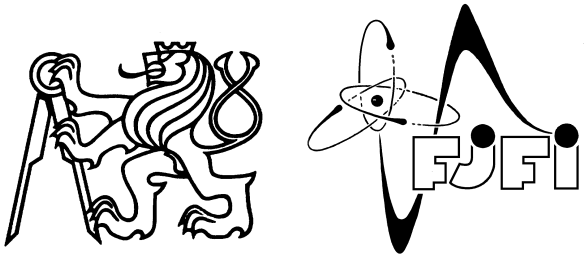
CZECH TECHNICAL UNIVERSITY IN PRAGUE

FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING

DEPARTMENT OF PHYSICS



Trace dynamics and quantum probability

DIPLOMA THESIS

2011 Kamil Červenka

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Acknowledgements

I would like to express my gratitude to Dr. Petr Jizba for his suggestions and support.

Název práce: **Trace dynamics a kvantová pravděpodobnost**

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Abstrakt: Trace dynamics je označení pro zobecnění klasické Hamiltonovské mechaniky pro nekomutující dynamické proměnné. Takto zobecněná teorie byla nedávno navržena jako možné řešení problému odvození kvantové mechaniky z prvních princip, a tím i k nalezení její smysluplné interpretace. Relativistická kvantová teorie pole se v rámci této teorie objevuje jako efektivní aproximace ke statistické mechanice jisté třídy maticových model v režimu nízkých energií. Tato práce se zabývá souvislostmi mezi trace dynamics s kvantovou pravděpodobností, a logickou konzistencí argumentu vedoucího k odvození kvantové teorie.

Klíčová slova: trace dynamics, kvantová pravděpodobnost, maticové modely, interpretace kvantové mechaniky

Title: **Trace dynamics and quantum probability**

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Abstract: Trace dynamics or generalized quantum dynamics is a recently proposed approach to the problem of derivation of quantum theory from first principles, and to find its satisfactory interpretation. It is a generalization of classical Hamiltonian mechanics to use noncommutative dynamical variables. relativistic quantum field theory is expected to emerge as an effective low energy aproximation to statistical mechanics of a particular class of classical matrix models. We investigate the connection of trace dynamics with quantum probability and the validity of the argument for emergence of quantum theory. We point out some drawbacks of the argument.

Keywords: trace dynamics, quantum probability, matrix models, interpretation of quantum mechanics

Contents

Introduction

Quantum theory, as a mathematical device to provide statistical predictions about the results of experiments, is an extraordinary successful theory. It describes a wide range of physical phenomena, without any significant bounds of the accuracy of its predictions. But it is still an open problem in the foundations of physics to find its satisfactory physical interpretation. Quantum mechanics is inherently indeterministic, but in such a way that the predicted statistics cannot be subsumed in the framework of classical probability. It is then necessary to generalize the concept of probability from commutative to noncommutative, and we get the the quantum probability.

The measurement problem suggests a necessity of division between microscopic and macroscopic objects, but without any clear boundary between the two worlds. Also the phenomena of the collapse of the wave packet, with its "spooky action at a distance", is apparently incompatible with special relativity. These, and other problems, have motivated the suspicion that quantum theory is not the most fundamental theory of nature. There have been many attempts to find such a fundamental theory, which would explain all the paradoxes of quantum theory, reproduce all its predictions, and hopefully provide some new ones. But despite a great effort, there is still no satisfactory alternative to quantum theory.

The aim of this work was to investigate the validity of another attempt in this direction, to derive quantum theory from first principles. It is called trace dynamics, and it was proposed by Stephen Adler and his collaborators in a book and a series of papers including [1] through [4]. The relativistic quantum field theory, and then also quantum mechanics as its nonrelativistic limit, is claimed to emerge from the statistical mechanics of a particular class of matrix models. If it turned to be correct, it would be of interest to analyze the issues concerning indeterminism and non-locality, and to draw some conclusions about the physical meaning and purpose of quantum probability. But unfortunately, trace dynamics is appears as just another unsuccessful attempt to rederive quantum theory. In this work, the structure of trace dynamics is analyzed, and the drawbacks of the argument for emergence of quantum theory is pointed out.

Chapter 1

Quantum probability

There are two notions of the concept of probability, which nevertheless eventually lead to mathematically equivalent theories. The *objective* view regards a probability of an event as an objective property of that event, a measure of potentiality of occurrence of that specific event in a random experiment. This probability can be at least in principle measured in all aspects by observing relative frequencies in a sufficient long run of independent random experiments. The aim is to find a probability distribution good enough, such that the predicted probabilities match the observed relative frequencies.

On the other hand, the *subjective* view regards probabilities merely as an information the observer has about the object prior to observation. It is an expression of our ignorance of specific details of the experiment, which, if they were known exactly, would uniquely determine the measurement outcome. The purpose of the probability theory is to allow us to form at least some conclusions prior to the experiment, in case when only limited knowledge is available, and definite conclusions about the result are not possible. Unlike the objective view the probability distribution is strongly dependent on the actual knowledge of the observer about the object. The observer should use all information available to obtain the best estimate of what is going to happen in the experiment. The probability distribution, not being an objective property of the events, is not expected to be verifiable in all aspects. A test of a good distribution is whether it correctly represents our prior knowledge of the object.

It is also possible to contemplate a concept of probability that combines both views. The occurrence of a specific event is then associated with an "irreducible" objective probability distribution forming the core of all the uncertainty. However, this distribution might not not be available due to incomplete knowledge of the object, and thereby the subjective probability has to be employed. But the relative frequencies of the objective view can always be reinterpreted in terms of the subjective probability by regarding the measurement outcome itself as the missing knowledge. In this way the subjective view can be perceived as a broader concept. Even quantum mechanics can be included into the subjective probability framework by means of, in the classification of [21], a hidden-variables theory of the *first* kind.

1.1 General structure of probability theory

In general, a probability theory involves the dual concepts of algebra $\mathcal{A}$ and state ϕ. The algebra contains elements that represent the objects which can be measured, and are therefore called (random) observables. The state determines probabilities of various events, that occur as a result of a random experiment, e.g. an observable taking a specific value. For the introduction of state

it is further required, that the algebra $\mathcal{A}$ contains a subset $\mathcal{A}_+$, called the positive cone of $\mathcal{A}$. An important algebras relevant for probability theory are the unital *-algebras. A unital *-algebra is a complex vector space with the properties

1. A product of any two elements $A, B \in \mathcal{A}$ is defined, which is associative and distributive. There is a neutral element with respect to multiplication, the identity element I.

2. For any $A \in \mathcal{A}$, there is a conjugated element $A^* \in \mathcal{A}$. The mapping $A \mapsto A^*$ is complex antilinear.

3. There is a norm, which has the additional property $\|A^*A\| = \|A\|^2$.

If the space is complete in the topology given by this norm, then it is called the C^*-algebra. The positive cone of $\mathcal{A}$ is the set $\mathcal{A}_+ = \{A \in \mathcal{A} \mid A = B^*B,\ B \in \mathcal{A}\}$. The state is then any functional on the algebra $\mathcal{A}$, $\phi : \mathcal{A} \mapsto \mathbb{C}$, with the properties

1. ϕ is complex linear,

2. ϕ is positive, i.e. $\phi(A) \geq 0$ for any $A \in \mathcal{A}_+$,

3. ϕ is normalized, i.e. $\phi(I) = 1$.

Usually the set of observables is identified with the subspace of hermitean elements of $\mathcal{A}$. An element $\mathcal{O} \in \mathcal{A}$ is hermitean, if it satisfies $\mathcal{O}^* = \mathcal{O}$. For an observable $\mathcal{O} \in \mathcal{A}$, $\phi(\mathcal{O})$ is interpreted as the average value of $\mathcal{O}$ in the state ϕ. Since the algebra $\mathcal{A}$ contains all powers of $\mathcal{O}$, the state determines also all moments of $\mathcal{O}$, $\phi(\mathcal{O}^k)$, the characteristic function of the observable $\mathcal{O}$, $\phi(\exp(i\mathcal{O}))$, and the probability distribution on the space of this values. Hence the state ϕ, that assigns expectation value to all elements of $\mathcal{A}$, is sufficient to determine probability distribution and statistical properties of all observables.

The algebra $\mathcal{A}$ further contains a distinguished subset $\mathcal{A}_0$ of observables, which represent set of all possible events associated with random experiments. It consists of those elements of $\mathcal{A}$ which are positive, hermitean, and satisfy $\mathcal{O}^2 = O$. For $\mathcal{O} \in \mathcal{A}_0$, the number $\phi(\mathcal{O})$ is interpreted as the probability of occurrence of the event represented by $\mathcal{O}$.

The examples of *-algebras are the set $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space or the algebra $C(X)$ of all continuous bounded functions on a compact Hausdorff space[1]. The former is in general a non-commutative algebra, whereas the latter is commutative, and it can be regarded as a special case of the former with the set $\mathcal{B}(\mathcal{H})$ replaced by its suitable commutative subset. It is even true, that any commutative C^*-algebra is isomorphic to $C(X)$, for some compact Hausdorff space X [8], [10], [26]. For $\mathcal{A} = \mathcal{B}(\mathcal{H})$ the subset $\mathcal{A}_0$ is formed by the set of all projections on $\mathcal{H}$, whereas for $\mathcal{A} = C(X)$ it is a set of characteristic functions of subsets of X.

Concrete and abstract algebras

There is an important connection between abstract *-algebras and the concrete algebras of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. The algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space is a *-algebra, and in turn an abstract *-algebra is associated with an algebra of bounded operators on a Hilbert space $\mathcal{H}$. The Hilbert space $\mathcal{H}$ is given by the GNS construction, a theorem of Gelfand, Naimark and Segal, [10],[8],[26]. It shows, that there is no loss of generality to work with concrete algebras of operators on a Hilbert space instead of the abstract ones.

[1] with the supremum norm, $\|f\| = \sup_{x \in X} |f(x)|$, $C(X)$ is a C^*-algebra

Given an abstract *-algebra $\mathcal{A}$ and a state ϕ on $\mathcal{A}$, there is a Hilbert space $\mathcal{H}_\phi$ with a distinguished unit vector Ψ_0, such that there is a *-algebra homomorphism[2] $\pi_\phi : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}_\phi) : A \mapsto \pi_\phi(A)$, and the state ϕ is determined by the scalar product on $\mathcal{H}$ as

$$\phi(A) = \langle \Psi_0, \pi_\phi(A)\Psi_0 \rangle_{\mathcal{H}_\phi}. \tag{1.1}$$

The homomorphism assigns a bounded operator $\pi_\phi(\mathcal{O})$ on the Hilbert space $\mathcal{H}_\phi$ to any element $\mathcal{O}$ of the C^*-algebra $\mathcal{A}$. The elements of the C^*-algebra is then said to be represented by bounded linear operators on $\mathcal{H}_\phi$.

The Hilbert space is formed starting from the vector space $\mathcal{A}$, with $\Psi_0 = I$. The scalar product on A is defined using the state ϕ as

$$\langle A, B \rangle_{\mathcal{A}} = \phi(A^*B). \tag{1.2}$$

By the properties of the state, the scalar product is a positive sesquilinear form on $\mathcal{A}$, and defines a norm on $\mathcal{A}$. The vector Ψ_0 is of unit length with respect to this norm,

$$\|\Psi_0\|^2 = \langle \Psi_0, \Psi_0 \rangle = \phi(I) = 1. \tag{1.3}$$

If the scalar product is not positive definite, the subset of zero norm form an ideal[3] $\mathcal{I}$ of $\mathcal{A}$, and we can replace $\mathcal{A}$ by its quotient space modulo this ideal. The Hilbert space $\mathcal{H}_\phi$ is then obtained by completing of $\mathcal{A}$ (or the quotient space) in the norm $\|.\|$. The representation of an algebra element $A \in \mathcal{A}$ as the operator $\pi_\phi(A)$ on $\mathcal{H}_\phi$ is defined by

$$\pi_\phi(A)B = AB, \qquad \text{for any } \mathcal{B} \in \mathcal{H}_\phi. \tag{1.4}$$

As seen from the construction, $\pi_\phi(\mathcal{A})\Psi_0$ is a dense set in $\mathcal{H}_\phi$, and hence all observables can be extended by continuity to the entire Hilbert space. The representation of $\mathcal{A}$ is then said cyclic with Ψ_0 the cyclic vector. Moreover, the dense set $\pi_\phi(\mathcal{A})\Psi_0$ is invariant with respect to the action of $\pi_\phi(A)$ for any $A \in \mathcal{A}$.

Operator algebras

The algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, with $*$ as the operator conjugation, is a *-algebra. A *-subalgebra $\mathbb{C}$ of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra, if it is closed with respect to the uniform operator topology (given by the operator norm). If it is closed also with respect to the strong operator topology[4], then it is called the von Neumann algebra. In particular, the algebra $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra. The notions of von Neumann algebra and C^*-algebra coincide on a finite-dimensional Hilbert space, since the strong and the uniform operator topology are the same. But in infinite dimensions the strong operator topology is weaker, and the closure is then larger

[2]Given *-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, a *-algebra homomorphism π is a mapping $\mathcal{A}_1 \to \mathcal{A}_2$ satisfying

1. $\pi(\lambda A + B) = \lambda\pi(A) + \pi(B)$ (linearity),
2. $\pi(AB) = \pi_\phi(A)\pi(B)$,
3. $\pi(A^*) = (\pi(A))^*$.

[3]This follows from the Schwarz and the triangle inequalities. For A, B in $\mathcal{I}$, by the triangle inequality $\|A+B\| \leq \|A\| + \|B\| = 0$, hence $A + B \in \mathcal{I}$. For $A \in \mathcal{I}$ and $B \in \mathcal{A}$, by the Schwarz inequality $\|AB\| \leq \|A\| \cdot \|B\| = 0$, and $AB \in \mathcal{I}$.

[4]A sequence of operators $\{A_n\}$ converges in the strong operator topology, if for any $\Psi \in \mathcal{H}$ the sequence of vectors $\{A_n\Psi\}$ converges in the Hilbert space norm.

(there are more limit points). Therefore any von Neumann algebra is also a C^* algebra, but the converse is not in general true. An important subset of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ (but not subalgebra) is the set $\mathcal{P}$ of all projection operators on $\mathcal{H}$. This is the set $\mathcal{A}_0$ mentioned above, that represents random events in a probability theory. A normal state on $\mathcal{B}(\mathcal{H})$ is a state with the additional continuity property

$$\phi\left(\sum_{n\in\mathbb{N}} P_n\right) = \sum_{n\in\mathbb{N}} \phi(P_n), \tag{1.5}$$

where $\{P_n\}$ is an arbitrary sequence of mutually orthogonal projection operators from $\mathcal{P}$. By a theorem of Gleason[5] ([21],[26],[10]), every normal state ϕ is given by

$$\phi(A) = \mathrm{Tr}(\rho A), \qquad \text{for } A \in \mathcal{B}(\mathcal{H}), \tag{1.6}$$

where $\rho \in \mathcal{B}(\mathcal{H})$ some hermitean operator with $\mathrm{Tr}\rho = 1$, determined uniquely by the state ϕ. A special case of a normal state is the vector state, where ρ is a one-dimensional projector P_ψ given by a vector $\psi \in \mathcal{H}$,

$$\phi(A) = \mathrm{Tr}(P_\psi A) = \langle \psi, A\psi \rangle, \qquad \text{for } A \in \mathcal{B}(\mathcal{H}). \tag{1.7}$$

Since the state[6] on the algebra (as defined above) can be equivalently given by density operator ρ, or in a special case by a Hilbert space vector ψ, the term 'state' can be used also when referring to ρ or ψ.

1.2 Classical probability

A classical probability theory is given by a triple (Ω, Σ, μ) (the probability space), where Ω is the space of individual outcomes (the sample space), Σ is a σ-algebra of μ-measurable subsets, and μ is a probability measure $\mu : \Sigma \mapsto \mathbb{R}$ satisfying the probability axioms of Kolmogorov,

(c1) $\mu(A) \geq 0$, for all $A \in \Sigma$,

(c2) $\mu(\Omega) = 1$,

(c3) $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$, for any sequence $\{A_n\}$ of mutually disjoint sets from Σ.

The system of μ-measurable sets represent the set random events, and form a Boolean algebra with the operation of set-theoretic union $\cup$, intersection $\cap$, and complement $^{\complement}$. The algebra is associative and commutative. Impossible event is represented by the empty set, the sure event by Ω, two disjoint events (those that can not occur simultaneously) by two disjoint subsets, and if $A \subset B$ then the event B occurs whenever occurs A.

A random variable (observable) is any μ-measurable function $f : \Omega \mapsto \mathbb{R}$, it represents an observable quantity in a random experiment. A distinguished subset of random variables is the set $L^\infty(\Omega, \Sigma)$, which is the von Neumann algebra of essentially bounded measurable functions on Ω. Any random variable $g \in L^\infty(\Omega, \Sigma)$ define a linear multiplication operator L_g on the Hilbert space $\mathcal{H} = L^2(\Omega, \Sigma, \mu)$ [7], by $L_g f = gf$ for any $f \in L^2(\Omega, \Sigma, \mu)$. This subset contains the subset $\mathcal{P}$ of characteristic functions of the sets in Σ,

$$\chi_A(x) = 1 \quad \text{for } x \in A, \qquad \text{and} \qquad \chi_A(x) = 0 \quad \text{for } x \notin A, \tag{1.8}$$

[5] Actually, the theorem asserts that the most general normal state on $\mathcal{P}$ with the property (1.5), is given by a density operator.

[6] We shall consider normal states only.

[7] By the Schwarz inequality, $\mathcal{H} = L^2(\Omega, \Sigma, \mu) \subset L^1(\Omega, \Sigma, \mu)$.

which is the set of orthogonal projections in the commutative Hilbert space $\mathcal{H}$. The projections are in one-to-one correspondence with then subsets in Σ, and represent the random events. A probability of occurrence of an event given by $A \in \Sigma$ is given by the state as $\phi(\chi_A)$. The state ϕ is defined for any μ-integrable random variable f from $L^1(\Omega, \Sigma, \mu)$ by

$$\phi(f) = \int_\Omega f(x) d\mu. \tag{1.9}$$

Due to the σ-additivity of the measure, the state ϕ satisfy the condition (1.5), and it is a normal state on $\mathcal{H}$. Then the classical probability theory can be reformulated in terms of the Hilbert space $\mathcal{H} = L^2(\Omega, \Sigma, \mu)$ with the set of bounded operators from $\mathcal{B}(\mathcal{H}) = L^\infty(\Omega, \Sigma)$ acting on it.

If μ is absolutely continuous measure with respect to the Lebesgue measure ν on Ω, then by the Radon-Nikodym theorem [8] there is a probability density ρ, such that $d\mu = \rho d\nu$. Then ρ is the density operator on the Hilbert space $\mathcal{H} = L^2(\Omega, \Sigma, \nu)$, since it is hermitean and

$$\mathrm{Tr}\rho = \int_\Omega \rho d\nu = \int_\Omega d\mu = \mu(\Omega) = 1.$$

Therefore classical probability is in this way subsumed into the more general framework of non-commutative probability theory.

1.3 Quantum probability

Quantum probability is the general probability theory given by the triple $(\mathcal{A}, \mathcal{P}, \phi)$, with $\mathcal{A}$ the full noncommutative *-algebra (which is now represented by a *-algebra of operators on a Hilbert space), $\mathcal{P}$ the subset of projections, and ϕ a normal state on $\mathcal{A}$. The set of projection operators $\mathcal{P}$, which is a subset of the subalgebra of bounded operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$, has an important role in quantum probability, since it represents the set of random events (the quantum propositions). Moreover, the projections determine the operator structure of all other observables, and the structure of $\mathcal{P}$ determines the properties of $\mathcal{B}(\mathcal{H})$ and of the full algebra $\mathcal{A}$.

The set $\mathcal{P}$ is a counterpart of the set of measurable subsets Σ of classical probability. For $P_1, P_2 \in \mathcal{P}$, the set-theoretic intersection becomes replaced by $P_1 \wedge P_2$ (the projection onto the intersection of ranges of P_1 and P_2), the set union becomes $P_1 \vee P_2$ (the projection onto the linear span of the union of ranges of P_1 and P_2, or equivalently onto the sum of the subspaces represented by P_1 and P_2), and the set complement is now an orthogonal complement. If P_1 and P_2 are compatible, then $P_1 \wedge P_2 = P_1 P_2$, and if they are orthogonal, $P_1 \vee P_2 = P_1 + P_2$. The impossible event is given be the zero projection, and the sure event by the identity I. If $P_1 \leq P_2$ (i.e. the range of P_1 is contained in those of P_2) then the occurrence of P_1 implies P_2. If $P_1 \wedge P_2 = 0$, then the events cannot occur simultaneously. With the operations $\wedge$ and $\vee$, the algebra $(\mathcal{P}, \wedge, \vee)$ is no longer a boolean algebra of classical random events, but it forms a lattice of quantum propositions. It is associative and commutative, but in general not distributive,

$$P_1 \vee (P_2 \wedge P_3) \neq (P_1 \wedge P_2) \vee (P_2 \wedge P_3).$$

The state ϕ assigns each proposition $P \in \mathcal{P}$ its probability $\phi(P)$, and it has the properties

(q1) $0 \leq \phi(P) \leq 1$ for any $P \in \mathcal{P}$,

(q2) $\phi(I) = 1$,

(q3) $\mu\left(\bigvee_{n\in\mathbb{N}} P_n\right) = \sum_{n\in\mathbb{N}} \mu(P_n)$, for any sequence $\{P_n\} \subset \mathcal{P}$ of mutually orthogonal projections.

These properties correspond to the probability axioms (c1)-(c3) of Kolmogorov, with the exception that (q3) requires $\{P_n\}$ to be mutually orthogonal. The orthogonality of the projections P_1 and P_2 is a stronger requirement than the disjointness of the corresponding events, which is given by the condition $P_1 \wedge P_2 = 0$, unless the projections are mutually compatible. This is in contrast with a commutative (classical) probability, where the disjointness of events already means the orthogonality of the corresponding projections.

By the von Neumann spectral theorem, a general observable can be constructed out of the projection operators. If A is an operator on $\mathcal{H}$ (possibly unbounded), then there exist a one-parameter set of projections $\lambda \mapsto P(\lambda) \in \mathcal{P}$, such that

$$A = \int \lambda dP(\lambda), \tag{1.10}$$

where the integration is over the spectrum $\sigma(A)$ of A, and it converges in the strong operator topology on the domain of A. Furthermore, for any bounded measurable function $f \in \mathcal{F}$ on the spectrum of A,

$$f(A) = \int f(\lambda) dP(\lambda), \tag{1.11}$$

and image of $\mathcal{F}$ under the mapping $f \mapsto f(A)$ is a commutative subalgebra of the algebra of operators on $\mathcal{H}$. Restricting ourselves to this commutative subalgebra, we obtain an algebra of observables of classical probability as embedded into the full noncommutative algebra of quantum probability, with the probability measure given by the state ϕ as $\mu(A) = \phi(P_A)$, with

$$P_A = \int_A dP(\lambda),$$

and $d\mu(\lambda) = \phi(dP(\lambda))$. Therefore the quantum probability can be seen as a generalization of classical probability by transition from a commutative algebra of the random variables to the noncommutative algebra of quantum observables.

An important difference between the two types of probability is, that quantum probability does not admit dispersion free states. A state ϕ on $\mathcal{A}$ is dispersion free, if $\phi(A^2) = \phi(A)^2$ for all observables A. In a commutative algebra, the dispersion free states are the pure states, the multiplicative states satisfying $\phi(AB) = \phi(A)\phi(B)$. In general a spectrum of algebra is the set of all multiplicative states. Setting $A = B$, all elements of the spectrum are dispersion free states. For example, given a n-dimensional commutative matrix *-algebra C_n, it can be diagonalized by a suitable unitary transformation $C_n \mapsto U^+ C_n U$ (applied to all matrices of the algebra). Then there are n multiplicative states, $\phi_i(e_j) = \delta_{ij}$ for $i = 1, 2, ..., n$, where e_j is the matrix with the jj element 1 and the rest zero. These states are clearly dispersion free, and for any matrix $A \in C_n$ the set $\{\phi_i(A) \mid i = 1, 2, ..., n\}$ is the spectrum of A.

The fact that there are no dispersion free states on a noncommutative algebras, does not imply that there can be no rule, that could assign a definite outcome to any observable of the algebra. In particular, it does not mean that there could not be any underlying hidden variables theory, that would reproduce all predictions of quantum theory. A state ϕ on the algebra has been defined as a linear functional on $\mathcal{A}$, so for any two observables $A, B \in \mathcal{A}$,

$$\phi(A + B) = \phi(A) + \phi(B). \tag{1.12}$$

With this definition, a theorem of von Neumann[8] shows, that an assumption of existence of a dispersion free state leads to a contradiction. But this proof is not conclusive, since a hidden-variables state s need not be committed to the linearity property (1.12). If the operators in (1.12) are

[8]This is the so called von Neumann proof of impossibility of hidden variables theories, e.g. [21].

represented by random functions f_A, f_B and f_{A+B} on the hidden-state space of a hidden-variables theory (which is a sample space classical probability theory), then f_{A+B} need not be given as f_A+f_B. If the observables A and B are not compatible, they refer to two measurements that require different experimental set-up and cannot be carried out simultaneously. In a noncommutative probability theory, the observables do not represent the particular outcomes of an experiment, but rather the statistical properties of the ensemble of outcomes of a sufficiently long run of repetitions of the experiment associated with a given observable. It is then admissible to represent the sum of averages of A and B by the sum $A + B$, since the average value is linear functional, and so (1.12) gives the correct result. But since a hidden-variables theory deals directly with individual outcomes, there is no reason that the individual outcomes of three different and incompatible experiments, represented by f_A, f_B and f_{A+B}, satisfy

$$f_{A+B}(\xi) = f_A(\xi) + f_B(\xi) \tag{1.13}$$

for a hidden variable ξ, hence the dispersion free hidden-variables state need not satisfy the property (1.12) of a quantum state.

1.4 The problem of measurement

Quantum mechanics is obtained from the quantum probability, if we identify observables (of a suitable algebra $\mathcal{A}$) with observed physical quantities, and specify their dynamics. The dynamics is given either for the elements of the algebra $\mathcal{A}$ (the Heisenberg picture) of equivalently for the states on $\mathcal{A}$ (the Schrödinger picture). The time evolution is determined by a unitary operator U_t, as $\rho(t) = U_t\rho U_t^+$ for a density operator ρ, and $\Psi \mapsto \Psi(t) = U_t\Psi$ for a statevector Ψ. In particular, the time evolution is linear. If the initial state $\Psi = \Psi(0)$ is given by a superposition

$$\Psi = \alpha\Psi_1 + \beta\Psi_2,$$

then the statevector at time t is given by the time evolved superposition $\Psi(t) = \alpha\Psi_1(t) + \beta\Psi_2(t)$.

Suppose we want to perform a measurement on a quantum system S, whose initial state is described by the statevector $|S\rangle$. The measurement process is an interaction between the measured system and a measurement device M. The result of measurement depend on the initial state of the system S, but it is not necessarily a property that S possesses prior to the measurement. Let the initial state of the measuring apparatus be $|M\rangle$, and let the statevectors $|M_i\rangle$ correspond to the different macroscopic state of the apparatus, which is directly observable by an observer O and indicate the result of the measurement. The initial state of combined system of the measured system and the apparatus is described by the statevector $\Psi = |S\rangle \otimes |M\rangle$. If the quantum mechanics has universal validity, then the quantum mechanical evolution law should apply also to the combined system. Then the dynamics of the measurement process is governed by the Schrödinger equation, and the combined system evolves as

$$\Psi(0) = |S\rangle \otimes |M\rangle \mapsto \Psi(t) = U_t(|S\rangle \otimes |M\rangle) = \sum_i |S_i(t)\rangle \otimes |M_i\rangle, \tag{1.14}$$

where the vectors $|S_i(t)\rangle$ are just the coefficients of the particular expansion of the final statevector. We get a superposition of the final macroscopic states of the measurement apparatus. But in a real measurement situation, the observer O always finds the combined system in a state given by only one of the terms of the superposition. The particular results can not be predicted, but in a series of experiments they occur with definite relative frequencies, determined on the basis of the coefficients

$|S_i\rangle$. Therefore the linear evolution given by the Schrödinger equation, from the point of view of the observer O, is apparently suspended at the moment of measurement, and the statevector $\Psi(t)$ collapses into one of the statevectors of the superposition.

The act of observation by O is also a kind of interaction between O and the combined system of S and M. The problem is, what is the real state of the system between the two measurements, whether it is a superposition or a collapsed state. Suppose there is another observer O'. Since the combined system of the measured system S, the apparatus M and the first observer O should obey the quantum laws as well, we could apply the quantum evolution law also in this case, and according to O' no collapse due to O happens, the state is given by a superposition. The state collapses with respect to the second observer O', only when O' asks O, what was the measurement result. Before that, according to O the measurement has a definite outcome, but for O it is just a superposition. Therefore, the reality seems to be relative to the particular observer.

Quantum mechanics tells us, that after a measurement the state of the measured system should be updated according to the measurement result (the projection postulate). In case when the measurement has been carried out but results have not been recorded, the state of the system has to be still collapsed to the corresponding mixed state. Within the subjective approach to probability, this would correspond to the strategy of using the maximum information available to predict the state of the system. The fact that the measurement took place, could be also a relevant piece of information, even without knowing the result. But in a quantum measurement, the change of the state of the system is not only subjective (updating one's knowledge of a property of the system independent of the act of measurement), but it is also objective. It is even possible to determine from subsequent measurements, whether the first one took place or not, e.g. in the double slit experiment.

If we admit that the collapse occurs upon measurement, we have the problem of determining, what process in nature is a measurement, and what is an ordinary unitary evolution according to the Schrödinger equation. Usually, the process is regarded as a measurement, when a microscopic system interacts with a macroscopic system (e.g. the measurement apparatus), and it is a unitary evolution when two microscopic interact between each other. But since no strict boundary between these two cases is known, such a division is somewhat obscure. The theory of quantum decoherence [23][24] attributes the collapse to the interaction of the measured system with the apparatus and the surrounding environment with many degrees of freedom. The evolution of the combined system is regarded as unitary at any time. When the internal degrees of freedom of the apparatus and the environment are discarded as being uninteresting, the time evolution of the system in the measurement situation (when coupled to the environment) become effectively non-unitary, and in a short time period its state becomes one of the eigenstates of the measured observable. However, decoherence still cannot explain the projection postulate, since the superfluous degrees of freedom are discarded by tracing over them, but this already involves a collapse of the state vector (of the apparatus and environment). Hence the apparatus measuring a quantum system shows a definite result only when it is observed (and the projection postulate is employed).

1.4.1 The EPR experiment

A problem closely related to the measurement problem is that of the apparent non-locality of quantum mechanics. In the early days after the discovery of quantum mechanics, it was realized, that the description by the wave function combined with the projection postulate, though perfectly agreeing with observations, has a sort of non-locality hidden in it. The wave function of a single particle can be spread over large portion of space, and when an observation is made, the particle appears sharply localized at a definite (but unpredictable) point in space. The probability of

occurrence at a specific point is determined on the basis of knowledge of the wave function. By the act of measurement, as the wave function collapses, the probability to find the particle in the rest of the space suddenly vanishes. This would not cause any problems, if the probabilities could be regarded as epistemic only, i.e. if the position of particle has been somehow determined prior to the measurement, and the measurement just revealed this position. Then there is no non-locality in the immediate vanishing of the probability in the rest of the space. But some phenomena, e.g. the two-slit experiment or the EPR experiment combined with the Bell inequalities, show that it is not possible to regard probabilities in quantum mechanics as merely epistemic. If the position of the particle is decided only at the moment of the measurement, then it is suspected, there is some kind of non-local influence, which assures the particle does not appear yet on another place than it actually did.

The well known thought experiment of Einstein, Podolsky and Rosen [17] shows, that if the locality principle should be valid even for the phenomena described by quantum mechanics, then the probabilities must be epistemic. Suppose we have two quantum systems, which have interacted in the past, but then they have been separated, and now they are far apart and can no longer interact with each other. The idividual systems are associated with the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and the combined system can be described by the vectors from the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The state, described by the statevector $\Psi \in \mathcal{H}$, of the combined system at the time of separation could be the (pure) entangled one, i.e. such a vector Ψ, that cannot be written as the tensor product $\Psi = \Psi_1 \otimes \Psi_2$ with $\Psi_1, \Psi_2 \in \mathcal{H}_1$ the statevectors of the individual subsystems. The state Ψ evolves undisturbed according to the Schrödinger equation until a measurement take place; we denote the statevector at the moment of measurement again by Ψ.

Now we perform a measurement on the first system, we can measure an observable (represented by the operator) A with eigenstates $\{a_n\}$, or another observable B with eigenstates $\{b_n\}$. The actual observable, that is measured on the combined system, is $A \otimes I$ or $B \otimes I$. This is a quantum mechanical expression of the principle of locality, a general observable for the first system is of the form $A \otimes I$ and for the second system $I \otimes B$. Then no measurement of an observable on the first system can influence the statistical properties predicted by quantum theory for the second system. Since $A \otimes I$ and $I \otimes B$ commute, they have the same systems of eigenvectors, which also represent the quantum events. The statistical properties of the two observables are determined by the probabilities assigned to these events, and it follows from the projection postulate, that they are independent of the order of measurement of $A \otimes I$ and $I \otimes B$.

The statevector of the combined system Ψ can be always written in the form

$$\Psi = \sum_n a_n \otimes \psi_n = \sum_m b_m \otimes \varphi_m, \tag{1.15}$$

where the vectors $\{\psi_n\}$ and $\{\varphi_n\}$ are regarded as coefficients of the expansions of Ψ into the basis given by $\{a_n\}$ and $\{b_n\}$ respectively. If we measure the observable A and find the eigenvalue corresponding to an eigenvector a_k, by the projection postulate we can infer that the state of the second system is described by the statevector Ψ_k. But we can also chose to measure B, then similarly when we find b_k, the second system collapses into the state described by φ_k. Since both systems can be separated by an arbitrarily large distance, by the principle of locality, the choice between A or B to measure on the first system cannot influence the state of the second one.

Quantum mechanics does not assign simultaneous values to two incompatible observables. This could be a consequence of an irreducible indeterminism in nature, that does not allow to tell in advance the result of measurement in both cases[9]. The EPR argument against such an irreducible

[9]This does not refer to our actual ability to make such a prediction, but rather to whether the result is determined

indeterminism now goes as follows. It is possible, that the two sets of vectors $\{\psi_n\}$ and $\{\varphi_n\}$ form two systems of eigenvectors of two mutually incompatible observables, A' and B' respectively, of the second system. Then by choosing between the measurement of A or B on the first system, we can predict the measurement outcome of either A' or B' with certainty without disturbing the second system. Then since the measurement results for A' or B' can be told in advance, they must be determined prior to the measurement.

1.4.2 The quantum non-locality

The EPR experiment can be carried out with a correlated pair of electrons or photons [8],[22]. In the case with electrons, the measured observables are their spin orientations with respect to two different directions. The observables for the first particle are denoted by A and B, and for the second one by A' and B'. Using the Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, (2.101), they are given by

$$A = \vec{a} \cdot \vec{\sigma} \otimes I, \quad B = \vec{b} \cdot \vec{\sigma} \otimes I, \quad A' = I \otimes \vec{a}' \cdot \vec{\sigma}, \quad B' = I \otimes \vec{b}' \cdot \vec{\sigma}, \tag{1.16}$$

where the spatial vectors $\vec{a}, \vec{b}, \vec{a}', \vec{b}'$ determine the direction of spin measurement. The (spin structure of) initial state of the system of two electrons is the singlet Bell state,

$$\Psi = \frac{1}{\sqrt{2}}(\psi^+ \otimes \psi^- - \psi^- \otimes \psi^+). \tag{1.17}$$

This state is spherically symmetric, it does not depend on any spatial direction. Moreover, since (1.17) is already in the form of (1.15), we see that any measurement of spin in the same direction always yields opposite spin orientations of the electrons.

Suppose there are some hidden variables ξ, that determine the result of measurement of the spins of electrons *locally*, i.e. a spin orientation of one particle is predicted without any reference to the particular observable being measured on the other particle[10]. The result predicted by such a hidden-variables theory of measurement of A is denoted by $A(\xi)$, and similarly for the other observables. the locality principle requires, that the observable A is represented by the same random variable $A(\xi)$, regardless whether A' or B' is chosen to be measured on the second particle. In other words, it is assumed that there is a classical probability theory, such that all the quantum observables A through B' can be uniquely represented by some random variables denoted by $A(\xi)$ through $B'(\xi)$. If the quantum observables are not compatible, this is not possible, and the Bell inequalities will show a possible contradiction.

The empirical correlation for measurement outcomes of the observables A and A' is obtained as an average value of their product,

$$\langle AA' \rangle_E = \frac{1}{N} \sum_{n=1}^{N} A_n A'_n, \tag{1.18}$$

with A_n and A'_n the measurement outcomes in n-th out of N repetitions of the experiment, $+1$ for spin up and -1 for spin down. Both quantum mechanics and a hidden-variables theory are required to predict this correlation. According to quantum mechanics,

$$\langle AA' \rangle_Q = \langle \psi, AA'\psi \rangle = -\vec{a} \cdot \vec{a}', \tag{1.19}$$

in advance or not.

[10] According to the classification of [21], it is a hidden-variables theory of the *second* kind.

and when A and A' are represented by their corresponding random variables $A(\xi)$ and $A'(\xi)$, a hidden-variables theory predicts

$$\langle AA'\rangle_H = \int A(\xi)A'(\xi)d\mu(\xi). \tag{1.20}$$

Consider the quantity

$$C = AA' + AB' + BA' - BB' = A(A' + B') + B(A' - B'). \tag{1.21}$$

When C is represented using the random variables $A(\xi)$ through $B'(\xi)$, its value for any ξ is either $+2$ or -2. Then by (1.20) the Bell-CHSH inequality follows,

$$|\langle C\rangle_H| = |\langle AA'\rangle_H + \langle AB'\rangle_H + \langle BA'\rangle_H - \langle BB'\rangle_H| \leq |\langle AA' + AB' + BA' - BB'\rangle_H| = 2. \tag{1.22}$$

A similar reasoning cannot be used for the quantum mechanical prediction of C, since unless $[A, A'] = 0$ and $[B, B'] = 0$, the two terms on the right hand side of (1.21) are not observables. The average value of C predicted by quantum theory is instead committed to the Tsirelson inequality [19], $\langle C\rangle_Q \leq 2\sqrt{2}$. Bell showed, that for a specific choice of the directions $\vec{a}...\vec{b}'$, the upper bound can be attained, contradicting (1.21).

Now suppose that there are no hidden variables determining the results of the spin measurements, but the results are nevertheless determined prior to the measurements, while still no interaction between the two particles is possible [25]. The experiment is carried out by measuring spin of both particles in three coplanar directions making angle $2\pi/3$ with each other. The measurement result in the n-th repetition of the experiment is denoted $A(i, n)$ for the first and $A'(j, n)$ for the second particle, where $i, j \in \{1, 2, 3\}$ numbers the three directions. The strict correlations, predicted by quantum mechanics and observed in real experiments, require

$$A(i, n) = -A'(i, n), \quad \text{for any } i \in \{1, 2, 3\}. \tag{1.23}$$

The pairs of directions (i, j) is chosen randomly in each repetition, there are nine possible choices, and six of them with $i \neq j$. Now there are two possible cases for the n-th repetition. In the first case, the measurement result is predetermined such that $A(i, n)$ would be the same for all i (if the spin was measured in that direction). Then, by (1.23) and the locality principle, the results for A and A' are always opposite irrespective of the (i, j) chosen. In the second case, $A(i, n)$ would not be always the same, and there are two out of the six different pairs of directions that lead to the opposite sign of A and A'. Therefore if $i \neq j$, the spins are opposite in at least $1/3$ of the repetitions. Quantum theory by (1.19) predicts the correlation $\langle AA'\rangle_Q = 1/2$ when the directions are unequal (with the angle $2\pi/3$). This implies the probability of opposite spin to be $1/4$, which contradicts the prediction above.

The apparently only way to establish the strict correlations in spin measurements while maintaining locality is to let the results be determined by a common cause, arising from the interaction of the particles in the past. If this is not possible, then either the locality must be violated, or the measurement outcomes are not a part of objective reality. In the latter case, the statevector would always collapse only with respect to the particular observer. Hence in the above experiment, the spin relations between the results, obtained by two observers independently measuring the spins of the two electrons, would be established only after one of the observers communicates his results to the other. Quantum theory is not committed to the Bell and similar no-go theorems, since it employs the noncommutative quantum probability instead of the commutative classical one, *and* it avoids most statements about results of individual experiments. Therefore quantum theory as a description of nature is allowed to be a local theory, but it does not imply that also the physical processes, incompletely described by quantum theory, must be local.

Chapter 2

Trace dynamics

The basic idea of trace dynamics is to generalize classical Hamiltonian and Lagrangian mechanics for non-commutative dynamical variables. The dynamical variables are operators on an N-dimensional underlying vector space $\mathcal{M}$, and are represented by $N \times N$ matrices. Every dynamical variable of classical mechanics, such as position of conjugate momentum, is replaced by a matrix valued quantity, which is nevertheless regarded as a single degree of freedom. No commutativity properties are assumed at this stage, the canonical algebra of quantum theory should appear only as an approximate property of a statistical ensemble. The dynamics and other physical properties is assumed to be be invariant with respect to a global unitary transformations of the form $A \mapsto U^+AU$ for any $U \in \mathcal{U}$.

The basic elements of a trace dynamics theory are dynamical variables represented by matrices composed either of ordinary or of Grassmann numbers. Both cases of complex and real ordinary or Grassmann numbers are considered, but ultimately we would like to choose the reals since they are more plausible physically. It turns out that trace dynamics gives some possibilities to recover the complex Hilbert space of quantum theory, even when started with the real case.

2.1 Grassmann algebra

The Grassmann numbers are used to include fermionic degrees of freedom into the theory. Grassmann algebra is built up from products of a set of basic Grassmann anticommuting elements $\{e_i\}$, $e_ie_j + e_je_i = 0$, together with an unit element e of the algebra. The basic elements or product of odd number of them are called odd grade (or fermionic) elements, a product of even number of basic elements or the unit is called an even grade (or bosonic) element of the Grassmann algebra. We denote the odd grade elements by e_i and even grade elements by b_i. General (either even or odd grade) basic element is a_i with its grade g_i, where $g_i = 0$ for even and $g_i = 1$ for odd grade. Then

$$\{e_i, e_j\} = 0, \qquad [e_i, b_j] = 0, \qquad [b_i, b_j] = 0, \qquad [e, a_i] = 0. \tag{2.1}$$

where $\{.,.\}$ stands for matrix anticommutation and $[.,.]$ for matrix commutation.

The full Grassmann algebra $\mathbb{G}$ is finally given by linear span of the set of all Grassmann elements over the complex or real numbers $\mathbb{F}$. The field $\mathbb{F}$ can be viewed as (isomorphic to) a subalgebra of $\mathbb{G}$ given by the linear span of the unit element e.

2.2 Dynamical variables

The dynamical variables of trace dynamics are operators on $\mathcal{M} = \mathbb{G}^N$, they are represented by $N \times N$ matrices with elements from the Grassman algebra $\mathbb{G}$. Any operator can be decomposed into the odd grade and the even grade part. The even grade (or bosonic) matrices will be denoted by B_i, odd grade (or fermionic) matrices by χ_i, and odd or even grade by A_i with g_i its grade ($g_i = 0$ for even and $g_i = 1$ for odd grade).

2.2.1 Trace quantities

The dynamical variables are operators rather than their representing matrices, hence there is an implicit invariance of the dynamical variables with respect to the *passive* transformations $A_i \mapsto R^{-1}A_iR$, for all $R \in \mathcal{G} = GL(G, N)$, i.e. those induced by a change of coordinates on the underlying space $\mathcal{M}$. But in trace dynamics it is assumed that the dynamics is also invariant with respect to (global, i.e. applied to all A_is simultaneously) *active* transformations of the form $A_i \mapsto R^{-1}A_iR$, for all $R \in \mathcal{G}' \subset \mathcal{G} = GL(G, N)$. An invariant function of dynamical variables $A \mapsto F(A) = F(R^{-1}AR)$ is given by a function of the coefficients of the characteristic polynomial of A. An invariant function which as also linear in A is function of one of the coefficients only, the trace of A. Generally, a linear functional on the set of matrices $\mathcal{A}$ is given by

$$F(A) = F_\rho(A) = \sum_{i,j} \rho^T_{ij} A_{ij} = Tr\rho A,$$

where ρ is a matrix. For constant ρ the invariance requirement $F(A) = F(R^{-1}AR)$ yields $\rho = R\rho R^{-1}$ for all $R \in \mathcal{G}$. Hence, up to c-number multiple, $\rho = I$, the unit matrix, and $F(A) = \mathrm{Tr}A$. If ρ is created from the dynamical variables using c-number coefficients only, then upon transformation $\rho \mapsto R^{-1}\rho R$, and there is no restriction on ρ. For trace dynamics, it is considered the case of constant ρ. The traces of operators will be further denoted by bold letters,

$$F(A) = \mathrm{Tr}A \equiv \mathbf{A}. \tag{2.2}$$

Therefore, in trace dynamics there are two important types of quantities, operator quantities given by matrices $A_i \in \mathcal{A}$, and trace quantities (or trace functionals) given by their traces.

2.2.2 Trace identities and cyclic identities

Although there are no general commutation properties for operator quantities, some properties arise for operators under the trace. For any two even grade bosonic matrices B_1 and B_2 we have

$$\mathrm{Tr}B_1B_2 = \sum_{m,n} (B_1)_{mn}(B_2)_{nm} = \sum_{mn} (B_2)_{nm}(B_1)_{mn} = \mathrm{Tr}B_2B_1, \tag{2.3}$$

but for any two odd grade fermionic matrices χ_1 and χ_2

$$\mathrm{Tr}\chi_1\chi_2 = \sum_{m,n} (\chi_1)_{mn}(\chi_2)_{nm} = -\sum_{mn} (\chi_2)_{nm}(\chi_1)_{mn} = -\mathrm{Tr}\chi_2\chi_1,$$

and for one even and the other odd grade matrix, $\mathrm{Tr}\mathcal{B}\chi = \mathrm{Tr}\chi\mathcal{B}$. This means that it is possible to perform a cyclic permutation under the trace, but permuting an odd grade element may change the sign of the trace,

$$\mathrm{Tr}A_1A_2 \ldots A_n = (-1)^{(g_1g_2 + \ldots g_1g_n} \mathrm{Tr}A_2 \ldots A_nA_1. \tag{2.4}$$

Conventionally the trace quantities $\mathbf{P}$ are constructed such that they are real and even grade element of the Grassmann algebra. Hence permuting an odd grade element always changes the sign,

$$\mathrm{Tr}A_1A_2\dots A_n = (-1)^{g_1}\mathrm{Tr}A_2\dots A_nA_1. \tag{2.5}$$

Using (2.4) we obtain useful trilinear cyclic identities, denoting $\{.,.\}_\pm$ as matrix anticommutator/commutator,

$$\mathrm{Tr}A_1\{A_2,A_3\}_\pm = (-1)^{g_1(g_2+g_3)}\mathrm{Tr}\left[A_2(A_3A_1 \pm (-1)^{g_3(g_1+g_2)}A_1A_3)\right] = \tag{2.6}$$

$$= (-1)^{g_3(g_1+g_2)}\mathrm{Tr}\left[A_3(A_1A_2 \pm (-1)^{g_2(g_1+g_3)}A_2A_1)\right],$$

which, when written out for bosonic and fermionic variables for the relevant (even grade) trace quantities, become

$$\mathrm{Tr}B_1\{B_2,B_3\}_\pm = \mathrm{Tr}B_2\{B_3,B_1\}_\pm = \mathrm{Tr}B_3\{B_1,B_2\}_\pm \tag{2.7}$$

$$\mathrm{Tr}B_1\{\chi_2,\chi_3\}_\pm = \mathrm{Tr}\chi_2\{\chi_3,B_1\}_\mp = -\mathrm{Tr}\chi_3\{B_1,\chi_2\}_\mp \tag{2.8}$$

$$\mathrm{Tr}\chi_1\{B_2,\chi_3\}_\pm = -\mathrm{Tr}B_2\{\chi_3,\chi_1\}_\mp = -\mathrm{Tr}\chi_3\{\chi_1,B_2\}_\pm \tag{2.9}$$

$$\mathrm{Tr}\chi_1\{\chi_2,B_3\}_\pm = -\mathrm{Tr}\chi_2\{B_3,\chi_1\}_\pm = -\mathrm{Tr}B_3\{\chi_1,\chi_2\}_\mp \tag{2.10}$$

2.2.3 Operator conjugation

There is a scalar product $\langle .,.\rangle$ on the space $\mathcal{M} = G^N$, such that for $x, y \in \mathcal{M}$

$$\langle x, y\rangle = (x, My), \tag{2.11}$$

where $(.,.)$ is the standard dot product and M the (generally symmetric) matrix corresponding to $\langle .,.\rangle$. The operator conjugation $A \mapsto A^*$ is defined in the usual way by Riesz lemma and the requirement $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y \in \mathcal{M}$. For the standard dot product ($M = I$) this amounts to transpose and complex conjugation $A \mapsto (A^g)^+ = (\overline{A^g})^T$, irrespective of the grade of A, which is denoted by the g superscript,

$$(A^g)^*_{mn} = (A^g)^+_{mn} \equiv \overline{(A^g)_{nm}}. \tag{2.12}$$

Note that $A \mapsto A^+$ denotes always the conjugation with respect to standard dot product, whereas $A \mapsto A^*$ is the conjugation with respect to the actual metric M.

The grade is important for conjugation of product of two matrices $A_1^{g_1}$ and $A_2^{g_2}$, since

$$(A_1A_2)^+_{mn} = \overline{(A_1A_2)}_{nm} = \sum_k \overline{(A_1)_{nk}}\,\overline{(A_2)_{km}} = (-1)^{g_1g_2}\sum_k \overline{(A_2)}_{km}\overline{(A_1)}_{nk} =$$

$$= (-1)^{g_1g_2}\sum_k (A_2^+)_{mk}(A_1^+)_{kn} = (-1)^{g_1g_2}\left(A_2^+A_1^+\right)_{mn}, \tag{2.13}$$

and in matrix notation,

$$(A_1A_2)^+ = (-1)^{g_1g_2}A_2^+A_1^+, \tag{2.14}$$

hence conjugation changes sign in case of two odd grade matrices.

For a general dot product the conjugation is given by

$$(A^+M^+x, y) = (x, MAy) = \langle x, Ay\rangle = \langle A^*x, y\rangle = (A^*x, My) = (M^+A^*x, y),$$

$$A^* = (M^{-1})^+ A^+ M^+, \tag{2.15}$$

and (2.13) holds also in the general case. Upon change of coordinates

$$x \mapsto R^{-1}x, \qquad A \mapsto R^{-1}AR, \qquad M \mapsto R^+MR, \qquad A^* \mapsto R^{-1}A^*R,$$

and the transformation preserves all adjointness properties of the matrices (with respect to the corresponding dot products). The basis in $\mathcal{M}$ is chosen such that $\langle .,. \rangle$ becomes the standard dot product. The assumed active transformations in general change the adjointness properties. Since adjointness properties of the dynamical variables are of interest for the emergence of quantum theory, the active transformations are restricted to unitary ones, i.e. we restrict the transformations $A \mapsto R^{-1}AR$ from the group $\mathcal{G}$ to its unitary (with respect to the standard dot product) subgroup $\mathcal{G}' = \mathcal{U}$, so that $A \mapsto U^+AU$ for $U \in \mathcal{U}$.

2.2.4 Properties of trace quantities

The trace quantities $\mathbf{P} = \mathrm{Tr}P$ are chosen real and of even grade. It follows that if $P = AO$, then A and O are of the same grade.

If $\mathrm{Tr}AO$ vanishes for all O (of the respective grade), then it implies also $P = 0$ as an operator. Write $P = \sum_n C_n K_n$, with K_n distinct c-number mononomials in Grassmann elements and C_n their real or complex matrix coefficients, $C_n \in \mathbb{F}^{N\times N}$. All K_n are of the same grade as P. For arbitrary but fixed C_p from the sum, take $O = \alpha \mathbb{C}_p^+$ with α a real number for K_p bosonic, and a Grassmann element not contained in K_p for K_p fermionic (otherwise $K_p\alpha = 0$), to obtain

$$0 = \mathrm{Tr}\left(\sum_n C_n C_p^+ K_n \alpha\right). \tag{2.16}$$

The coefficients of the distinct Grasmann monomials $K_n\alpha$ have to vanish separately, in particular $C_pC_p^+ = 0$, $C_p = 0$, an this is true for all p giving $P = 0$.

The assertion still holds, if O is restricted to all self-adjoint or all anti-self-adjoint and either bosonic or fermionic matrices. For example consider the case of all self-adjoint bosonic O. Decompose P into its self-adjoint and anti-self-adjoint part, $P = P^{sa} + P^{asa}$. Since

$$\overline{\mathrm{Tr}P^{sa}O} = \mathrm{Tr}(P^{sa}O)^+ = \mathrm{Tr}OP^{sa} = \mathrm{Tr}P^{sa}O,$$
$$\overline{\mathrm{Tr}P^{asa}O} = \mathrm{Tr}(P^{asa}O)^+ = -\mathrm{Tr}OP^{asa} = -\mathrm{Tr}P^{sa}O,$$

the trace $\mathrm{Tr}P^{sa}O$ is real and $\mathrm{Tr}P^{asa}O$ is imaginary, and so they must vanish separately in $\mathrm{Tr}PO = 0$. Taking $O = C_p^{sa}$ for the first trace and $O = iC_p^{asa}$ for the second in (2.16), both P^{sa} and P^{asa} must vanish giving $P = 0$.

If PO is such that $\mathrm{Tr}PO$ is real and bosonic for any self-adjoint or any anti-self-adjoint O of either odd or even grade, then P has the same grade and adjointness properties as O. For example, for all O self-adjoint bosonic matrices, using (2.5) and (2.13),

$$0 = \mathrm{Im}\mathrm{Tr}PO = \mathrm{Tr}PO - \overline{\mathrm{Tr}PO} = \mathrm{Tr}PO - \mathrm{Tr}(PO)^+ = \mathrm{Tr}(PO - OP^+) = \mathrm{Tr}((P - P^+)O),$$

which implies $P = P^+$, i.e. P is self-adjoint. For O anti-self-adjoint fermionic matrices, analogously,

$$0 = \mathrm{Im}\mathrm{Tr}PO = \mathrm{Tr}PO - \mathrm{Tr}(PO)^+ = \mathrm{Tr}PO - \mathrm{Tr}OP^+ = \mathrm{Tr}PO + \mathrm{Tr}P^+O = \mathrm{Tr}((P + P^+)O),$$

and since O as arbitrary anti-self-adjoint, $P + P^+$ must vanish, and P is anti-self-adjoint.

2.2.5 Operator derivative of a trace quantity

Let P is a polynomial constructed from the dynamical variables including an operator O. The derivation of trace quantity $\mathbf{P} = \mathrm{Tr}P$ with respect to O is defined by an infinitesimal variation of P in O, $O \mapsto O + \delta O$, discarding the higher order terms in δO, and cyclic permutation to make δO stand in the trace on the right,

$$\delta\mathbf{P} = \mathrm{Tr}\frac{\delta\mathbf{P}}{\delta O}\delta O, \tag{2.17}$$

which for arbitrary infinitesimal δO gives the derivative $\delta\mathbf{P}/\delta O$. This definition is useful together with (2.5), when P is a polynomial or a rational function of the dynamical variables. Rewriting (2.17) by using matrix indices

$$\delta\mathbf{P} = \sum_{m,n}\left(\frac{\delta\mathbf{P}}{\delta O}\right)_{mn}(\delta O)_{nm},$$

we get for the matrix components of the derivative

$$\left(\frac{\delta\mathbf{P}}{\delta O}\right)_{mn} = \frac{\partial\mathbf{P}}{\partial O_{nm}}. \tag{2.18}$$

According to a remark above, since we choose P such that $\mathrm{Tr}P$ is real and bosonic, the derivative of $\mathbf{P}$ with respect to an operator O is always of the same grade, and shares the same adjointness properties as O. From (2.18) it also follows that the derivative with respect to an operator obeys the Leibniz rule, for any two trace quantities of general functions $\mathbf{P}$ and $\mathbf{Q}$,

$$\frac{\delta(\mathbf{PQ})}{\delta x_r} = \frac{\delta\mathbf{P}}{\delta x_r}\mathbf{Q} + \mathbf{P}\frac{\delta\mathbf{Q}}{\delta x_r}. \tag{2.19}$$

2.3 Dynamics of matrix models

Let the configuration space of a system with n degrees of freedom be described by the set $\{q_r(t)\}$, for $r = 1, 2, \ldots, n$, of dynamical variables with t as a time parameter. The variables are even grade (bosonic) or odd grade (fermionic) matrices representing bosonic or fermionic degrees of freedom respectively. The dynamics is given by the trace Lagrangian $\mathbf{L}$,

$$\mathbf{L}(\{q_r\}, \{\dot{q}_r\}) = \mathrm{Tr}L(\{q_r\}, \{\dot{q}_r\}), \tag{2.20}$$

where L is the Lagrangian operator, which is chosen such that $\mathbf{L}$ is real and Grassmann even grade. This allows us to define conjugate momentum operator p_r to each q_r of the same grade and adjointness properties by

$$p_r = \frac{\delta\mathbf{L}}{\delta\dot{q}_r}\ .$$

The variation of the trace action

$$\mathbf{S} = \int_{t_1}^{t_2}\mathbf{L}dt \tag{2.21}$$

with respect to q_r with fixed $q_r(t_1)$ and $q_r(t_2)$ gives

$$\delta\mathbf{S} = \int_{t_1}^{t_2}\sum_r \mathrm{Tr}\left(\frac{\delta\mathbf{L}}{\delta q_r} - \frac{d}{dt}\frac{\delta\mathbf{L}}{\delta\dot{q}_r}\right)\delta q_r,$$

and for stationary action $\delta\mathbf{S} = 0$ and arbitrary independent variations δq_r we obtain the operator Euler-Lagrange equations

$$\frac{\delta\mathbf{L}}{\delta q_r} - \frac{d}{dt}\frac{\delta\mathbf{L}}{\delta\dot{q}_r} = 0, \qquad r = 1, 2, \ldots, n. \tag{2.22}$$

The Legendre transformation of the trace Lagrangian defines the trace Hamiltonian function $\mathbf{H}(\{q_r\}, \{p_r\})$ as

$$\mathbf{H} = \mathrm{Tr}\sum_r p_r\dot{q}_r - \mathbf{L}. \tag{2.23}$$

By variations of $\mathbf{H}$ with the use of the Euler-Lagrange equations (2.22),

$$\delta\mathbf{H} = \mathrm{Tr}\sum_r \left(\delta p_r\dot{q}_r + p + r\delta\dot{q}_r - \dot{p}_r\delta q_r - p_r\delta\dot{q}_r\right) = \mathrm{Tr}\sum_r \left(\epsilon_r\dot{q}_r\delta p_r - \dot{p}_r\delta q_r\right),$$

with $\epsilon_r = (-1)^{g_r}$ is $+1$ for bosonic and -1 for fermionic variable. Then the Hamilton equations reads

$$\dot{p}_r = -\frac{\delta\mathbf{H}}{\delta q_r}, \qquad \dot{q}_r = \epsilon_r\frac{\delta\mathbf{H}}{\delta p_r}. \tag{2.24}$$

It is convenient to introduce a compact notation. Set $x_{2k-1} = q_k$ and $x_{2k} = q_k$ for $k = 1, 2, \ldots n$, and define a $2n \times 2n$ matrix ω by

$$\omega = \mathrm{diag}(\Omega_1, \Omega_2, \ldots \Omega_n), \tag{2.25}$$

where $\Omega_k = \Omega_B$ for r bosonic and $\Omega_k = \Omega_F$ for r fermionic with

$$\Omega_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega_F = \begin{pmatrix} 0 & -1 \\ -1 & 0. \end{pmatrix} \tag{2.26}$$

For future reference, we note that the matrix ω has the following properties,

$$(\omega^2)_{rs} = -\epsilon_r\delta_{rs}, \qquad \omega_{sr} = -\epsilon_r\omega_{rs} = -\epsilon_s\omega_{rs}, \qquad \sum_t \omega_{tr}\omega_{ts} = \sum_t \omega_{rt}\omega_{st} = \delta_{rs}. \tag{2.27}$$

The Hamilton equations (2.24) now become

$$\dot{x}_r = \sum_{s=1}\omega_{rs}\frac{\delta\mathbf{H}}{\delta x_s}, \qquad r = 1, 2, \ldots, 2n.$$

We use them to determine the time derivative of a bosonic trace functional $\mathbf{A}$,

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \sum_{r=1}^{2n}\mathrm{Tr}\left(\frac{\delta\mathbf{A}}{\delta x_r}\dot{x}_r\right) = \frac{\partial\mathbf{A}}{\partial t} + \sum_{r,s=1}^{2n}\left(\frac{\delta\mathbf{A}}{\delta x_r}\omega_{rs}\frac{\delta\mathbf{H}}{\delta x_s}\right) = \frac{\partial\mathbf{A}}{\partial t} + \{\mathbf{A}, \mathbf{H}\}. \tag{2.28}$$

The last term on the right hand side is the *generalized Poisson bracket*. For any two bosonic trace quantities $\mathbf{A}$ and $\mathbf{B}$ it is defined by

$$\{\mathbf{A}, \mathbf{B}\} = \sum_{r,s=1}^{2n}\left(\frac{\delta\mathbf{A}}{\delta x_r}\omega_{rs}\frac{\delta\mathbf{H}}{\delta x_s}\right) = \mathrm{Tr}\sum_{r=1}^{n}\epsilon_r\left(\frac{\delta\mathbf{A}}{\delta q_r}\frac{\delta\mathbf{B}}{\delta p_r} - \frac{\delta\mathbf{B}}{\delta q_r}\frac{\delta\mathbf{A}}{\delta p_r}\right). \tag{2.29}$$

The generalized Poisson bracket has all the properties of ordinary Poisson bracket, it is antisymmetric in its arguments, satisfies the Jacobi identity

$$\{\mathbf{A},\{\mathbf{B},\mathbf{C}\}\}+\{\mathbf{C},\{\mathbf{A},\mathbf{B}\}\}+\{\mathbf{B},\{\mathbf{C},\mathbf{A}\}\}=0, \tag{2.30}$$

and obeys the Leibniz rule

$$\{\mathbf{AB},\mathbf{C}\}=\mathbf{A}\{\mathbf{B},\mathbf{C}\}+\{\mathbf{A},\mathbf{C}\}\mathbf{B}. \tag{2.31}$$

This follows by expanding (2.29) into the matrix components with the use of (2.18),

$$\{\mathbf{A},\mathbf{B}\}=\sum_{i,j=1}^{N}\sum_{r=1}^{n}\epsilon_r\left(\left(\frac{\partial\mathbf{A}}{\partial q_r}\right)_{ij}\left(\frac{\partial\mathbf{B}}{\partial p_r}\right)_{ji}-\left(\frac{\partial\mathbf{B}}{\partial q_r}\right)_{ij}\left(\frac{\partial\mathbf{A}}{\partial p_r}\right)_{ji}\right)$$

and using the Jacobi identity for the ordinary Poisson bracket (extended to the Grassmann algebra) on each term of the sum over i,j.

The equation (2.28) implies, that if $\mathbf{A}$ has no explicit time dependence, it is conserved by the time evolution if and only if its Poisson bracket with the trace Hamiltonian $\mathbf{H}$ vanishes. The trace Hamiltonian itself is a conserved quantity, since by assumption $\mathbf{L}$ nor $\mathbf{H}$ has explicit time dependence. From the generalized Jacobi identity it follows, that the Poisson bracket of two conserved quantities is also conserved, hence the algebra of symmetries form a Lie algebra under the action of the generalized Poisson bracket.

2.4 Symplectic structure of trace dynamics

Trace dynamics has a natural symplectic structure of classical mechanics[1], since at the level of the matrix components it is essentially classical mechanics. The configuration manifold M of trace dynamics is coordinatized by the operators (matrices) $q=(q_1,\ldots,q_n)$. Any curve on the configuration manifold $\gamma:\tau\mapsto q(\tau)$ determines a tangent vector V_p^γ at the point $p=\gamma(0)$ by its action on a trace functional $\mathbf{A}=\mathbf{A}(\{q_r\})$,

$$V_p^\gamma A=\frac{d}{d\tau}(\mathbf{A}\circ\gamma)(0)=\sum_{rij}\frac{\partial\mathbf{A}}{\partial(q_r)_{ij}}\frac{d\gamma_{r,ij}}{d\tau}(0)=\mathrm{Tr}\sum_r\frac{\delta\mathbf{A}}{\delta q_r}\frac{d\gamma_r}{d\tau}(0)=\mathrm{Tr}\sum_r\left(\epsilon_r\frac{d\gamma_r}{d\tau}(0)\frac{\delta}{\delta q_r}\right)\mathbf{A}. \tag{2.32}$$

Denote the coordinates of the tangent vector at the point p by $\{X_r(p)\}$,

$$X_r(p)=\epsilon_r\frac{d\gamma_r}{d\tau}(0), \tag{2.33}$$

then we see that the action of a general vector field X on the configuration manifold M, given by some curve γ with $\gamma(0)=p$, at a trace functional $\mathbf{A}$ at the point $p\in M$ can be written as

$$X|_p\mathbf{A}=\mathrm{Tr}\sum_r\left(X_r(p)\frac{\delta\mathbf{A}}{\delta q_r}\right). \tag{2.34}$$

Given a vector field X (or the corresponding vector X_p at the point $p\in M$) and a trace functional $\mathbf{A}$, we can define a 1-form $F_\mathbf{A}$ by the action of X at the functional $\mathbf{A}$ by

$$F_\mathbf{A}X=X\mathbf{A}=\mathrm{Tr}\sum_r\left(X_r\frac{\delta\mathbf{A}}{\delta q_r}\right)=\sum_{rij}\sum_{smn}\left((X_r)_{ij}\left(\frac{\partial}{\partial(x_r)_{ij}},(dx_s)_{mn}\right)\frac{\partial\mathbf{A}}{\partial(q_r)_{mn}}\right)= \tag{2.35}$$

[1]The case of symplectic structure of classical mechanics is treated for example in [5].

$$= \sum_{rij}\sum_{smn}\left((X_r)_{ij}\left(\left(\frac{\delta}{\delta x_r}\right)_{ji},(\delta x_s)_{mn}\right)\left(\frac{\delta \mathbf{A}}{\delta q_r}\right)_{nm}\right) = \mathrm{Tr}\sum_r\left(X_r\frac{\delta}{\delta x_r}\right)\mathrm{Tr}\sum_s\left(\frac{\delta \mathbf{A}}{\delta x_s}\delta x_s\right),$$

where we have introduced the symbols and their mutual relation

$$\left(\frac{\delta}{\delta x_r}\right)_{ji} = \frac{\partial}{\partial (x_r)_{ij}}, \quad (\delta x_s)_{mn} = (dx_s)_{mn}, \quad \left(\left(\frac{\delta}{\delta x_r}\right)_{ji},(\delta x_s)_{nm}\right) = \delta_{rs}\delta_{im}\delta_{jn}. \tag{2.36}$$

Denoting the coordinates of the form $F_{\mathbf{A}}$ at p by

$$F_s = \frac{\delta \mathbf{A}}{\delta q_s}, \qquad (F_s)_{mn} = \left(\frac{\delta \mathbf{A}}{\delta q_s}\right)_{nm}, \tag{2.37}$$

we can infer the shape of a general differential form F on the manifold M, and from (2.34) of a general vector field X,

$$F|_p = \mathrm{Tr}\sum_s\left(F_s(p)\delta x_s\right), \qquad X|_p = \mathrm{Tr}\sum_r\left(X_r(p)\frac{\delta}{\delta q_r}\right). \tag{2.38}$$

The configuration space of trace dynamics is M, the tangent bundle of M is TM, and the cotangent bundle is T^*M. The cotangent bundle of the configuration manifold is a natural space for the formulation of classical symplectic mechanics, and it is called the symplectic space. To extend the symplectic formulation to trace dynamics, we introduce coordinates $\{x_r\}$ on T^*M, which split to the coordinates $\{q_r\}$ of a point on the manifold M and the coordinates $\{p_r\}$ of the 1-forms at this point. The notation is the same as in previous section. The vector fields and differential forms on T^*M have the same shape as in (2.38) with q_r replaced by x_r. There is a natural projection π that takes a point in T^*M with coordinates $x = (q, p)$ to the point q on the configuration manifold M, and its derivative π_* maps $T(T^*M)$ to TM. It is used to define a distinguished differential 1-form $\omega^1 \in T^*(T^*M)$ on the symplectic manifold T^*M. Given a point $x = (q, p) \in T^*M$, i.e. a 1-form $F = \sum_r Tr(p_r\delta q_r) \in T^*M$, the 1-form ω^1 is defined by its action at a vector $\xi \in T(T^*M)$ by

$$\omega^1(\xi) = F(\pi_*\xi). \tag{2.39}$$

The form ω^1 can be written in coordinates as

$$\omega^1 = \sum_r \mathrm{Tr}(p_r\delta q_r). \tag{2.40}$$

Given a form $\omega \in T^*(T^*M)$,

$$\omega = \sum_r \mathrm{Tr}(\omega_r\delta x_r) = \sum_{rij}(\omega_r)_{ji}(\delta x_r)_{ij} = \sum_{rij}(\omega_r)_{ij}(dx_r)_{ij},$$

and taking advantage of the decomposition to matrix components, we can define the differential operator δ acting on the form ω by

$$\delta\omega = \sum_{rij}\sum_{smn}\frac{\partial(\omega_s)_{nm}}{\partial(x_r)_{ij}}(dx_r)_{ij}\wedge(dx_s)_{mn} = \mathrm{Tr}\sum_{rs}\left(\frac{\delta\omega_s}{\delta x_r}\delta x_r\wedge\delta x_s\right). \tag{2.41}$$

Applying this result to the 1-form ω^1, we obtain the symplectic 2-form ω^2,

$$\omega^2 = \delta\omega^1 = \mathrm{Tr}\sum_{rs}\left(\frac{\delta p_s}{\delta x_r}\delta x_r\wedge\delta q_s\right) = \mathrm{Tr}\sum_r \delta p_r\wedge\delta q_r. \tag{2.42}$$

This 2-form has a close connection to the Hamiltonian formulation of trace dynamics. It establishes an isomorphism between vector fields ξ and 1-forms ω^1_ξ on T^*M by

$$\xi \mapsto \omega^1_\xi(\chi) = \omega^2(\chi, \xi), \qquad \chi \in T(T^*M). \tag{2.43}$$

Denote the decomposition of the coordinates of ξ and χ to the q and p components, $\xi_r = (\xi_{q,s}, \xi_{p,s})$ and $\chi = (\chi_{q,s}, \chi_{p,s})$, where $r = 1, 2, \ldots 2n$ and $s = 1, 2, \ldots, n$. Then by (2.41),

$$\omega^2(\xi, \chi) = (\delta\omega^1)(\chi, \xi) = \mathrm{Tr} \sum_s \left(\epsilon_s \chi_{q,s} \xi_{p,s} - \chi_{p,s} \xi_{q,s}\right). \tag{2.44}$$

This gives the mapping $\xi \mapsto \omega^1_\xi$ the component representation $(\xi_{q,s}, \xi_{p,s}) \mapsto (-\xi_{p,s}, \epsilon_s \xi_{q,s})$. The inverse mapping $I : \omega^1_\xi \mapsto \xi$ is in the q and p components given by

$$I : (\omega^1_{\xi,q,s}, \omega^1_{\xi,p,s}) \mapsto (\epsilon_s \omega^1_{\xi,p,s}, -\omega^1_{\xi,q,s}), \tag{2.45}$$

which corresponds just to the matrix ω from previous section, (2.25) and (2.26)

Given a bosonic trace functional $\mathbf{H}(\{x_r\}) = \mathbf{H}(\{q_s\}, \{p_s\})$ on the symplectic space T^*M, its differential is

$$\delta\mathbf{H} = \mathrm{Tr} \sum_r \left(\frac{\delta\mathbf{H}}{\delta x_r} \delta x_r\right) = \mathrm{Tr} \sum_s \left(\frac{\delta\mathbf{H}}{\delta q_s} \delta q_s + \frac{\delta\mathbf{H}}{\delta p_s} \delta p_s\right), \tag{2.46}$$

and it is a 1-form on T^*M. Its image under the mapping I, $I\delta\mathbf{H} \in T(T^*M)$, is a vector on the symplectic space T^*M. Denoting the integral curve of this vector field by $x_r(t)$, then it obeys the equation

$$\dot{x}_r(t) = I\delta H, \tag{2.47}$$

or in coordinates

$$(\dot{q}_s, \dot{p}_s) = \left(\epsilon_s \frac{\delta\mathbf{H}}{\delta q_s}, -\frac{\delta\mathbf{H}}{\delta p_s}\right), \qquad \dot{x}_r = \sum_s \omega_{rs} \frac{\delta\mathbf{H}}{\delta x_s}. \tag{2.48}$$

If the trace functional $\mathbf{H}$ is interpreted as the trace Hamiltonian, then (2.48) are just the Hamilton equations of motion (2.24). The vector field $I\delta\mathbf{H}$ is then called the Hamiltonian vector field. For a general (bosonic) trace functional, it represents the one-parameter canonical transformations (2.56) on the phase space - the symplectic manifold T^*M.

Denote the vector field associated with a trace functional $\mathbf{A}$ through the mapping I by $X_\mathbf{A} = I\delta\mathbf{A}$. Any trace functional $\mathbf{H}$ generate a flow $t \mapsto x_r(t)$ on the phase space T^*M. The derivative of a trace functional $\mathbf{A}$ along this flow at the point $p = x_r(0)$ is, with the use of the definition of ω^2 (2.42),

$$\frac{d}{dt}\mathbf{A}(\{x_r(t)\})|_0 = (X_\mathbf{H})|_p \mathbf{A} = \delta\mathbf{A}(X_\mathbf{H}) = \omega^2(X_\mathbf{H}, X_\mathbf{A}).$$

Evaluating $\omega^2(X_\mathbf{H}, X_\mathbf{A})$ with the use of (2.44), (2.45) and (2.46), we find

$$\omega^2(X_\mathbf{H}, X_\mathbf{A}) = -\mathrm{Tr} \sum_{r=1}^n \epsilon_r \left(\frac{\delta\mathbf{H}}{\delta q_r} \frac{\delta\mathbf{A}}{\delta p_r} - \frac{\delta\mathbf{A}}{\delta q_r} \frac{\delta\mathbf{H}}{\delta p_r}\right) = \{\mathbf{A}, \mathbf{H}\},$$

which is just the generalized Poisson bracket (2.29). Hence the derivative of $\mathbf{A}$ in the direction of the vector field $X_\mathbf{H} = I\delta\mathbf{H}$ is given by the Poisson bracket,

$$X_\mathbf{H}\mathbf{A} = \{A, H\}.$$

The other way round, the Poisson bracket associates a vector field $X_{\mathbf{H}}$ to any (bosonic) trace functional $\mathbf{H}$ by

$$X_{\mathbf{H}} = -\{H, .\} = -\mathrm{Tr} \sum_{rs} \left(\frac{\delta \mathbf{H}}{\delta x_r} \omega_{rs} \frac{\delta}{\delta x_r} \right). \tag{2.49}$$

As a consequence of the Jacobi identity for the generalized Poisson bracket, $X_{\mathbf{H}}$ obeys the Leibniz product rule

$$X_{\mathbf{H}}\{\mathbf{A}, \mathbf{B}\} = \{X_{\mathbf{H}}\mathbf{A}, \mathbf{B}\} + \{\mathbf{A}, X_{\mathbf{H}}\mathbf{B}\},$$

and the commutator of vector fields satisfies

$$[X_{\mathbf{A}}, X_{\mathbf{B}}]C = X_{\mathbf{A}}X_{\mathbf{B}}C - X_{\mathbf{B}}X_{\mathbf{A}}C = X_{\{\mathbf{A},\mathbf{B}\}}.$$

for any trace functionals $\mathbf{A}, \mathbf{B}, \mathbf{C}$. The last identity implies, that the vector fields $X_{\mathbf{H}}$ form a Lie algebra under commutation, which is isomorphic to the Lie algebra of trace functionals $\mathbf{H}$ under the generalized Poisson bracket. The Poisson bracket is independent of the time evolution generated by the Hamiltonian vector field $X_{\mathbf{H}}$, or more generally, of any one-parameter canonical transformation generated by a trace functional $\mathbf{G}$. Hence the symplectic 2-form ω^2 is invariant with respect to these transformations. It follows that also all the exterior powers of ω^2 are invariant. In particular its nN-th power (which corresponds, up to a constant multiple, to the operator phase space element volume) is invariant under canonical transformation.

For $\mathbf{A}$ not explicitly time dependent bosonic functional the equation of motion is

$$\dot{\mathbf{A}} = \{\mathbf{A}, \mathbf{H}\} = -\{\mathbf{H}, \mathbf{A}\}.$$

If we define the trace quantity $\mathbf{X}_r = \mathrm{Tr} j_r x_r$, the equations of motion can be *formally* integrated,

$$\mathbf{X}_r(t) = \exp(-tX_{\mathbf{H}})\mathbf{X}_r(0), \qquad x_r(t) = \frac{\delta}{\delta j_r} \left[\exp(-tX_{\mathbf{H}}) \mathrm{Tr}(j_r x_r(0)) \right],$$

with the expansion for small t,

$$\mathbf{X}_r(t) = \mathbf{X}_r(0) - t\{\mathbf{H}, \mathbf{X}_r\}_{t=0} + \frac{1}{2}t^2\{\mathbf{H}, \{\mathbf{H}, \mathbf{X}_r\}\}_{t=0} - \frac{1}{6}t^3\{\mathbf{H}, \{\mathbf{H}, \{\mathbf{H}, \mathbf{X}_r\}\}\}_{t=0} + \ldots.$$

Note that, up to some special cases, the evolution generated by the trace Hamiltonian is not unitary. It is expected to become approximately and effectively unitary in statistical averages only.

2.5 Conserved quantities

Conserved quantities are of interest in statistical mechanical treatment, since they are used to condition the statistical ensemble on their expectations. Besides the trace Hamiltonian, which is always conserved, there is an important operator conserved due to the global unitary invariance of $\mathbf{L}$ and $\mathbf{H}$, the C operator, and in case of r as the infinitesimal spatial box index in a field theory there are conserved trace Poincaré generators. These additional conserved quantities appear when certain assumptions about the Lagrangian and Hamiltonian operators are made.

2.5.1 The conserved C operator

When the Hamiltonian operator H is constructed from the dynamical variables with c-number coefficients only, then the trace Hamiltonian $\mathbf{H}$ is invariant with respect to global unitary transformations on the underlying vector space $\mathcal{M}$,

$$q_r \mapsto U^+ q_r U, \qquad p_r \mapsto U^+ p_r U,$$

the invariance of $\mathbf{H}$ means that

$$\mathbf{H}(\{q_r\},\{p_r\}) \mapsto \mathbf{H}(\{U^+ q_r U\},\{U^+ p_r U\}) = \mathbf{H}(\{q_r\},\{p_r\}). \tag{2.50}$$

The conserved operator C is obtained, if we consider infinitesimal unitary transformations $U = \exp(\varepsilon\Lambda)$ with Λ a constant even grade anti-self-adjoint matrix, and ε an infinitesimal c-number parameter. Then $U = I + \varepsilon\Lambda + o(\varepsilon)$ and

$$q_r \mapsto U^+ q_r U = q_r - \varepsilon[\Lambda, q_r] + o(\varepsilon), \qquad p_r \mapsto U^+ p_r U = p_r - \varepsilon[\Lambda, p_r] + o(\varepsilon). \tag{2.51}$$

The invariance of $\mathbf{H}$ implies

$$o(\varepsilon) = \delta\mathbf{H} = -\varepsilon \mathrm{Tr} \sum_r \left(\frac{\delta H}{\delta q_r}[\Lambda, q_r] + \frac{\delta H}{\delta p_r}[\Lambda, p_r] \right) + o(\varepsilon).$$

Since $\delta\mathbf{H}$ is of order less than ε, the coefficient of ε must vanish. Using the Hamilton equations (2.24) and cyclic identities under the trace,

$$0 = \mathrm{Tr} \sum_r \left(\dot{p}_r[\Lambda, q_r] - \epsilon_r \dot{q}_r[\Lambda, p_r] \right) = \mathrm{Tr} \sum_r \left(\epsilon_r q_r \dot{p}_r - \dot{p}_r q_r - p_r \dot{q}_r + \epsilon_r \dot{q}_r p_r \right) \Lambda = \frac{d}{dt} \mathrm{Tr} \sum_r \left(\epsilon_r q_r p_r - p_r q_r \right) \Lambda,$$

and since Λ is arbitrary anti-self-adjoint, the operator

$$C = \sum_r \left(\epsilon_r q_r p_r - p_r q_r \right) \tag{2.52}$$

is conserved by the evolution generated by the trace Hamiltonian $\mathbf{H}$. The operator can be also equivalently written as

$$C = \sum_{r \in B} [q_r, p_r] - \sum_{r \in F} \{p_r, q_r\} = \sum_{rs} \left(x_r \omega_{rs} x_s \right), \tag{2.53}$$

where the first sum on the right is over bosonic and the second over fermionic degrees of freedom.

The conserved operator C does not change, if general transformations of the form $A \mapsto R^{-1} A R$ for $R \in \mathcal{G} = GL(G, N)$ are considered. In that case the generator Λ is an arbitrary matrix, and the infinitesimal transformation becomes (2.51) with U replaced by R and U^+ by R^{-1},

$$R = \exp(\varepsilon\Lambda) = I + \varepsilon\Lambda + o(\varepsilon), \qquad R = \exp(-\varepsilon\Lambda) = I - \varepsilon\Lambda + o(\varepsilon).$$

The C operator is the Noether charge associated with the global unitary invariance. Generally, if the trace Lagrangian $\mathbf{L}$ is invariant under the infinitesimal transformations,

$$q_r \mapsto q_r + \varepsilon\Delta_r,$$
$$\dot{q}_r \mapsto \dot{q}_r + \varepsilon\dot{\Delta}_r,$$

then with the use of the Euler-Lagrange equations (2.22)

$$0 = \delta\mathbf{L} = \mathrm{Tr}\sum_r \left(\frac{\delta\mathbf{L}}{\delta q_r}\varepsilon\Delta_r + \frac{\delta\mathbf{L}}{\delta\dot{q}_r}\varepsilon\dot{\Delta}_r\right) = \frac{d}{dt}\mathrm{Tr}\sum_r \left(\frac{\delta\mathbf{L}}{\delta\dot{q}_r}\Delta_r\right),$$

and there is a conserved Noether trace charge

$$\mathbf{J} = \mathrm{Tr}\sum_r \left(\frac{\delta\mathbf{L}}{\delta\dot{q}_r}\Delta_r\right). \tag{2.54}$$

In the case of unitary transformations (2.51), the conserved trace charge reads

$$J = -\mathrm{Tr}\sum_r \left(\frac{\delta\mathbf{L}}{\delta\dot{q}_r}[\Lambda, q_r]\right) = \mathrm{Tr}\sum_r \left(p_r\Lambda q_r - p_r q_r\Lambda\right) = \mathrm{Tr}\sum_r \left(p_r q_r - \epsilon_r q_r p_r\right)\Lambda = \mathrm{Tr}C\Lambda,$$

which together with the arbitrariness of Λ again implies the conservation of the operator C, (2.52).

Usually, conserved quantities can be used as generators of those transformations, with respect to which they are conserved. The conserved operator C can be used as a generator of global unitary canonical transformations on the operator phase space. General infinitesimal canonical transformations in trace dynamics are defined for any self-adjoint bosonic operator G by

$$\delta q_r = \varepsilon\epsilon_r\frac{\delta\mathbf{G}}{\delta q_r}, \qquad \delta q_r = -\varepsilon\frac{\delta\mathbf{G}}{\delta p_r}, \qquad \delta x_r = \varepsilon\sum_s \omega_{rs}\frac{\delta\mathbf{G}}{\delta x_s}, \tag{2.55}$$

with ε an infinitesimal parameter. If $q_r(\tau)$ and $p_r(\tau)$ is a curve in the operator phase space parametrized by τ and letting $\varepsilon \to 0$ with

$$\frac{1}{\varepsilon}\delta x_r \equiv \frac{1}{\varepsilon}\left(x_r(\tau+\varepsilon) - x_r(\tau)\right) \to \frac{\partial x_r}{\partial\tau},$$

the infinitesimal transformation (2.55) becomes

$$\frac{\partial}{\partial\tau}q_r(\tau) = \epsilon_r\frac{\delta\mathbf{G}}{\delta q_r}, \qquad \frac{\partial}{\partial\tau}q_r(\tau) = -\frac{\delta\mathbf{G}}{\delta p_r}, \qquad \frac{\partial}{\partial\tau}x_r(\tau) = \sum_s \omega_{rs}\frac{\delta\mathbf{G}}{\delta x_s}. \tag{2.56}$$

It is a Hamiltonian phase flow (2.24) parametrized by τ and generated by the trace functional $\mathbf{G}$. We get an infinitesimal unitary transformation with the special choice

$$\mathbf{G}_\Lambda = \mathrm{Tr}\Lambda C, \tag{2.57}$$

for Λ some bosonic anti-self-adjoint operator. Inserting $\mathbf{G}_\Lambda$ into (2.55),

$$\delta x_r = \varepsilon\sum_s \omega_{rs}\frac{\delta\mathrm{Tr}(\Lambda C)}{\delta x_s} = \varepsilon\sum_s \omega_{rs}\frac{\delta}{\delta x_s}\mathrm{Tr}\left(\Lambda\sum_{m,n} x_m\omega_{mn}x_n\right) =$$

$$= \varepsilon\sum_{s,n}\mathrm{Tr}\left(\omega_{rs}\omega_{sn}\epsilon_n x_n\Lambda + \omega_{rs}\omega_{ns}\Lambda x_n\right) = \varepsilon[\Lambda, x_r],$$

where we have used the properties (2.27) of ω in the last step. The resulting formula corresponds (up to the sign which can be nevertheless included into Λ) to the unitary transformation (2.51) and can be written $x_r \mapsto \exp(\varepsilon\Lambda)x_r$.

The infinitesimal transformation of a trace functional $\mathbf{A}$ generated by another trace functional $\mathbf{G}$ (bosonic, even grade) can be written in direct analogy with (2.28) as

$$\mathbf{A} \mapsto \mathbf{A} + \delta\mathbf{A} = \mathbf{A} + \varepsilon \sum_{r=1}^{2n} \mathrm{Tr}\left(\frac{\delta \mathbf{A}}{\delta x_r}\delta x_r\right)\mathbf{A} + = \varepsilon \sum_{r,s=1}^{2n}\left(\frac{\delta \mathbf{A}}{\delta x_r}\omega_{rs}\frac{\delta \mathbf{G}}{\delta x_s}\right) = \mathbf{A} + \varepsilon\{\mathbf{A},\mathbf{G}\},$$

hence the difference and the phase flow is given by the generalized Poisson bracket as

$$\delta\mathbf{A} = \varepsilon\{\mathbf{A},\mathbf{G}\}, \qquad \frac{\partial \mathbf{A}(\tau)}{\partial \tau} = \{\mathbf{A},\mathbf{G}\}. \tag{2.58}$$

For the case of unitary transformations, $\mathbf{G} = \mathbf{G}_\Lambda$, which leave the trace functional $\mathbf{A}$ unchanged, (2.58) implies

$$0 = \delta\mathbf{A} = \varepsilon\{\mathbf{A},\mathbf{G}_\Lambda\}.$$

But this can be also interpreted as that $\mathbf{G}_\Lambda$ is invariant under canonical transformations generated by any $\mathbf{A}$ which is unitary invariant,

$$\delta\mathbf{G}_\Lambda = \varepsilon\{\mathbf{G}_\Lambda,\mathbf{A}\} = 0,$$

and since Λ in (2.57) is arbitrary anti-self-adjoint, the operator C is invariant under any canonical transformation with unitary invariant generator (when Λ is anti-self-adjoint).

The generalized Poisson bracket of two generators of unitary transformations $\mathbf{G}_\Lambda$ and $\mathbf{G}_\Sigma$ given by (2.57) can be evaluated with the help of (2.51) and (2.56) as

$$\{\mathbf{G}_\Lambda,\mathbf{G}_\Sigma\} = \mathrm{Tr}\sum_{rs}\frac{\delta \mathbf{G}_\Lambda}{\delta x_r}\omega_{rs}\frac{\delta \mathbf{G}_\Sigma}{\delta x_s} = \mathrm{Tr}\sum_{r}\left(\frac{\delta \mathbf{G}_\Lambda}{\delta x_r}[\Sigma,x_r]\right) = \mathrm{Tr}\sum_{rs}[\Lambda,x_r]\omega_{rs}[\Sigma,x_s]) =$$

$$= \mathrm{Tr}\sum_{rs}\omega_{rs}\left(\Lambda x_r\Sigma x_s + x_r\Lambda x_s\Sigma - \Lambda x_r x_s\Sigma - x_r\Lambda\Sigma x_s\right) = \mathrm{Tr}\sum_{rs}\omega_{rs}\left(-\Sigma\Lambda x_r x_s - \Lambda\Sigma x_s x_r\epsilon_r\right) =$$

$$= \mathrm{Tr}[\Lambda,\Sigma]C = \mathbf{G}_{[\Lambda,\Sigma]},$$

where we were using the cyclic identities under the trace and the properties of the ω matrix (2.27). The algebra of unitary transformation generators $\mathbf{G}_\Lambda$ under the generalized Poisson bracket (a subalgebra of generators of general canonical transformations) is therefore isomorphic to the subalgebra of anti-self-adjoint matrices Λ under matrix commutation.

The conserved operator C plays a central role in the emergence of quantum theory from the statistical mechanics of matrix models with trace dynamics as the underlying deterministic theory. For that sake it is needed that the statistical ensemble average of C is anti-self-adjoint or at least with only small self-adjoint contribution. This can be achieved by assigning suitable adjointness properties to the operators representing bosonic and fermionic degrees of freedom. If bosonic and fermionic variables are represented by self-adjoint or anti-self-adjoint, the resulting C is always anti-self-adjoint. An alternative assignment is to take the bosonic (anti-)self-adjoint and the fermionic variables arbitrary with the constraint $p_r = q_r^+$. This has of course implications for the kinetic part of the fermionic Lagrangian $\mathbf{L}_{kin,F}$, for example it could have the form

$$\mathbf{L}_{kin,F} = \mathrm{Tr}\sum_{r,s\in F} q_r^+ A_{rs}\dot{q}_s, \tag{2.59}$$

where A_{rs} for $r,s=1,2,\ldots,n$ are constant even grade matrices. Generally the trace Lagrangian is required to be real and bosonic up to a time derivative, which vanishes in the action integral. This condition holds, if the matrices A_{rs} are chosen such that $A_{rs}^+ = A_{sr}$, since then

$$\mathbf{L}_{kin,F}^+ = -\mathrm{Tr}\sum_{r,s\in F}\left(\dot{q}_s^+ A_{rs}^+ q_r\right) = -\frac{d}{dt}\left(\sum_{r,s\in F} q_s^+ A_{rs}^+ q_r\right) + \mathrm{Tr}\sum_{r,s\in F} q_r A_{sr}^+ \dot{q}_r = \mathbf{L}_{kin,F} + \frac{d}{dt}\left(\ldots\right).$$

With this Lagrangian, the canonical momentum is

$$p_r = \frac{\delta \mathbf{L}}{\delta \dot{q}_r} = \sum_{r\in F} q_r^+ A_{rs},$$

and the resulting Hamiltonian is independent of the constant matrices A_{rs} and so (as a function of $\{q_r\}$ and $\{p_r\}$) it is unitary invariant. The bosonic part of C is anti-self-adjoint and the fermionic is

$$C_F = -\sum_{s\in F}\{q_s, p_s\} = -\sum_{r,s\in F}\{q_s, q_r^+ A_{rs}\} = -\sum_{r,s\in F}\left(q_s q_r^+ A_{rs} + q_r^+ A_{rs} q_s\right),$$

This, together with

$$C_F^+ = \sum_{r,s\in F}\left(A_{sr} q_r q_s^+ + q_s^+ A_{sr} q_r\right),$$

gives the self-adjoint part of C as

$$C_F^{sa} = \frac{1}{2}\left(C_F + C_F^+\right) = \sum_{r,s\in F}\left[A_{sr}, q_r q_s^+\right]. \tag{2.60}$$

For the special choice $A_{rs} = \delta_{rs} I$, the C operator is purely anti-self-adjoint, and for more general choice it can have a self-adjoint part. The trace of C vanishes with any choice of adjointness assignments for the dynamical variables, and the self-adjoint and anti-self-adjoint part of C is conserved separately.

2.5.2 The conserved "number" quantities

Apart from the trace Hamiltonian, there are additional conserved trace quantities in the case, when the Hamiltonian operator is constructed from equal number of the dynamical q_r and p_r variables. Denote the number of occurrences of q_r in $\mathbf{H}$ as n_{q_r} and the number of p_r as n_{p_r}. Then from the definition of the derivative of a trace quantity with respect to an operator,

$$\mathrm{Tr}\frac{\delta \mathbf{H}}{\delta q_r} q_r = n_{q_r}\mathbf{H}, \qquad \mathrm{Tr}\frac{\delta \mathbf{H}}{\delta p_r} p_r = n_{p_r}\mathbf{H}, \tag{2.61}$$

since each monomial with k operators q_r yields k terms in the differential $\delta\mathbf{H}$ with respect to variations of q_r, and this differential is obtained from (2.61) by $q_r \mapsto \delta q_r$ on the right. Then if the number of fermionic q-s and p-s in the trace Hamiltonian are equal, there is conserved trace "fermion number" N_F, and similarly if the bosonic variables are balanced, there is conserved trace "boson number" N_B,

$$N_F = \frac{1}{2}\mathrm{Tr}\sum_{r\in F}[q_r, p_r] = \mathrm{Tr}\sum_{r\in F} q_r p_r, \qquad N_B = \frac{1}{2}\mathrm{Tr}\sum_{r\in B}\{q_r, p_r\} = \mathrm{Tr}\sum_{r\in B} q_r p_r. \tag{2.62}$$

The conservation follows from taking the time derivatives,

$$\dot{N}_F = \mathrm{Tr}\sum_{r\in F}(\dot{q}_r p_r + q_r\dot{p}_r) = \mathrm{Tr}\sum_{r\in F}\left(-\frac{\delta\mathbf{H}}{\delta p_r}p_r + \frac{\delta\mathbf{H}}{\delta q_r}q_r\right) = \sum_r(-n_{p_r} + n_{q_r})\mathbf{H} = 0,$$

where we have used the Hamilton equations for fermionic degrees of freedom, the fact that the fermionic variables anticommute under the trace, and (2.61). Similarly for the bosonic case,

$$\dot{N}_B = \mathrm{Tr}\sum_{r\in F}(\dot{q}_r p_r + q_r\dot{p}_r) = \mathrm{Tr}\sum_{r\in F}\left(\frac{\delta\mathbf{H}}{\delta p_r}p_r - \frac{\delta\mathbf{H}}{\delta q_r}q_r\right) = \sum_r(n_{p_r} - n_{q_r})\mathbf{H} = 0.$$

It is obvious, that if the numbers of q_r and p_r are balanced separately for some r, then the quantity $N_r = q_r p_r$ is conserved too. Also in case of local field theories, if the numbers are balanced locally, there are local conserved "number" charges.

In the case of complex underlying vector space and the fermionic adjointness assignment $p_r = q_r^+$, the conserved trace "fermion number" N_F is the Noether charge associated with c-number rephasings of the dynamical variables,

$$p_r \mapsto e^{i\varepsilon}p_r, \qquad q_r \mapsto e^{i\varepsilon}q_r.$$

If the trace Lagrangian $\mathbf{L}$ is invariant with respect to these rephasings, that is, if the number of fermionic q-s and p-s are balanced in the trace Lagrangian, then the corresponding conserved Noether charge is (2.54) with $\Delta_r = iq_r$,

$$-iN_F = -i\sum_{r\in F}\left(\frac{\delta\mathbf{L}}{\delta\dot{q}_r}iq_r\right) = -\mathrm{Tr}\sum_{r\in F}p_r q_r = \mathrm{Tr}\sum_{r\in F}q_r p_r = \mathrm{Tr}\sum_{r\in F}q_r q_r^+.$$

2.5.3 Conserved quantities in a trace dynamics field theory

If the trace dynamics is treated as a continuum spacetime field theory, the degrees of freedom become labeled by composite indices $r = (x^\mu, l)$, where $x = (x_0, \vec{x})$ is the spacetime point, and l numbers the field components. The Greek letters denote Lorentz indices, and they are lowered and raised by the Minkowski metric tensor $\eta = \mathrm{diag}(1, -1, -1, -1)$. The Einstein summation convention will be used, the summation over repeating upper and lower Greek indices is understood.

To deal with the uncountably infinite degrees of freedom that arise in this way, it is customary to impose a kind of locality condition by linking the spacetime points with the field amplitudes by a continuous and differentiable function $q_l : x^\mu \mapsto q_l(x^\mu)$. The continuity of q_l gives the topology of the spacetime to the infinite dimensional configuration space consisting of $\{q_{x^\mu,l}\}_{x^\mu,l}$ for all x and l. The state of the system (field) is now given by a correlation between the variables x^μ and the values of q_l, i.e. by a surface in a finite dimensional space. The interaction between individual degrees of freedom enters the dynamics only through the spacetime derivatives $\partial_\mu q_l(x)$, which allows to formulate the dynamics through the trace Lagrangian density $\mathcal{L}$,

$$\mathbf{L} = \int d^3x\mathcal{L}(\{q_l(x)\}, \{\partial_\mu q_l(x)\}). \tag{2.63}$$

The variation of the action (2.21) with respect to $q_l(x)$, with $q_l(x)$ vanishing in spatial infinity,

$$\delta\mathbf{S} = \int d^4x\mathrm{Tr}\sum_l\left(\frac{\delta\mathcal{L}}{\delta q_l(x)}\delta q_l(x) + \frac{\delta\mathcal{L}}{\delta\partial_\mu q_l(x)}\delta\partial_\mu q_l(x)\right),$$

gives the local operator Euler-Lagrange equations,

$$\frac{\delta \mathcal{L}}{\delta q_l(x)} - \frac{\partial}{\partial x^\mu} \frac{\delta \mathcal{L}}{\delta \partial_\mu q_l(x)} = 0. \tag{2.64}$$

In general any field, regarded as a mapping from spacetime to some target space T, is associated with a representation of the Poincaré group on T,

$$\{a^\mu, \Lambda^{\mu\nu}\} \mapsto S,$$

where a is the fourvector of a spacetime translation, Λ represents a Lorentz transformation, and S acts on the field component indices. In the case of a trace dynamics field theory, $\{S_{kl}\}$ is a set of c-number multiples of the identity matrix I on the underlying vector space. Upon a Poincaré transformation $x^\mu \mapsto x'^\mu = \theta^\mu_{.\nu} x^\nu + a^\nu$, the field $x \mapsto q_k(x)$ becomes

$$q_k(x) \mapsto q'_k(x') = \sum_l S(\theta^{-1})_{kl} q_l(\theta^{-1}(x-a)). \tag{2.65}$$

There are conserved quantities associated with the transformation properties of the local trace Lagrangian density $\mathcal{L}$ with respect to the Poincaré transformations. The trace Lagrangian density is required to be a scalar field (a scalar function on the spacetime), i.e. upon transformation $x \mapsto x'$ the Lagrangian density $x \mapsto \mathcal{L}(x)$ becomes $x' \mapsto \mathcal{L}'(x') = \mathcal{L}(x(x'))$.

If the trace Lagrangian density is a scalar density with respect to spacetime translations, the conserved current is the trace energy-momentum tensor $\mathcal{T}^{\mu\nu}$ and the conserved charge is the energy-momentum trace four-vector $\mathbf{P}$. A spacetime translation is given by an infinitesimal constant four-vector a^μ,

$$x^\mu \mapsto x^{\mu'} = x^\mu + a^\mu,$$

which induces the transformation (2.65) of the fields q_l and their spacetime derivatives $\partial_\mu q_l$,

$$\begin{aligned} q_l(x) &\mapsto q_l(x+a) = q_l(x) + \partial_\nu q_l(x) a^\nu + o(a) \\ \partial_\mu q_l(x) &\mapsto \partial_\mu q_l(x+a) = \partial_\mu q_l(x) + \partial_\mu \partial_\nu q_l(x) a^\nu + o(a). \end{aligned}$$

The trace Lagrangian

$$\mathcal{L}(x) = \mathrm{Tr} \sum_l \mathcal{L}(\{q_l(x)\}, \{\partial_\mu q_l(x)\}) \tag{2.66}$$

changes to

$$\mathcal{L}(x'(x)) = \mathcal{L}(x) + \mathrm{Tr} \sum_l \left(\frac{\delta \mathcal{L}}{\delta q_l} \partial_\mu q_l a^\mu + \frac{\delta \mathcal{L}}{\delta \partial_\mu q_l} \partial_\mu \partial_\nu q_l a^\nu \right) + o(a).$$

Subtracting $\mathcal{L}(x)$ from $\mathcal{L}(x'(x))$, using the Euler-Lagrange equations, and dropping the higher order terms in a,

$$\partial_\mu \mathcal{L} a^\mu = \mathrm{Tr} \sum_l \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu q_l} \partial_\nu q_l \right) a^\nu.$$

Stripping away the arbitrary constant a_μ, we see that the trace energy-momentum tensor,

$$\mathcal{T}^{\mu\nu} = -\eta^{\mu\nu} \mathcal{L} + \mathrm{Tr} \sum_l \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu q_l} \partial^\nu q_l \right), \tag{2.67}$$

is a conserved current (with respect to μ),

$$\partial_\mu \mathcal{T}^{\mu\nu} = 0. \tag{2.68}$$

The conserved trace charge is then the trace energy-momentum four-vector,

$$\mathbf{P}^\nu = \int d^3x \mathcal{T}^{0\nu}, \tag{2.69}$$

with the trace Hamiltonian $\mathbf{H}$ being the time component $\mathbf{P}^0$ in the chosen Lorentz reference frame,

$$\mathbf{P}^0 = \int d^3x \left(-\mathcal{L} + \mathrm{Tr} \sum_l \frac{\delta \mathcal{L}}{\delta \partial_0 q_l} \partial^0 q_l \right) = \int d^3x \left(-\mathcal{L} + p_l q_l \right) = \mathbf{H},$$

since the conjugate momentum p_l is defined using the x^0 derivative as

$$p_l(x) = \frac{\delta \mathcal{L}}{\delta \partial_0 q_l}. \tag{2.70}$$

The components of the trace energy-momentum four-vector can be used as the trace generators of canonical transformations - spacetime translations, of which is the time evolution a special case.

If the trace Lagrangian density is a scalar density with respect to spacetime rotations, we get the spacetime rotation trace generators $\mathbf{M}^{\mu\nu}$ as the conserved charge. An infinitesimal spacetime rotation is given by

$$x^\mu \mapsto x'^\mu = x^\mu + \theta^\mu_{\ .\nu} x^\nu,$$

with $\theta^{\mu\nu}$ the infinitesimal rotation parameters given by an antisymmetric matrix. The rotation induces an infinitesimal transformation (2.65) on the field components q_l,

$$q_l(x) \mapsto \sum_m S_{lm}(\theta) q_m(x') = q_l(x) + \partial^\mu q_l(x) \theta_{\mu\nu} x^\nu - \frac{1}{2} \theta_{\mu\nu} \sum_m \Sigma^{\mu\nu}_{lm} q_m(x) + o(\theta), \tag{2.71}$$

and their spacetime derivatives, with the use of antisymmetry of $\theta_{\mu\nu}$,

$$\partial_\lambda q_l(x) \mapsto \partial_\lambda q_l(x) + \partial_\lambda \partial_\mu q_l(x) \theta_{\mu\nu} x^\nu - \frac{1}{2} \theta_{\mu\nu} \sum_m \Sigma^{\mu\nu}_{lm} \partial_\lambda q_m(x) + o(\theta). \tag{2.72}$$

The mapping S represents the action of the rotation group on the field components, and $\Sigma^{\mu\nu}_{lm}$ for $\mu, \nu = 0, 1, 2, 3$ represent its generators. The trace Lagrangian (2.66) changes to

$$\mathcal{L}(x') = \mathcal{L}(x) + \mathrm{Tr} \sum_l \left(\frac{\delta \mathcal{L}}{\delta q_l} \delta q_l + \frac{\delta \mathcal{L}}{\delta \partial_\mu q_l} \delta \partial_\mu q_l \right) + o(\theta),$$

and after the substitution for δq_r and $\delta \partial_\mu q_l$ from (2.71, (2.72), using the Euler-Lagrange equations, and some computation, we get for the difference $\delta_x \mathcal{L} = \mathcal{L}(x') - \mathcal{L}(x)$,

$$\partial_\mu (\mathcal{L} \theta^\mu_{\ .\nu} x^\nu) = \delta_x \mathcal{L} = \partial_\mu \left[\theta^\mu_{\ .\nu} x^\nu \mathcal{L} - \mathrm{Tr} \sum_l \frac{\delta \mathcal{L}}{\delta \partial_\mu q_l} \left(\partial^\sigma q_l \theta_{\sigma\nu} x^\nu - \frac{1}{2} \theta_{\sigma\nu} \sum_m \Sigma^{\sigma\nu}_{lm} q_m \right) \right].$$

Removing the arbitrary rotation parameters $\theta_{\mu\nu}$, and regrouping some terms with the use of the definition of trace energy-momentum tensor $\mathcal{T}^{\mu\nu}$ (2.67), we get the conserved trace current

$$M^{\mu\alpha\beta} = x^\alpha \mathcal{T}^{\mu\beta} - x^\beta \mathcal{T}^{\mu\alpha} + \mathrm{Tr} \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu q_l} \sum_m \Sigma^{\alpha\beta}_{lm} q_m \right), \tag{2.73}$$

$$\partial_\mu M^{\mu\alpha\beta} = 0. \tag{2.74}$$

Spatial integrations then give the conserved trace charges,

$$M^{\mu\nu} = \int d^3x M^{0\mu\nu}, \tag{2.75}$$

which are generators of the canonical transformations - spacetime rotations.

Note that the generators of Poincaré transformations (2.67) and (2.73) are both trace quantities, thus global unitary invariant, and therefore, according to a remark in section 2.5.1, the conserved C operator is Poincaré invariant.

2.6 Statistical treatment of trace dynamics

The deterministic evolution equation of trace dynamics gives a definite state of system at all times t, if the state at an initial time t_0 is known. If this is not the case, if the knowledge of the initial state is limited, we have to resort to statistical methods. The dynamical equations and commutation relations of quantum theory is expected ([1],[3]) to emerge, when statistical methods are applied to trace dynamics evolution and some further assumptions and "low energy" approximations are made. Although quantum mechanics is non-classical, the assumed underlying theory of such an emergent quantum theory is essentially a classical deterministic theory, formulated in the framework of trace dynamics. The probabilities in the statistical treatment of trace dynamics are therefore also classical. The state of a (deterministic[2]) system can be equivalently given either by a point in operator phase space at a given (e.g. initial) time slice, or by the whole graph $(\{q_r(t)\}, \{p_r(t)\}, t)$ of the motion. The state space is therefore taken as the operator phase space Ω at a given time slice, or a spacelike hypersurface in case of trace dynamics as a field theory.

Let us assume the system has n degrees of freedom, each represented by $N \times N$ matrices $\{q_r\}$ and their conjugate momenta $\{p_r\}$. Then the operator phase space can be represented by the space $\Omega = \mathbb{G}^{2nN^2}$, in any case it is required to be invariant under infinitesimal shifts $x_r \mapsto x_r + \delta x_r$. Each operator degree of freedom x_r, $r = 1, 2, \ldots, 2n$, is expected to spawn exactly one (quantum-mechanical) operator in the emergent quantum theory. The probability measure on Ω is estimated through the standard maximum entropy principle methods of statistical mechanics.

2.6.1 Operator phase space measure

Since we are dealing with deterministic systems, the state space is given by the operator phase space Ω. The natural measure on Ω is

$$d\mu(\{x_r\}) = \prod_{\epsilon rmn} d(x_r)^\epsilon_{mn}, \tag{2.76}$$

where $(x_r)^\epsilon_{mn} = (x_r)_{mn} + i(x_r)^\epsilon_{mn}$ is the decomposition of x_r into real ($\epsilon = 1$) and imaginary part ($\epsilon = -1$)[3]

$$dP(\{x_r\}) = \rho(\{x_r\}) d\mu(\{x_r\}). \tag{2.77}$$

According to the remark at the end of section 2.4, an analog of Liouville theorem apply also for trace dynamics, the operator phase space volume element is invariant with respect to any canonical

[2]Were the system not deterministic, we would be limited to the latter option only.

[3]In case of the real variables, ϵ is restricted to $\epsilon = 1$ only.

transformation (generated by any bosonic trace functional $\mathbf{G}$), in particular the time evolution, a Poincaré transformation or a global unitary transformation. This remains true, even when specific restrictions associated with the adjointness properties of $\{x_r\}$ are imposed on the measure (2.76).

The expansion of $\delta\mathbf{G}$ into components is

$$\delta\mathbf{G} = \mathrm{Tr}\frac{\delta\mathbf{G}}{\delta x_r}\delta x_r = \sum_{mn}\left(\frac{\delta\mathbf{G}}{\delta x_r}\right)_{nm}(\delta x_r)_{mn} = \sum_{mn}\left(\frac{\delta\mathbf{G}}{\delta x_r}\right)^{0}_{nm}(\delta x_r)^{0}_{mn} - \sum_{mn}\left(\frac{\delta\mathbf{G}}{\delta x_r}\right)^{1}_{nm}(\delta x_r)^{1}_{mn},$$

hence with the definition of the real/imaginary marker ϵ from above and the definition of the derivative with respect to an operator (2.17),

$$\left(\frac{\delta\mathbf{G}}{\delta x_r}\right)^{\epsilon}_{mn} = \epsilon\frac{\partial\mathbf{G}}{\partial(x_r)^{\epsilon}_{nm}}. \tag{2.78}$$

For a direct check of the Liouville theorem, consider a general infinitesimal canonical transformation (2.55),

$$x_r \mapsto x_r + \varepsilon\sum_s \omega_{rs}\frac{\delta\mathbf{G}}{\delta x_s},$$

with a trace functional $\mathbf{G}$ as its generator, which in components reads,

$$(x_r)^{\epsilon}_{mn} \mapsto (x_r)^{\epsilon}_{mn} + \varepsilon\sum_s \epsilon\omega_{rs}\left(\frac{\partial\mathbf{G}}{\partial x_s}\right)^{\epsilon}_{nm}.$$

The Jacobi matrix of this transformation is

$$J = \frac{\partial(x_{r'})^{\epsilon'}_{m'n'}}{\partial(x_r)^{\epsilon}_{mn}} = \delta_{\epsilon\epsilon'}\delta_{rr'}\delta_{mm'}\delta_{nn'} + \varepsilon\sum_s \epsilon'\omega_{r's}\frac{\partial^2\mathbf{G}}{\partial(x'_r)^{\epsilon'}_{m'n'}\partial(x_r)^{\epsilon}_{mn}},$$

and since in general the determinant of a matrix $J = 1 + \epsilon X$ is $\det J = 1 + \varepsilon\mathrm{Tr}X + o(\varepsilon)$,

$$\det J = 1 + \varepsilon\sum_{rsmn\epsilon}\epsilon\omega_{rs}\frac{\partial^2\mathbf{G}}{\partial(x_r)^{\epsilon}_{mn}\partial(x_s)^{\epsilon}_{nm}} + o(\varepsilon). \tag{2.79}$$

For bosonic variables the trace is zero, because ω_{rs} ((2.25) and (2.26)) is antisymmetric, whereas the second derivative is symmetric in r, s. For fermionic variables it is also zero, since ω_{rs} is symmetric, but the derivative is antisymmetric in r and s. Hence the Liouville theorem holds in the unrestricted case. Since with our definition of matrix conjugation $(x^{+}_r)^{\epsilon}_{mn} = \epsilon(x_r)^{\epsilon}_{nm}$, the determinant (2.79) can be rewritten also as

$$\det J = 1 + \varepsilon\sum_{rsmn\epsilon}\omega_{rs}\frac{\partial^2\mathbf{G}}{\partial(x_r)^{\epsilon}_{mn}\partial(x^{+}_s)^{\epsilon}_{mn}} + o(\varepsilon). \tag{2.80}$$

If the dynamical variables are restricted to self-adjoint or anti-self-adjoint, the measure is appropriately restricted to exclude the redundant elements. The m, n indices in the measure (2.76) is chosen such, that for self-adjoint variables $m > n$ and $m = n$ only with $\epsilon = 1$, and for anti-self-adjoint variables $m > n$ and $m = n$ only with $\epsilon = -1$. Since the trace term in the expansion of $\det J$ vanishes only by virtue of the symmetry and antisymmetry in r and s, it follows from (2.80) that the Liouville theorem holds even with these adjointness restrictions.

Another assignment is to let the bosonic variables be self-adjoint or anti-self-adjoint, an the fermionic arbitrary with $p_r = q^{+}_r$. Then measure in the fermionic sector is restricted to the q_r

fermionic variables only, and the bosonic sector is as above. Since ω_{rs} is nonzero (and equal to -1) for conjugate r and s only, the fermionic part of the trace term in (2.79) becomes

$$-\varepsilon \sum_{rmn\epsilon} \epsilon \frac{\partial^2 \mathbf{G}}{\partial (q_r)^{\epsilon}_{mn} \partial (q_r)^{\epsilon}_{nm}} = 0, \tag{2.81}$$

by the property of Grassmann odd grade numbers, and so the Liouville theorem holds for this assignment too.

2.6.2 Statistical ensembles

The generalized Liouville theorem implies, that the phase space measure is invariant under a global unitary transformation, and the time evolution of probability measure is given solely by the time dependence of the probability density. If the probability measure happen to be time independent, then ρ can depend on conserved quantities only. For trace dynamics, these are mainly the C operator and the trace Hamiltonian $\mathbf{H}$ or the trace Poincaré generators $\mathbf{P}^{\nu}$ and $\mathbf{M}^{\mu\nu}$ (in case of trace dynamics field theory with Poincaré invariant trace Lagrangian). For specific operator Hamiltonians, when the number of fermionic and/or bosonic $\{q_r\}$ are balanced with $\{p_r\}$, there are the additional conserved "number" trace quantities $\mathbf{N}_F$ and/or $\mathbf{N}_B$. Hence the general *static* ensemble is of the form

$$\rho = \rho(\lambda, C, \tau_{\mu}, \mathbf{P}^{\mu}, \kappa_{\mu\nu}, M^{\mu\nu}, \alpha, \mathbf{N}_F, \beta, \mathbf{N}_B). \tag{2.82}$$

The operator-valued parameter λ and the c-number parameters τ_{μ}, $\kappa_{\mu\nu}$, α, and β are the respective stochastic conjugate quantities. Note that the trace Hamiltonian $\mathbf{H}$ is already included in (2.82) as $\mathbf{P}^0$, and it turns out that λ and C appear in ρ only through $\mathrm{Tr}(\lambda C)$, hence we have

$$\rho = \rho(\mathrm{Tr}\lambda C, \tau_{\mu}\mathbf{P}^{\mu}, \kappa_{\mu\nu}M^{\mu\nu}, \alpha\mathbf{N}_F, \beta\mathbf{N}_B). \tag{2.83}$$

The parameters τ_{μ} are constants of the ensemble, but τ transforms as a covector upon a Lorentz transformation, and similarly for $\kappa_{\mu\nu}$. The statistical ensemble is therefore Lorentz covariant.

Moreover, since the matrix elements of the operator parameter λ are statistically conjugated to the matrix elements of C, the tracelessness and adjointness properties of λ is carried over from C to λ, i.e. $\mathrm{Tr}\lambda = 0$, and if C is anti-self-adjoint, so is the parameter λ. If C has dimension of action, the dimension of λ is inverse action, and its statistical ensemble average is conjugated to be associated with the Planck constant in the emergent quantum theory.

The ensemble averages of an operator quantity $A = A(\{x_r\})$ and of a trace quantity $\mathbf{A}$ are given by

$$\langle A \rangle_{\rho} = \int_{\Omega} A\rho d\mu, \qquad \langle \mathbf{A} \rangle_{\rho} = \int_{\Omega} \mathbf{A}\rho d\mu. \tag{2.84}$$

There is an important observation. If A is constructed from the dynamical variables $\{x_r\}$ using c-number coefficients only, the matrix structure of its average $\langle A \rangle_{\rho}$ is a function of the operator parameter λ only. This follows, if we make a unitary transformation $x_r \mapsto U^{+}x_r U$ of all the dynamical variables, with U an even grade unitary matrix. The operator phase space measure is always unitary invariant, $d\mu \mapsto d\mu$. The assumption that A does not contain matrix-valued constant coefficients implies $A \mapsto U^{+}AU$. Since the probability density ρ, (2.82), contains the fixed operator λ, we have for general U,

$$\rho(\{x_r\}, \lambda) \mapsto \rho(\{U^{+}x_r U\}, \lambda) = \rho(\{x_r\}, U\lambda U^{+}). \tag{2.85}$$

The unitary transformation of integration variables with general U does not change the value of the integral (2.84),

$$F_A(\lambda) = \int_\Omega A\rho d\mu \mapsto U^+ \int_\Omega A\rho(\{x_r\}, U\lambda U^+)d\mu U = U^+ F_A(U\lambda U^+)U = F_A(\lambda), \tag{2.86}$$

hence the matrix structure of F_A is given only by the operator λ. In particular $[\lambda, F_A(\lambda)] = 0$, and $[U_e, F_A(\lambda)] = 0$ for all unitary U_e commuting with λ.

2.6.3 A static ensemble

For simplicity, the Lorentz frame is chosen such that the statistical ensemble is at rest, i.e. there is no mean spatial translation ($\mathbf{P}^k = 0$ for $k = 1, 2, 3$), mean rotation ($\mathbf{J}^k = \varepsilon^{kmn}\mathbf{M}_{mn} = 0$) nor mean acceleration ($\mathbf{K}^k = \mathbf{M}^{0k} = 0$). This allows to simplify the static ρ to

$$\rho = \rho(\lambda, C, \tau, \mathbf{H}, \alpha, \mathbf{N}_F, \beta, \mathbf{N}_B), \tag{2.87}$$

but it also apparently breaks the relativistic covariance of the statistical ensemble. However, this will not do any harm as long as we keep the chosen coordinate frame fixed; otherwise the general ρ, (2.82), should be used instead.

The probability distribution is estimated using the maximum entropy principle. The entropy

$$S = -\langle \log \rho \rangle_\rho = \int_\Omega \rho \log \rho d\mu, \tag{2.88}$$

is maximized subject to the constraints imposed by giving the average values $\langle C \rangle_\rho$, $\langle \mathbf{H} \rangle_\rho$ and $\langle \mathbf{N}_{B/F} \rangle_\rho$, together with the normalization condition $\langle 1 \rangle_\rho = 1$. The maximum of the functional

$$F(\rho) = S - \theta\langle 1 \rangle_\rho - \sum_{mn} \lambda^T_{mn} \langle C_{mn} \rangle_\rho - \tau \langle \mathbf{H} \rangle_\rho - \alpha \langle \mathbf{N}_B \rangle_\rho - \beta \langle \mathbf{N}_F \rangle_\rho,$$

where λ^T, τ, α and β are the Lagrange multipliers, is found to be

$$\rho = \frac{1}{Z} \exp\left(-\tau \mathbf{H} - \mathrm{Tr}(\lambda C) - \alpha \mathbf{N_F} - \beta \mathbf{N_B}\right), \tag{2.89}$$

with Z the partition function,

$$Z = \int_\Omega \rho d\mu = \int_\Omega \exp\left(-\tau \mathbf{H} - \mathrm{Tr}(\lambda C) - \alpha \mathbf{N_F} - \beta \mathbf{N_B}\right) d\mu. \tag{2.90}$$

The Lagrange multipliers are to be chosen such that their statistically conjugated quantities have their prescribed average values. Depending on the specific form of the Hamiltonian, $\mathbf{N}_F$ or $\mathbf{N}_B$ may not be a conserved quantity, and should be dropped, if one wants to maintain a static distribution. The trace Hamiltonian has a secondary role to ensure the convergence of Z. It should be positive and grow sufficiently fast in the q and p variables, since the C operator is not positive definite. In general, the partition function Z could become negative due to the fermionic integrations, but this only sets the correct sign to the probability density after the fermionic variables are integrated out. No constraints on the fermionic variables and/or the fermionic part of the Hamiltonian is placed by the requirement $Z > 0$.

The distribution ρ has the usual properties as in classical statistical mechanics, appropriately generalized to trace dynamics. From (2.82) it follows that the ensemble averages of C, $\mathbf{H}$, $\mathbf{N}_F$ and $\mathbf{N}_B$ can be expressed using the partition function as

$$\langle C \rangle_\rho = -\frac{\delta \log Z}{\delta \lambda}, \quad \langle \mathbf{H} \rangle_\rho = -\frac{\partial \log Z}{\partial \tau}, \quad \langle \mathbf{N}_F \rangle_\rho = -\frac{\partial \log Z}{\partial \alpha}, \quad \langle \mathbf{N}_B \rangle_\rho = -\frac{\partial \log Z}{\partial \beta}. \tag{2.91}$$

2.6.4 Ensemble average of the C operator

The C operator is always given by a traceless even grade matrix. If the adjointness properties of the dynamical variables are such, that the C operator is anti-self-adjoint, so is its ensemble average. In the case when the Grassmann algebra is over complex numbers, there is always a unitary matrix U, for which $U^+\langle C\rangle_\rho U$ is purely imaginary diagonal matrix. This matrix can be decomposed into a product of a real positive "magnitude" matrix D and an imaginary "phase" matrix $i_{\rm eff}$,

$$\langle C\rangle_\rho \mapsto U^+\langle C\rangle_\rho U = Di_{\rm eff}, \tag{2.92}$$

with $i_{\rm eff}^+ i_{\rm eff} = I$ and $i_{\rm eff}^2 = -I$.

The diagonal matrix $Di_{\rm eff}$ is determined uniquely up to a permutation of the vector space basis. A crucial assumption of trace dynamics is made at this point. Because of symmetry reasons - the physics and the statistical distribution of the dynamical variables should be such, that no direction in the underlying vector space $\mathcal{M}$ would be preferred among others. Therefore the ensemble average of C is assumed to be a c-number times an anti-self-adjoint *unitary* matrix. This is the most symmetric anti-self-adjoint matrix, i.e. not preferring any direction in the underlying vector space, and determined up to unitary equivalence. Since the C operator is traceless, this assumption also implies ${\rm Tr} i_{\rm eff} = 0$. In other words, the "magnitude" matrix is assumed to be a c-number times the identity matrix I, $D = \hbar I$.

The matrix $i_{\rm eff}$ is composed from equal number of $+i$ and $-i$, from which it follows, that the dimension N of the underlying vector space $\mathcal{M}$ has to be even. Therefore $\langle C\rangle_\rho$ in the diagonal basis becomes

$$U^+\langle C\rangle_\rho U = \hbar i_{\rm eff} I = \hbar i_{\rm eff}, \qquad i_{\rm eff} = \begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix} \otimes I_{N/2}. \tag{2.93}$$

The notation in (2.93) is chosen in this particular way, because $\hbar$ will appear as the reduced Planck constant in the emergent quantum theory, and $i_{\rm eff}$ will effectively give rise to the imaginary unit associated with the emergent complex Hilbert space[4]. The imaginary unit appears even in the physically more appealing case of the Grassmann algebra over *real* numbers. Although an anti-self-adjoint (now skew-symmetric) matrix can no longer be diagonalized over real numbers, we can still obtain an analog of decomposition (2.93). The "symmetry" assumption from above gives (up to a c-number multiple) the condition $\langle C\rangle_\rho^+\langle C\rangle_\rho = I$ (unitarity) and $\langle C\rangle_\rho^2 = -1$ (unitarity with self-adjointness). This implies that $\langle C\rangle_\rho$ is a c-number times a skew-symmetric orthogonal matrix, the most symmetric matrix in the sense from above. The dimension of $\mathcal{M}$ is even, and there exists an orthogonal matrix O (in fact a permutation matrix), such that in the corresponding new basis the $\langle C\rangle_\rho$ operator becomes represented by

$$O^+\langle C\rangle_\rho O = \hbar i_{\rm eff} I_N = \hbar i_{\rm eff}, \qquad i_{\rm eff} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes I_{N/2}, \tag{2.94}$$

with I_N the $N \times N$ identity matrix. This is one of a few differences between the real and complex underlying vector space $\mathcal{M}$ of trace dynamics. In both cases we have

$$\langle C\rangle_\rho = i_{\rm eff}\hbar, \quad {\rm Tr}\, i_{\rm eff} = 0, \quad i_{\rm eff}^+ i_{\rm eff} = 1, \quad i_{\rm eff}^2 = -1. \tag{2.95}$$

As noted above, the matrix structure of the ensemble average of any operator is a function of a single operator, the operator parameter λ of ρ. It follows that all operators which are ensemble

[4]That is why the anti-self-adjointness of C is sought for. Then the anti-self-adjoint unitary matrix $i_{\rm eff}$ is preferred in the above argument over an even more symmetric self-adjoint unitary matrix, a multiple of identity.

averages commute among themselves, and there is a basis of $\mathcal{M}$ in which all averages are diagonal. In particular the ensemble average of C is

$$\langle C\rangle_\rho = \int_\Omega C\rho d\mu = \hbar i_{\rm eff} = F_C(\lambda), \tag{2.96}$$

and all averages commute with $i_{\rm eff}$. The operator parameter acquires the matrix structure from its statistically conjugated operator $\langle C\rangle_\rho$, hence it is an anti-self-adjoint unitary matrix (or antisymmetric orthogonal in the real case), which is up to a c-number multiple equal to $i_{\rm eff}$,

$$\lambda = \lambda_0 i_{\rm eff}. \tag{2.97}$$

The ensemble average of any operator is then given by a linear combination of just two operators, $i_{\rm eff}$ and the unit operator I.

The dimension of the vector space $\mathcal{M}$ is $N = 2K$. Any matrix of an operator M on $\mathcal{M}$ can be decomposed as

$$M = \tau_+ M_+ + \tau_- M_- + \tau_a M_a + \tau_b M_b, \tag{2.98}$$

with

$$\tau_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.99}$$

or (in the case of complex $\mathcal{M}$) as

$$M = \frac{1}{2}(\sigma_0 + \sigma_3) \otimes M_+ + \frac{1}{2}(\sigma_0 - \sigma_3) \otimes M_- + \sigma_1 \otimes M_1 + \sigma_2 \otimes M_2, \tag{2.100}$$

where $\{\sigma_i\}$ are the Pauli σ-matrices,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.101}$$

with M_+, M_-, M_1, M_2, M_a and M_b some $K \times K$ matrices, and

$$M_1 = \frac{1}{2}(M_a + M_b), \qquad M_2 = \frac{i}{2}(M_a - M_b). \tag{2.102}$$

Denote the first two terms in (2.98) by $M_{\rm eff}$ and the last two by $M_{ab} = M_{12}$ so that

$$M = M_{\rm eff} + M_{12}, \quad M_{\rm eff} = \tau_+ M_+ + \tau_- M_-, \quad M_{ab} = \tau_a M_a + \tau_b M_b. \tag{2.103}$$

This splits any operator M into the part $M_{\rm eff}$ which commutes with $i_{\rm eff}$ and the part $M_{ab} = M_{12}$ which anticommutes with $i_{\rm eff}$,

$$\begin{aligned} [M_{\rm eff}, i_{\rm eff}] &= 0 \qquad & \{M_{\rm eff}, i_{\rm eff}\} &= 2i\tau_+ \otimes M_+ - 2i\tau_- \otimes M_- = 2i_{\rm eff} M_{\rm eff}, \\ \{M_{12}, i_{\rm eff}\} &= 0 \qquad & [M_{12}, i_{\rm eff}] &= -2i\tau_a \otimes M_a + 2i\tau_b \otimes M_b = -2i_{\rm eff} M_{12}. \end{aligned} \tag{2.104}$$

Therefore the ensemble average through $i_{\rm eff}$ effectively splits the vector space $\mathcal{M}$ into two sectors, in which the averaged operators act with opposite sign. The part of an operator M, which mixes these two sectors, M_{12}, vanishes upon taking the average. The anticommutator or commutator with $i_{\rm eff}$ can be used to form the projection $M \mapsto M_{\rm eff}$ or $M \mapsto M_{12}$ respectively,

$$M_{\rm eff} = -\frac{1}{2} i_{\rm eff}\{M, i_{\rm eff}\}, \qquad M_{12} = \frac{1}{2} i_{\rm eff}[M, i_{\rm eff}]. \tag{2.105}$$

The projection $M \mapsto M_{\rm eff}$ removes a part M_{12} of M which vanishes in the average over the statistical ensemble (2.89), hence it is called the effective projection. Note that only the effective projection $M_{\rm eff}$ of M affects the trace of M,

$$\mathrm{Tr}M = \mathrm{Tr}M_{\rm eff}, \qquad \mathrm{Tr}M_{12} = 0. \tag{2.106}$$

However, although the part M_{12} of M does not influence the ensemble average $\langle M\rangle_\rho$ nor ensemble average of another operator A, it nevertheless contributes to the covariance of M and A,

$$\mathrm{Cov}(M,A) = \langle (M - \langle M\rangle_\rho)(A - \langle A\rangle_\rho)\rangle_\rho,$$

since $[M_{12}A_{12}, i_{\rm eff}] = 0$.

2.6.5 Unitarily fixed ensembles

The presence of the fixed operator λ in the probability density breaks the original unitary invariance of trace dynamics, and picks a preferred basis in which all averages are diagonal matrices. A global unitary transformation $x_u \mapsto U^+ x_u U$ of the dynamical variables changes the average values of $\{x_u\}$, unless U commutes with λ. The assumed global unitary invariance can be viewed either as a kind of gauge invariance, the actual physical situation does not change upon taking such a global unitary transformation, or as a real change of the physical situation, which nevertheless has no influence on the dynamics. In the first case it is not a priori clear, whether the integrations should be carried out over the whole gauge symmetry group, or a unitary fixing should be introduced into the operator phase space measure.

A unitarily fixed phase space measure $d\widehat{\mu}$ is given by the decomposition of the unfixed measure $d\mu$,

$$d\mu = d[\widehat{U}]d\widehat{\mu}, \tag{2.107}$$

with $d[\widehat{U}]$ the Haar measure for integration over a given subgroup of the full group of global unitary transformations. The fixed averages are then evaluated with this subgroup factored away from the integration measure. It can be for example carried out by picking one or two dynamical variables, and restricting the original integration measure $d\mu$ and the integration range Ω, such that these dynamical variables are fixed and not integrated over. The fixed probability density $\widehat{\rho}$ and the fixed averages may depend on the particular selection of the variables, whose overall unitary transformation is fixed in the integrations.

The equation (2.86) admits that the unitarily fixed averages may involve constant matrix coefficients commuting with the subgroup, which is *not* fixed in the integration. If this subgroup was taken to be the whole group, i.e. the measure was completely unitarily fixed, the matrix structure of unitarily fixed averages would no longer be a function of λ only, but it could involve arbitrary constant matrix coefficients. If only the subgroup of all those operators that commute with λ is fixed, then all terms in ρ, including $\mathrm{Tr}\lambda C$, are invariant with respect to a global unitary transformation $x_u \mapsto U^+ x_u U$ for any U from this subgroup. Hence the unitarily fixed average of an operator A (constructed from the dynamical variables using c-number coefficients only),

$$\int_{\widehat{\Omega}} A\widehat{\rho}d\widehat{\mu}, \tag{2.108}$$

is the same as the unfixed one,

$$\int_\Omega A\rho d\mu = \int_\Omega A\rho d[U]d\widehat{\mu}, \tag{2.109}$$

unless A contains the variables x_f, on which the unitary fixing is performed.

The derivation of Ward identities in the next section requires the possibility to make arbitrary shifts of selected integration variables x_r. When unitary fixing is employed, the fixed variables x_f have to be chosen to be different from the variables x_r in the Ward identity (2.116). Therefore the unitary fixing can affect only the average values of those operators, which involve the variables x_f, such as the 'global' operators C, H or $N_{F/B}$. It is nevertheless assumed by [1], since the fixing is performed on one or two variables only, that the unitarily fixed average of the extensive C operator is assumed approximately unchanged. Moreover, the ensemble average of the C operator is a more physically fundamental quantity than its statistically conjugate operator λ. Therefore $\langle C\rangle_\rho$ should remain unchanged by the fixing, and from symmetry reasons, the λ parameter should then be unchanged too. By (2.109), any trace of fixed average of an operator is the same as the trace of unfixed average, and independent of the choice of the variables x_f. In particular, the trace quantities $\mathbf{H}$, $\mathbf{N}_F$ and $\mathbf{N}_B$ are preserved by any unitary fixing. As a result, the above unitary fixing has not any significant effect on the statistical properties of the trace dynamics matrix models, and it will not be considered further.

2.7 Ward identities

The quantum-theoretic behavior is expected to be an emergent property connected with statistical averages of the dynamical variables over a suitable ensemble. We are interested mainly with the time evolution law of the ensemble averages of x_r, and with the ensemble averages of two point correlations of the form $\langle x_r x_s\rangle_\rho$.

For future convenience, let us introduce sources j_r into the statistical ensemble distribution density (2.89),

$$\rho_j = \frac{1}{Z_j}\exp\left(-\tau\mathbf{H} - \mathrm{Tr}(\lambda C) - \alpha\mathbf{N_F} - \beta\mathbf{N_B} - \mathrm{Tr}\sum_r j_r x_r\right), \tag{2.110}$$

with Z_j the partition function with sources,

$$Z_j = \int_\Omega \rho_j d\mu = \int_\Omega \exp\left(-\tau\mathbf{H} - \mathrm{Tr}(\lambda C) - \alpha\mathbf{N_F} - \beta\mathbf{N_B} - \mathrm{Tr}\sum_r j_r x_r\right) d\mu. \tag{2.111}$$

This allows to express the ensemble average $\langle x_r\rangle_\rho$ of a dynamical variable x_s, the coefficient of j_r, by taking the derivative of Z with respect to the operator j_s, and setting all sources to zero, $j \mapsto 0$,

$$\langle x_s\rangle_\rho = \langle x_s\rangle_{\rho_{j=0}} = \left(\frac{\delta Z_j}{\delta j_s}\right)_{j=0}. \tag{2.112}$$

The Ward identities are based on the possibility of making an infinitesimal shift[5] $x_r \mapsto x_r + \delta x_r$ of the integration variables $\{x_r\}$ in

$$\int_\Omega \mathbf{A}(\{x_r\}) Z_j \rho_j(\tau, \lambda, \ldots, \{x_r\}) d\mu(\{x_r\}) \tag{2.113}$$

without changing the value of the integral. Since the integration measure $d\mu$ is invariant with respect to this shift, the difference of shifted and the original integral gives

$$0 = \int_\Omega \left[(\mathbf{A}Z_j\rho_j)(\{x_r+\delta x_r\}) - (\mathbf{A}Z_j\rho_j)(\{x_r\})\right] d\mu(\{x_r\}) = \int_\Omega \sum_s \mathrm{Tr}\left(\frac{\delta(\mathbf{A}Z_j\rho_j)}{\delta x_s}\delta x_s\right) d\mu. \tag{2.114}$$

[5] As always, assuming all δx_r of the same grade and adjointness properties as x_r.

The variation δx_s of x_s is arbitrary, hence we obtain the equations,

$$\int_\Omega \frac{\delta(\mathbf{A}Z_j\rho_j)}{\delta x_s}d\mu = 0 \tag{2.115}$$

for all $s = 1, 2, 3, \ldots, 2n$. Using the Leibniz property (2.19) of the derivative with respect to an operator, and dividing out the nonzero Z_j factor which appears in both terms, we can rewrite the equations (2.115) to obtain the general Ward identities,

$$\left\langle \frac{\delta \mathbf{A}}{\delta x_r} + \mathbf{A}\frac{\delta \log(Z_j\rho_j)}{\delta x_r} \right\rangle_{\rho_j} = 0. \tag{2.116}$$

These identities can give us an information about the statistical properties of our system for some suitable choices of A.

2.7.1 Variation of sources

Inserting a particular trace functional $\mathbf{A}$ into the general Ward identity (2.116), and evaluating the derivatives, we obtain an expression of the form,

$$\langle Dx_u + \mathbf{A}\sum_r \omega_{ur} j_r \rangle_{\rho_j} = 0, \tag{2.117}$$

where Dx_u denote all terms not containing the sources $\{j_r\}$, and it depends on the particular choice of $\mathbf{A}$. We now use the sources to extend this identity further. Such an extended formulation of Ward identity is of interest for the argument of emergence of quantum theory. By variation of the sources before setting to zero, it can be arranged that (2.117) holds even when inserted between two polynomials[6] in the dynamical variables S_L and S_R before taking the zero source average,

$$\langle S_L Dx_u S_R \rangle_\rho = 0.$$

This is an average of an operator quantity, and it is convenient to contract the two open matrix indices by multiplying by an arbitrary matrix denoted by δj_u and taking the trace. By picking a suitable value for δj_u, it is always possible to recover any matrix component of the original operator, hence also the entire matrix. So multiplying (2.117) by δj_u and taking the trace,

$$\left\langle \mathrm{Tr}(\delta j_u Dx_u) - \mathbf{A}\sum_r \omega_{ur}\mathrm{Tr}(\delta j_u j_r) \right\rangle_{\rho_j} = 0. \tag{2.118}$$

The result can be multiplied further with the nonzero bosonic c-number Z_j to rewrite (2.118) equivalently as an integral

$$\int_\Omega \left[\mathrm{Tr}(\delta j_u Dx_u) - \mathbf{A}\sum_r \omega_{ur}\mathrm{Tr}(\delta j_u j_r)\right] \exp\left(-\tau\mathbf{H} - \mathrm{Tr}(\lambda C) - \alpha\mathbf{N_F} - \beta\mathbf{N_B} - \mathrm{Tr}\sum_r j_r x_r\right) = 0. \tag{2.119}$$

Now we perform independent sequential variations of all the sources $\{j_s\}$ corresponding to the dynamical variables $\{x_s\}$ contained in the polynomials S_L and S_R (due to additivity, it is enough to consider monomials only). Each variation yields a bosonic c-number term $\mathrm{Tr}(\delta^{(k)} j_s\, x_s)$ multiplying the rectangular bracket in (2.119), with $s = k_i$ the index of a desired x_s to appear in S_L or S_R, and

[6] But not entirely arbitrary, there are some restrictions.

i labeling the variations (which are all distinct and independent). In such a way, for $i = 1, 2, \ldots, m$ with m the number of factors in S_R and S_L, we get

$$\int_\Omega (\mathrm{Tr}\delta^{(1)} j_{k_1} x_{k_1})(\mathrm{Tr}\delta^{(k_2)} j_{k_2} x_{k_2}) \ldots (\mathrm{Tr}\delta^{(m)} j_{k_m} x_{k_m})(\mathrm{Tr}\delta j_u D x_u) \exp\left(\cdots - \mathrm{Tr} \sum_r j_r x_r \right). \quad (2.120)$$

If $s = k_i$ in the i^{th} step is such that $\omega_{us} \neq 0$, i.e. the operator x_s is canonically conjugated to x_u, there appears an additional term under the integral sign,

$$(\mathrm{Tr}\delta^{(1)} j_{s_1} x_{k_1})(\mathrm{Tr}\delta^{(k_{i-1})} j_{s_{i-1}} x_{k_2})(\omega_{u,k_i} \mathbf{A} \mathrm{Tr}(\delta j_u \delta^{(i)} j_{k_i})) \exp\left(\cdots - \mathrm{Tr} \sum_r j_r x_r \right). \quad (2.121)$$

After performing all the variations, setting all sources to zero, and dividing by the Z_j factor, we obtain

$$\left\langle (\mathrm{Tr}\delta^{(1)} j_{k_1} x_{k_1}) \ldots (\mathrm{Tr}\delta^{(m)} j_{k_m} x_{k_m})(\mathrm{Tr}\delta j_u D x_u) + \sum_{i=1}^{m} \omega_{u,k_i} bfA \prod_{l:k_l \neq i} (\mathrm{Tr}\delta^{(l)} j_{k_l} x_{k_l})(\mathrm{Tr}\delta j_u \delta j_{k_i}) \right\rangle_\rho = 0. \quad (2.122)$$

Now we strip away the arbitrary $\delta^{(i)} j_{k_i}$ and δj_u in such a way that resulting matrices x_{k_i} become linked in the appropriate order to create the desired structure of the regular term $S_L D x_u S_R$,

$$\sum \mathrm{Tr}(\delta^{(1)} j_{k_1} x_{k_1}) \ldots \mathrm{Tr}(\delta^{(l)} j_{k_l} x_{k_l}) \mathrm{Tr}(\delta j_u D x_u) \mathrm{Tr}(\delta^{(l+1)} j_{k_{l+1}} x_{k_{l+1}}) \ldots \mathrm{Tr}(\delta^{(l+r)} j_{k_{l+r}} x_{k_{l+r}}), \quad (2.123)$$

with l and r the numbers of matrices in the monomials S_L and S_R respectively. The sum ranges over the subset of $(l + r + 1)$-tuples of source matrices which is needed to reconstruct the desired matrix structure. In this process the superfluous terms (2.121) on the right in (2.122) yield a similar structure as (2.123), but in each of them the factor with $D x_u$ is missing, and in i^{th} term the factor $\mathrm{Tr}(\delta^{(i)} j_{k_i} x_{k_i})$ become substituted by $\mathrm{Tr}(\delta j_u \delta^{(i)} j_{k_i})$,

$$\sum \mathrm{Tr}(\delta^{(1)} j_{k_1} x_{k_1}) \ldots \mathbf{A}\mathrm{Tr}(\omega_{uk_i} \delta j_u \delta^{(i)} j_{k_i}) \ldots \mathrm{Tr}(\delta^{(l)} j_{k_l} x_{k_l})\, 1\, \mathrm{Tr}(\delta^{(l+1)} j_{k_{l+1}} x_{k_{l+1}}) \ldots \mathrm{Tr}(\delta^{(l+r)} j_{k_{l+r}} x_{k_{l+r}}). \quad (2.124)$$

This results into a term of the form of the regular term $S_L D x_u S_R$, in which the operators $D x_u$ and x_{k_i} are removed, and the operators between them are contracted into a trace, we denote it as

$$S_{L,\widehat{k_i}} S_R \quad \text{or} \quad S_L S_{R,\widehat{k_i}}, \quad (2.125)$$

depending on whether x_{k_i} comes from S_L or S_R. A closer examination of the influence of fermionic variables, when stripping away the *graded* source matrices from the regular and superfluous terms (2.123) and (2.124), shows, that an *odd* number of fermionic variables between $D x_u$ and x_{k_i} causes a relative change of sign with respect to the regular term, otherwise the relative sign is left unchanged.

Having in mind all these facts, we introduce a shorthand expression for the superfluous terms

$$\sum_{v \in S_L \cup S_R} \omega_{uv} \mathbf{A}(S_L D x_u S_R)_{\widehat{v}} \equiv \sum_{i: x_{k_i} \in S_L} \pm \omega_{uk_i} \mathbf{A}(S_{L,\widehat{k_i}} S_R) + \sum_{j: x_{k_j} \in S_R} \pm \omega_{uk_j} \mathbf{A}(S_L S_{R,\widehat{k_j}}) \quad (2.126)$$

where v is an index going through *all* occurrences of *all* the dynamical variables in S_L or S_R, and the possible sign changes are understood to be included. In this way the extended Ward identity becomes,

$$\left\langle S_L D x_u S_R + \sum_{v \in S_L \cup S_R} \omega_{uv} \mathbf{A}(S_L D x_u S_R)_{\widehat{v}} \right\rangle_\rho = 0. \quad (2.127)$$

From (2.123) and (2.124) it also follows, that the source variations can be used to create the structure $S_L Dx_u(\mathrm{Tr}S_T)S_R$, with $\mathbf{S}_T = \mathrm{Tr}S_T$ the trace of a monomial in the dynamical variables (extension by additivity to a polynomial is straightforward). The Ward identity (2.127) then becomes

$$\left\langle S_L Dx_u \mathbf{S}_T S_R + \sum_{v \in S_L \cup S_T \cup S_R} \omega_{uv} \mathbf{A} (S_L Dx_u \mathbf{S}_T S_R)_{\widehat{v}} \right\rangle_\rho = 0, \tag{2.128}$$

where the index v belongs to a variable x_v from S_T,

$$S_T = x_1 \dots x_l x_v x_{l+1} \dots x_{l+r},$$

and the shorthand notation means that (for $v \in \mathbf{S}_T$)

$$(S_L Dx_u \mathbf{S}_T S_R)_{\widehat{v}} = \pm S_L x_{l+1} \dots x_{l+r} x_1 \dots x_l S_R.$$

Hence the trace is opened at the place of x_v, and inserted into the regular term instead of Dx_u, discarding both Dx_u and x_v. The sign is determined as above according to the number of fermionic variables between Dx_u and x_v in the regular term. By additivity this construction is readily extended from monomials to polynomials. Note that the polynomials S_L, S_R and S_T may even involve constant matrix coefficients. From the definition of the derivative of a trace with respect to an operator, it follows that the sum over indices $v \in S_T$ in (2.128) is given simply by

$$\sum_{v \in S_T} (S_L Dx_u \mathbf{S}_T S_R)_{\widehat{v}} = \sum_v S_L \frac{\delta \mathbf{S}_T}{\delta x_v} S_R, \tag{2.129}$$

where v on the right hand side sweeps through all *distinct* variables x_v used to construct S_T, instead through all *occurrences* as before.

In addition, if we define DS for any polynomial S in the dynamical variables by the Leibniz rule,

$$D(x_1 x_2 \dots x_m) = Dx_1 x_2 \dots x_m + x_1 Dx_2 \dots x_m + \dots + x_1 x_2 \dots Dx_m, \tag{2.130}$$

then the Ward identity (2.128) can be extended further to

$$\left\langle S_L \; DS \; \mathbf{S}_T S_R + \sum_{u \in S} \sum_{v \in S_L \cup S_T \cup S_R} \omega_{uv} \mathbf{A} (S_L S_{\widehat{u}} \mathbf{S}_T S_R)_{\widehat{v}} \right\rangle_\rho = 0, \tag{2.131}$$

with the index u going through all occurrences of all the dynamical variables x_u used to create S, the corresponding x_u from S now being picked to be Dx_u in (2.127), and the remaining variables of S included in the left or right monomial in (2.127). This extension follows from additivity and the possibility to include the extra variables from (2.130) into S_L or S_R. No additional contribution to the superfluous terms in (2.127) appears, since for any two x_{u_1} and x_{u_2} from S (even if they happen to be canonically conjugated, i.e. $\omega_{u_1 u_2} \neq 0$), the contribution to (2.121) is

$$\omega_{u_1 u_2} \prod_{l: k_l \neq u_1} (\mathrm{Tr}\delta^{(l)} j_{k_l} x_{k_l})(\mathrm{Tr}\delta j_{u_1} \delta j_{k_2}) + \omega_{u_2 u_1} \prod_{l: k_l \neq u_2} (\mathrm{Tr}\delta^{(l)} j_{k_l} x_{k_l})(\mathrm{Tr}\delta j_{u_2} \delta j_{k_1}). \tag{2.132}$$

The conjugated variables x_{u_1} and x_{u_2} can be either both fermionic or both bosonic. If they are fermionic(bosonic), $\omega_{u_1 u_2}$ is symmetric(antisymmetric), and the products in (2.132) are antisymmetric(symmetric) in the indices u_1 and u_2, hence the two terms always cancel each other.

2.7.2 Trace dynamics Ward identities

A choice for A in the general identity (2.116), suggested by [1] and [3] as the relevant one for the argument of emergence of quantum-theoretic structures from the statistical thermodynamics of trace dynamics matrix models, is

$$A = \{C, i_{\text{eff}}\}W, \qquad \mathbf{A} = \text{Tr}\left(\{C, i_{\text{eff}}\}W\right), \tag{2.133}$$

where W is a bosonic operator constructed from the dynamical variables. For specific choices of W, the Ward identity (2.116) is expected to yield the canonical commutation relations, the Heisenberg evolution equations, and general unitary canonical transformations of the emergent quantum theory.

Using (2.104) and (2.106), the trace quantity $\mathbf{A}$, (2.133), can be written also as

$$\mathbf{A} = \text{Tr}\left(\{i_{\text{eff}}, C\}W\right) = \text{Tr}\left(\{i_{\text{eff}}, W\}C\right) = 2\text{Tr}\left(i_{\text{eff}}C_{\text{eff}}W_{\text{eff}}\right). \tag{2.134}$$

We now proceed to evaluate the general Ward identity, (2.116), for $\mathbf{A}$ and the probability density with sources ρ_j, given by (2.110). According to the definition of the derivative with respect to an operator, we first consider the variations δ_{x_r} of $\mathbf{A}$ and $\log(Z_j\rho_j)$ with respect to same type variations of x_r. For the first term in (2.110),

$$\delta_{x_r}\mathbf{A} = \text{Tr}\left(\{i_{\text{eff}}, W\}\delta_{x_r}C\right) + \text{Tr}\left(\{i_{\text{eff}}, C\}\delta_{x_r}W\right). \tag{2.135}$$

The variation of the C operator is

$$\delta_{x_r}C = \delta_{x_r}\left(\sum_{ab}\omega_{ab}x_a x_b\right) = \sum_a \omega_{ra}\delta x_r x_a + \omega_{ar}x_a\delta_{x_r}, \tag{2.136}$$

hence by cyclic permutations and the properties of ω_{rs}, (2.27), the first term in (2.135) becomes

$$\text{Tr}\left(\{i_{\text{eff}}, W\}\delta_{x_r}C\right) = \text{Tr}\sum_a\left(\omega_{ar}\left[\{i_{\text{eff}}, W\}, x_a\right]\delta_{x_r}\right). \tag{2.137}$$

For the second term in (2.135), we write the W operator as

$$W = \sum_k x_{l_1^k}x_{l_2^k}\ldots x_{l_{n_k}^k}, \tag{2.138}$$

where $x_{l_i^k}$ is the i^{th} factor of the k^{th} term (a monomial with n_k factors) of the expansion of W into a polynomial in the dynamical variables $\{x_s\}$. Then its variation is

$$\delta_{x_r}W = \sum_k\sum_{j=1}^{n_k} x_{l_1^k}\ldots x_{l_{j-1}^k}(\delta_{x_r}x_{l_j^k})x_{l_{j+1}^k}\ldots x_{l_{n_k}^k} = \sum_l W_r^{Ll}\delta x_r W_r^{Rl}, \tag{2.139}$$

with W_r^{Ll} denoting all nonzero (those for which $l_j^k = r$) terms indexed by the composite index $l = l_i^k$ standing on the left, and W_r^{Rl} the corresponding terms on the right from δx_r. Then by cyclic permutation,

$$\text{Tr}\left(\{i_{\text{eff}}, C\}\delta_{x_r}W\right) = \text{Tr}\sum_l\left(\epsilon_l W_r^{Rl}\{C, i_{\text{eff}}\}W_r^{Ll}\delta x_r\right). \tag{2.140}$$

This gives

$$\frac{\delta\mathbf{A}}{\delta x_r} = \sum_a \omega_{ar}\left[\{i_{\text{eff}}, W\}, x_a\right] + \sum_l \epsilon_l W_r^{Rl}\{C, i_{\text{eff}}\}W_r^{Ll}. \tag{2.141}$$

The second term in (2.110) is a derivative of

$$\log(Z_j\rho_j) = -\tau\mathbf{H} - \mathrm{Tr}\lambda C - \alpha\mathbf{N}_F - \beta\mathbf{N}_B - \mathrm{Tr}\sum_s j_s x_s, \tag{2.142}$$

and we have for the variation of the first and the last term,

$$\delta_{x_r}\mathbf{H} = \mathrm{Tr}\frac{\delta\mathbf{H}}{\delta x_r}\delta x_r, \qquad \delta_{x_r}\mathrm{Tr}\sum_s j_s x_s = \mathrm{Tr} j_r \delta_{x_r}. \tag{2.143}$$

It is convenient to rewrite the "fermionic number" trace quantity $\mathbf{N}_F$ as

$$\mathbf{N}_F = \frac{1}{2}\mathrm{Tr}\sum_{r\in F}[q_r, p_r] = \frac{1}{2}\mathrm{Tr}\sum_{r\in F}(q_r p_r - p_r q_r) = \frac{1}{2}\sum_{a,b\in F}\alpha_{ab}x_a x_b, \qquad \alpha = (0,\ldots,0,\Omega_B,\ldots,\Omega_B), \tag{2.144}$$

and the "bosonic number" trace quantity $\mathbf{N}_B$ as

$$\mathbf{N}_B = \frac{1}{2}\mathrm{Tr}\sum_{r\in F}\{q_r, p_r\} = \frac{1}{2}\mathrm{Tr}\sum_{r\in F} q_r p_r + p_r q_r = \frac{1}{2}\sum_{a,b\in F}\beta_{ab}x_a x_b, \qquad \beta = (-\Omega_F,\ldots,-\Omega_F,0,\ldots,0), \tag{2.145}$$

with notation of (2.25) and (2.26). Then the variations of $\mathbf{N}_F$ and $\mathbf{N}_B$ are

$$\delta_{x_r}\mathbf{N}_F = \frac{1}{2}\mathrm{Tr}\sum_a(\alpha_{ar}x_a\delta x_r - \alpha_{ra}x_a\delta x_r) = \mathrm{Tr}\sum_a \alpha_{ar}x_a\delta x_r, \tag{2.146}$$

$$\delta_{x_r}\mathbf{N}_B = \frac{1}{2}\mathrm{Tr}\sum_a(\beta_{ar}x_a\delta x_r + \beta_{ra}x_a\delta x_r) = \mathrm{Tr}\sum_a \beta_{ar}x_a\delta x_r, \tag{2.147}$$

by antisymmetry of α and symmetry of β. Collecting (2.141) and (2.143) through (2.147) together, we get the relevant term of the general Ward identity (2.116),

$$\sum_a \omega_{ar}\left[\{i_{\mathrm{eff}}, W\}, x_a\right] + \sum_l \epsilon_l W_r^{Rl}\{i_{\mathrm{eff}}, C\}W_r^{Ll} - \mathbf{A}\left(\tau\frac{\delta\mathbf{H}}{\delta x_r} + \sum_s \omega_{rs}[\lambda, x_s] + \sum_a(\alpha\alpha_{ar} - \beta\beta_{ar})\,x_a\right), \tag{2.148}$$

which becomes zero when averaged over the statistical ensemble with sources given by the distribution (2.110). It can be further simplified multiplying by ω_{ur} and summing over r, to finally get the identity,

$$\left\langle [\{i_{\mathrm{eff}}, W\}, x_u] + \sum_{lr}\omega_{ur}\epsilon_l W_r^{Rl}\{i_{\mathrm{eff}}, C\}W_r^{Ll} + \left(-\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u - \sum_r \omega_{ur}j_r\right)\mathbf{A}\right\rangle_{\rho_j} = 0 \tag{2.149}$$

with $\mathbf{A}$ given by (2.134), since

$$\dot{x}_u = \sum_r \omega_{ur}\frac{\delta\mathbf{H}}{\delta x_r}, \qquad \sum_r \omega_{ur}\omega_{rs} = -\epsilon_r\omega_{us}, \qquad \sum_r \omega_{ur}\alpha_{ar} = \varepsilon_u^F, \qquad \sum_r \omega_{ur}\beta_{ar} = -\varepsilon_u^B,$$

with the notation $\varepsilon_u^{F/B} = +1$ for x_u a fermionic/bosonic q, $\varepsilon_u^{F/B} = -1$ for x_u a fermionic/bosonic p, $\varepsilon_u^{F/B} = 0$ zero for x_r not being a fermionic/bosonic variable, and

$$\gamma_u = \alpha\varepsilon_u^F - \beta\varepsilon_u^B. \tag{2.150}$$

Denoting all the terms not containing sources inside the average brackets in (2.149) by Dx_u,

$$Dx_u = [\{i_{\rm eff}, W\}, x_u] + \sum_{lr} \omega_{ur}\epsilon_l W_r^{Rl}\{i_{\rm eff}, C\}W_r^{Ll} + (-\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u)\,{\rm Tr}(\{i_{\rm eff}, C_{\rm eff}\}W_{\rm eff}), \tag{2.151}$$

the basic zero-sources Ward identity (2.149) becomes

$$\langle Dx_u\rangle_\rho = 0, \tag{2.152}$$

and with the definition of DS, (2.130), from the extended Ward identity (2.131) we obtain,

$$\langle S^L(\{x_r\})DS(\{x_r\})S^R(\{x_r\})\rangle_\rho = -\left\langle \sum_{u\in S}\sum_{v\in S_L\cup S_R} \omega_{uv}\mathbf{A}(S_L(\{x_r\})DS_{\widehat{u}}(\{x_r\})S_R(\{x_r\}))_{\widehat{v}}\right\rangle_\rho , \tag{2.153}$$

for any polynomials S, S_L and S_R in the dynamical variables with c-number coefficients. Moreover with the restriction, that S does not contain any variable conjugated with some variable in S_R or S_L, the Ward identity reduces to

$$\langle S_L(\{x_r\})\; DS(\{x_r\})\; S_R(\{x_r\})\rangle_\rho = 0. \tag{2.154}$$

Ward identity for effective variables

As noted above, the zero source average of any operator constructed of dynamical variables (here also the integration variables) using only c-number coefficients is a function of the constant operator parameter λ, and for that reason $[\lambda, \langle x_u\rangle_\rho] = 0$. Then the term $\epsilon_u[\lambda, x_u]$ in (2.149) vanishes in the average over the zero source ensemble, and we obtain the identity

$$\left\langle [\{i_{\rm eff}, W\}, x_u] + \sum_{lr} \omega_{ur}\epsilon_l W_r^{Rl}\{C, i_{\rm eff}\}W_r^{Ll} + (-\tau\dot{x}_u + \gamma_u x_u)\,\mathbf{A}\right\rangle_\rho = 0. \tag{2.155}$$

Since $i_{\rm eff}$ commutes with λ (as being also its function with c-number coefficients), the zero source average already involves the effective projection. Hence the identity (2.155) without sources could be equally well written in terms of the effective projection of the dynamical variables,

$$\left\langle [\{i_{\rm eff}, W_{\rm eff}\}, x_{u\rm eff}] + \sum_{lr} \omega_{ur}\epsilon_l \left(W_r^{Rl}\{C, i_{\rm eff}\}W_r^{Ll}\right)_{\rm eff} + (-\tau\dot{x}_{u\rm eff} + \gamma_u x_{u\rm eff})\,\mathbf{A}\right\rangle_\rho = 0. \tag{2.156}$$

Although the term with commutator $\epsilon_u[\lambda, x_u]$ vanishes in the basic Ward identity (2.152), in general it does not in the extended one (2.153).

The commutator $\epsilon_u[\lambda, x_u]$ can be completely removed, if we consider the effective projections of the variables $x_{u\rm eff}$ instead of the variables x_u themselves. Apart form the vanishing of $\epsilon_u[\lambda, x_u]$, the effective variables $x_{u\rm eff}$ obey the same Ward identities as x_u. This can be seen, if we first take the effective projection of (2.149) using (2.105) prior to the variations. Then we get

$$\left\langle [\{i_{\rm eff}, W_{\rm eff}\}, x_{u\rm eff}] + \sum_{lr} \omega_{ur}\epsilon_l \left(W_r^{Rl}\{C, i_{\rm eff}\}W_r^{Ll}\right)_{\rm eff} + \left(-\tau\dot{x}_{u\rm eff} + \gamma_u x_{u\rm eff} - \sum_r \omega_{ur} j_{r,\rm eff}\right)\mathbf{A}\right\rangle_{\rho_j} = 0, \tag{2.157}$$

which (up to the sources) is the basic Ward identity (2.156), equivalent to (2.155). Starting from (2.157) instead of (2.149), and performing the variations with respect to the effective projections $j_{r\mathrm{eff}}$ of the sources in the same way as before, we eventually arrive at the extended Ward identity (2.131), but with Dx_u, (2.151), replaced by $D_{x_{u\mathrm{eff}}}$,

$$Dx_{u\mathrm{eff}} = [\{i_{\mathrm{eff}}, W\}, x_{u\mathrm{eff}}] + \sum_{lr} \omega_{ur}\epsilon_l W_r^{Rl}\{C, i_{\mathrm{eff}}\}W_r^{Ll} + (-\tau\dot{x}_{u\mathrm{eff}} + \gamma_u x_{u\mathrm{eff}})\,\mathrm{Tr}(\{i_{\mathrm{eff}}, C_{\mathrm{eff}}\}W_{\mathrm{eff}}). \tag{2.158}$$

The only difference in the derivation leading to (2.131) is that the variations are performed with respect to the effective projection $j_{r\mathrm{eff}}$ of the sources j_r. This brings only the effective projection $x_{r\mathrm{eff}}$ from the exponential, since

$$\mathrm{Tr} j_r x_r = \mathrm{Tr} j_{r\mathrm{eff}} x_{r\mathrm{eff}} + j_{r12} x_{r12}.$$

Hence the extended Ward identity holds for the effective variables $x_{r\mathrm{eff}}$ as well,

$$\langle S^L(\{x_{r\mathrm{eff}}\})DS(\{x_{r\mathrm{eff}}\})S^R(\{x_{r\mathrm{eff}}\})\rangle_\rho = -\left\langle \sum_{u\in S}\sum_{v\in S_L\cup S_R} \omega_{uv}\mathbf{A}(S_L DS_{\widehat{u}} S_R)_{\widehat{v}} \right\rangle_\rho , \tag{2.159}$$

with $DS(\{x_{r\mathrm{eff}}\})$ given by the Leibniz rule (2.130) with $Dx_{u\mathrm{eff}}$ in place of Dx_u.

2.8 Consequences of Ward identities

The trace dynamics Ward identity $\langle Dx_u\rangle_\rho = 0$ with

$$Dx_u = [\{i_{\mathrm{eff}}, W\}, x_u] + \sum_{lr} \omega_{ur}\epsilon_l W_r^{Rl}\{i_{\mathrm{eff}}, C\}W_r^{Ll} + (-\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u)\,\mathrm{Tr}(\{i_{\mathrm{eff}}, C_{\mathrm{eff}}\}W_{\mathrm{eff}}), \tag{2.160}$$

(2.152) and (2.151), evaluated for the simplified static ensemble (2.89), already suggests the possible emergence of quantum theory in the trace dynamics framework. However, some rather stringent conditions are to be met before the structure of quantum theory could actually appear. There are still some disturbing superfluous terms present in (2.160), namely the last three terms on the right,

$$(-\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u)\,\mathrm{Tr}(\{i_{\mathrm{eff}}, C_{\mathrm{eff}}\}W_{\mathrm{eff}}). \tag{2.161}$$

The argument for emergence of quantum theory of [1] and [3] requires these terms to effectively vanish. Therefore some additional conditions on trace dynamics are to be found, that would allow to make these odd terms vanish, or at least negligible with respect to the other terms. This has to happen even when inserted between any two polynomials before averaging over the ensemble. Since *all* the superfluous terms come from the variation of the probability density, such conditions could be satisfied by some kind of rigidity of the distribution, i.e. that it changes little when the x_r are varied. First the suggested conditions, assumptions and approximations are reviewed, and the consequences of their validity are drawn. Then some consistency issues associated with the possibility of imposing such assumptions are discussed.

The conditions and approximations suggested by [1] and [3] are:

1. The quantum field operators are expected to emerge from the effective projections $x_{u\mathrm{eff}}$ instead of the x_u dynamical variables. This cancels the commutator $[\lambda, x_u]$, even if it is inserted between any two polynomials and then averaged over the ensemble. Further reasons for making this assumption are given below.

2. There can be such an underlying trace dynamics theory, that the term $\tau \dot{x}_{u\text{eff}} \mathrm{Tr}(\{i_{\text{eff}}, C_{\text{eff}}\} W_{\text{eff}})$ can be made approximately zero with respect to the other terms. It is supposed that the $x_{u\text{eff}}$ variable can be split into a slow and a fast part. The fast part is expected to be effectively unaccessible to observation due to its rapid fluctuations, and/or its time derivative vanishes due to the disjoint operator phase space supports of $\dot{x}_{u\text{eff}}$ and of the C operator. The slow part of $\dot{x}_{u\text{eff}}$ is expected to become effectively suppressed by the parameter τ, which is assumed to be small.

3. The coefficient γ_u is set to zero. It effectively removes the condition on the average values of N_B and N_F. This is supposed to correspond to approximately zero ensemble averages of N_F and N_B. If N_B (or N_F) is not a conserved quantity, it is dropped anyway.

4. The extensive conserved C operator in the second term in (2.160) can be replaced by its average value $\langle C \rangle_\rho = \hbar i_{\text{eff}}$, and its fluctuations over this average can be neglected.

5. The properties of the underlying trace dynamics theory allows to drop the terms on the right hand side of the Ward identity (2.159), hence the Ward identity approximately holds when Dx_u is inserted between arbitrary polynomials S_L and S_R in the dynamical variables.

2.8.1 The emergence of quantum-theoretic structures

Taking granted that the underlying trace dynamics theory can be made such that the above assumptions are satisfied, and unwanted terms in (2.151) vanish or become negligible, we evaluate some of its consequences. With these assumptions and (2.134), the Ward identity $\langle Dx_u \rangle_\rho = 0$ with (2.160) becomes

$$\langle S_L Dx_u S_R \rangle_\rho = 0, \qquad Dx_u \approx 2\left[i_{\text{eff}} W_{\text{eff}}, x_u\right] + 2 \sum_{lr} \omega_{ur} \epsilon_l W_r^{Rl} i_{\text{eff}} C_{\text{eff}} W_r^{Ll}, \tag{2.162}$$

up to some restrictions for any two polynomials S_L and S_R in the dynamical variables, and

$$\langle S_L Dx_u S_R \rangle_\rho \approx -\left\langle \sum_{v \in S_L \cup S_R} \omega_{uv} \mathbf{A}(S_L Dx_u S_R)_{\widehat{v}} \right\rangle_\rho, \tag{2.163}$$

in the general case without any restrictions. With the assumption, that the conserved C operator can be replaced by its average value $\langle C \rangle_\rho = \hbar i_{\text{eff}}$, Dx_u in (2.162) simplifies to (by going back from (2.140) to (2.139))

$$Dx_u \approx 2\left[i_{\text{eff}} W_{\text{eff}}, x_u\right] - 2\hbar \sum_{lr} \omega_{ur} \epsilon_l W_r^{Rl} W_r^{Ll} = 2\left[i_{\text{eff}} W_{\text{eff}}, x_u\right] - 2\hbar \sum_r \omega_{ur} \frac{\delta \mathbf{W}}{\delta x_r}. \tag{2.164}$$

Canonical commutation relations

Specializing the operator W to $\sigma_v x_v$, where σ_v is an auxiliary odd grade c-number[7], and even grade unity otherwise, (2.164) becomes

$$Dx_u \approx 2\left[i_{\text{eff}} \sigma_v x_{v\text{eff}}, x_u\right] - 2\hbar \omega_{uv} \sigma_v. \tag{2.165}$$

Stripping away the auxiliary σ_v and reordering, the basic Ward identity gives,

$$\langle [x_u, i_{\text{eff}} x_{v\text{eff}}]_{-\epsilon_u} + \hbar \epsilon_u \omega_{uv} \rangle_\rho \approx 0, \tag{2.166}$$

[7]Its purpose is to make W even grade, if x_v is fermionic.

which is the same as with the effective projections of the dynamical variables x_u and x_v,

$$\langle i_{\rm eff}\,[x_{u\rm eff}, x_{v\rm eff}]_{-\epsilon_u} + \hbar\epsilon_u\omega_{uv}\rangle_\rho \approx 0. \tag{2.167}$$

But only the latter expression allows us to pull the $i_{\rm eff}$ matrix out of the commutator, even when inserted between two polynomials S_L and S_R, and averaged over the statistical ensemble,

$$\langle S_L([x_{u\rm eff}, x_{v\rm eff}]_{-\epsilon_u} - i_{\rm eff}\hbar\epsilon_u\omega_{uv})S_R\rangle_\rho \approx 0, \tag{2.168}$$

where $i_{\rm eff}$ has been included into S_L. With the restriction on the polynomials S_L and S_R not to contain the variable canonically conjugated to $x_{u\rm eff}$, this equation implies that the averages of the effective dynamical variables $x_{u\rm eff}$ satisfy the usual canonical commutation relations of quantum field theory in a weak sense. This is one of the reasons to suggest the identification of the dynamical variables in the emergent quantum theory with the *effective* projections $\{x_{r\rm eff}\}$ of trace dynamics variables, rather than with the dynamical variables $\{x_r\}$ themselves.

It is convenient to introduce the notation

$$A \overset{\rho,S}{\approx} B,$$

when the operators A and B become approximately equal, when inserted between polynomials S_L and S_R, and averaged over the statistical ensemble, with the polynomials S_L and S_R restricted to not contain any variable canonically conjugated to some variable used to construct A or B. With this definition, (2.168) is rewritten as

$$[x_{u\rm eff}, x_{v\rm eff}]_{-\epsilon_u} \overset{\rho,S}{\approx} i_{\rm eff}\hbar\epsilon_u\omega_{uv}. \tag{2.169}$$

Expanding the compact notation using the definition of the ω matrix, (2.25), for bosonic variables we obtain

$$[q_{r\rm eff}, p_{s\rm eff}] \overset{\rho,S}{\approx} i_{\rm eff}\hbar\delta_{rs}, \quad [q_{r\rm eff}, q_{s\rm eff}] \overset{\rho,S}{\approx} [p_{r\rm eff}, p_{s\rm eff}] \overset{\rho,S}{\approx} 0, \tag{2.170}$$

for fermionic variables,

$$\{q_{r\rm eff}, p_{s\rm eff}\} \overset{\rho,S}{\approx} i_{\rm eff}\hbar\delta_{rs}, \quad \{q_{r\rm eff}, q_{s\rm eff}\} \overset{\rho,S}{\approx} \{p_{r\rm eff}, p_{s\rm eff}\} \overset{\rho,S}{\approx} 0, \tag{2.171}$$

and for mixed bosonic and fermionic variables,

$$[q_{r\rm eff}, p_{s\rm eff}] \overset{\rho,S}{\approx} [q_{r\rm eff}, q_{s\rm eff}] \overset{\rho,S}{\approx} [p_{r\rm eff}, p_{s\rm eff}] \overset{\rho,S}{\approx} 0, \tag{2.172}$$

since no fermionic variable is canonically conjugated to any bosonic one.

The operator $i_{\rm eff}$ has the role of the imaginary unit i in the complex Hilbert space of quantum theory. It arises naturally, even if the underlying vector space of trace dynamics is taken over real numbers. In that case, $i_{\rm eff}$ is an antisymmetric matrix with vanishing trace, and $i_{\rm eff}^2 = -I$. Due to the tracelessness of $i_{\rm eff}$, there appears no contradiction when taking the trace of the relation

$$[q_{r\rm eff}, p_{r\rm eff}] \overset{\rho,S}{\approx} i_{\rm eff}\hbar, \tag{2.173}$$

in spite of the fact that the matrices $q_{r\rm eff}$ and $p_{r\rm eff}$ are finite dimensional. But due to the restrictions on S_L and S_R, the commutation relations cannot be used freely as in ordinary quantum theory. There is an example [1] of a possible contradiction. When we evaluate the expression $q^2p^2 + p^2q^2 -$

$2qp^2q$ for a canonically conjugated bosonic pair q, p using the commutation relations (2.173) for $q = q_{r\text{eff}}$ and $p = p_{r\text{eff}}$, we obtain

$$q^2p^2 + p^2q^2 - 2qp^2q = q[q,p]p - p[q,p]q + qp[q,p] - [q,p]pq \overset{\rho,S}{\approx} -2\hbar^2, \tag{2.174}$$

which is not consistent with the vanishing trace of the left hand side. The suggested resolution [1] to this apparent contradiction is to enlarge the underlying vector space by taking $N \to \infty$ in such a way that the cyclic permutation under the trace ceases to hold. But this would lead to entire breakdown of trace dynamics formalism, as it heavily relies on this property. Since the condition on S_L and S_R not to contain a canonically conjugated variable to at least one variable in the commutator is violated in (2.174), it would seem that the contradiction is brought in by unjustified using (2.173) for the last equality in (2.174). However, a more precise calculation with the use of (2.163) points rather towards some consistency issues with the approximations made in getting to (2.164). This point will be discussed in more detail below.

Heisenberg evolution equations

Specializing W to the operator Hamiltonian H and using the effective projection of dynamical variables, (2.164) implies,

$$Dx_{u\text{eff}} \approx 2i_{\text{eff}}\,[H_{\text{eff}}, x_{u\text{eff}}] - 2\hbar\dot{x}_{u\text{eff}}, \tag{2.175}$$

where the i_{eff} operator can be pulled out of the commutator due to the commutativity of i_{eff} with H_{eff}. It is further assumed that $H_{\text{eff}}(\{x_u\})$, the effective projection of $H(\{x_u\})$, can be effectively represented as a function of the *effective* projections $\{x_{u\text{eff}}\}$ of the dynamical variables only, $H_{\text{eff}}(\{x_{u\text{eff}}\})$. This approximation is being justified by the hypothesis, that quantum theory emerges only as a low energy approximation to a more complete theory, and in this low energy regime the non-effective components $\{x_{u12}\}$ of the dynamical variables do not make a significant contribution to the dynamics. This assumption enables us to use the Ward identity with the effective projections $x_{u\text{eff}}$, (2.159). It tells us, that $Dx_{u\text{eff}}$ is approximately zero when inserted between S_L and S_R (regarded as functions of $\{x_{r\text{eff}}\}$ with the usual restrictions not to contain the canonically conjugated variable to $x_{u\text{eff}}$), and averaged over the ensemble. In this way we obtain the effective Heisenberg evolution equations for $x_{u\text{eff}}$,

$$\dot{x}_{u\text{eff}} \overset{\rho,S}{\approx} \frac{i_{\text{eff}}}{\hbar}\,[H_{\text{eff}}, x_{u\text{eff}}] \quad \text{or} \quad \langle S_L\,(\hbar\dot{x}_{u\text{eff}} - i_{\text{eff}}\,[H_{\text{eff}}, x_{u\text{eff}}])\,S_R\rangle_\rho \approx 0. \tag{2.176}$$

Let $S = S(\{x_{u\text{eff}}\})$ be an arbitrary polynomial in the effective dynamical variables. The operator DS is defined using the Leibniz rule by (2.130). Substituting $Dx_{u\text{eff}}$ from (2.175) into the expansion of DS, and using the Leibniz property of the commutator and time derivative in (2.176), we obtain

$$\dot{S} \overset{\rho,S}{\approx} \frac{i_{\text{eff}}}{\hbar}\,[H_{\text{eff}}, S] \quad \text{or} \quad \langle S_L\,(\hbar\dot{x}_{u\text{eff}} - i_{\text{eff}}\,[H_{\text{eff}}, S])\,S_R\rangle_\rho \approx 0. \tag{2.177}$$

Note that this is the case when the Ward identity $\langle S_L DS S_R\rangle_\rho = 0$ holds without any restrictions on the dynamical variables used to construct S. Therefore any operator function of the dynamical variables, up to the approximations considered, obeys an effective Heisenberg evolution law of quantum theory. Moreover, by choosing $S = H_{\text{eff}}$, we learn, that the effective Hamiltonian H_{eff} is conserved under the time evolution it generates, as required for consistency of its interpretation as an effective Hamiltonian.

Given a value of a dynamical variable $x_{u\mathrm{eff}}(t_0)$ at the time t_0, its value at the time t is

$$x_{u\mathrm{eff}}(t) \overset{\rho,S}{\approx} U_{\mathrm{eff}}^{+}(t-t_0)x_{u\mathrm{eff}}(t_0)U_{\mathrm{eff}}(t-t_0), \tag{2.178}$$

with the effective evolution operator U_{eff} given by

$$U_{\mathrm{eff}}(s) = \exp\left(-\frac{i_{\mathrm{eff}}}{\hbar}sH_{\mathrm{eff}}\right). \tag{2.179}$$

This is readily verified by taking the time derivative of $x_{u\mathrm{eff}}(t)$ and comparing with the effective Heisenberg equations (2.176). Hence, although the evolution of the underlying trace dynamics theory is not unitary, it becomes approximately unitary in the emergent effective theory. By (2.177) this result is immediately extended to any operator function S of the dynamical variables,

$$S(t) \overset{\rho,S}{\approx} U_{\mathrm{eff}}^{+}(t-t_0)S(t_0)U_{\mathrm{eff}}(t-t_0). \tag{2.180}$$

Given two such operator functions $S_1(t)$ and $S_2(t)$ at equal time, the ensemble average of their product $S_1(t)S_2(t+\Delta t)$ at distinct times (or vice versa) can be evaluated by

$$\langle S_1(t)S_2(t+\Delta t)\rangle_\rho \approx \langle S_1(t)U_{\mathrm{eff}}^{+}(\Delta t)S_2(t)U_{\mathrm{eff}}(\Delta t)\rangle_\rho, \tag{2.181}$$

as long as S_1 does not contain variables canonically conjugated to those of S_2. This identity is later used to define Wightman functions by the ensemble averages of polynomials in dynamical variables.

General canonical transformations

The time evolution is only a special case of a general canonical transformations, in which the parameter of the one-parameter group of time evolution transformations in the chosen reference frame is interpreted as time. Therefore the results from previous sections can be carried over to general canonical transformations, generated by a bosonic operator G, through the general relations (2.56). The infinitesimal canonical transformations are represented by the effective infinitesimal unitary transformations $x_{u\mathrm{eff}} \mapsto x_{u\mathrm{eff}} + \delta x_{u\mathrm{eff}}$,

$$\delta x_{u\mathrm{eff}} \overset{\rho,S}{\approx} \frac{i_{\mathrm{eff}}}{\hbar}\left[\varepsilon G_{\mathrm{eff}}, x_{u\mathrm{eff}}\right] \quad \text{or} \quad \langle S_L\left(\hbar\delta x_{u\mathrm{eff}} - i_{\mathrm{eff}}\left[\varepsilon G_{\mathrm{eff}}, x_{u\mathrm{eff}}\right]\right)S_R\rangle_\rho \approx 0, \tag{2.182}$$

and the corresponding one parameter group of canonical transformation is given by the effective unitary operator $U_{\mathbf{G}\mathrm{eff}}$

$$U_{\mathbf{G},\mathrm{eff}}(\tau) = \exp\left(-\frac{i_{\mathrm{eff}}}{\hbar}\tau G_{\mathrm{eff}}\right). \tag{2.183}$$

Similarly as the Hamiltonian operator above, the G_{eff} operator is assumed to be effectively given (with some approximation) as a function of the effective dynamical variables only.

In case of a field theory with the field operators $x \mapsto q_k(x)$, the spacetime translations and rotations are also special cases of general canonical transformations. The generators of translation are given by the effective part P^μ_{eff} of the operators corresponding to the components of trace energy-momentum four-vector $\mathbf{P}^\mu$, (2.69). Then the translation $q_{k\mathrm{eff}}(x) \mapsto q_{k\mathrm{eff}}(x+a)$ of the ensemble averaged effective fields is represented by an effective unitary transformation given by $U_{\mathbf{P}\mathrm{eff}}(a)$,

$$q_{k\mathrm{eff}}(x) \mapsto q_{k\mathrm{eff}}(x+a) \overset{\rho,S}{\approx} U_{\mathbf{P}\mathrm{eff}}^{+}(a)q_{k\mathrm{eff}}(x)U_{\mathbf{P}\mathrm{eff}}(a), \text{ with } U_{\mathbf{P}\mathrm{eff}}(a) = \exp\left(-\frac{i_{\mathrm{eff}}}{\hbar}a_\mu P^\mu_{\mathrm{eff}}\right), \tag{2.184}$$

and with the parameter τ included in a. Spatial translation and time evolution is then a special case of (2.184), for which a is purely spatial or purely temporal vector in the given Lorentz frame. The spacetime rotations of the fields are similarly represented by the unitary transformation with $U_{\mathbf{M}\mathrm{eff}}(\Lambda)$,

$$q_{k\mathrm{eff}}(x) \mapsto q_k(\Lambda x) \overset{\rho,S}{\approx} \sum_l S_{kl}(\Lambda) U^+_{\mathbf{M}\mathrm{eff}}(\Lambda) q_{l\mathrm{eff}}(x) U_{\mathbf{M}\mathrm{eff}}(\Lambda), \tag{2.185}$$

with

$$U_{\mathbf{M}\mathrm{eff}}(\Lambda) = \exp\left(-\frac{i_{\mathrm{eff}}}{\hbar}\Lambda_{\mu\nu} M^{\mu\nu}_{\mathrm{eff}}\right), \tag{2.186}$$

and $M^{\mu\nu}_{\mathrm{eff}}$ the effective part of the operator associated with the trace quantity $\mathbf{M}^{\mu\nu}$, (2.75). This gives the relation,

$$\sum_l S_{kl}(\Lambda^{-1}) q_l(\Lambda x) \overset{\rho,S}{\approx} U^+_{\mathbf{M},\mathrm{eff}}(\Lambda) q_{k\mathrm{eff}}(x) U_{\mathbf{M},\mathrm{eff}}(\Lambda). \tag{2.187}$$

Taking both cases together, the general Poincaré transformation of the fields induced by the transformation $x \mapsto \{a, \Lambda\}x = \Lambda x + a$ is given by

$$q_{k\mathrm{eff}}(x) \mapsto \{a,\Lambda\}^{-1} q_{k\mathrm{eff}}(x) = q_k(\Lambda x + a) \overset{\rho,S}{\approx} \sum_l S_{kl}(\Lambda^{-1}) U^+_{\{a,\Lambda\},\mathrm{eff}} q_{l\mathrm{eff}}(x) U_{\{a,\Lambda\},\mathrm{eff}}, \tag{2.188}$$

with the unitary operator $U_{\{a,\Lambda\},\mathrm{eff}}$ given by

$$U_{\{a,\Lambda\},\mathrm{eff}} = \exp\left(-\frac{i_{\mathrm{eff}}}{\hbar}\left(a_\mu P^\mu_{\mathrm{eff}} + \Lambda_{\mu\nu} M^{\mu\nu}_{\mathrm{eff}}\right)\right). \tag{2.189}$$

2.8.2 Correspondence with quantum field theory

The assumptions on trace dynamics from previous section, that are needed (e.g. by [1]) to recover the usual quantum-theoretic structures, can be summed up in the following points.

1. It is possible to find an underlying trace dynamics theory such that the superfluous terms (2.161) in the Ward identities can be made zero of negligible comparing to the other terms, when inserted between two polynomials in the dynamical variables and averaged over the statistical ensemble.

2. The extensive conserved operator quantity C can be replaced by its ensemble average value in the last term of Dx_u, (2.162), even when inserted between two polynomials and averaged.

3. The effective projection of the operator Hamiltonian H_{eff} or generators of general canonical transformations, in a low energy approximation for which quantum theory is expected to emerge, can be represented using the effective projections of dynamical variables only.

4. The additional assumptions on the polynomials S_L and S_R not to contain a canonically conjugated variable to x_u of Dx_u or to any variable in S of DS, can be with some approximation dropped. Then the Ward identity (2.163) holds with approximately zero right hand side, and the relevant equalities from previous section hold in a weak sense as operator equalities.

Provided these assumptions are satisfied by an underlying trace dynamics theory, the averaged effective projections of the trace dynamics dynamical variables $x_{u\mathrm{eff}}$ are seen to acquire (in a specific sense) the properties of quantum operators - they obey the usual commutation relations and the unitary evolution law of quantum theory.

Ensemble averages and Wightman functions

As remarked in section 2.5.1, the ensemble average of any operator is a linear combination of two fixed operators $i_{\rm eff}$ and I. Therefore the averages are uniquely determined by the two coefficients of the linear combination, and these coefficients are expected to correspond in some sense to observed values. Since both operators $i_{\rm eff}$ and I commute with all averaged dynamical variables $\langle x_{u\rm eff}\rangle_\rho$, the matrix $i_{\rm eff}$ could be identified with the imaginary (c-number) unit i in the emergent complex quantum Hilbert space. This applies even in case of the real vector space of trace dynamics, since the pair I and $i_{\rm eff}$ have all the algebraic properties of the generators 1 and i of the complex field $\mathbb{C}$. With the identification

$$I \leftrightarrow 1 \in \mathbb{C} \qquad i_{\rm eff} \leftrightarrow i \in \mathbb{C}, \tag{2.190}$$

the ensemble average $\langle x_{u\rm eff}\rangle_\rho$ of the effective projection of a trace dynamics variable $x_{u\rm eff}$ is essentially given by a complex number. In this way trace dynamics could offer a natural explanation of the appearance of complex numbers in quantum physics. In both cases of complex and Hilbert vector space of trace dynamics, we can assign any matrix-valued ensemble average $\langle x_{u\rm eff}\rangle_\rho$ of $x_{u\rm eff}$ a complex number $\langle x_{u\rm eff}\rangle^c_\rho$ by the identification (2.190). Equivalently, the identification can be made by means of a trace of the averages with the operator $I - i_{\rm eff}i$,

$$\langle x_{u\rm eff}\rangle^c_\rho = \frac{1}{N}\,{\rm Tr}\left((I - i_{\rm eff}i)\langle x_{u\rm eff}\rangle_\rho\right). \tag{2.191}$$

The same reasoning apply also to any polynomial S in the effective or non-effective dynamical variables with c-number coefficients. Hence the ensemble average $\langle S\rangle_\rho$ of S corresponds to a complex number given by the trace

$$\langle S\rangle^c_\rho = \frac{1}{N}{\rm Tr}\left((I - i_{\rm eff}i)\langle S\rangle_\rho\right). \tag{2.192}$$

Given a probability distribution ρ and a set of trace dynamics variables $\{x_{u\rm eff}\}$, the averages of any polynomial $S(\{x_{u\rm eff}\})$ can be evaluated, yielding a complex number $\langle S(\{x_{u\rm eff}\})\rangle^c_\rho$. The correspondence between the trace dynamics variables $x_{u\rm eff}$ and the field operators X_u of the emergent quantum field theory,

$$\{x_{u\rm eff}\} \leftrightarrow \{X_u\}, \tag{2.193}$$

is established by the identification of these ensemble averages of polynomials in $x_{u\rm eff}$ with vacuum expectation values of the corresponding polynomials in X_u. The Wightman reconstruction theorem can be used for this purpose. The ensemble average of a polynomial S defines the Wightman function W_S,

$$W_S(\{X_{u\rm eff}\}) = \langle S(\{x_{u\rm eff}\})\rangle^c_\rho. \tag{2.194}$$

The properties of Wightman functions and their relation to quantum field theory are summarized in the appendix. Several conditions have to be satisfied by the hierarchy of ensemble averages,

$$\langle S(\{x_{u\rm eff}\})\rangle^c_\rho \quad \text{for all mononomials } S,$$

before the Wightman reconstruction theorem can be used to reconstruct the corresponding local quantum field theory. Then it provides a separable Hilbert space $\mathcal{H}$ with a scalar product $\langle,\rangle_{\mathcal{H}}$, a distinguished normalized vector state Ψ_0 (the vacuum state), and operators X_u acting on it, such that the trace dynamics ensemble averages are given by the vacuum expectation values of polynomials in X_u,

$$\langle S(\{x_{u\rm eff}\})\rangle^c_\rho = W_S(\{X_{u\rm eff}\}) = \langle\Psi_0, S(\{X_{u\rm eff}\})\Psi_0\rangle_{\mathcal{H}}. \tag{2.195}$$

If S is a monomial $S(\{x_u\}) = x_1 x_2 \dots x_n$, then expanding the composite index $u = (i_u, x_u)$ into the infinitesimal space box label x_u and the field component label i_u, (2.195) becomes

$$\langle x_{i_1\text{eff}}(x_1) x_{i_2\text{eff}}(x_2) \dots x_{i_n\text{eff}}(x_n)\rangle^c_\rho = W^{(n)}_{i_1 i_2 \dots i_n}(x_1, x_2, \dots, x_n) = \langle \Psi_0, X_{i_1}(x_1) X_{i_2}(x_2) \dots X_{i_n}(x_n)\Psi_0\rangle_{\mathcal{H}}. \tag{2.196}$$

which is of the form of the Wightman function (A.18).

The identification between ensemble averages and Wightman functions is at a first sight carried out in a different way by [1]. Since trace Lagrangian density is already assumed to be Poincaré invariant, there exists the conserved trace four-vector $\mathbf{P}^\mu$ and the corresponding three-vector $\vec{\mathbf{P}}$. Then some additional assumptions are made, which again are to be understood as constraints on the underlying trace dynamics theory. The effective Hamiltonian operator $H_{\text{eff}}(\{x_{s\text{eff}}\})$ is assumed to be bounded from below, and there should be a unique eigenvector ψ_0 with the lowest eigenvalue of H_{eff} and zero eigenvalues of $\vec{P}_{\text{eff}}$. Then the proposed correspondence with Wightman functions is

$$\psi_0^+ \langle S(\{x_{u\text{eff}}\})\rangle_{\widehat{\rho}} \psi_0 = W_S(\{X_u\}), \tag{2.197}$$

where the unitarily fixed average given by the fixed density $\widehat{\rho}$ (section 2.6.5) is used instead of the unfixed one. But as remarked in the section concerning unitarily fixed ensembles, both average differ only for the operator quantities involving those variables whose overall unitary transformation $x_u \mapsto U_{\text{eff}} x_u U_{\text{eff}}$ has been fixed. This is the case of the conserved C operator, but the difference is considered negligible [1]. But if the variables on which the unitary fixing is performed are different that those in S, both averages give the same result. This is the case of the choice $S(\{x_{r\text{eff}}\})$ to be $x_{u\text{eff}}$ or a monomial in $\{x_{r\text{eff}}\}$. Since ψ_0 is an eigenvector of H_{eff}, and $[H_{\text{ieff}}, i_{\text{eff}}] = 0$, it is also an eigenvector of $\lambda = \lambda_0 i_{\text{eff}}$ (2.97). Therefore, up to a possible constant factor,

$$\psi_0^+ \langle S(\{x_{u\text{eff}}\})\rangle_{\widehat{\rho}} \psi_0 \propto \langle S(\{x_{u\text{eff}}\})\rangle^c_\rho,$$

and both identifications lead to the same emergent quantum theory.

Properties of Wightman functions

The Wightman functions defined by the ensemble averages in (2.196) are expected to be complex valued distributions. Furthermore, the conditions (a) through (f) of section A.2 have to be satisfied by the ensemble averages:

(a) Relativistic transformation law

The relativistic transformation law for the ensemble averages of the effective variables $q_{u\text{eff}}$ is (2.188) with the effective unitary transformation $U = U_{\{a,\Lambda\},\text{eff}}$ given by (2.189). Then we obtain for the Wightman functions (2.196),

$$\begin{aligned} W^{(n)}_{i_1 \dots i_n}(x_1, ..., x_n) &= \langle q_{i_1}(x_1) ... q_{i_n}(x_n)\rangle^c_\rho = \langle U^+ q_{i_1}(x_1) U U^+ q_{i_2}(x_2) U ... U^+ q_{i_n}(x_n) U\rangle^c_\rho = \\ &= \sum_{j_1,\dots,j_n} S_{i_1 j_1}(\Lambda^{-1}) ... S_{i_n j_n}(\Lambda^{-1}) \langle q_{j_1}(x_1 + a) q_{j_2}(x_2 + a) ... q_{j_n}(x_n + a)\rangle^c_\rho = \\ &= \sum_{j_1,\dots,j_n} S_{i_1 j_1}(\Lambda^{-1}) ... S_{i_n j_n}(\Lambda^{-1}) W^{(n)}_{j_1,\dots j_n}(\Lambda x_1 + a, ..., \Lambda x_n + a). \end{aligned} \tag{2.198}$$

The second equality follows from the observation, that U commutes with i_{eff}, and by (2.191) for an operator A the quantity $\langle A\rangle^c_\rho$ is $\text{Tr}(I - i_{\text{eff}} i)A$, so the two outermost U operators cancel each other by the cyclicity property, and the equality becomes an identity. Hence the Wightman functions have the required relativistic transformation property (A.8).

(b) Hermiticity

The hermiticity condition (A.9) is satisfied for all adjointness assignment for the trace dynamics variables and for arbitrary combinations of fermionic and bosonic variables, since the averages of monomials have the same properties with respect to conjugation as the vacuum expectation values,

$$W^{(n)}_{i_1...i_n}(x_1,...,x_n) = \overline{\langle (q_{i_1}(x_1)...q_{i_n}(x_n))^+\rangle^c_\rho} = \overline{\langle q_{i_n}(x_n)^+...q_{i_1}(x_1)^+\rangle^c_\rho} = W^{(n)}_{i^*_n,...i^*_1}(x_n,...,x_1). \tag{2.199}$$

The bosonic trace dynamics variables $q_{i_1},...q_{i_n}$ are assumed to be self-adjoint, hence

$$W^{(n)}_{i_1...i_n}(x_1,...,x_n) = W^{(n)}_{i_1...i_n}(x_n,...,x_1). \tag{2.200}$$

For fermionic variables we have either the self-adjointness or the assignment: $q_i(x)$ arbitrary and $p_i(x) = q_i(x)^+$. The latter option leads to non-self-adjoint field operators.

(c) Local commutativity

The local commutativity condition, (A.12), requires the arguments of Wightman functions to commute or anticommute, when they refer to mutually spacelike separated points. This condition is satisfied due to the commutation/anticommutation relations for the effective variables, which hold when inserted between any two polynomials in the variables $q_l(x)$ and averaged over the statistical ensemble. For instance,

$$\begin{aligned} W^{(n)}_{i_1...i_j i_{j+1}...i_n}(x_1,...,x_j,x_{j+1},...x_n) &= \langle q_{i_1}(x_1)...q_{i_j}(x_j)q_{i_{j+1}}(x_{j+1})...q_{i_n}(x_n)\rangle^c_\rho = \\ &= \pm\langle q_{i_1}(x_1)...q_{i_{j+1}}(x_{j+1})q_{i_j}(x_j)...q_{i_n}(x_n)\rangle^c_\rho = \pm W^{(n)}_{i_1...i_{j+1}i_j...i_n}(x_1,...,x_{j+1},x_j,...x_n), \end{aligned} \tag{2.201}$$

with the sign varying according to the exchange of bosonic or fermionic variables. Since the canonical conjugate momenta are not among the field variables $\{q_l\}$ considered for the definition of the Wightman functions, there are no additional restrictions on the polynomials.

(d) Positivity property

For any sequence $\{f_j\}$ and $\{g_j\}$ of test functions with $f_j \in S(\mathbb{R}^{4j})$, in which only finite number of entries are different from zero,

$$\sum_{j,k}\sum_{j_1...k_k}\int \overline{f_{j,j^*_j...j^*_1}} W^{(k+j)}_{j^*_k...j^*_1 k_1...k_k} f_{k,k_1...k_k} = \left\langle \sum_{j,k}\sum_{j_1...k_k}\int \overline{f_{j,j^*_j...j^*_1}} q^*_{j_j}...q^*_{j_1} q_{k_1}...q_{k_k} f_{k,k_1...k_k} d\mu \right\rangle^c_\rho \geq 0,$$

which can be seen by noting that the right hand side is $\langle A^+A\rangle^c_\rho$ with

$$A = \sum_k \sum_{k_1,...,k_k}\int f_{k,k_1...k_k}(x_1,...x_k) q_{k_1}(x_1)...q_{k_k}(x_k) dx_1...dx_k.$$

Therefore the Wightman functions have the positivity property required by (A.13).

(e) Spectral condition

The spectral condition requires that any Wightman function $W^{(n)}_{i_1...i_n}(x_1,...,x_n)$ can be expressed as a function $\mathbb{W}^{(n-1)}_{i_1...i_n}(x_1-x_2,...,x_{n-1}-x_n)$ of the differences of $\{x_k\}$. By the unitary

representation of spacetime translations (2.187),

$$W^{(n)}_{i_1...i_n}(x_1,...,x_n) = \langle q_{i_1}(x_1)...q_{i_n}(x_n)\rangle^c_\rho = \tag{2.202}$$
$$= \left\langle e^{\frac{i_{\rm eff}}{\hbar}P_\mu x_1^\mu} q_{i_1}(0) e^{-\frac{i_{\rm eff}}{\hbar}P_\mu (x_1-x_2)^\mu} q_{i_2}(0)...q_{i_{n-1}}(0) e^{-\frac{i_{\rm eff}}{\hbar}P_\mu (x_{n-1}-x_n)^\mu} q_{i_n}(0) e^{-\frac{i_{\rm eff}}{\hbar}P_\mu x_n^\mu}\right\rangle^c_\rho .$$

Unless $x_1 = x_n$, the outer exponentials do not cancel each other as before. To establish the spectral property, it is necessary these exponentials to effectively vanish upon taking the average. This corresponds to the requirement of [1] mentioned above, that there is a common eigenvector ψ_0 of the operators P_μ with zero eigenvalue, and the connection between ensemble averages and Wightman functions being made by (2.197). Assuming there is a trace dynamics theory, in which it is possible, we obtain the function $\mathbb{W}$ of differences $\xi_k = x_{k+1} - x_k$ of the coordinates,

$$\mathbb{W}^{(n-1)}_{i_1...i_n}(\xi_1,...,\xi_{n-1}) = \left\langle q_{i_1}(0) e^{\frac{i_{\rm eff}}{\hbar}P_\mu \xi_1^\mu} q_{i_2}(0)...q_{i_{n-1}}(0) e^{\frac{i_{\rm eff}}{\hbar}P_\mu \xi_{n-1}^\mu} q_{i_n}(0)\right\rangle^c_\rho . \tag{2.203}$$

This is a distribution acting in the variables $\xi_1...\xi_{n-1}$. Its Fourier image,

$$\widetilde{\mathbb{W}}^{(n-1)}_{i_1...i_n}(p_1,...,p_{n-1}) = \int \mathbb{W}^{(n-1)}_{i_1...i_n}(\xi_1,...,\xi_{n-1}) e^{-i\sum_k p_k\xi_k} d\xi_1...d\xi_k, \tag{2.204}$$

is, since the matrix $i_{\rm eff}$ acts effectively as an imaginary unit,

$$\widetilde{\mathbb{W}}^{(n-1)}_{i_1...i_n}(p_1,...,p_{n-1}) = (2\pi\hbar)^{4(n-1)} \left\langle q_{i_1}(0)\delta(P-p_1) q_{i_2}(0)...q_{i_{n-1}}(0)\delta(P-p_{n-1}) q_{i_n}(0)\right\rangle^c_\rho . \tag{2.205}$$

Thus $\widetilde{\mathbb{W}}^{(n-1)}$ vanishes, if one of the p_k lies outside the spectrum of all P of the statistical ensemble, in particular outside the forward light cone. The relation (A.16) between $\widetilde{\mathbb{W}}^{(n-1)}$ and the Fourier image $\widetilde{W}^{(n)}$ of $W^{(n)}$ follows, by the use of the Abel summation formula,

$$\sum_{j=1}^n p_j x_j = (p_1 + ... + p_n)x_n - \sum_{j=1}^{n-1}(p_1 + ... + p_j)(x_j - x_j),$$

from

$$\widetilde{W}^{(n)}_{i_1...i_n}(p_1,...,p_n) = \int W^{(n)}_{i_1...i_n}(x_1,...,x_n) e^{-i\sum_j p_j x_j} dx_1...dx_k = \tag{2.206}$$
$$= (2\pi)^4 \delta\left(\sum_{j=1}^n p_j\right) \widetilde{\mathbb{W}}^{(n-1)}_{i_1...i_n}(p_1, p_1+p_2, ..., p_1+p_2+...+p_{n-1}).$$

(f) Cluster decomposition property

The cluster property can be established by introducing yet another assumption, that the fields at sufficiently separated points become statistically independent. Then if there are two mononomials S_1 and S_2 constructed of fields of sufficiently separated supports, the average of their product by the independence satisfy

$$\langle S_1 S_2\rangle_\rho = \langle S_1\rangle_\rho \langle S_2\rangle_\rho. \tag{2.207}$$

The matrix structure of both averages on the right hand side is given by a function of the ensemble parameter λ, and according to the discussion around (2.190) it can be represented by

a single c-number. It follows, that the Wightman function defined by the average of $S_1 S_2$ also splits into a product of Wightman functions associated with S_1 and S_2, $W_{S_1 S_2} = W_{S_1} W_{S_2}$. Then

$$\lim_{\lambda\to+\infty} \langle q_{i_1}(x_1)...q_{i_j}(x_j)\ q_{i_{j+1}}(x_{j+1}+\lambda)...q_{i_n}(x_n+\lambda)\rangle_\rho^c = \tag{2.208}$$

$$= \lim_{\lambda\to+\infty} \langle q_{i_1}(x_1)...q_{i_j}(x_j)\rangle_\rho^c\ \langle q_{i_{j+1}}(x_{j+1}+\lambda)...q_{i_n}(x_n+\lambda)\rangle_\rho^c =$$

$$= \langle q_{i_1}(x_1)...q_{i_j}(x_j)\rangle_\rho^c\ \langle q_{i_{j+1}}(x_{j+1})...q_{i_n}(x_n)\rangle_\rho^c, \tag{2.209}$$

with the last equality due to the invariance of Wightman functions with respect to spacetime translation. Therefore the cluster property required by (A.17) is also satisfied by the Wightman functions derived from the trace dynamics ensemble averages.

Reconstruction of quantum field theory

Since all the conditions are satisfied, the Wightman functions given by the ensemble averages have all the properties of Wightman functions of a local quantum field theory. The Wightman reconstruction theorem can now be used to reconstruct such a local quantum field theory corresponding to the statistical ensemble of the classical trace dynamics fields, provided all the various assumptions stated above can be met by a specific trace dynamics theory.

The construction of the Hilbert space $\mathcal{H}$ and field operators is described in more detail in the appendix. The Hilbert space is the completion (in the norm given by the scalar product) of a set of equivalence classes (f, g are in the same class, if $\|f-g\|=0$) of sequences

$$f = (f_0, f_1, f_2, f_3, \dots) \tag{2.210}$$

of test functions $f_k \in \mathcal{S}(\mathbb{R}^{4k})$. The functions are complex valued for a single-component (scalar) field, and take values in an appropriate tensor algebra for multi-component field. The action of field operators on the vectors (2.210) is given by (A.22) or (A.30), and the scalar product is defined using the Wightman functions by (A.20) or (A.27). There is distinguished cyclic vector $\Psi_0 = (1,0,0,...)$, the vacuum state, which is unique up to a complex multiple. The vectors of the form $(0, f_1, 0, ...)$ (which depend on a single spacetime point), correspond to the one particle states within the standard interpretation of quantum theory. Similarly the vector $(0,0,f_2,0,...)$ corresponds to a two-particle state, and $(0,...,f_k,0,...)$ to a k-particle state. The designation "k-particle state" is adopted also in the context of trace dynamics, although the particle interpretation has not been established there. The Hilbert space $\mathcal{H}$, together with Ψ_0 and the field operators, are determined uniquely up to a unitary equivalence, hence it is unitarily equivalent to the Fock space.

The Wightman functions are defined using the trace dynamics field variables $\{q_{r\text{eff}}\}$ only, and they correspond to the vacuum expectation values of the field operators φ_r. The canonically conjugated momenta $p_{r\text{eff}}$ can be nevertheless expressed using the field variables $q_{r\text{eff}}$ and their spacetime derivatives. Therefore both field variables and their momenta are carried over to the emergent quantum theory, together with their canonical commutation relations and Heisenberg evolution equations. The corresponding variables of trace dynamics and of quantum field theory are denoted by

$$q_{r\text{eff}}(x) \leftrightarrow \varphi_r(x), \qquad p_{r\text{eff}}(x) \leftrightarrow \pi_r(x), \qquad x_{r\text{eff}}(x) \leftrightarrow X_r(x), \tag{2.211}$$

where r is the field component index. The commutation relations of trace dynamics for bosonic fields (2.170) then become

$$[\varphi_r(x), \varphi_s(y)] = 0, \quad [\pi_r(x), \pi_s(y)] = 0, \quad [\varphi_r(x), \pi_s(y)] = i\hbar\delta_{rs}\delta(x-y), \tag{2.212}$$

the anticommutation relations for fermionic variables,

$$\{\varphi_r(x), \varphi_s(y)\} = 0, \quad \{\pi_r(x), \pi_s(y)\} = 0, \quad \{\varphi_r(x), \pi_s(y)\} = i\hbar\delta_{rs}\delta(x-y), \tag{2.213}$$

and all fermionic variables commute with any bosonic one. For simplicity of notation, we further consider only the one component (scalar) field, and drop the component indices r, s.

The Hilbert space obtained by the reconstruction theorem has a non-standard scalar product, given by the Wightman functions. For example, the one particle space is formed by (the norm closure of) the set of vectors

$$\{\Phi_n(x) = (0, \phi_n(x), 0, ...) | \phi_n \in \mathcal{S}(R^4)\},$$

and the scalar product is

$$\langle \Phi_n, \Phi_m \rangle_{\mathcal{H}} = \int \overline{\phi_n(x)} W^{(2)}(x,y) \phi_m(y) dx dy = \langle \phi_n, \phi_m \rangle_{\mathcal{H}_1}, \tag{2.214}$$

hence the one particle space is effectively (the closure of) $\mathcal{S}(R^4)$. Sometimes it is convenient to omit the completion step of the reconstruction of $\mathcal{H}$. Then the resulting space is not a Hilbert space, since it is not complete in the norm given by the scalar product. We obtain rather the structure of *rigged* Hilbert space of quantum theory, [6]. For any $\phi \in S(\mathbb{R}^4)$, the scalar product can be used to define a functional on $S(\mathbb{R}^4)$, a generalized function $F_\phi \in S'(\mathbb{R}^4)$,

$$\phi \mapsto F_\phi(\psi) = \langle \phi, \psi \rangle_{\mathcal{H}_1}. \tag{2.215}$$

Then the rigged Hilbert space is given by the triple $S(\mathbb{R}^4) \subset L^2(\mathbb{R}^4) \subset S'(\mathbb{R}^4)$, with $S'(\mathbb{R}^4)$ the set of distributions associated with $S(\mathbb{R}^4)$. In quantum theory, the operators are defined on (a subset of) $L^2(\mathbb{R}^4)$, and they can be extended even beyond $L^2(\mathbb{R}^4)$ to include the vectors associated with the continuous part of the spectra.

To obtain the one particle space with the usual scalar product, we have to find a basis in which W diagonalizes. We can introduce the field operators $A(x)$ and $A^+(y)$,

$$A = \frac{1}{\sqrt{2}}(\varphi + i\pi), \qquad A^+ = \frac{1}{\sqrt{2}}(\varphi - i\pi), \tag{2.216}$$

and the relations (2.212) and (2.213) can be rewritten in terms of the operator A as

$$[A(x), A(y)]_\pm = 0 = [A^+(x), A^+(y)]_\pm, \quad [A(x), A^+(y)]_\pm = \hbar\delta(x-y), \tag{2.217}$$

with the plus sign for fermionic, and the minus sign for bosonic operators. If $A(x)$ annihilate the vacuum, then $A(x)$ and $A(x)^+$ can be regarded as annihilation and creation operator respectively. The creation and annihilation operators can be used to form one or multiple-particle states with the standard scalar product. For one particle subspace formed by all the one particle states, by the annihilation property of A and (2.217),

$$\langle \Psi_0, A(x) A(y)^+ \Psi_0 \rangle_{\mathcal{H}} = \hbar\delta(x-y). \tag{2.218}$$

Inserting a complete orthonormal (with respect to the scalar product on $\mathcal{H}$) set $\{\phi_n\}$, we obtain

$$\sum_n \hbar^{-1/2} \langle A(x)^+ \Psi_0, \phi_n \rangle_{\mathcal{H}} \, \hbar^{-1/2} \langle \phi_n, A(y)^+ \Psi_0 \rangle_{\mathcal{H}} = \sum_n \overline{\psi_n(x)} \psi_n(y) = \delta(x-y), \tag{2.219}$$

with

$$\psi_n(x) = \hbar^{-1/2}\langle \phi_n, A(x)^+\Psi_0\rangle_{\mathcal{H}}, \tag{2.220}$$

a one particle wavefunction, which is a generalized function from $\mathcal{S}'(\mathbb{R}^4)$. Multiplying (2.219) by $\psi(x)$ and integrating over x [8], we learn that the set $\{\psi_n(x)\}_n$ is an orthogonal set with respect to the standard scalar product on $\mathcal{S}(\mathbb{R}^4)$,

$$\int \overline{\psi_n(x)}\psi_m(x)dx = \delta_{mn}, \tag{2.221}$$

and it is the basis (of the one particle space), in which the scalar product becomes the standard one.

The reconstruction theorem provides also the Hamiltonian operator (denoted again H_{eff}), since it is given by a polynomial in the dynamical variables, and the Heisenberg evolution operator U_{eff}. Both are operators acting on the Hilbert space $\mathcal{H}$ of the emergent quantum field theory, and

$$U_{\text{eff}}(t) = \exp\left(-\frac{i}{\hbar}tH_{\text{eff}}\right). \tag{2.222}$$

The unitary operator U_{eff} acts on those vectors, whose support is confined onto the spacelike surface orthogonal to the time direction in the chosen Lorentz frame, as it comes from the operator (2.179). Changing the reference frame changes the Hamiltonian and hence the evolution operator, since the conserved energy-momentum trace four-vector transforms as a vector. These vectors are the Heisenberg picture vectors. Let Φ_{t_0} be such a vector. Then $t \mapsto \Phi_t = U_{\text{eff}}(t-t_0)\Phi_{t_0}$ is the corresponding Schrödinger picture vector, from which it follows that Φ_t obeys the Schrödinger equation

$$i\hbar\frac{\partial\Phi_t}{\partial t} = H_{\text{eff}}\Phi_t. \tag{2.223}$$

This is an evolution equation on the multi-particle space of quantum field theory. For a suitable Hamiltonian (with the one particle space as its invariant subspace) it can be specialized to the one particle Schrödinger equation by restriction to the one particle space. With the notation $\Phi_t = (0, \phi_t, 0, ...)$, and $H^{(1)}_{\text{eff}}$ the restriction of H_{eff} to the one particle space, the one particle Schrödinger equation reads

$$i\hbar\frac{\partial\phi_t}{\partial t} = H^{(1)}_{\text{eff}}\phi_t. \tag{2.224}$$

Therefore on the basis of validity of the assumptions made in the simplification of the Ward identities in section 2.8, the averages of trace dynamics variables are associated with quantum field operators, and their dynamics is governed by the usual quantum-theoretic dynamical laws.

2.9 Consistence of approximations

All the above results can be derived as soon as the effective canonical commutation/anticommutation relations, the Heisenberg evolution law, and the general unitary canonical transformation equations are established for the ensemble averages of trace dynamics variables. But this could be made only after some rather far reaching conditions have been imposed on those underlying trace dynamics theory. It is not a priori clear, whether there exists such a theory, that would meet all the conditions and assumptions. Therefore there appears a necessity to examine the consistency of all the conditions and assumptions, that lead to the emergence of the structure of quantum theory. These assumptions are mostly summarized in section (2.8.2).

[8] This is not entirely mathematically correct, since the action of the distribution $\psi(x)$ is defined only for the test functions from $\mathcal{S}(\mathbb{R}^+)$. On the other hand, any generalized function can be approximated by a function from $\mathcal{S}(\mathbb{R}^4)$.

2.9.1 Commutation relations

Some consistency issues with the assumptions and approximations leading to the emergence of quantum theory were already encountered in the discussion around the equation (2.174). Using the emergent canonical commutation relations (2.170) (for bosonic variables) leads to the equation,

$$q^2p^2 + p^2q^2 - 2qp^2q \overset{\rho,S}{\approx} -2\hbar^2. \tag{2.225}$$

This should in average hold as a weak operator identity, i.e. the difference of both sides should be zero, when inserted between arbitrary polynomials and averaged over the zero source ensemble. This raises doubts about the existence of a trace dynamics theory with all the required properties to meet the various assumptions.

Consider the Ward identity given by (2.160),

$$\left\langle [\{i_{\rm eff}, W\}, x_u] + \sum_{lr} \omega_{ur}\epsilon_l W_r^{Rl}\{i_{\rm eff}, C\}W_r^{Ll} + (-\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u)\,{\rm Tr}(\{i_{\rm eff}, C_{\rm eff}\}W_{\rm eff}) \right\rangle_\rho = 0. \tag{2.226}$$

Taking $\mathbf{A} = 1$ in the general Ward identity (2.116), and using the result of variation of the term (2.142), we get

$$\langle Dx_u\rangle_{\rho_j} = 0, \quad \text{with} \quad Dx_u = -\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u - \sum_r \omega_{ur} j_r. \tag{2.227}$$

The variation of the sources j_r can be used to obtain the ensemble average of the superfluous term,

$$(-\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u)\,{\rm Tr}(\{i_{\rm eff}, C_{\rm eff}\}W_{\rm eff}), \tag{2.228}$$

in (2.226), that is assumed to be approximately zero for the argument of emergence of quantum theory. Setting $W = \sigma_s x_s$ as in the derivation of the commutation relations, and using the most general form of the extended Ward identity (2.153), we learn that the second variation terms given by the right hand side of this Ward identity yield exactly the first two terms in (2.226) with $W = \sigma_s x_s$. This indicates that the superfluous term (2.228) is not approximately zero with respect to the remaining terms (leaving the latter also zero, as needed for the emergent commutation relations to hold), but they are both nonzero and rather cancel each other.

Moreover, the average value of the commutator of effective projections of any two bosonic dynamical variables $x_{u\rm eff}$ and $x_{v\rm eff}$ is always zero. This follows from the fact, that the average of $[x_{u\rm eff}, x_{v\rm eff}]$ is a function of the ensemble operator parameter $\lambda = \lambda_0 i_{\rm eff}$, and by (2.191) it can be expressed as

$$\langle [x_{u\rm eff}, x_{v\rm eff}]\rangle_\rho^c = \frac{1}{N}\,{\rm Tr}\left((I - i_{\rm eff}i)\langle [x_{u\rm eff}, x_{v\rm eff}]\rangle_\rho\right). \tag{2.229}$$

Since all effective variables commute with $i_{\rm eff}$, the average of the commutator is zero. Similarly, the average of the anticommutator of effective projections of any two fermionic dynamical variables is also zero. Then the Ward identity (2.226) with $W = \sigma_v x_v$ becomes

$$\langle \omega_{uv}\sigma_v\{C, i_{\rm eff}\} + (-\tau\dot{x}_u + \gamma_u x_u)\,{\rm Tr}(\{i_{\rm eff}, C_{\rm eff}\}\sigma_v x_{v\rm eff})\rangle_\rho = 0. \tag{2.230}$$

Since $\langle \omega_{uv}\sigma_v\{C, i_{\rm eff}\}\rangle_\rho = -2\hbar\omega_{uv}\sigma_v$, it is not possible to neglect the average value of the superfluous term (2.228). In particular, the assumption for vanishing the term with τ due to different support properties of $\dot{x}_u$ and C cannot be true. This spoils the argument for the emergence of commutation

relations and similarly also for the emergence of Heisenberg evolution equations for the effective dynamical variables. Therefore the identification of quantum variables with the averages of effective projections of trace dynamics variables is not viable.

However, without the effective projection, the ensemble averaged commutators/anticommutators of the bosonic/fermionic variables x_u and x_v can in general be nonzero, since they no longer commute with $i_{\rm eff}$. We can derive another Ward identities by taking $A = CW$ in the general Ward identity (2.116), with the use of the derivation of (2.225) replacing $\{i_{\rm eff}, C\}$ by C,

$$\left\langle [W, x_u] + \sum_{lr} \omega_{ur}\epsilon_l W_r^{Rl} C W_r^{Ll} + (-\tau\dot{x}_u + \gamma_u x_u)\,{\rm Tr}CW \right\rangle_\rho = 0. \tag{2.231}$$

For $W = \sigma_v x_v$ it becomes

$$\left\langle [x_v, x_u]_{-\epsilon_u} + \omega_{uv} i_{\rm eff}\hbar + (-\tau\dot{x}_u + \gamma_u x_u)\,{\rm Tr}Cx_v \right\rangle_\rho = 0, \tag{2.232}$$

with the auxiliary c-number σ_v stripped away, and with the use of

$$[\sigma_v x_v, x_u] = \sigma_v [x_v, x_u]_{-\epsilon_u}.$$

Hence we obtain the commutation relations

$$\epsilon_u \left[x_u, x_v\right]_{\epsilon_u} \stackrel{\rho}{=} \omega_{uv} i_{\rm eff}\hbar + (-\tau\dot{x}_u + \gamma_u x_u)\,{\rm Tr}Cx_v, \tag{2.233}$$

where the $\stackrel{\rho}{=}$ sign means, that the equality holds exactly when averaged over the ensemble, but without inserting between polynomials. For bosonic variables it becomes, with the definition of γ_u in (2.150), and ω_{uv} in (2.25),

$$[q_r, p_s] \stackrel{\rho}{=} \delta_{rs} i_{\rm eff}\hbar + (-\tau\dot{q}_r + \alpha q_r)\,{\rm Tr}Cp_s \stackrel{\rho}{=} \delta_{rs} i_{\rm eff}\hbar + (\tau\dot{p}_s + \alpha p_s)\,{\rm Tr}Cq_r \tag{2.234}$$

and for fermionic variables,

$$\{q_r, p_s\} \stackrel{\rho}{=} \delta_{rs} i_{\rm eff}\hbar + (\tau\dot{q}_r + \beta q_r)\,{\rm Tr}Cp_s \stackrel{\rho}{=} \delta_{rs} i_{\rm eff}\hbar + (\tau\dot{p}_s - \beta p_s)\,{\rm Tr}Cq_r. \tag{2.235}$$

If the commutation/anticommutation relations were to hold, then the terms with traces should effectively vanish or become negligible. From the derivation of the C operator (2.53) it follows that

$$i_{\rm eff}\hbar = \langle C\rangle_\rho \stackrel{\rho}{=} i_{\rm eff}\hbar(n_B - n_F) - \sum_{r\in B\cup F} (\tau\dot{q}_r - \gamma_r q_r){\rm Tr}Cp_r, \tag{2.236}$$

with n_B and n_F denoting the total number of bosonic and fermionic variables respectively. Hence in any trace dynamics theory, in which the superfluous terms approximately vanish, there has to be approximately equal number of fermionic and bosonic variables.

In the general case, when the relations are to hold even when inserted between two polynomials in the dynamical variables, we have to reintroduce the $\epsilon_u[\lambda, x_u]$ term and the C operator, and use the extended Ward identity (2.127) to obtain

$$\epsilon_u \left[x_u, x_v\right]_{\epsilon_u} \stackrel{\rho,S}{=} \omega_{uv} C + (-\tau\dot{x}_u + \epsilon_u[\lambda, x_u] + \gamma_u x_u)\,{\rm Tr}Cx_v + \sum_{v\in S_L\cup S_R} \omega_{uv}(S_L D x_u S_R)_{\widehat{v}}. \tag{2.237}$$

Expanding the compact symplectic notation, the *exact* commutation relations for bosonic variables become

$$[q_r, p_s] \overset{\rho,S}{=} \delta_{rs} C + (-\tau\dot{q}_r + [\lambda, q_r] + \alpha q_r)\,\mathrm{Tr}Cp_s + \sum_{p_s \in B\cap(S_L\cup S_R)} (S_L Dx_s S_R)_{\widehat{s}}, \tag{2.238}$$

and the anticommutation relations for fermionic variables,

$$\{q_r, p_s\} \overset{\rho,S}{=} \delta_{rs} C + (\tau\dot{q}_r + [\lambda, q_r] + \beta q_r)\,\mathrm{Tr}Cp_s + \sum_{p_s \in F\cap(S_L\cup S_R)} (S_L Dx_s S_R)_{\widehat{s}}. \tag{2.239}$$

Using these commutation relations in (2.225), and denoting $B = (\tau\dot{q} - [\lambda, q] + \alpha q)\mathrm{Tr}Cp$, we obtain,

$$q^2p^2 - p^2q^2 - 2qp^2q \overset{\rho,S}{=} q(C+B)p - p(C+B)q + qp(C+B) - (C+B)pq, \tag{2.240}$$

which removes the previous contradiction arising from different traces of the left and right hand sides of (2.225). This nevertheless shows, that even if the superfluous terms could be made negligible and C replaced by its ensemble average $i_{\mathrm{eff}}\hbar$, the averaged (anti)commutation relations are not (neither approximately) equivalent to those of ordinary quantum field theory. With such assumptions, up to the approximations involved,

$$(2.240) \overset{\rho,S}{\approx} qi_{\mathrm{eff}}p - pi_{\mathrm{eff}}q + qpi_{\mathrm{eff}} - i_{\mathrm{eff}}pq = \hbar(q\widetilde{p} - p\widetilde{q} + q\widetilde{p} - \widetilde{p}\widetilde{q})i_{\mathrm{eff}} = \hbar(qp_{\mathrm{eff}} - p_{\mathrm{eff}}\widetilde{q})i_{\mathrm{eff}} \overset{\rho}{=} 0,$$

since $[i_{\mathrm{eff}}, x] = \widetilde{x}$, for $x \in \{q, p\}$, and with the notation $\widetilde{x} = x_{\mathrm{eff}} - x_{12}$ for $x = x_{\mathrm{eff}} + x_{12}$. The last equality follows from the fact that all the "12" components of any operator and the commutator $[q_{\mathrm{eff}}, p_{\mathrm{eff}}]$ vanish in the ensemble average.

2.9.2 Heisenberg evolution

Taking $W = H$ in the Ward identity $\langle Dx_u\rangle_\rho$ with Dx_u given by (2.226), we get the identity

$$\left\langle (-\tau\dot{x}_u + \gamma_u x_u)\mathrm{Tr}CH + [H, x_u] + \sum_{s,l} \omega_{us} H_s^{Rl} C H_s^{Ll} \right\rangle_\rho = 0. \tag{2.241}$$

Since the operators H_s^{Rl} in general commute neither with C nor with i_{eff}, the last term no longer (approximately) equals to $i_{\mathrm{eff}}\dot{x}_u$. Moreover, if we admit that the γ_u can be set to zero, from (2.227) it follows, that $\langle\dot{x}_u\rangle_\rho = 0$ independently of the trace dynamics operator Hamiltonian H. Therefore the time evolution of the averages of trace dynamics variables is *not* governed by the Heisenberg evolution law $\dot{x}_u \overset{\rho,S}{\approx} [H, x_u]$ in the static ensemble, (2.89).

From all these remarks it can be concluded, that the necessary conditions for emergence of quantum theory apparently cannot be satisfied by the static ensemble averages associated with any underlying trace dynamics theory. It nonetheless cannot be used to completely exclude the possibility, that the commutation relations and unitary canonical transformations could be a feature of a trace dynamics theory with another operator phase space probability distribution, or even of some non trace dynamics matrix model (a matrix model whose dynamics is not determined by a trace Hamiltonian). It is therefore worthwhile to keep assuming that such an underlying theory could exist, and try to draw some conclusions about the possible emergence of quantum probability.

2.10 Trace dynamics and quantum probability

In this section it is taken for granted, that there exists a theory, whose statistical properties give rise to the emergence of quantum field structures as in the previous sections. The ensemble averages of its dynamical variables obeys the appropriate commutation relations and unitary transformation laws, and they define the Wightman functions satisfying all the axioms of a local quantum field theory. This then allows their identification with the vacuum expectation values and the reconstruction of the Hilbert space formalism of quantum field theory. We are concerned with whether the quantum probability already emerges from such an identification, as the effective probability associated with the statistical description of the deterministic underlying theory.

The construction of quantum field theory from the Wightman functions (section A.3) suggests[9], that the field theory is complete, i.e. any vector of the emergent Hilbert space $\mathcal{H}$ can be obtained by the action of a polynomial in the field operators on the vacuum state Ψ_0. The field operator X_u correspond to a statistical ensemble of trace dynamics operators x_u, given by the operator phase space distribution. We can reinterpret the moments of x_u as the moments of the corresponding field operator X_u, and similarly all the cross-moments of $\{x_u\}$ with their corresponding cross-moments of $\{X_u\}$. The moments and cross-moments of $\{X_u\}$ are not given by a probability distribution, but directly by the Wightman functions. For example, we interpret $\langle\Psi_0, X_1\Psi_0\rangle_{\mathcal{H}} = \langle x_1\rangle_\rho^c$ as the average value of X_1, $\langle\Psi_0, X_1^2\Psi_0\rangle_{\mathcal{H}} = \langle x_1^2\rangle_\rho^c$ as the second moment of X_1, and $\langle\Psi_0, X_1X_2\Psi_0\rangle_{\mathcal{H}} = \langle x_1x_2\rangle_\rho^c$ as the covariance of X_1 and X_2.

If we know all the Wightman functions, we know also all the moments of an observable $\mathcal{O}$ (a function of the field operators $\{X_u\}$), and they can be used to recover the probability distribution on the space of values the observable could take. These possible values have to be actually inferred only from the values of the moments of $\mathcal{O}$, since the correspondence between values of trace dynamics variables and the individual observed values has not been established. Since all the moments of $\mathcal{O}$ are c-numbers, the recovered values of $\mathcal{O}$ are c-numbers too, whereas the trace dynamics variables are matrices. Such a theory then could give only a statistical prediction of results of experiments, the set of possible values and their probability distribution.

Consider two observables $\mathcal{O}$ and $\mathcal{O}'$ that correspond to trace dynamics operators A and A' respectively[10]. If A effectively commutes with A' when averaged over the statistical ensemble[11], then there is a common classical probability space of the values of $\mathcal{O}$ and $\mathcal{O}'$, on which probability distributions for both observables, including the joint probability distribution, can be recovered from the moments and cross-moments. Thus the moments and cross-moments of any fixed commutative set of operators $\{O_k\}$ are compatible with the moments of some random variables of a classical probability theory. Then it is in principle possible to find a classical probability space (the space of individual outcomes and a probability distribution), on which the operators are represented by random variables, such that their statistical properties are the same as those of the operators.

But it could also happen, than A and A' do not commute in the average due to the commutation relations (2.170) and (2.171). In this case there exists no classical probability space on which both $\mathcal{O}$ and $\mathcal{O}'$ could be represented by random variables, since otherwise their correlation would have to be commutative, but that contradicts

$$\langle\Psi_0, OO'\Psi_0\rangle_{\mathcal{H}} = \langle AA'\rangle_\rho^c \neq \langle A'A\rangle_\rho^c = \langle\Psi_0, O'O\Psi_0\rangle_{\mathcal{H}}. \tag{2.242}$$

[9]This actually holds, and it is proved for example in [7].

[10]If $\mathcal{O}$ is given by a polynomial S in the field operators, then A is given by the same polynomial S, but in the corresponding trace dynamics variables.

[11]This means that $AA' \overset{\rho,S}{=} AA'$, and it happens, if A does not contain a canonically conjugated variable of some variable in A'. Then also the corresponding observables $\mathcal{O}$ and $\mathcal{O}'$ commute.

Therefore the statistics associated with $\{O\}$ are in general described by a noncommutative probability. The trace dynamics (random) variables are operator valued, but their ensemble averages are given by the c-numbers $\langle x_u \rangle_\rho^c$. The ensemble averages are nevertheless still noncommutative in the sense of (2.242). Let $\mathcal{A}$ be the algebra of the trace dynamics variables, and define a state ϕ on $\mathcal{A}$ by (2.192) with subsequent averaging over the ensemble. Then for any polynomial in the dynamical variables S, $\phi(S)$ is the Wightman function W_S. Therefore the emergent noncommutative probability is just a consequence of the fact, that another noncommutative probability has been effectively employed at the trace dynamics level.

The physical interpretation of the trace dynamics fields as the hidden-variables and the emergence of definite measurement outcomes does not immediately follow from the underlying theory. If the underlying trace dynamics determined all the quantum measurement outcomes, it would either have to be non-local or in some way deny the reality of the measurement outcomes in order to avoid the Bell theorem. The connection between locality and commutativity of observables (i.e. two spacelike separated observables in a local theory have to commute) is only a consequence of the projection postulate and the Born probability rule. But these quantum-theoretic rules are not available for an underlying trace dynamics theory. Hence although the trace dynamics variables are given by completely noncommutative matrices, it still does not imply, that the underlying trace dynamics theory is necessarily non-local. Actually, there is no non-locality in trace dynamics, unless it is explicitly inserted into the dynamics of the underlying theory.

The underlying trace dynamics theory is Lorentz covariant, since the trace dynamics fields are classical fields with an assumed Lorentz covariant Lagrangian. The non-locality need not necessarily be in a conflict with Lorentz covariance and causality. In a deterministic theory with reversible dynamics, the cause and the corresponding effect are interchangeable, and the possible reversed time ordering of both events in a specific Lorentz frame makes no problem. On the other hand, a non-local stochastic theory has to pick a preferred Lorentz frame, in which the non-local interactions propagate instantaneously. Such a theory then cannot be Lorentz covariant, but this does not exclude the Lorentz covariance of some statistical approximation, of which quantum theory might be an example.

2.10.1 Born probability rule

The probability interpretation and the Born probability rule is proposed [1] to be a consequence of the statistical ensemble fluctuations of the C operator in (2.162). These fluctuations have been neglected in the passage from (2.162) to (2.164), which subsequently yielded the effective Heisenberg unitary evolution equation (2.176). Keeping these fluctuation terms, the effective evolution equation become modified, and if the C operator is not purely anti-self-adjoint, the evolution may cease to be unitary. Then also the Schrödinger equation (2.224) become modified by the fluctuation terms. The equation is still linear, but it is no longer unitary, hence it does not preserve the norm of the statevector.

The fluctuation terms are modeled by time dependent classical stochastic process, which is added to the Schrödinger equation, turning it into a stochastic Schrödiger equation. Taking an average over the realizations of the stochastic process, it becomes again the original Schrödiger equation without any fluctuation terms. These fluctuation terms causes the time evolution of a statevector ψ not to be deterministic, and the norm of ψ may change with time. It is then suggested to replace ψ by the normalized vector Ψ,

$$\psi \mapsto \Psi = \psi / \langle \psi, \psi \rangle_{\mathcal{H}}. \tag{2.243}$$

As a consequence, the time evolution of the normalized vector Ψ becomes governed by a nonlinear Schrödinger equation.

In general, a nonlinearity in the Schrödinger equation is responsible for non-local effects, such as the instantaneous action-at-distance [15], [16]. But the nonlinearity[12] can also be made responsible for the dynamical reduction of the statevector [18] during the measurement of an observable $\mathcal{O}$. This is a process, in which (with a suitable non-linear Schrödinger equation) an initial statevector continuously evolves into exactly one of the eigenstates of $\mathcal{O}$, with the particular eigenstate determined by the actual trajectory of some stochastic process. Since the nonlinearity is present only in the fluctuations, which vanish when averaged with respect to the (unobservable) stochastic process, the non-locality is not directly observable through the statistical predictions of the theory.

With some additional assumptions, the Schrödinger equation for the normalized statevector Ψ is shown ([1],[2],[4]), to be of a suitable form to trigger the dynamical reduction of Ψ into one of the eigenvectors φ_k of $\mathcal{O}$, when the system interacts with a measurement apparatus associated with an observable $\mathcal{O}$. Moreover, the probability of transition $\Psi \to \varphi_k$ turns out to be given by the Born rule, $|\langle \Psi, \varphi_k \rangle_{\mathcal{H}}|^2$.

The Born probability rule also follows already from the structure of the probability algebra (as a set of linear operators on a Hilbert space), the interpretation of the state on the algebra as the expectation functional, and the assumption, that the collapse takes place during the measurement process. By the Gleason theorem (section 1.1), any (normal) state ϕ is given by a density operator ρ, and $\phi(\mathcal{O}) = \mathrm{Tr}\rho\mathcal{O}$. After the collapse, the observable $\mathcal{O}$ takes definite value, hence ρ has to be the one-dimensional projection P_{φ_k} given by an eigenvector φ_k of $\mathcal{O}$. The initial state ϕ_0 is given by the density operator $\rho_0 = P_\Psi$, due to a statevector collapse in the preparation stage of the experiment. Then from the interpretation of the state, the probability of the transition $\Psi \to \varphi_k$ is $\phi_0(P_{\varphi_k}) = \mathrm{Tr}P_\Psi P_{\varphi_k} = |\langle \Psi, \varphi_k \rangle_{\mathcal{H}}|^2$.

The correspondence between trace dynamics variables and field operators provides the probability algebra, the state on this algebra, and the interpretation of the state. The last thing to establish the Born rule is the reduction of the statevector, and this is not possible, when the dynamics of the measurement process is given by a linear Schrödinger equation. Unfortunately, the renormalization of the statevectors $\psi \to \Psi$, (2.243), which has brought the nonlinearity into the stochastic Schrödinger equation, is not well justified and presumably invalid. The statevectors are obtained by the action of field operators on the vacuum state, and there is no a priori reason for them to be normalized. And even if the initial statevector is normalized, its norm need not be conserved by the evolution with the stochastic Schrödinger equation, since the stochastic terms disturb the unitarity of the evolution[13]. Moreover, the renormalization artificially introduces non-locality, which is not (or need not to be) present in the underlying trace dynamics theory. This is a kind of non-locality, that appears even in the case, when the underlying theory is local. This suggests that the renormalization of the statevector is not correct, and consequently the fluctuating terms in the stochastic Schrödinger equation do not actually lead to the statevector reduction.

[12] As noted in the first chapter, the linear Schrödinger equation cannot lead to a collapse of the statevector.

[13] The norm is conserved, only when averaged over the realizations of the stochastic process. In this case the stochastic terms are averaged out, and the stochastic Schrödinger equation becomes the ordinary deterministic one.

Conclusion

The effective projections of trace dynamics variables, at a first sight, could provide the correct Wightman functions of an emergent quantum field theory. But we have seen that some of the the relevant terms in the Ward identities have a tendency to vanish, while the the unwanted superfluous terms are likely to persist. If we use the full dynamical variables, we get non-zero results, and we also spare the assumption, that all the effective operators are composed only from the effective parts of the dynamical variables. But then the familiar structure of quantum theory does not appear, since the emergent imaginary unit no longer commutes with the other operators.

Furthermore, the argument effectively requires, at least in some approximation, that the averaged effective dynamics and commutation relations to be independent of the probability distribution of the trace dynamics fields. This condition is likely not to be possible to satisfy, and it also spoils the interpretation of the state of a quantum system as a specific configuration of the trace dynamics fields. The quantum probability is recovered anyway, but in a trivial way, since it was effectively imposed on the underlying theory by the specific way, in which the correspondence with Wightman functions has been established. The appearance of definite outcomes of measurements and the particle interpretation of the fields have remained unexplained. The underlying theory would have to be non-local, in order to imply an effective dynamical state vector reduction, but trace dynamics is not. Therefore trace dynamics cannot explain the collapse of the state vector. Unless a way to overcome all these problems is found, the emergence of quantum theory from trace dynamics is likely to be a lost cause.

Appendix A

Quantum fields and Wightman functions

This appendix, reviews some basic properties of Wightman functions, [7] and [10], their relation to vacuum expectation values and the Hilbert space formalism of quantum field theory. Such a correspondence can be used to reconstruct the Hilbert space and the quantum field operators acting on it from the statistical ensemble averages of the operators of a trace dynamics theory, provided certain requirements placed on this averages are satisfied.

A.1 Axiomatic quantum field theory

A classical field is a function defined on spacetime and taking values in $\mathbb{R}$ or $\mathbb{R}^n$. The natural generalization of the notion of classical field to quantum field would be a function defined on spacetime, and taking values in the set of operators on a Hilbert space. However, it turns out that the fields in their dependence on spacetime points are more singular than the classical fields. This singularity is an inevitable consequence of the requirement of relativistic covariance of the function $x \mapsto \phi(x)$. It is even true, that if the field was a relativistic covariant operator function on the spacetime, the corresponding Hilbert space would have to be trivial, i.e. consisting of the vacuum state only, [13]. To remedy this, the quantum field ϕ can be defined as an operator valued distribution, a linear functional from a space of test functions from $\mathcal{S}(\mathbb{R}^4)$ with values in the set of linear operators on a Hilbert space. The space $\mathcal{S}(\mathbb{R}^4)$ can be either the Schwartz space or the space of infinitely differentiable functions with compact support. Unlike the field $x \mapsto \phi(x)$, the smeared field $f \in \mathcal{S}(\mathbb{R}^4) \mapsto \phi(f)$ is a well defined operator.

A field theory is required to have the following properties (the Wightman axioms, [7]):

1. General properties

The states of the quantum field theory are described by unit rays in a separable Hilbert space $\mathcal{H}$ with the scalar product

$$(\varphi, \psi) \in \mathcal{H} \times \mathcal{H} \mapsto \langle \varphi, \psi \rangle \in \mathbb{C}.$$

For any two fixed vectors $\varphi, \psi \in \mathcal{H}$, the expression

$$f \in \mathcal{S} \mapsto \langle \varphi, \phi_k(f)\psi \rangle$$

defines a complex valued distribution (generalized function) acting on $\mathcal{S}(\mathbb{R}^4)$, and $\phi_k(f)$ is then an operator valued distribution. There is a dense subset D of the Hilbert space $\mathcal{H}$, such

that for any $f \in \mathcal{S}(\mathbb{R}^4)$ the operators $\phi_k(f)$ and their adjoints $\phi_k(f)^+$ are defined on D, and D is their common invariant subset,

$$\phi_k(f)D \subset D, \quad \phi_k(f)^+D \subset D.$$

This subset is large enough to ensure that any symmetric operator defined on D has a unique self-adjoint extension. There is a distinguished vector Ψ_0, the *vacuum state*, that is a cyclic vector for the smeared fields, i.e. the set $D_0 \subset D \subset \mathcal{H}$,

$$D_0 = \bigcup_P P(\{\phi_k(f_l)\})\Psi_0,$$

is dense in $\mathcal{H}$ (the completeness property of the field theory), where the union is over all polynomials P in the field operators.

2. Relativistic properties

The Poincaré transformation law is given by a continuous unitary representation of the Poincaré group,

$$\{a, \Lambda\} \mapsto U(a, \Lambda),$$

where a is a four-vector of spacetime translation, and Λ an element of the proper Lorentz group.[1] The field operator $\phi_i(f)$ transforms as

$$\phi_i(f) \mapsto U_{\{a,\Lambda\}}\phi_i(f)U^+_{\{a,\Lambda\}} = \sum_{ij} S_{ij}(\Lambda^{-1})\phi_j(\{a, \Lambda\}f), \tag{A.1}$$

with the action of the Poincaré group element $\{a, \Lambda\}$ on $f \in \mathcal{S}(\mathbb{R}^4)$ given by

$$\{a, \Lambda\}f(x) = f(\Lambda^{-1}(x - a)), \tag{A.2}$$

and where $\Lambda \mapsto S_{ij}(\Lambda)$ the representation of the Lorentz group on the field components. The vacuum state Ψ_0 and the set D are invariant under the action of $U_{\{a,\Lambda\}}$,

$$U_{\{a,\Lambda\}}D \subset D, \quad U_{\{a,\Lambda\}}\Psi_0 = \Psi_0. \tag{A.3}$$

The spacetime translations are given by $U_{\{a,I\}} = \exp(iP_\mu a^\mu)$, with the generator P^μ an unbounded self-adjoint operator. The operator $P^\mu P_\mu = m^2 I$ represents a squared mass of a particle associated with the particular representation of Poincaré group. The spectrum of P^μ is restricted to be contained in the forward light cone, $P^\mu P_\mu \geq 0$ (the spectral condition).

3. Causality properties

For any two test functions $f, g \in \mathcal{S}$ with spacelike separated support, i.e. $f(x)g(y) = 0$ for x and y spacelike separated points, the fields $\phi_i(f)$ and $\phi_j(g)$ either commute or anticommute (not both),

$$[\phi_i(f), \phi_j(g)]_\pm = 0.$$

[1] The proper Lorentz group is the subgroup of the full Lorentz group, which is connected with the unit element. In the notation, the Poincaré group elements and the corresponding matrices Λ or four-vectors a (associated with the representation of the group on the spacetime) are identified.

It is often convenient to represent the distribution $f \mapsto \phi(f)$ by the symbol $\phi(x)$, where x indicates that $\phi(x)$ acts on the test functions of the form $x \mapsto f(x)$. The relativistic transformation law (A.1) then can be symbolically restated as

$$\phi_i(x) \mapsto U_{\{a,\Lambda\}}\phi_i(x)U^+_{\{a,\Lambda\}} = \sum_j S_{ij}(\Lambda^{-1})\phi_j(\Lambda x + a). \tag{A.4}$$

It is generally required that the field theory contains enough field operators to express any statevector using fields and functions of fields. This can be achieved by requiring the vacuum vector to be a cyclic vector (as in the first property above), by the requirement the field operators to form an irreducible set of operators on $\mathcal{H}$, or by postulating the commutation/anti-commutation relations,

$$[\phi_i(x), \pi_j(y)]_\pm = i\delta(\vec{x} - \vec{y})\delta_{ij}, \tag{A.5}$$

with π_j a definite linear combination of the fields and their spacetime derivatives (the canonically conjugated field operator to ϕ_i).

A.2 Wightman functions and vacuum expectations

The Wightman functions are defined as the vacuum expectation values of monomials in the field operators,

$$W^{(n)}_{i_1 i_2 \ldots i_n} = \langle \Psi_0, \phi_{i_1}(x_1)\phi_{i_2}(x_2)\ldots\phi_{i_n}(x_n)\Psi_0\rangle, \tag{A.6}$$

with the superscript denoting the number of arguments, and the $(i_1, i_2, \ldots, i_n)$ is an ordered n-tuple of field component indices. All monomials for all combinations of the field components ϕ_{i_k} are considered, giving a hierarchy of Wightman functions labeled by the n-tuples $(i_1, \ldots, i_n)$ for all positive integers n. For $n = 0$ we define $W^{(0)} = 1$. We admit also Wightman functions with non-self-adjoint operators, in that case the indices corresponding to conjugated operators will be denoted by an asterix, i.e. for example

$$W^{(n)}_{i_1^* i_2 i_3} = \langle \Psi_0, \phi^+_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)\Psi_0\rangle.$$

The term $\phi_{i_1}(x_1)\phi_{i_2}(x_2)\ldots\phi_{i_n}(x_n)$ on the right hand side is regarded as the tensor product,

$$\widehat{W}^{(n)}_{i_1 i_2 \ldots i_n}(x_1, x_2, \ldots, x_n) = \phi_{i_1}(x_1) \otimes \phi_{i_2}(x_2) \otimes \cdots \otimes \phi_{i_n}(x_n).$$

of the operator valued distributions ϕ_{i_k}. It is an operator valued distribution on $\mathcal{S}(\mathbb{R}^{4n})$, and the Wightman function W, (A.6), is a complex valued distribution on $\mathcal{S}(\mathbb{R}^{4n})$. The action of W on a special test function $f \in \mathcal{S}(\mathbb{R}^{4n})$ of the form $f(x_1, \ldots, x_n) = f_1(x_1)f_2(x_2)\ldots f_n(x_n)$ is defined by

$$W^{(n)}_{i_1 i_2 \ldots i_n}(f) = \langle \Psi_0 \phi_{i_1}(f_1)\phi_{i_n}(f_n)\ldots\phi_{i_n}(f_n)\Psi_0\rangle,$$

which is a separately continuous (i.e. with respect to the functions f_i individually) linear functional, and by the properties of the space of test functions $S(\mathbb{R}^4)$ it is extended to a continuous linear functional on $\mathcal{S}(\mathbb{R}^{4n})$,

$$W^{(n)}_{i_1 i_2 \ldots i_n} : f \in \mathcal{S}(\mathbb{R}^{4n}) \mapsto \mathbb{C}.$$

This functional is then represented by the symbol $W^{(n)}_{i_1 i_2 \ldots i_n}(x_1, x_2, \ldots, x_n)$ on the left hand side of (A.6), and it is a well defined complex valued distribution.

There are several properties of the vacuum expectation values of a quantum field theory, which are carried over to the Wightman functions. These properties are not separate postulates, they follow ([7], or [10]) from the axioms listed in the previous section. Given a set of Wightman functions $\{W^{(n)}_{i_1\dots i_n}(x_1,\dots,x_n)\}$ for all possible n, $(i_1,i_2,\dots,i_n)$ and $(x_1,x_2,\dots,x_n)$ with all that properties, it is possible to recover the corresponding quantum field theory from the Wightman functions. These properties of W are:

(a) Relativistic transformation law

The law (A.1) of Poincaré transformation, represented by the unitary operator $U = U_{\{a,\Lambda\}}$, carries over to the vacuum expectation values as, in the symbolic notation of (A.4),

$$\langle\Psi_0,\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)\rangle \mapsto \langle\Psi_0,U\phi_{i_1}U^+\dots U\phi_{i_n}U^+\Psi_0\rangle = \tag{A.7}$$
$$= \left\langle \Psi_0, \sum_{j_1,\dots,j_n} S_{i_1j_1}(\Lambda^{-1})\dots S_{i_nj_n}(\Lambda^{-1})\phi_{j_1}(\Lambda x_1+a)\dots\phi_{j_n}(\Lambda x_n+a)\Psi_0\right\rangle,$$

All vacuum expectation values are Poincaré invariant by virtue of the property of the vacuum state Ψ_0,

$$\Psi_0 \mapsto U\Psi_0 = \Psi_0, \qquad \Psi_0 \mapsto U^+\Psi_0 = \Psi_0.$$

It follows immediately from (A.7), that the Wightman functions satisfy

$$W^{(n)}_{i_1\dots i_n}(x_1,\dots,x_n) = \sum_{j_1,\dots,j_n} S_{i_1j_1}(\Lambda^{-1})\dots S_{i_nj_n}(\Lambda^{-1})W^{(n)}_{j_1,\dots,j_n}(\Lambda x_1+a,\dots,\Lambda x_n+a). \tag{A.8}$$

(b) Hermiticity

The vacuum expectation values satisfy

$$\langle\Psi_0,\phi_{i_1}(x_1)\phi_{i_2}(x_2)\dots\phi_{i_n}(x_n)\Psi_0\rangle = \overline{\langle\Psi_0,\phi_{i_n}(x_n)^+\dots\phi_{i_2}(x_2)^+\phi_{i_1}(x_1)^+\Psi_0\rangle}, \tag{A.9}$$

hence we have the condition for the Wightman functions,

$$W^{(n)}_{i_1i_2\dots i_n}(x_1,x_2,\dots,x_n) = \overline{W^{(n)}_{i_n^*\dots i_2^*i_1^*}(x_n,x_{n-1},\dots,x_1)}, \tag{A.10}$$

and in case of self-adjoint field operators it reduces to

$$W^{(n)}_{i_1i_2\dots i_n}(x_1,x_2,\dots,x_n) = \overline{W^{(n)}_{i_n\dots i_2i_1}(x_n,x_{n-1}\dots,x_1)}. \tag{A.11}$$

(c) Local commutativity

For any permutation π of the set $\{1,2,\dots n\}$,

$$W^{(n)}_{i_{\pi(1)}i_{\pi(2)}\dots i_{\pi(n)}}(x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)}) = (-1)^{m(\pi)}W^{(n)}_{i_1i_2\dots i_n}(x_1,x_2,\dots,x_n), \tag{A.12}$$

if the spacetime points x_i are all mutually spacelike separated. The sign of the left hand side is determined by the number $m(\pi)$ of exchanges of anticommuting fields in the permutation.

(d) Positivity property

Denote $f = \{f_j \mid j = 0, 1, 2, ...\}$ a sequence of test functions, with $f_0 \in \mathbb{C}$ and $f_k \in \mathcal{S}(\mathbb{R}^{4k})$ for $k > 0$. Then

$$\sum_{k,l=0}^{+\infty} \int dx_1...dx_k dy_1...dy_l \overline{f_k(x_1, ..., x_j)} W^{(k+l)}_{i^*_{kk}...i^*_{k1} i_{l1}...i_{ll}}(x_k, ..., x_1, y_1, ..., y_l) f_l(y_1, ..., y_l) \geq 0, \tag{A.13}$$

for all f and all possible sets $\{i_{kj} | k, j = 0, 1, 2, \dots\}$ of field indices. The equation (A.13) corresponds to the Hilbert space norm of the vector

$$\phi = (f_0 + \phi_{i_{11}}(x_1)(f_1) + (\phi_{i_{21}}\phi_{i_{22}})(f_2) + (\phi_{i_{31}}\phi_{i_{32}}\phi_{i_{33}})(f_3) + \dots)\,\Psi_0,$$

indexed by the test function $f = (f_0, f_1, f_2, \dots)$ [2]. The positiveness of (A.13) is then implied by the positiveness of the Hilbert space scalar product, and in turn enables us to define a positive scalar product using the Wightman functions.

(e) Spectral condition

For a given Wightman function W, an operator valued distribution on $\mathcal{S}(\mathbb{R}^{4n})$,

$$W^{(n)}(x_1, \dots, x_n) \equiv W^{(n)}_{i_1...i_n}(x_1, \dots, x_n) = \langle \Psi_0, \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n)\Psi_0 \rangle, \tag{A.14}$$

there exists an operator valued distribution $\mathbb{W}^{(n-1)}$ on $\mathcal{S}(\mathbb{R}^{4(n-1)})$, such that

$$W^{(n)}(x_1, x_2, \dots, x_n) = \mathbb{W}^{(n-1)}(x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n). \tag{A.15}$$

This follows from the relativistic transformation law of the fields, (A.4) with

$$a = x_i, \quad \Lambda = I, \quad U_{\{a,\Lambda\}} = \exp(iP_\mu x^\mu_i), \quad i = 1, 2, \dots, n,$$

and the invariance of the vacuum state (A.3), which together gives the vacuum expectation vales in terms of the coordinate differences only,

$$\langle \Psi_0, \phi_{i_1}(x_1)...\phi_{i_n}(x_n)\Psi_0 \rangle = \langle \Psi_0, \phi_{i_1}(0) e^{iP_\mu (x_2 - x_1)^\mu} \phi_{i_2}(0)...\phi_{i_{n-1}}(0) e^{iP_\mu (x_n - x_{n-1})^\mu} \phi_{i_n}(0)\Psi_0 \rangle.$$

The Fourier transforms of the two distributions $\widetilde{W}^{(n)}(p_1, \dots, p_n)$ and $\widetilde{\mathbb{W}}^{(n-1)}(q_1, \dots, q_{n-1})$ are related by

$$\widetilde{W}^{(n)}(p_1, \dots, p_n) = (2\pi)^4 \delta \left(\sum_{j=1}^{n} p_j \right) \widetilde{\mathbb{W}}^{(n-1)}(p_1, p_1 + p_2, p_1 + p_2 + p_3, \dots, p_1 + p_2 + \dots + p_{n-1}). \tag{A.16}$$

Furthermore, $\widetilde{\mathbb{W}}^{(n-1)}(q_1, \dots, q_{n-1}) = 0$ whenever one of the q_k lies outside the energy-momentum spectrum, in particular outside the forward light cone.

(f) Cluster decomposition property

For any spacelike vector a and any $j = 1, 2, \dots, n$,

$$\lim_{\lambda \to +\infty} W^{(n)}_{i_1...i_n}(x_1, \dots, x_j, x_{j+1} + \lambda a, \dots, x_n + \lambda a) = W^{(k+l)}_{i_1...i_j}(x_1, \dots, x_j) W^{(k+l)}_{i_{j+1},...,i_n}(x_{j+1}, \dots, x_n). \tag{A.17}$$

This property is interpreted as no interaction between two points separated by a sufficiently large spacelike interval.

[2] It is generally assumed that the sequence contains only finite number of nonzero entries, hence the sum in (A.13) is over a finite number of terms.

A.3 Wightman reconstruction theorem

Given a set of complex valued distributions $\{W^{(n)}_{i_1\dots i_n}(x_1,\dots,x_n)\}$, for all n and all ordered n-tuples of indices i_k[3], with the properties (a) through (f) of the Wightman functions, the Wightman reconstruction theorem[4] asserts, that a local quantum field theory can be recovered from the knowledge of this set $\{W^{(n)}_{i_1\dots i_n}\}$. There exists a separable Hilbert space $\mathcal{H}$, field operators $\{\phi_{i_k}\}$ defined on a dense subspace $D \subset \mathcal{H}$, and a vacuum state Ψ_0, such that all Wightman functions can be obtained as the corresponding vacuum expectation values,

$$W^{(n)}_{i_1 i_2\dots i_n}(x_1,x_2,\dots,x_n) = \langle \Psi_0, \phi_{i_1}(x_1)\phi_{i_2}(x_2)\dots\phi_{i_n}(x_n)\Psi_0\rangle. \tag{A.18}$$

Moreover, there is a unitary representation $\{a,\Lambda\} \mapsto U_{\{a,\Lambda\}}$ on $\mathcal{H}$ of the Poincaré group, and the vacuum state Ψ_0 is invariant under the action of the group, $U_{\{a,\Lambda\}}\Psi_0 = \Psi_0$ for any $\{a,\Lambda\}$. Such a quantum field theory is unique up to a unitary equivalence, i.e. if there is another Hilbert space $\mathcal{H}'$ with vacuum state Ψ'_0, field operators $\{\phi'_{i_k}\}$ with domain D, unitary representation of the Poincaré group $U'_{\{a,\Lambda\}}$, and with all the vacuum expectation values given by the same Wightman functions as above,

$$\langle \Psi'_0, \phi'_{i_1}(x_1)\phi'_{i_2}(x_2)\dots\phi'_{i_n}(x_n)\Psi'_0\rangle = W^{(n)}_{i_1 i_2\dots i_n}(x_1,x_2,\dots,x_n),$$

then there exists a unitary isomorphism $V : \mathcal{H} \to \mathcal{H}'$, such that

$$\Psi'_0 = V\Psi_0, \quad U'_{\{a,\lambda\}} = VU_{\{a,\Lambda\}}V^{-1}, \quad \phi'_{i_k} = V\phi_{i_k}V^{-1}, \quad D' = VD.$$

The proof of the theorem is constructive, and it is illustrative to sketch its main ideas for the case of a self-adjoint scalar field. To construct the Hilbert space of the quantum field theory, we start from a vector space $\mathcal{H}_0$, which is formed by sequences f of test functions of the form

$$f = (f_0, f_1, f_2, \dots), \tag{A.19}$$

where the components are test functions, $f_k \in \mathcal{S}(\mathbb{R}^{4k})$ for $k > 0$, and $f_0 \in \mathbb{C}$. Only the sequences with at most finite number of nonzero components are considered. The addition of two vectors $f, g \in H_0$ and multiplication of f by a scalar $\alpha \in \mathbb{C}$ is defined component-wise. The vector space $\mathcal{H}_0$ is then given by linear span of the vectors (A.19). The Wightman functions are used to define scalar product on $\mathcal{H}_0$, with the definition $W^{(0)} = 1$, by

$$\langle f, g\rangle = \sum_{k,l=0}^{+\infty} \int dx_1 ... dx_k dy_1 ... dy_l \overline{f_k(x_1,...,x_j)} W^{(k+l)}(x_k,...,x_1,y_1,...,y_l) g_l(y_1,...,y_l), \tag{A.20}$$

which is sesquilinear, and positive by virtue of the positivity property of Wightman functions. By the hermiticity property of Wightman functions, $\langle f, g\rangle = \overline{\langle g, f\rangle}$.

The representation of Poincaré group is introduced by

$$U_{\{a,\Lambda\}}f = U_{\{a,\Lambda\}}(f_0, f_1, f_2, \dots) = (f_0, \{a,\Lambda\}f_1, \{a,\Lambda\}f_2, \dots), \tag{A.21}$$

[3] In case of a quantum field theory with non-self-adjoint operators, also the Wightman function with the asterisked indices are needed. If the Wightman functions does not depend on the choice between i_k and i_k^*, they correspond to a self-adjoint field, in that case we set $i_k = i_k^*$. If only the i_k indices are given, only the self-adjoint parts of the quantum field operators can be recovered. For the simplicity of notation, only the i_k indices will be indicated.

[4] The reconstruction theorem is related to the GNS construction, which assigns a representation to a C^* algebra by operators on a Hilbert space, but it does not need an a priori knowledge of the C^* algebra.

where the action of the group element $\{a,\Lambda\}$ on the function f_k is given by

$$\{a,\Lambda\}f(x_1,\dots,x_k) = f\left(\Lambda^{-1}(x_1-a),\dots,\Lambda^{-1}(x_k-a)\right),$$

and the constant number $f_0 = \{a,\Lambda\}f_0$ is left unchanged. It is indeed a representation ot the Poincaré group, the inverse transformation is given by $U_{\{-a,\Lambda^{-1}\}}$, and the composition of $U_{\{a_1,\Lambda_1\}}$ and $U_{\{a_2,\Lambda_2\}}$ is

$$U_{\{a_1,\Lambda_1\}}U_{\{a_1,\Lambda_1\}} = U_{\{a_1+\Lambda_1 a_2,\Lambda_1\Lambda_2\}}.$$

Its unitarity follows from the relativistic transformation properties of the Wightman functions, $W^{(n)}(\{a,\Lambda\}x) = W^{(n)}(x)$,

$$\begin{aligned}\langle U_{\{a,\Lambda\}}f, U_{\{a,\Lambda\}}g\rangle &= \sum_{j,k}\int d^{4j}xd^{4k}y\{a,\Lambda\}\overline{f_j(x)}W^{(j+k)}(x,y)\{a,\Lambda\}g_k(y) = \\ &= \sum_{j,k}\int d^{4j}xd^{4k}y\{a,\Lambda\}\overline{f_j(x)}W^{(j+k)}(\{a,\Lambda\}x,\{a,\Lambda\}y)g_k(y) = \langle f,g\rangle,\end{aligned}$$

for all $f=(f_0,f_1,f_2,...), g=(g_0,g_1,g_2,...)\in\mathcal{H}_0$. The vacuum state is $\Psi_0=(1,0,0,\dots)$, and it is clearly Poincaré invariant.

The field operator $\phi(h)$, indexed by the test function $h\in\mathcal{S}(\mathbb{R}^4)$, is defined by its action on the vector $f=(f_0,f_1,f_2,\dots)\in\mathcal{H}_0$,

$$\phi(h)f = \phi(h)(f_0,f_1,f_2,\dots) = (0,h\otimes f_0, h\otimes f_1, h\otimes f_2,\dots). \tag{A.22}$$

The definitions of $U_{\{a,\Lambda\}}$ and $\phi(f)$ together imply, that $\phi(f)$ transforms according to the rule,

$$\left(U_{\{a,\Lambda\}}\phi(h)U^{+}_{\{a,\Lambda\}}\right)f = \phi(\{a,\Lambda\}h)f, \qquad \forall f=(f_0,f_1,f_2,\dots)\in\mathcal{H}_0. \tag{A.23}$$

With the definition of scalar product (A.20), the vacuum state $\Psi_0=(1,0,0,...)$, and the field operator ϕ, (A.22), the Wightman functions are given by the vacuum expectation values of the reconstructed field theory,

$$\int dx_1\dots dx_n f_1(x_1)\dots f_n(x_n)W^{(n)}(x_1,\dots,x_n) = \langle\Psi_0,\phi(f_1)\dots\phi(f_n)\Psi_0\rangle, \tag{A.24}$$

or symbolically in the unsmeared form,

$$W^{(n)}(x_1,\dots,x_n) = \langle\Psi_0,\phi(x_1)\dots\phi(x_n)\Psi_0\rangle.$$

The space $\mathcal{H}_0$ is in general only a pre-Hilbert space, since the scalar product may not be positive definite or the space may not be complete. If the scalar product is not positive definite, the set of zero-norm vectors form a subspace of $\mathcal{H}$, in which all vectors are mutually orthogonal (a consequence of the triangle and Schwarz inequalities). This allows us to consistently define an equivalence relation on $\mathcal{H}$ (two vectors are equivalent if they differ by a zero-norm vector), and factorize the space $\mathcal{H}_0$ modulo this equivalence. If the resulting quotient space is not a complete space, the standard completion procedure can be used. We consider all sequences of vectors from $\mathcal{H}_0$, and choose all Cauchy sequences with respect to the norm given by the scalar product in $\mathcal{H}_0$. Two Cauchy sequences are regarded equivalent, if the norm of their difference converges to zero. Then the quotient space of the space of all Cauchy sequences (of sequences of test functions) with respect to this equivalence becomes the completed space.

The subspace $D \subset \mathcal{H}$ is formed from those equivalence classes that correspond to the vectors of $\mathcal{H}_0$ (i.e. equivalent to constant sequences). This is the dense subspace $D \subset \mathcal{H}$ required by the first axiom of field theory, which is contained in the domain of all the field operators. Since the space $\mathcal{H}_0$ is invariant with respect to all the original operators defined on $\mathcal{H}_0$, after the completion the field operators as extended to $\mathcal{H}$ (or to a subspace of $\mathcal{H}$ containing D) map D to D, in particular we have

$$\phi(f)D \subset D, \qquad U_{\{a,\Lambda\}}D \subset D, \qquad \Psi_0 \in D.$$

The subspace $D_0 \subset \mathcal{H}$ is obtained by acting by all polynomials $P(\{\phi(h)\})$ (indexed by arbitrary test functions) on the vacuum state Ψ_0. Eventually, we end up with a separable Hilbert space $\mathcal{H}$ with the unitary representation $U_{\{a,\Lambda\}}$ of Poincaré group extended by continuity from $\mathcal{H}_0$ to $\mathcal{H}$, and continuous in a and Λ.

For any $f, g \in D$, $f = (f_0, f_1, f_2, \dots)$ and $g = (g_0, g_1, g_2, \dots)$, and with the definition $g_{-1} = 0$ and $f_{-1} = 0$, we obtain

$$\begin{aligned}\langle f, \phi(h)g\rangle &= \sum_{k,l=0}^{+\infty} \int \overline{f_k(x_1, \dots x_k)} W^{(k+l)}(x_k, \dots x_1, z, y_1, \dots y_{l-1})(\phi(h)g)_l(z, y_1, \dots y_{l-1}) d^{4k}x d^4 z d^{4(l-1)}y = \\ &= \sum_{k,l=0}^{+\infty} \int \overline{f_k(x_1, \dots x_k)} W^{(k+l)}(x_k, \dots x_1, z, y_1, \dots y_{l-1}) g_{l-1}(y_1, \dots y_{l-1}) d^{4k}x d^4 z d^{4(l-1)}y h(z) = \\ &= \sum_{k',l'=0}^{+\infty} \int \overline{(\phi(\overline{h})f)_{k'}} W^{(k'+l')} g_{l'} d^{4k'}x d^4 z d^{4l'}y = \langle \phi(\overline{h})f, g\rangle = \langle \phi(h)^+ f, g\rangle.\end{aligned}$$

Therefore the field operator $\phi(h)$ for any $h \in \mathcal{S}(\mathbb{R}^4)$ is symmetric on D,

$$\phi^+(h)f = \phi(\overline{h})^+ f = \phi(h)f, \qquad f \in D.$$

The commutativity of the fields at spacelike separated regions of spacetime, represented by the supports of the test functions h_1 and h_2, follows from the corresponding commutativity property of the Wightman functions. For any $f, g \in \mathcal{S}(\mathbb{R}^4)$,

$$\begin{aligned}\langle f, \phi(h_1)\phi(h_2)g\rangle &= \sum_{k,l=0}^{+\infty} \int f_k(x) W(x, z_1, z_2, y) h_1(z_1) h_2(z_2) g_l(y) d^{4k}x dz_1 dz_2 d^{4l}y = \\ &= \sum_{k,l=0}^{+\infty} \int f_k(x) W(x, z_2, z_1, y) h_1(z_1) h_2(z_2) g_l(y) d^{4k}x dz_1 dz_2 d^{4l}y = \langle f, \phi(h_2)\phi(h_1)g\rangle.\end{aligned}$$

The vacuum state Ψ_0 is invariant with respect to any Poincaré transformation, $U_{\{a,\Lambda\}}\Psi_0 = \Psi_0$, and up to a complex multiple of Ψ_0 there is no other such vector. For if there was such a vector Ψ_0' (we can assume Ψ_0' to be normalized and orthogonal to Ψ_0), then with the use of invariance of $\Psi_0' = (f_0, f_1, f_2, \dots)$ and picking a spacelike vector a,

$$\langle \Psi_0', \Psi_0'\rangle = \lim_{\lambda \to +\infty} \langle \Psi_0', U_{\{\lambda a, I\}}\Psi_0'\rangle = \langle \Psi_0', \Psi_0\rangle\langle \Psi_0, \Psi_0'\rangle = 0,$$

by the assumed orthogonality, hence the vector Ψ_0' must be zero in $\mathcal{H}$. The second equality follows, by definition of the scalar product (A.20), from the cluster decomposition property (A.17),

$$\lim_{\lambda \to +\infty} \sum_{k,l=0}^{+\infty} \int dx_1 \dots dx_k dy_1 \dots dy_l \overline{f_k(x_1, \dots, x_j)} W^{(k+l)}(x_k, \dots, x_1, y_1, \dots, y_l)\{\lambda a, I\} f_l(y_1, \dots, y_l) =$$

$$= \sum_{k,l=0}^{+\infty} \int dx_1...dx_k \overline{f_k(x_1,...,x_j)} W^{(k)}(x_k,...,x_1) \int dy_1...dy_l W^{(l)}(y_1,...,y_l) f_l(y_1,...,y_l),$$

and this is $\langle \Psi_0', \Psi_0 \rangle \langle \Psi_0, \Psi_0' \rangle$, recalling $\Psi_0 = (1,0,0,\dots)$ and the hermiticity property of the Wightman functions,

$$\overline{\langle \Psi_0', \Psi_0 \rangle} = \sum_{l=0}^{+\infty} \int dx_1...dx_k 1 W^{(l)}(x_1,...,x_l) f_l(x_1,...,x_l) = \langle \Psi_0, \Psi_0' \rangle.$$

The reconstruction of Hilbert space of a field theory from a given hierarchy of Wightman functions can be carried through also in case of multi-component and non-hermitean fields $\phi_k(x)$, with n vector and/or spinor components. The relativistic transformation properties are in general given by the representation $\{a,\Lambda\} \mapsto S(a,\Lambda)$ of the Poincaré group on the space of field components. These properties are already encoded in the given set of Wightman functions. The initial vector space is composed of linear combinations of the vectors f,

$$f = (f_0, f_1, f_2, \dots) \tag{A.25}$$

where the f_k are test functions defined on $\mathcal{S}(\mathbb{R}^k)$ with values in the k-th tensor power of $\mathbb{C}^n$. The Poincaré transformation law is defined through (A.21) with

$$\{a,\Lambda\} f_k(x) = \big(S(\Lambda^{-1}) \otimes ... \otimes S(\Lambda^{-1})\big)\, f_k(\{a,\Lambda\}x). \tag{A.26}$$

The scalar product is introduced similarly as (A.20),

$$\langle f, g \rangle = \sum_{k,l=0}^{+\infty} \int f_k^+(x_1,...x_k) W_{*-}^{(k+l)}(x_k,...,x_1,y_1,...,y_l) g_l(y_1,...,y_l) dx_1...dx_k dy_1...dy_l. \tag{A.27}$$

or in the the tensor components,

$$\langle f, g \rangle = \sum_{k,l=0}^{+\infty} \int \sum_{i_{k1}...i_{kk}} \sum_{i_{l1}...i_{ll}} \overline{f_{k,i_{kk}...i_{k1}}(x_k,...,x_1)} W^{(k+l)}_{i^*_{kk}...i^*_{k1} g_{l1}...g_{ll}}(x_k,...,x_1,y_1,...,y_l) g_{l,i_{l1}...i_{ll}}(y_1,...,y_l) d\mu, \tag{A.28}$$

with the definition of f_k^+,

$$f^+_{k,i_{k1}...i_{kk}}(x_1,...,x_k) = \overline{f_{k,i_{kk}...i_{k1}}(x_k,...,x_1)}. \tag{A.29}$$

It has all the properties as in the case of scalar field, in particular it is positive and sesquilinear. The unitarity follows from the fact, that the transformation law of Wightman functions (A.8) is the opposite of those for f, (A.25) and (A.26). Finally, the field operator $\phi(h)$, indexed by a $\mathbb{C}^n$-valued test function h, is defined by the action on f as

$$\phi(h) f = (0, h \otimes f_0, h \otimes f_1, h \otimes f_1 \dots). \tag{A.30}$$

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