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Faculty of Nuclear Sciences and Physical Engeneering

Nehermitovské $\mathcal{PT}$-symetrické Sturm-Liouvillovy operátory a jim podobné samosdružené Hamiltoniány

Non-Hermitian $\mathcal{PT}$-Symmetric Sturm-Liouvill Operators and Similar Self-Adjoint Hamiltonians

Výzkumný úkol

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Abstract

In papers [1], [2] a new very simple $\mathcal{PT}$-symmetric model was introduced and closed formula for the metric operator was found. We present a new, integral form of the metric operator, which is more suitable for practical computations and use this new form for analyzing the spectrum of the metric.

Contents

1 Introduction

In quantum mechanics, observable quantities (usually called observables) are described by linear operators on some Hilbert space and possible outcomes of measurements correspond to points from their spectra. One of the basic axioms of quantum mechanics states that these operators are self-adjoint.[1] The motivation for this axiom is the fact (among others) that self-adjoint operators have real spectrum. It is, however, possible to construct quantum mechanics using the set of non-self-adjoint observables $\{A_\iota\}$ (see [4], [5] and [6]). This is done using the so called metric operator θ, which is a bounded, positive operator with bounded inversion, such that all observables A_ι satisfy

$$A_\iota^* \theta = \theta A_\iota. \tag{1}$$

In this case we can consider a new Hilbert space, which coincides with the original one as a vector space, but is endowed with a modified scalar product

$$(\phi, \psi)_\theta = (\phi, \theta\psi), \tag{2}$$

where (ϕ, ψ) is the scalar product of the original Hilbert space. The observables are self-adjoint in this Hilbert space and one can, therefore, use standard quantum mechanics. Non-self-adjoint operator for which there exists a metric operator is called quasi-Hermitian.

This approach does not bring in any new physics because the observables A are similar to self-adjoint operators $A_F = \sqrt{\theta} A \sqrt{\theta}^{-1}$[2] and if we consider these observables instead of the non self-adjoint ones, we will get exactly the same predictions. The possible advantage of this approach lies however in the fact that some or all of the observables A can be much simpler than their self-adjoint counterparts.

For simplicity, often only the observable corresponding to the energy of the system, the so called Hamiltonian is considered. This is usually the most important observable because it governs the time evolution of the system. In the past decades, there has been great interest in the so-called $\mathcal{PT}$-symmetric Hamiltonians because it was found that these Hamiltonians often have real spectrum despite being non-self-adjoint, which has led to development of the

[1] Here we use the definition of self-adjoint operator from e.g. [3]. In physical texts these operators are often called Hermitian, we use the term Hermitian only for bounded operators.

[2] $\sqrt{\theta}$ is a positive operator which satisfies $\sqrt{\theta}^2 = \theta$, for positive operators it exists and is unique.

so called $\mathcal{PT}$-symmetric quantum mechanics (see [7], [8], [9] and proceedings of the *International workshops on Pseudo-Hermitian Hamiltonians in Quantum Physics* [10]- [17]). $\mathcal{PT}$ is an operator on Hilbert space $L^2(\mathbb{R}^n)$, composed of an operator of spatial inversion $\mathcal{P}$: $\mathcal{P}\psi(x) = \psi(-x)$ and of an operator of complex conjugation $\mathcal{T}$: $\mathcal{T}\psi = \overline{\psi}$. Note that $\mathcal{T}$(and therefore also $\mathcal{PT}$) is not a linear operator. An operator H on Hilbert space $L^2(\mathbb{R}^n)$ is then $\mathcal{PT}$-symmetric if it commutes with the $\mathcal{PT}$ operator:

$$H\mathcal{PT} = \mathcal{PT}H, \quad \forall\psi \in D(H). \tag{3}$$

For the case of a self-adjoint Hamiltonian, this definition has a direct physical meaning. The Hamiltonian is $\mathcal{PT}$ symmetric if and only if the time evolution of the system is invariant under simultaneous spatial and time inversion. In the case of quasi-Hermitian Hamiltonians this no longer applies because x will generally no longer correspond to the position operator and $\mathcal{P}$ will, therefore, no longer be a spatial inversion.

There have been many attempts to find the metric in various $\mathcal{PT}$-symmetric models (see [18]- [22]), but the metric is usually a complicated non-local operator and the results are therefore mostly formal and approximative. For this reasons, new very simple model was introduced (in paper [2] by D. Krejčiřík, M. Znojil and H. Bíla), in which it is possible to find the metric in a closed form. In this thesis we will present a new form of this metric, more suitable for practical calculation, and use this form to analyze the spectrum of the metric.

2 The Model

In this chapter the definition of the model and main results from the papers [2], [1] will be recalled.

The Hilbert space of the model is

$$\mathcal{H} = L^2((0, d)), \tag{4}$$

where d is a positive number and only one observable is considered, namely the Hamiltonian H_α, which is Laplacian with complex boundary conditions:

$$H_\alpha\psi = -\psi'',$$
$$D(H_\alpha) = \{\psi \in W^{2,2}((0, d)) | \psi'(0) + i\alpha\psi(0) = \psi'(d) + i\alpha\psi(d) = 0\},$$

where α is a real constant. Note that the boundary values make sense because $W^{2,2}((0,d))$ is embedded into $C^0([0,d])$ (see [23, section V]). For $\alpha \neq 0$ it is not a self-adjoint operator; in fact it satisfies

$$H_\alpha^* = H_{-\alpha}. \tag{5}$$

The spectrum of H_α and H_α^* is

$$\sigma(H_\alpha) = \sigma(H_\alpha^*) = \{\alpha^2\} \cup \{k_j^2\}_{j=1}^\infty, \text{ where } k_j = \frac{j\pi}{d}, \tag{6}$$

and the eigenfunctions of H_α resp. H_α^* are

$$\psi_j^\alpha = \begin{cases} A_0^\alpha e^{-i\alpha x} & \text{if} \quad j = 0, \\ A_j^\alpha \left(\cos(k_j x) - i\frac{\alpha}{k_j}\sin(k_j x)\right) & \text{if} \quad j \geq 1, \end{cases}$$

resp. (7)

$$\phi_j^\alpha = \begin{cases} \sqrt{1/d}\, e^{i\alpha x} & \text{if} \quad j = 0, \\ \sqrt{2/d}\left(\cos(k_j x) + i\frac{\alpha}{k_j}\sin(k_j x)\right) & \text{if} \quad j \geq 1, \end{cases}$$

If the following condition is satisfied

$$\alpha \neq \frac{j\pi}{d}, \ \forall j \in \mathbb{Z}, \tag{8}$$

and if we choose the normalisation constants A_j^α according to the equations:

$$1 = A_0^\alpha \sqrt{1/d}\frac{1 - e^{-2i\alpha d}}{2i\alpha}, \tag{9}$$

$$1 = A_j^\alpha \sqrt{2/d}\frac{(k_j^2 - \alpha^2)d}{2k_j^2} \quad j \geq 1, \tag{10}$$

then functions $\{\psi_j, \phi_k\}$ form a biorthonormal basis, i.e.

$$(\phi_j, \psi_k) = \delta_{jk}, \quad \forall j,k \geq 0, \tag{11}$$

$$\psi = \sum_{j=0}^\infty \psi_j(\phi_j, \psi), \quad \forall \psi \in \mathcal{H}. \tag{12}$$

If the condition (8) is satisfied, then a metric operator can be constructed using the eigenfunctions of H_α; the series

$$\theta_\alpha = \sum_{j=0}^\infty \phi_j^\alpha(\phi_j^\alpha, \cdot), \tag{13}$$

strongly converges and it is a metric operator, i.e. it is a positive bounded operator with bounded inversion, satisfying

$$H_\alpha^* \theta_\alpha = \theta_\alpha H_\alpha. \tag{14}$$

The sum (13) converges even if the condition (8) is not satisfied and it is still a bounded Hermitian operator, however, it is no longer positive, so it is not a metric. The series (13) can be summed up using the spectral theorem (see [2] for details), the result is

$$\theta_\alpha = I + P_0^\alpha - P_0^N + \alpha p(-\Delta_D)^{-1} + \alpha p^*(-\Delta_N^\perp)^{-1} + \alpha^2(-\Delta_D)^{-1}, \tag{15}$$

where

- P_0^α is an orthogonal projector on the first eigenfunction of H_α^* (see eq. (7)):
$$P_0^\alpha = \phi_0^\alpha(\phi_0^\alpha, \cdot) \tag{16}$$

- P_0^N is an orthogonal projector on the first eigenfunction of the Neumann Laplacian (see Appendix A):
$$P_0^N = \chi_0^N(\chi_0^N, \cdot). \tag{17}$$
Note that $P_0^N = P_0^0$.

- p is a "momentum" operator defined as:
$$\begin{aligned} p\psi &= -i\psi', \\ D(p) &= \{\varphi \in W^{1,2}((0,d)) \,|\, \varphi(0) = \varphi(d) = 0\}. \end{aligned} \tag{18}$$

- p^* is the adjoint of p, it acts in the same way, but has a larger domain:
$$\begin{aligned} p^*\psi &= -i\psi', \\ D(p^*) &= W^{1,2}((0,d)). \end{aligned} \tag{19}$$

- $(-\Delta_D)^{-1}$ is an inversion of the Dirichlet Laplacian (see Appendix A for the definition of the Dirichlet Laplacian).

- $(-\Delta_N^\perp)^{-1}$ is the so-called reduced resolvent of the Neumann Laplacian (see [24, section III-§6.5] for the definition of the reduced resolvent and

Appendix A for definition of Neumann Laplacian), which can be defined as

$$(-\Delta_N^{\perp})^{-1} = \sum_1^{\infty} k_j^{-2} \chi_j^N (\chi_j^N, \cdot). \tag{20}$$

Note that despite the notation, this is not inversion of any operator densely defined on $\mathcal{H}$.

3 Writing the metric as an integral operator

As was suggested in [2, remark 2], metric operator (15) can be rewritten in another form using the fact that both $(-\Delta_D)^{-1}$ and $(-\Delta_N^{\perp})^{-1}$ are integral operators with very simple kernels (P_0^{α} and P_0^N are also integral operators with kernels $\phi_0^{\alpha}(x)\bar{\phi}_0^{\alpha}(y)$ resp. $\frac{1}{d}$).

Throughout this chapter we will need the following property of 1-dimensional Sobolev spaces (see [25, section 7.1] for proof):

$$\begin{aligned} W^{1,2}((0,d)) &= \{\phi \in L^2((0,d)) | \phi \in AC[0,d], \phi' \in L^2((0,d))\}, \\ W^{2,2}((0,d)) &= \{\phi \in L^2((0,d)) | \phi, \phi' \in AC[0,d], \phi'' \in L^2((0,d))\}. \end{aligned} \tag{21}$$

Here $AC[0,d]$ denotes the set of all absolutely continuous functions over the interval $[0,d]$.

3.1 Dirichlet Laplacian

$(-\Delta_D)^{-1}$ is an integral operator whose kernel is Green's function of equation $-\Delta_D\psi = f$. It is known to be

$$G_D(x,y) = \begin{cases} \frac{x(d-y)}{d} & \text{for} \quad x < y, \\ \frac{y(d-x)}{d} & \text{for} \quad y < x. \end{cases} \tag{22}$$

We will not derive this result, but we will prove it. Derivation can be found e.g. in [26, section II-§6].

Proposition 3.1. *Let $\varphi \in L^2((0,d))$, then $u_{\varphi}(x) := \int_0^d G_D(x,y)\varphi(y)dy \in D(-\Delta_D)$ and $-\Delta_D u_{\varphi} = \varphi$.*

Proof. Writing u_{φ} as

$$\begin{aligned} u_{\varphi}(x) &= \int_0^x \frac{y(d-x)}{d}\varphi(y)dy + x\int_x^d \frac{(d-y)}{d}\varphi(y)dy \\ &= \int_0^x y\varphi(y)dy - \frac{x}{d}\int_0^x y\varphi(y)dy + x\int_x^d \frac{(d-y)}{d}\varphi(y)dy, \end{aligned}$$

one can easily verify that u_φ satisfies Dirichlet boundary conditions. Furthermore we see that u_φ is an absolutely continuous function and for almost every $x \in [0, d]$ its derivative reads

$$u'_\varphi(x) = x\varphi(x) - \frac{x}{d}x\varphi(x) - \frac{1}{d}\int_0^x y\varphi(y)dy + \int_x^d \frac{d-y}{d}\varphi(y)dy - x\frac{d-x}{d}\varphi(x)$$
$$= -\frac{1}{d}\int_0^d y\varphi(y)dy + \int_x^d \varphi(y)dy. \tag{23}$$

This is again an absolutely continuous function and

$$u''_\varphi = -\varphi. \tag{24}$$

This is a function from $L^2((0, d))$ and u_φ therefore lies in the domain of $-\Delta_D$ (see (95) and (21)), moreover $-\Delta_D u_\varphi = \varphi$. □

$\alpha^2(-\Delta_D)^{-1}$ is, therefore, an integral operator with kernel $\alpha^2 G_D(x, y)$. $\alpha p(-\Delta_D)^{-1}$ is an integral operator too, because

$$(\alpha p(-\Delta_D)^{-1}\psi)(x) = -i\alpha\frac{d}{dx}\int_0^d G_D(x, y)\psi(y)dy, \tag{25}$$

and from eq. (23) we see that this is equal to

$$\int_0^d K_1(x, y)\psi(y)dy,$$

where

$$K_1(x, y) = i\alpha\frac{y}{d} + \begin{cases} -i\alpha & \text{for} \quad x < y, \\ 0 & \text{for} \quad y < x. \end{cases} \tag{26}$$

3.2 Neumann Laplacian

The term $\alpha p^*(-\Delta_N^\perp)^{-1}$ in (15) is also an integral operator. We will first show that $(-\Delta_N^\perp)^{-1}$ is an integral operator.

Using spectral decomposition (97) and the definition (20)

$$(-\Delta_N)(-\Delta_N^\perp)^{-1}\psi = (-\Delta_N)\sum_{j=1}^\infty k_j^{-2}(\chi_j^N, \psi)\chi_j^N = \sum_{j=1}^\infty (\chi_j^N, \psi)\chi_j^N, \tag{27}$$

$$(-\Delta_N^\perp)^{-1}(-\Delta_N)\psi = (-\Delta_N^\perp)^{-1}\sum_{j=0}^\infty k_j^2(\chi_j^N, \psi)\chi_j^N = \sum_{j=1}^\infty (\chi_j^N, \psi)\chi_j^N. \tag{28}$$

$(-\Delta_N^{\perp})^{-1}\psi$ lies in the domain of $(-\Delta_N)$ because we see from (27) that series $\sum_{j=0}^{\infty} k_j^2 (\chi_j^N, (-\Delta_N^{\perp})^{-1}\psi)\chi_j^N$ converges. If we denote the subspace orthogonal to χ_j^N by $\{\chi_0^N\}^{\perp}$ and by P an orthogonal projector on this subspace, then

$$(-\Delta_N)(-\Delta_N^{\perp})^{-1}\psi = P\psi, \tag{29}$$

$$(-\Delta_N^{\perp})^{-1}(-\Delta_N)\psi = P\psi, \tag{30}$$

for all $\psi \in \mathcal{H}$. It follows that $-\Delta_N$ restricted to $\{\chi_0^N\}^{\perp}$ is an injective operator and

$$(-\Delta_N^{\perp})^{-1} = (-\Delta_N|_{\{\chi_0^N\}^{\perp}})^{-1}P. \tag{31}$$

This means that $(-\Delta_N^{\perp})^{-1}\varphi$ is a solution of differential equation

$$-\Delta_N\psi = P\varphi. \tag{32}$$

This equations has more solutions because $-\Delta_N$ is not injective; $\ker(-\Delta_N) = \operatorname{span}\{\chi_0^N\}$. If ψ is a solution of this equation, then $\psi + C\chi_0^N$, where C is any complex number, is also a solution. $(-\Delta_N^{\perp})^{-1}$ gives the solution that is orthogonal to χ_0^N. Solution of equation (32) can again be found using Green's function. In this case, it is called modified Green's function, see [27, section 5.2] for more information. Appropriate Green's function is

$$G_N^{\perp}(x,y) = \begin{cases} \frac{d}{3} - y + \frac{x^2+y^2}{2d} & \text{for} \quad x < y, \\ \frac{d}{3} - x + \frac{x^2+y^2}{2d} & \text{for} \quad y < x. \end{cases} \tag{33}$$

Proposition 3.2. *Let $\varphi \in L^2((0,d))$, then $u_\varphi(x) := \int_0^d G_N^{\perp}(x,y)\varphi(y)dy$ is orthogonal to χ_0^N, $u_\varphi \in D(-\Delta_N)$ and $-\Delta_N u_\varphi = P\varphi$.*

Proof. Function $G_N^{\perp}(x,y)$ satisfies

$$\int_0^d G_N^{\perp}(x,y)dx = 0. \tag{34}$$

This implies

$$\begin{aligned} (\chi_0^N, u_\varphi) = \sqrt{1/d}\int_0^d u_\varphi(x)dx = \sqrt{1/d}\int_0^d \left(\int_0^d G_N^{\perp}(x,y)\varphi(y)dy\right)dx \qquad (35) \\ = \sqrt{1/d}\int_0^d \varphi(y)\cdot\left(\int_0^d G_N^{\perp}(x,y)dx\right)dy = 0. \end{aligned}$$

We have used the fact that $G_N^{\perp}(x,y)\varphi(y)$ is integrable in $(0,d)\times(0,d)$ because

$$\int_0^d\int_0^d |G_N^{\perp}(x,y)\varphi(y)|dxdy \le \left(\int_0^d\int_0^d \left|G_N^{\perp}(x,y)\right|^2 dxdy\right)^{1/2}\left(\int_0^d\int_0^d |\varphi(y)|^2 dxdy\right)^{1/2}$$
$$= \left(\int_0^d\int_0^d \left|G_N^{\perp}(x,y)\right|^2 dxdy\right)^{1/2}\sqrt{d}\left(\int_0^d |\varphi(y)|^2 dy\right)^{1/2}, \tag{36}$$

and $G_N^{\perp}$ is square integrable in $(0,d)\times(0,d)$. u_φ is an absolutely continuous function:

$$u_\varphi(x) = (\frac{d}{3} - x + \frac{x^2}{2d})\int_0^x \varphi(y)dy + \frac{1}{2d}\int_0^x y^2\varphi(y)dy$$
$$+ (\frac{d}{3} + \frac{x^2}{2d})\int_x^d \varphi(y)dy + \int_x^d (-y + \frac{y^2}{2d})\varphi(y)dy, \tag{37}$$

and, therefore, it has a derivative for almost every $x\in[0,d]$

$$u'_\varphi(x) = \frac{x}{d}\int_0^d \varphi(y)dy - \int_0^x \varphi(y)dy. \tag{38}$$

This is again an absolutely continuous function and furthermore we see that u_φ satisfies Neumann boundary conditions: $u'_\varphi(0) = 0$ and $u'_\varphi(d) = 0$. The derivative of u'_φ is for almost every $x\in[0,d]$

$$u''_\varphi(x) = \frac{1}{d}\int_0^d \varphi(y)dy - \varphi(x) = \sqrt{(}1/d)\int_0^d \sqrt{1/d}\varphi(y)dy - \varphi(x) \tag{39}$$
$$= -\big(I - \chi_0^N(\chi_0^N,\cdot)\big)\varphi(x) = -(I - P_0^N)\varphi(x) = -P\varphi(x),$$

and we have

$$-\Delta_N u_\varphi = P\varphi. \tag{40}$$

□

$(-\Delta_N^{\perp})^{-1}$ is, therefore, an integral operator with kernel $G_N^{\perp}(x,y)$. We are interested in operator $\alpha p^*(-\Delta_N^{\perp})^{-1}$. From (38) we see that it is also an integral operator with kernel:

$$K_2(x,y) = -i\alpha\frac{x}{d} + \begin{cases} 0 & \text{for} \quad x<y, \\ i\alpha & \text{for} \quad y<x. \end{cases} \tag{41}$$

3.3 Kernel of the metric

Putting all previous results together we find that the kernel of $\theta_\alpha - I$ is given by

$$K_\alpha(x,y) = \frac{e^{i\alpha(x-y)}-1}{d} + i\alpha(\frac{y}{d}-\frac{x}{d}) - \alpha^2\frac{xy}{d} + \begin{cases} -i\alpha+\alpha^2 x & \text{for} \quad x<y, \\ i\alpha+\alpha^2 y & \text{for} \quad y<x. \end{cases} \tag{42}$$

Thus, we conclude with:

Theorem 3.3. *The metric operator θ_α has the form $\theta_\alpha = I + \mathcal{K}_\alpha$, where $\mathcal{K}_\alpha$ is an integral operator with kernel* (42)*.*

4 Properties of the metric

In [1] it has been shown that operator θ_α is a metric operator if the following condition is satisfied

$$\alpha \neq \frac{j\pi}{d},\ j \in \mathbb{Z}. \tag{43}$$

Therefore, also integral operator (42) must be a metric, since these two operators are equal. We can however show this directly without using results of [2] and [1]. Throughout this chapter $\mathcal{K}_\alpha$ will denote the integral operator with kernel (42) and θ_α will denote operator $I + \mathcal{K}_\alpha$, i.e.

$$(\theta_\alpha\psi)(x) = \psi(x) + \int_0^d \mathcal{K}_\alpha(x,y)\psi(y)dy. \tag{44}$$

The kernel of $\mathcal{K}_\alpha$ is a square integrable function in $L^2((0,d)\times(0,d))$ (because it is a piecewise continuous function on a compact set) and $\mathcal{K}_\alpha$ is therefore bounded and compact operator on $L^2((0,d))$ (see [3, section VI.6]). Adjoint of integral operator with kernel $H(x,y)$ is also an integral operator with kernel $\overline{H(y,x)}$; it follows that $\mathcal{K}_\alpha$ (and therefore also θ_α) is a Hermitian operator, because $K_\alpha(x,y) = \overline{K_\alpha(y,x)}$. For operator with these properties we can use the following two theorems (see [3, section VI.5] for proofs), which will be needed later:

Theorem 4.1 (Riesz-Shauder theorem)**.** *Let A be a compact operator on some Hilbert space $\mathcal{H}$, then only possible limit point of the spectrum is* 0 *and all points from the spectrum except 0 are eigenvalues with finite multiplicities.*

Theorem 4.2 (Hilbert-Schmidt theorem)**.** *Let A be a Hermitian compact operator on some Hilbert space $\mathcal{H}$, then eigenfunctions of A form an orthonormal basis of $\mathcal{H}$.*

Operator θ_α is not compact, but these theorems are still useful, because:

$$\lambda \in \sigma(\theta_\alpha) \Leftrightarrow \lambda - 1 \in \sigma(\mathcal{K}_\alpha), \tag{45}$$

$$\theta_\alpha \psi = \lambda \psi \Leftrightarrow \mathcal{K}_\alpha \psi = (\lambda - 1)\psi. \tag{46}$$

The second equivalence is trivial, the first one can be easily proved using the Weyl's criterion(see [3, section VII.3] for proof):

Theorem 4.3 (Weyl's criterion)**.** *Let A be a self-adjoint operator on some Hilbert space $\mathcal{H}$, then $\lambda \in \sigma(A)$ if and only if there exist a sequence $\{\psi_n\}_{n=1}^{\infty} \subset D(A)$ such that*

$$\|\psi_n\| = 1, \quad \forall n \in \mathbb{N},$$
$$\lim_{n\to\infty} \|(A - \lambda)\psi_n\| = 0.$$

Summarizing:

Proposition 4.4. *Eigenfunctions of θ_α form an orthonormal basis (such operators are said to have a purely point spectrum), the only possible limit point of its spectrum is 1 and all points from the spectrum except 1 are eigenvalues.*

4.1 Proof of identity (14)

We will prove that θ_α satisfies:

$$H_\alpha^* \theta_\alpha \varphi = \theta_\alpha H_\alpha \varphi, \quad \forall \varphi \in D(H_\alpha). \tag{47}$$

First, we have to show that $\theta_\alpha D(H_\alpha) \subset D(H_\alpha^*)$. Since $H_\alpha^* = H_{-\alpha}$, the domain of H_α^* differs from $D(H_\alpha)$ only in the boundary conditions:

$$D(H_\alpha^*) = \{\varphi \in W^{2,2}((0,d)) | \varphi'(0) - i\alpha\varphi(0) = \varphi'(d) - i\alpha\varphi(d) = 0\}. \tag{48}$$

For every $\varphi \in D(H_\alpha)$:

$$\begin{aligned}(\theta_\alpha \varphi)(x) = \varphi(x) &+ \frac{e^{i\alpha x}}{d}\int_0^d e^{-\alpha y}\varphi(y)dy - \frac{1}{d}\int_0^d \varphi(y)dy \\ &+ \frac{i\alpha}{d}\int_0^d y\varphi(y)dy - \frac{i\alpha x}{d}\int_0^d \varphi(y)dy - \frac{\alpha^2 x}{d}\int_0^d y\varphi(y)dy \\ &+ \int_0^x (i\alpha + \alpha^2 y)\varphi(y)dy - i\alpha\int_x^d \varphi(y)dy + \alpha^2 x\int_x^d \varphi(y)dy.\end{aligned} \tag{49}$$

This is an absolutely continuous function, because on closed bounded interval, product and sum of two absolutely continuous functions is an absolutely continuous function, therefore, we can compute its derivative for almost all x:

$$
\begin{aligned}
(\theta_\alpha \varphi)'(x) = \varphi'(x) + \frac{i\alpha}{d} e^{i\alpha x} \int_0^d e^{-\alpha y} \varphi(y) dy - \frac{i\alpha}{d} \int_0^d \varphi(y) dy \\
- \frac{\alpha^2}{d} \int_0^d y\varphi(y) dy + (i\alpha + \alpha^2 x)\varphi(x) + i\alpha\varphi(x) \\
+ \alpha^2 \int_x^d \varphi(y) dy - \alpha^2 x \varphi(x).
\end{aligned}
\tag{50}
$$

This is again an absolutely continuous function and one can easily verify that $\theta_\alpha \varphi$ indeed satisfies boundary conditions in (48). Differentiating again, we find that for almost all x:

$$
(\theta_\alpha \varphi)''(x) = \varphi''(x) - \frac{\alpha^2}{d} e^{i\alpha x} \int_0^d e^{-i\alpha y} \varphi(y) dy + 2i\alpha\varphi'(x) - \alpha^2 \varphi(x). \tag{51}
$$

This is a square integrable function in $L^2((0,d))$ and $\theta_\alpha \varphi$ therefore lies in $D(H_\alpha^*)$ with $-H_\alpha^* \varphi$ equal to (51). Using an integration by parts, it is simple, but lengthy, to show that the right hand side of (47) has the form (51) too.

4.2 Positivity

An operator A is said to be positive if $(\psi, A\psi) > 0$, $\forall \psi \in D(A)$. To show the positivity of θ_α we will use the eigenfunctions of H_α and H_α^*: $\{\psi_j, \phi_k\}$ (see eq. (7)). We can show by direct computation that

$$
\theta_\alpha \psi_j^\alpha = \phi_j^\alpha, \qquad \forall j \in \mathbb{N}. \tag{52}
$$

Let ψ be a non-zero function from $L^2((0,d))$, and suppose the non-degeneracy condition (43) is satisfied so that the functions $\{\psi_j, \phi_k\}$ form a biorthonormal basis. Then

$$
\begin{aligned}
(\psi, \theta_\alpha \psi) &= (\sum_{j=0}^\infty (\phi_j^\alpha, \psi)\psi_j^\alpha, \sum_{k=0}^\infty (\phi_k^\alpha, \psi)\theta_\alpha \psi_k^\alpha) \\
&= \sum_{j=0}^\infty \sum_{k=0}^\infty \overline{(\phi_j^\alpha, \psi)} (\phi_k^\alpha, \psi)(\psi_j^\alpha, \phi_k^\alpha) = \sum_{j=0}^\infty \left|(\phi_j^\alpha, \psi)\right|^2 > 0,
\end{aligned}
\tag{53}
$$

and θ_α is, therefore, positive.

We have already proven that θ_α is a positive, bounded operator, satisfying the identity (14). To prove that it is a metric operator we have yet to prove that it has a bounded inversion. It follows from (53) that it is injective, so it has an inversion and the inversion must be bounded because if it was not bounded, then 0 would lie in the continuous spectrum of the metric, but that cannot happen since only 1 can lie in its the continuous spectrum (see proposition 4.4).

5 The Spectrum of the Metric

Since operator θ_α has a purely point spectrum (see section 4), it follows from the spectral theorem (see e.g. [24, section VI-§5]) that it admits a decomposition

$$\theta_\alpha = \sum_{n=0}^{\infty} \lambda_j^\alpha \varphi_j^\alpha (\varphi_j^\alpha, \cdot), \tag{54}$$

where λ_j^α denotes the eigenvalues of θ_α and φ_j^α the corresponding eigenfunctions and that for any complex-valued continuous function bounded on the spectrum

$$f(\theta_\alpha) = \sum_{n=0}^{\infty} f(\lambda_j^\alpha) \varphi_j^\alpha (\varphi_j^\alpha, \cdot). \tag{55}$$

In particular,

$$\sqrt{\theta_\alpha} = \sum_{n=0}^{\infty} \sqrt{\lambda_j^\alpha} \varphi_j^\alpha (\varphi_j^\alpha, \cdot). \tag{56}$$

We are interested in the operator $\sqrt{\theta_\alpha}$, because using this operator we could construct a self-adjoint Hamiltonian similar to H_α by setting

$$H_\alpha^F = \sqrt{\theta_\alpha} H_\alpha \sqrt{\theta_\alpha}^{-1}. \tag{57}$$

With this operator one could give physical meaning to our model because it is equivalent to a model with Hamiltonian H_α^F and H_α^F is a self-adjoint operator acting on $L^2((0,d))$ and can, therefore, have physical meaning. For this reason we tried to find the eigenvalues and the eigenfunctions of θ_α. Though we did not find the eigenvalues in explicit form, we found an implicit equation for eigenvalues, which could be easily solved numerically.

We can deduce some basic information about the spectrum from the Hilbert-Schmidt and Riesz-Schauder theorems (see section 4). Since θ_α has a purely point spectrum and all eigenvalues have finite multiplicities it must have infinitely many eigenvalues. It is a positive operator so its eigenvalues must be positive and since it is bounded, they must all lie in the bounded interval $(0, \|\theta_\alpha\|)$. Thus the spectrum must have a limit point and from Riesz-Schauder theorem it follows that the limit point is 1.

5.1 Numerical calculation of the spectrum

Just for the overview of the spectrum's behavior we will do a simple numerical calculation of the spectrum's dependence on the parameter $c := \alpha d$. We will show in next section that spectrum depends only on this parameter. The eigenvalue equation

$$\theta_\alpha \psi = \lambda \psi, \tag{58}$$

can be discretized using the biorthonormal basis $\{\psi_j, \phi_k\}$ (see equation (7)). The advantage of this basis is that it satisfies the following relation

$$\theta_\alpha \psi_k = \phi_k. \tag{59}$$

The function ψ can be decomposed as

$$\psi = \sum_{j=0}^{\infty} a_j \psi_j, \tag{60}$$

where a_j are some complex numbers. The eigenvalue equation then becomes

$$\sum_{j=0}^{\infty} a_j \phi_j = \lambda \sum_{n=0}^{\infty} a_j \psi_j \tag{61}$$

we multiply this equation with ϕ_k and use the biorthonormality

$$\sum_{j=0}^{\infty} a_j (\phi_k, \phi_j) = \lambda a_k. \tag{62}$$

This is an eigenvalue equation for an infinite-dimensional matrix

$$(\phi_k, \phi_j) \;\equiv\; (\widetilde{\theta}_\alpha)_{kj}. \tag{63}$$

We can find an approximative solution of this equation by considering finite n-dimensional matrix, which is composed of first n rows and n columns of the matrix (ϕ_k, ϕ_j), i.e. we consider a matrix $\widetilde{\theta}_\alpha^n$ such that

$$(\widetilde{\theta}_\alpha^n)_{kj} = (\phi_k, \phi_j), \quad k, j \in \{1, ..., n\}. \tag{64}$$

The dependence of the spectrum of $\widetilde{\theta}_\alpha^n$ on the parameter c for $n = 10$ resp. $n = 20$ is shown on the figure 1 resp. 2.

We can see that the spectrum is virtually the same for both values of n, the only difference is that the spectrum of $\widetilde{\theta}_\alpha^{20}$ contains more eigenvalues. One can, therefore, expect that the spectrum of $\widetilde{\theta}_\alpha^n$ is a good approximation of the spectrum of θ_α.

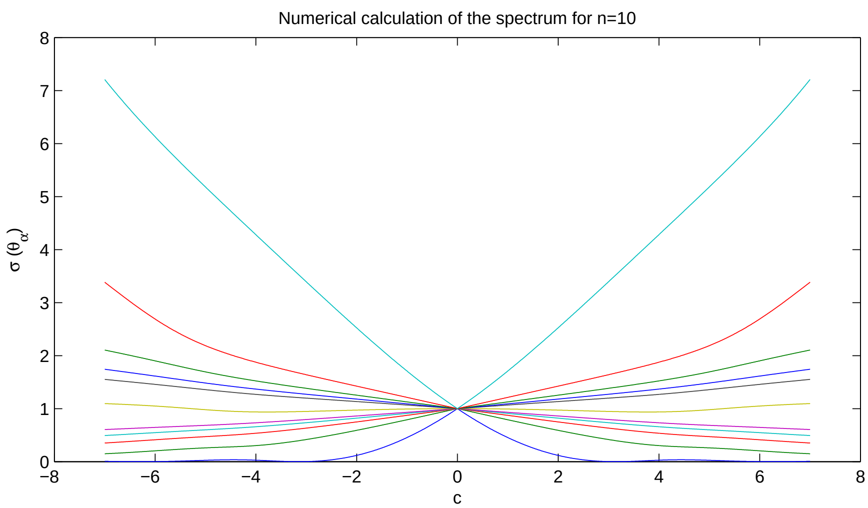

Figure 1: Dependence of the spectrum of θ_α on parameter c for $n = 10$

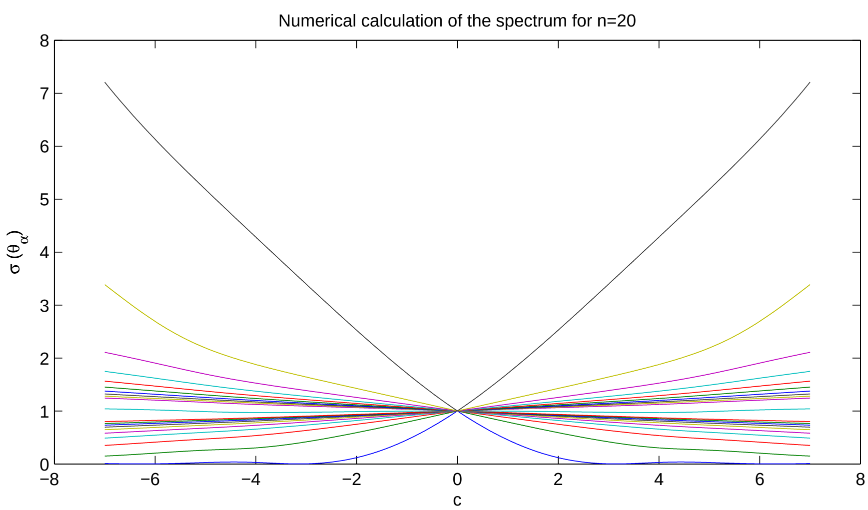

Figure 2: Dependence of the spectrum of θ_α on parameter c for $n = 20$

5.2 Implicit equation for eigenvalues

In this section the implicit equation for eigenvalues of θ_α will be derived. We will be looking for eigenvalues of $\mathcal{K}_\alpha$ instead of θ_α; eigenvalues of θ_α can be obtained from eigenvalues of $\mathcal{K}_\alpha$ just by adding 1. Throughout this chapter, we will suppose $\alpha \neq 0$. The eigenvalue equation is rather complicated integral equation

$$\begin{aligned} \frac{e^{i\alpha x}}{d}\int_0^d e^{-i\alpha y}\varphi(y)dy - \frac{1}{d}\int_0^d e^{-i\alpha y}\varphi(y)dy + \frac{i\alpha}{d}\int_0^d y\varphi(y)dy \\ -\frac{i\alpha}{d}x\int_0^d \varphi(y)dy - \frac{\alpha^2 x}{d}\int_0^d y\varphi(y)dy - i\alpha\int_x^d \varphi(y)dy \\ +\alpha^2 x\int_x^d \varphi(y)dy + \int_0^x (i\alpha+\alpha^2 y)\varphi(y)dy = \lambda\varphi(x). \end{aligned} \tag{65}$$

The left-hand side of this equation is an absolutely continuous function. If φ solve this equation it must also be an absolutely continuous function, therefore, we can differentiate this equation, arriving at

$$\begin{aligned} \frac{i\alpha}{d}e^{i\alpha x}\int_0^d e^{-i\alpha y}\varphi(y)dy - \frac{i\alpha}{d}\int_0^d \varphi(y)dy - \frac{\alpha^2}{d}\int_0^d y\varphi(y)dy \\ +(i\alpha+\alpha^2 x)\varphi(x) + i\alpha\varphi(x) + \alpha^2\int_x^d \varphi(y)dy - \alpha^2 x\varphi(x) = \lambda\varphi'(x). \end{aligned} \tag{66}$$

The left-hand side of this equation is again an absolutely continuous function so we can differentiate again

$$-\frac{\alpha^2}{d}e^{i\alpha x}\int_0^d e^{-i\alpha y}\varphi(y)dy + 2i\alpha\varphi'(x) - \alpha^2\varphi(x) = \lambda\varphi''(x). \tag{67}$$

Every solution of equation (65) must be therefore twice differentiable almost everywhere and it must solve equation (67) as well.

To solve equation (67) we first put $M = \frac{\alpha^2}{d}\int_0^d e^{-i\alpha y}\varphi(y)dy$ and treat it as a constant (i.e. we suppose it does not depend on φ). Equation (67) now becomes inhomogeneous linear differential equation of second order:

$$-\lambda\varphi''(x) + 2i\alpha\varphi'(x) - \alpha^2\varphi(x) = e^{i\alpha x}M, \tag{68}$$

which has two linearly independent solutions for every complex $\lambda \neq -1, 0, 3$ (we will deal with the special values of λ later) and for every complex M:

$$\varphi(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x} + \frac{M}{\alpha^2(\lambda-3)}e^{i\alpha x}, \tag{69}$$

where

$$k_1 = \frac{i\alpha - \sqrt{-\alpha^2 - \lambda\alpha^2}}{\lambda}, \tag{70}$$

$$k_2 = \frac{i\alpha + \sqrt{-\alpha^2 - \lambda\alpha^2}}{\lambda}. \tag{71}$$

Function (69) is a solution of (67) if and only if

$$M = \frac{\alpha^2}{d}\int_0^d e^{-i\alpha y}\varphi(y)dy, \tag{72}$$

which gives the following equation:

$$M = C_1\frac{\alpha^2}{d}\frac{1}{-i\alpha + k_1}(e^{(-i\alpha+k_1)d} - 1) + C_2\frac{\alpha^2}{d}\frac{1}{-i\alpha + k_2}(e^{(-i\alpha+k_2)d} - 1) + \frac{M}{(\lambda - 3)}. \tag{73}$$

If $\lambda \neq 4$, we get

$$M = \frac{(\lambda-3)}{(\lambda-4)}\frac{\alpha^2}{d}\left(C_1\frac{1}{-i\alpha + k_1}(e^{(-i\alpha+k_1)d} - 1) + C_2\frac{1}{-i\alpha + k_2}(e^{(-i\alpha+k_2)d} - 1)\right). \tag{74}$$

If we choose M according to this equation, function (69) will be a solution of equation (67). Equation (67) therefore have two linearly independent solutions for every complex $\lambda \neq -1, 0, 3, 4$. Furthermore, the function φ depends on C_1 and C_2 linearly so the solutions of (67) form a two-dimensional subspace of $\mathcal{H}$.

The solution of equation (67) will not be a solution of equation (65) for every λ, however we know that there must be countably many λ_i for which the solution of equation (67) is also a solution of equation (65). To find these solutions we insert function (69) into the equation (65) and find for which λ this equation is satisfied. This results into the following equation

$$\begin{aligned}
&\frac{x}{d}\Bigg[C_1\Bigg((e^{k_1 d} - 1)(\frac{i\alpha}{k_1} - \lambda) - \frac{2}{d}\frac{1}{-i\alpha + k_1}\frac{1}{\lambda - 4}(e^{(-i\alpha+k_1)d} - 1)(e^{i\alpha d} - 1)\Bigg) \\
&+ C_2\Bigg((e^{k_2 d} - 1)(\frac{i\alpha}{k_2} - \lambda) - \frac{2}{d}\frac{1}{-i\alpha + k_2}\frac{1}{\lambda - 4}(e^{(-i\alpha+k_2)d} - 1)(e^{i\alpha d} - 1)\Bigg)\Bigg] \\
&+ C_1\Bigg(-\frac{1}{d}(e^{k_1 d} - 1)(\frac{i\alpha}{k_1^2} + \frac{1}{k_1}) - \frac{2}{d}\frac{1}{-i\alpha + k_1}\frac{1}{\lambda - 4}(e^{(-i\alpha+k_1)d} - 1) \\
&\quad + \frac{2i}{\alpha d^2}\frac{1}{-i\alpha + k_1}\frac{1}{\lambda - 4}(e^{(-i\alpha+k_1)d} - 1)(e^{i\alpha d} - 1) + (\frac{i\alpha}{k_1} - \lambda)\Bigg) \\
&+ C_2\Bigg(-\frac{1}{d}(e^{k_2 d} - 1)(\frac{i\alpha}{k_2^2} + \frac{1}{k_2}) - \frac{2}{d}\frac{1}{-i\alpha + k_2}\frac{1}{\lambda - 4}(e^{(-i\alpha+k_2)d} - 1) \\
&\quad + \frac{2i}{\alpha d^2}\frac{1}{-i\alpha + k_2}\frac{1}{\lambda - 4}(e^{(-i\alpha+k_2)d} - 1)(e^{i\alpha d} - 1) + (\frac{i\alpha}{k_2} - \lambda)\Bigg) = 0.
\end{aligned} \tag{75}$$

This equation will be satisfied only if both the absolute term and the term proportional to x are equal to 0. Thus we have two equations linear in C_1 and C_2 (here, for simplicity, we do not write out the coefficients explicitly):

$$C_1 a_1(\lambda, \alpha, d) + C_2 a_2(\lambda, \alpha, d) = 0,$$
$$C_1 b_1(\lambda, \alpha, d) + C_2 b_2(\lambda, \alpha, d) = 0. \tag{76}$$

We will be able to find C_1 and C_2 such that these equations are satisfied if and only if the determinant of this system of equations is equal to 0, i.e. if

$$a_1(\lambda, \alpha, d) b_2(\lambda, \alpha, d) - a_2(\lambda, \alpha, d) b_1(\lambda, \alpha, d) = 0. \tag{77}$$

In the special case when all of the coefficients are 0, any constants C_1 and C_2 will solve this equation and we will thus have two linearly independent solutions of the equation (65). If at least one coefficient is non-zero (and the determinant is zero) we will have only one solution. After inserting the coefficients from (75) into (77), we will get the following equation

$$\begin{aligned}
&\big(\frac{i\alpha}{k_1} - \lambda\big)\big(\frac{i\alpha}{k_2} - \lambda\big)\big(e^{k_1 d} - e^{k_2 d}\big) + \frac{2}{d}\frac{1}{\lambda - 4} e^{i\alpha d}(e^{(-i\alpha + k_1)d} - 1) \\
&\times (e^{(-i\alpha + k_2)d} - 1)\big(\frac{1}{-i\alpha + k_1}(\frac{i\alpha}{k_2} - \lambda) - \frac{1}{-i\alpha + k_2}(\frac{i\alpha}{k_1} - \lambda)\big) = 0.
\end{aligned} \tag{78}$$

We can further simplify this equation if we make use of the fact that the spectrum of $\mathcal{K}_\alpha$ lies in $(-1, \infty)$ so that we can restrict ourselves only on $\lambda \in (-1, \infty)$. If we also do the substitution $\lambda \to \lambda - 1$ (because we are interested in eigenvalues of θ_α), we will get the following equation

$$\begin{aligned}
f(\lambda) = \frac{4}{c(\lambda - 5)(\lambda - 4)} e^{ic}(e^{-ic(1 - \frac{1 + \sqrt{\lambda}}{\lambda - 1})} - 1)(e^{-ic(1 - \frac{1 - \sqrt{\lambda}}{\lambda - 1})} - 1) \\
- \sqrt{\lambda} e^{\frac{ic}{\lambda - 1}} \sin(\frac{\sqrt{\lambda}}{\lambda - 1} c) = 0,
\end{aligned} \tag{79}$$

$$\tag{80}$$

where $c = \alpha d$. This equation is an implicit equation for eigenvalues of θ_α, it probably cannot be solved explicitly, but we know that it must have countably many solutions with limit point in 1. The equation (and therefore also the spectrum) depends only on the product of α and d denoted here by c. The plot of the absolute value of the implicit equation for specific choice of c is on the figure 3. The zeros of the graph corresponds to eigenvalues of θ_α. You can see that 1 indeed is a limit point of the spectrum.

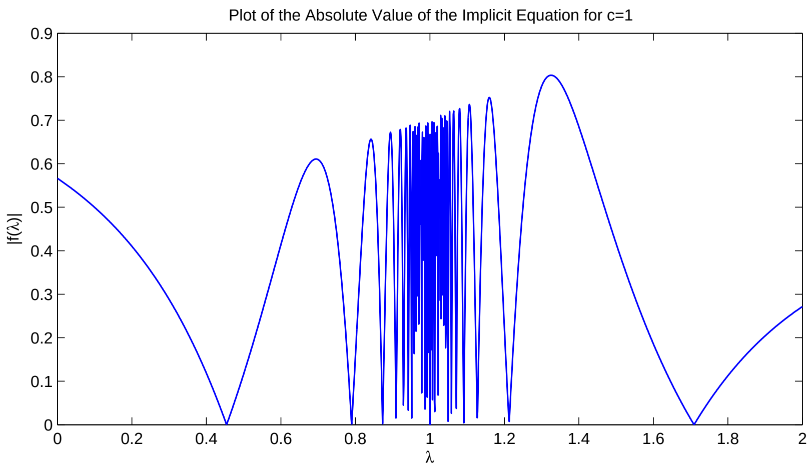

Figure 3: Plot of the absolute value of the implicit equation for $c = 1$

5.2.1 Special values of λ

In this section the values of λ for which the derivation in the previous section does not apply will be examined.

a) $\lambda = -1$

If -1 is an eigenvalue of $\mathcal{K}_\alpha$, then 0 is an eigenvalue of θ_α, but we have already shown that θ_α is positive, if the non-degeneracy condition (8) is satisfied, so 0 can be an eigenvalue of θ_α only if $\alpha d = k\pi$. One can easily verify that in such case 0 indeed is an eigenvalue of θ_α with the corresponding eigenfunction being

$$\varphi(x) = e^{-i\alpha x}. \tag{81}$$

b) $\lambda = 0$

If $\lambda = 0$, then eq. (67) has only one linearly independent solution

$$\varphi(x) = e^{-\frac{1}{2}i\alpha x} + \frac{1}{6i\alpha d}(e^{-\frac{3}{2}i\alpha d} - 1)e^{i\alpha x}, \tag{82}$$

which satisfy eq. (65) if and only if the following two equations are satisfied

$$-2(e^{-\frac{1}{2}i\alpha d} - 1) + \frac{1}{6i\alpha d}(e^{-\frac{3}{2}i\alpha d} - 1)(e^{i\alpha d} - 1) = 0, \tag{83}$$

$$\frac{1}{6i\alpha d}(e^{-\frac{3}{2}i\alpha d} - 1) = 2. \tag{84}$$

If we insert second equation into first, we find that αd must satisfy

$$e^{\frac{3}{2}i\alpha d} = 1. \tag{85}$$

Such αd cannot, however, satisfy the equation (84), so we conclude that $\lambda = 0$ can never be an eigenvalue of $\mathcal{K}_\alpha$.

c) $\lambda = 3$

If $\lambda = 3$, then eq. (67) has two linearly independent solutions

$$\varphi(x) = C_1 e^{-\frac{1}{3}i\alpha x} + C_2 e^{i\alpha x} + \left(-C_1 \frac{3}{2d}\frac{1}{8-i\alpha d}(e^{-\frac{4}{3}i\alpha d} - 1) - C_2 \frac{2\alpha}{i(8-i\alpha d)} \right) x e^{i\alpha x}. \tag{86}$$

After inserting this function into eq. (65) we find

$$\begin{aligned} &C_1 \Bigg(-6(e^{-\frac{1}{3}i\alpha d} - 1) - 6 - -\frac{3}{2d}\frac{1}{8-i\alpha d}(e^{-\frac{4}{3}i\alpha d} - 1) \\ &\qquad \times \left(-\frac{3}{\alpha^2 d}(e^{i\alpha d} - 1) - \frac{3i}{\alpha} + \frac{2i}{\alpha}e^{i\alpha d} \right) \Bigg) \\ &+ C_2 \Bigg(-2(e^{i\alpha d} - 1) - 2 - \frac{2\alpha}{i(8-i\alpha d)} \left(-\frac{3}{\alpha^2 d}(e^{i\alpha d} - 1) - \frac{3i}{\alpha} + \frac{2i}{\alpha}e^{i\alpha d} \right) \Bigg) = 0, \end{aligned} \tag{87}$$

$$\begin{aligned} &C_1 \Bigg(-6(e^{-\frac{1}{3}i\alpha d} - 1) - \frac{3}{2d}\frac{1}{8-i\alpha d}(e^{-\frac{4}{3}i\alpha d} - 1) \left(-\frac{3}{i\alpha}(e^{i\alpha d} - 1) - 2de^{i\alpha d} \right) \Bigg) \\ &\quad + C_2 \Bigg(-2(e^{i\alpha d} - 1) - \frac{2\alpha}{i(8-i\alpha d)} \left(-\frac{3}{i\alpha}(e^{i\alpha d} - 1) - 2de^{i\alpha d} \right) \Bigg) = 0. \end{aligned} \tag{88}$$

Determinant of this set of equations is

$$\begin{aligned} &12(e^{-\frac{1}{3}i\alpha d} - e^{i\alpha d}) \\ &\left(\frac{12\alpha d}{i(8-i\alpha d)}(e^{-\frac{1}{3}i\alpha d} - 1) - \frac{3}{(8-i\alpha d)}(e^{i\alpha d} - 1)(e^{-\frac{4}{3}i\alpha d} - 1) \right) \\ &\quad \times \left(-\frac{6}{\alpha^2 d^2}(e^{i\alpha d} - 1) - \frac{3i}{\alpha d} + \frac{2i}{\alpha d}e^{i\alpha d} + \frac{2}{i\alpha d}e^{i\alpha d} \right) \\ &+ \left(-\frac{3}{i\alpha d}(e^{i\alpha d} - 1) - 2e^{i\alpha d} \right) \left(-\frac{12\alpha d}{i(8-i\alpha d)} + \frac{3}{(8-i\alpha d)}(e^{-\frac{4}{3}i\alpha d} - 1) \right). \end{aligned} \tag{89}$$

$\lambda = 3$ is an eigenvalue of $\mathcal{K}_\alpha$ if and only if αd satisfies this equation.

d) $\lambda = 4$

In this case eq. (67) has two linearly independent solutions

$$\varphi(x) = C_1\left(e^{i\alpha\frac{-3+\sqrt{5}}{4}x} - (-7+3\sqrt{5})\frac{e^{i\alpha d\frac{-3+\sqrt{5}}{4}}-1}{e^{i\alpha d\frac{-3-\sqrt{5}}{4}}-1}e^{i\alpha\frac{-3-\sqrt{5}}{4}x}\right)+C_2e^{i\alpha x}. \quad (90)$$

This function is an eigenfunction of $\mathcal{K}_\alpha$ only if

$$C_1\Bigg((e^{i\alpha d\frac{-3+\sqrt{5}}{4}}-1)4\frac{\sqrt{5}}{1+\sqrt{5}}$$
$$-(-7+3\sqrt{5})4\frac{\sqrt{5}}{1-\sqrt{5}}\frac{e^{i\alpha d\frac{-3+\sqrt{5}}{4}}-1}{e^{i\alpha d\frac{-3-\sqrt{5}}{4}}-1}(e^{i\alpha d\frac{-3-\sqrt{5}}{4}}-1)\Bigg) - 3C_2(e^{i\alpha d}-1) = 0, \quad (91)$$

$$C_1\Bigg((e^{i\alpha\frac{-3+\sqrt{5}}{4}d}-1)4\frac{\sqrt{5}}{1+\sqrt{5}}\frac{1}{i\alpha d} + 4\frac{\sqrt{5}}{1+\sqrt{5}}$$
$$-(-7+3\sqrt{5})4\frac{\sqrt{5}}{1-\sqrt{5}}\frac{e^{i\alpha d\frac{-3+\sqrt{5}}{4}}-1}{e^{i\alpha d\frac{-3-\sqrt{5}}{4}}-1}((e^{i\alpha d\frac{-3-\sqrt{5}}{4}}-1)\frac{1}{i\alpha d}+1\Bigg)$$
$$+3C_2(\frac{1}{i\alpha d}(e^{i\alpha d}-1)+1) = 0. \quad (92)$$

There exists C_1 and C_2 solving this equation if and only if the determinant of this set of equations is zero, which leads to the following equation

$$\frac{1}{1+\sqrt{5}}(e^{i\alpha d} - e^{i\alpha d\frac{-3+\sqrt{5}}{4}}) \quad (93)$$

$$+\frac{1}{1-\sqrt{5}}(-7+3\sqrt{5})\frac{e^{i\alpha d\frac{-3+\sqrt{5}}{4}}-1}{e^{i\alpha d\frac{-3-\sqrt{5}}{4}}-1}(e^{i\alpha d} - e^{i\alpha d\frac{-3-\sqrt{5}}{4}}) = 0, \quad (94)$$

therefore, $\lambda = 4$ is an eigenvalue of $\mathcal{K}_\alpha$ if and only if αd solves this equation.

5.2.2 Summary

We can summarize:

Theorem 5.1. *The spectrum of θ_α for depends only on αd and for $\alpha \neq 0$ it satisfies:*

1. *$\lambda \neq 0, 1, 4, 5$ is an eigenvalue of θ_α if and only if it is a positive number solving the equation* (79). *For λ solving this equation the corresponding eigenfunctions are given by the function* (69), *where M is given by the equation* (74) *and the constants C_1 and C_2 are solutions of the system of equations* (76).

2. $\lambda = 0$ *is an eigenvalue of* θ_α *if and only if the non-degeneracy condition* (8) *is not satisfied.*

3. $\lambda = 1$ *can never be an eigenvalue of* θ_α.

4. $\lambda = 4$ *is an eigenvalue of* θ_α *if and only if* αd *satisfies eq.* (89)*, the corresponding eigenfunction is the function* (86) *with constants* C_1, C_2 *given by the solutions of the system of equations* (88).

5. $\lambda = 5$ *is an eigenvalue of* θ_α *if and only if* αd *solves the equation* (94)*, in such case the function* (90)*, with constants* C_1, C_2 *given by the solution of the system of equations* (92)*, is the corresponding eigenfunction.*

6 Conclusion

We found a new, simple, integral-type formula for the metric operator in a $\mathcal{PT}$-symmetric model and we used this new formula for analyzing the spectrum of the metric. While we were unable to find the spectrum explicitly we found an implicit equation for eigenvalues, which reduces the eigenvalue problem to the problem of finding solutions of some algebraic equation. Unfortunately, this result cannot be used for finding the explicit formula for the square root of the metric, which was our original motivation for analyzing the spectrum. It would be possible to solve the implicit equation numerically and use the result to find the approximation of the metric. We do not plan to use this method, instead, in a future work, we are going to do a perturbative calculation of the square root for small values of α.

A Dirichlet and Neumann Laplacians

Dirichlet and Neumann Laplacians, denoted by $-\Delta_D$ and $-\Delta_N$ respectively are operators on $L^2(0,d)$, which both acts as minus second derivative, but have different boundary conditions

$$D(\Delta_D) = \{\varphi \in W^{1,2}((0,d)) \,|\, \varphi(0) = \varphi(d) = 0\}, \tag{95}$$

$$D(\Delta_N) = \{\varphi \in W^{1,2}((0,d)) \,|\, \varphi'(0) = \varphi'(d) = 0\}. \tag{96}$$

They are self-adjoint operators and they both have purely point spectrum, thus they can be decomposed as

$$-\Delta_D = \sum_{j=0}^{\infty} \lambda_j^D \chi_j^D (\chi_j^D, \cdot) \qquad - \Delta_N = \sum_{j=0}^{\infty} \lambda_j^N (\chi_j^N)(\chi_j^N, \cdot), \tag{97}$$

where λ_j^D and χ_j^D resp. λ_j^N and χ_j^N denotes eigenvalues and eigenfunctions of Dirichlet resp. Neumann Laplacian. In this text only eigenvalues and eigenfunctions of Neumann Laplacian are needed:

$$\lambda_j^N \equiv k_j^2 = \left(\frac{j\pi}{d}\right)^2, \tag{98}$$

$$\chi_j^N = \begin{cases} \sqrt{1/d} & \text{for } j = 0, \\ \sqrt{2/d}\cos(k_j x) & \text{for } j \geq 1. \end{cases} \tag{99}$$

An important fact is that the domain of Δ_N coincides with the set of all functions for which series (97) converges.

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