CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering
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RESEARCH TASK

Přenos informace pro více excitací

State transfer for several excitations

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0.1 Introduction

In this work we want to focus on perfect state transfer (PST) on trapped ions. The state transfer is an issue addressed by previous work in the field [1–5], here we want to present the former results and suggest a path the generalization might take in order to cover the most simple ion trap computers as well as it has served many different quantum systems used for computation.

We are focusing our attention on trapped ions because it is one of the most promising quantum systems that could be used for computation, recent results have gone as far as to trapping 14 ions that can be used for quantum computation [6]. A stunning result because only few tens of qubits could become useful for computation beyond the reach of today's classical computers. Multiple methods for solving the issues of ion trap computers have been presented and experimentally verified [7–11], most important of which is the possibility of using radio frequency (rf) or microwave (mw) radiation for driving the ions in the traps instead of using lasers working on frequencies of visible light that have to be focused on individual ions in the trap. The technology for working with rf or mw radiation has been eminently developed in the past decades and is very sophisticated and accessible. Throughout this work we shall, however, limit ourselves to traps that have to be driven with optical radiation and present how to perform the PST without the need of additional magnetic fields that allow the computation with mw or rf radiation.

The text is divided into sections devoted solely to presenting the general approach of previous articles [3–5] with additional notes about classes of Hamiltonians derived and into sections covering the most simple possibilities of how to transfer a state over trapped ions. Positive results concerning illumination of single ions in the chain are shown as well as some positive results with illuminating multiple ions in the trap, in both cases we will show how it is possible to attain the state transfer.

0.2 Notation

Throughout this work we shall use notation standard for quantum physics, the Dirac notation where $\mathscr{H}$ signifies the Hilbert space of physical systems, $|x\rangle \in \mathscr{H}$ is then a vector from the mentioned Hilbert space and $\langle x|$ is a dual vector to $|x\rangle$. We will not use the hat $\hat{A}$ sign over a letter to denote operators. This symbol will be reserved for notation of the following set: $\hat{n} \equiv \{1, 2, \ldots, n\}, n \in \mathbb{N}$. Operators will be denoted by capital Latin letters only: A. The only exception will be the creation and annihilation operators for which $a^\dagger$ and a symbols are reserved respectively. We will denote vectors of real or complex numbers exclusively by an arrow above the letter $\vec{v}$. As we will be concerned with permutations, it is good to state that we will be using the cycle notation of permutations. For example permutation P that acts on the set $\{1, 2, 3\}$ and turns it into $\{2, 1, 3\}$ would be written in the cycle notation as $(1\,2)\,(3)$. Other than that we will

be using standard notation from commonly known quantum systems as harmonic oscillators, spin states or Coulomb interaction, which the reader should be familiar with.

Throughout this work we set $\hbar = 1$. We use scaled time, instead of using τ in time units we will use dimensionless quantity $t \equiv \tau/\mathrm{J}$, where J is our time units. That will allow us to simplify formulas and work in a very general framework for deriving the Hamiltonians that lead to perfect state transfer.

1 State transfer

Every quantum computer, similarly to classical computer, needs means of communication to be able to transport information between different components of the computer as well as between different quantum computers. Nowadays no one can image computers without hard drives or the Internet. And the state transfer could provide an analogue of these that would be respecting the limitations of the quantum computer. We can imagine the quantum systems that allow us to transfer a state as wires connecting quantum systems used for computation.

The approach that is being explored greatly [1–5] are the so called passive quantum wires. The idea behind them is simple. We take a quantum system which consists of several nodes, quantum objects, which interact with each other and we try to find parameters of the interactions (couplings) that lead after a well defined time t through unitary time evolution to a transfer of states of one or more source nodes onto destination nodes. This is a simplified statement and the exact definition will follow in the text.

The great advantage of this approach compared to for example encoding the state of an input node onto a photon and then transmitting the photon over optical fiber and encoding the information onto the destination node is very high compatibility with the quantum computer itself. The system we are most concerned about in this work are the trapped ions and we will show how to perform a state transfer on trapped ions that can be used as a quantum computer themselves. The previous statement is, of course, valid for many different designs of quantum computers as usually the state transfer is a permutation on certain states chosen to be the computational basis; permutations are unitary transformations and usually the very first fact to be theoretically shown with any quantum computer is that there can be performed any unitary operation (computation)[12].

Usually we tend to image the quantum computers as a linear chain or a different well organized planar structure, that helps us to comprehend the system as if it were a network of connected nodes.

1.1 Transfer of a single excitation

In complete analogy to previous work in the field [3–5] we will be at first working in a very general framework describing a wide variety of quantum computers and

later we will focus on trapped ions and investigate the state transfer on them and comment on the compatibility of trapped ions with presented framework of the state transfer. Let us first focus on single excitation present in the network.

We will consider a network of n sites labeled by $\mathbb{S}_n \equiv \hat{n} = \{1, 2, 3, \ldots, n\}$, where the node 1 is our input node prepared at time $t = 0$ in the input state $|1\rangle$ (in the excited state) and all the remaining nodes are at their ground states $|0\rangle$ and we want to achieve transfer of the excitation to the last site. Therefore the overall state at time $t = 0$ of the network is $|1\rangle_1 |0\rangle_2 \ldots |0\rangle_n$. If we now denote by

$$|\alpha\rangle \equiv |0\rangle_1 \ldots |1\rangle_\alpha \ldots |0\rangle_n , \tag{1.1}$$

the state where excitation is present at site α, the set

$$\{|\alpha\rangle |\, \alpha \in \hat{n}\} \tag{1.2}$$

will form our computational basis. And we can define exactly what perfect state transfer means to us. It is a unitary transformation, permutation in the computational basis:

$$P = \begin{pmatrix} 0 & & & \\ \vdots & & \widetilde{P} & \\ 0 & & & \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \tag{1.3}$$

where $\widetilde{P}$ is also a permutation in the computational basis. And we will be looking for Hamiltonians that will lead after exactly time t to this matrix through time evolution, which expressed by an equation looks like:

$$U(t) \equiv e^{-iHt} = P, \tag{1.4}$$

where we need to keep in mind that the equation is expressed for operators acting only in the single excitation subspace. Note now that if we are not interested in systems where multiple excitations interact or where multiple excitations can be present at one site, solving this problem will solve transfer of multiple excitations as well as we can use the transfer of one excitation multiple times. In the following we will divide the theory to one-cycle permutations P and many-cycle permutations P as that has been proven useful for many reasons.

1.2 One-cycle permutations

Imagine a one cycle permutation of the form (1.3). It is easy to see that we only need to consider the following permutation:

$$P = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \tag{1.5}$$

as the remaining $(n-2)!-1$ can be achieved from this one just by relabeling the sites $\{2,\ldots,n-1\}$. We will now derive the whole class $\mathfrak{C}_H$ of Hamiltonians that lead to this permutation using the theory of representations. It is straightforward to show that the eigenvalues of the permutation (1.5) are:

$$\sigma = \left\{ \lambda_j \middle| \lambda_j = \exp\left(i2\pi\frac{j}{n}\right), j \in \widehat{(n-1)} \cup \{0\} \right\}. \tag{1.6}$$

The corresponding eigenvectors expressed in the computational basis are:

$$\left|y_{\lambda_j}\right\rangle = \frac{1}{\sqrt{n}} \sum_{\alpha\in\mathbb{S}_n} \lambda_j^{\alpha-1} \left|\alpha\right\rangle = \frac{1}{\sqrt{n}} \left(1, \lambda_j^1, \lambda_j^2, \ldots, \lambda_j^{n-1}\right). \tag{1.7}$$

We can write the spectral decomposition of the permutation (1.5) as:

$$P = \sum_{\lambda_j\in\sigma} \lambda_j \left|y_{\lambda_j}\right\rangle \left\langle y_{\lambda_j}\right|. \tag{1.8}$$

From these equations and using the fact that for finite-dimensional matrices it is true that if matrix A is in its diagonal basis of the form:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \tag{1.9}$$

then its exponential is in the same basis of the form:

$$\exp A = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}, \tag{1.10}$$

the obvious construction of the first Hamiltonian (one particular choice from many) to fulfill the equation (1.4) is:

$$H = \frac{1}{t} \sum_{\lambda_j\in\sigma} \arg\left(\lambda_j\right) \left|y_{\lambda_j}\right\rangle \left\langle y_{\lambda_j}\right|, \tag{1.11}$$

where $\arg\left(\lambda_j\right)$ is the complex phase of λ_j. We can shift each eigenenergy by an arbitrary multiple of 2π (resulting from periodicity of $\exp\left(i\varphi\right)$) and we will achieve infinitely many more Hamiltonians that as well lead to aforementioned permutation after exactly time t:

$$H_{\vec{l}} = \frac{1}{t}H + \frac{1}{t} \sum_{\lambda_j\in\sigma} \frac{2\pi l_{\lambda_j}}{t} \left|y_{\lambda_j}\right\rangle \left\langle y_{\lambda_j}\right| = \frac{1}{t} \sum_{\lambda_j\in\sigma} \left[\arg\left(\lambda_j\right) + 2\pi l_{\lambda_j}\right] \left|y_{\lambda_j}\right\rangle \left\langle y_{\lambda_j}\right|, \tag{1.12}$$

where $\vec{l} \in \mathbb{Z}^n$. Let us denote the whole class by

$$\mathfrak{C}_H = \left\{ H_{\vec{l}} | l \in \mathbb{Z}^n \right\}, \tag{1.13}$$

with $H_{\vec{l}}$ taken from eq. (1.12). It is good to look further into the form of the Hamiltonian we just derived and rewrite it as:

$$H_{\vec{E}} = \sum_{\lambda_j \in \sigma} \epsilon_{\lambda_j} \left| y_{\lambda_j} \right\rangle \left\langle y_{\lambda_j} \right| = \sum_{\lambda_j \in \sigma} \epsilon_{\lambda_j} \Pi_{\lambda_j}, \tag{1.14}$$

where the Π_{λ_j} is a projector onto the eigen-subspace associated with eigenvalue λ_j and

$$\epsilon_{\lambda_j} = \frac{\arg\left(\lambda_j\right) + 2\pi l_{\lambda_j}}{t} \in \mathbb{R} \tag{1.15}$$

is the component of energy vector $\vec{E} \in \mathbb{R}^n$, which is just a different parametrization of the class $\mathfrak{C}_H$. It is now easy to express the Hamiltonian parametrized by the energy vector in the computational basis as:

$$H_{\vec{E}} = \sum_{\alpha,\beta \in \mathbb{S}_n} \sum_{\lambda_j \in \sigma} \epsilon_{\lambda_j} \lambda_j^{\alpha-\beta} \left| \alpha \right\rangle \left\langle \beta \right|. \tag{1.16}$$

Usually this is written in the form

$$H_{\vec{E}} = H^0_{\vec{E}} + \nu_{\vec{E}}, \tag{1.17}$$

where $H^0_{\vec{E}}$ is the diagonal part

$$H^0_{\vec{E}} = \sum_{\alpha \in \mathbb{S}_n} E_\alpha \left| \alpha \right\rangle \left\langle \alpha \right|, \tag{1.18}$$

and the interaction components of the Hamiltonian

$$\nu_{\vec{E}} = \sum_{\alpha \neq \beta \in \mathbb{S}_n} \mathrm{G}\left(\alpha, \beta\right) \left| \alpha \right\rangle \left\langle \beta \right|. \tag{1.19}$$

Where we have defined the energies and couplings

$$E_\alpha = \sum_{\lambda_j \in \sigma} \epsilon_{\lambda_j}, \tag{1.20}$$

$$\mathrm{G}\left(\alpha, \beta\right) = \sum_{\lambda_j \in \sigma} \epsilon_{\lambda_j} \lambda_j^{\alpha-\beta}. \tag{1.21}$$

From these equations it is easy to see a very important fact, which we later apply to every cycle of a multi-cycle permutation, that Hamiltonians associated with aforementioned permutation must involve the same energy levels for all the sites!

Next let us mention that the following theorem has been proven where by dimension the authors mean the number of qubits n (we will be using it later in the section about PST on trapped ions)[3]:

Theorem 1. *For networks of arbitrary dimension ($n > 2$), there exists no nearest-neighbor-interaction Hamiltonian satisfying condition* (1.4) *in the framework of permutation* (1.5).

We will later be discussing consequences of the following theorem for trapped ions as well:

Theorem 2. *The interaction part of* (1.17) *with couplings* (1.21) *is symmetric along the anti-diagonal. That is* $\mathrm{G}(\alpha,\beta) = \mathrm{G}(n-\beta+1, n-\alpha+1)$.

Proof. It is easy to see that (1.21) depends only on $\alpha - \beta$ and from that the equality of elements is easy to see, because

$$N - \beta + 1 - (N - \alpha + 1) = \alpha - \beta, \tag{1.22}$$

□

1.3 Many-cycle permutations

Now we will describe the many-cycle permutations using the tools listed in the previous section. The first idea that comes to mind is to decompose the permutation to individual (disjoint!) cycles involving a subset of nodes and treat each cycle as a one-cycle permutation along the lines of the previous section. Let us consider a cycle of length $d < n$. This cycle acts as a permutation on a subset of nodes, we denote by

$$\mathbb{S}_d \subset \mathbb{S}_n. \tag{1.23}$$

The eigenvalues of such permutation will be (as before)

$$\lambda_j = \exp\left(i2\pi\frac{j}{d}\right) \text{ for } j \in \widehat{(d-1)} \cup \{0\}, \tag{1.24}$$

and the projections on the computational basis of corresponding eigenvectors are

$$\left\langle \alpha \,\middle|\, v_{\lambda_j}^{(k)} \right\rangle = \begin{cases} \frac{\lambda_j^{\alpha-1}}{\sqrt{d}} & for\, \alpha \in \mathbb{S}_d \\ 0 & otherwise \end{cases}. \tag{1.25}$$

We can see from (1.24) that the spectrum of a many cycle permutation is always degenerate as the eigenvalue λ_0 appears as many times as the total number of cycles. We denote the number of distinct vectors corresponding to eigenvalue λ_j by δ_{λ_j}, i.e. the δ_{λ_j} is degeneracy of the value λ_j. And we denote the subspace spanned by the eigenvectors belonging to λ_j by $\mathcal{E}_{\lambda_j}$.

Now using the Gram-Schmidt orthonormalization process we choose in every subspace $\mathcal{E}_{\lambda_j}$ orthonormal basis

$$\mathcal{E}_{\lambda_j} = \mathrm{span}\left(|y_{\lambda_j}^{(k)}\rangle\right), \tag{1.26}$$

where $|y_{\lambda_j}^{(k)}\rangle$ are linear combinations of the original eigenvectors belonging to λ_j and therefore also eigenvectors with eigenvalue λ_j.

As a direct generalization of the derivation (1.13), here we also can shift the phase of each eigenvalue of the Hamiltonian by 2π, however, here we can do it for every vector from every (1.26) and we write:

$$\begin{aligned}\mathcal{H}_{\vec{l}} &= \frac{1}{\tau}\sum_{\lambda_j}\sum_{k=1}^{\delta_{\lambda_j}}\left[arg(\lambda_j)+2\pi l_{\lambda_j}^{(k)}\right]\left|y_{\lambda_j}^{(k)}\right\rangle\left\langle y_{\lambda_j}^{(k)}\right| \\ &= \frac{1}{\tau}\sum_{\lambda_j}\sum_{k=1}^{\delta_{\lambda_j}}\varepsilon_{\lambda_j}^{(k)}\left|y_{\lambda_j}^{(k)}\right\rangle\left\langle y_{\lambda_j}^{(k)}\right|,\end{aligned} \tag{1.27}$$

where $\vec{l}\in\mathbb{Z}^n \equiv \left\{(l_{\lambda_0}^{(1)},\ldots,l_{\lambda_0}^{(\delta_{\lambda_0})};l_{\lambda_1}^{(1)},\ldots,l_{\lambda_1}^{(\delta_{\lambda_1})};\ldots)\,|\,l_{\lambda_j}^{(\delta_{\lambda_j})}\in\mathbb{Z}\right\}$.

As we mentioned before, the transfer of single a excitation will serve the purpose of transmitting multiple excitations if there cannot be present multiple excitations at one site and if the interactions cannot interact. Except for deriving the class of suitable Hamiltonians, two important facts have been mentioned: in this framework for each cycle the Hamiltonian must involve the same energy levels on all nodes that the cycle involves and that in order to achieve PST for more then two nodes more then nearest neighbor interactions have to be considered.

We will only mention here that the previous formalism was developed further [5] and the Hamiltonians have been classified deeper as for example one can ask if the Hamiltonian that leads to the desired permutation after time t, will lead to the same permutation after some additional time (for example the case of integer multiples of t) or if it leads to transfer perhaps even in some earlier time, answers to these can be found in [5]. Also it is possible to find Hamiltonians that lead to sending the information to a subset of nodes at exactly defined times, we can design Hamiltonians that lead to transfer of the excitation to for example nodes $5, 13, 9, n$ in this order using previous formalism and theorems from [5].

1.4 Transfer of multiple excitations

A generalization of the previous framework has been presented in [4], the generalization lies in the fact that one can design Hamiltonians that transfer excitations in systems where multiple excitations can be present at one site and the excitations can even interact with each other even beyond nearest-neighbor or on-site interactions in this model. That suits multiple quantum systems that can be used for quantum computation, for example quantum dots where multiple electrons (excitations in this case) can be present at one site [4]. Also introducing more excitations into the system lets us consider differences between bosons and fermions, which gives us more constraints on the forms of suitable Hamiltonians.

Real systems are usually composed of objects with several properties (spin, energy, angular momentum), only some of them are used for encoding information but we can consider more of them. Typically as in previous section, the nodes that do not contain information are prepared in their ground states $|0\rangle$ and a subset of nodes used for encoding of the information contain some excitations. Also typically the Hamiltonians used for computation preserve the total number of excitations [4].

We will need to alter our notation to better suit this generalized model. Because we are concerned only with systems that preserve the total number of excitations, we are now working in N-excitation Hilbert space $\mathscr{H}_N$ on M nodes with basis:

$$\mathscr{H}_N = \text{span}\left\{ |n_{1,\sigma}, n_{2,\sigma}, \ldots, n_{M,\sigma}\rangle \middle| \sum_{j,\sigma} n_{j,\sigma} = N \right\}, \tag{1.28}$$

where $n_{j,\sigma}$ denotes the number of excitations in the degree of freedom σ at the jth site and σ goes through all the degrees of freedom that can be used for information encoding. We will denote the set of orthonormal basis states by \$. Note now, that for $N = 1$ the frameworks for single excitation and multiple excitations coincide.

The generalization lies in using second quantization to talk about the states (both initial and the desired ones), in general we can write the initial state as

$$|\psi(0)\rangle = f\left(a^\dagger_{j,\sigma}\right)|0\rangle, \tag{1.29}$$

where $f\left(a^\dagger_{j,\sigma}\right)$ is a function of creation operators $a^\dagger_{j,\sigma}$ that create excitations in the degree of freedom σ at site j. The same generality will be preserved for the desired state if we require

$$|\psi(t)\rangle = f\left(a^\dagger_{P(j),\sigma}\right)|0\rangle, \tag{1.30}$$

where P is a permutation. We can see now that we will be requiring the permutation to act on the basis states instead on the nodes. The difference is that the rank of permutation that acts on the nodes is M while the rank of the permutation that acts on the basis states is inevitably M^N, which is easily seen is the same for $N = 1$. What one can do now is to consider the permutation as a permutation in one excitation space but in M^N dimensional space (we only have permutation of basis states) and use directly the tools provided by previous sections. For the purposes of simplicity we will now present the results for $N = 2$ from [4]. The main results from the case of $N = 2$ will of course be valid for $N > 2$ as well.

The space of two excitations $\mathscr{H}_2$ is spanned by the states $\{|i, \mu; j, \nu\rangle\}$, where $\mu, \nu \in$ \$, with

$$|i, \mu; j, \nu\rangle \equiv |1_{i,\mu}\rangle\,|1_{j,\nu}\rangle. \tag{1.31}$$

We will denote the initial state by $|s_1, \mu; s_2, \nu\rangle$, where the respective position of each excitation is important. We can see that we can find three subspaces in $\mathscr{H}_2$, which will be important for the nature of PST

$$\mathscr{H}_2^{(<)}, \text{spanned by } \{|i, \mu; j, \nu\rangle |\, i < j\}\,, \tag{1.32}$$

$$\mathscr{H}_2^{(=)}, \text{spanned by } \{|i, \mu; j, \nu\rangle |\, i = j\}\,, \tag{1.33}$$

$$\mathscr{H}_2^{(>)}, \text{spanned by } \{|i, \mu; j, \nu\rangle |\, i > j\}\,. \tag{1.34}$$

Very special case when these subspaces are decoupled — there is no interaction between them — can be considered. In this case we would solve the PST in each subspace separately as the excitations can not jump from one subspace to another. We can imagine the permutation in the block-diagonal form as

$$P = \begin{pmatrix} P^{(<)} & & \\ & P^{(>)} & \\ & & P^{(=)} \end{pmatrix}. \tag{1.35}$$

For networks that are insensitive to different degrees of freedom the subspaces $\mathscr{H}^{(<)}$ and $\mathscr{H}^{(>)}$ are basically equivalent. After finding the eigenvalues of permutations $P^{(.)}$ one can work along the lines of previous sections and solve the PST problem in each subspace separately. In general case, however, the spaces are not always decoupled. Using the above methodology one can solve the PST in very general scenarios such as decoupled spaces, interacting and non-interacting excitations (bosons or fermions). Next we will show how to formalize the difference between interacting and non-interacting excitations.

In the case of non-interacting excitations the total energy is simply

$$E_{i,j}^{(\mu,\nu)} = \epsilon_{i,\mu} + \epsilon_{j,\nu}, \tag{1.36}$$

whereas in the case of interacting excitations, the total energy reads

$$\widetilde{E}_{i,j}^{(\mu,\nu)} = E_{i,j}^{(\mu,\nu)} + U_{i,j}^{(\mu,\nu)}. \tag{1.37}$$

The additional energy can take into account different degrees of freedom as well as different positions of excitations in the network, it can describe the on-site interactions as well as inter-site interactions.

Several results have been presented for Hamiltonians of the form

$$\begin{aligned} H = &\sum_{i,\sigma} \epsilon_{i,\sigma} n_{i,\sigma} + \frac{1}{2} \sum_{i,k} \sum_{\sigma,\sigma'} U_{i,k}^{(\sigma,\sigma')} n_{i,\sigma} \left(n_{k,\sigma'} - \delta_{i,k}\delta_{\sigma,\sigma'}\right) \\ &+ \sum_{i<k} \sum_{\sigma} J_{i,k} \left(a_{i,\sigma}^{\dagger} a_{k,\sigma} + H.c.\right), \end{aligned} \tag{1.38}$$

where $n_{i,\sigma} = a^{\dagger}_{i,\sigma} a_{i,\sigma}$. The last term in (1.38) describes the coupling between different nodes and is responsible for the very possibility of state transfer. This Hamiltonian is met in various designs of quantum computers. Results for decoupled subspaces and nearest neighbor interactions and beyond nearest neighbor interactions with coupled subspaces have been presented in [4]. The solutions always followed the methodology from above, however sometimes exact analytical solution could not be found and numerical methods had to be used to gain insight.

2 Trapped ions

Let us now move to a concrete physical implementation of quantum computer and attempt to solve the problem of PST for it. As the name of the section suggests, we are interested in trapped ions. We will be talking about ions trapped in classical Paul traps, description of which can be found in for example [11–13]. When one ion is introduced into the trapping potential, it can be shown [12] that the resulting trapping can be separated into slow (on the time scales of interest) motion and fast oscillations, the quickly oscillating part can be neglected as its time average is zero and we are interested in the slow motion. It can be shown [12] that the equation of motion for the slow part is of the form of the harmonic oscillator. If we cool the ion enough, its quantum features become significant and we need quantum mechanics to describe its motion and other properties (spin...).

That alone would not be much interesting, but we can introduce multiple ions into the trap and observe what happens. If the trapping is very strong in the longitudinal direction, the ions will form a linear chain [7–10]. If we now include the Coulomb force into the equations of motion, using approximation of small oscillations and transforming into normal coordinates we can show that individual motion of each ion in the trap is a subject to collective modes of oscillation and the entire system behaves approximately as one harmonic oscillator. Which is very important as that will be the way the ions can "communicate" with each other. We can store one qubit into the first two (ground and excited) states of the oscillator. This is the so called "information bus". As we will show, every ion in the trap can be coupled to this qubit and that is how the ions communicate (they will not be coupled to each other directly).

Another essential part of computing with trapped ions is their internal spin state, every ion in the trap can either have projection of spin in z direction $+\frac{1}{2}$ or $-\frac{1}{2}$, we will denote the $+\frac{1}{2}$ state by $|e\rangle$ as the excited state and the other one by $|g\rangle$ as a ground state. These internal states are used for information encoding. It is very illustrational example of a quantum computer as we can imagine very well how to encode number 01001 onto internal spin states of a chain of 5 ions...

The ions in the trap and their internal state can be coupled with the overall harmonic oscillator state via interaction with laser focused on each ion. The coupling is described by Jaynes-Cummings interaction resulting from common dipole interacting with electromagnetic field [11–13]. The interaction will be our

next stop.

2.1 Dipole interaction

We now want to focus on standard dipole interaction for single ion in the trap

$$H_I = -\vec{\mu} \cdot \vec{B}, \tag{2.1}$$

where the dipole moment $\vec{\mu}$ is proportional to spin operator

$$\vec{\mu} = \mu_m \vec{S}, \tag{2.2}$$

with $\mu_m$being the respective magneton and $\vec{S}$ is proportional to the Pauli vector

$$\vec{S} = \frac{\vec{\sigma}}{2}. \tag{2.3}$$

And we are interested in magnetic field of the form

$$\vec{B} = B_1 \vec{x} \cos\left(kz - \omega t + \varphi\right), \tag{2.4}$$

with B_1 characterizing the field strength, k is its momentum in the z-direction, ω its frequency, φ its phase and $\vec{x}$is a unit vector in the x-direction.

Except for interaction with spin there is even interaction with the modes of harmonic oscillations as the form of the field the ion sees depends on his position too. That is the intuitive way to image it. We know [13] as we have previously mentioned that the spin is confined within a harmonic trapping, let us denote its energy scale by ω_z as $\hbar = 1$. The ion has been cooled enough that its position becomes quantized and we need to describe it using a Hermitian operator

$$z = z_0 \left(a^\dagger + a\right), \tag{2.5}$$

where $a^\dagger$ and a are the creation and annihilation operators of phonons [13] associated with harmonic oscillations. Let us now assume that the the ion is near its ground state and the width of its oscillation is small compared to the wavelength of the incident light. Expressing this more correctly means that the Lamb-Dicke parameter $\eta \equiv kz_0$ is small. If we now define a real coupling constant g as

$$g \equiv \mu_m \frac{B_1}{2}, \tag{2.6}$$

and using that [13]

$$S_x = \frac{S_+ + S_-}{2}, \tag{2.7}$$

where S_+, S_- are atomic raising and lowering operators, we rewrite the interaction Hamiltonian for small η to be

$$\begin{aligned} H_I &= -\vec{\mu} \cdot \vec{B} \\ &\approx \left[\frac{g}{2}\left(S_+ e^{i(\varphi-\omega t)} + S_- e^{-i(\varphi-\omega t)}\right)\right] \\ &+ \left[\frac{i\eta g}{2}\left(S_+ a + S_- a^\dagger + S_+ a^\dagger + S_- a\right)\left(e^{i(\varphi-\omega t)} - e^{-i(\varphi-\omega t)}\right)\right]. \end{aligned} \tag{2.8}$$

The first bracket describes the Jaynes-Cummings interaction model that would be sufficient if the ion wasn't oscillating, but its position is not constant and so the second term describing the coupling between the harmonic state of the ion and the internal spin state [13] arises.

Now we can transform into the interaction picture with the free particle Hamiltonian being

$$H_0 = \omega_0 S_z + \omega_z a^\dagger a, \tag{2.9}$$

which is a result of previous description of the ion (with spin) being trapped by electromagnetic field in a harmonic potential. This transformation generates the following time evolutions for the operators used in the previous formula

$$\begin{array}{ll} S_+(t) = S_+ e^{i\omega_0 t} & S_-(t) = S_- e^{-i\omega_0 t} \\ a^\dagger(t) = a^\dagger e^{i\omega_z t} & a(t) = a e^{-i\omega_z t} \end{array} \tag{2.10}$$

Using these we can rewrite the Hamiltonian (2.8) in specific cases of the frequency of the incident light as

$$H_I' = \begin{cases} i\frac{\eta g}{2}\left(S_+ a^\dagger e^{i\varphi} - S_- a e^{-i\varphi}\right) & \text{for } \omega = \omega_0 + \omega_z \\ i\frac{\eta g}{2}\left(S_+ a e^{i\varphi} - S_- a e^{\dagger - i\varphi}\right) & \text{for } \omega = \omega_0 - \omega_z \end{cases}, \tag{2.11}$$

where only the dominant terms in the Hamiltonian have been written (so called rotating wave approximation). From now on we will only be using the second possibility in (2.11) and we will transform it back into the Schrödinger picture, where if we rewrite the raising and lowering operators into the computational basis, we get

$$H_I = \frac{g}{2}\left(a\left|e\right\rangle\left\langle g\right| + a^\dagger\left|g\right\rangle\left\langle e\right|\right), \tag{2.12}$$

for a particular choice of the phase φ. If we now rewrite the free Hamiltonian using the same notation, we can write the complete Hamiltonian for frequency $\omega = \omega_0 - \omega_z$ of the incident light as:

$$H = \omega_e\left|e\right\rangle\left\langle e\right| + \omega_g\left|g\right\rangle\left\langle g\right| + \omega_z a^\dagger a + \frac{g}{2}\left(a\left|e\right\rangle\left\langle g\right| + a^\dagger\left|g\right\rangle\left\langle e\right|\right), \tag{2.13}$$

where ω_e and ω_g respond to energies of eigenstates of $\omega_0 S_z$ with projection of spin $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively ($\omega_0 = \omega_e - \omega_g$). This Hamiltonian can be generalized directly to the case of N particles in the trap by reassigning $g \longrightarrow \frac{g}{\sqrt{N}}$, that is due to the fact that as we mentioned, when multiple ions are introduced into the trap, they tend to oscillate as one harmonic oscillator, which acts as an informational bus. Lastly we need to mention that we did not take into account the affect the interaction has on the electromagnetic field that interacts with ion in the trap. That is a valid approximation to a very good degree as the light does not tend to become entangled with the system it interacted with to a very deep level [13].

3 State transfer on trapped ions

In this section we find various solutions of the problem of the perfect state transfer on trapped ions using the tools previously derived and showing in which cases they cannot be used in the form they are presented and why.

3.1 Illuminating single ion

Here we will want to show how to perform a perfect state transfer by illuminating always one ion in the trap and we will find a concrete sequence of pulses that lead to state transfer from the first ion in the trap to the last one. That this can be done is no surprise as we know that using this method we can achieve any unitary transformation on the ions [11–13].

As we derived earlier, the complete Hamiltonian for one ion in a trap reads:

$$H = \left(\omega_e \left|e\right\rangle \left\langle e\right| + \omega_g \left|g\right\rangle \left\langle g\right| + \omega_z a^\dagger a\right) + \frac{g}{2}\left(a \left|e\right\rangle \left\langle g\right| + a^\dagger \left|g\right\rangle \left\langle e\right|\right), \tag{3.1}$$

We will now be working in a single excitation space where the excitation can be present either in the internal spin state of an ion in the trap or in the oscillation mode of the ions. We will denote the vectors (as previously) by

$$\left|i\right\rangle \equiv \left|g\right\rangle_1 \ldots \left|e\right\rangle_i \ldots \left|g\right\rangle_{N+1},$$

where the right hand side means that excitation is present on the j-th qubit, where the last $(N+1)$th qubit is the oscillation of the entire ion string and the first N qubits stand for internal states of the qubits. The one thing that we need to be careful with is that the last qubits excitation has different energy level, we will see that come into play later.

When illuminating one ion in the trap, we can focus on the subspace of the internal state of the ion and the overall oscillation state as the remaining internal state spaces are decoupled with these two (there is no laser acting on them). We will now show how to perform certain unitary transformations on this subspace and for now we will only consider one ion in the trap. We can see that if we denote the most important part of the interaction (3.1), we will call it coupling, by

$$\Sigma = a \left|e\right\rangle \left\langle g\right| + a^\dagger \left|g\right\rangle \left\langle e\right|, \tag{3.2}$$

we can by simple relabeling the constants write that this coupling (when laser is pointed to our ion) generates the transformation

$$U_\theta = \exp\left(-i\theta\Sigma\right), \tag{3.3}$$

where $\theta \in \mathbb{R}$ stands for the coupling strength (the coupling constant g) and the time — the length of the interaction, where we have chosen one particular phase φ. From the way exponential (time evolution) is defined, it is easy to see that

$$U_\theta \left|g\right\rangle \left|g\right\rangle = \left|g\right\rangle \left|g\right\rangle, \tag{3.4}$$

where the notation is the same as for N ions and the second term stands for the ground state of the oscillations, sometimes this is denoted by $|g,0\rangle$ [12]. If we now calculate

$$\Sigma^2 = aa^\dagger |e\rangle \langle e| + aa^\dagger |g\rangle \langle g| , \tag{3.5}$$

we can write how coupling operator and its square act on the basis states

$$\begin{array}{ll} \Sigma^2 |e\rangle |g\rangle = |e\rangle |g\rangle & \Sigma |e\rangle |g\rangle = |g\rangle |e\rangle \\ \Sigma^2 |g\rangle |e\rangle = |g\rangle |e\rangle & \Sigma |g\rangle |e\rangle = |e\rangle |g\rangle \end{array} . \tag{3.6}$$

That comes to be particularly useful when we expand the time evolution as

$$\begin{aligned} \exp(-i\theta\Sigma) &= \sum_{k=0}^{\infty} \frac{(-i\theta)^k}{k!} \Sigma^k \\ &= \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k}}{(2k)!} \Sigma^{2k} + \sum_{k=0}^{\infty} \frac{(-i\theta)^{2k+1}}{(2k+1)!} \Sigma^{2k+1}, \end{aligned} \tag{3.7}$$

as we can see now that

$$U_\theta |g\rangle |e\rangle = \cos\theta |g\rangle |e\rangle - i \sin\theta |e\rangle |g\rangle , \tag{3.8}$$

and

$$U_\theta |e\rangle |g\rangle = \cos\theta |e\rangle |g\rangle - i \sin\theta |g\rangle |e\rangle . \tag{3.9}$$

Thus a π pulse ($2\theta = \pi$) will cause the transformation

$$|g\rangle |e\rangle \longrightarrow -i |e\rangle |g\rangle , \tag{3.10}$$

$$|e\rangle |g\rangle \longrightarrow -i |g\rangle |e\rangle , \tag{3.11}$$

and respectively the transformation for a 2π pulse ($2\theta = 2\pi$) is

$$|g\rangle |e\rangle \longrightarrow - |g\rangle |e\rangle , \tag{3.12}$$

$$|e\rangle |g\rangle \longrightarrow - |e\rangle |g\rangle , \tag{3.13}$$

It is worth noting that the resonant frequency and all other parameters depend on the overall oscillation state and have to be chosen appropriately to achieve a pulse actually corresponding to some θ.

Let us now come back to N ions in the trap, everything we presented for illumination of a single ion in the trap is valid for N ions as well, we only need to change the coupling constant g to $\frac{g}{\sqrt{N}}$ [13], which means that θ stands for a different constant, but a constant nevertheless. We will now denote a θ pulse applied to ion k by U_θ^k, so for example a π pulse applied to ion 13 would be U_π^{13}. Let us now perform a following sequence of pulses, we want to attempt to move

the excitation from qubit 1 to qubit N (the $(N+1)$th one is the oscillation qubit) and perform the permutation (1.5):

$$\begin{array}{cccc} & |e\rangle_1 \dots |g\rangle_N |g\rangle_{N+1} & |g\rangle_1 \dots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \dots |g\rangle_N |e\rangle_{N+1} \\ U_\pi^1 & \downarrow & \downarrow & \downarrow \\ & -i |g\rangle_1 \dots |g\rangle_N |e\rangle_{N+1} & |g\rangle_1 \dots |e\rangle_N |g\rangle_{N+1} & -i |e\rangle_1 \dots |g\rangle_N |g\rangle_{N+1} \\ U_\pi^N & \downarrow & \downarrow & \downarrow \\ & |g\rangle_1 \dots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \dots |e\rangle_N |g\rangle_{N+1} & -i |e\rangle_1 \dots |g\rangle_N |g\rangle_{N+1} \\ U_\pi^1 & \downarrow & \downarrow & \downarrow \\ & |g\rangle_1 \dots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \dots |e\rangle_N |g\rangle_{N+1} & |g\rangle_1 \dots |g\rangle_N |e\rangle_{N+1} \end{array} \tag{3.14}$$

This expressed in the basis of the single excitation space presented at the beginning (1.2) reads as transformation U,

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & \\ & 0 & \ddots & & \vdots & \\ \vdots & \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \\ 1 & 0 & \cdots & & 0 & 0 \\ & & \cdots & & 0 & 1 \end{pmatrix}. \tag{3.15}$$

This is exactly the sought permutation on the N sites. However this procedure has not been covered by the presented framework of finding Hamiltonians that lead to PST for one reason. Because this is not one Hamiltonian that would belong to any of the classes derived, but rather a sequence of Hamiltonians, which we can order as we wish in time, that generate different unitary transformations. Also we need to be a little careful with the requirements we laid on the Hamiltonians in the general sections, the Hamiltonian of trapped ions does not necessarily conserve the total number of excitations when interacting with light, the reasons why the transformation we found conserved the number of excitations were choosing the appropriate pulses (and frequency of the incident light before that). One more problem may arise in generalizing the theory to be applied to trapped ions, the Hamiltonian of the string does not involve the same energy levels for ions and the qubit of oscillation.

3.2 Illuminating two ions

In this part we will first focus on a theoretical case of illuminating all of the N ions in the trap by light on resonant frequency as that is the case that comes to mind as first when thinking about illumination of ions - lets illuminate multiple ions. Word "theoretical" has been used intentionally as that is not how the current experiments are usually done. The more standard approach is illuminating only one ion or all the ions but on resonant frequency of only one ion in a little different model, which we do not consider in this work.

We will now express the Hamiltonian of the system in the computational basis formed by eigenstates of the free particle Hamiltonian in the single excitation subspace and we will denote by

$$|e\rangle_n \equiv |g\rangle_1 \dots |e\rangle_n \dots |g\rangle_N |g\rangle_{N+1}, \tag{3.16}$$

and by

$$|g\rangle_n \equiv |g\rangle_1 \dots |g\rangle_n \dots |g\rangle_N |e\rangle_{N+1}, \tag{3.17}$$

we will denote state where the excitation is present in the oscillation qubit and the index is used to underline the fact that n-th ion is in a ground state.

For the case of illuminating all the ions on their resonant frequencies, following coupling arises in the Hamiltonian

$$\Sigma = a \sum_{n=1}^{N} |e\rangle_n \langle g|_n + a^\dagger \sum_{n=1}^{N} |g\rangle_n \langle e|_n, \tag{3.18}$$

where $a^\dagger, a$ act on the vibrational level and then the complete Hamiltonian reads

$$H = \omega_e \sum_{n=1}^{N} |e\rangle_n \langle e|_n + \omega_g \sum_{n=1}^{N} |g\rangle_n \langle g|_n + \omega_z a^\dagger a + \frac{g}{2}\Sigma. \tag{3.19}$$

In order to be able to talk about this Hamiltonian in the previously presented general framework, we need to express it in the basis of single excitation subspace (1.2) and it helps to write the vectors of the basis explicitly

$$\begin{array}{c} |e\rangle_1 |g\rangle_2 \dots |g\rangle_N |g\rangle_{N+1}, \\ |g\rangle_1 |e\rangle_2 \dots |g\rangle_N |g\rangle_{N+1}, \\ \vdots \\ |g\rangle_1 |g\rangle_2 \dots |e\rangle_N |g\rangle_{N+1}, \\ |g\rangle_1 |g\rangle_2 \dots |g\rangle_N |e\rangle_{N+1}. \end{array} \tag{3.20}$$

And now it is easy to see that the Hamiltonian will be of the form

$$H = \begin{pmatrix} \omega_e & 0 & \cdots & 0 & \frac{g}{2} \\ 0 & \omega_e & & & \\ \vdots & & \ddots & & \vdots \\ 0 & & & \omega_e & \frac{g}{2} \\ \frac{g}{2} & & \cdots & \frac{g}{2} & \omega_z \end{pmatrix}, \tag{3.21}$$

which is a case where all of the first N nodes are coupled to the last "bus" node and this Hamiltonian might be a member of one of the classes we derived. It certainly fulfills the criteria for one-cycle permutations - it is not nearest neighbor only.

Nevertheless it cannot be associated with one cycle permutations after all, because every cycle must involve sites with the same energy level and condition $\omega_e = \omega_z$ would experimentally be very hard to achieve and hold. Therefore we need to consider only permutations that contain a cycle involving the last oscillation qubit only. Note now that the condition of anti-diagonal symmetry is met by the Hamiltonian now as the symmetry is valid only for each cycle and because we consider only permutations with cycle $(N+1)$, we can see that the rest is perfectly symmetric along the anti-diagonal. Note also that the fact that we will not be able to transfer the excitation to the last qubit is not a severe restriction as we are using only the internal states for information encoding. Now we will focus only on the case of two qubits in the trap, both illuminated.

Considering this case is sufficient because if we achieve PST on these two ions, in the case of N ions we can illuminate only pairs of them consequently and achieve PST of multiple excitations one-by-one, because if we illuminate only two ions, the rest will not be coupled to them by any means and the same solution we might achieve for two ions in the trap can directly be used for pairs of N ions. In this case the Hamiltonian is of the form

$$H = \begin{pmatrix} \omega_e & 0 & \frac{g}{2} \\ 0 & \omega_e & \frac{g}{2} \\ \frac{g}{2} & \frac{g}{2} & \omega_z \end{pmatrix}, \tag{3.22}$$

and the only two-cycle permutation we can consider is

$$P = (1,2)\,(3) \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.23}$$

because it is the only many-cycle permutation except for identity that involves cycle $(N+1) = (3)$. It is exactly what we are looking for in the space of internal states of the ions ($\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Now we will work along the lines of the section (1.3) to see the constraints on the form of the Hamiltonian (3.22).

First we have to find the eigenvalues and eigenvectors of the cycles of (3.23) with dimension $d_1 = 2$ and $d_2 = 1$, the eigenvalues follow from (1.24) to be

$$\lambda_0 = e^{i0} = 1, \tag{3.24}$$

and

$$\lambda_1 = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1, \tag{3.25}$$

where λ_0 is doubly degenerate and λ_1 is non-degenerate and the corresponding

eigenvectors from (1.25) are

$$\left|v_1^{(1)}\right\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}, \tag{3.26}$$

$$\left|v_1^{(2)}\right\rangle = \frac{|3\rangle}{1}, \tag{3.27}$$

$$\left|v_{-1}^{(1)}\right\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}. \tag{3.28}$$

Therefore the corresponding subspaces are

$$\mathcal{E}_1 = \left(\left|v_1^{(1)}\right\rangle, \left|v_1^{(2)}\right\rangle\right)_\lambda, \tag{3.29}$$

$$\mathcal{E}_{-1} = \left(\left|v_{-1}^{(1)}\right\rangle\right)_\lambda, \tag{3.30}$$

where $()_\lambda$ stands for linear span of respective vectors.

The only freedom we have now in choosing the basis of these subspaces is the choice of the basis for $\mathcal{E}_1$, we can choose it as

$$\left|y_1^{(1)}\right\rangle = \mu \left|v_1^{(1)}\right\rangle + \nu \left|v_1^{(2)}\right\rangle, \tag{3.31}$$

$$\left|y_1^{(2)}\right\rangle = \nu^* \left|v_1^{(1)}\right\rangle - \mu^* \left|v_1^{(2)}\right\rangle, \tag{3.32}$$

where

$$|\mu|^2 + |\nu|^2 = 1. \tag{3.33}$$

And therefore the eigen-energies can be chosen as (1.27):

$$\mathcal{E}_1 : \begin{cases} \epsilon_1^{(1)} = 0 + 2\pi l_1^{(1)} & , l_1^{(1)} \in \mathbb{Z} \\ \epsilon_1^{(2)} = 0 + 2\pi l_1^{(2)} & , l_1^{(2)} \in \mathbb{Z} \end{cases}, \tag{3.34}$$

and

$$\mathcal{E}_{-1} : \quad \epsilon_{-1}^{(1)} = \pi + 2\pi l_{-1}^{(1)} \quad , l_{-1}^{(1)} \in \mathbb{Z}. \tag{3.35}$$

And we can write the whole class of Hamiltonians that lead to this permutation as

$$H_{\vec{l}} = \frac{1}{t}\left(\epsilon_1^{(1)} \left|y_1^{(1)}\right\rangle \left\langle y_1^{(1)}\right| + \epsilon_1^{(2)} \left|y_1^{(2)}\right\rangle \left\langle y_1^{(2)}\right| + \epsilon_{-1}^{(1)} \left|v_{-1}^{(1)}\right\rangle \left\langle v_{-1}^{(1)}\right|\right). \tag{3.36}$$

Now we will ask this Hamiltonian to be of the form (3.22), which expressed in the computational basis gives several equations for the free parameters $\mu, \epsilon_1^{(1)}, \epsilon_1^{(2)}, \epsilon_{-1}^{(1)}$. Firstly we will write the homogeneous one

$$\langle 1| H_{\vec{l}} |2\rangle = 0, \tag{3.37}$$

which when expressed in the free parameters can be written as

$$\begin{aligned}\langle 1|\left(H_{\vec{l}}\right)|2\rangle =&\frac{1}{t}\langle 1|\left[\epsilon_1^{(1)}\left(\mu\left|v_1^{(1)}\right\rangle+\nu\left|v_1^{(2)}\right\rangle\right)\left(\mu^*\left\langle v_1^{(1)}\right|+\nu^*\left\langle v_1^{(2)}\right|\right)\right. \qquad (3.38)\\ &+\epsilon_1^{(2)}\left(v^*\left|v_1^{(1)}\right\rangle-\mu^*\left|v_1^{(2)}\right\rangle\right)\left(\nu\left\langle v_1^{(1)}\right|-\mu\left\langle v_1^{(2)}\right|\right)\\ &\left.+\epsilon_{-1}^{(1)}\left(\left|v_{-1}^{(1)}\right\rangle\right)\left(\left\langle v_-^{(1)}\right|\right)\right]|2\rangle\\ =&\frac{1}{2t}\left(\epsilon_1^{(1)}|\mu|^2+\epsilon_1^{(2)}|\nu|^2-\epsilon_{-1}^{(1)}\right)=0. \qquad (3.39)\end{aligned}$$

this equation obviously yields

$$\epsilon_1^{(1)}|\mu|^2+\epsilon_1^{(2)}|\nu|^2-\epsilon_{-1}^{(1)}=0. \qquad (3.40)$$

This can be rewritten with the normalizing condition (3.33) as

$$\left(\epsilon_1^{(2)}-\epsilon_1^{(1)}\right)|\nu|^2+\epsilon_1^{(1)}-\epsilon_{-1}^{(1)}=0, \qquad (3.41)$$

and similarly to example in [3] we see that if we have a solution for $\left(\epsilon_1^{(1)},\epsilon_1^{(2)},\epsilon_{-1}^{(1)}\right)$, then there is a solution for $\left(\epsilon_1^{(1)}+2\pi j,\epsilon_1^{(2)}+2\pi j,\epsilon_{-1}^{(1)}+2\pi j\right), j\in\mathbb{Z}$, because the equation depends only on differences between eigen-energies. And we see that the homogeneous equation restricted the choice of the basis as the absolute value of the parameter must be

$$|\nu|=\sqrt{\frac{\epsilon_{-1}^{(1)}-\epsilon_1^{(1)}}{\epsilon_1^{(2)}-\epsilon_1^{(1)}}}. \qquad (3.42)$$

For this to be a solution, one of the following conditions has to be met

$$\epsilon_1^{(1)}<\epsilon_{-1}^{(1)}\wedge\epsilon_1^{(1)}<\epsilon_1^{(2)}, \qquad (3.43)$$

$$\epsilon_1^{(1)}>\epsilon_{-1}^{(1)}\wedge\epsilon_1^{(1)}>\epsilon_1^{(2)}. \qquad (3.44)$$

It means that we have found a solution for the case when the Hamiltonian has non-degenerate spectrum and these conditions restrict possible choices of energies. The opposite case when the spectrum is degenerate will be discussed later.

Now it is time to use the remaining equations

$$\begin{aligned}&\langle 1|H_{\vec{l}}|1\rangle=\omega_e,\quad \langle 1|H_{\vec{l}}|3\rangle=\tfrac{g}{2},\\ &\langle 2|H_{\vec{l}}|2\rangle=\omega_z,\end{aligned} \qquad (3.45)$$

these equations would give us estimation of the parameters $\omega_e,\omega_z,\frac{g}{2}$ for any choice of energies satisfying one of the conditions (3.43) or (3.44).

The choice of non-degenerate spectrum was well justified as when we choose for example $\epsilon_1^{(1)} = \epsilon_1^{(2)}$, the equation for $\frac{g}{2}$ reads

$$\frac{1}{t}\left(\epsilon_1^{(1)}\frac{\mu\nu^*}{\sqrt{2}} - \epsilon_1^{(2)}\frac{\mu\nu^*}{\sqrt{2}}\right) = \frac{g}{2}, \tag{3.46}$$

which for this case gives $g = 0$, in other words no coupling would be present. Also from (3.41) we would see that this condition would yield $\epsilon_1^{(1)} = \epsilon_{-1}^{(1)}$ and the spectrum would be completely degenerate.

The degenerate case of $\epsilon_1^{(1)} = \epsilon_{-1}^{(1)}$ would from (3.41) mean that $\epsilon_1^{(1)} = \epsilon_1^{(2)}$, which we have discussed or that $|\nu|^2 = 0$, which would mean $|\mu|^2 = 1$ and that would give $\omega_e = \omega_z$, which is the case we do not consider for previously mentioned reasons (mainly this Hamiltonian would be experimentally complicated).

4 Conclusions

The main conclusion from section 3.2 is that illuminating multiple ions leads to PST in the presented framework. The illumination of two ions can be directly used for transfer of multiple excitations. For example we can have a chain of N ions in the state

$$|e\rangle_1 |g\rangle_2 |e\rangle_3 |g\rangle_4 \dots |g\rangle_{N-2} |g\rangle_{N-1} |g\rangle_N |g\rangle_{N+1}, \tag{4.1}$$

in this case we would first illuminate the first ion and the ion on position $N-2$ simultaneously. We would choose the constants of the trapping and of the driving field so that the resulting Hamiltonian would lead to permutation as in section 3.2. Then consequently we would do the same procedure with ions 2 and $N-1$ and then 3 and N. And we see that we would transfer the information encoded on the first three ions to the last three positions. Condition for this to work is obviously that the positions of the source ions and the target ions cannot overlap. One could ask if it is possible to transfer the excitation by illuminating all ions in the trap, not just two of them. The answer could be positive, but experimentally the system would absorb much more heat in this case then in the case of illumination of only two ions. One interesting question that would deserve further exploration is whether the possibility of presence of multiple excitations in the oscillation qubit could give together with illuminating all ions positive results concerning PST.

From section 3.1 we know that PST can be achieved by sequences of pulses onto certain subsets of ions in the trap. Very important property of the transfer of single excitation from section 3.1 is that it can be directly used for transfer of multiple excitations, let us illustrate this on transfer of two excitations

$$|e\rangle_1 |g\rangle_2 |e\rangle_3 |g\rangle_4 \dots |g\rangle_{N-2} |g\rangle_{N-1} |g\rangle_N |g\rangle_{N+1}, \tag{4.2}$$

applying the sequence (3.14) first to ions 1 and $N-2$ and then the same sequences to ions 2 and $N-1$ and 3 and N will cause the system to evolve into the state

$$|g\rangle_1 |g\rangle_2 |g\rangle_3 |g\rangle_4 \dots |e\rangle_{N-2} |g\rangle_{N-1} |e\rangle_N |g\rangle_{N+1}, \tag{4.3}$$

which we can see is a transport of information from nodes $\{1,2,3\}$ to $\{N-2, N-1, N\}$, exactly the same as previously. The same thing could be done for any subsets of nodes that do not overlap. For overlapping subsets we would need an extra node, which would serve as a temporary swap container of a state.

One also has to think about transporting multiple excitations to the last qubit, the oscillation state, at once... in the example above if we changed the ordering of the pulses to make the excitations first jump from ions to the oscillation mode and only then attempted to write them back onto the internal states, we would get the final result

$$|g\rangle_1 |g\rangle_2 |g\rangle_3 |g\rangle_4 \dots |e\rangle_{N-2} |e\rangle_{N-1} |g\rangle_N |g\rangle_{N+1}, \tag{4.4}$$

where an obvious loss of information occurred. So this approach does not allow for multiple excitations to be present at the last qubit (where it is only possible).

The general framework presented in this work is a framework used in presence by authors working on generalizing their approach to perfect state transfer. The method of achieving the perfect state transfer on trapped ions from section 3.1 by a sequence of Hamiltonians might be an idea how to generalize it even further.

The next step of this research should be an attempt to find Hamiltonians from classes presented in this work in models of trapped ions that allow for usage of mw and rf radiation instead of lasers that have Hamiltonians of the form (1.38), where spin-to-spin couplings are present [7–10] and the formalism from section 1.4 could therefore be used effectively.

Comparing the two approaches to PST from sections 3.2 and 3.1 gives some advantages and disadvantages for both. The main advantage of the illumination of two ions is that is covered by current formalism of PST and therefore can be easily compared to other systems and the same numerical methods that can be used for finding the right parameters of other systems can be used for trapped ions as well. The positive property of illuminating a single ion, however, is that similar operations have been previously implemented and therefore implementation of such transformation should not be more complicated then previous experiments, because the language of π-pulses etc. is common both for PST and any other calculation of trapped ions.

References

[1] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, *Perfect state transfer in quantum spin networks*, Phys. Rev. Lett. **92**, 187902 (2004).

[2] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Phys. Rev. A **71**, 032312 (2005).

[3] V. Košťák, G. M. Nikolopoulos, and I. Jex, *Perfect state transfer in networks of arbitrary topology and coupling configuration*, Phys. Rev. A **75**, 042319 (2007).

[4] T. Brougham, G. M. Nikolopoulos, and I. Jex, *Perfect transfer of multiple excitations in quantum networks*, Phys. Rev. A **83**, 022323 (2011).

[5] T. Brougham, G. M. Nikolopoulos, and I. Jex, *Communication in quantum networks of logical bus topology*, Phys. Rev. A **80**, 052325 (2009).

[6] T. Monz, P. Schindler, J. T. Barreiro, M. Chwalla, D. Nigg, W. A. Coish, et al., *14-qubit entanglement: Creation and coherence*, Phys. Rev. Lett. **106**, 130506 (2011).

[7] M. Johanning, A. Braun, N. Timoney, V. Elman, W. Neuhauser, and Ch. Wunderlich, *Individual adressing of trapped ions and coupling of motional and spin states using rf radiation*, Phys. Rev. Lett. **102**, 073004 (2009).

[8] F. Mintert and Ch. Wunderlich, *Ion-trap quantum logic using long-wavelength radiation*, Phys. Rev. Lett. **87**, 257904 (2001).

[9] D. Mc Hugh and J. Twamley, *Quantum computer using a trapped-ion spin molecule and microwave radiation*, Phys. Rev. A **71**, 012315 (2005).

[10] H. Wunderlich, Ch. Wunderlich, K. Singer, and F. Schmidt-Kaler, *Two-dimensional cluster-state preparation with linear ion traps*, Phys. Rev. A **79**, 052324 (2009).

[11] H. Figger, D. Meschede, and C. Zimmermann, *Laser Physics at the Limit* (Springer, Berlin, 2001), pp. 261–271.

[12] C. Macchiavello, G. M. Palma, and A. Zeilinger, *Quantum Computation and Quantum Information Theory* (World Scientific Publishing Co., Singapore, 2000), pp. 277–278, 331–315, 401–402, 465–468.

[13] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information Theory* (Cambridge University Press, Cambridge, 2000), pp. 17–120, 283–287.