if matrix A is in its diagonal basis of the form

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \tag{7.12}$$

then the exponential expressed in the same basis is

$$\exp A = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}, \tag{7.13}$$

we can directly construct the first Hamiltonian, solution of (7.7)

$$H = \frac{1}{t} \sum_{\lambda_j \in \sigma} \arg(\lambda_j) |\lambda_j\rangle \langle\lambda_j| , \tag{7.14}$$

where $\arg(\lambda_j)$ is the complex phase of λ_j. To get the whole class of Hamiltonians from this one, we can use the periodicity of $\exp(i\varphi)$ and shift each eigenenergy by an arbitrary multiple of 2π. Members of the class then can be expressed to be

$$H_{\vec{l}} = \frac{1}{t}H + \frac{1}{t}\sum_{\lambda_j\in\sigma}\frac{2\pi l_{\lambda_j}}{t}\left|y_{\lambda_j}\right\rangle\left\langle y_{\lambda_j}\right| = \frac{1}{t}\sum_{\lambda_j\in\sigma}\left[\arg(\lambda_j)+2\pi l_{\lambda_j}\right]\left|y_{\lambda_j}\right\rangle\left\langle y_{\lambda_j}\right|, \tag{7.15}$$

where $\vec{l}\in\mathbb{Z}^n$. And the class is

$$\mathfrak{C}_H = \left\{H_{\vec{l}}\,|\,\vec{l}\in\mathbb{Z}^n\right\}. \tag{7.16}$$

To get a grip on which Hamiltonians we have found, let us rewrite the Hamiltonian and denote

$$H_{\vec{E}} = \sum_{\lambda_j\in\sigma}\epsilon_{\lambda_j}\left|y_{\lambda_j}\right\rangle\left\langle\lambda_j\right| = \sum_{\lambda_j\in\sigma}\epsilon_{\lambda_j}\Pi_{\lambda_j}, \tag{7.17}$$

where Π_{λ_j} is a projector onto the subspace spanned by the eigenvectors of λ_j and

$$\epsilon_{\lambda_j} = \frac{\arg(\lambda_j)+2\pi l\lambda_j}{t}\in\mathbb{R}, \tag{7.18}$$

is a different parametrization of the class by the real energy vector. Now it is easy to express the Hamiltonian in the computational basis

$$H_{\vec{E}} = \sum_{\alpha,\beta\in\mathbb{S}_n}\sum_{\lambda_j\in\sigma}\epsilon_{\lambda_j}\lambda_j^{\alpha-\beta}\left|\alpha\right\rangle\left\langle\beta\right|, \tag{7.19}$$

and divide it into two parts

$$H_{\vec{E}} = H^0_{\vec{E}} + \nu_{\vec{E}}. \tag{7.20}$$

$H^0_{\vec{E}}$ is the diagonal part

$$H_{\vec{E}} = \sum_{\alpha\in\mathbb{S}_n}E_\alpha\left|\alpha\right\rangle\left\langle\alpha\right|, \tag{7.21}$$

and

$$\nu_{\vec{E}} = \sum_{\alpha\neq\beta\in\mathbb{S}_n}\mathrm{G}(\alpha,\beta)\left|\alpha\right\rangle\left\langle\beta\right|, \tag{7.22}$$

is the interaction part. The coefficients are defined as $E_\alpha = \sum_{\lambda_j\in\sigma}\epsilon_{\lambda_j}$ and $\mathrm{G}(\alpha,\beta) = \sum_{\lambda_j\in\sigma}\epsilon_{\lambda_j}\lambda_j^{\alpha-\beta}$. From how the coefficients are defined it is possible to see a very important fact, the sites involved in the same cycle (or one-cycle permutation) must have the same energy levels!

Let us also mention the theorem from reference [15]

Theorem 7.1. *For networks of arbitrary dimension ($n>2$), there exists no nearest-neighbor-interaction Hamiltonian satisfying condition* (7.7) *in the framework of permutation* (7.8)*.*

Proof. The proof has been published in [15], where interested reader can find it. □

Many-cycle permutations

We want to use the tools of previous section for dealing with many-cycle permutations. Each such a permutation can be decomposed into disjoint sub-cycles that involve a subset of nodes instead of all of them. We can treat each sub-cycle as if it were a one-cycle permutation from the previous section 7.1 and then compose the results. Let us consider a cycle of length $d<n$. The subset of nodes it acts on we will denote $\mathbb{S}_d\subset\mathbb{S}_n$.

This cycle will have exactly the same eigenvalues as before, explicitly

$$\sigma_d = \left\{\lambda_j\mid\lambda_j=\exp\left(2\pi i\frac{j}{d}\right), j=0,1,\ldots d-1\right\}, \tag{7.23}$$

and the the corresponding eigenvectors projected onto the computational basis vectors are

$$\left\langle \alpha \left| v_{\lambda_j}^{(k)} \right\rangle = \begin{cases} \frac{\lambda_j^{\alpha-1}}{\sqrt{d}} & \text{for } \alpha \in \mathbb{S}_d \\ 0 & \text{otherwise} \end{cases} . \right. \tag{7.24}$$

From the expression of the spectrum we can clearly see that the spectrum of a many-cycle permutation is always degenerate as at least the λ_0 is present as many times as we have cycles. The degeneracy of the eigenvalue λ_j we will denote δ_{λ_j} and the subspace spanned by eigenvectors belonging to λ_j - $\mathcal{E}_{\lambda_j}$. It follows from the Gram-Schmidt orthonormalization process that we can choose an orthonormal basis in every subspace $\mathcal{E}_{\lambda_j}$, lets call the vectors $\left\{ \left| y_{\lambda_j}^{(k)} \right\rangle \right\}$. Since these vectors are linear combinations of eigenvectors belonging to λ_j, they are also eigenvectors belonging to λ_j. And again, we can shift each eigenenergy of the Hamiltonian by 2π, but this time we can do it for every vector from the bases for every eigenvalue, resulting in a wider class of Hamiltonians

$$\begin{aligned} \mathcal{H}_{\vec{l}} &= \frac{1}{\tau} \sum_{\lambda_j} \sum_{k=1}^{\delta_{\lambda_j}} \left[arg(\lambda_j) + 2\pi l_{\lambda_j}^{(k)} \right] \left| y_{\lambda_j}^{(k)} \right\rangle \left\langle y_{\lambda_j}^{(k)} \right| \\ &= \frac{1}{\tau} \sum_{\lambda_j} \sum_{k=1}^{\delta_{\lambda_j}} \varepsilon_{\lambda_j}^{(k)} \left| y_{\lambda_j}^{(k)} \right\rangle \left\langle y_{\lambda_j}^{(k)} \right| , \end{aligned} \tag{7.25}$$

for $\vec{l} \in \mathbb{Z}^n \equiv \left\{ (l_{\lambda_0}^{(1)}, \ldots, l_{\lambda_0}^{(\delta_{\lambda_0})}; l_{\lambda_1}^{(1)}, \ldots, l_{\lambda_1}^{(\delta_{\lambda_1})}; \ldots) \mid l_{\lambda_j}^{(\delta_{\lambda_j})} \in \mathbb{Z} \right\}$.

7.2 Transfer of multiple excitations

Extension to the previous research has been published [17] in that it allows for transfer of multiple excitations. The core difference between the two is that if we consider multiple indistinguishable excitations, we can introduce interactions between the excitations themselves, which can be on-site or even inter-site. We can consider various effects from the nature of excitations like bosonic or fermionic characteristics. All of this directly applies to for example quantum dots, which we have presented in the section 6, because there the excitations can be electrons and these both tend to interact via Coulombic interaction and are fermions that can be present on the same site (same dot).

The systems considered for quantum computation - as we have seen - usually have more degrees of freedom, but only some of those are used for information encoding. Typically, the nodes of network that are not used for information encoding are prepared in the ground states and the subset of nodes used for information encoding contains some excitations. Again, we are interested in number-of-excitations-preserving Hamiltonians.

We need to alter our notation a little to contain this generalized model. Firstly, the Hilbert space we are working on is now N-excitation Hilbert space (which suffices because we only work with energy conserving Hamiltonians) of M nodes spanned by vectors

$$\mathbb{S} = \left\{ |n_{1,\sigma}, n_{2,\sigma}, \ldots, n_{M,\sigma}\rangle \mid \sum_{j,\sigma} = N \right\}, \tag{7.26}$$

where $n_{j,\sigma}$ stands for number of excitations in the state σ at the j-th site. And σ indexes all degrees of freedom that can be used for information encoding. For $N = 1$ we should be getting the same results as previously.

The idea of solving the problem relies on second quantization. We want to benefit from how we were always writing the Hamiltonians with creation and annihilation operators. Generally the initial state of the network can be written

$$|\psi(0)\rangle = f(a_{j,\sigma}^{\dagger}) |0\rangle , \tag{7.27}$$

where $a^\dagger_{j,\sigma}$ are creation operators that create excitation in the state σ at site j and f is some function of these. With the same notation we can write the desired result

$$|\psi(t)\rangle = f\left(a^\dagger_{P(j),\sigma}\right)|0\rangle\,, \tag{7.28}$$

where P is the permutation similarly to previous sections, only this time the permutation is a permutation acting on basis states and not the nodes (it does not only switch i and j in the states). The rank of this matrix is inevitably M^N, which is giving correct results for $N = 1$. To be able to use the tools from previous text, we can think of the M^N dimensional state space as of a state space of single excitation (that is just relabeling the states). We will now present the results for $N = 2$ from [17], which are easily valid for $N > 2$ as well.

The state space $\mathscr{H}_2$ for two excitations is spanned by vectors, where two positions of the excitations need to be specified and the state they are in, therefore

$$\mathscr{H}_2 = \mathrm{span}\,\{|i,\mu;j,\nu\rangle\}\,, \tag{7.29}$$

where $|i,\mu;j,\nu\rangle \equiv |1_{i,\mu}\rangle\,|1_{j,\nu}\rangle$. Now we will fix the initial state of the network to be

$$|\psi(0)\rangle = |s_1,\mu;s_2,\nu\rangle\,. \tag{7.30}$$

It is important to notice the ordering of the excitations. We can even divide the Hilbert space to three subspaces, which will be important later

$$\mathscr{H}_2^{(<)}, \text{spanned by } \{|i,\mu;j,\nu\rangle|\, i<j\}\,, \tag{7.31}$$

$$\mathscr{H}_2^{(=)}, \text{spanned by } \{|i,\mu;j,\nu\rangle|\, i=j\}\,, \tag{7.32}$$

$$\mathscr{H}_2^{(>)}, \text{spanned by } \{|i,\mu;j,\nu\rangle|\, i>j\}\,. \tag{7.33}$$

One of the cases we can consider is when all these subspaces are decoupled and the excitation cannot jump between them. Possible permutations consequently have a block-diagonal form of

$$P = \begin{pmatrix} P^{(<)} & & \\ & P^{(>)} & \\ & & P^{(=)} \end{pmatrix}, \tag{7.34}$$

in which case we can solve the problem for each block separately from knowledge of the eigenvalues for each $P^{(\cdot)}$.

In general case, however, the spaces are not decoupled and one has to work as mentioned as if with single excitation, but in a state space of dimension M^N at worse. It has been presented how to work with differently decoupled spaces and interacting and non-interacting excitations (bosons or fermions) in [17].

We will present how to formalize differences between interacting and non-interacting excitations from [17]. In case of two non-interacting excitations, the total energy is simply

$$E^{(\mu,\nu)}_{i,j} = \epsilon_{i,\mu} + \epsilon_{j,\nu}, \tag{7.35}$$

while for interacting excitations it can be written

$$\widetilde{E}^{(\mu,\nu)}_{i,j} = E^{(\mu,\nu)}_{i,j} + U^{(\mu,\nu)}_{i,j}. \tag{7.36}$$

The additional term can represent all kinds of interactions, starting with on-site and inter-site interactions over to various interactions between different degrees of freedom.

Several results have even been published for the Hamiltonians of a very general form, which have been investigated thoroughly in past articles [38, 39]

$$H = \sum_{i,\sigma}\epsilon_{i,\sigma}n_{i,\sigma} + \frac{1}{2}\sum_{i,k}\sum_{\sigma,\sigma'}U^{(\sigma,\sigma')}_{i,k}n_{i,\sigma}\left(n_{k,\sigma'} - \delta_{i,k}\delta_{\sigma,\sigma'}\right) + \sum_{i<k}\sum_{\sigma}J_{i,k}\left(a^\dagger_{i,\sigma}a_{k,\sigma} + \text{h.c.}\right), \tag{7.37}$$

where

$$n_{i,\sigma} \equiv a^{\dagger}_{i,\sigma} a_{i,\sigma}. \tag{7.38}$$

The last part of (7.37) is the interaction between different nodes and is the reason that PST is possible at all. This Hamiltonian is met under various conditions, even in the ideas of quantum computers we presented. Results with this Hamiltonian and the tools presented were published in [17] that concerned decoupled spaces, coupled spaces with both nearest-neighbor and beyond-nearest-neighbor interactions. The solutions always followed the methodology described, but sometimes analytical solutions were not obtained and instead numeric methods had to be used.

8 State transfer on trapped ions

In this section we apply the previous method to the system of trapped ions. These are our results, previously described in the reference [41].

8.1 Illuminating single ion

Using the theory that we presented in the section 5 about trapped ions, one could immediately show that any permutation on the internal states of ions can be achieved because we have shown any unitary operation can be performed and permutations are unitary. The universal procedure would however rely on decomposing the transformation into $\mathrm{C-NOT}$ gate and single qubit rotations, performing these is certainly possible and we can perform any unitary transformation exactly, but it does not have to be the most efficient way of doing so.

Instead we found a sequence of pulses that can be one-by-one applied in the designed order to achieve the transformation. Let us remind here how the pulses work on trapped ions. The π-pulse moves the excitation between the collective oscillation and internal level of the ion while adding $-i$ overall factor to the state and the 2π-pulse does not move the excitation, but adds a prefactor of -1 to the state.

If we use the introduced notation for the overall state, where first N qubits are the internal spin states of ions and the $(N+1)$-th qubit is the collective oscillation and denote $U^k_{2\theta}$ a respective pulse applied to ion k, we can write down following sequence of pulses and its result

$$\begin{array}{cccc}
\hline
 & |e\rangle_1 \ldots |g\rangle_N |g\rangle_{N+1} & |g\rangle_1 \ldots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \ldots |g\rangle_N |e\rangle_{N+1} \\
U^1_\pi & \downarrow & \downarrow & \downarrow \\
 & -i |g\rangle_1 \ldots |g\rangle_N |e\rangle_{N+1} & |g\rangle_1 \ldots |e\rangle_N |g\rangle_{N+1} & -i |e\rangle_1 \ldots |g\rangle_N |g\rangle_{N+1} \\
U^N_\pi & \downarrow & \downarrow & \downarrow \\
 & |g\rangle_1 \ldots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \ldots |e\rangle_N |g\rangle_{N+1} & -i |e\rangle_1 \ldots |g\rangle_N |g\rangle_{N+1} \\
U^1_\pi & \downarrow & \downarrow & \downarrow \\
 & |g\rangle_1 \ldots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \ldots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \ldots |g\rangle_N |e\rangle_{N+1} \\
\hline
\end{array} \tag{8.1}$$

If we express the transformation in the basis of single excitation subspace (7.5), we get

$$U = \begin{pmatrix}
0 & & & \cdots & & & 0 \\
 & 0 & 1 & 0 & \cdots & 0 & \\
 & & 0 & \ddots & & \vdots & \\
\vdots & \vdots & \vdots & & \ddots & 0 & \vdots \\
 & 0 & 0 & \cdots & 0 & 1 & \\
 & 1 & 0 & \cdots & & 0 & 0 \\
0 & & & \cdots & & 0 & 1
\end{pmatrix}. \tag{8.2}$$

which if we focus on the top-right $N \times N$ block, is exactly the permutation (7.8).

This procedure is not covered by the tools from section 7, because this Hamiltonian does not belong to any of the classes derived. Reason behind this is that this procedure does not involve a

single Hamiltonian for the whole time but rather a sequence of Hamiltonians as the laser is turned on and off, this is a possible path for generalization of the theory - to sequences of Hamiltonians that one can move in time. It however might not be beneficial, because this procedure somewhat violates the condition on minimal external control. Another problem with the generalization might be that the interaction with light does not always conserve the number of excitations and energy of the system, we only achieved that by choosing appropriate frequencies of light and durations of the pulses.

Also because we know that nodes involved by any sub-cycle of the permutation must have the same energy levels, there can be no permutation in the presented framework that would move the excitation to the $(N+1)$-th qubit, because it has different energy level from the rest and we need to keep that in mind.

8.2 Illuminating multiple ions

We were, however, investigating Hamiltonians that could belong to one of the classes presented amongst the trapped ions, because they have constant couplings between all the nodes. Theoretically we can illuminate all the ions in the trap by light on the resonant frequency (5.25). We intentionally wrote theoretically, because that is not the current way of working with trapped ions and it could bring heat into the system and cause instabilities.

For illumination of all the ions we need to write down the Hamiltonian of such a thought experiment. Let us write the basis states of a single excitation subspace in a following way

$$|e\rangle_n \equiv |g\rangle_1 \dots |e\rangle_n \dots |g\rangle_N |g\rangle_{N+1}\,, \tag{8.3}$$

and by

$$|g\rangle_n \equiv |g\rangle_1 \dots |g\rangle_n \dots |g\rangle_N |e\rangle_{N+1} \tag{8.4}$$

we denote the (last basis) state with the excitation in the oscillation qubit, this notation is chosen to underline the fact that all the ions are in the ground state. The interaction can now be written in the nice form of

$$\Sigma = a \sum_{n=1}^{N} |e\rangle_n \langle g|_n + a^\dagger \sum_{n=1}^{N} |g\rangle_n \langle e|_n\,, \tag{8.5}$$

where $a^\dagger$ and a are harmonic raising and lowering operators for the collective oscillation. The complete Hamiltonian from the dipole interaction of every ion reads

$$H = \omega_e \sum_{n=1}^{N} |e\rangle_n \langle e|_n + \omega_g \sum_{n=1}^{N} |g\rangle_n \langle g|_n + \omega_z a^\dagger a + \frac{\Omega}{2}\Sigma. \tag{8.6}$$

To understand better how this Hamiltonian works, we would like to rewrite it into the computational basis vectors, listed with the particular ordering below

$$\begin{aligned} |\psi_1\rangle &= |e\rangle_1 |g\rangle_2 \dots |g\rangle_N |g\rangle_{N+1}\,, \\ |\psi_2\rangle &= |g\rangle_1 |e\rangle_2 \dots |g\rangle_N |g\rangle_{N+1}\,, \\ &\vdots \\ |\psi_N\rangle &= |g\rangle_1 |g\rangle_2 \dots |e\rangle_N |g\rangle_{N+1}\,, \\ |\psi_{N+1}\rangle &= |g\rangle_1 |g\rangle_2 \dots |g\rangle_N |e\rangle_{N+1}\,. \end{aligned} \tag{8.7}$$

In this basis the Hamiltonian is of the form

$$H = \begin{pmatrix} \omega_e & 0 & \cdots & 0 & \frac{\Omega}{2} \\ 0 & \omega_e & & & \\ \vdots & & \ddots & & \vdots \\ 0 & & & \omega_e & \frac{\Omega}{2} \\ \frac{\Omega}{2} & & \cdots & \frac{\Omega}{2} & \omega_z \end{pmatrix}, \tag{8.8}$$

which is a Hamiltonian where all the qubits are coupled to the last one - the *bus*, which is a general class of Hamiltonians that might not be applicable just for the trapped ions. And we know that the theorem 7.1 does not limit us to many-cycle permutations, because the Hamiltonian is not nearest-neighbor.

We also know that nodes involved in any sub-cycle of any permutation must have the same energy levels and any permutation that would transport the excitation to the N-th qubit must have one sub-cycle of length > 1 that involves the $(N+1)$-th bus qubit, because there are no other couplings. Therefore we can only consider Hamiltonians further restricted to

$$H = \begin{pmatrix} \omega_e & 0 & \cdots & 0 & \frac{\Omega}{2} \\ 0 & \omega_e & & & \\ \vdots & & \ddots & & \vdots \\ 0 & & & \omega_e & \frac{\Omega}{2} \\ \frac{\Omega}{2} & & \cdots & \frac{\Omega}{2} & \omega_e \end{pmatrix}, \tag{8.9}$$

which limits our choices regarding trapping to $\omega_e = \omega_z$. We will discuss this condition later.

From the peculiar choice of trapping we now know that we only need to consider one-cycle permutations, because any many-cycle permutation, if we had the solution for one-cycle permutations, could be achieved by sequential illumination of the subsets of nodes that each sub-cycle involves. Particularly we will only consider the permutation (7.8) for the same reasons as before. We have previously denoted the interaction part of the Hamiltonians related to this permutation by

$$\mathrm{G}(\alpha, \beta) = \sum_{\lambda_j \in \sigma} \epsilon_{\lambda_j} \lambda_j^{\alpha-\beta}, \tag{8.10}$$

where $\lambda_j \neq 0$. For further argumentation we need to proof the following easy theorem

Theorem 8.1. *The Hamiltonians with couplings given by* (8.10) *have the interaction part "anti-diagonally symmetric".*

Proof. "Anti-diagonal symmetry" is defined: if we have an arbitrary $N \times N$ matrix A, then A is "anti-diagonally symmetric" if for all $\alpha, \beta \in \{1, \ldots, N\}$ is valid

$$A_{\alpha,\beta} = A_{N-\beta+1, N-\alpha+1}, \tag{8.11}$$

where $A_{\alpha,\beta}$ is the element of A in row α and column β.

From the form of (8.10) we know that each element depends only on the difference $\alpha - \beta$. If we now calculate this difference for the element on the anti-diagonally symmetric position to (α, β), we get

$$N - \beta + 1 - (N - \alpha + 1) = \alpha - \beta, \tag{8.12}$$

and we see that the difference is the same for both elements and therefore the interaction (off-diagonal) part of any such Hamiltonian is symmetric as defined. □

The Hamiltonian (8.9) clearly would be a violation to this theorem. And therefore we know that there is no PST in the sense of the framework presented possible for the case of illuminating all the ions.

The condition $\omega_e = \omega_z$ just might be met by careful engineering of the network, because the energy spectrum of harmonic states of trapped cold ions is comparable to the energy spectrum of the internal spin states. However it might be difficult to hold this condition for a long enough time.

Finally let us mention that the proposed scenario of the sequential illuminating single ions can easily be used for transfer of multiple excitations as well, note however that one has to be careful of the ordering of procedures, because the experimentalist would need to transfer excitations through the bus qubit and do it by pairs of source-target qubits, because if he tried transferring excitations from multiple qubits onto the oscillation state and only then onto the target qubits, he would not achieve the desired state transfer at all.

Part IV
Quantum networks with bending

This part of the thesis is dedicated to our recent research of the linear networks with bending. It is related to the thought that any large scale computers will need more complex techniques of manipulation with the information like routing, splitting or switching [16, 25]. It is very natural to move into two dimensions for doing so. We will first present two nearest neighbor protocols that are known to be able to transfer information faithfully and then introduce a defect or a perturbation into the network resulting from bending of the linear chain, not from any disturbance or random noise. This perturbation will from the nature of things be beyond nearest neighbor. The effect of this new coupling has not been investigated before, despite that nearest neighbor approximation is in many experiments related to spatial separation, trapped ions or quantum dots are excellent examples. We expect that if we introduce the bending into equidistant chain at site j, new coupling can appear between $j-1$ and $j+1$ sites which will depend on the angle of bending.

We wanted to implement approach with minimal external control. One way of dealing with the new coupling would be to redesign the couplings in the network taking into account the new interaction, by the procedures described above. This problem does not have to be solvable, not many beyond nearest neighbor models are known to transfer information faithfully. Instead we applied numerical optimization algorithms for finding optimal shifts in energy of one site to optimize the probability of the transfer. Changing energy of a single site within reasonable margins should not be a problem. In quantum dots, for example, it would require only change of voltage on the dot. The results of this analysis have been submitted for publication [42].

9 Investigated protocols

Both investigated protocols work on networks depicted in the figure 9.1 and are centrosymmetric. That

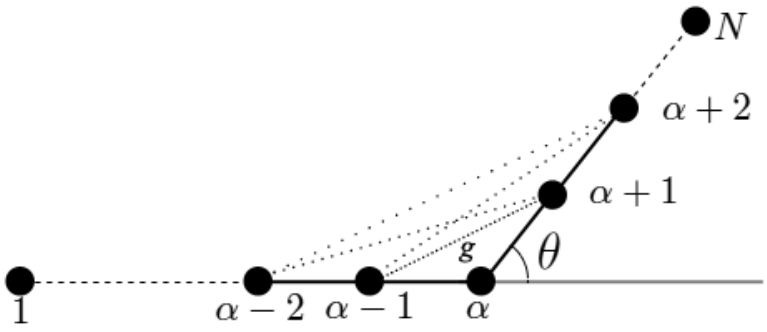


Figure 9.1: Bent network

is on an equidistant chain of N permanently coupled qubits with nearest neighbor interactions, which we bent at site α; we already know how to obtain solutions for $\theta = 0$, when the network is straight. Let us write the Hamiltonian of the system divided into two parts

$$H = H_0 + V, \tag{9.1}$$

where H_0 corresponds to the Hamiltonian of the chain with $\theta = 0$ and V is the additional coupling introduced by the bend. We only considered

$$\theta \in \langle 0, \frac{2\pi}{3} \rangle, \tag{9.2}$$

because we do not want the new coupling to be dominant, only a perturbation. And we only wanted to consider the coupling g, not the other dotted interactions in the figure 9.1. Justification of this assumption can be seen in one particular system. When we consider the interaction between the neighboring sites to be of the form

$$V(r) = A\exp(-r\lambda), \tag{9.3}$$

where A and λ are constants describing the interaction and r is the distance between qubits, the interaction to be considered nearest neighbor has to fulfill the condition

$$A\exp(-2r\lambda) \ll A\exp(-r\lambda), \tag{9.4}$$

for some typical distance r. This type of interaction is met for example in quantum dots if we consider electronic excitations and the interaction is tunneling. And if we put A, λ and r typical for quantum dot experiments, the upper bound for θ was very close to $\frac{2\pi}{3}$ (if we did not want to consider additional couplings between $\alpha-2$ and $\alpha+1$ sites and the symmetric ones). Certainly for smaller θ, g is the first additional coupling that will come into the picture.

We expect H_0 of the following form, which as we have seen is met in various physical systems

$$H_0 = \sum_{j=1}^{N} \epsilon_j a_j^\dagger a_j + \sum_{j=1}^{N-1} \Omega_{j,j+1}\left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right), \tag{9.5}$$

where $a_j^\dagger$ is the creation operator of an excitation at site j with energy ϵ_j and $\Omega_{j,j+1}$ is a coupling between two adjacent sites j and $j+1$. This Hamiltonian is very general, it does not assume the same energy levels for all the sites and allows for different nearest neighbor couplings in the network. Let us now focus on the unperturbed case and present previous results concerning the two protocols.

9.1 Protocol 1: Uniform couplings

This protocol was first proposed in [43]. It relies on choosing

$$\epsilon_j = \epsilon, \forall j, \tag{9.6}$$

i.e. same energy levels for all the sites and the couplings are engineered so that

$$\Omega_{j,j+1} = \begin{cases} \Omega_0 & \text{for } j = 2,\dots N-2 \\ \Omega & \text{for } j = 1, N-1 \end{cases}. \tag{9.7}$$

This protocol unlike the others is suitable for designing long chains, because in the other protocols (one of them we shall present in the next section) the couplings typically depend on the position in the chain and could be either too small or too large to fabricate for a long chain. The Ω coupling of the two outermost sites is supposed to be controllable and the first and last qubit are naturally expected to be the write-in and read-out qubits. The idea is to choose optimal ratio $\frac{\Omega}{\Omega_0}$ for the faithful transfer to occur. That this is possible is not a trivial fact and the procedure described in [43] relies on working with dispersion relation for general wave-packets traveling through the chain that optimizes the occurrence of peaks.

The reference [43] presented how to implement the procedure for $N+1$ spins in a chain (their indexing was slightly different from ours). And the network was prepared in the initial state where in both ends was sitting an excitation and the rest was in the ground state. An excellent illustration of how successful the procedure is, is the figure 9.2. Where the optimization has been done for a network of $N = 50$ spin states. The result is after choosing the right ratio that the excitations have swapped positions after a well defined time.

We have verified the results from [43] by a slightly different procedure. We assumed our network to be in the initial state as before, where single excitation is present at the 1 site and the rest is in the

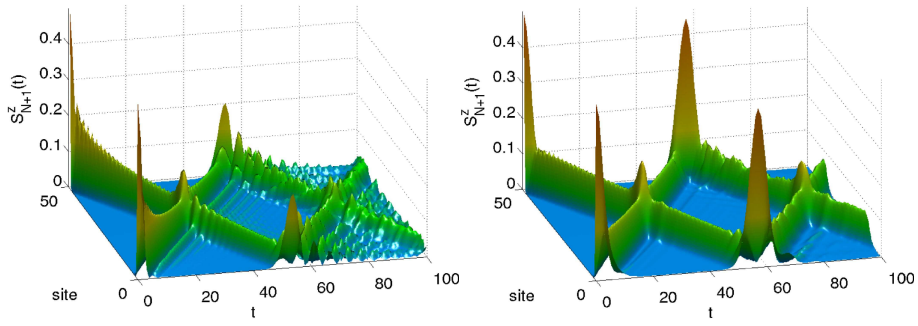


Figure 9.2: Time evolution of the projection of the spin into the $\vec{z}$ direction for $N = 50$, ratios of the couplings $\frac{\Omega}{\Omega_0} = 1$ (left) and $\frac{\Omega}{\Omega_0}$=0.58 (right), the images were taken from [43]

ground state. In the basis of the single excitation subspace the general state of the network can be expressed as

$$|\psi\rangle = \sum_j A_j(t)\,|j\rangle\,, \tag{9.8}$$

where $A_j(t)$ are the complex probability amplitudes and $|A_j(t)|^2$ gives the probability of finding the excitation at site j at any time. The Hamiltonian (9.5) results in the set of differential equations for the amplitudes

$$i\frac{dA_j(t)}{dt} = \epsilon_j A_j(t) + \Omega_{j-1,j}A_{j-1}(t) + \Omega_{j,j+1}A_{j+1}(t)\,,\ j = 1,\ldots,N.,\tag{9.9}$$

where we have specified the energies and the couplings by the equations (9.6) and (9.7).

We then used the Fehlberg fourth-fifth order Runge-Kutta method or *rfk45* [44] to obtain the numerical solution for this system. We ran optimization algorithm that maximized the height of the first peak in $|A_N|^2$ arriving at the last site on a reasonable time scale by varying the ratio $\frac{\Omega}{\Omega_0}$. We were able to find a "reasonable" optimal ratio so that the height of the first peak was closer to 1 than 0.03 and by increasing the precision we would be able to find even better ratios. The ratios for $N = 3,\ldots,12$ are shown in the figure 9.3. The typical property of the ratio is $\frac{\Omega}{\Omega_0} < 1$.

9.2 Protocol 2: Optimal couplings

The second protocol we discussed was first proposed in [13], it is covered by the tools of section 7 [15]. It relies on choosing

$$\epsilon_j = \epsilon,\tag{9.10}$$

and

$$\Omega_{j,j+1} = \Omega_0\sqrt{(N-1)\,j}.\tag{9.11}$$

In contrast to the previous protocol it is optimal in the sense that after $t = \frac{\pi}{2\Omega_0}$ the $|A_N|^2$ amplitude equals one. However the couplings are chosen based on the position in the chain and therefore this protocol is more suitable for short wires.

To see why the procedure works, one needs to put the choices (9.10) and (9.11) into the differential equations (9.9) and perform a simple Laplace transformation. The linear set of equations can be solved, transformed back to get the solutions

$$A_j(t) = \left(\begin{array}{c} N-1 \\ j-1 \end{array}\right)^{\frac{1}{2}} [-i\sin(\Omega_0 t)]^{j-1}\,[\cos(\Omega_0 t)]^{N-j}\,,\tag{9.12}$$

which gives that all the amplitudes $|A_j(t)|^2$ oscillate on commensurate frequencies

$$\lambda_j = \Omega_0\,(2j - N - 1)\,.\tag{9.13}$$

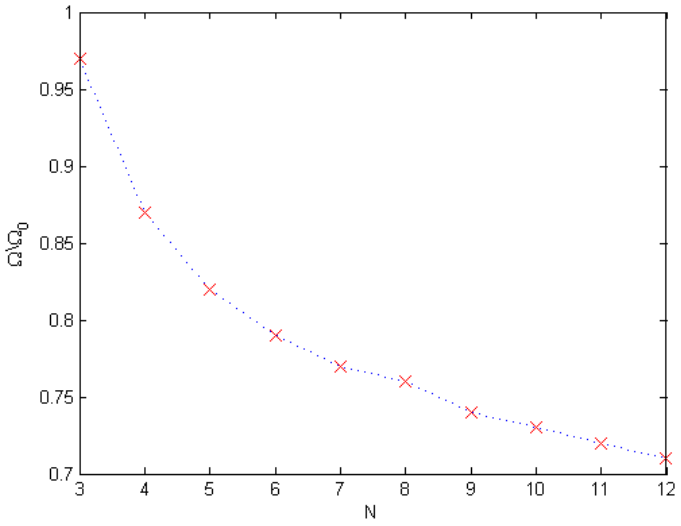


Figure 9.3: Optimal ratios of the couplings in the uniform case

This causes the excitation to travel through the chain back and forth, because the probabilities of occupation for the first and the last site are respectively

$$|A_1|^2(t) = [\cos(\Omega_0 t)]^{2(N-1)}, \tag{9.14}$$

$$|A_N|^2(t) = [\sin(\Omega_0 t)]^{2(N-1)}. \tag{9.15}$$

The solutions are plotted in the figure 9.4 for $N = 10$ and time goes from 0 to $\frac{\pi}{\Omega_0}$. It is clear that we have achieved state transfer after $\frac{\pi}{2\Omega_0}$. The plot has been given for the choice $\Omega_0 = 1$. These two protocols are the two main investigated and most promising ideas in the state transfer [11–17, 21, 25, 26, 36–40, 43]. Let us note that simulations of the transfer of entangled states, mixed states and simulations of various noises and their influences were performed.

10 Analysis of bending loses

Let us present our recent results with introducing the bending into the two protocols. Let bending be present at the site $\alpha \in \{2, \ldots, N-1\}$, which does not change the couplings before bending, only introduces a new one. As we commented before, we are only considering additional coupling g from the figure 9.1, because we expect the interactions to depend on spatial separations of qubits and drop with distance, therefore there will always be region of θ where this assumption will be valid.

We can now write the non-nearest neighbor interaction part from (9.1) explicitly

$$V(\theta) = g(\theta)\left(a^\dagger_{\alpha-1}a_{\alpha+1} + a^\dagger_{\alpha+1}a_{\alpha-1}\right), \tag{10.1}$$

where the dependence of g on θ will be different from realization to realization and it might even be related to residual stress in optical waveguides. We expect the performance of the network to depend on the ratio of the disturbance g to the local couplings $\Omega_{\alpha,\alpha\pm1}$ and for $g \ll \Omega_{\alpha,\alpha\pm1}$ we expect the network to work almost the same as before the introduction of perturbation.

We parametrized the strength of perturbation by the parameter

$$\kappa = \frac{g}{\Omega_{max}}, \tag{10.2}$$

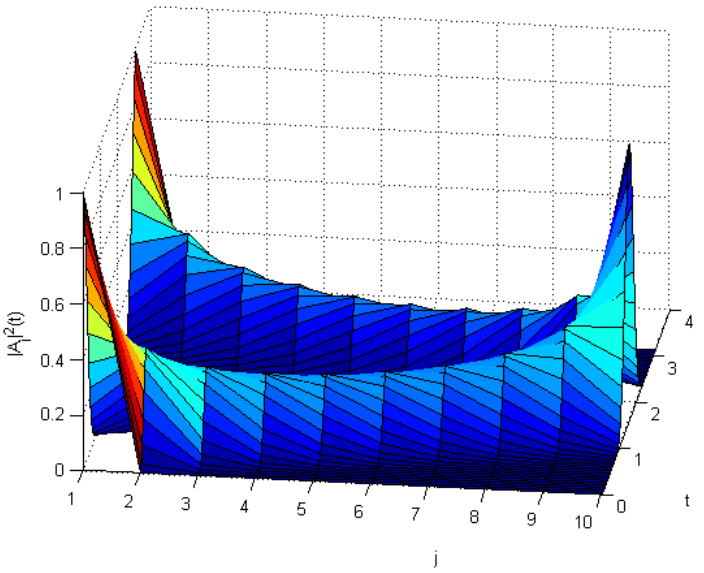


Figure 9.4: Network with optimal couplings, probability of finding the excitation moves from the first site to the last one and back

where

$$\Omega_{max} \equiv \max_{i,j} \{\Omega_{i,j}\} . \tag{10.3}$$

As before we have the network prepared so that the excitation is present at the first site and the rest is initially in the ground state (the Hamiltonian (9.1) with (10.1) preserve the total number of excitations, so achieving this is again enough for the transfer of arbitrary state).

Because of the new term (10.1) we need to change the differential equation (9.9) for the $\alpha \pm 1$ sites to become

$$i\frac{dA_{\alpha-1}(t)}{dt} = \epsilon_{\alpha-1}A_{\alpha-1}(t) + \Omega_{\alpha-2,j}A_{\alpha-2}(t) + \Omega_{\alpha}A_{\alpha} + gA_{\alpha+1}(t) , \tag{10.4}$$

and perform the respective change for the $A_{\alpha+1}$ amplitude.

10.1 Robustness of protocols vs. bending

These equations are no longer solvable by simple Laplace transform and the existence of exact solutions is not known in the literature. Therefore we used the rfk45 method to integrate the equations numerically for both protocols for various $\kappa \in \langle 0, 1\rangle$, various numbers of sites and various positions α to simulate the behavior. We investigated the first peak in the probability of the excitation being at the last site $|A_N|^2(t)$ by noting its position in time T_i and magnitude P_i, where indexes correspond to protocols. The corresponding data are known from previous work for unperturbed networks with $\theta = 0$. Let us denote these by $T_i^{(0)}$ and $P_i^{(0)}$. The illustration of how the perturbation affects the state transfer can be done by plotting the ratios

$$Q_i \equiv \frac{P_i}{P_i^{(0)}}, \tag{10.5}$$

$$S_i \equiv \frac{T_i}{T_i^{(0)}}. \tag{10.6}$$

They are presented in the figures 10.1 and 10.2. From these one can see that both protocols are

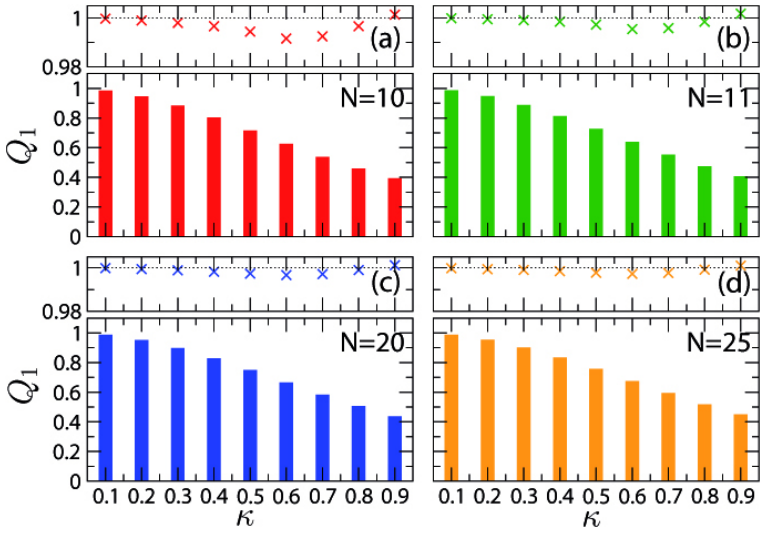


Figure 10.1: Robustness of protocol 1 vs. bending, (a-d) belong to different numbers of qubits in the chain, the bars in (a-d) denote the ratios Q_i and the $\times$'s stand for probabilities of the transfer after the optimization, positions of the bendings were (a-b) $\alpha = 6$, (c) $\alpha = 11$, (d) $\alpha = 13$

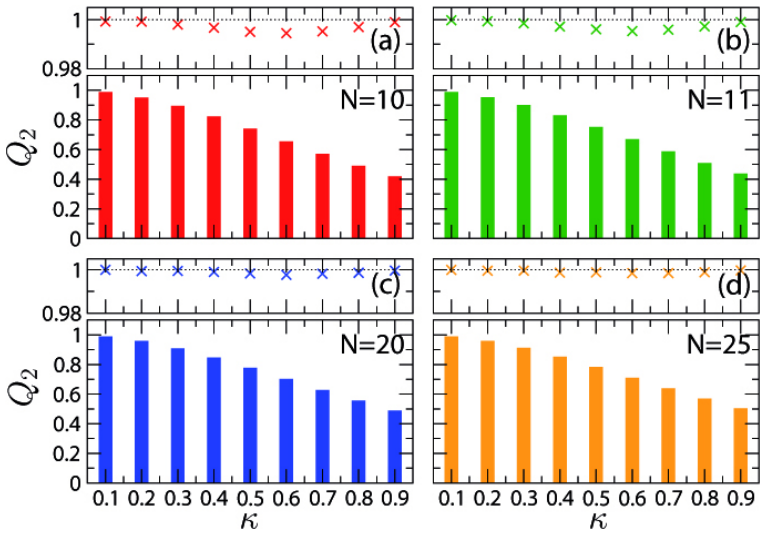


Figure 10.2: Robustness of protocol 2 vs. bending, (a-d) belong to different numbers of qubits in the chain, the bars in (a-d) denote the ratios Q_i and the $\times$'s stand for probabilities of the transfer after the optimization, positions of the bendings were (a-b) $\alpha = 6$, (c) $\alpha = 11$, (d) $\alpha = 13$

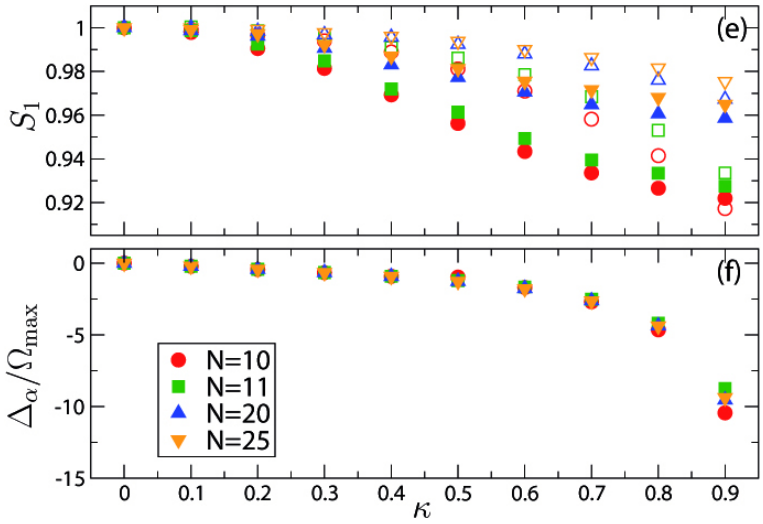


Figure 10.3: Ratios of times of the transfer for perturbed network vs. the unperturbed one, protocol 1, (e) open symbols stand for the perturbed, unoptimized networks; closed symbols correspond to optimized networks, (f) detuning of the corner site over the energy of remaining sites

relatively stable vs. the perturbation up to the ratio $\kappa \approx 0.2$ and then the probability starts dropping for both protocols. The values suggest that protocol 2 performs slightly better vs. the perturbation. For both protocols it is valid that networks with larger N are more resistant to small perturbations. This is due to localization of the perturbation to small areas around the corner.

The influence of the perturbation on the time of the transfer can be illustrated similarly with ratios S_i as in the figures 10.3 and 10.4 (only the open symbols). In both cases the transfer is somewhat accelerated and the values suggest that for protocol 1 the acceleration is more pronounced. Similarly to robustness in probabilities, larger numbers of qubits tend to be less influenced in the time of the transfer due to localization.

All the figures have been presented for the cases with the bending α present at the middle of the chain, our simulations suggest no differences for other positions. The only exception from this is the behavior for the protocol 2, if we move the bending towards the edges. The mentioned effects become more pronounced, because the relative strength of the new coupling is larger as the couplings are the lowest at the edges.

10.2 Optimization of the transfer

The idea we implemented theoretically for minimization of the losses is detuning the energy of the corner site ϵ_α by a margin Δ_α while leaving all the other parameters untouched. This certainly is underlined by the thought of minimal external control and for example in the case of quantum dots can easily be achieved by changes in the voltage of the corner dot.

The procedure went as before with optimizing the couplings for the protocol 1. We used the rfk45 method to numerically integrate the equations (10.4) and the unchanged ones (9.9), and then optimized (maximized) the first peak arriving to the N-th site by varying Δ_α (or $\epsilon_\alpha$equivalently) for various configurations.

Results of these optimizations were excellent as all the values were very close to 1 after the optimization, these are the $\times$ symbols at the top of the figures 10.1 and 10.2, all very close to one. The effect on time of the transfer has been explored as well, it is depicted by the closed symbols in the figures 10.3 and 10.4.

The magnitude of necessary detuning is illustrated by the plots in figures 10.3 and 10.4 (f). It is apparent that for reasonable perturbations the detunings are within the order of the energies of

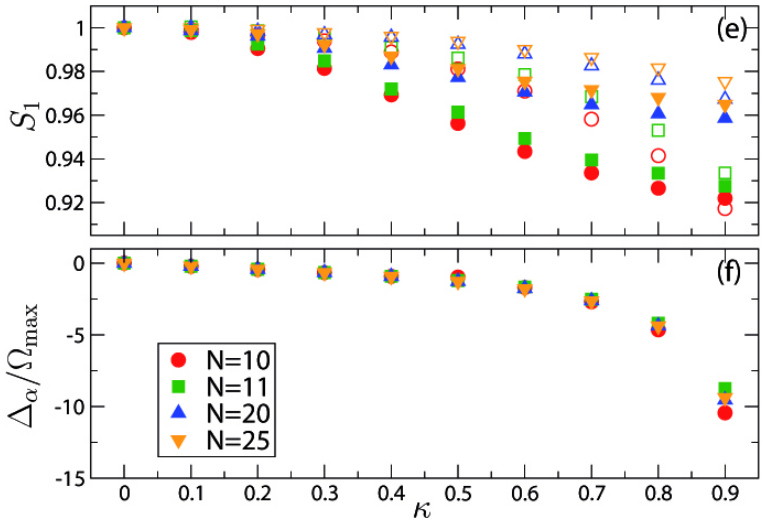


Figure 10.4: Ratios of times of the transfer for perturbed network vs. the unperturbed one, protocol 2, (e) open symbols stand for the perturbed, unoptimized networks; closed symbols correspond to optimized networks, (f) detuning of the corner site over the energy of remaining sites

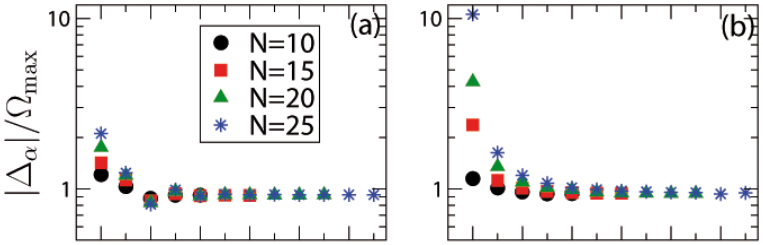


Figure 10.5: Influence of position α on the optimal detuning for fixed $\kappa = 0.4$, Index is the position α from 2 to $\frac{N}{2}$, (a) protocol 1, (b) protocol 2

remaining sites and therefore achievable, but for κ close to one, the values of detuning jump to ten folds of the other energies and would probably be not attainable.

We investigated the influence of the position α on the detuning as well, results are depicted in the figure 10.5.

For larger networks the optimized detuning is very similar for all the positions. Problems are caused by the bending present at the edges of the chain, where again the detuning in energy jumps far away from the other energies in the network and these setups are not obtainable.

Part V
Conclusions

The first sections are devoted to an overview of quantum computation. This is followed by a summary of quantum computer candidates. Then we focused on the problem of PST in a very general framework that describes various physical systems. The two most important parts of the thesis are contained in the sections 8-10. They contain my investigations of PST on trapped ions and the analysis of quantum networks with bending. Let us summarize in some detail.

In section 8 we investigated two ideas how to perform state transfer on trapped ions, which is a very promising quantum computer candidate. We have used previously known tools of sequential pulses to achieve it in a relatively efficient way. We attempted to find Hamiltonians provided by the framework of known procedures of constructing PST Hamiltonians on the trapped ions. We were able to prove that in the setup described there is not a PST Hamiltonian that would be a member of any of the classes presented. We suggested possible ways of generalizing the framework further to encapsulate it with possible limitations it might bring.

A possible next step of research would be to attempt to find PST Hamiltonians from one of the presented classes in trapped ions systems that have an additional magnetic field with non-zero gradients [22–24, 27]. It seems this might be possible. These systems have the advantage of using microwave and radio-frequency pulses for driving the ions. It is a technology which has been developed even for commercial use in nuclear magnetic resonance methods in medicine and other areas. Thus it is cheaper than manufacturing laboratory lasers that are capable of focusing on a single atom in the trap with coherent light.

In sections 9-10 we have summarized recent research on which we cooperated with Georgios M. Nikolopoulos. We investigated thoroughly linear networks with bendings, which are expected to be at the core of any more advanced technique of manipulation quantum information in more than one dimension. We compared them with known solutions of unbent networks both in time and efficiency of the transfer. We have presented results of numerical analysis of two most important protocols known for PST, when the defined perturbation is introduced into the system. Generally the perturbation tends to accelerate the transfer in time and the quality of the transfer was proportional to the relative size of the new coupling.

We proposed a method of compensating for imperfections by a relatively simple procedure. Numerical simulations suggest considerable improvement for both protocols. We have also discovered some limitations of the procedure in dependence on the position of the bending and the magnitude of the additional interaction. The procedure followed the requirement of minimal external control compared to other possible methods and from what we require from a quantum computer it should be easily implementable.

The framework in which we have worked is very general and it could work for many concrete physical realizations of a quantum computer. This might be the reason why exact solutions for the problem are difficult to find. Nevertheless the numerics suggested that there might exist a rule for choosing the right detuning, it was not completely random. Especially for smaller perturbations the dependence appeared to be linear.

The next possible course of research in this area might be discussing more complex geometrical arrangements in two dimensions that would allow for information routing between multiple quantum wires. Attempts to find the minimal requirements for doing PST by means of adding qubits and interactions to the system could be carried out. It might be beneficial to investigate systems similar to the ones already explored, but considering further interactions from the corner of bending and see if the method proposed would prove useful even for these systems.

References

[1] R. P. Feynman, *Simulating physics with computers*, Int. J. Theor. Phys. **21**, 467 (1982).

[2] R. P. Feynman, *Quantum-mechanical computers*, J. Opt. Soc. Am. B **1**, 464 (1984).

[3] R. P. Feynman, *Quantum-mechanical computers*, Found. Phys. **16**, 507 (1986).

[4] C. H. Bennet and G. Brassard, *Quantum cryptography: Public key distribution and coin tossing*, Proc. IEEE Int. Conference on Computers, Systems and Signal Processing, Bangalore, India pp. 175–179 (1984).

[5] D. W. Mount, *Bioinformatics: Sequence and Genome Analysis* (Cold Spring Harbor Laboratory Press, New York, 2004).

[6] T. Monz, P. Schindler, J. T. Barreiro, M. Chwalla, D. Nigg, W. A. Coish, et al., *14-qubit entanglement: Creation and coherence*, Phys. Rev. Lett. **106**, 130506 (2011).

[7] J. T. Barreiro, M. Müller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, et al., *An open-system quantum simulator with trapped ions*, Nature **470**, 486 (2011).

[8] B. P. Lanyon, C. Hempel, D. Nigg, M. Müller, R. Gerritsma, F. Zähringer, et al., *Universal digital quantum simulation with trapped ions*, Science **334**, 57 (2011).

[9] J.W. Britton, B.C. Sawyer, A. Keith, C.-C. J. Wang, J.K. Freericks, H. Uys, et al., *Engineered 2d ising interactions on a trapped-ion quantum simulator with hundreds of spins*, Nature **484**, 489 (2012).

[10] D. P. DiVincenzo, *The physical implementation of quantum computation*, arXiv:quant-ph/0002077v3 (2000).

[11] S. Bose, *Quantum communication through an unmodulated spin chain*, Phys. Rev. Lett. **91**, 207901 (2003).

[12] G. M. Nikolopoulos, D. Petrosyan, and P. Lambropoulos, *Coherent electron wavepacket propagation and entaglement in array of coupled quantum dots*, Europhys. Lett. **65**, 297 (2004).

[13] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, *Perfect state transfer in quantum spin networks*, Phys. Rev. Lett. **92**, 187902 (2004).

[14] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Phys. Rev. A **71**, 032312 (2005).

[15] V. Košťák, G. M. Nikolopoulos, and I. Jex, *Perfect state transfer in networks of arbitrary topology and coupling configuration*, Phys. Rev. A **75**, 042319 (2007).

[16] T. Brougham, G. M. Nikolopoulos, and I. Jex, *Communication in quantum networks of logical bus topology*, Phys. Rev. A **80**, 052325 (2009).

[17] T. Brougham, G. M. Nikolopoulos, and I. Jex, *Perfect transfer of multiple excitations in quantum networks*, Phys. Rev. A **83**, 022323 (2011).

[18] J.Formánek, *Úvod do kvantové teorie I.* (ACADEMIA, Praha, 2004).

[19] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[20] S. Stenholm and K. Suominen, *Quantum Approach to Informatics* (John Wiley & Sons, Inc., New Jersey, 2005).

[21] G. M. Nikolopoulos, D. Petrosyan, and P. Lambropoulos, *Electron wavepacket propagation in a chain of coupled quantum dots*, J. Phys.: Condens. Matter **16**, 4991 (2004).

[22] M. Johanning, A. Braun, N. Timoney, V. Elman, W. Neuhauser, and Ch. Wunderlich, *Individual adressing of trapped ions and coupling of motional and spin states using rf radiation*, Phys. Rev. Lett. **102**, 073004 (2009).

[23] F. Mintert and Ch. Wunderlich, *Ion-trap quantum logic using long-wavelength radiation*, Phys. Rev. Lett. **87**, 257904 (2001).

[24] D. Mc Hugh and J. Twamley, *Quantum computer using a trapped-ion spin molecule and microwave radiation*, Phys. Rev. A **71**, 012315 (2005).

[25] D. Zueco, F. Galve, S. Kohler, and P. Hänggi, *Quantum router based on ac control of qubit chains*, Phys. Rev. A **80**, 042303 (2009).

[26] M. I. Makin, J. H. Cole, C. D. Hill, and A. D. Greentree, *Spin-guides and spin-splitters: Waveguide analogies in one-dimensional spin chains*, Phys. Rev. Lett. **108**, 017207 (2012).

[27] H. Wunderlich, Ch. Wunderlich, K. Singer, and F. Schmidt-Kaler, *Two-dimensional cluster-state preparation with linear ion traps*, Phys. Rev. A **79**, 052324 (2009).

[28] H. Figger, D. Meschede, and C. Zimmermann, *Laser Physics at the Limit* (Springer, Berlin, 2001), pp. 261–271.

[29] C. Macchiavello, G. M. Palma, and A. Zeilinger, *Quantum Computation and Quantum Information Theory* (World Scientific Publishing Co., Singapore, 2000).

[30] S. Aaronson, *Quantum computing, postselection, and probabilistic polynomial-time*, Proc. of the Royal Society A **461**, 3473 (2005).

[31] A. Schreiber, A. Gábris, P. P. Rohde, K. Laiho, M. Štefaňák, V. Potoček, et al., *A 2d quantum walk simulation of two-particle dynamics*, Science **336**, 6077 (2012).

[32] A. Hoskovec, *Perfect state transfer for trapped ions* (Bakalářská práce, ČVUT v Praze, Fakulta jaderná a fyz. inženýrská, Katedra fyziky, Praha, 2010).

[33] L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinenann, Oxford, 1976).

[34] J. Adamowski, S. l. Bednarek, and B. l. Szafran, *Quantum computing with quantum dots*, Schedae Informaticae **14**, 95 (2005).

[35] S. Bednarek, B. Szafran, and J. Adamowski, *Theoretical description of electronic properties of vertical gated quantum dots*, Phys. Rev. B **64**, 195303 (2001).

[36] G. M. Nikolopoulos, *Directional coupling for quantum computing and communication*, Phys. Rev. Lett. **101**, 200502 (2008).

[37] L. Banchi, A. Bayat, P. Verrucchi, and S. Bose, *Non-perturbative entangling gates between distant qubits using uniform cold atom chains*, Phys. Rev. Lett. **106**, 140501 (2011).

[38] S. Bose, *Quantum communication through spin chain dynamics: an introductory overview*, Contemp. Phys. **48**, 13 (2006).

[39] A. Kay, *A review of perfect, efficient, state transfer and its application as a constructive tool*, Int. J. Quantum Inf. **8**, 641 (2010).

[40] D. Cvetkovič and I. Gutman, *Selected topics on applications of graph spectra* (Matematički institut SANU, Beograd, 2011), chap. Applications of Graph Spectra in Quantum Physics (Dragan Stevanovič).

[41] A. Hoskovec, *State transfer for several excitations* (Výzkumný úkol, ČVUT v Praze, Fakulta jaderná a fyz. inženýrská, Katedra fyziky, Praha, 2011).

[42] G. M. Nikolopoulos, A. Hoskovec, and I. Jex, *submitted*, Phys. Rev. A (2012).

[43] L. Banchi, T. J. G. Apollaro, A. Cuccoli, R. Vaia, and P. Verrucchi2, *Optimal dynamics for quantum-state and entanglement transfer through homogeneous quantum wires*, Phys. Rev. A **82**, 052321 (2010).

[44] E. Hairer, Syvert Nørsett, and Gerhard Wanner, *Solving Ordinary Differential Equations I: Non-stiff Problems* (Springer-Verlag, Berlin, 1993).