RESEARCH WORK

Feynman summation in finite-dimensional quantum mechanics

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Contents

Introduction

This paper is summary and enhancement of existing rather scattered literature regarding finite-dimensional quantum mechanics. In the later parts Feynman's path summation is discussed.

Purpose of chapter 1 is to get familiar with finite-dimensional appoximation operator using discrete Fourier transformation as an example. In chapter 2 idea of inducing discrete kinematics using pair of mappings is discussed for special case of Schwinger approximation on flat configuration manifold $\mathbb{R}$. In chapter 3 convergence question for defined Hilbert space imbedding of Swinger approximation on $\mathbb{R}$ is discussed. In chapter 4 most intuitive discrete-time evolution definitions are discussed. Special attention is paid to Feynman's path integral. Feynman's checkerboard problem closely connected to Feynman's path integral is also included in this chapter.

Chapter 1

Discrete fourier transformation

Definition 1 ((odd) Discrete fourier transformation)**.**

$$\langle\rho_1|\Big(F_N|\rho_2\rangle\Big) = \sqrt{\frac{2\pi}{2N+1}}.\frac{\exp(i\frac{2\pi}{2N+1}\rho_1\rho_2)}{\sqrt{2\pi}},$$

where dimension in which we define this *unitary* operator is $2N+1$ i.e. odd and $\{|\rho\rangle\}$ is orthonormal basis denoted $\rho \in \{-N, -N+1\ldots N\}$, $N \in \mathbb{N}$

From the following example its similarity to the fourier transform on $\mathbb{R}$ or finite intervals can be easily seen.

Example 1.

$$\langle\rho_1|\Big(F_N|\psi\rangle\Big) = \sum_{\rho_2=-N}^{N}\sqrt{\frac{2\pi}{2N+1}}\quad\frac{\exp(i\frac{2\pi}{2N+1}\rho_1\rho_2)}{\sqrt{2\pi}}\langle\rho_2|\psi\rangle$$

Theorem 1 (Basic theorems)**.**

(1.) $F_N^2 = \mathrm{antidiag}(1,1,\ldots), F_N^3 = F_N^*, F_N^4 = 1$

(2.) $\sigma(F_N) = \{1,-1,i,-i\} = \{i^k|k \in \{0,1,2,3\}\}$

(3.) $F_N|h_k^D\rangle = i^k|h_k^D\rangle$, where $\langle\rho|h_k^D\rangle := \sum_{p=-\infty}^{+\infty} h_k\Bigg(\sqrt{\frac{2\pi}{2N+1}}(\rho+(2N+1)p)\Bigg)$

and h_k are Hermite functions i.e. orthonormal basis on $L^2(\mathbb{R}, dx)$.

(4.) $\langle\rho|h_k^D\rangle = \sum_{l=-\infty}^{+\infty}\exp(i\frac{2\pi}{2N+1}l\rho)h_k\Bigg(\sqrt{\frac{2\pi}{2N+1}}l\Bigg)$

(5.) $\{|h_k^D\rangle|k \in \{0,1,\ldots 2N\}\}$ is non-orthogonal basis of Hilbert space.

Proof. Done in [12]. □

In the following section it will be further mentioned that in the model employed in this paper, spacial and momentum eigenvectors are connected through the discrete Fourier transformation, i.e.

$$\langle \rho | k \rangle = \sqrt{\frac{2\pi}{2N+1}} . \frac{\exp(i \frac{2\pi}{2N+1} \rho k)}{\sqrt{2\pi}} .$$

A theorem of convergence of discrete Fourier transformation to the Fourier transform on $L^2(\mathbb{R}, dx)$ will be proven in chapter. 3.

Chapter 2

Induced discrete kinematics

Idea of this chapter is to introduced cannonical way of using specific pair of mappings between continuous functions on configuration manifold $C(M)$ and functions defined on lattice approximating this manifold $C(M^D)$ to induce approximation of kinematics. Since only case $M = \mathbb{R}$ is covered it is possible to use Weyl tranformation to obtain quantum observables from classical ones.

2.1 Induced classical kinematics on discrete space

At first configuration sets will be defined: discrete ordered set and manifold from which we will induce kinematics. There are two simplest confuguration manifolds on which we can introduce model used in this paper. They are $\mathbb{R}$ and the circle. In this paper only former possibility will be covered.

Definition 2.

$M := \mathbb{R}$, $\mathcal{H} := L^2(\mathbb{R}, dx)$,
$M^D := \{-N, -N+1, \dots, N\} \subset \mathbb{Z}$, where $N \in \mathbb{N}, \tilde{N} := 2N+1$,
$\{|\rho\rangle : \rho \in M^D\}$ is orthonormal basis of finite dimensional hilbert space $\mathcal{H}_N$
$C^\infty(M^D, T) := \{f \colon M^D \to T\}$

Definition 3 (modulo with sign and floor)**.**

$$(m \bmod \tilde{N}) = l \in M^D \Leftrightarrow \exp(i\frac{2\pi}{\tilde{N}}m) = \exp(i\frac{2\pi}{\tilde{N}}l)$$
$$\lfloor y \rceil := l \in \mathbb{Z} : l - y \leq 0.5 \vee y - l < 0.5, y \in \mathbb{R}$$

Let us define natural flat imbedding with real parameter a.

Definition 4 (natural flat imbedding)**.**

$$\mathrm{i_M}\colon \mathbb{Z} \to M : \mathrm{i_M}(\rho) := \eta_N \rho$$
$$\mathrm{c_M} : M \to M^D : \mathrm{c_M}(x) := \eta_N(\lfloor \frac{x}{\eta_N} \rfloor \bmod \tilde{N}),$$
$$\text{where } \eta_N := \sqrt{\frac{2\pi}{\tilde{N}}} a, \quad a > 0$$

For both set there is natural group acting transitively - groups of translations:

Definition 5 (groups of translations)**.**

$$G := (\mathbb{R}, +) : \sigma(g, x) := x + g,$$
$$G^D := \{-N, -N+1, ..., N\}, \ : \sigma(j, \rho) := (\rho + j \bmod \tilde{N})$$

Since $T_m M \simeq M \simeq G$ for every point $m \in M$ and there is isomorfism $\phi : T_m M \to G : \phi(v) := exp(v)$, we define:

Definition 6.

$$T_\rho M^D \simeq T^*_\rho M^D \simeq G^D$$
$$\mathrm{i_T} : \mathbb{Z} \to T^*_{\mathrm{i_M}(\rho)} M : \mathrm{i_T}(j) := \eta_N j,$$
$$\mathrm{i_{T^*}} : \mathbb{Z} \to T^*_{\mathrm{i_M}(\rho)} M : \mathrm{i_{T^*}}(j) := \eta_N j,$$
$$\mathrm{c_T} : T_{\mathrm{i_M}(\rho)} M \to T_\rho M^D : \mathrm{c_T}(p) := \eta_N(\lfloor \frac{x}{\eta_N} \rfloor \bmod \tilde{N}),$$
$$\mathrm{c_{T^*}} : T^*_{\mathrm{i_M}(\rho)} M \to T^*_\rho M^D : \mathrm{c_T}(p) := \xi_N(\lfloor \frac{x}{\xi_N} \rfloor \bmod \tilde{N}),$$
$$\text{where } \xi_N := \sqrt{\frac{2\pi}{\tilde{N}}} \frac{1}{a}.$$

Imbedding into trivial tangent bundle can be defined in analogy, but will not be needed. I will now introduce *special mappings that are essential to model* that will be treated in this paper.

Definition 7 (Schwinger approximation)**.**

$$\mathrm{app}_S[f] :=$$
$$= \begin{cases} \sum_{j,\rho=-N}^{N} \eta_N \xi_N \, \mathrm{e}^{i \, \mathrm{i_{T^*}}(j_2)(q - \mathrm{i_M}(\rho))} \, f(k, \rho) \in C^\infty(M) & f \in C(M^D) \\ \sum_{j_1, j_2, k, \rho=-N}^{N} \eta_N \xi_N \, \mathrm{e}^{i \, \mathrm{i_T}(j_1)(p - \mathrm{i_{T^*}}(k))} \, \eta_N \xi_N \, \mathrm{e}^{i \, \mathrm{i_{T^*}}(j_2)(q - \mathrm{i_M}(\rho))} \, f(k, \rho) & f \in C(T^* M) \end{cases}$$

$$\mathrm{red}_S[f] := \begin{cases} f(\mathrm{i_M}(\rho)) \in C(M^D) & f \in C(M) \\ f(\mathrm{i_{T^*}}(k), \mathrm{i_M}(\rho)) \in C(T^* M^D) & f \in C(T^* M) \end{cases}$$

Other models utilize other mappings e.g. spline interpolation. This approximation is not covered in this paper, but can be easily modified for this.

It is evident that both introduced mappings differ for every $N \in \mathbb{N}$. Mapping $\mathrm{app}[.]$ is linear and functions $\mathrm{app}[f]$ are periodic i.e.:

$$\mathrm{app}_S[f](p,q) = \mathrm{app}_S[f](p + \tilde{N}\xi_N j, q + \tilde{N}\eta_N \rho), \text{ where } j, \rho \in \mathbb{Z}.$$

Approximation of derivation and vector action will be defined in following:

Definition 8.

$$\partial_y f := \mathrm{red}_S[\partial_y[\mathrm{app}_S(f)]]$$

$$v(f) := \left(\sum_{j=-N}^{N} \mathrm{i_T}(v_j^{(q)})\partial_q + \mathrm{i_M}(v_j^{(p)})\partial_p \right) f, \text{ where } f \in C^\infty(T^*M^D),\ v \in T(T^*M^D)^D$$

Interesting feature of Schwinger approximation of derivative is that it has "non-local character" i.e. value of Schwinger derivative in every point *depends on value of wave-function at all points*, althought with weight falling quickly with rising distance.

There is convergence theorem to this definition of derivation in chapter 3.

Definition 9 (phase-space)**.**

$$\Gamma := T^*M \simeq \mathbb{R}^2, \quad \Gamma^D := T^*M^D \simeq \{-N, -N+1, ..., N\}^2 \subset \mathbb{Z}^2, \text{ where } N \in \mathbb{N}$$

To obtain approximation of observables $f \in C(\Gamma, \mathbb{R})$ on discrete phase-space reduction mapping on Γ will be used.

Definition 10 (Induced observables on discrete phase-space)**.**

$$(.)_D : C(\Gamma, \mathbb{R}) \to C^\infty(\Gamma^D, \mathbb{R}) : f_D := \mathrm{red}[f], \\ \text{where } f \in C(\Gamma, \mathbb{R})$$

On discrete phase-space we can introduce induced Poisson bracket, thus induce approximation of clasical kinematics on discrete space.

Definition 11 (discrete Poisson bracket)**.**

$$\{.,.\}_P^D : C^\infty(\Gamma^D, \mathbb{R}) \times C^\infty(\Gamma^D, \mathbb{R}) \to C^\infty(\Gamma^D, \mathbb{R}) : \{f, g\}_P^D := \mathrm{red}[\{\mathrm{app}[f], \mathrm{red}[g]\}_P]$$

Following theorem is related to hamiltonian equations of motion.

Theorem 2 (discrete Poisson bracket properties)**.**

$$\{p, H(p,q)\}_P^D = -\partial_q H(p,q), \quad \{q, H(p,q)\}_P^D = \partial_p H(p,q)$$

Proof. Using unitarity of discrete fourier transformation. □

Some convergence theorems regarding pervious matter are included in section 3.

2.2 Induced quantum kinematics on discrete space

Only available mapping between spaces are approximation and reduction mappings, which has to be modified to be well defined L^2 mappings converging to original ones.

Definition 12 (Schwinger reduction)**.**

$$\hat{O}_D := \lim_{\delta\to 0}\Big(\sum_{\rho=-N}^{N}|\rho\rangle\langle \mathrm{i}_{\mathrm{M}}(\rho)\pm\delta/2|\Big)\hat{O}\Big(\sum_{k=-N}^{N}|\mathrm{i}_{\mathrm{T}^*}(k)\pm\delta/2\rangle\langle k|\Big)$$

where $|\mathrm{i}_{\mathrm{M}}(\rho)\pm\delta/2\rangle := \hat{E}^{\hat{Q}}_{(\mathrm{i}_{\mathrm{M}}(\rho)+\frac{\delta}{2},\mathrm{i}_{\mathrm{M}}(\rho)-\frac{\delta}{2})}L^2(\mathbb{R},dx)$,

$|\mathrm{i}_{\mathrm{T}^*}(k)\pm\delta\rangle := \hat{E}^{\hat{P}}_{(\mathrm{i}_{\mathrm{T}^*}(k)+\frac{\delta}{2},\mathrm{i}_{\mathrm{T}^*}(k)+\frac{\delta}{2})}L^2(\mathbb{R},dx)$,

$E^{\hat{P}}$ and $E^{\hat{Q}}$ are spectral measures of corresponding operators,

$\hat{O}$ is operator on $L^2(\mathbb{R},dx)$ for which above limit exists.

Corollary 1 (Asymptotic properties of Schwinger reduction)**.**

$$\partial_x^n\langle x|\textstyle\sum_{k=-N}^{N}|\mathrm{i}_{\mathrm{T}^*}(k)\pm\delta/2\rangle\langle k|f\rangle \xrightarrow[\delta\to 0]{} \partial_x^n\,\mathrm{app}_S[f](x),$$
$$\langle\rho_2|\textstyle\sum_{\rho=-N}^{N}|\rho\rangle\langle \mathrm{i}_{\mathrm{M}}(\rho)\pm\delta/2|\psi\rangle \xrightarrow[\delta\to 0]{} \mathrm{red}_S[\psi](\rho_2),$$
where $f : M^D\to\mathbb{C}$, $\psi\in C(\mathbb{R},\mathbb{C})$.

Name for this reduction is justified by following definition and theorem.

Definition 13 (Schwinger operators)**.**

$\hat{Q}_{Schwinger}|\rho\rangle = \mathrm{i}_{\mathrm{M}}(\rho)|\rho\rangle$, where $U(\alpha)|\rho\rangle = \mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\rho)\,\mathrm{i}_{\mathrm{T}}(\alpha)}\,|\rho\rangle$,

$\hat{P}_{Schwinger}|k\rangle = \mathrm{i}_{\mathrm{T}^*}(k)|k\rangle$, where $V(\beta)|k\rangle = \mathrm{e}^{i\,\mathrm{i}_{\mathrm{T}^*}(k)\,\mathrm{i}_{\mathrm{T}}(\beta)}\,|k\rangle$,

where U, V are operators obtained from analogue of Mackey's quantization method [5] in finite dimension [6], $\rho,k\in M^D$

It is sufficient to define scalar product of base vec. :$\langle\rho|k\rangle := \sqrt{\dfrac{2\pi}{\tilde{N}}}\dfrac{\mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\rho)\,\mathrm{i}_{\mathrm{T}^*}(k)}}{\sqrt{2\pi}}$

Note that: $U(\alpha)V(\beta) = V(\beta)U(\alpha)\,\mathrm{e}^{\mathrm{i}_{\mathrm{M}}(\alpha)\,\mathrm{i}_{\mathrm{T}^*}(\beta)}$.

Theorem 3 (Schwinger reduction of position and momentum operators)**.**

$$\hat{Q}_D = \hat{Q}_{Schwinger},\ \hat{P}_D = \hat{P}_{Schwinger}$$

Proof. Directly from definitions. □

That means that pervious procedure using special pair of mappings delivered same spacial and momentum observable as Mackey's quantization method in finite dimension [6, 5].

Theorem 4 (Schwinger reduction of Fourier transformation)**.**

$$\hat{\mathrm{F}}_D = \mathrm{F}_N \text{ which was defined in chapter 1.}$$

Proof. Directly from definitions. □

Since configuration space $M = \mathbb{R}$ is flat, we can use Weyl quatization method. I will simply use approximation of observables from discrete space to create self-adjoint operators on $L^2(\mathbb{R}, dx)$ and then by approximation and reduction mapping reduce this operator to space $\mathcal{H}_N$.

This method is particulary suitable for Schwinger quantum kinematic on discrete space. Also I am proposing general convergence theorem for special class of classical observables on phase-space, which is how ever not yet proven.

Let us define operators to special set of real functions on phase-space.

Definition 14 (Weyl transformation)**.**

$$(\Phi_W[f(p,q)]\psi)(y) := \left(\mathbf{F}[g(q)](\beta), \mathrm{e}^{i\hat{Q}\beta}\left(\mathbf{F}[h(p)](\alpha), \mathrm{e}^{i\frac{\alpha\beta}{2}}\,\mathrm{e}^{i\hat{P}\alpha}\,\psi\right)\right)$$

where $f(p,q) := h(p) \otimes g(q) : h, g \in \{s \in C^\infty(\mathbb{R},\mathbb{R}) | \exists c_n : |s(x)| < \sum_{n=0}^{C} c_n x^n\}$,

$\psi \in C_0^\infty(\mathbb{R})$ and **(,)** has common sense used for Schwartz space distributions.

Using pervious, we can define Weyl tranformation on linear span of above set.

Theorem 5 (Simplification of Weyl transformation)**.**

$$(\Phi_W[h(p) \otimes g(q)]\psi)(y) = \left(h(\hat{P})g\left(\frac{\hat{Q}+y}{2}\right)\psi\right)(y)$$

Proof. Using theory of distributions and fourier transform on Schwartz space. □

Example 2 (common Hamiltonian)**.**

$$H(p,q) := p^2 \otimes 1 + 1 \otimes V(q)$$
$$(\Phi_W[H(p,q)]\psi)(y) = (-\partial_q^2\psi)(y) + V(y)\psi(y)$$

Following example demonstrates problems that may occure, when using Weyl transformation.

Example 3.

$$(\Phi_W[\mathrm{e}^{-p^2+q^2}]\psi)(y) = (\Phi_W[\mathrm{e}^{-p^2}\otimes\mathrm{e}^{-q^2}]\psi)(y) = [F^{-1}\,\mathrm{e}^{-\hat{Q}^2}\,F\,\mathrm{e}^{-(\frac{\hat{Q}+y}{2})^2}\,\psi](y) =$$
$$\int_{\mathbb{R}} dx \frac{1}{\sqrt{2}}\,\mathrm{e}^{-\frac{(x-y)^2}{4}}\,\mathrm{e}^{-\frac{(x+y)^2}{4}}\,\psi(x) = \frac{1}{\sqrt{2}}\,\mathrm{e}^{-\frac{y^2}{2}}\int_{\mathbb{R}} dx\,\mathrm{e}^{-\frac{x^2}{2}}\,\psi(x) = \frac{1}{\sqrt{2}}|h_0\rangle\langle h_0|\psi\rangle$$

Theorem 6 (Weyl transformation of Schwinger approximated functions)**.**

$$(\Phi_W\circ\mathrm{app}_S)[f] := \sum_{j_1,j_2,k,\rho=-N}^{N} (\eta_N\xi_N)\,\mathrm{e}^{i\,\mathrm{i}_{\mathrm{T}}(j_1)(\hat{P}-\mathrm{i}_{\mathrm{T}^*}(k))}$$
$$(\eta_N\xi_N)\,\mathrm{e}^{i\,\mathrm{i}_{\mathrm{T}^*}(j_2)(\hat{Q}-\mathrm{i}_{\mathrm{M}}(\rho))}\,\mathrm{e}^{i\frac{\mathrm{i}_{\mathrm{T}}(j_1)\,\mathrm{i}_{\mathrm{T}^*}(j_2)}{2}}\,f(k,\rho),$$

where Φ_W is Weyl transformation defined above for all $f\in C(\Gamma^D,\mathbb{R})$

Proof. Using theory of distributions and fourier transform on Schwartz space. □

We have now obtained opertator on $L^2(M,dx)$. To reduce it to $\mathcal{H}_N$ we need suitable unitary mappings.

Corollary 2 (discrete Weyl transformation)**.**

$$(\Phi_W\circ\mathrm{app}_S)_D[f] := \sum_{j_1,j_2,\rho_1,\rho_2=-N}^{N} (\eta_N\xi_N)\,\mathrm{e}^{i\,\mathrm{i}_{\mathrm{T}}(j_1)(\hat{P}_D-\mathrm{i}_{\mathrm{T}^*}(\rho_1))}$$
$$(\eta_N\xi_N)\,\mathrm{e}^{i\,\mathrm{i}_{\mathrm{T}^*}(j_2)(\hat{Q}_D-\mathrm{i}_{\mathrm{M}}(\rho_2))}\,\mathrm{e}^{i\frac{\mathrm{i}_{\mathrm{T}}(j_1)\,\mathrm{i}_{\mathrm{T}^*}(j_2)}{2}}\,f(\rho_1,\rho_2)$$

Φ_W is Weyl transformation.

Theorem 7 (Simplification of discrete Weyl transformation)**.**

$$(\Phi_W[h(\rho_1)\otimes g(\rho_2)]\psi)_D(\rho) = \left(h(\hat{P}_D)\,\mathrm{app}_S[g]\left(\frac{\hat{Q}_D+\mathrm{i}_{\mathrm{M}}(\rho)}{2}\right)\psi\right)(\rho)$$

Proof. Using theory of discrete fourier transformatin 1. □

Thus we have quantization mechanism for obtaining operators of observables form their classical couterparts on flat phase-space Γ. Possible general operator convergence theorem may be consequence appeareance of only two kinds of operators on the right-hand side i.e. position and momentum operators.

Example 4 (common Hamiltonian)**.**

$$H(p,q) := p^2\otimes 1 + 1\otimes V(q)$$
$$((\Phi_W\circ\mathrm{app}_S)_D[H(p,q)] = \hat{P}_D^2 + V(\hat{Q}_D)$$

Chapter 3

Convergence

3.1 Imbedding

Let us have sequence of finite dimensional hilbert spaces and operators of certain observables on it and their counterparts on $\mathcal{H}_N$ space. We expect that physics on discrete space would be same as on continuous one for sufficiently large N i.e. observations on one state $\psi \in L^2(M, dx)$ are in limit $N \to \infty$ the same. This is question of convergence of sequilinear forms and of mapping sequilinear forms that are defined on discrete space $\mathcal{H}_N$ to their $L^2(\mathbb{R})$ counterpart. This can be solved by analogue of approximation and reduction mappings. Approximation and reduction mapping of smooth functions on flat spaces in principle are badly defined on $L^2(\mathbb{R})$.

Definition 15 (L^2-convergence characterictic function map.)**.**

$$(\hat{O}_C f)(x) := \eta_N \Big(\sum_{\rho_2=-N}^{N} |\mathrm{i}_{\mathrm{M}}(\rho_2) \pm \eta_N/2\rangle\langle\rho_2| \Big) \hat{O} \Big(\sum_{\rho_1=-N}^{N} |\rho_1\rangle\langle \mathrm{i}_{\mathrm{M}}(\rho_1) \pm \eta_N/2| \Big)$$

$$\mathrm{app}_{\mathrm{L}^2}[f, N](x) := \langle x| \sum_{\rho_2=-N}^{N} |\mathrm{i}_{\mathrm{M}}(\rho_2) \pm \eta_N/2\rangle\langle\rho_1|f\rangle,$$

$$\mathrm{red}_{\mathrm{L}^2}[f, N](\rho) := |\rho\rangle \sum_{\rho_1=-N}^{N} |\rho_1\rangle\langle \mathrm{i}_{\mathrm{M}}(\rho_1) \pm \eta_N/2|f\rangle,$$

where $\hat{O}$ is operator on $\mathcal{H}_N$

Some basic properties of above mapping are listed below.

Corollary 3.

1.) $\mathrm{red}_{\mathrm{L}^2} \circ \mathrm{app}_{\mathrm{L}^2} = \mathrm{id}_{L^2(M^D, d\mu)}$

2.) $\mathrm{app}_{\mathrm{L}^2} \circ \mathrm{red}_{\mathrm{L}^2} =$ orthogonal projector on $L^2(M, dx)$

3.) $\hat{O}_C$ is degenerate Hilbert-Schmidt integral operator,

4.) $\hat{O}_C = \hat{O}_C^* \Leftrightarrow \hat{O} = \hat{O}^*$, where $[.]^*$ denotes adjoint operator,

5.) $\sigma(\hat{O}_C) \setminus \{0\} = \sigma_p(\hat{O}_C) \setminus \{0\} = \sigma(\hat{O}) \setminus \{0\}$, where σ denotes spectra of operator,

6.) $0 \in \sigma_p(\hat{O}_C)$ and has infinite multiplicity,

7.) $\left\| \hat{O}_C \right\|_2 = \left\| \hat{O} \right\|_2$, where $\|.\|_2$ are Hilbert-Schmidt norms,

3.2 Strong resolvent convergence

Definition 16 (strong resolvent convergence)**.**

A_N, A are self-adjoint operators.

$A_N \xrightarrow[N\to\infty]{s.res.} A \Leftrightarrow R_\lambda(A_N) \xrightarrow[N\to\infty]{s.} R_\lambda(A)$ (strongly) $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$

Theorem 8 (strong resolvent convengence for self-adjoint operators)**.**

A_N, A are essentially self-adjoint on set $D \wedge \forall \psi \in D, A_N\psi \xrightarrow[N\to\infty]{s.} A\psi$

$\Rightarrow A_N \xrightarrow[N\to\infty]{s.res.} A$

Proof. Proof done in [7]. □

Pervious work [11] to this subject regarded only special cases of follwing theorems and proving in complicated way using older version of the book. Here second edition of the book [8] is used and we have:

Theorem 9.

$\{A_N\}_{N=0}^{\infty}, A$ self-adjoint operators $\wedge\, A_N \xrightarrow[N\to\infty]{s.res.} A$

1.) $(\forall N, (a,b) \cap \sigma(A_N) = \emptyset) \Rightarrow (a,b) \cap \sigma(A) = \emptyset$, where $a, b \in \mathbb{R}$

2.) $\lambda \in \sigma(A) \Rightarrow \exists \lambda_N \in \sigma(A_N) : \lambda_N \to \lambda$

3.) $a, b \notin \sigma_p(A) \Rightarrow E_{(a,b)}(A_N) \xrightarrow[N\to\infty]{s.} E_{(a,b)}(A)$, where $a, b \in \mathbb{R}$

4.) $f(A_N) \xrightarrow[N\to\infty]{s.res.} f(A)$, where f is bounded continuous function on $\mathbb{R}$

Proof. Proof done in [8]. □

Corollary 4.

1.) $\exists \lambda, \epsilon : (\lambda-\epsilon, \lambda+\epsilon) \cap \sigma_p(A) = \{\lambda\} \Rightarrow \exists \{|\lambda_N^{(1)}\rangle, |\lambda_N^{(2)}\rangle, ... |\lambda_N^{(\dim \mathbf{RanE}_\lambda)}\rangle\} :$

$$A_N|\lambda_N^{(j)}\rangle = \lambda_N^{(j)}|\lambda_N^{(j)}\rangle \wedge \lambda_N^{(j)} \to \lambda \wedge \sum_{j=1}^{\dim \mathbf{RanE}_\lambda} |\lambda_N^{(j)}\rangle\langle\lambda_N^{(j)}| \xrightarrow[N\to\infty]{s.} \mathbf{E}_\lambda,$$

where $\mathbf{E}_\lambda := \mathbf{E}_{(\lambda-\epsilon,\lambda+\epsilon)}$

2.) $\dim \mathbf{RanE}_\lambda = 1 \Rightarrow \exists \phi : \quad \mathrm{e}^{i\phi}\,|\lambda_N\rangle \xrightarrow[N\to\infty]{s.} |\lambda\rangle$

Theorem 10.

$$\{A_N\}_{N=0}^{\infty}, A \text{ self adjoint operators } \Rightarrow$$
$$(A_N \xrightarrow[N\to\infty]{s.res.} A) \Leftrightarrow (\mathrm{e}^{itA_N} \xrightarrow[N\to\infty]{s.res.} \mathrm{e}^{itA}, \forall t \in \mathbb{R})$$

Proof. Proof done in [8]. □

3.3 Convergence theorems

In [11] was proven following strong convergence theorem for discrete Fourier transformation:

Theorem 11 (strong convergence of discrete Fourier transformation)**.**

$$(\mathrm{red}_S(q^l))_C[F_N]_C \xrightarrow[N\to\infty]{s.} q^l F,$$

where F_N is defined in chapter 1 and F is corresponding operator on $L^2(\mathbb{R})$.

Pervious theorem was than used to prove following:

Theorem 12 (Schwinger strong resolvent convergence)**.**

$$h \in \{s \in C^\infty(\mathbb{R},\mathbb{R}) | \exists c_n : |s(x)| < \sum_{n=0}^{C} c_n x^n\}\}, \quad g \in C(\mathbb{R},\mathbb{R})$$

$$V_1 \in C(\mathbb{R},\mathbb{R}) \wedge \int_{\mathbb{R}} |V_1(x)|^2 dx \leq \infty, \quad V_3 \in C(\mathbb{R},\mathbb{R}) \wedge \exists C_2 : \quad C_2 \leq V_3(x)$$

$$V_4 \in C^\infty(\mathbb{R},\mathbb{R})$$

$$\Rightarrow (1.)\ h((\hat{P}_D))_C \xrightarrow[N\to\infty]{s.res.} h(\hat{P})$$

$$(2.)\ g((\hat{Q}_D))_C \xrightarrow[N\to\infty]{s.res.} g(\hat{Q})$$

$$(3.)\ \frac{(\hat{P}_D)_C^2}{2m} + V_1((\hat{Q}_D))_C + V_2((\hat{Q}_D))_C \xrightarrow[N\to\infty]{s.res.} \frac{\hat{P}^2}{2m} + V_1(\hat{Q}) + V_2(\hat{Q})$$

$$(4.)\ (\hat{P}_D)_C^2 \sigma_z - m\sigma_x + V_4((\hat{Q}_D))_C \xrightarrow[N\to\infty]{s.res.} \hat{P}^2\sigma_z - m\sigma_x + V_4(\hat{Q}),$$

where σ_x, σ_z are Pauli matrices.

Proof. (1.) Operator $h(\hat{P})$ is essentially self-adjoint on Schwartz space $\mathcal{S}(\mathbb{R})$ since it is unitary (fourier trf.) equivalent of operator which is. Now is only left to prove $h(\hat{P}_D)_C\psi \to h(\hat{p})\psi : \psi \in \mathcal{S}(\mathbb{R})$.

$$\left\| F_N^{-1} g((\hat{Q}_D)_C) F_N \psi - F^{-1} g(\hat{Q}) F\psi \right\| \leq$$

$$\leq C \left\| g((\hat{Q}_D)_C) F_N \psi - g(\hat{Q}) F\psi \right\| + \left\| (F_N^{-1} - F^{-1}) g(\hat{Q}) F\psi \right\| \xrightarrow[N\to\infty]{} 0$$

(2.) is trivial. (3.) Operator is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ [9]. Convergence follows trivially. (4.) Operator is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ [10]. Convergence follows trivially. □

Example 5 (0-spin relativistic hamiltonian)**.**

$$H(p,q) := \sqrt{p^2 + m^2}$$

$$((\Phi_W \circ \mathrm{app}_S)_D[H(p,q)] = \sqrt{(\hat{P}_D)_C^2 + m^2} \xrightarrow[N\to\infty]{s.res.} \sqrt{\hat{P}^2 + m^2}$$

Theorem 13 (Schwinger Weyl systems operator convergence)**.**

$$\mathrm{e}^{i\alpha[[\hat{P}]_D]_C}\, \mathrm{e}^{i\beta[[\hat{Q}]_D]_C} \xrightarrow[N\to\infty]{} \mathrm{e}^{i\alpha\hat{P}}\, \mathrm{e}^{i\beta\hat{Q}}\text{, where } \alpha, \beta \in \mathbb{R}$$

Proof. Using previous and uniform boundeness principle. □

Following proposal for theorem might offer, if it holds, convergence for large set of observables. *However proof is yet not known* .

$$((\Phi_W \circ \mathrm{app}_S \circ \mathrm{red}_S[f])_D)_C \xrightarrow[N\to\infty]{s.res.} \Phi_W[f]$$

$$f \in \{h(p) \otimes g(q) | h, g \in \{s \in C^\infty(\mathbb{R},\mathbb{R}) | \exists c_n : |s(x)| < \sum_{n=0}^{C} c_n x^n\}\}$$

Φ_W is Weyl transformation.

Following proposal for theorem might also offer some additional insight into character of convergence of discrete to continuum quantum kinematics.

Although proof is not available I expect that order of sumation in

$$\sum_{p=-\infty}^{+\infty} \sum_{l=0}^{+\infty} a(g)_{l,k} h_l\left(\sqrt{\frac{2\pi}{\tilde{N}}}(\rho + \tilde{N}p)\right)$$

can be exchanged and thus following theorem would hold:

Let $g \in C^\infty \wedge g\left(\sqrt{\frac{2\pi}{\tilde{N}}}(\rho + \tilde{N}p)\right) = g\left(\sqrt{\frac{2\pi}{\tilde{N}}}(\rho)\right)$

expecting that order of summation can be exchanged in

$$\sum_{p=-\infty}^{+\infty} \sum_{l=0}^{+\infty} a(g)_{l,k} h_l\left(\sqrt{\frac{2\pi}{\tilde{N}}}(\rho + \tilde{N}p)\right) \Rightarrow$$

(1.) $g(\hat{Q}_D)|\psi\rangle = \sum_{l=0}^{+\infty} a_l |h_l^D\rangle,$

where $a_l := \langle h_l | g(\hat{Q}) | \psi\rangle$

(2.) $g(\hat{P}_D)|\psi\rangle = \sum_{l=0}^{+\infty} a_l |h_l^D\rangle,$

where $a_l := \langle h_l | g(\hat{P}) | \psi\rangle$

Chapter 4

Time evolution

4.1 Ambiguity of time evolution definition

Having some observables we turn our attention to time evolution on discrete configuration space.

If we assume continuous time than evolution is trivially defined by Schödinger equation with discrete hamiltonian operator $i\partial_t\psi(t,\rho) = \hat{H}\psi(t,\rho)$. However if we assume discrete time, there is ambiguity in definition. There are three most intuitive definitions of time evolution:

1.) Sampling of continuous time evolution $\psi_\tau = \mathrm{e}^{i\,\mathrm{i_M}(\tau)H}\,\psi_o$, where $\tau \in \mathbb{N}$.

2.) Swinger approximation of Schödinger equation $P^0_D\Psi =_{\mathrm{mod}} H\Psi$,
where $P^0 := i\partial_t$ differential operator on $L^2(\mathbb{R}, dx)$ and $\Psi : M^D \times M^D \to \mathbb{C}$

3.) Feynman's short time evolution operator based on Trotter's formula:
$U(\tau) := \left[\mathrm{e}^{i\epsilon H_0(\hat{p})}\,\mathrm{e}^{i\epsilon V(\hat{q})}\right]^\tau$, where $\epsilon > 0$ is time slicing parameter
and $H = H_0(p) + V(x)$ is Hamiltonian.

Configuration imbedding is also used for time imbedding because I expect that some discrete analogue of Poincaré group may be instroduced. This idea however is not covered in this paper. *Interesting feature of Swinger approximation of derivative is that it has "non-local character"* i.e. value of swinger derivative at every point *depends on value of wavefunction at all points*, althought with weight falling quickly with rising distance.

Definition 17 (operator modulo)**.**

$$\hat{A} =_{\mathrm{mod}} \hat{B} \Leftrightarrow \sigma(\hat{A}), \sigma(\hat{B}) \subset \mathrm{i_{T^*}}(\mathbb{Z}) \wedge \mathrm{c_{T^*}}(\hat{A}) = \mathrm{c_{T^*}}(\hat{B})$$

Definition 18 (evolution operator)**.**

1.) sampling type: $U(\tau) := \mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\tau)H}$

2.) swinger type: $\tau \to U(\tau)$ is operator solution of $P^0_D U =_{\mathrm{mod}} HU$,

3.) feynman type: $U(\tau) := \left[\mathrm{e}^{i\epsilon H_0(\hat{p})}\,\mathrm{e}^{i\epsilon V(\hat{q})}\right]^{\tau}$

where $\tau \in \mathbb{Z}$

Next theorem can be put simply: Both definitions give same result for Hamiltonias with spectrum "on lattice".

Theorem 14.

1.) 1.1) $P^0_D U =_{\mathrm{mod}} HU$ has solution $\Leftrightarrow \sigma(H) \subset \mathrm{i}_{\mathrm{T}^*}(\mathbb{Z})$

2.) $\sigma(H) \subset \mathrm{i}_{\mathrm{T}^*}(M^D) \Rightarrow$ 2.1) $\mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\tau)H}$ is **unique** operator solution of $P^0_D U =_{\mathrm{mod}} HU$

2.2) $U(\tau)$ are unitary operators

2.3) evolution is periodic i.e. $U(\tau \bmod \tilde{N}) = U(\tau)$

Proof.

1.1) Due to spectra of P^0_D and following proof.

2.1) $\sigma(H) \subset \mathrm{i}_{\mathrm{T}^*}(M^D) \Rightarrow P^0_D \sum_{\tau \in M^D} \mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\tau)H} \otimes |\tau\rangle\langle\tau|$

$$= \sum_{n:\mathrm{i}_{\mathrm{T}^*}(n)\in\sigma(H)} |n\rangle\langle n| \otimes \left(P^0_D|\tau\rangle\langle\tau|\,\mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\tau)\,\mathrm{i}_{\mathrm{T}^*}(n)}\right) =_{\mathrm{mod}} H\,\mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\tau)H},$$

where hamiltonian spectrum decomposition have been used.

For uniqueness let us have two solution U, V with property $U(0), V(0) = 1$.

$\langle n_1|P^0_D(U-V)|n_2\rangle = P^0_D\langle n_1|(U-V)|n_2\rangle = \mathrm{i}_{\mathrm{T}^*}(n_1)\langle n_1|(U-V)|n_2\rangle$

Using spectral decomposition of momentum

$P^0_D = \sum_l l|l\rangle\langle l|$, $\langle l|\tau\rangle = \eta_N\,\mathrm{e}^{i\,\mathrm{i}_{\mathrm{M}}(\tau)\,\mathrm{i}_{\mathrm{T}^*}(l)}$,

we rewrite above vector equation into sum of equations:

$\mathrm{i}_{\mathrm{T}^*}(l-n_1)\langle l|\langle n_1|(U-V)|n_2\rangle\rangle = 0$.

So only free parameter is $\langle l = n_1|\langle n_1|(U-V)|n_2\rangle\rangle$

which is set by initial condition.

(2.2,3)are evident from definition and spectral decomposition.

□

Pervious theorems can be put in following way: *If discrete analogue of Schödinger equation is to have solution, spectrum of Hamiltonian has to lie on same lattice as momentum, thus we can expect some discrete analogue of Poincaré group also on energy-momentum space.*

Easiest way of obtaining hamiltonian with spectrum on momentum lattice is using floor, expecting that in $N \to \infty$ it converges in strong resolvent sence to former hamiltonian limit.

Theorem 15.

$$\tilde{H} := \mathrm{c}_{\mathrm{T}^*}(\hat{H}) \Rightarrow \sigma(\tilde{H}) \subset \mathrm{i}_{\mathrm{T}^*}(\mathbb{Z})$$

If there is some analogue of Poincaré symmetry we may also introduce analogues of Klein-Gordon, Dirac equation which are strainghtforward. However spectrum would not lie on momentum lattice since $E = \pm\sqrt{p^2 + m^2}$ containes square root. However for pairs $(p/xi_N, m/\xi_N)$ that are *Pythagorean* i.e. $(E, p, m)/\xi_N \in \mathbb{Z}^3 \wedge E^2 = p^2 + m^2$. would spectra lie on momentum lattice. For large N density of pythagorean pairs $(p, m)/\xi_N$ for small E/ξ_N grows significantly thus I expect convergence to continuum case.

4.2 Feynman's path integral

As defined above evolution operator of Feynmann type is multiplication of two unitary operators generated by self-adjoint operators which in sum give total hamiltonian of system $H = H_0(p) + V(x)$. Two separate parts of Hamiltonian depend always only on one of the parameters q or p. This is usually used as an approximation based on Trotter's formula ($\mathrm{e}^{A+B} = (\mathrm{e}^{\frac{A}{N}}\,\mathrm{e}^{\frac{B}{N}})^N$) when Feynman path integral is being derived. Thus we can define approximation of our evolution in this manner expecting convergence as $\epsilon \to 0$ and similarities when deriving propagator.

Corollary 5.

$$\langle \rho_2 | \Big(\hat{U}(\tau) | \rho_1 \rangle \Big) = \sum_{\rho_2, \rho_3 \ldots \rho_{\tau-1} \in (M^D)^{\tau-2}} \; \sum_{k_2, k_3 \ldots k_{\tau-1} \in (T^* M^D)^{\tau-2}} \frac{1}{(\sqrt{\tilde{N}})^{\tau-2}}$$
$$\exp\Big(i\epsilon \sum_{l=0}^{\tau-1} - \mathrm{i}_{\mathrm{T}^*}(k_l)\,\mathrm{i}_{\mathrm{M}}(\rho_{l+1} - \rho_l) + H_0(\mathrm{i}_{\mathrm{T}^*}(k_l)) + V(\rho_l) \Big)$$

For special hamiltonian it is possible to carry out summation over momentum space as in continuous case we obtain lagrangian.

Theorem 16.

$$\langle \rho_2 | \left(\mathrm{e}^{i\epsilon \frac{\hat{p}(p - \mathrm{i}_{\mathrm{T}^*}(1))}{2m}} \mathrm{e}^{i\epsilon V(\hat{q})} |\rho_1\rangle \right) = \frac{\exp\left(i\frac{\epsilon}{2} \left(\frac{\mathrm{i}_{\mathrm{M}}(\rho_2 - \rho_1 \bmod \tilde{N}) + \frac{1}{2}\mathrm{i}_{\mathrm{M}}(1)}{\epsilon} \right)^2 + \epsilon V(\mathrm{i}_{\mathrm{M}}(\rho_1)) \right)}{\sqrt{\tilde{N}}}$$

where $\epsilon = \dfrac{a^2}{m}, m > 0$.

Proof. Done in [1] using formula derived in [2, 3, 4]. □

This however is problematic since time-slicing $\epsilon = \frac{a^2}{m}$ remains constant when $N \to \infty$ thus proper time evolution is not obtained in continuum limit.

Other possibility is to add some additional N-dependent time slicing. As mentioned above simplest way is to use space imbedding also for time i.e. $\epsilon := \mathrm{i}_{\mathrm{M}}(1)$ as in following example:

Example 6.

$$H_0 := \sqrt{p^2 + \mathrm{i}_{\mathrm{T}^*}(\mathrm{c}_{\mathrm{T}^*}(m))^2}, \epsilon := \mathrm{i}_{\mathrm{M}}(1) \Rightarrow H_0\epsilon = \sum_{k=-N}^{N} \frac{2\pi}{\tilde{N}} \sqrt{k^2 + \mathrm{c}_{\mathrm{T}^*}(m)^2}$$

4.2.1 Feynman's checkerboard

Feynman proposed problem [13] closely connected to Feynmnan's path integral on lattice futher developed in [16] and whose simplified solution [14] I will include into this paper.

Suppose particle traveling by jumping at speed of light on space-time $\mathbb{Z}^2$ lattice. At every step particle jump left or right in space and one step forward in time. Let use define propagator function on this lattice in special way so that it's continuum limit will be Dirac particle propagator in 1+1 space-time.

Definition 19.

$$K_{\alpha,\beta}(x_f, t_f; x_0, t_0) := \frac{1}{2\epsilon} \sum_{C \in \mathbb{N}_0} \Phi_{\alpha,\beta}(C)(i\epsilon m)^C,$$

where $\Phi_{\alpha,\beta}(C)$ is number of histories with start-ending points $(x_0, t_0), (x_f, t_f)$
with C changes in path direction and starting resp. ending
with right/left direction $\alpha \in \{+, -\}$ resp. $\beta \in \{+, -\}$
$\epsilon = \dfrac{t_f - t_0}{N}$ is space-time-slicing parameter ($c = 1$)
$N \in \mathbb{N}$ and $m > 0$ is mass.

Theorem 17.

$$\Phi_{-,+}(C) = \Phi_{+,-}(C) = \begin{cases} \binom{R-1}{\frac{C-1}{2}}\binom{L-1}{\frac{C-1}{2}} \sim_{N\to\infty} \frac{(RL)^{\frac{1}{2}(C-1)}}{[(\frac{1}{2}(C-1))!]^2} & \text{for } C \text{ odd} \\ 0 & \text{for } C \text{ even.} \end{cases}$$

$$\Phi_{+,+}(C) = \begin{cases} 0 & \text{for } C \text{ odd,} \\ \binom{R-2}{\frac{C}{2}-1}\binom{L-1}{\frac{C}{2}} \sim_{N\to\infty} L\frac{(RL)^{\frac{C}{2}}}{(\frac{C}{2}-1)!(\frac{C}{2})!} & \text{for } C \text{ even.} \end{cases}$$

$$\Phi_{-,-}(C) = \begin{cases} 0 & \text{for } C \text{ odd,} \\ \binom{R-1}{\frac{C}{2}}\binom{L-2}{\frac{C}{2}-1} \sim_{N\to\infty} R\frac{(RL)^{\frac{C}{2}}}{(\frac{C}{2}-1)!(\frac{C}{2})!} & \text{for } C \text{ even.} \end{cases}$$

where R, L are numbers of steps right, left. using approximation

$$\binom{n}{k} \sim_{n\to\infty} \frac{n^k}{k!}$$

Proof. First part is simple combinatorics where $C \ll N, R, L$. □

Theorem 18.

$$K_{+,-}(x_f, t_f; x_0, t_0) = K_{-,+}(x_f, t_f; x_0, t_0) \sim_{N\to\infty} \frac{im}{2}\sum_{k=0}^{+\infty}(-1)^k\frac{(z/2)^{2k}}{(k!)^2} = \frac{im}{2}J_0(z),$$

$$K_{+,+}(x_f, t_f; x_0, t_0) \sim_{N\to\infty} im\frac{t+x}{z}\sum_{k=0}^{+\infty}(-1)^k\frac{(z/2)^{2k+1}}{(k!)(k+1)!} = im\frac{t+x}{z}J_1(z),$$

$$K_{-,-}(x_f, t_f; x_0, t_0) \sim_{N\to\infty} im\frac{t-x}{z}\sum_{k=0}^{+\infty}(-1)^k\frac{(z/2)^{2k+1}}{(k!)(k+1)!} = im\frac{t-x}{z}J_1(z),$$

where $z := m\sqrt{(t_f - t_0)^2 - (x_f - x_0)^2}$ and J_0, J_1 are Bessel functions.

Proof. Using $RL = \frac{1}{4}(N + R - L)(N - R - L) = z/2$, oddness or eveness of R and definitions. □

Theorem 19.

$$K(x_f, t_f; x_0, t_0) = \begin{pmatrix} K_{+,+} & K_{+,-} \\ K_{-,+} & K_{-,-} \end{pmatrix} \sim_{N\to\infty} \begin{pmatrix} im\frac{t+x}{z}J_1(z) & \frac{im}{2}J_0(z) \\ \frac{im}{2}J_0(z) & im\frac{t-x}{z}J_1(z) \end{pmatrix},$$

where right side is propagator of dirac particle in 1+1 dimension

Proof. Propagator of Dirac particle in 1+1 dimension can be found easily using properties of Pauli matrices, trigonometrical identities and integral definition of Bessel functions. Note that $H = \hat{p}\sigma_z - m\sigma_x$. □

Concluding remarks

Quantum mechanics can be effectively approximated on finite-dimensional Hilbert spaces, which is sometimes used for deriving propagators in Feynmann's path integral. Further research could be directed towards studying detailes of Feynman's checkerboard problem, proving complementary theorems and applications on particular quantum systems. It might also be interesting investigating approximation maping first $\tilde{N}$ discrete hermite functions to corresponding hermite fuctions and as reduction to map all hermite function to corresponding discrete hermite functions with scalar product defined such that first $\tilde{N}$ discrete hermite functions would be orthonormal basis.

Bibliography

[1] F. J. Tolar and G. Chadzitaskos 2009 *Feynman's path integral and mutually unbiased bases* J.Phys. A: Math. Theor. 42 245306

[2] Siegel C L 1960 *Über das quadratische Reziprozitä tsgesetz in algebraichen Zahlkörpem* Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1-16

[3] Berndt B C and Evans R J 1981 *The determination of Gauss sums* Bull. Am. Math. Soc. (New Series **5** 107-29 (formula (2.8))

[4] Berndt B C, Evans R J and Williams K S 1998 *Gauss and Jacobi Sums (Canadian Mathematical Society, Series of Monographs and Advanced Texts* vol 21) (New York: Wiley-Interscience) (Theorem 1.2.2)

[5] Mackey, G. W. 1970 *Induced representations and quantum mechanics.* Benjamin, New York

[6] Šťovicek P and Tolar J 1984 *Quantum Mechanics in a discrete space-time* Rep. Math. Phys. **20** 157-70

[7] Reed M, and Simon B 1980 *Methods of Modern Mathematical Physics 2nd Edition vol 1* Academic Press **8** 292 (theorem 25)

[8] Reed M, and Simon B 1980 *Methods of Modern Mathematical Physics 2nd Edition vol 1* Academic Press **8** 287,290 (theorem 21,24)

[9] Reed M, and Simon B 1975 *Methods of Modern Mathematical Physics vol 2* Academic Press **10** 185 (theorem 29)

[10] Chernoff P R 1977 *Schrödinger and Dirac operators with singular potentials and hyperbolic equations* Pacific J. Math. **72** 361-382

[11] Digernes T, Varadarajan V S and Varadhan S R S 1994 Finite approximations to quantum systems *Rev. Math. Phys.* **6** 621-48 (proposition 2.2.1)

[12] Mehta M L 1986 *Eigenvalues and eigenvectors of the finite Fourier transform* J. Math. Phys. **28** (4) 781-785

[13] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* McGraw-Hill, Ney York 34-6 (problem 2-6)

[14] Plavchan P *Feynman's Checkerboard, the Dirac Equation and Spin* University of California, Los Angeles

[15] Fuksa J 2009 *Feynman path integral and its approximations* Czech Technical University, Prague

[16] Jacobson T 1984 *J. Phys. A: Math. Gen.* **17** 2433-2451