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Poisson structures on Lie groups

DIPLOMA THESIS

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Abstrakt:

Poissonovy variety představují přirozený způsob zavedení Poissonovy závorky na obecné diferencovatelné varietě. Poisson-Lieovy grupy jsou netriviální kombinací Lieovy grupy a Poissonovy variety. Teorie Lieových bialgeber zkoumá možnost definice Lieovy závorky na duálním prostoru dané Lieovy algebry, a to způsobem, který je v jistém smyslu reflexivní. Ukazuje se, že ke každé Poisson-Lieově grupě existuje odpovídající Lieova bialgebra a naopak.

Tato práce se po nezbytném a podrobném uvedení do problematiky Lieových bialgeber a Poissonových variet zabývá zejména přímočarou konstrukcí Poisson-Lieovy grupy odpovídající dané Lieově bialgebře. Tento postup je obecně známý pro speciální třídu tzv. kohraničních Lieových bialgeber. S využitím adjungované reprezentace Lieovy grupy a Drinfel'dova doublu lze však sestrojit Poisson-Lieovu grupu pro zcela obecnou Lieovu bialgebru.

Poissonovy sigma modely představují zajímavé využití Poissonových variet pro konstrukci klasické polní teorie. V této práci ukážeme jejich definici v jazyce fibrovaných prostorů a odvodíme příslušné polní rovnice pomocí variačního principu. Odvodíme elegantní formu pohybových rovnic pro tzv. Poisson-Lie sigma modely.

Klíčová slova: Poissonova varieta, Lieova bialgebra, kohomologie, Poisson-Lieova grupa, multiplikativní tenzorové pole, Poissonův sigma model.

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Abstract:

A natural way to define Poisson bracket on general differentiable manifold is the concept of Poisson manifolds. Poisson-Lie groups are non-trivial combinations of Lie groups and Poisson manifolds. A theory of Lie bialgebras examines the possibility of Lie bracket definition on the vector space dual to the given Lie algebra, which is reflexive in a certain meaning. It turns out that to every Poisson-Lie group there exists a corresponding Lie bialgebra and vice versa.

After the necessary and detailed introduction into the theory of Lie bialgebras and Poisson manifolds, this work focuses especially on the straightforward construction of the Poisson-Lie group corresponding to a given Lie bialgebra. This procedure is widely known only for a special class of so called coboundary Lie bialgebras. However, using the adjoint representation of Lie group and Drinfel'd double, Poisson-Lie group can be constructed for general Lie bialgebra.

Poisson sigma models represent an interesting use of Poisson manifolds for the construction of a classical field theory. In this work their definition in the language of fibre bundles is shown and the corresponding field equations are derived using a variational principle. The elegant form of equations of motion for so called Poisson-Lie groups is derived.

Keywords: Poisson manifold, Lie bialgebra, cohomology, Poisson-Lie group, multiplicative tensor field, Poisson sigma model.

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Contents

Notation index

By M we always mean a differentiable manifold M. By G we always mean a Lie group and by $\mathfrak{g}$ its Lie algebra.

$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers (positive integers)
$\mathbb{N}_0$	set of natural numbers with zero included
$C^\infty(M)$	space of smooth functions on M
$\mathcal{T}^p_q(M)$	space of p-times contravariant and q-times covariant smooth tensor fields on M
$\Omega^p(M)$	space of differential p-forms on M
$L_p(M)$	space of p-vector fields on M
$\mathfrak{X}(M)$	space of smooth vector fields on M
$\Omega(M)$	Cartan algebra of differential forms on M
$L(M)$	multivector field algebra on M
T_pM	tangent space at the point p of M
T^*_pM	cotangent space at the point p of M
f_*	pushforward corresponding to the map f
f^*	pullback corresponding to the map f
L_g, R_g	left (right) translation on G by $g \in G$
L_X, R_X	left-(right-)invariant vector field on G generated by $X \in \mathfrak{g}$
$+cyclic\{X, Y, Z\}$	add a cyclic permutation of X, Y, Z
$\mathcal{L}_V$	Lie derivative along the vector field V
$\langle \alpha, X \rangle$	canonical pairing between the dual space and the vector space
V^*	dual space to the vector space V
$\bigotimes^k V$	k-th tensor power of the vector space V
S^kV	subspace of symmetric k-linear maps in $\bigotimes^k V$
$\bigwedge^k V$	subspace of completely skew-symmetric k-linear maps in $\bigotimes^k V$
$span\{V_1, \ldots, V_n\}$	linear span of the set of vectors $\{V_1, \ldots V_n\}$

For $T \in \mathcal{T}^p_q(M)$ and $k \in M$, by $T(k)$ we mean T evaluated at the point k of M, i.e. the tensor $T(k)$ in the tensor algebra above $T_k(M)$. For the convenience, we will sometimes use the notation $T|_k$ or T_k if there is too many brackets present. It should be commented or obvious from the context.

Chapter 1

Introduction

More than two hundred years ago (1809), Siméon Poisson came with the idea of bracket

$$\{f,g\} = \sum_{j=1}^{n} \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j} - \frac{\partial g}{\partial p_j}\frac{\partial f}{\partial q^j}, \tag{1.1}$$

where f and g are functions of coordinates q^j and of conjugated momenta $p_j := \frac{\partial L}{\partial \dot{q}^j}$. He found that $\{f,h\} = 0$ and $\{g,h\} = 0$ implies $\{\{f,g\},h\} = 0$. This is nowadays a direct consequence of the famous Jacobi identity for Poisson bracket, which was proved by Carl Jacobi more than thirty years later (1842). Meanwhile, William Hamilton introduced his description of the mechanics and found a huge use for Poisson bracket.

Due to the Hamiltonian mechanics, Poisson bracket can be viewed as the bilinear map of phase space functions, giving again a function on the phase space ($\mathbb{R}^{2n}$). This map is skew-symmetric with respect to the interchanging of two functions, satisfies the Jacobi identity and moreover it acts like a differential operator on the product of two functions (this property is called Leibniz rule).

The first idea of generalization came in the year 1890 from Sophus Lie himself [1]. Lie examined, according to Alan Weinstein [2], a possibility to define a bracket

$$\{\cdot,\cdot\} : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

of the same properties as the Poisson bracket, using the functions $\mathcal{P}^{ij}(x^1,\dots,x^n) \in C^\infty(\mathbb{R}^n)$, having the properties

$$\mathcal{P}^{ij} = -\mathcal{P}^{ji}, \tag{1.2}$$

$$\mathcal{P}^{ri}\frac{\partial \mathcal{P}^{jk}}{\partial x^r} + \mathcal{P}^{rj}\frac{\partial \mathcal{P}^{ki}}{\partial x^r} + \mathcal{P}^{rk}\frac{\partial \mathcal{P}^{ij}}{\partial x^r} = 0. \tag{1.3}$$

Lie then defined his bracket as

$$\{f,g\} := \mathcal{P}^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \tag{1.4}$$

for all $f,g \in C^\infty(\mathbb{R}^n)$. This is in fact a local coordinate form of Poisson bracket as we think of it now (see e.g. (3.26)).

A development in the differential geometry and the interest in the curved spaces led naturally to the following generalization of Poisson bracket. We take an arbitrary even-dimensional manifold M and we intend to have a Poisson bracket on M, which has in some local coordinates the form (1.1). It turned out, that this can be done using an arbitrary non-degenerate closed 2-form

$\omega \in \Omega^2(M)$. Using this form, we can assign to each smooth function $f \in C^\infty(M)$ a smooth vector field $\zeta_f \in \mathfrak{X}(M)$ and define a Poisson bracket on M as

$$\{f, g\} := \omega(\zeta_f, \zeta_g),$$

for all $f, g \in C^\infty(M)$. It can be shown (see section (3.5)) that such defined bracket satisfies all demanded properties of Poisson bracket. A pair (M, ω) is called a symplectic manifold. The study of such manifolds gave rise to many parts of the modern differential geometry and theoretical mechanics (e.g. symplectic mechanics, symplectic topology).

The more general concept of Poisson manifold came in the early 1970's in various works on various topics, the modern language of the Poisson geometry was introduced in the work of Alan Weinstein [2]. Similarly as in the symplectic case, we can encode the Poisson bracket into the bivector field $\mathcal{P} \in L_2(M)$, called Poisson bivector, as

$$\{f, g\} := \mathcal{P}(df, dg),$$

for all $f, g \in C^\infty(M)$. It turns out that so defined bracket satisfies the Jacobi identity, if and only if $[\mathcal{P}, \mathcal{P}] = 0$, where $[\cdot, \cdot]$ denotes the Schouten-Nijenhuis bracket, a straightforward generalization of the vector field commutator, discovered in 1940 by the Dutch mathematician Jan Schouten [3] and further examined by his student Albert Nijenhuis in 1955.

The idea of Poisson-Lie groups, Lie bialgebras and Yang-Baxter equations was introduced in the year 1983 by the Ukrainian mathematician Vladimir Drinfel'd in [4].

The main purpose of my diploma thesis is to give a compact overview of the theory of Poisson-Lie groups, focusing on their relation to Lie bialgebras. Our goal is to present proofs of the most important statements and to give enough examples. The outline of the work is following:

In the chapter 2 we will first introduce the elements of Chevalley-Eilenberg cohomology of Lie algebras, needed for the definition of Lie bialgebra. Then the basic properties of Lie bialgebras will be introduced, including their correspondence to special triples of Lie algebras (Manin triples). This will become very useful in the construction of simple examples. As in every cohomology theory, we can discuss the possibility of finding a "potential" 0-cochain, such that Lie bialgebra 1-cocycle is its coboundary. This will lead us to the concept of r-matrix and Yang-Baxter equations.

Chapter 3 summarizes the elementary Poisson geometry. We will give the definition of Poisson manifold, introduce a Schouten-Nijenhuis bracket and use it for the "bivector language" in the description of Poisson manifolds. We will show the idea of Hamiltonian fields, a characteristic distribution and then the most interesting property of Poisson structures - the symplectic foliation of a Poisson manifold. Finally, we define a Poisson-Lie group and show that its bivector lies in the class of so called multiplicative tensor fields.

In the chapter 4 we will in detail examine the properties of multiplicative tensor fields and use them to find the important relation between Lie algebras and Poisson-Lie groups. We will in fact find a direct way how to construct every Poisson-Lie group bivector on connected and simply connected Lie groups.

In the final chapter 5 we will introduce a theory of Poisson sigma models, classical field theory using Poisson manifolds as target manifolds. Fields of Poisson sigma model can be described using the language of fibre bundles. The geometric interpretation of the fields variation will be shown. We will prove the existence of the local solution for general Poisson sigma model. Special cases of Poisson sigma models will be examined, including the models constructed using the Poisson-Lie groups.

Chapter 2

Lie bialgebras

2.1 Introduction

In this chapter we will introduce the elements of the theory of Lie bialgebras. Lie bialgebras are strongly entangled (by definition) with a Chevalley-Eilenberg Lie algebra cohomology, and the question "when closed is exact?" naturally comes to the play. This will lead us to the discovery of so called r-matrix and the equation describing it, (classical) Yang-Baxter equation.

There is an interesting remark about the origin of such theory - Lie bialgebras and r-matrices were found as a classical limit of the structures involved in a certain quantum theory (Quantum inverse scattering method), mostly in the works of Ukrainian mathematician Vladimir Drinfel'd [4].

All sections of this chapter are based mostly on the great lecture notes of Yvette Kosmann-Schwarzbach [5], we added only some technical details into some of the proofs. We will also bring some examples of presented structures.

2.2 Lie algebra cohomology

Before we can introduce Lie bialgebras, we have to show the basics of the Lie algebra cohomology. It is constructed using the chosen representation of Lie algebra in the way very similar to ordinary de Rham cochain complex (differentiable forms on manifolds). Instead of the completely skew-symmetric tensor fields, completely skew-symmetric k-linear maps from $\mathfrak{g}$ to the representation vector space V are used. For further details and examples, see [6].

Definition 2.2.1. Let $\mathfrak{g}$ be a Lie algebra with a Lie bracket $[\cdot, \cdot]$, V a vector space. The linear map $\rho : \mathfrak{g} \to End(V)$ is called a **representation** of $\mathfrak{g}$ on V, if for all $X, Y \in \mathfrak{g}$

$$\rho([X, Y]) = [\rho(X), \rho(Y)] \equiv \rho(X) \cdot \rho(Y) - \rho(Y) \cdot \rho(X), \tag{2.1}$$

where $\cdot$ denotes the map composition. $End(V)$ denotes the space of linear endomorphisms of V. For $X \in \mathfrak{g}, v \in V$ we denote $\rho(X)(v)$ as $X.v$.

Definition 2.2.2. For every finite-dimensional Lie algebra $\mathfrak{g}$ we can define a **generalized adjoint representation** of the Lie algebra $\mathfrak{g}$. For fixed $p \in \mathbb{N}$ it is a representation of $\mathfrak{g}$ on $\otimes^p \mathfrak{g}$. For arbitrary basis $(T_1, \ldots, T_n)$ of $\mathfrak{g}$ we can decompose every element $A \in \otimes^p \mathfrak{g}$ as

$$A = A^{i_1 \cdots i_p} T_{i_1} \otimes \cdots \otimes T_{i_p}. \tag{2.2}$$

The representation map $\rho : \mathfrak{g} \to End(\otimes^p \mathfrak{g})$ is then for $X \in \mathfrak{g}$ and $A \in \otimes^p \mathfrak{g}$ defined as

$$\rho(X)(A) \equiv ad_X^{(p)}(A) := A^{i_1 \cdots i_p} \sum_{k=1}^{n} T_{i_1} \otimes \cdots \otimes [X, T_{i_k}] \otimes \cdots \otimes T_{i_p}. \tag{2.3}$$

It is easy to verify that ρ is indeed a representation of $\mathfrak{g}$.

Remark 2.2.3. One can ask, if there exists a representation of a Lie group G, such that $ad^{(p)}$ is its derived representation. The answer is positive. Usual adjoint representation of Lie group is defined as the pushforward of the inner automorphism and there is no problem to use it on the tensor product of the vectors from $\mathfrak{g}$, that is we define

$$Ad_g^{(p)}(A) := I_{g*}(A),$$

for $A \in \otimes^p \mathfrak{g}$, $g \in G$ and I_g denotes the inner automorphism in G. In components we get

$$Ad_g^{(p)}(A) = A^{i_1 \cdots i_p} Ad_g(T_{i_1}) \otimes \cdots \otimes Ad_g(T_{i_p}).$$

By the same arguments as in the case of $p = 1$ this is a representation of a Lie group G. Then of course $ad_X^{(p)} = \frac{d}{dt}\big|_{t=0} Ad_{e^{tX}}$.

Definition 2.2.4. Let $\mathfrak{g}$ be a Lie algebra. For arbitrary representation ρ of $\mathfrak{g}$ on V and $k \in \mathbb{N}_0$ we define a ***k*-cochain** of $\mathfrak{g}$ with values in V as a k-linear completely skew-symmetric map from $\mathfrak{g}$ to V. The 0-cochain is just an element of V.

We denote the space of k-cochains of $\mathfrak{g}$ with values in V as $c^k(\mathfrak{g}, V)$. We define a linear map $\Delta : c^k(\mathfrak{g}, V) \to c^{k+1}(\mathfrak{g}, V)$, called the **coboundary operator**, as

$$\Delta(u)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i.(u(X_0, \cdots, \widehat{X_i}, \cdots X_k)) + \tag{2.4}$$

$$+ \sum_{i<j} (-1)^{i+j} u([X_i, X_j], X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k),$$

for $X_0, \ldots, X_k \in \mathfrak{g}$. For example for $k = 0$ and $u \in V, X \in \mathfrak{g}$ we have $\Delta(u)(X) = X.u$. For $k = 1$ and v a 1-cochain we have for $X, Y \in \mathfrak{g}$

$$\Delta(v)(X, Y) = X.v(Y) - Y.v(X) - v([X, Y]). \tag{2.5}$$

Remark 2.2.5. Notice the resemblance to the formula for the explicit computation of the exterior derivative of a differential k-form.

Theorem 2.2.6. *For arbitrary k-cochain u of $\mathfrak{g}$ with values in V there holds*

$$\Delta(\Delta(u)) = 0. \tag{2.6}$$

Proof. The proof is not difficult, but rather long to show here. For details see [6]. However, for our purpose it is sufficient to know, that the theorem holds for $k = 0$. This is easy to show using (2.1). ∎

Definition 2.2.7. A k-cochain u is called a ***k*-cocycle** if $\Delta(u) = 0$.

A k-cochain u is called a ***k*-coboundary** if there exists $(k-1)$-cochain w, such that $\Delta(w) = u$.

For k-cochain u the $(k+1)$-cochain $\Delta(u)$ is called a **coboundary** of u.

Remark 2.2.8. It follows from the theorem 2.2.6 that every k-coboundary is also a k-cocycle.

The resulting cochain complex is sometimes called a *Chevalley-Eilenberg complex*.

2.3 Lie bialgebra, dual of a Lie bialgebra

Every Lie algebra $\mathfrak{g}$ is by definition a vector space. Every vector space (for simplicity finite-dimensional) has its dual vector space of the same dimension. It is natural to ask, whether it is possible to have some Lie algebra on the dual space either. This is of course always possible. But there is another interesting property of every finite-dimensional vector space, called the reflexivity. It says that the dual space of the dual space is canonically isomorphic to the original vector space.

We would like to retain this property for our Lie algebra on the dual space $\mathfrak{g}^*$ (in a certain meaning). In the vector space case we have the natural action of the covectors on the vectors, which assures us the reflexivity. The reflexivity of Lie algebras is provided by an additional condition on the Lie bracket in $\mathfrak{g}^*$, namely the 1-cocycle condition.

We start with the definition of the transpose map (with respect to the canonical pairing). Let us also remark some important benefits of finite-dimensional spaces.

Definition 2.3.1. Let V and W be the vector spaces, $A : V \to W$. We define a **transpose map** to the map A, denoted as $A^* : W^* \to V^*$, by the relation

$$\langle A^*(\alpha), X\rangle := \langle \alpha, A(X)\rangle, \tag{2.7}$$

$x \in V$, $\alpha \in W^*$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing of V and V^*.

We should remark that for a finite-dimensional vector space V there holds

$$(V \otimes V)^* \equiv V^* \otimes V^* \tag{2.8}$$

and every linear map on $V^* \otimes V^*$ can be viewed as a bilinear map on V^*.

Now we can proceed to the promised definition of the "special" Lie algebra on $\mathfrak{g}^*$. We do not directly set the Lie bracket on $\mathfrak{g}^*$, but instead of it we supplement an additional structure δ on $\mathfrak{g}$ and call the pair $(\mathfrak{g}, \delta)$ a Lie bialgebra.

Definition 2.3.2. Lie bialgebra $(\mathfrak{g}, \delta)$ is a Lie algebra $\mathfrak{g}$ equipped by an additional structure, a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, such that

1. $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$,
2. δ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, where $\rho = ad^{(2)}$, i.e. for every $X, Y \in \mathfrak{g}$

$$\Delta(\delta)(X,Y) \equiv ad^{(2)}_X \delta(Y) - ad^{(2)}_Y \delta(X) - \delta([X,Y]) = 0. \tag{2.9}$$

The linear map δ is usually called a **cocommutator** on $\mathfrak{g}$.

Remark 2.3.3. In the case of infinite-dimensional $\mathfrak{g}$ one must restrict the result of the transposition, because in general $\mathfrak{g}^* \otimes \mathfrak{g}^* \subset (\mathfrak{g} \otimes \mathfrak{g})^*$. For simplicity (and for our purposes) let us consider only finite-dimensional Lie bialgebras.

If we denote the Lie bracket on $\mathfrak{g}^*$ as $[\cdot,\cdot]_{\mathfrak{g}^*}$, we have by definition

$$[\xi, \eta]_{\mathfrak{g}^*} = \delta^*(\xi \otimes \eta), \tag{2.10}$$

for $\xi, \eta \in \mathfrak{g}^*$.

Remark 2.3.4. If we choose a basis $(T_1, \ldots, T_n)$ in $\mathfrak{g}$ and write down $\mathfrak{g}^*$ in the dual basis $(T^i)_{i=1}^n$ using the structure constants $[T^i, T^j]_{\mathfrak{g}^*} = f^{ij}{}_k T^k$, we have by simple computation

$$\delta(T_i) = f^{kl}{}_i T_k \otimes T_l. \tag{2.11}$$

Moreover, if $[T_i, T_j] = c_{ij}{}^k T_k$, the 1-cocycle condition (2.9) reads

$$c_{ij}{}^k f^{mn}{}_k = c_{ik}{}^m f^{kn}{}_j + c_{ik}{}^n f^{mk}{}_j - c_{jk}{}^m f^{kn}{}_i - c_{jk}{}^n f^{mk}{}_i. \tag{2.12}$$

The 1-cocycle condition (2.9) is often written in the form

$$\delta([X, Y]) = (ad_X \otimes 1 + 1 \otimes ad_X)(\delta(Y)) - (ad_Y \otimes 1 + 1 \otimes ad_Y)(\delta(X)), \tag{2.13}$$

$X, Y \in \mathfrak{g}$, where $(ad_X \otimes 1)(Y_1 \otimes Y_2) = ad_X(Y_1) \otimes Y_2$. In some literature the 1-cocycle condition is written as

$$\delta([X, Y]) = [X \otimes 1 + 1 \otimes X, \delta(Y)] - [Y \otimes 1 + 1 \otimes Y, \delta(X)]. \tag{2.14}$$

The meaning of $[X \otimes 1 + 1 \otimes X, u]$ is obvious.

To show the meaning of the intended and promised reflexivity of the Lie algebras in the Lie bialgebra, we introduce a very effective approach, where no particular coordinates in the particular basis are needed. This approach is based mostly on the representation of $\mathfrak{g}$ on the dual space $\mathfrak{g}^*$, induced by the ordinary (and always fully functional) adjoint representation.

Definition 2.3.5. Coadjoint representation of $\mathfrak{g}$ is a representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ defined by the relation

$$\langle ad^*_X(\xi), V \rangle := -\langle \xi, ad_X(V) \rangle, \tag{2.15}$$

$X, V \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$. One should easily check that it is indeed a representation of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$.

From the definition of the tensor product it is obvious that one can transpose the *ad* map between the left and right hand side of the canonical pairing bracket with no fear even for arbitrary tensor product, for example

$$\langle ad_Z(X) \otimes Y, \xi \otimes \eta \rangle = -\langle X \otimes Y, ad^*_Z(\xi) \otimes \eta \rangle,$$

for every $Z, X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$. This fact becomes very useful in the following lines.

Condition (2.9) can be written in a way emphasising the symmetry between $\mathfrak{g}$ and $\mathfrak{g}^*$ with respect to their roles in a Lie bialgebra structure. First of all, we can write

$$\langle ad^{(2)}_X \delta(Y), \xi \otimes \eta \rangle = \langle ad^{(2)}_X \big(\delta(Y)^{ij} T_i \otimes T_j\big), \xi \otimes \eta \rangle =$$

$$= \delta(Y)^{ij} \langle ad_X(T_i) \otimes T_j, \xi \otimes \eta \rangle + \delta(Y)^{ij} \langle T_i \otimes ad_X(T_j), \xi \otimes \eta \rangle =$$

$$= -\delta(Y)^{ij} \langle T_i \otimes T_j, ad^*_X(\xi) \otimes \eta \rangle - \delta(Y)^{ij} \langle T_i \otimes T_j, \xi \otimes ad^*_X(\eta) \rangle =$$

$$= -\langle \delta(Y), ad^*_X(\xi) \otimes \eta \rangle - \langle \delta(Y), \xi \otimes ad^*_X(\eta) \rangle =$$

$$= -\langle [ad^*_X(\xi), \eta]_{\mathfrak{g}^*}, Y \rangle - \langle [\xi, ad^*_X(\eta)]_{\mathfrak{g}^*}, Y \rangle,$$

for $\xi, \eta \in \mathfrak{g}^*$, $X, Y \in \mathfrak{g}$. The condition (2.9) can be then rewritten as

$$\langle [\xi, \eta]_{\mathfrak{g}^*}, [X, Y] \rangle = \langle [ad^*_Y(\xi), \eta]_{\mathfrak{g}^*}, X \rangle + \langle [\xi, ad^*_Y(\eta)]_{\mathfrak{g}^*}, X \rangle -$$

$$- \langle [ad^*_X(\xi), \eta]_{\mathfrak{g}^*}, Y \rangle - \langle [\xi, ad^*_X(\eta)]_{\mathfrak{g}^*}, Y \rangle.$$

If we set
$$ad_\xi(\eta) := [\xi, \eta]_{\mathfrak{g}^*} \tag{2.16}$$
and
$$\langle \eta, ad^*_\xi(X)\rangle := -\langle ad_\xi(\eta), X\rangle, \tag{2.17}$$
$\xi, \eta \in \mathfrak{g}^*$, $X \in \mathfrak{g}$, we get the adjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}^*$ and the coadjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}^{**} \cong \mathfrak{g}$. Using this, we can write
$$\langle [ad^*_Y(\xi), \eta]_{\mathfrak{g}^*}, X\rangle = -\langle ad_\eta(ad^*_Y(\xi)), X\rangle = \langle ad^*_Y(\xi)), ad^*_\eta(X))\rangle$$
and et cetera for the other terms. Thus we finally get
$$\begin{aligned} \langle [\xi, \eta]_{\mathfrak{g}^*}, [X, Y]\rangle + \langle ad^*_X(\xi)), ad^*_\eta(Y))\rangle - \langle ad^*_X(\eta)), ad^*_\xi(Y))\rangle - \\ -\langle ad^*_Y(\xi)), ad^*_\eta(X))\rangle + \langle ad^*_Y(\eta)), ad^*_\xi(X))\rangle = 0. \end{aligned} \tag{2.18}$$
The symmetry of a Lie bialgebra with respect to interchanging $\mathfrak{g}$ and $\mathfrak{g}^*$ is then fairly obvious. This leads us to the intended reflexivity of both Lie algebras. A true meaning of this magic world should be fully revealed in the following proposition.

Proposition 2.3.6. *If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, μ is a Lie multiplication in $\mathfrak{g}$, then $(\mathfrak{g}^*, \mu^*)$ is also a Lie bialgebra, called the **dual of a Lie bialgebra**.*

Proof. On the left-hand side of (2.18) we have
$$\langle [\xi, \eta]_{\mathfrak{g}^*}, [X, Y]\rangle = \langle \mu^*([\xi, \eta]_{\mathfrak{g}^*}), X \otimes Y\rangle.$$
On the right-hand side we just repeat backwards all the computation above, only interchanging $X, Y \leftrightarrow \xi, \eta$, to get
$$\langle ad^{(2)}_\xi(\mu^*(\eta)), X \otimes Y\rangle - \langle ad^{(2)}_\eta(\mu^*(\xi)), X \otimes Y\rangle,$$
for all $X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ and therefore
$$\mu^*([\xi, \eta]_{\mathfrak{g}^*}) = ad^{(2)}_\xi(\mu^*(\eta)) - ad^{(2)}_\eta(\mu^*(\xi)),$$
for all $\xi, \eta \in \mathfrak{g}^*$. This is exactly a 1-cocycle condition (2.9) for a Lie bialgebra $(\mathfrak{g}^*, \mu^*)$. ∎

2.4 The double of a Lie bialgebra, Manin triple

In this section we introduce a mathematical object called Manin triple and show that it is in fact equivalent to the Lie bialgebra structure. Manin triple is just a Lie algebra with some additional structures, and it is very useful for the construction of low-dimensional Lie bialgebras.

Definition 2.4.1. Let V be a vector space over the field $\mathbb{T}$. A **natural inner product** on $V \oplus V^*$ is a symmetric bilinear map $(\cdot|\cdot) : (V \oplus V^*) \times (V \oplus V^*) \to \mathbb{T}$ defined as
$$(v|w) := 0, \ (v|\alpha) := \langle \alpha, V\rangle, \ (\alpha|\beta) := 0, \tag{2.19}$$
for $v, w \in V$, $\alpha, \beta \in V^*$.

Proposition 2.4.2. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. There exists a unique Lie algebra structure on $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are its subalgebras and the natural inner product on $\mathfrak{d}$ is ad-invariant in the sense*
$$([A, B]_{\mathfrak{d}}|C) + (B, |[A, C]_{\mathfrak{d}}) = 0, \tag{2.20}$$
for all $A, B, C \in \mathfrak{d}$.

Proof. A Lie bracket on $\mathfrak{d}$ will be denoted as $[\cdot,\cdot]_{\mathfrak{d}}$. We set

$$[X,Y]_{\mathfrak{d}} := [X,Y]_{\mathfrak{g}},\ [\xi,\eta]_{\mathfrak{d}} := [\xi,\eta]_{\mathfrak{g}^*} = \delta^*(\xi\otimes\eta), \tag{2.21}$$

for $X,Y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$. We get the mixed commutation relations uniquely, using the (intended) ad-invariance property of the inner product (2.20). For $X,Y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$ we have

$$(Y|[X,\xi]_{\mathfrak{d}}) = ([Y,X]_{\mathfrak{d}}|\xi) = \langle\xi,[Y,X]\rangle = \langle ad^*_X(\xi),Y\rangle = (Y|ad^*_X(\xi)).$$

In the same way we get

$$(\eta|[X,\xi]_{\mathfrak{d}}) = -(\eta|ad^*_\xi(X)).$$

Therefore we have to set

$$[X,\xi]_{\mathfrak{d}} = -ad^*_\xi(X) + ad^*_X(\xi). \tag{2.22}$$

To finish the proof we have to show that $[\cdot,\cdot]_{\mathfrak{d}}$ satisfies the Jacobi identities. We choose a basis $(T_1,\dots,T_n)$ in $\mathfrak{g}$ and write

$$[T_i,T_j] = c_{ij}{}^kT_k,\ [T^i,T^j]_{\mathfrak{g}^*} = f^{ij}{}_kT^k.$$

Because $\mathfrak{g}$ and $\mathfrak{g}^*$ are by definition subalgebras of $\mathfrak{d}$, we have to prove just

$$[T_i,[T^j,T^k]_{\mathfrak{d}}]_{\mathfrak{d}} + cyclic\{i,j,k\} = 0, \tag{2.23}$$

$$[T^k,[T_i,T_j]_{\mathfrak{d}}]_{\mathfrak{d}} + cyclic\{i,j,k\} = 0. \tag{2.24}$$

The coordinate version of (2.22) reads

$$[T_i,T^k]_{\mathfrak{d}} = f^{kl}{}_iT_l - c_{il}{}^kT^l.$$

Direct computation of (2.23) gives

$$[T_i,[T^j,T^k]_{\mathfrak{d}}]_{\mathfrak{d}} + cyclic\{i,j,k\} = -(f^{km}{}_lf^{jl}{}_i + f^{jk}{}_lf^{ml}{}_i + f^{mj}{}_lf^{kl}{}_i)T_m +$$
$$+(c_{il}{}^jf^{lk}{}_m + c_{il}{}^kf^{jl}{}_m - c_{ml}{}^jf^{lk}{}_i - c_{ml}{}^kf^{jl}{}_i - c_{im}{}^lf^{jk}{}_l)T^m.$$

The term proportional to T_m vanishes due to Jacobi identities in $\mathfrak{g}^*$ and the term proportional to T^m equals zero because of (2.12). The result of the computation of (2.24) is very similar and the right hand side vanishes because of the Jacobi identities in $\mathfrak{g}$ and (2.12).

Hence $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}^*$ has a unique Lie algebra structure $[\cdot,\cdot]_{\mathfrak{d}}$. ■

Definition 2.4.3. Let $(\mathfrak{g},\delta)$ be a Lie bialgebra. A Lie algebra on $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}^*$ from the preceding proposition, defined by relations (2.21) and (2.22), is called the **double of a Lie bialgebra** $\mathfrak{g}$ and denoted as $\mathfrak{g}\bowtie\mathfrak{g}^*$.

Remark 2.4.4. See that $\mathfrak{g}\bowtie\mathfrak{g}^* \equiv \mathfrak{g}^*\bowtie\mathfrak{g}$.

Definition 2.4.5. A **Manin triple** $(\mathfrak{p},\mathfrak{p}^+,\mathfrak{p}^-)$ is a triple of Lie algebras $\mathfrak{p}$, $\mathfrak{p}^+$ and $\mathfrak{p}^-$, such that $\mathfrak{p} = \mathfrak{p}^+\oplus\mathfrak{p}^-$ as vector spaces, $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are Lie subalgebras of $\mathfrak{p}$ and as vector subspaces they are isotropic with respect to a non-degenerate, symmetric, ad-invariant bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ on $\mathfrak{p}$.

Proposition 2.4.6. *Finite-dimensional Manin triples are in one-to-one correspondence with finite-dimensional Lie bialgebras.*

Proof. Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra. If we set $\mathfrak{p}^+ = \mathfrak{g}$, $\mathfrak{p}^- = \mathfrak{g}^*$, $\mathfrak{p} = \mathfrak{g} \bowtie \mathfrak{g}^*$ and

$$\langle V, W \rangle_{\mathfrak{p}} := (V|W),$$

where $V, W \in \mathfrak{g} \bowtie \mathfrak{g}^*$ and $(\cdot|\cdot)$ is the natural inner product, we get Manin triple by definition of the double of a Lie bialgebra.

Conversely, if $(\mathfrak{p}, \mathfrak{p}^+, \mathfrak{p}^-)$ is a Manin triple with the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$, we set $\mathfrak{g} = \mathfrak{p}^+$. A non-degeneracy of $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ implies $\dim(\mathfrak{p}^+) = \dim(\mathfrak{p}^-)$. It is not difficult to see that for the arbitrary fixed basis $(T_i)_{i=1}^n$ of $\mathfrak{g} \equiv \mathfrak{p}^+$ there exists a unique basis $(\widetilde{T}^j)_{j=1}^n$ in $\mathfrak{p}^-$, such that

$$\langle T_i, \widetilde{T}^j \rangle_{\mathfrak{p}} = \delta_i^j.$$

Since $\dim(\mathfrak{g}^*) = \dim(\mathfrak{p}^-)$, there exists a unique linear bijection $\mathbf{P} : \mathfrak{p}^- \to \mathfrak{g}^*$ such that $\mathbf{P}(\widetilde{T}^j) = T^j$, where $(T^j)_{j=1}^n$ is the basis of $\mathfrak{g}^*$ dual to $(T_j)_{j=1}^n$. We define a Lie bracket $[\cdot, \cdot]_{\mathfrak{g}^*}$ in $\mathfrak{g}^*$ using a Lie bracket $[\cdot, \cdot]$ in $\mathfrak{p}$:

$$[T^i, T^j]_{\mathfrak{g}^*} = \mathbf{P}([\widetilde{T}^i, \widetilde{T}^j]).$$

If $[\widetilde{T}^i, \widetilde{T}^j] = f^{ij}{}_k \widetilde{T}^k$, then by definition $[T^i, T^j]_{\mathfrak{g}^*} = f^{ij}{}_k T^k$. If we choose $(T_i, \widetilde{T}^j)_{i=1,j=1}^{n,n}$ as the basis in $\mathfrak{p}$ and denote $[T_i, T_j] = c_{ij}{}^k T_k$, we can compute from the ad-invariance of $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ that

$$[T_i, \widetilde{T}^j] = f^{jk}{}_i T_k - c_{ik}{}^j \widetilde{T}^k. \tag{2.25}$$

We set $\delta^* = [\cdot, \cdot]_{\mathfrak{g}^*}$. Its transpose map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ satisfies by definition the first property of a Lie bialgebra. If we write the condition (2.9) in the coordinates with respect to $(T_i)_{i=1}^n$, we get directly the expression (2.12). If we now express the mixed Jacobi identities of $\mathfrak{p}$ in the coordinates with respect to the basis $(T_i, \widetilde{T}^j)_{i=1,j=1}^{n,n}$,

$$[T_i, [\widetilde{T}^j, \widetilde{T}^k]] + cyclic\{i, j, k\} = 0, \tag{2.26}$$

$$[\widetilde{T}^k, [T_i, T_j]] + cyclic\{i, j, k\} = 0, \tag{2.27}$$

we will find out, just like in the proof of 2.4.2, that they imply the satisfying of the condition (2.12). Thus $(\mathfrak{g}, \delta)$ is a Lie bialgebra. ■

2.5 Coboundary Lie bialgebra, r-matrix

It is natural to ask, if for given Lie algebra structure $\mathfrak{g}$ there exists a 0-cocycle r of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, i.e. $r \in \mathfrak{g} \otimes \mathfrak{g}$, such that $\delta = \Delta(r)$ defines a Lie bialgebra structure on $\mathfrak{g}$. r will be called an r-matrix. The answer is positive. For many of the 1-cocycles one can find their "potential" in $\mathfrak{g} \otimes \mathfrak{g}$.

An interesting part of the question is whether the condition on $\Delta(r)$ to be a Lie algebra cocommutator can be somehow transformed to the condition on $r \in \mathfrak{g} \otimes \mathfrak{g}$ itself. It will turn out that there indeed exist simple algebraic equations for r, called the Yang-Baxter equations (in different variants).

Definition 2.5.1. Lie bialgebra $(\mathfrak{g}, \delta)$ is called a **coboundary Lie bialgebra** if there exists $r \in \mathfrak{g} \otimes \mathfrak{g}$, such that $\delta = \Delta(r)$.

What are the necessary and sufficient conditions for r, if we want $\Delta(r)$ to be a cocommutator on $\mathfrak{g}$? By definition $\Delta(r)$ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. Thus there remains the question, for which r the $\Delta(r)^*$ is a Lie bracket on $\mathfrak{g}^*$.

We denote (for general r it is not necessarily a Lie bracket on $\mathfrak{g}^*$)

$$[\xi, \eta]^r := \Delta(r)^*(\xi, \eta), \tag{2.28}$$

for $\xi, \eta \in \mathfrak{g}^*$. The bracket $[\cdot, \cdot]^r$ is skew-symmetric if and only if $\Delta(r)(X) \in \bigwedge^2 \mathfrak{g}$. Indeed

$$\langle [\xi, \eta]^r, X \rangle = \langle \xi \otimes \eta, \Delta(r)(X) \rangle = -\langle \eta \otimes \xi, \Delta(r)(X) \rangle = -\langle [\eta, \xi]^r, X \rangle,$$

for $\xi, \eta \in \mathfrak{g}^*$, $X \in \mathfrak{g}$.

Let us denote $s \in S^2\mathfrak{g}$ the symmetric part of r and $a \in \bigwedge^2 \mathfrak{g}$ the skew-symmetric part of r. Of course, by definition, $r = s + a$.

The (skew-)symmetry of the elements of $\mathfrak{g} \otimes \mathfrak{g}$ is preserved by the coboundary operator. The converse is obviously not true.

Lemma 2.5.2. *For arbitrary $r \in \mathfrak{g} \otimes \mathfrak{g}$, $r = s + a$, $X \in \mathfrak{g}$ there holds*

$$\Delta(a)(X) \in \bigwedge^2 \mathfrak{g}, \tag{2.29}$$

$$\Delta(s)(X) \in S^2\mathfrak{g}. \tag{2.30}$$

Proof. We will show only the proof of (2.29), the second one is completely the same. For $\xi, \eta \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$ we have using the skew-symmetry of a

$$\langle \Delta(a)(X), \xi \otimes \eta \rangle = \langle ad_X^{(2)}(a), \xi \otimes \eta \rangle = -\langle a, ad_X^*(\xi) \otimes \eta + \xi \otimes ad_X^*(\eta) \rangle =$$

$$= \langle a, \eta \otimes ad_X^*(\xi) + ad_X^*(\eta) \otimes \xi \rangle = -\langle ad_X^{(2)}(a), \eta \otimes \xi \rangle = -\langle \Delta(a)(X), \eta \otimes \xi \rangle.$$

■

To every element r of $\mathfrak{g} \otimes \mathfrak{g}$ we can associate the linear map $\underline{r} : \mathfrak{g}^* \to \mathfrak{g}$, defined as

$$\langle \eta, \underline{r}(\xi) \rangle \equiv \underline{r}(\xi)(\eta) := r(\xi, \eta). \tag{2.31}$$

It is easy to show that

$$\underline{s} = \frac{1}{2}(\underline{r} + \underline{r}^*), \ \underline{a} = \frac{1}{2}(\underline{r} - \underline{r}^*). \tag{2.32}$$

Using these facts we can find the sufficient and necessary condition for r to define a skew-symmetric bracket $[\cdot, \cdot]^r$. In the components with respect to some chosen basis this is a system of *linear algebraic equations* for the components of the symmetric part s.

Proposition 2.5.3. *Let $r \in \mathfrak{g} \otimes \mathfrak{g}$, $[\cdot, \cdot]^r$ defined by (2.28). $r = s + a$, where $s \in S^2\mathfrak{g}$, $a \in \bigwedge^2 \mathfrak{g}$.*

Then $[\cdot, \cdot]^r$ is skew-symmetric if and only if s is ad-invariant, i.e. $\Delta(s) = 0$.

Proof. For arbitrary $X \in \mathfrak{g}$ we use the linearity of Δ acting on 0-cycles to get

$$\Delta(r)(X) = \Delta(s + a)(X) = \Delta(s)(X) + \Delta(a)(X).$$

From the preceding lemma we know that $\Delta(s)(X) \in S^2\mathfrak{g}$. From the discussion under (2.28) we get

$$[\cdot, \cdot]^r \text{ is skew-symmetric} \Longleftrightarrow \Delta(r)(X) \in \bigwedge^2 \mathfrak{g} \Longleftrightarrow \Delta(s)(X) = 0.$$

■

Remark 2.5.4. The ad-invariance condition $\Delta(s) = 0$ can be rewritten as

$$ad_X \circ \underline{s} = \underline{s} \circ ad^*_X, \tag{2.33}$$

for all $X \in \mathfrak{g}$. Indeed, for $X \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ we have

$$0 = \langle \Delta(s)(X), \xi \otimes \eta \rangle = \langle ad^{(2)}_X(s), \xi \otimes \eta \rangle = -\langle s, ad^*_X(\xi) \otimes \eta \rangle - \langle s, \xi \otimes ad^*_X(\eta) \rangle =$$
$$= -\langle \eta, \underline{s}(ad^*_X(\xi)) \rangle - \langle ad^*_X(\eta), \underline{s}(\xi) \rangle = \langle \eta, -\underline{s}(ad^*_X(\xi)) + ad_X(\underline{s}(\xi)) \rangle.$$

Lemma 2.5.5. *For arbitrary $\xi \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$ there holds*

$$\langle ad^*_X(\xi), Y \rangle = -\langle ad^*_Y(\xi), X \rangle. \tag{2.34}$$

Proof.

$$\langle ad^*_X(\xi), Y \rangle = -\langle \xi, ad_X(Y) \rangle = \langle \xi, ad_Y(X) \rangle = -\langle ad^*_Y(\xi), X \rangle.$$

■

The bracket $[\cdot,\cdot]^r$ on $\mathfrak{g}^*$ can be for skew-symmetric $r \in \mathfrak{g} \otimes \mathfrak{g}$ rewritten in the words of coadjoint representation of $\mathfrak{g}$.

Proposition 2.5.6. *Let $r \in \bigwedge^2 \mathfrak{g}$. Then for $\xi, \eta \in \mathfrak{g}^*$ we have*

$$[\xi, \eta]^r = ad^*_{\underline{r}(\xi)}(\eta) - ad^*_{\underline{r}(\eta)}(\xi). \tag{2.35}$$

Proof. In the proof we use the skew-symmetry of r, definition of $\underline{r}$ and the preceding lemma. We find that

$$\langle [\xi, \eta]^r, X \rangle = \langle \Delta(r)^*(\xi, \eta), X \rangle = \langle ad^{(2)}_X(r), \xi \otimes \eta \rangle = -\langle r, ad^*_X(\xi) \otimes \eta \rangle - \langle r, \xi \otimes ad^*_X(\eta) \rangle =$$
$$= \langle r, \eta \otimes ad^*_X(\xi) \rangle - \langle r, \xi \otimes ad^*_X(\eta) \rangle = \langle ad^*_X(\xi), \underline{r}(\eta) \rangle - \langle ad^*_X(\eta), \underline{r}(\xi) \rangle \overset{(2.34)}{=}$$
$$\overset{(2.34)}{=} -\langle ad^*_{\underline{r}(\eta)}(\xi), X \rangle + \langle ad^*_{\underline{r}(\xi)}(\eta), X \rangle,$$

for arbitrary $X \in \mathfrak{g}$ and therefore we have finished the proof. ■

Definition 2.5.7. Let $r \in \bigwedge^2 \mathfrak{g}$. Then the **algebraic Schouten bracket** $[\![r, r]\!]$ is an element of $\bigwedge^3 \mathfrak{g}$ defined as

$$[\![r, r]\!](\xi, \eta, \zeta) := 2\langle \zeta, [\underline{r}(\xi), \underline{r}(\eta)] \rangle + cyclic\{\xi, \eta, \zeta\}, \tag{2.36}$$

for $\xi, \eta, \zeta \in \mathfrak{g}^*$.

Remark 2.5.8. For given $r \in \bigwedge^2 \mathfrak{g}$ we can define the bivector field L_r on the corresponding Lie group G to be the left translation of r, that is

$$L_r(g) = L_{g*}(r),$$

for $g \in G$. Then one can find that for all $g \in G$

$$[L_r, L_r](g) = L_{g*}([\![r, r]\!]),$$

where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket on the Lie group G, see section 3.2. This remark also shows that $[\![r, r]\!]$ is indeed the element of $\bigwedge^3 \mathfrak{g}$.

$[\![r,r]\!]$ can be viewed as 0-cochain of $\mathfrak{g}$ with values in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$. Therefore we define

$$\Delta([\![r,r]\!])(X) := ad_X^{(3)}[\![r,r]\!]. \tag{2.37}$$

The most important lemma of this section shows that the ad-invariance of the algebraic Schouten bracket is equivalent to the satisfaction of the Jacobi identities for $[\cdot,\cdot]^r$. This condition on the algebraic Schouten bracket is a *system of the homogeneous quadratic algebraic equations* for the components of $r\in\mathfrak{g}\otimes\mathfrak{g}$ in a chosen basis.

Lemma 2.5.9. *Let $r\in\bigwedge^2\mathfrak{g}$. Then for arbitrary $\xi,\eta,\zeta\in\mathfrak{g}^*$ and $X\in\mathfrak{g}$ there holds*

$$\langle[[\xi,\eta]^r,\zeta]^r,X\rangle + cyclic\{\xi,\eta,\zeta\} = -\frac{1}{2}\Delta([\![r,r]\!])(X)(\xi,\eta,\zeta) \tag{2.38}$$

Proof. Let us start from the left hand side.

$$\langle[[\xi,\eta]^r,\zeta]^r,X\rangle \overset{(2.35)}{=} \langle[ad^*_{\underline{r}(\xi)}(\eta)-ad^*_{\underline{r}(\eta)}(\xi),\zeta]^r,X\rangle \overset{(2.35)}{=}$$

$$\overset{(2.35)}{=} \langle ad^*_{\underline{r}(ad^*_{\underline{r}(\xi)}(\eta)-ad^*_{\underline{r}(\eta)}(\xi))}(\zeta),X\rangle - \langle ad^*_{\underline{r}(\zeta)}(ad^*_{\underline{r}(\xi)}(\eta)-ad^*_{\underline{r}(\eta)}(\xi)),X\rangle \overset{(2.34)}{=}$$

$$\overset{(2.34)}{=} -\langle ad^*_X(\zeta),\underline{r}(ad^*_{\underline{r}(\xi)}(\eta)-ad^*_{\underline{r}(\eta)}(\xi))\rangle + \langle ad^*_X(ad^*_{\underline{r}(\xi)}(\eta)-ad^*_{\underline{r}(\eta)}(\xi)),\underline{r}(\zeta)\rangle = \otimes.$$

We will use the representation property of ad^* for

$$ad^*_X(ad^*_{\underline{r}(\xi)}(\eta)) = ad^*_{[X,\underline{r}(\xi)]}(\eta) + ad^*_{\underline{r}(\xi)}(ad^*_X(\eta))$$

and the skew-symmetry of r in the form $\langle\underline{r}(\alpha),\beta\rangle = -\langle\alpha,\underline{r}(\beta)\rangle$ to get

$$\otimes = \langle\underline{r}(ad^*_X(\zeta)),ad^*_{\underline{r}(\xi)}(\eta)\rangle - \langle\underline{r}(ad^*_X(\zeta)),ad^*_{\underline{r}(\eta)}(\xi)\rangle + \langle ad^*_{\underline{r}(\xi)}(ad^*_X(\eta)),\underline{r}(\zeta)\rangle +$$

$$+\langle ad_{[X,\underline{r}(\xi)]}(\eta),\underline{r}(\zeta)\rangle - \langle ad^*_X(ad^*_{\underline{r}(\eta)}(\xi)),\underline{r}(\zeta)\rangle = \otimes.$$

Now we will use

$$\langle ad^*_{[X,\underline{r}(\xi)]}(\eta),\underline{r}(\zeta)\rangle \overset{(2.34)}{=} -\langle ad^*_{\underline{r}(\zeta)}(\eta),[X,\underline{r}(\xi)]\rangle = \langle ad^*_X(ad^*_{\underline{r}(\zeta)}(\eta)),\underline{r}(\xi)\rangle,$$

$$\langle\underline{r}(ad^*_X(\zeta)),ad^*_{\underline{r}(\xi)}(\eta)\rangle = \langle\eta,[\underline{r}(ad^*_X(\zeta)),\underline{r}(\xi)]\rangle,$$

$$-\langle\underline{r}(ad^*_X(\zeta),ad^*_{\underline{r}(\eta)}(\xi)\rangle = \langle\xi,[\underline{r}(\eta),\underline{r}(ad^*_X(\zeta)]\rangle,$$

$$\langle ad^*_{\underline{r}(\xi)}(ad^*_X(\eta)),\underline{r}(\zeta)\rangle = \langle ad^*_X(\eta),[\underline{r}(\zeta),\underline{r}(\xi)]\rangle,$$

to obtain

$$\otimes = \langle\eta,[\underline{r}(ad^*_X(\zeta)),\underline{r}(\xi)]\rangle + \langle\xi,[\underline{r}(\eta),\underline{r}(ad^*_X(\zeta)]\rangle + \langle ad^*_X(\eta),[\underline{r}(\zeta),\underline{r}(\xi)]\rangle +$$

$$+\langle ad^*_X(ad^*_{\underline{r}(\zeta)}(\eta)),\underline{r}(\xi)\rangle - \langle ad^*_X(ad^*_{\underline{r}(\eta)}(\xi)),\underline{r}(\zeta)\rangle.$$

Last two terms vanish in the cyclic summation and therefore we have

$$\langle[\xi,\eta]^r,\zeta]^r,X\rangle + cyclic\{\xi,\eta,\zeta\} =$$

$$= \langle\eta,[\underline{r}(ad^*_X(\zeta)),\underline{r}(\xi)]\rangle + \langle\xi,[\underline{r}(\eta),\underline{r}(ad^*_X(\zeta)]\rangle + \langle ad^*_X(\eta),[\underline{r}(\zeta),\underline{r}(\xi)]\rangle + cyclic\{\xi,\eta,\zeta\}. \tag{2.39}$$

If we start from the right hand side of (2.38) we can write

$$-\frac{1}{2}\Delta([\![r,r]\!])(X)(\xi,\eta,\zeta) = -\frac{1}{2}\langle ad_X^{(3)}([\![r,r]\!]),\xi\otimes\eta\otimes\zeta\rangle =$$

$$= \frac{1}{2}\langle [\![r,r]\!], \eta\otimes\zeta\otimes ad_X^*(\xi)\rangle + \frac{1}{2}\langle [\![r,r]\!], \zeta\otimes\xi\otimes ad_X^*(\eta)\rangle + \frac{1}{2}\langle [\![r,r]\!], \xi\otimes\eta\otimes ad_X^*(\zeta)\rangle =$$

$$= \frac{1}{2}\langle [\![r,r]\!], \xi\otimes\eta\otimes ad_X^*(\zeta)\rangle + cyclic\{\xi,\eta,\zeta\} =$$

$$= \langle ad_X^*(\zeta), [\underline{r}(\xi), \underline{r}(\eta)]\rangle + \langle \eta, [\underline{r}(ad_X^*(\zeta)), \underline{r}(\xi)]\rangle + \langle \xi, [\underline{r}(\eta), \underline{r}(ad_X^*(\zeta))]\rangle + cyclic\{\xi,\eta,\zeta\}.$$

Comparing this to the right hand side of (2.39) we find out that we have just proved the lemma. ■

Proposition 2.5.10. *Let $r \in \bigwedge^2 \mathfrak{g}$. Then $[\cdot,\cdot]^r$ is a Lie bracket on $\mathfrak{g}^*$, if and only if $[\![r,r]\!]$ is ad-invariant:*

$$\Delta([\![r,r]\!]) = 0. \tag{2.40}$$

Lie bracket $[\cdot,\cdot]_{\mathfrak{g}^*} \equiv [\cdot,\cdot]^r$ is then given by (2.35).

Proof. This proposition is an immediate result of proposition 2.5.3, because $s = 0$, and the preceding lemma. ■

All these lemmas and propositions can be summarized in the following theorem.

Theorem 2.5.11. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Let $r \in \mathfrak{g}\otimes\mathfrak{g}$. $r = s + a$, where s is the symmetric part of r and a the skew-symmetric part of r. Then $\Delta(r)$ defines the coboundary Lie bialgebra structure on $\mathfrak{g}$, if and only if*

$$\Delta(s) = 0, \tag{2.41}$$

$$\Delta([\![a,a]\!]) = 0. \tag{2.42}$$

A Lie bracket in $\mathfrak{g}^*$ is then given as

$$[\xi,\eta]_{\mathfrak{g}^*} = ad^*_{\underline{a}(\xi)}(\eta) - ad^*_{\underline{a}(\eta)}(\xi), \tag{2.43}$$

$\xi,\eta \in \mathfrak{g}^*$. $r \in \mathfrak{g}\otimes\mathfrak{g}$ satisfying (2.41) and (2.42) is called the ***classical r-matrix*** *or simply the r-matrix. The condition $\Delta([\![a,a]\!]) = 0$ is called a* ***generalized Yang-Baxter equation****.*

Proof. Only the antisymmetric part of r plays role in the $[\cdot,\cdot]^r$, because

$$\langle [\xi,\eta]^r, X\rangle = \langle \Delta(r)(X), \xi\otimes\eta\rangle = \langle ad_X^{(2)}(r), \xi\otimes\eta\rangle \overset{(2.41)}{=} \langle ad_X^{(2)}(a), \xi\otimes\eta\rangle = \cdots = \langle [\xi,\eta]^a, X\rangle.$$

Because $a \in \bigwedge^2 \mathfrak{g}$ and $\Delta([\![a,a]\!]) = 0$, we know from proposition 2.5.10 that $[\cdot,\cdot]^r \equiv [\cdot,\cdot]^a$ is the Lie bracket on $\mathfrak{g}^*$. By definition $\Delta(r)$ is a 1-cocycle on $\mathfrak{g}$ with the values in $\mathfrak{g}\otimes\mathfrak{g}$ and $(\mathfrak{g}, \Delta(r))$ is then a coboundary Lie bialgebra. ■

2.6 The classical Yang-Baxter equation

For $r \in \mathfrak{g}\otimes\mathfrak{g}$ we can define a bilinear map $\underline{\langle r,r\rangle} : \mathfrak{g}^*\otimes\mathfrak{g}^* \to \mathfrak{g}$ by setting

$$\underline{\langle r,r\rangle}(\xi,\eta) := [\underline{r}(\xi), \underline{r}(\eta)] - \underline{r}([\xi,\eta]^r), \tag{2.44}$$

for $\xi,\eta \in \mathfrak{g}^*$. To this linear map we can associate $\langle r,r\rangle \in \otimes^3\mathfrak{g}$, defined as

$$\langle r,r\rangle(\xi,\eta,\zeta) := \langle \zeta, \underline{\langle r,r\rangle}(\xi,\eta)\rangle. \tag{2.45}$$

Proposition 2.6.1. *(i) Let $a \in \bigwedge^2 \mathfrak{g}$. Then $\langle a, a\rangle \in \bigwedge^3 \mathfrak{g}$ and*

$$\langle a, a\rangle = \frac{1}{2}[\![a, a]\!]. \tag{2.46}$$

(ii) Let $s \in S^2\mathfrak{g}$ and $\Delta(s) = 0$. Then $\langle s, s\rangle \in \bigwedge^3 \mathfrak{g}$ and

$$\underline{\langle s, s\rangle}(\xi, \eta) = [\underline{s}(\xi), \underline{s}(\eta)], \tag{2.47}$$

$\xi, \eta \in \mathfrak{g}^*$. Moreover $\langle s, s\rangle$ is also ad-invariant, that is $\Delta(\langle s, s\rangle) = 0$.

(iii) Let $r \in \mathfrak{g} \otimes \mathfrak{g}$, $r = s + a$, where $s \in S^2\mathfrak{g}$, $\Delta(s) = 0$ and $a \in \bigwedge^2 \mathfrak{g}$. Then $\langle r, r\rangle \in \bigwedge^3 \mathfrak{g}$ and

$$\langle r, r\rangle = \langle s, s\rangle + \langle a, a\rangle. \tag{2.48}$$

Proof. (i) For $\xi, \eta, \zeta \in \mathfrak{g}^*$ we have

$$\langle a, a\rangle(\xi, \eta, \zeta) = \langle \zeta, [\underline{a}(\xi), \underline{a}(\eta)]\rangle - \langle \zeta, \underline{a}([\xi, \eta]^a)\rangle \overset{(2.35)}{=}$$

$$\overset{(2.35)}{=} \langle \zeta, [\underline{a}(\xi), \underline{a}(\eta)]\rangle - \langle \zeta, \underline{a}(ad^*_{\underline{a}(\xi)}(\eta))\rangle + \langle \zeta, \underline{a}(ad^*_{\underline{a}(\eta)}(\xi))\rangle =$$
$$= \langle \zeta, [\underline{a}(\xi), \underline{a}(\eta)]\rangle + \langle \underline{a}(\zeta), ad^*_{\underline{a}(\xi)}(\eta)\rangle - \langle \underline{a}(\zeta), ad^*_{\underline{a}(\eta)}(\xi)\rangle =$$
$$= \langle \zeta, [\underline{a}(\xi), \underline{a}(\eta)]\rangle + \langle \eta, [\underline{a}(\zeta), \underline{a}(\xi)]\rangle + \langle \xi, [\underline{a}(\eta), \underline{a}(\zeta)]\rangle = \frac{1}{2}[\![a, a]\!](\xi, \eta, \zeta).$$

This finishes the proof of the part (i), because for $a \in \bigwedge^2 \mathfrak{g}$ we know that $[\![a, a]\!] \in \bigwedge^3 \mathfrak{g}$.

(ii) Since $[\cdot, \cdot]^s = 0$, we have $\underline{\langle s, s\rangle}(\xi, \eta) = [\underline{s}(\xi), \underline{s}(\eta)]$, for $\xi, \eta \in \mathfrak{g}^*$. The skew-symmetry of $\langle s, s\rangle$ in first two variables is then obvious. To prove the skew-symmetry in the last two variables we write using the symmetry of s

$$\langle s, s\rangle(\xi, \eta, \zeta) = \langle \zeta, ad_{\underline{s}(\xi)}(\underline{s}(\eta))\rangle \overset{(2.33)}{=} \langle \zeta, \underline{s}(ad^*_{\underline{s}(\xi)}(\eta))\rangle = \langle \underline{s}(\zeta), ad^*_{\underline{s}(\xi)}(\eta)\rangle =$$
$$= -\langle \eta, [\underline{s}(\xi), \underline{s}(\zeta)]\rangle = -\langle s, s\rangle(\xi, \zeta, \eta).$$

We have just proved that $\langle s, s\rangle \in \bigwedge^3 \mathfrak{g}$. One can easily find that the condition $\Delta(\langle s, s\rangle) = 0$ is equivalent to the validity of the equation

$$\underline{\langle s, s\rangle}(ad^*_X(\xi), \eta) + \underline{\langle s, s\rangle}(\xi, ad^*_X(\eta)) = ad_X(\underline{\langle s, s\rangle}(\xi, \eta),$$

for all $\xi, \eta \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. When we start from the left hand side, we have

$$\underline{\langle s, s\rangle}(ad^*_X(\xi), \eta) + \underline{\langle s, s\rangle}(\xi, ad^*_X(\eta)) = [\underline{s}(ad^*_X(\xi)), \underline{s}(\eta)] + [\underline{s}(\xi), \underline{s}(ad^*_X(\eta))] \overset{(2.33)}{=}$$

$$\overset{(2.33)}{=} [ad_X(\underline{s}(\xi)), \underline{s}(\eta)] + [\underline{s}(\xi), ad_X(\underline{s}(\eta))] \overset{Jacobi}{=} ad_X([\underline{s}(\xi), \underline{s}(\eta)]).$$

(iii) To prove the last part we should recall that $[\cdot, \cdot]^r = [\cdot, \cdot]^a$. For $r = a + s$ we write

$$\underline{\langle r, r\rangle}(\xi, \eta) = [\underline{a}(\xi), \underline{a}(\eta)] - \underline{a}([\xi, \eta]^a) + [\underline{a}(\xi), \underline{s}(\eta)] + [\underline{s}(\xi), \underline{a}(\eta)] - \underline{s}([\xi, \eta]^a) + [\underline{s}(\xi), \underline{s}(\eta)].$$

If we now use the ad-invariance of s and (2.35) we get

$$\underline{s}([\xi, \eta]^a) = \underline{s}(ad^*_{\underline{a}(\xi)}(\eta) - ad^*_{\underline{a}(\eta)}(\xi)) = ad_{\underline{a}(\xi)}(\underline{s}(\eta)) - ad_{\underline{a}(\eta)}(\underline{s}(\xi)).$$

Therefore we finally obtain $\langle r, r\rangle = \langle s, s\rangle + \langle a, a\rangle$.

∎

Theorem 2.6.2. *Let* $r \in \mathfrak{g} \otimes \mathfrak{g}$, $r = s + a$, *where* $s \in S^2\mathfrak{g}$, $\Delta(s) = 0$ *and* $a \in \bigwedge^2 \mathfrak{g}$. *Then* r *is the classical* r*-matrix, if and only if*

$$\Delta(\langle r, r \rangle) = 0. \tag{2.49}$$

Proof. The theorem is an immediate consequence of the preceding proposition and of the theorem 2.5.11, because

$$\Delta(\langle r, r \rangle) = \Delta(\langle s, s \rangle + \langle a, a \rangle) = \Delta(\langle s, s \rangle) + \Delta(\langle a, a \rangle) = 0 + \frac{1}{2}\Delta([\![a, a]\!]).$$

■

Definition 2.6.3. The equation (2.49) is obviously satisfied, if for $r \in \mathfrak{g} \otimes \mathfrak{g}$

$$\langle r, r \rangle = 0. \tag{2.50}$$

This condition is called the **classical Yang-Baxter equation**, abbreviated as CYBE. A general solution of CYBE is called a **quasi-triangular** r-matrix.

If the symmetric part s of quasi-triangular r-matrix becomes a non-degenerate symmetric bilinear form on $\mathfrak{g}^*$, we call such r-matrix **factorizable**.

A skew-symmetric solution of CYBE is called a **triangular** r-matrix. CYBE for the skew-symmetric r by proposition 2.6.1 reads

$$[\![r, r]\!] = 0. \tag{2.51}$$

Example 2.6.4. Let us give a simple example of the CYBE and GYBE solution. Let $\mathfrak{g}$ be a Bianchi $VIII$ algebra ($\cong sl(2, \mathbb{R})$), that is $\mathfrak{g} = span\{T_1, T_2, T_3\}$ and

$$[T_1, T_2] = -T_3,\ [T_2, T_3] = T_1,\ [T_3, T_1] = T_2.$$

We define a Lie bracket on $\mathfrak{g}^*$ using the dual basis (T^1, T^2, T^3) as

$$[T^1, T^2] = -T^2,\ [T^3, T^1] = -T^3,$$

zero otherwise. The cocommutator of this Lie bialgebra is therefore

$$\delta(T_1) = 0,$$

$$\delta(T_2) = T_2 \otimes T_1 - T_1 \otimes T_2 \equiv -T_1 \wedge T_2,$$

$$\delta(T_3) = T_1 \otimes T_3 - T_3 \otimes T_1 \equiv T_1 \wedge T_3.$$

After checking the 1-cocycle conditions one finds that it is indeed a Lie bialgebra isomorphic to the Manin triple (**8**|**5.i**|**1**) (see Appendix B of [7]).

First we should find the most general symmetric element $s \in \mathfrak{g} \otimes \mathfrak{g}$, common to all Lie bialgebras with given $\mathfrak{g}$. It is natural to search for the components s^{ij} in the expansion $s = s^{ij} T_i \otimes T_j$. We find (and write it in the form of matrix), that

$$s^{ij} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & -k \end{pmatrix},\ k \in \mathbb{R}.$$

Now we solve the equation $\Delta([\![a,a]\!]) = 0$ together with the condition $[\cdot,\cdot]^a = [\cdot,\cdot]_{\mathfrak{g}^*}$ for $a \in \bigwedge^2 \mathfrak{g}$. Again we look for its components a^{ij} in $a = a^{ij} T_i \otimes T_j$ and we find the solution

$$a^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Hence the most general r-matrix $r = r^{ij} T_i \otimes T_j$ corresponding to $(\mathfrak{g}, \delta)$ is

$$r^{ij} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 1 \\ 0 & -1 & -k \end{pmatrix}, \; k \in \mathbb{R}.$$

By founding the r-matrix we have proved that $(\mathfrak{g}, \delta)$ is a coboundary Lie bialgebra (δ is 1-coboundary). Furthermore we can try to look for the solution of CYBE, that is $\langle r, r \rangle = 0$. Obviously, in this case (not in general) this can only narrow the choice of k which appears in the symmetric part s. In fact, to satisfy CYBE we must set $k = \pm 1$.

Thus we get two possible quasi-triangular (and also factorizable) r-matrices corresponding to $(\mathfrak{g}, \delta)$:

$$r^{ij} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 1 \\ 0 & -1 & \mp 1 \end{pmatrix}.$$

By this example we have shown that in general the r-matrix (solution of GYBE or CYBE) is not by far unique. It may seem that it is just because s is always a 0-cochain. However, they are also Lie bialgebras with ambiguity in a. For a simple example see the Lie bialgebra isomorphic to the Manin triple $(\mathbf{2}|\mathbf{1})$.

2.7 R-matrix, double Lie algebra

In this section we develop the notation very useful particularly for semisimple Lie algebras. For a given coboundary Lie algebra $(\mathfrak{g}, \delta)$ it enables us to identify its dual bracket $[\cdot,\cdot]_{\mathfrak{g}^*}$ with another Lie algebra structure on $\mathfrak{g}$, called a double Lie algebra.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$. Let $R : \mathfrak{g} \to \mathfrak{g}$ be an arbitrary linear operator on $\mathfrak{g}$. Then we can define a bilinear, skew-symmetric bracket $[\cdot,\cdot]_R$ on $\mathfrak{g}$ as

$$[X, Y]_R := [R(X), Y]_{\mathfrak{g}} + [X, R(Y)]_{\mathfrak{g}}, \tag{2.52}$$

for all $X, Y \in \mathfrak{g}$.

We can ask, for which R the bracket $[\cdot,\cdot]_R$ defines a Lie algebra structure on $\mathfrak{g}$. The answer is given in the following lemma:

Lemma 2.7.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $R : \mathfrak{g} \to \mathfrak{g}$ a linear operator on $\mathfrak{g}$. Let us define a bilinear form $\langle R, R \rangle_\alpha$ on $\mathfrak{g}$ with values in $\mathfrak{g}$ as*

$$\langle R, R \rangle_\alpha(X, Y) := [R(X), R(Y)]_{\mathfrak{g}} - R\big([R(X), Y]_{\mathfrak{g}} + [X, R(Y)]_{\mathfrak{g}}\big) + \alpha [X, Y]_{\mathfrak{g}}, \tag{2.53}$$

for all $X, Y \in \mathfrak{g}$, $\alpha \in \mathbb{R}$.

Then $[\cdot,\cdot]_R$ is a Lie bracket on $\mathfrak{g}$, if and only if

$$[Z, \langle R, R \rangle_\alpha(X, Y)]_{\mathfrak{g}} + cyclic\{X, Y, Z\} = 0, \tag{2.54}$$

for all $X, Y, Z \in \mathfrak{g}$.

Proof. The proof is very straightforward, only Jacobi identities for $[\cdot,\cdot]_{\mathfrak{g}}$ are used. ■

From the previous lemma it is clear, that it is sufficient to satisfy the condition $\langle R,R\rangle_\alpha = 0$ for some real number α.

Definition 2.7.2. Condition

$$\langle R,R\rangle_\alpha = 0 \tag{2.55}$$

is called a **modified Yang-Baxter equation** with coefficient α. R is called a **classical R-matrix**, or R-matrix. It is called **factorizable**, if $\alpha \neq 0$.

Lie algebra structure $[\cdot,\cdot]_R$, where R is an R-matrix, is called a **double Lie algebra**.

The most interesting thing about R-matrices is their relation to r-matrices of coboundary Lie bialgebras. It will become completely clear after the following proposition:

Proposition 2.7.3. *Let $r = a + s$ be an arbitrary element of $\mathfrak{g}\otimes\mathfrak{g}$, such that its symmetric part $s \in S^2(\mathfrak{g})$ is ad-invariant ($\Delta(s) = 0$) and the induced linear map $\underline{s} : \mathfrak{g}^* \to \mathfrak{g}$ is invertible. We set $R := \underline{a} \circ \underline{s}^{-1}$. R is clearly a linear operator on $\mathfrak{g}$. Then*

$$[X,Y]_R = \underline{s}\big([\underline{s}^{-1}(X), \underline{s}^{-1}(Y)]^r\big) = \underline{s}\big([\underline{s}^{-1}(X), \underline{s}^{-1}(Y)]^a\big), \tag{2.56}$$

for all $X,Y \in \mathfrak{g}$. $[\cdot,\cdot]^r$ is a bilinear bracket on $\mathfrak{g}^*$ defined by (2.28).

Proof.

$$\begin{aligned}[X,Y]_R &\equiv [R(X),Y]_{\mathfrak{g}} + [X,R(Y)]_{\mathfrak{g}} = ad_{R(X)}(Y) - ad_{R(Y)}(Y) = \\ &= ad_{(R\circ\underline{s})\underline{s}^{-1}(X)}(Y) - ad_{(R\circ\underline{s})\underline{s}^{-1}(Y)}(X) \overset{(2.33)}{=} \\ &\overset{(2.33)}{=} \underline{s}\big(ad^*_{\underline{a}(\underline{s}^{-1}(X))}(\underline{s}^{-1}(Y)) - ad^*_{\underline{a}(\underline{s}^{-1}(Y))}(\underline{s}^{-1}(X))\big) \overset{(2.35)}{=} \\ &\overset{(2.35)}{=} \underline{s}\big([\underline{s}^{-1}(X), \underline{s}^{-1}(Y)]^r\big).\end{aligned}$$

■

Thus, if $[\cdot,\cdot]^r$ is a Lie bracket on $\mathfrak{g}^*$, the Lie bracket $[\cdot,\cdot]_R$ is just an isomorphic image of $[\cdot,\cdot]^r$.

This particular fact can be very well used for a semisimple Lie algebra $\mathfrak{g}$ equipped with coboundary Lie bialgebra defined by r-matrix $r = a+s$. Because the changing of s does not change the Lie algebra structure on $\mathfrak{g}^*$, we can choose $s := K^{-1}$, where K^{-1} denotes the inverse of Killing form K of $\mathfrak{g}$. It is not difficult to show that $\Delta(K^{-1}) = 0$, most easily it follows directly from (5.45).

We can then use the previous lemma and set $R := \underline{a} \circ \underline{K}$ to get

$$[X,Y]_R = \underline{K}^{-1}[\underline{K}(X), \underline{K}(Y)]^r. \tag{2.57}$$

Thus $[X,Y]_R$ is just the "lowering of indices" in $[\cdot,\cdot]^r$.

Example 2.7.4. Let $\mathfrak{g}$ be a Lie algebra $so(3) = span\{T_1, T_2, T_2\}$, with relations

$$[T_i, T_j]_{\mathfrak{g}} = \epsilon_{ijk} T_k, \tag{2.58}$$

where ϵ_{ijk} is the usual Levi-Civita symbol. This algebra can be paired with Bianchi algebra V to form a Lie bialgebra isomorphic to Manin triple (**IX**|**V**|**1**) (cf. [7]). The commutation relations on $\mathfrak{g}^*$ in the dual basis $(T^i)_{i=1}^3$ take the form

$$[T^1,T^2]_{\mathfrak{g}^*} = -T^2,\ [T^2,T^3]_{\mathfrak{g}^*} = 0,\ [T^3,T^1]_{\mathfrak{g}^*} = T^3. \tag{2.59}$$

Lie algebra $\mathfrak{g}$ is semisimple, Lie bialgebra defined above is coboundary. The most general r-matrix has the form

$$s^{ij} = \kappa \cdot \mathbf{1}_3, \kappa \in \mathbb{R}, \tag{2.60}$$

$$a^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{2.61}$$

It never solves CYBE, r cannot be chosen as quasi-triangular r-matrix. Symmetric part s, satisfying $\Delta(s) = 0$, has the most general form (2.60). Since this has to be true also for the inverse of the Killing form K^{-1}, we find that $s = \alpha K^{-1}$ for some $\alpha \in \mathbb{R}$.

Hence it is easy to calculate the Killing form - its matrix K_{ij} in $K_{ij} T^i \otimes T^j$ has to be a real multiple of the unit matrix, one finds that

$$K_{ij} = -2 \cdot \mathbf{1}_3.$$

It is easy to get the matrix $R_{\mathcal{X}}$ of the map $R := \underline{a} \circ \underline{K}$, because $(R_{\mathcal{X}})^i{}_j = K_{jl} a^{li}$. Therefore

$$R_{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Finally, we calculate the bracket $[\cdot, \cdot]_R$ as

$$[T_1, T_2]_R = -2T_2, \; [T_2, T_3]_R = 0, \; [T_3, T_1]_R = 2T_3. \tag{2.62}$$

One should immediately see, that $[\cdot, \cdot]_R$ is indeed isomorphic to $[\cdot, \cdot]_{\mathfrak{g}^*}$.

We have shown, that for given r-matrix of coboundary Lie bialgebra (with semisimple $\mathfrak{g}$), we are able to construct the linear map R, defining a Lie algebra $[\cdot, \cdot]_R$. Conversely, one can ask, if for given linear map R satisfying (2.54) there exists an r-matrix, such that Lie algebra $[\cdot, \cdot]^r$ on $\mathfrak{g}^*$ is isomorphic to $[\cdot, \cdot]_R$. Moreover, there arises a question whether the solutions of CYBE (2.50) somehow correspond to the solutions of MYBE (2.55).

It is obvious that in general, the answer is negative. We can for example consider $R = Id_{\mathfrak{g}}$. This linear map satisfies MYBE for $\alpha = 1$, and $[\cdot, \cdot]_R$ is thus a Lie bracket, isomorphic to $\mathfrak{g}$ itself. Lie algebra $[\cdot, \cdot]^r$ should be then isomorphic to $\mathfrak{g}$. But for example for $\mathfrak{g} = sl(2, \mathbb{R})$ (which is semisimple), there exists no Lie bialgebra with $[\cdot, \cdot]_{\mathfrak{g}^*}$ isomorphic to $sl(2, \mathbb{R})$ ([7]).

However, for special classes of linear maps R, we can find such r-matrix.

Definition 2.7.5. Let $\mathfrak{g}$ be a Lie algebra equipped with a symmetric ad-invariant bilinear form s on $\mathfrak{g}^*$, that is $s \in S^{(2)}(\mathfrak{g})$ and $\Delta(s) = 0$. A linear map $R : \mathfrak{g} \to \mathfrak{g}$ is called a **skew-symmetric endomorphism** of $(\mathfrak{g}, s)$, if

$$\langle \eta, (R \circ \underline{s})(\xi) \rangle = -\langle \xi, (R \circ \underline{s})(\eta) \rangle, \tag{2.63}$$

for all $\xi, \eta \in \mathfrak{g}^*$.

Lemma 2.7.6. *Let $c \in \bigwedge^3 \mathfrak{g}$, We define a bilinear form $\underline{c}$ on $\mathfrak{g}^*$ with values in $\mathfrak{g}$ as*

$$\langle \zeta, \underline{c}(\xi, \eta) \rangle := c(\xi, \eta, \zeta), \tag{2.64}$$

for all $\xi, \eta, \zeta \in \mathfrak{g}^*$. Then

$$\Delta(c) = 0 \Leftrightarrow \langle \zeta, [Z, \underline{c}(\xi, \eta)]_{\mathfrak{g}} \rangle + cyclic\{\xi, \eta, \zeta\} = 0, \tag{2.65}$$

for all $\xi, \eta, \zeta \in \mathfrak{g}^*$ and all $Z \in \mathfrak{g}$.

Proof.

$$0 = \langle ad_Z^{(3)}(c), \xi \otimes \eta \otimes \zeta \rangle = -\big(\langle c, \xi \otimes \eta \otimes ad_Z^*(\zeta)\rangle + cyclic\{\xi, \eta, \zeta\}\big) =$$
$$= \langle \zeta, [Z, \underline{c}(\xi, \eta)]_{\mathfrak{g}} \rangle + cyclic\{\xi, \eta, \zeta\}.$$

∎

We have prepared all necessary terms to state the following:

Proposition 2.7.7. *Let $\mathfrak{g}$ be a Lie algebra equipped with a non-degenerate symmetric ad-invariant bilinear form s on $\mathfrak{g}^*$. Let R be a skew-symmetric endomorphism of $(\mathfrak{g}, s)$, satisfying the condition (2.54) for $\alpha > 0$.*

Then we can define $a \in \bigwedge^2 \mathfrak{g}$ as

$$a(\xi, \eta) = \langle \eta, \underline{a}(\xi) \rangle := \langle \eta, (R \circ \underline{s})(\xi) \rangle,$$

for $\xi, \eta \in \mathfrak{g}^*$.

Bilinear form $r_\alpha := a + \sqrt{\alpha} s$ on $\mathfrak{g}^*$ is an r-matrix of $\mathfrak{g}$ and $[\cdot,\cdot]^r$ is isomorphic to $[\cdot,\cdot]_R$. We can write $\underline{r_\alpha} = (R + \sqrt{\alpha} Id_{\mathfrak{g}}) \circ \underline{s}$.

Moreover, r_α satisfies CYBE (2.50) if and only if R satisfies MYBE (2.55) with coefficient α.

Proof. Let $X, Y \in \mathfrak{g}$. We have shown in the proof of 2.7.3, that

$$[X, Y]_R \equiv [R(X), Y]_{\mathfrak{g}} + [X, R(Y)]_{\mathfrak{g}} = \underline{s}\big([\underline{s}^{-1}(X), \underline{s}^{-1}(Y)]^a\big).$$

This means that if $[\cdot,\cdot]^a$ is a Lie bracket, $\underline{s}$ is a required isomorphism of the two Lie brackets. Applying R on the previous equation, we get

$$R([R(X), Y]_{\mathfrak{g}} + [X, R(Y)]_{\mathfrak{g}}) = \underline{a}\big([\underline{s}^{-1}(X), \underline{s}^{-1}(Y)]^a\big).$$

Thus, according to definitions and the proposition (2.6.1), we have

$$\langle R, R \rangle_\alpha(X, Y) = [\underline{a}(\underline{s}^{-1}(X)), \underline{a}(\underline{s}^{-1}(Y))]_{\mathfrak{g}} - \underline{a}\big([\underline{s}^{-1}(X), \underline{s}^{-1}(Y)]^a\big) + \alpha[\underline{s}(\underline{s}^{-1}(X)), \underline{s}(\underline{s}^{-1}(Y))]_{\mathfrak{g}} =$$
$$= \underline{\langle a, a \rangle}(\underline{s}^{-1}(X), \underline{s}^{-1}(Y)) + \underline{\langle \sqrt{\alpha} s, \sqrt{\alpha} s \rangle}(\underline{s}^{-1}(X), \underline{s}^{-1}(Y)) = \underline{\langle r_\alpha, r_\alpha \rangle}(\underline{s}^{-1}(X), \underline{s}^{-1}(Y)).$$

It is clear now, that R satisfies MYBE with coefficient α if and only if r_α satisfies CYBE.

To prove that r_α is indeed an r-matrix (not quasi-triangular in general), it is by (2.49) sufficient to show that $\Delta(\langle r_\alpha, r_\alpha \rangle) = 0$. But lemma 2.7.6 says it is equivalent to

$$\langle \zeta, [U, \underline{\langle r_\alpha, r_\alpha \rangle}(\xi, \eta)]_{\mathfrak{g}} \rangle + cyclic\{\xi, \eta, \zeta\} = 0,$$

for all $\xi, \eta, \zeta \in \mathfrak{g}^*$ and all $U \in \mathfrak{g}$. Putting $\xi = \underline{s}^{-1}(X), \eta = \underline{s}^{-1}(Y)$ and $\zeta = \underline{s}^{-1}(Z)$ and using the previous computation, we get that r_α is an r-matrix, if and only if

$$\langle \underline{s}^{-1}(Z), [U, \langle R, R \rangle_\alpha(X, Y)]_{\mathfrak{g}} \rangle + cyclic\{X, Y, Z\} = 0,$$

for all $X, Y, Z, U \in \mathfrak{g}$ ($\underline{s}$ is a linear isomorphism). Using the ad-invariance of s in the form (2.33), we can write

$$\langle \underline{s}^{-1}(Z), [U, \langle R, R \rangle_\alpha(X, Y)]_{\mathfrak{g}} \rangle = -\langle \underline{s}^{-1}(Z), ad_{\langle R, R \rangle_\alpha(X,Y)}(U) \rangle =$$
$$= \langle ad^*_{\langle R, R \rangle_\alpha(X,Y)}(\underline{s}^{-1}(Z)), U \rangle = \langle \underline{s}^{-1}\big(ad_{\langle R, R \rangle_\alpha(X,Y)}(Z)\big), U \rangle =$$
$$= -\langle \underline{s}^{-1}\big([Z, \langle R, R \rangle_\alpha(X, Y)]_{\mathfrak{g}}\big), U \rangle.$$

Since U is arbitrary and $\underline{s}^{-1}$ is non-degenerate, we finally obtain that r_α is an r-matrix if and only if

$$[Z, \langle R, R \rangle_\alpha(X, Y)]_{\mathfrak{g}} + cylic\{X, Y, Z\} = 0,$$

for all $X, Y, Z \in \mathfrak{g}$. But this is exactly the condition (2.54), which we have supposed. Thus r_α is an r-matrix and we have finished the proof. ∎

Example 2.7.8. Let $(\mathfrak{g}, \delta)$ be an arbitrary Lie bialgebra, $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$ the corresponding Manin triple. We know, that we have Lie algebra $\mathfrak{d}$ equipped with symmetric, bilinear, ad-invariant and non-degenerate form $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$.

If we denote it like $\mathcal{D}$, we can take $s = \mathcal{D}^{-1}$, where $\mathcal{D}^{-1}$ is the "inverse" of $\mathcal{D}$. From the ad-invariance of $\mathcal{D}$ it follows (just like in the case of Killing form), that $\Delta(\mathcal{D}^{-1}) = 0$.

We define $R := \frac{1}{2}\big(P - \widetilde{P}\big)$, where P and $\widetilde{P}$ are projectors on $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ respectively. It is easy to show that it is indeed a skew-symmetric endomorphism of $(\mathfrak{d}, s)$. Moreover, R satisfies MYBE with coefficient $\alpha = \frac{1}{4}$. Thus we can define a quasi-triangular and factorizable r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$, such that $\underline{r} = (R + \frac{1}{2} Id_{\mathfrak{d}}) \circ \underline{\mathcal{D}}^{-1}$.

This interesting fact will have a very important consequence at the end of chapter 4, see the example 4.9.4.

Chapter 3

Poisson-Lie groups

3.1 Introduction

The aim of this chapter is to introduce the elements of the Poisson geometry, a huge branch of differential geometry with many applications in mathematical physics.

The central point of the Poisson geometry lies in the study of Poisson manifolds - differentiable manifolds equipped with a Poisson bracket. The convenient way to describe such structures is the language of multivector fields, completely skew-symmetric contravariant tensor fields on manifolds. This is why we have included a brief introduction to multivector fields in section 3.2.

In the following sections we introduce the basic terms involved in the study of Poisson manifolds, such as Hamiltonian fields, symplectic manifolds, Poisson maps and finally one of their most important properties - the symplectic foliation of a Poisson manifold.

In the last section of this chapter we are then able to give a definition of Poisson-Lie group, a Poisson manifold which is also a Lie group. The most interesting fact about Poisson-Lie groups is their relation to Lie bialgebras, which were deeply discussed in the previous chapter. This will be the main aim of chapter 4.

The section (3.2) is based on the proceedings from the winter school in Srní of Peter Michor [8] and the book of Izu Vaisman [9]. The section about symplectic manifolds (3.5) is based on the statements which can be found in the book of Marián Fecko [10]. We have used excerpts from the books [9], [11] and from the article of Alan Weinstein [2] in the section about the symplectic foliation 3.7. The rest of this chapter can be in fact found in the book [12].

3.2 Multivector field algebra, Schouten-Nijenhuis bracket

Definition 3.2.1. Let M be a differentiable manifold of dimension n. We denote $\mathcal{T}_0^k(M)$ the space of k-times contravariant tensor fields on M and $L_k(M)$ its subspace of completely skew-symmetric tensor fields. We define the **multivector field algebra** as a direct sum

$$L(M) := \bigoplus_{k=-\infty}^{+\infty} L_k(M) \tag{3.1}$$

equipped with the exterior product $\wedge$, where $L_0 \equiv C^\infty(M)$ and $L_k = 0$ for $k < 0$ and $k > n$. Multivectors lying in the particular $L_k(M)$ subspace are called **homogeneous**. The **grade** of a homogeneous non-zero element X of some $L_k(M)$ is defined as k and denoted $|X| = k$.

The elements of $L_k(M)$ in the form of $X_1 \wedge \cdots \wedge X_k,\ X_i \in L_1(M) \equiv \mathfrak{X}(M)$, are called **simple**.

Definition 3.2.2. Schouten-Nijenhuis bracket $[\cdot,\cdot] : L(M) \times L(M) \to L(M)$ is the $\mathbb{R}$-bilinear mapping defined on homogeneous simple elements of $L(M)$ as

$$[X_1 \wedge \cdot\cdot \wedge X_n, Y_1 \wedge \cdot\cdot \wedge Y_m] := \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdot\cdot \wedge \widehat{X_i} \wedge \cdot\cdot \wedge X_n \wedge Y_1 \wedge \cdot\cdot \wedge \widehat{Y_j} \wedge \cdot\cdot \wedge Y_m, \tag{3.2}$$

where $\widehat{X_i}$ denotes the omission of the vector field in the product and $[X_i, X_j]$ is an ordinary commutator of the vector fields. For $f \in L_0(M)$ and homogeneous $X \in L(M)$ the Schouten-Nijenhuis bracket is defined as

$$[f, X] := -i_{df} X, \tag{3.3}$$

$$[X, f] := (-1)^{(|X|+1)} i_{df} X, \tag{3.4}$$

where i is the common interior product operator (insertion operator).

It is obvious that the Schouten-Nijenhuis bracket of two multivector fields is again a multivector field, it is just the sum of exterior products of vector fields on M.

Remark 3.2.3. Schouten-Nijenhuis bracket enjoys the following properties (on the homogeneous elements):

$$(\forall X, Y \in L(M))\ (|[X, Y]| = |X| + |Y| - 1). \tag{3.5}$$

$$(\forall f, g \in L_0(M))\ ([f, g] = 0). \tag{3.6}$$

$$(\forall X, Y \in L(M))\ ([X, Y] = -(-1)^{(|X|-1)(|Y|-1)} [Y, X]). \tag{3.7}$$

$$(\forall X, Y, Z \in L(M))\ ([X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{|Y|(|X|-1)} Y \wedge [X, Z]). \tag{3.8}$$

$$(\forall X, Y, Z \in L(M))\ ([X \wedge Y, Z] = X \wedge [Y, Z] + (-1)^{|Y|(|Z|-1)} [X, Z] \wedge Y). \tag{3.9}$$

$$(\forall f, g \in L_0(M))(\forall X, Y \in L(M))\ ([fX, gY] = fg[X, Y] + gX \wedge [f, Y] + (-1)^{|X|} f[g, X] \wedge Y). \tag{3.10}$$

$$(\forall X, Y, Z \in L(M))\ ((-1)^{(|X|-1)(|Z|-1)} [X, [Y, Z]] + \mathit{cyclic}\{X, Y, Z\} = 0). \tag{3.11}$$

$$(\forall X \in L_1(M))(\forall Y \in L(M))\ ([X, Y] = \mathcal{L}_X Y). \tag{3.12}$$

All these properties can be proven directly from the definition, although it could be technically quite difficult (for example (3.11), see [9]).

We can ask how to express Schouten-Nijenhuis bracket in some local coordinates $(x^1, \ldots, x^n)$ on the manifold M. Let $P \in L_p(M)$ and $Q \in L_q(M)$. We can express them in the local coordinates as

$$P = \frac{1}{p!} P^{i_1 \cdots i_p} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}},$$

$$Q = \frac{1}{q!} Q^{j_1 \cdots j_q} \frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_q}}.$$

We start directly from the definition (3.2) and write

$$[P, Q] = \frac{1}{p!} \frac{1}{q!} [P^{i_1 \cdots i_p} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}}, Q^{j_1 \cdots j_q} \frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_q}}] =$$

$$= \frac{1}{p!} \frac{1}{q!} \Big\{ [P^{i_1 \cdots i_p} \frac{\partial}{\partial x^{i_1}}, Q^{j_1 \cdots j_q} \frac{\partial}{\partial x^{j_1}}] \wedge \frac{\partial}{\partial x^{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_p}} \wedge \frac{\partial}{\partial x^{j_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_q}} +$$

$$+\sum_{s=2}^{q}(-1)^{s+1}Q^{j_1\cdots j_q}[P^{i_1\cdots i_p}\frac{\partial}{\partial x^{i_1}},\frac{\partial}{\partial x^{j_s}}]\wedge\frac{\partial}{\partial x^{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\widehat{\frac{\partial}{\partial x^{j_s}}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}+$$

$$+\sum_{t=2}^{p}(-1)^{t+1}P^{i_1\cdots i_p}[\frac{\partial}{\partial x^{i_t}},Q^{j_1\cdots j_q}\frac{\partial}{\partial x^{j_1}}]\wedge\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\widehat{\frac{\partial}{\partial x^{i_t}}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}\Big\}=\otimes.$$

One should easily check that (remember that the coordinate vector fields commute)

$$[P^{i_1\cdots i_p}\frac{\partial}{\partial x^{i_1}},Q^{j_1\cdots j_q}\frac{\partial}{\partial x^{j_1}}]=P^{i_1\cdots i_p}\frac{\partial Q^{j_1\cdots j_q}}{\partial x^{i_1}}\frac{\partial}{\partial x^{j_1}}-Q^{j_1\cdots j_q}\frac{\partial P^{i_1\cdots i_p}}{\partial x^{j_1}}\frac{\partial}{\partial x^{i_1}},$$

$$[P^{i_1\cdots i_p}\frac{\partial}{\partial x^{i_1}},\frac{\partial}{\partial x^{j_s}}]=-\frac{\partial P^{i_1\cdots i_p}}{\partial x^{j_s}}\frac{\partial}{\partial x^{i_1}},$$

$$[\frac{\partial}{\partial x^{i_t}},Q^{j_1\cdots j_q}\frac{\partial}{\partial x^{j_1}}]=\frac{\partial Q^{j_1\cdots i_q}}{\partial x^{i_t}}\frac{\partial}{\partial x^{j_1}}$$

and substitute it into the term above.

$$\otimes=\frac{1}{p!}\frac{1}{q!}\Big\{\sum_{s=1}^{q}(-1)^{s}Q^{j_1\cdots j_q}\frac{\partial P^{i_1\cdots i_p}}{\partial x^{j_s}}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\widehat{\frac{\partial}{\partial x^{j_s}}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}+$$

$$+\sum_{t=1}^{p}(-1)^{t+1}(-1)^{p-1}P^{i_1\cdots i_p}\frac{\partial Q^{j_1\cdots j_q}}{\partial x^{i_t}}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\widehat{\frac{\partial}{\partial x^{i_t}}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}\Big\}=$$

$$=\frac{1}{p!}\frac{1}{q!}\Big\{(-1)qQ^{uj_2\cdots j_q}\frac{\partial P^{i_1\cdots i_p}}{\partial x^{u}}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}+$$

$$+(-1)^{p+1}pP^{ui_2\cdots i_p}\frac{\partial Q^{j_1\cdots j_q}}{\partial x^{u}}\frac{\partial}{\partial x^{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}\Big\}=$$

$$=\frac{-1}{p!(q-1)!}Q^{uj_2\cdots j_q}\frac{\partial P^{i_1\cdots i_p}}{\partial x^{u}}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}+$$

$$+\frac{(-1)^{p+1}}{(p-1)!q!}P^{ui_2\cdots i_p}\frac{\partial Q^{j_1\cdots j_q}}{\partial x^{u}}\frac{\partial}{\partial x^{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_p}}\wedge\frac{\partial}{\partial x^{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{j_q}}.$$

In the local coordinates we then have

$$[P,Q]=\frac{1}{(p+q-1)!}[P,Q]^{k_1\cdots k_{p+q-1}}\frac{\partial}{\partial x^{k_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{k_{p+q-1}}},\tag{3.13}$$

where

$$[P,Q]^{k_1\cdots k_{p+q-1}}=\frac{-1}{p!(q-1)!}Q^{uj_2\cdots j_q}\frac{\partial P^{i_1\cdots i_p}}{\partial x^{u}}\delta^{k_1\cdots k_{p+q-1}}_{i_1\cdots i_p j_2\cdots j_q}+\frac{(-1)^{p+1}}{(p-1)!q!}P^{ui_2\cdots i_p}\frac{\partial Q^{j_1\cdots j_q}}{\partial x^{u}}\delta^{k_1\cdots k_{p+q-1}}_{i_2\cdots i_p j_1\cdots j_q}.\tag{3.14}$$

The generalized Kronecker delta $\delta^{i\cdots j}_{k\cdots l}$ gives $+1$ if the indices $i\cdots j$ are an even permutation of $k\cdots l$, -1 if the indices $i\cdots j$ are an odd permutation of $k\cdots l$ and 0 otherwise.

3.3 Poisson manifolds

During a study of the classical Hamiltonian mechanics there naturally arises an interesting geometric structure, a Poisson bracket of two observable quantities (functions on a phase space). It is a Lie algebra bracket in the space of smooth functions $C^\infty(\mathbb{R}^{2n})$ on the phase space manifold $\mathbb{R}^{2n}$, defined as

$$\{f,g\} := \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x^k} - \frac{\partial g}{\partial p_k}\frac{\partial f}{\partial x^k}, \tag{3.15}$$

where $f,g \in C^\infty(\mathbb{R}^{2n})$ and $(x^1,\ldots,x^n,p_1,\ldots,p_n)$ are canonical coordinates on a phase space. One particular smooth function on a phase space is extremely important for the mechanics - Hamilton function H. We know that the real motion of the mechanical system can be viewed as a curve in the phase space, on which the Hamilton equations of motion are satisfied, that is

$$\dot{p}_k = -\frac{\partial H}{\partial x^k},\ \dot{x}^j = \frac{\partial H}{\partial p_j}.$$

These equations are nothing but the differential equations for integral curves of the vector field

$$\zeta_H := \frac{\partial H}{\partial p_k}\frac{\partial}{\partial x^k} - \frac{\partial H}{\partial x^k}\frac{\partial}{\partial p_k}.$$

This vector field can be written in a coordinate-free way as

$$\zeta_H = \{H,\cdot\},\ \text{or}\ \zeta_H(f) = \{H,f\},$$

for $f \in C^\infty(\mathbb{R}^{2n})$. Thus we can study a mechanics on a phase space using only the Poisson bracket on its space of observable functions and a distinguished function, Hamilton function H.

This point of view leads us naturally to the idea of generalization of this approach. We can try to define Poisson bracket on an arbitrary differentiable manifold M. By choosing a Hamilton function we can find the equations of motion and do the "mechanics" on M. These special manifolds M are called Poisson manifolds.

Definition 3.3.1. Differentiable manifold M is called a **Poisson manifold** if it is equipped by an additional structure $\{\cdot,\cdot\}$ called a **Poisson bracket**, which is a bilinear map $C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ having the following properties:

$$(\forall f,g \in C^\infty(M))\ (\{f,g\} = -\{g,f\}). \tag{3.16}$$

$$(\forall f,g,h \in C^\infty(M))\ (\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0). \tag{3.17}$$

$$(\forall f,g,h \in C^\infty(M))\ (\{fg,h\} = f\{g,h\} + \{f,h\}g). \tag{3.18}$$

In the other words, $\{\cdot,\cdot\}$ adds on $C^\infty(M)$ the Lie algebra structure and for every $h \in C^\infty(M)$ the map $\{\cdot,h\} : C^\infty(M) \to C^\infty(M)$ lies in $Der(C^\infty(M))$.

Remark 3.3.2. The property (3.17) is usually called Jacobi identity, the property (3.18) is for obvious reasons called Leibniz rule.

The Poisson bracket on M can be easily encoded into the special bivector field $\mathcal{P}$ on M, called (not surprisingly) a Poisson bivector on M.

The skew-symmetry of the Poisson bracket will lead onto a skew-symmetry of $\mathcal{P}$ (this is why we talk about a bivector field). The Leibniz rule (3.18) will be satisfied "for free" by every bivector field $\mathcal{P}$. The only problem arises with the Jacobi identity for the Poisson bracket. It turns out that it can be encoded into the words of Schouten-Nijenhuis bracket, which we have canonically defined on every manifold M.

Proposition 3.3.3. *Let M be a differentiable manifold. Every Poisson bracket $\{\cdot,\cdot\}$ on M corresponds to the unique bivector field $\mathcal{P} \in L_2(M)$, such that*

$$[\mathcal{P},\mathcal{P}] = 0. \tag{3.19}$$

Conversely, every bivector $\mathcal{P} \in L_2(M)$ satisfying (3.19) can be used to define a Poisson bracket on M. The bivector field $\mathcal{P}$ is called a ***Poisson bivector (field)*** *on M.*

Proof. The correspondence between the Lie bracket $\{\cdot,\cdot\}$ and the Poisson bivector $\mathcal{P}$ is given by relation

$$\mathcal{P}(df, dg) = \{f, g\}, \tag{3.20}$$

for $f, g \in C^\infty(M)$. The skew-symmetry of $\mathcal{P}$ corresponds to the skew-symmetry of $\{\cdot,\cdot\}$. The Poisson bracket defined using $\mathcal{P}$ and (3.20) always satisfies the Leibniz rule, because $\mathcal{P}(d(fg), dh) = \mathcal{P}((df)g + f(dg), dh) = f\mathcal{P}(dg, dh) + g\mathcal{P}(df, dh)$.

The left hand side of (3.20) depends only on the differentials of the functions. Therefore one has to verify that $df = dh$ implies

$$\{f, g\} = \{h, g\},$$

for all $g \in C^\infty(M)$. Using the Leibniz rule one finds that if $c \in \mathbb{R}$, then $\{f, c\} = 0$ for all $f \in C^\infty(M)$. $df = dh$ implies $f = h + c$, $c \in \mathbb{R}$. Then

$$\{f, g\} = \{h + c, g\} = \{h, g\} + \{c, g\} = \{h, g\}.$$

The only unsolved question is the satisfaction of the Jacobi identities for the Poisson bracket (3.17). We will show that they are equivalent to (3.19).

Let f, g, h be the arbitrary smooth functions. First of all see that

$$\{f, g\} = \mathcal{P}(df, dg) = i_{dg} i_{df} \mathcal{P} = [g, [f, \mathcal{P}]],$$

and consequently

$$\{f, \{g, h\}\} = [[h, [g, \mathcal{P}]], [f, \mathcal{P}]]. \tag{3.21}$$

We compute some useful relations using the properties (3.11) and (3.7) of the Schouten-Nijenhuis bracket:

$$[h, [\mathcal{P}, \mathcal{P}]] = -2[\mathcal{P}, [\mathcal{P}, h]]. \tag{3.22}$$

$$[g, [\mathcal{P}, [\mathcal{P}, h]]] = -[[\mathcal{P}, h], [g, \mathcal{P}]] + [\mathcal{P}, [[\mathcal{P}, h], g]]. \tag{3.23}$$

Using these equations and the definition of the Schouten-Nijenhuis bracket, we can write

$$[\mathcal{P}, \mathcal{P}](df, dg, dh) = [f, [g, [h, [\mathcal{P}, \mathcal{P}]]]] \overset{(3.22)}{=} -2[f, [g, [\mathcal{P}, [\mathcal{P}, h]]]] \overset{(3.23)}{=}$$

$$\overset{(3.23)}{=} 2\underbrace{[f, [[h, \mathcal{P}], [g, \mathcal{P}]]]}_{A} - 2\underbrace{[f, [\mathcal{P}, [[\mathcal{P}, h], g]]]}_{B} = \otimes.$$

If we denote $X := [h, \mathcal{P}]$, $Y := [g, \mathcal{P}]$, we can write

$$A = [f, [[h, \mathcal{P}], [g, \mathcal{P}]]] = [f, [X, Y]] \overset{(3.11)}{=} -[Y, [f, X]] - [X, [Y, f]] =$$

$$= -[[g, \mathcal{P}], [f, [h, \mathcal{P}]]] - [[h, \mathcal{P}], [[g, \mathcal{P}], f]] = [[f, [h, \mathcal{P}]], [g, \mathcal{P}]] - [[f, [g, \mathcal{P}]], [h, \mathcal{P}]] \overset{(3.21)}{=}$$

$$\overset{(3.21)}{=} \{g, \{h, f\}\} - \{h, \{g, f\}\}.$$

We define $q := [g, [h, \mathcal{P}]]$ and then

$$B = [f, [\mathcal{P}, [[\mathcal{P}, h], g]]] = -[f, [\mathcal{P}, q]] \overset{(3.11)}{=} [q, [f, \mathcal{P}]] = [[g, [h, \mathcal{P}]], [f, \mathcal{P}]] \overset{(3.21)}{=} \{f, \{h, g\}\}.$$

$\otimes = 2A - 2B = 2(\{g, \{h, f\}\} - \{h, \{g, f\}\} - \{f, \{h, g\}\}) = 2(\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\})$. Hence we have

$$[\mathcal{P}, \mathcal{P}](df, dg, dh) = [f, [g, [h, [\mathcal{P}, \mathcal{P}]]]] = 2(\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\})$$

for arbitrary smooth functions f, g, h and the proof of the theorem done. ∎

Since now, we can denote a Poisson manifold not as a pair $(M, \{\cdot,\cdot\})$, but as a pair $(M, \mathcal{P})$. We will not distinguish between these two points of view. Sometimes it is convenient to use a Poisson bracket, sometimes a Poisson bivector. Especially in the case of Poisson-Lie groups we will talk only in the language of bivector fields on M.

If we write $\mathcal{P}$ in local coordinates $(x^1, \dots, x^n)$ on M, we have

$$\mathcal{P} = \mathcal{P}^{ij}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}. \tag{3.24}$$

In this local coordinates we can then rewrite the Poisson bracket as

$$\{f, g\}(x) = \mathcal{P}^{ij}(x) \left. \frac{\partial f}{\partial x^i} \right|_x \left. \frac{\partial g}{\partial x^j} \right|_x . \tag{3.25}$$

The Jacobi identities (3.17) are equivalent to the coordinate expression

$$\mathcal{P}^{ri} \frac{\partial \mathcal{P}^{jk}}{\partial x^r} + \mathcal{P}^{rj} \frac{\partial \mathcal{P}^{ki}}{\partial x^r} + \mathcal{P}^{rk} \frac{\partial \mathcal{P}^{ij}}{\partial x^r} = 0. \tag{3.26}$$

Example 3.3.4. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with a Lie bracket $[\cdot,\cdot]$. We can consider its dual space $\mathfrak{g}^*$ as a differentiable manifold.

We can use an arbitrary (but fixed) basis $(T_i)_{i=1}^n$ of $\mathfrak{g}$ as the (global) coordinates on $\mathfrak{g}^*$. The tangent space $T_\xi(\mathfrak{g}^*)$ at any point $\xi \in \mathfrak{g}^*$ can be identified with $\mathfrak{g}^*$ itself, there exists the linear isomorphism $\mathbf{A}_\xi : \mathfrak{g}^* \to T_\xi(\mathfrak{g}^*)$, such that $\mathbf{A}_\xi(T^i) = \left.\frac{\partial}{\partial T_i}\right|_\xi$, where (T^i) is a basis of $\mathfrak{g}^*$ dual to (T_i).

We get the same map $\mathbf{A}_\xi$ for every basis of $\mathfrak{g}$, that is if $(Y_i)_{i=1}^n$ is also a basis of $\mathfrak{g}$, then $\mathbf{A}_\xi(Y^i) = \left.\frac{\partial}{\partial Y_i}\right|_\xi$. We can then define a Poisson bracket of $f, g \in C^\infty(\mathfrak{g}^*)$ as

$$\{f, g\}(\xi) := \langle \xi, [\mathbf{A}^*_\xi((df)_\xi), \mathbf{A}^*_\xi((dg)_\xi)] \rangle, \tag{3.27}$$

for all $\xi \in \mathfrak{g}^*$. $A^*_\xi : T^*_\xi(\mathfrak{g}^*) \to \mathfrak{g}$ is the linear map dual to A_ξ.

One should easily check, that for arbitrary smooth functions f, g on $\mathfrak{g}^*$ and $\xi \in \mathfrak{g}^*$

$$\{f, g\}(\xi) = \mathcal{P}_{ij}(\xi) \left. \frac{\partial f}{\partial T_i} \right|_\xi \left. \frac{\partial g}{\partial T_j} \right|_\xi , \tag{3.28}$$

where

$$\mathcal{P}_{ij}(\xi) = c_{ij}{}^k \langle \xi, T_k \rangle \equiv c_{ij}{}^k \xi_k \tag{3.29}$$

and c^k_{ij} are the structure coefficients of $\mathfrak{g}$ with respect to $(T_i)_{i=1}^n$, i.e.

$$[T_i, T_j] = c_{ij}{}^k T_k. \tag{3.30}$$

A skew-symmetry and a derivation property of the Poisson bracket can be seen from the definition. Coordinate expression of the Jacobi identities (3.26) reads

$$(c_{jk}{}^r c_{ri}{}^s + c_{ki}{}^r c_{rj}{}^s + c_{ij}{}^r c_{rk}{}^s) \langle \xi, T_s \rangle = 0, \tag{3.31}$$

which is true because of the Jacobi identities for the Lie bracket on $\mathfrak{g}$. $\mathfrak{g}^*$ with the Poisson bracket (3.28) is then a Poisson manifold. It is called a **Lie-Poisson structure**, or sometimes for obvious reasons a **linear Poisson structure** (components of $\mathcal{P}$ are linear functions on $\mathfrak{g}^*$).

3.4 Hamiltonian fields, characteristic spaces

Definition 3.4.1. Let $(M, \mathcal{P})$ be a Poisson manifold. We can assign to each smooth function $f \in C^\infty(M)$ a vector field ζ_f, called a **Hamiltonian field** generated by function f:

$$\zeta_f := \mathcal{P}(df, \cdot) \equiv \{f, \cdot\}. \tag{3.32}$$

Note that for $g \in C^\infty(M)$ we have

$$\zeta_f(g) = \mathcal{P}(df, dg) \equiv \{f, g\}. \tag{3.33}$$

Proposition 3.4.2. *A Poisson bivector $\mathcal{P}$ is invariant under a (infinitesimal) Lie transport along an arbitrary Hamiltonian field, i.e. for an arbitrary $f \in C^\infty(M)$ there holds*

$$\mathcal{L}_{\zeta_f}\mathcal{P} = 0. \tag{3.34}$$

Proof.

$$\mathcal{L}_{\zeta_f}\mathcal{P} = [\zeta_f, \mathcal{P}] = [\mathcal{P}(df, \cdot), \mathcal{P}] = [i_{df}(\mathcal{P}), \mathcal{P}] = -[[f, \mathcal{P}], \mathcal{P}] \overset{(3.11)}{=} \frac{1}{2}[f, [\mathcal{P}, \mathcal{P}]] \overset{(3.19)}{=} 0.$$

■

Remark 3.4.3. Every Hamiltonian field thus generates a local flow preserving the Poisson structure: Let $x \in M$ and ϕ^t is the local flow corresponding to ζ_f, such that $\phi^0(x) = x$. Then (for t where $\phi^t(x)$ is well defined) we get

$$\phi^t_*(\mathcal{P}(x)) = \mathcal{P}(\phi^t(x)). \tag{3.35}$$

Lemma 3.4.4. *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. Let $f, g \in C^\infty(M)$. Then*

$$[\zeta_f, \zeta_g] = \zeta_{\{f,g\}}. \tag{3.36}$$

Proof. Let $\phi \in C^\infty(M)$ arbitrary. Then

$$[\zeta_f, \zeta_g](\phi) = \zeta_f(\zeta_g(\phi)) - \zeta_g(\zeta_f(\phi)) = \zeta_f(\{g, \phi\}) - \zeta_g(\{f, \phi\}) = \{f, \{g, \phi\}\} + \{g, \{\phi, f\}\} \overset{(3.17)}{=}$$

$$\overset{(3.17)}{=} \{\{f, g\}, \phi\} = \zeta_{\{f,g\}}(\phi).$$

■

From the lemma we see that Hamiltonian fields form an infinite-dimensional subalgebra of $\mathfrak{X}(M)$. This fact is very important for the symplectic foliation of a Poisson manifold, which will be examined in the section 3.7.

Definition 3.4.5. Let $(M, \mathcal{P})$ be a Poisson manifold. For every $x \in M$ we can define a map $\#_x : T_x^*(M) \to T_x(M)$ as

$$\#_x(\alpha_x) := \mathcal{P}(x)(\alpha_x, \cdot), \tag{3.37}$$

for $\alpha_x \in T_x^*(M)$. The subspace $S(x) := \#_x(T_x^*(M))$ of $T_x(M)$ is called a **characteristic space** of $(M, \mathcal{P})$ at x. $\rho_{\mathcal{P}}(x) := \dim S(x)$ is called the **rank of the Poisson structure** $\mathcal{P}$ at x and $\rho_{\mathcal{P}} := \max_{x \in M}(\rho_{\mathcal{P}}(x))$ is called the rank of $\mathcal{P}$.

In the case of $\rho_{\mathcal{P}}(x)$ constant on the whole M we call $\mathcal{P}$ a **regular Poisson structure**.

We can also define a map $\# : \Omega^1(M) \to \mathfrak{X}(M)$ as

$$\#(\alpha) := \mathcal{P}(\alpha, \cdot), \tag{3.38}$$

for $\alpha \in \Omega^1(M)$. Clearly $\zeta_f = \#(df)$.

Remark 3.4.6. Note that at every point $x \in M$ the characteristic space is spanned by the values of Hamiltonian vector fields at x. However, this does not mean that every vector field tangent to characteristic distribution (cf. example 3.7.6) is a Hamiltonian vector field!

Remark 3.4.7. Note that the rank of $\mathcal{P}$ at arbitrary $x \in M$ is always even-dimensional. This is a consequence of the skew-symmetry of $\mathcal{P}$.

The direct consequence of the Leibniz rule is $\{c, f\} = 0$ for an arbitrary constant $c \in \mathbb{R}$ and $f \in C^\infty(M)$. In the case of $\rho_{\mathcal{P}} < \dim M$ one may find non-constant functions with this property.

Definition 3.4.8. Let $(M, \mathcal{P})$ be a Poisson manifold. $f \in C^\infty(M)$ is called a **Casimir function** on M, if

$$\{f, \cdot\} \equiv \zeta_f = 0. \tag{3.39}$$

3.5 Symplectic manifolds, symplectic form

Definition 3.5.1. Let us have a Poisson manifold $(M, \mathcal{P})$. When the map $\#$ (see definition (3.4.5)) is a $C^\infty(M)$-linear bijection between $\Omega^1(M)$ and $\mathfrak{X}(M)$, we call M a **symplectic manifold**. Obviously then $\rho_{\mathcal{P}}(x) = \dim M$ for all $x \in M$. Equation

$$\alpha = \omega(\cdot, \mathcal{P}(\alpha, \cdot)) \tag{3.40}$$

where α is arbitrary 1-form on M, then defines a non-degenerate 2-form ω on M, called a **symplectic form**.

On the symplectic manifold we can rewrite a Poisson bracket using the symplectic form ω. For arbitrary smooth functions f, g we have

$$\omega(\zeta_f, \zeta_g) = \langle \omega(\cdot, \zeta_g), \zeta_f \rangle = \langle \omega(\cdot, \mathcal{P}(dg, \cdot)), \zeta_f \rangle \overset{(3.40)}{=} \langle dg, \zeta_f \rangle = \zeta_f(g) = \{f, g\}. \tag{3.41}$$

Recalling the fact that $\#$ is bijective on symplectic manifolds, we can equivalently define a Hamiltonian field ζ_f by the equation

$$i_{\zeta_f} \omega = -df. \tag{3.42}$$

Indeed, for $\zeta_f = \mathcal{P}(df, \cdot)$ we have also $\omega(\cdot, \zeta_f) = \omega(\cdot, \mathcal{P}(df, \cdot)) \overset{(3.40)}{=} df$. Conversely, if $\omega(\cdot, \zeta_f) = df$, we rewrite the right side in the same way and use the non-degeneracy of ω.

Remark 3.5.2. Note that ω is not the "inverse" of $\mathcal{P}$, but its opposite. In some local coordinates $(x^1, \ldots, x^n)$, where $\mathcal{P} = \mathcal{P}^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ and $\omega = \omega_{ij} dx^i \otimes dx^j$, we have

$$\omega_{ik} \mathcal{P}^{kj} = -\delta_i^j.$$

Proposition 3.5.3. *Every manifold M equipped with a non-degenerate closed 2-form ω can be made into a Poisson manifold. Its Poisson bivector is non-degenerate and ω is its symplectic form. Therefore M is a symplectic manifold.*

We will prove this important statement in a few steps. **For a moment we will denote** ζ_f every vector field satisfying the equation (3.42) for $f \in C^\infty(M)$ and call it a Hamiltonian field (despite the fact we have no Poisson structure at the moment).

Lemma 3.5.4. *Let M be a manifold equipped with a non-degenerate closed 2-form ω. Then for arbitrary $f \in C^\infty(M)$*

$$\mathcal{L}_{\zeta_f} \omega = 0. \tag{3.43}$$

Proof. Using the Cartan identity for a Lie derivative

$$\mathcal{L}_V = i_V d + d i_V, \tag{3.44}$$

for any $V \in \mathfrak{X}(M)$, we have

$$\mathcal{L}_{\zeta_f} \omega = i_{\zeta_f} d\omega + d i_{\zeta_f} \omega = i_{\zeta_f} 0 - ddf = 0.$$

■

Lemma 3.5.5. *For arbitrary $f, g \in C^\infty(M)$, where M is equipped with a closed non-degenerate 2-form ω, we have*

$$[\zeta_f, \zeta_g] = \zeta_{\omega(\zeta_f, \zeta_g)}. \tag{3.45}$$

Proof. We will again use (3.44) and the identity

$$\mathcal{L}_V i_W - i_W \mathcal{L}_V = i_{[V,W]}. \tag{3.46}$$

For $f, g \in C^\infty(M)$ we can write

$$i_{[\zeta_f, \zeta_g]} \omega = \mathcal{L}_{\zeta_f}(i_{\zeta_g} \omega) - i_{\zeta_g}(\mathcal{L}_{\zeta_f} \omega) = d(i_{\zeta_f} i_{\zeta_g} \omega) = -d(\omega(\zeta_f, \zeta_g)).$$

■

Proof of Theorem 3.5.3. Because ω is non-degenerate, we can define the Poisson bivector $\mathcal{P}$ by equation

$$V = \mathcal{P}(\omega(\cdot, V), \cdot). \tag{3.47}$$

for all $V \in \mathfrak{X}(M)$. For arbitrary $f, g \in C^\infty(M)$ we then have

$$\{f, g\} = \mathcal{P}(df, dg) = \mathcal{P}(\omega(\cdot, \zeta_f), dg) = \langle \mathcal{P}(\omega(\cdot, \zeta_f), \cdot), dg \rangle = \langle dg, \zeta_f \rangle = \omega(\zeta_f, \zeta_g).$$

This also implies, that ζ_f defined by (3.42) can be written as $\zeta_f = \mathcal{P}(df, \cdot)$. Hence ζ_f is a Hamiltonian field generated by f, corresponding to the Poisson structure $\mathcal{P}$.

Only thing left to be proven is satisfying the Jacobi identities for the Poisson bracket, which, as we will show, is the consequence of the form ω being closed. We will use a well known identity which holds for every 2-form β and arbitrary vector fields U, V, W:

$$d\beta(U,V,W) = U(\beta(V,W)) - \beta([U,V],W) + cyclic\{U,V,W\}. \tag{3.48}$$

Using this identity we have

$$0 = d\omega(\zeta_f, \zeta_g, \zeta_h) = \zeta_f(\omega(\zeta_g, \zeta_h)) - \omega([\zeta_f, \zeta_g], \zeta_h) + cyclic\{\zeta_f, \zeta_g, \zeta_h) \overset{(3.45)}{=}$$

$$\overset{(3.45)}{=} \{f,\{g,h\}\} + \{h,\{f,g\}\} + cyclic\{f,g,h\} = 2(\{f,\{g,h\}\} + cyclic\{f,g,h\})$$

for arbitrary $f, g, h \in C^\infty(M)$. $\mathcal{P}$ is then a Poisson bivector and we have the proof done. ■

Proposition 3.5.6. *Let us have a Poisson manifold $(M, \mathcal{P})$ which is also symplectic, with a symplectic form ω. Then ω is closed.*

Proof. A proof is similar to the last part of the proof of the proposition 3.5.3, but instead of (3.45) we will use (3.36) which was proved using only the Jacobi identities. Hence we know that $d\omega(\zeta_f, \zeta_g, \zeta_h) = 0$ for arbitrary $f, g, h \in C^\infty(M)$. Since in some coordinate patch $(x^1, \dots, x^n)$ we can write every vector field as

$$V = \mathcal{P}(\#^{-1}(V), \cdot) = (\#^{-1}(V))_k \mathcal{P}(dx^k, \cdot) = (\#^{-1}(V))_k \zeta_{(x^k)},$$

we have the proof done. ■

We have shown that every manifold equipped with a non-degenerate closed 2-form is symplectic and every symplectic form is closed. The fact that every symplectic form is closed is equivalent to the Jacobi identities for the Poisson bracket.

Example 3.5.7. Symplectic structure on the cotangent bundle T^*M

Let M be a differentiable manifold. We define the set $T^*M = \bigcup_{x \in M} T^*_x(M)$, that is a set of all tangent covectors at all points of M. The set T^*M is called (the total space of) a **contangent bundle** of M. We define the projection $\pi : T^*M \to M$ as

$$\pi : \alpha \in T^*_x M \mapsto x \in M.$$

T^*M has a structure of a differentiable manifold, let $\mathcal{O} \subset M$ be an open subset of M with coordinates $(x^1, \dots, x^n)$, then on $\pi^{-1}(\mathcal{O}) =: \hat{\mathcal{O}} \subset T^*M$ we have the coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$, defined for $\alpha \in T^*_x M$, $x \in \mathcal{O}$ as follows:

$$x^i(\alpha) := x^i(x), \tag{3.49}$$

$$p_i(\alpha) := \langle \alpha, \frac{\partial}{\partial x^i}\Big|_x \rangle. \tag{3.50}$$

These coordinates are called the canonical coordinates on T^*M. One has to check the transition functions on the overlaps of two (three) coordinate patches. Everything follows from the fact that we use the original patches from the manifold M.

There exists a canonical 1-form θ on T^*M, defined at $p \in T^*M$ as

$$\langle \theta_p, V_p \rangle := \langle p, \pi_*(V_p) \rangle, \tag{3.51}$$

for every $V_p \in T_p(T^*M)$. One can easily show that in the canonical coordinates θ can be written as

$$\theta = p_k dx^k. \tag{3.52}$$

We can now define the 2-form ω as

$$\omega := d\theta = dp_k \wedge dx^k.$$

This form is obviously non-degenerate and closed by definition. Hence ω is by the previous theorem a symplectic form on T^*M. T^*M is thus always a symplectic manifold, no matter how obscure manifold M was. The corresponding Poisson bivector is

$$\mathcal{P} = \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial x^k}. \tag{3.53}$$

Therefore for $f, g \in C^\infty(T^*M)$ we get a well known form of Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x^k} - \frac{\partial f}{\partial x^k}\frac{\partial g}{\partial p_k}. \tag{3.54}$$

3.6 Poisson maps

Definition 3.6.1. M and N be Poisson manifolds with Poisson brackets $\{\cdot,\cdot\}_M$ and $\{\cdot,\cdot\}_N$. A smooth map $F : N \to M$ is called a **Poisson map**, if

$$(\forall f, g \in C^\infty(M)) \ (\{f, g\}_M \circ F = \{f \circ F, g \circ F\}_N) \tag{3.55}$$

Proposition 3.6.2. *The Poisson map property (3.55) is equivalent to*

$$F_*\mathcal{P}_N = \mathcal{P}_M, \tag{3.56}$$

where $\mathcal{P}_N$ and $\mathcal{P}_M$ are Poisson bivectors on N and M.

Proof. Let f, g be the arbitrary smooth functions on M, $x \in N$. Equation (3.55) then reads

$$\{f, g\}_M(F(x)) = \{f \circ F, g \circ F\}_N(x).$$

We can rewrite this using Poisson bivectors and go on with the right side:

$$\mathcal{P}_M(F(x))(df|_{F(x)}, dg|_{F(x)}) = \mathcal{P}_N(x)(d(f \circ F)|_x, d(g \circ F)|_x) =$$

$$= \mathcal{P}_N(x)(d(F^*f)|_x, d(F^*g)|_x) = \mathcal{P}_N(x)((F^*df)|_x, (F^*dg)|_x) =$$

$$= (F_*(\mathcal{P}_N(x)))(df|_{F(x)}, dg|_{F(x)}).$$

■

Remark 3.6.3. The concept of Poisson maps is obviously the most natural way to describe a "perserving of the Poisson structure".

Definition 3.6.4. Let us have a Poisson manifold $(M, \{\cdot,\cdot\})$. Let S be its included submanifold, which is also a Poisson manifold. S is called a **Poisson submanifold**, if the inclusion map $S \hookrightarrow M$ is a Poisson map.

Equivalently, for every point $x \in S$ of the Poisson submanifold S the Poisson bivector $\mathcal{P}$ of M lies in the subspace $T_x(S) \otimes T_x(S)$.

Having two Poisson manifolds M and N, we can easily make their Cartesian product $M \times N$ into a Poisson manifold, defining the Poisson bracket for arbitrary $f, g \in C^\infty(M \times N)$ and $(x, y) \in M \times N$ in a natural way as

$$\{f, g\}_{M\times N}(x, y) := \{f(\cdot, y), g(\cdot, y)\}_M(x) + \{f(x, \cdot), g(x, \cdot)\}_N(y). \tag{3.57}$$

Required properties of the Poisson bracket can be easily seen.

Definition 3.6.5. Let M and N be Poisson manifolds with Poisson brackets $\{\cdot,\cdot\}_M$ and $\{\cdot,\cdot\}_N$. Smooth map $F : N \to M$ is called **anti-Poisson map**, if

$$(\forall f, g \in C^\infty(M)) \ (\{f, g\}_M \circ F = -\{f \circ F, g \circ F\}_N) \tag{3.58}$$

3.7 The symplectic foliation of a Poisson Manifold

As we have seen, in many cases the rank of the Poisson structure depends on the position at manifold. This will be the case of Poisson-Lie groups, which always have a zero rank at the unit of the group. We may ask if there exist such submanifolds of M, where the rank is constant and moreover, the restriction of the Poisson bivector is non-degenerate. The answer is positive, but first we have to introduce the concept of singular distributions and foliations of a differentiable manifold.

Definition 3.7.1. Let M be an n-dimensional differentiable manifold. A **singular distribution** $\mathcal{D}$ on M is an assignment $x \in M \mapsto \mathcal{D}_x \Subset T_x(M)$, where $\mathcal{D}_x$ is a vector subspace of the tangent vector space $T_x(M)$. $\rho_{\mathcal{D}}(x) := \dim \mathcal{D}_x$ is called a rank of $\mathcal{D}$ at $x \in M$.

We call $\mathcal{D}$ a **smooth singular distribution** if for all $x \in M$ there exists a neighbourhood U_x, such that for every $v \in \mathcal{D}_x$ there exists a vector field $X \in \mathfrak{X}(M)$, such that $X(y) \in \mathcal{D}_y$ for all $y \in U_x$ and $X(x) = v$.

Remark 3.7.2. We use the word "singular" according to [11]. It is written to emphasize the fact, that it is not distribution in the sense found in many textbooks on differential geometry, where some fixed dimension of the subspace D_x is supposed.

Example 3.7.3. An important example of such smooth singular distribution is an assignment $x \mapsto S(x)$ for $x \in M$, $(M, \mathcal{P})$ a Poisson manifold, where $S(x)$ is the characteristic space at $x \in M$. The satisfaction of the smoothness condition is obvious from the definition of $S(x)$. Such smooth singular distribution is called a **characteristic distribution** of the Poisson manifold $(M, \mathcal{P})$.

We would like to cut the Poisson manifold into symplectic "slices", while the "cutting" should be in some sense smooth. For this purpose we should define properly this cutting, called a singular foliation of the manifold M.

Definition 3.7.4. Let M be a differentiable manifold. A **smooth singular foliation** $\mathcal{F}$ is a partition $\mathcal{F} = \{\mathcal{F}_\alpha\}$ of manifold M into a disjoint union of immersed connected differentiable submanifolds $\mathcal{F}_\alpha$ (called leaves), satisfying the **local foliation property**, which can be stated as follows:

Let $x \in M$. Then there exists a leaf $\mathcal{F}_x \in \mathcal{F}$ containing x. Denote $d = \dim \mathcal{F}_x$. There have to exist local coordinates $(y_1, \dots, y_m)$ in a neighbourhood U_x of x, such that $U_x = \{-\epsilon < y_1 < \epsilon, \dots, -\epsilon < y_n < \epsilon\}$ and the disk $\{y_{d+1} = \dots = y_n = 0\}$ coincides with the path-connected component of $\mathcal{F}_x \cap U$. Moreover, each disk $\{y_{d+1} = c_{d+1}, \dots, y_n = c_n\}$, where $-\epsilon < c_{d+1}, \dots, c_n < \epsilon$ are real constants, is wholly contained in some leaf of $\mathcal{F}$.

For illustration - in rough words, the local foliation property means that you can find a neighbourhood at every point, which looks as a bread sliced by the foliation.

Remark 3.7.5. If a dimension of all leaves in $\mathcal{F}$ is the same, we call $\mathcal{F}$ a **smooth regular foliation**.

Example 3.7.6. Another important example of smooth singular distribution is a **tangent distribution** $\mathcal{D}^{\mathcal{F}}$ of a smooth singular foliation $\mathcal{F}$. It is defined as $\mathcal{D}^{\mathcal{F}}_x := T_x(\mathcal{F}_x)$ for $x \in M$ and $\mathcal{F}_x$ a leaf containing x.

For given smooth singular distribution $\mathcal{D}$ on M and given point $x \in M$ we can ask, if there exists an immersed submanifold of M, such that $\mathcal{D}_x$ forms its tangent space at x. If the answer is positive at each point $x \in M$, we talk about an integrable distribution. Most of the distributions presented here are integrable, but there exists a plenty of non-integrable distributions with huge applications in physics - e.g. horizontal distributions in the theory of principal bundles.

Definition 3.7.7. A singular smooth distribution $\mathcal{D}$ on a differentiable manifold M is called an **integrable distribution** if for every $x \in M$ there exists an immersed connected differentiable submanifold N_x of M, such that for every $y \in N_x$ we have $T_y(N_x) = \mathcal{D}_y$.

It is obvious (from the fact that every submanifold is always a manifold), that a commutator of every two smooth vector fields on M, tangent to the integrable distribution $\mathcal{D}$, is again a smooth vector field on M tangent to $\mathcal{D}$.

A general smooth distribution $\mathcal{D}$ with such property is called an **involutive distribution**.

For regular smooth distributions there holds also an opposite statement (called Frobenius theorem). Every involutive regular smooth distribution on M is integrable. However, in the general case, that is no longer true. Instead of this, we have to find another simple sufficient condition for the integrability of a smooth distribution.

Example 3.7.8. We give a very simple example of Frobenius theorem failure for singular smooth distributions. Let $M = \mathbb{R}^2$. If we set

$$\mathcal{D}_{(x,y)} := T_{(x,y)}(\mathbb{R}^2),\ x > 0,$$

$$\mathcal{D}_{(x,y)} := span\{ \frac{\partial}{\partial x}\bigg|_{(x,y)} \},\ x \leq 0,$$

we obviously get a smooth singular distribution in $\mathbb{R}^2$.

This distribution is involutive, but non-integrable. See for example the point $(0, 1)$. Clearly there exists no submanifold N of $\mathbb{R}^2$, such that its tangent spaces in some neighbourhood of $(0, 1)$ coincide with the distribution $\mathcal{D}$.

Definition 3.7.9. Let $C \subset \mathfrak{X}(M)$ be some subset of all smooth vector fields on M.

We say that a smooth singular distribution $\mathcal{D}$ on M is generated by C, if for every $x \in M$ the subspace $\mathcal{D}_x$ of $T_x(M)$ is spanned by values of vector fields of C at x.

We say that smooth singular distribution $\mathcal{D}$ is invariant with respect to C, if for every $x \in M$ and every $X \in C$, we have

$$\phi^X_{t*}(\mathcal{D}_x) = \mathcal{D}_{\phi^X_t(x)},$$

where ϕ^X_t denotes the local flow of X, wherever $\phi^X_t(x)$ is well-defined.

This is the property of distributions we were looking for. The sketch of the proof of the following theorem is given in [11]. We completely omit it here.

Theorem 3.7.10 (Stefan-Sussman)**.** *Let $\mathcal{D}$ be a smooth singular distribution on a differentiable manifold M. The three following conditions are equivalent:*

(i.) $\mathcal{D}$ is an integrable distribution.

(ii.) $\mathcal{D}$ is generated by some $C \subset \mathfrak{X}(M)$ and it is invariant with respect to C.

(iii.) $\mathcal{D}$ is the tangent distribution $\mathcal{D}^{\mathcal{F}}$ of some smooth singular foliation $\mathcal{F}$ of M.

Proposition 3.7.11. *Let $(M, \mathcal{P})$ be a Poisson manifold. Let $x \in M \mapsto S(x)$ be its characteristic distribution. Then it is integrable and tangent to a smooth singular foliation $\mathcal{F}_{\mathcal{P}}$. Such foliation is called the* ***symplectic foliation of a Poisson manifold****.*

Proof. According to (3.7.10) it is sufficient to find $C \subset \mathfrak{X}(M)$, such that the characteristic distribution is generated by C, and verify that it is invariant with respect to C. Clearly we take C as the subalgebra of Hamiltonian fields (see lemma 3.4.4). Characteristic distribution is then by definition generated by C.

It remains to prove the invariance. Let $x \in M$, $S(x)$ is the characteristic space at x. This space is spanned by vectors $\zeta_f(x) \equiv \mathcal{P}(df, \cdot)(x)$, where $f \in C^\infty(M)$. It is then sufficient to prove that $\phi^g_{t*}(\zeta_f(x)) \in S(\phi^g_t(x))$ for all $f, g \in C^\infty(M)$, where ϕ^g_t is the local flow corresponding to ζ_g.

From the remark 3.4.3 we know that $\phi^g_{t*}(\mathcal{P}(x)) = \mathcal{P}(\phi^g_t(x))$. Let $\alpha \in T^*_{\phi^g_t(x)}(M)$. Then

$$\langle \phi^g_{t*}(\zeta_f(x)), \alpha \rangle = \langle \phi^g_{t*}\big(\mathcal{P}(x)(\,df|_x, \cdot)\big), \alpha \rangle = \mathcal{P}(x)(\,df|_x, \phi^{g*}_t(\alpha)\rangle = \otimes.$$

Since we have $df|_x = \phi^{g*}_t\big(\, d(f \circ \phi^g_{-t})\big|_{\phi^g_t(x)}\big)$, we can write

$$\otimes = \big(\phi^g_{t*}(\mathcal{P}(x))\big)(\,d(f \circ \phi^g_{-t})\big|_{\phi^g_t(x)}, \alpha) = \mathcal{P}(\phi^g_t(x))(\,d(f \circ \phi^g_{-t})\big|_{\phi^g_t(x)}, \alpha).$$

Hence

$$\phi^g_{t*}(\zeta_f(x)) = \zeta_{(f \circ \phi^g_{-t})}(\phi^g_t(x)) \in S(\phi^g_t(x)),$$

which was to be proved. ■

To give a meaning to the name "symplectic foliation", we present a theorem of Alan Weinstein [2], which says that locally every Poisson manifold is a Cartesian product of a symplectic leaf with a complementary Poisson manifold.

Theorem 3.7.12 (Splitting theorem). *Let $x \in M$ be a point of an n-dimensional Poisson manifold $(M, \mathcal{P})$. The rank of $\mathcal{P}$ at x is $\rho_{\mathcal{P}}(x) = 2s$. Let $\mathcal{F}_x$ be a leaf of $\mathcal{F}_{\mathcal{P}}$ containing x.*

Then there exists an $(n-2s)$-dimensional submanifold N of M, which is transversal to the characteristic space $S(x)$, and a neighbourhood U_x of x equipped with the coordinates

$$(x^1, \ldots, x^s, p_1, \ldots, p_s, z^1, \ldots, z^{(n-2s)}),$$

such that

a) $p_i(N \cap U_x) = x^i(N \cap U_x) = 0$.

b) $z^i(\mathcal{F}_x \cap U_x) = 0$.

c) $\{x^i, x^j\}(U_x) = 0$, $\{p_i, p_j\}(U_x) = 0$, $\{p_i, x^j\}(U_x) = \delta^i_j$.

d) $\{z^i, p_j\}(U_x) = \{z^i, x^j\}(U_x) = 0$.

e) $\{z^i, z^j\}(x) = 0$.

*These coordinates are called **local canonical coordinates**. Therefore for $f, g \in C^\infty(M)$ we have in these coordinates*

$$\{f, g\} = \sum_{i,j=1}^{m-2s} \frac{\partial f}{\partial z^i} \frac{\partial g}{\partial z^j} \{z^i, z^j\} + \sum_{i=1}^{s} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial x^i}. \tag{3.59}$$

Corollary 3.7.13. *For $\mathcal{P}$ a symplectic manifold ($\rho_{\mathcal{P}}(x) \equiv n$) we immediately get the well known Darboux theorem for symplectic manifolds, local canonical coordinates $(x^1, \ldots, x^s, p_1, \ldots, p_s)$ are then called **Darboux coordinates**. In these coordinates, we can write the Poisson bracket as*

$$\{f, g\} = \sum_{i=1}^{s} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial x^i}, \tag{3.60}$$

for all $f, g \in C^\infty(M)$. Thus on the symplectic manifold the Poisson bracket is locally always of the form (3.15).

Remark 3.7.14. It is important to observe that (in the notation given by the previous theorem) the functions $\{z^i, z^j\}$ *do not* depend on the coordinates p_i and q^j, respectively. This is the reason why we can consider N as the Poisson manifold "transversal" to $\mathcal{F}_x$.

Indeed, denote $h := \{z_i, z_j\}$. From the property d) in the splitting theorem and from the Jacobi identities for the Poisson bracket we get $\{p_k, h\} = 0$. Then from (3.59)

$$0 = \{p_k, h\} = \sum_{i,j=1}^{m-2s} \underbrace{\frac{\partial p_k}{\partial z^i}}_{=\,0} \frac{\partial h}{\partial z^j} \{z^i, z^j\} + \sum_{i=1}^{s} \underbrace{\frac{\partial p_k}{\partial p_i}}_{=\,\delta_{ik}} \frac{\partial h}{\partial x^i} - \frac{\partial h}{\partial p_i} \underbrace{\frac{\partial p_k}{\partial x^i}}_{=\,0} = \frac{\partial h}{\partial x^k}.$$

In the same way one can prove that $\frac{\partial h}{\partial p_k} = 0$.

This finally gets us to the important statement for Poisson manifolds, saying that every Poisson manifold can be viewed as "glued from" symplectic submanifolds, of which symplectic structure fully determines the global Poisson structure.

Proposition 3.7.15. *Let $(M, \mathcal{P})$ be a Poisson manifold. Then every leaf $\mathcal{F}_x$ of the symplectic foliation $\mathcal{F}_\mathcal{P}$ is an immersed symplectic Poisson submanifold of M. The Poisson structure is determined by the symplectic structures on the leaves of $\mathcal{F}_\mathcal{P}$.*

Proof. Let $x \in M$, $\mathcal{F}_x$ be a leaf of dimension $2s$ containing x. Let $(x^1, \dots, x^s, p_1, \dots, p_s, z^1, \dots, z^{n-2s})$ be local canonical coordinates in some neighbourhood of x. Then we define a (symplectic) Poisson structure on $\mathcal{F}_x$ as

$$\{f,g\}_{\mathcal{F}_x} := \sum_{i=1}^{s} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial x^i}.$$

for $f, g \in C^\infty(\mathcal{F}_x)$. Now we have to prove that it is indeed a Poisson submanifold. Let us take $f, g \in C^\infty(M)$. Then for every $y \in \mathcal{F}_x$ we can write

$$\{f,g\}(y) \overset{(3.59)}{=} \sum_{i,j=1}^{m-2s} \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}\{z^i, z^j\}(y) + \{f|_{\mathcal{F}_x}, g|_{\mathcal{F}_x}\}_{\mathcal{F}_x}(y) =$$

$$\frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}\underbrace{\{z^i, z^j\}(x)}_{=\,0} + \{f|_{\mathcal{F}_x}, g|_{\mathcal{F}_x}\}_{\mathcal{F}_x}(y) = \{f_{\mathcal{F}_x}, g|_{\mathcal{F}_x}\}_{\mathcal{F}_x}(y).$$

Thus we have proven that the immersion map is indeed a Poisson map and moreover, the definition of a symplectic structure on $\mathcal{F}_x$ does not depend on the choice of the local canonical coordinates. The rest of the statement is just the conclusion of the previous statements in this section. ■

Remark 3.7.16. It is clear from the definition, that every symplectic leaf can be described as the subset of M, of which every two points are piecewise connected by integral curves of Hamiltonian vector fields.

Example 3.7.17. We shall try to find the symplectic leaves of the linear Poisson structure from example 3.3.4. It will turn out that they are nothing but the orbits of the right coadjoint action of the corresponding Lie group.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. $\mathcal{P} \in L_2(\mathfrak{g}^*)$ is a linear Poisson structure on $\mathfrak{g}^*$. Let $(T_i)_{i=1}^n$ be an arbitrary basis of $\mathfrak{g}$, serving as global coordinates on $\mathfrak{g}^*$. Denote ${c_{ij}}^k$ the structure constants in this basis, i.e. $[T_i, T_j] = {c_{ij}}^k T_k$.

Let us define a right action of group G on $\mathfrak{g}^*$, $g \in G \mapsto Ad_g^* \in Aut(\mathfrak{g}^*)$, as the transpose map of the adjoint representation of G :

$$\langle Ad_g^*(\xi), X\rangle := \langle \xi, Ad_g(X)\rangle, \tag{3.61}$$

for all $g \in G$, $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. It is easy to verify that it is indeed a right action of the group G.

We want to show that orbits of this action coincide with the symplectic leaves of the linear Poisson structure on $\mathfrak{g}^*$. To do this, it is sufficient to show that the generators (fundamental fields) of the right action Ad^* coincide with the Hamiltonian fields. Generators form the tangent distribution to the orbits, as the Hamiltonian fields do so for the symplectic leaves.

Let $V_Z(\xi)$ be the generator corresponding to $Z \in \mathfrak{g}$ at the point $\xi \in \mathfrak{g}^*$. From the definition of the group action generator, we get

$$V_Z(\xi) = \frac{d}{dt}\bigg|_{t=0} Ad^*_{e^{tZ}}(\xi).$$

In particular, we want the components of $V_{T_k}(T^l) = [V_{T_k}(T^l)]_m \left.\frac{\partial}{\partial T_m}\right|_{T^l}$. Hence by definition

$$[V_{T_k}(T^l)]_m = \frac{d}{dt}\bigg|_{t=0} \langle T_m, Ad^*_{e^{tT_k}}(T^l)\rangle = \langle \frac{d}{dt}\bigg|_{t=0} Ad_{e^{tT_k}}(T_m), T^l\rangle = {c_{km}}^l.$$

We can now look for the components of the Hamiltonian vector field ζ_{T_k} at the point $T^l \in \mathfrak{g}^*$ of the same coordinate expansion:

$$[\zeta_{T_k}(T^l)]_m = \mathcal{P}_{km}(T^l) = \langle T^l, [T_k, T_m] \rangle = c_{km}{}^l.$$

Whence $V_{T_k} \equiv \zeta_{T_k}$ and since the both assignments $\mathfrak{g} \to \mathfrak{X}(\mathfrak{g}^*)$ are linear, we have just proved our statement.

3.8 Poisson-Lie groups

A concept of Lie group interconnects the abstract world of the group theory with the geometric world of differentiable manifolds. It merges the given group with a differentiable manifold in a compatible way, demanding group operations to be the smooth maps. Tools of differential geometry can be then used to study the group structure and vice versa.

We would like to add the third structure, a Poisson bracket, into a Lie group G. Properties of this bracket have to be of course somehow connected with the group multiplication. As we have seen, in the theory of Poisson manifolds appears the concept of "bracket preserving" Poisson maps. It is natural then to demand for the group multiplication map to be a Poisson map (with respect to Poisson brackets involved).

Definition 3.8.1. A Lie group G, which is also a Poisson manifold, is called a **Poisson-Lie group**, if the group multiplication map $\mu : G \times G \to G$ is a Poisson map, considering $G \times G$ endowed with the product Poisson structure (3.57).

A homomorphism of Poisson-Lie groups is a homomorphism of Lie groups, which is also a Poisson map.

The inversion map of a Poisson-Lie group *is not* a Poisson map in general, as same as the left or right translation on a group.

The condition on the multiplication map μ can be rewritten in a very elegant way using the Poisson bivector P on G. The Poisson map condition (3.55) reads

$$\{f_1, f_2\}(gh) = \{f_1 \circ L_g, f_2 \circ L_g\}(h) + \{f_1 \circ R_h, f_2 \circ R_h\}(g)$$

for all $f_1, f_2 \in C^\infty(G), g, h \in G$. If we rewrite this relation using the Poisson bivector $\mathcal{P}$ we have

$$\mathcal{P}(gh)((df_1)|_{gh}, (df_2)|_{gh}) = \mathcal{P}(h)(d(f_1 \circ L_g)|_h, d(f_2 \circ L_g)|_h) + \mathcal{P}(g)(d(f_1 \circ R_h)|_g, d(f_2 \circ R_h)|_g).$$

This can be rewritten as

$$\mathcal{P}(gh) = L_{g*}(\mathcal{P}(h)) + R_{h*}(\mathcal{P}(g)), \tag{3.62}$$

for all $g, h \in G$. Such bivector fields are called multiplicative. They have many interesting properties, as we will see in the chapter 4.

Proposition 3.8.2. *The rank of the Poisson structure of a Poisson-Lie group is always zero at the unit element e of the group G, i.e. $\rho_{\mathcal{P}}(e) = 0$. Poisson-Lie group is thus never a symplectic manifold.*

Proof. Put $g = h = e$ into (3.62). ■

Example 3.8.3. The zero Poisson bivector (we denote it as Θ) on arbitrary Lie group G is always multiplicative, making (G, Θ) into a Poisson-Lie group.

Example 3.8.4. Lie-Poisson structure (see example 3.3.4), where the vector space $\mathfrak{g}^*$ is viewed as the Abelian group, is a Poisson-Lie group. The condition (3.62) reads

$$\mathcal{P}(\xi + \eta) = L_{\xi *}(\mathcal{P}(\eta)) + R_{\eta *}(\mathcal{P}(\xi)), \tag{3.63}$$

for all $\xi, \eta \in \mathfrak{g}^*$. Let us choose a basis $(T_i)_{i=1}^n$ and use it as global coordinates on $\mathfrak{g}^*$. Let us denote $[T_i, T_j]_{\mathfrak{g}} = c_{ij}{}^k T_k$ the structure coefficients in this basis. Then the left hand side of (3.63) reads

$$\mathcal{P}(\xi + \eta) = (\xi_k + \eta_k) c_{ij}^k \left.\frac{\partial}{\partial T_i}\right|_{\xi+\eta} \otimes \left.\frac{\partial}{\partial T_j}\right|_{\xi+\eta}.$$

The first term on the right hand side can be written as

$$L_{\xi *}(\mathcal{P}(\eta)) = \eta_k c_{ij}{}^k L_{\xi *}(\left.\frac{\partial}{\partial T_i}\right|_{\eta}) \otimes L_{\xi *}(\left.\frac{\partial}{\partial T_j}\right|_{\eta}) = \eta_k c_{ij}{}^k \left.\frac{\partial}{\partial T_i}\right|_{\xi+\eta} \otimes \left.\frac{\partial}{\partial T_j}\right|_{\xi+\eta}.$$

The second term on the right hand side gives

$$R_{\eta *}(\mathcal{P}(\xi)) = \xi_k c_{ij}{}^k \left.\frac{\partial}{\partial T_i}\right|_{\xi+\eta} \otimes \left.\frac{\partial}{\partial T_j}\right|_{\xi+\eta}.$$

The condition (3.63) is thus satisfied.

More examples are given at the end of the chapter 4. We will in fact give a recipe how to construct *every possible* connected and simply connected Poisson-Lie group.

Chapter 4

Relation between Lie bialgebras and Poisson-Lie groups

4.1 Introduction

In this chapter we will in fact connect the preceding two chapters. The tangent structure at the unit of the Poisson-Lie group receives for free an additional structure, so called intrinsic derivative of the Poisson bivector $\mathcal{P}$. It will turn out that it is nothing but the Lie bialgebra cocommutator on a Lie algebra $\mathfrak{g}$.

The opposite direction is far more interesting. We can start from the Lie bialgebra $(\mathfrak{g}, \delta)$ and construct the (unique) Poisson-Lie group $(G, \mathcal{P})$, such that $\mathfrak{g}$ is its Lie algebra and the intrinsic derivative of $\mathcal{P}$ gives δ. This can be done in an amazingly simple way, using nothing but the adjoint representation of the group G.

This chapter originated from the footnote in the article of Ctirad Klimčík and Pavol Ševera [13], which claims the matrix ba^{-1}, where b, a are certain submatrices of the adjoint representation, to be a Poisson-Lie structure on the Lie group G. Using this and the definition of the Poisson-Lie group, we have successfully constructed a multiplicative bivector field on G, as is shown in the sections 4.2, 4.8 and 4.5.

Solving the technical difficulties during the computation of its Schouten-Nijenhuis bracket led us to the work of Alan Weinstein and Jiang-Hua Lu [14], where they develop the theory of multiplicative tensor fields, introduce the intrinsic derivative and give the proof of the important proposition 4.3.9. We introduce excerpts from their work in the section 4.3, including the proofs of all propositions (some of them are not present in the original article).

Using the properties of multiplicative tensor fields, we were able to prove that we have indeed discovered the Poisson-Lie bivector of the Poisson-Lie group G corresponding to the Lie bialgebra $(\mathfrak{g}, \delta)$, calculating its intrinsic derivative and the Schouten-Nijenhuis bracket.

In the section 4.8 we were then able to state and prove the most important theorem of this chapter, relating Lie bialgebras and Poisson-Lie groups (which was already well known), giving moreover a direct way to compute the Poisson bivector (for a general Lie bialgebra). However, as we have recognized later, this has been already done in a quite recent book of Jean-Paul Dufour and Nguyen Tien Zung [11].

In the section 4.9 we introduce the most known way of Poisson bivector construction for coboundary Lie bialgebras, called the Sklyanin bracket. We use it to do a very important observation, described in the example (4.9.4). Although it is not mentioned in most of the articles and books, it can be again (without proof) found in [11].

4.2 Notation

We have shown in the section 2.4 that every n-dimensional Lie bialgebra $(\mathfrak{g}, \delta)$ corresponds to the Manin triple $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$, where $\mathfrak{d}$ is the $2n$-dimensional Lie algebra, equipped with a symmetric, non-degenerate ad-invariant bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, the $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ are the subalgebras of $\mathfrak{d}$ and isotropic subspaces with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$.

Moreover $\mathfrak{d}$ is the direct sum of the subspaces $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$, that is $\mathfrak{d} = \mathfrak{g} \oplus \widetilde{\mathfrak{g}}$.

We define D to be a connected and simply connected Lie group, such that $\mathfrak{d}$ is its Lie algebra. Then $G, \widetilde{G}$ are the (connected and simply connected) Lie subgroups of D corresponding to the subalgebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$. Lie group D is called a **Drinfel'd double**.

We will construct a Poisson-Lie structure on the Lie group G. First we have to deal with some technical details.

We consider the basis $(T_i, \widetilde{T}^j)_{i,j=1}^{n,n}$ of $\mathfrak{d}$, where $(T_i)_{i=1}^n$ is the basis of $\mathfrak{g}$ and $(\widetilde{T}^j)_{j=1}^n$ is the basis of $\widetilde{\mathfrak{g}}$, such that

$$\langle T_i, \widetilde{T}^j \rangle_{\mathfrak{d}} = \delta_i^j. \tag{4.1}$$

Because of that, we can use $\widetilde{T}^j$ instead of the "real" dual basis $(T^j)_{i=1}^n$ in $\mathfrak{g}^*$, that is

$$T^i(T_j) \equiv \langle T^i, T_j \rangle = \langle \widetilde{T}^i, T_j \rangle_{\mathfrak{d}} = \delta_j^i,$$

and in the same manner for T_i and $\widetilde{T}_i$. Let us remark that $\langle \cdot, \cdot \rangle$ always denotes the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

We denote $(X_i)_{i=1}^{2n} = (T_i, \widetilde{T}^j)_{i,j=1}^{n,n}$. Let $L : \mathfrak{d} \to \mathfrak{d}$ be a linear operator on $\mathfrak{d}$. We can define its matrix $L_{\mathcal{X}}$ in the basis (X_i) as

$$(L_{\mathcal{X}})^i{}_j := X^i(L(X_j)) \equiv \langle X^i, L(X_j) \rangle,$$

where (X^i) is the dual basis to (X_i).

Let $A : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be a linear map. We define an $n \times n$ matrix $A_{\mathcal{X}}$ as

$$(A_{\mathcal{X}})^{ij} := \langle \widetilde{T}^i, A(T^j) \rangle_{\mathfrak{d}} \equiv \langle T^i, A(\widetilde{T}^j) \rangle.$$

In the same way we define the following matrices (all maps are considered linear)

$$B : \mathfrak{g} \to \widetilde{\mathfrak{g}},\ (B_{\mathcal{X}})_{ij} := \langle T_i, B(T_j) \rangle_{\mathfrak{d}},$$

$$C : \mathfrak{g} \to \mathfrak{g},\ (C_{\mathcal{X}})^i{}_j := \langle \widetilde{T}^i, C(T_j) \rangle_{\mathfrak{d}},$$

$$D : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}, (D_{\mathcal{X}})_i{}^j := \langle T_i, D(\widetilde{T}^j) \rangle_{\mathfrak{d}}.$$

To avoid confusions we declare that a left matrix index always counts rows and a right matrix index always counts columns.

Before we proceed to the definition of the Poisson bivector $\mathcal{P}$ on G, we have to do a more detailed examination of the structures playing the main role, multiplicative tensor fields.

4.3 Multiplicative tensor fields

We have seen in the chapter 3 that we are interested in the bivector fields satisfying

$$\mathcal{P}(gh) = L_{g*}(P(h)) + R_{h*}(P(g)).$$

We can of course demand this property for an arbitrary tensor field on a Lie group. It turns out that these tensor fields, we would call them multiplicative, have many interesting properties, which would be very useful in the proof of the main theorem of this chapter.

Definition 4.3.1. Let K be a tensor field on G. We call K a **multiplicative tensor field** on G if

$$K(gh) = L_{g*}(K(h)) + R_{h*}(K(g)), \tag{4.2}$$

for every $g, h \in G$.

Remark 4.3.2. Putting $g = h = e$ into (4.2) we find the important property of multiplicative tensor fields,

$$K(e) = 0. \tag{4.3}$$

Therefore every multiplicative tensor field vanishes at the unit of the group G.

From this remark we can see that a non-trivial multiplicative tensor field is never left-invariant. It would be then generated by its value at the unit of the group, that is always a zero tensor, and therefore identically zero. By the same arguments it is neither a right-invariant tensor field.

Although the multiplicative tensor cannot be constructed from its value at e, it can be constructed (on a connected Lie group) from the arbitrarily small neighbourhood of the unit element. To justify this statement, let us prove a very important property of a connected Lie group.

Proposition 4.3.3. *Let G be a connected Lie group. Let A be an arbitrary open subset of G containing the unit e of G.*

Then A generates G, that is $G = \langle A \rangle$, where

$$\langle A \rangle = \{a_1^{k_1} a_2^{k_2} \dots a_n^{k_n} | a_1, a_2, \dots, a_n \in A;\ k_1, k_2, \dots, k_n \in \{-1, +1\}; n \in \mathbb{N}\}.$$

In the other words, every element $g \in G$ can be written as a finite product of the elements of A (and their inversions). We should remark that in general it is not unique.

Proof. Let $g \in \langle A \rangle$. We suppose that A is open and thus there exists a neighbourhood U_e of $e \in A$, such that $U_e \subset A$. From the definition of Lie group we know that $U_g := L_g(U_e)$ is a neighbourhood of g. By definition

$$U_g = \{gh | h \in U_e \subset A\},$$

and therefore $U_g \subset \langle A \rangle$. Thus we have just found that $\langle A \rangle$ is an open subset of G.

Again by definition of G, we find that the coset $g\langle A \rangle \equiv L_g(\langle A \rangle)$ is open for arbitrary $g \in G$. We can write

$$\langle A \rangle = G - (\bigcup_{\substack{g \in G \\ g\langle A \rangle \neq \langle A \rangle}} g\langle A \rangle),$$

and hence $\langle A \rangle$ is a closed subset of G.

Finally, because G is connected and $\langle A \rangle \neq \emptyset$, we get $G = \langle A \rangle$. ∎

Now it is clear that on a connected Lie group G we can use the multiplicativity of the tensor field to find its values on the whole group G just from the knowledge of its values in some (arbitrarily small) neighbourhood of the unit of the group.

Multiplicative tensor fields themselves are never left-invariant. This does not hold for their change in the direction of left-invariant vector fields. Lie derivative of the multiplicative tensor field along every left-invariant vector field is always left-invariant. Interesting fact is that on connected Lie groups there holds an opposite statement. Tensor field with left-invariant Lie derivative along arbitrary left-invariant vector field, vanishing at the unit of G, is necessarily a multiplicative tensor field. Thus on connected Lie groups we get the powerful tool for recognizing the multiplicative tensor fields.

Lemma 4.3.4. *Let K be a tensor field on a connected Lie group G.*

K is a multiplicative tensor field on G (i.e. it satisfies (4.2)), if and only if

$K(e) = 0$ and $\mathcal{L}_V(K)$ is a left-invariant tensor field for all left-invariant vector fields V.

Proof. Implication $\Rightarrow$:

To see that $K(e) = 0$ it is enough to put $g = h = e$ into (4.2).

Let V be an arbitrary left-invariant vector field. By definition we can write $V(g) = L_{g*}(X)$, where $X \in \mathfrak{g}$. If we denote ϕ_t the flow belonging to V, there holds that $\phi_t(g) = ge^{tX} = R_{e^{tX}}(g)$. Then we can write

$$[\mathcal{L}_V(K)](gh) = \frac{d}{dt}\bigg|_{t=0} \phi_{-t*}(K(\phi_t(gh))) = \frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}(K(ghe^{tX})) \overset{(4.2)}{=}$$

$$\overset{(4.2)}{=} \frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}\big(L_{g*}(K(he^{tX})) + R_{he^{tX}*}(K(g))\big) = L_{g*}\frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}(K(he^{tX})) +$$

$$+ \frac{d}{dt}\bigg|_{t=0} R_{h*}(K(g)) = L_{g*}([\mathcal{L}_V(K)](h)),$$

thus $\mathcal{L}_V(K)$ is the left-invariant tensor field on G.

Implication $\Leftarrow$:

We can rewrite the left-invariance condition for $\mathcal{L}_V(K)$, where V is the left-invariant vector field generated by $X \in \mathfrak{g}$ as

$$\frac{d}{dt} R_{e^{-tX}*}(K(ghe^{tX})) = \frac{d}{dt} L_{\mathfrak{g}*}(R_{e^{-tX}*}(K(he^{tX}))).$$

Indeed,

$$\frac{d}{dt} R_{e^{-tX}*}(K(ghe^{tX})) = \frac{d}{ds}\bigg|_{s=0} R_{e^{-(t+s)X}*}(K(ghe^{(t+s)X})) =$$

$$= R_{e^{-tX}*}(\frac{d}{ds}\bigg|_{s=0} R_{e^{-sX}*}(K(ghe^{tX}e^{sX}))) = R_{e^{-tX}*}([\mathcal{L}_V(K)](ghe^{tX})) =$$

$$= R_{e^{-tX}*}(L_{g*}([\mathcal{L}_V(K)](he^{tX}))) = L_{g*}(R_{e^{-tX}*}(\frac{d}{ds}\bigg|_{s=0} R_{e^{-sX}*}(K(he^{tX}e^{sX})))) =$$

$$= \frac{d}{ds}\bigg|_{s=0} L_{g*}(R_{e^{-(t+s)X}*}(K(he^{(t+s)X}))) = \frac{d}{dt} L_{g*}(R_{e^{-tX}*}(K(he^{tX}))).$$

Now we can integrate it with respect to t to get

$$R_{e^{-tX}*}(K(ghe^{tX})) = L_{g*}(R_{e^{-tX}*}(K(he^{tX}))) + const(g,h), \tag{4.4}$$

for all $g, h \in G$ and all $X \in \mathfrak{g}$. For h from the range of the exponential map $exp(\mathfrak{g})$ we can choose $X \in \mathfrak{g}$, such that $h = e^{-tX}$. Putting this into (4.4) and using $K(e) = 0$ we get

$$R_{h*}(K(g)) = L_{g*}(R_{h*}(K(e))) + const(g,h) = const(g,h).$$

Now if we take $t = 0$ in (4.4) we get

$$K(gh) = L_{g*}(K(h)) + R_{h*}(K(g)).$$

If we now act on the both sides of (4.4) with $R_{e^{tX}*}$, we get

$$const(g, he^{tX}) = R_{e^{tX}*}(const(g,h)). \tag{4.5}$$

We have supposed that G is a connected Lie group. We can then use (4.3.3) by taking A as some open subset of the $exp(\mathfrak{g})$ and use the relation (4.5) to find $const(g,h) = R_{h*}(K(g))$ for all $g, h \in G$.

■

Remark 4.3.5. There holds also that K is the multiplicative tensor field on G if and only if $K(e) = 0$ and $\mathcal{L}_V(K)$ is right-invariant for all right-invariant vector fields V. The proof is completely the same.

It is a well known fact that a smooth vector field commutes with all left-invariant vector fields if and only if it is a right-invariant vector field. This can be observed as the vanishing of its Lie derivative along all left-invariant vector fields. But this fact can be considered as obvious, because the flow generated by a left-invariant vector field is just the right translation on the group. Lie derivative measures the change of the vector field under the action of this flow, therefore it should be zero by the definition of the right-invariant vector field.

One would expect this behaviour even for general tensor fields on a Lie group, which is stated and proved in the following lemma.

Lemma 4.3.6. *Let K be a tensor field on a connected Lie group G.*

Then K is right-invariant, if and only if $\mathcal{L}_V(K) = 0$ for every left-invariant vector field V.

Proof. Implication $\Rightarrow$:

We can write $K(g) = R_{g*}(K(e))$. Let V be an arbitrary left-invariant vector field. Hence $V = L_X$ for some $X \in \mathfrak{g}$. Then we can write

$$[\mathcal{L}_V(K)](g) = \frac{d}{dt}\bigg|_{t=0} \phi_{-t*}(K(\phi_t(g))) = \frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}(R_{ge^{tX}*}(K(e))) = \frac{d}{dt}\bigg|_{t=0} R_{g*}(K(e)) = 0.$$

Implication $\Leftarrow$:

Let $V(g) = L_{g*}(X)$. Vanishing of the Lie derivative of K along V at $g \in G$ means that

$$\frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}\big(K(R_{e^{tX}}(g))\big) = 0.$$

Then

$$\frac{d}{dt} R_{e^{-tX}*}\big(K(R_{e^{tX}}(g))\big) = R_{e^{-tX}*} \frac{d}{ds}\bigg|_{s=0} R_{e^{-sX}*}\big(K(R_{e^{sX}}(R_{e^{tX}}(g))\big) = 0.$$

If we integrate this identity, we get

$$R_{e^{-tX}*}\big(K(R_{e^{tX}}(g))\big) = K(g).$$

Thus

$$K(R_h(g)) = R_{h*}\big(K(g)\big),$$

for all $g \in G$ and $h \in exp(\mathfrak{g})$. We can now use the connectedness of the group G and the proposition 4.3.3 for open $A \subset exp(\mathfrak{g})$, $e \in A$, to prove the right-invariance on the whole group G by induction.

■

Remark 4.3.7. Again we can prove that K is the left-invariant tensor field if and only if $\mathcal{L}_V(K) = 0$ for all right-invariant vector fields V.

A crucial property of left-invariant vector fields is the left-invariance of their commutator. We find that left-invariant vector fields constitute a finite-dimensional Lie subalgebra of the infinite-dimensional Lie algebra of all vector fields on a Lie group G.

It is easy to extend this property to all left-invariant multivector fields and their Schouten-Nijenhuis bracket (see section 3.2), i.e. the Schouten-Nijenhuis bracket of two left-invariant multivector fields is a left-invariant multivector field. We leave the proof upon the reader, the proof is the same as in the well-known vector field case.

The interesting fact is that the same thing can be said about multiplicative multivector fields. They form a (infinite-dimensional) subalgebra in $\mathbb{Z}$-graded multivector field algebra (induced by the Schouten-Nijenhuis bracket). Beware that in general the exterior product of two multiplicative multivector fields is *not* multiplicative. This is a significant difference compared to left-invariant tensor fields.

Lemma 4.3.8. *Let K, L be multiplicative multivector fields on a connected Lie group G. Then their Schouten-Nijenhuis bracket $[K, L]$ is also a multiplicative multivector field.*

Proof. Let us recall that for $\mathfrak{X}(G)$ and $X \in L(G)$ we have $[V, X] = \mathcal{L}_V(X)$.

Let us choose V (W) to be an arbitrary left-(right-)invariant vector field. Using (3.11) we get the relation

$$\mathcal{L}_V([K, L]) = [\mathcal{L}_V(K), L] + [K, \mathcal{L}_V(L)],$$

and therefore

$$\begin{aligned}\mathcal{L}_W(\mathcal{L}_V([K, L])) = [\mathcal{L}_W(\mathcal{L}_V(K), L] + [\mathcal{L}_V(K), \mathcal{L}_W(L)] + \\ +[\mathcal{L}_W(K), \mathcal{L}_V(L)] + [K, \mathcal{L}_W(\mathcal{L}_V(L))].\end{aligned}$$

By preceding lemmas we know that $\mathcal{L}_V(K)$ and $\mathcal{L}_V(L)$ are left-invariant and $\mathcal{L}_W(K)$ and $\mathcal{L}_W(L)$ are right-invariant vector fields on G. Let us note that Schouten-Nijenhuis bracket of the left-invariant multivector field and the right-invariant multivector field always vanishes. This can by proven for example by writing these multivector fields in the left-invariant or right-invariant basis respectively.

Hence by lemma 4.3.6 and this generalized commutativity between left-invariant and right-invariant multivector fields all the terms vanish. Hence we have

$$\mathcal{L}_W(\mathcal{L}_V([K, L])) = 0.$$

Using the lemma 4.3.6 we get that $\mathcal{L}_V([K, L])$ is a left-invariant tensor field. One finds quickly that $K(e) = L(e) = 0$ implies $[K, L](e) = 0$. Thus by lemma 4.3.4 $[K, L]$ is a multiplicative multivector field. ■

We have already examined that multiplicative tensor fields are given by their value in the arbitrarily small neighbourhood of the unit of the group. Thus we could expect that there is some important information stored in the infinitesimal change of the multiplicative tensor field when we go away from the unit of the group. This expectation is correct, as shows the following proposition:

Proposition 4.3.9. *Let K be a multiplicative tensor field on a connected Lie group G. Then $[\mathcal{L}_V(K)](e) = 0$ for all left-invariant vector fields V, if and only if $K \equiv 0$.*

Proof. Implication $\Rightarrow$:

From lemma 4.3.4 we get that $\mathcal{L}_V(K)$ is a left-invariant tensor field. Hence

$$[\mathcal{L}_V(K)](g) = L_{g*}([\mathcal{L}_V(K)](e)) = 0.$$

From lemma 4.3.6 we know that K is then a right-invariant tensor field. From the multiplicativity we have $K(e) = 0$ and thus

$$K(g) = R_{g*}(K(e)) = 0.$$

The converse of the equivalence is trivial.

■

Remark 4.3.10. In fact, this proposition says that $DK \equiv 0 \Leftrightarrow K \equiv 0$. Cf. section 4.6.

4.4 Definition and the skew-symmetry of $\Pi(g)$

We can now return to the Drinfel'd double D and its subgroup G. Recall that D (and so also G) is connected and simply connected, thus we can use all the propositions from the preceding section.

At the start of our task we define the map $\Pi(g)$ between the subalgebras $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ as was done in [13]. We will show that it is equivalent to the form written in [15] and it is the skew-symmetric map with respect to the bilinear form of Drinfel'd double D.

When talking about a matrix in a basis, we always think arbitrary (but fixed) basis of $\mathfrak{d}$ (4.1).

For arbitrary $g \in G$ we can write the matrix of the adjoint representation $Ad_{g^{-1}}$ in the form:

$$(Ad_{g^{-1}})_{\mathcal{X}} = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix}, \tag{4.6}$$

where $a(g)$, $b(g)$, $d(g)$ are G-dependent $n \times n$ matrices. In the other words $a : G \to \mathbb{R}^{n,n}$ and in the same way for b and d. A zero in the bottom left corner is the consequence of G being a subgroup of D.

From the properties of Ad one simply gets that

$$(Ad_g)_{\mathcal{X}} = \begin{pmatrix} a(g)^{-T} & -a(g)^{-T} b(g)^T d(g)^{-T} \\ 0 & d(g)^{-T} \end{pmatrix}, \tag{4.7}$$

The equation (6) of [13] for given $g \in G$ defines the linear map $\Pi(g) : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ using its matrix in a basis as

$$(\Pi(g))_{\mathcal{X}} = b(g)a(g)^{-1}. \tag{4.8}$$

One easily finds that

$$b(g)^T = (P Ad_{g^{-1}} \widetilde{P})_{\mathcal{X}},$$
$$a(g)^{-T} = (P Ad_g P)_{\mathcal{X}},$$

where P and $\widetilde{P}$ are the projectors on $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ respectively. Using this we can rewrite the definition (4.8) as

$$(\Pi(g))_{\mathcal{X}} = b(g)a(g)^{-1} = (a(g)^{-T} b(g)^T)^T = ((P Ad_g P Ad_{g^{-1}} \widetilde{P})_{\mathcal{X}})^T.$$

On a connected group (e.g. Drinfel'd double D) the ad-invariance of symmetric bilinear forms coincides with their Ad-invariance, which is stated in the following lemma:

Lemma 4.4.1. *Let G be a connected Lie group G with the Lie algebra $\mathfrak{g}$. $\langle\cdot,\cdot\rangle_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ is a symmetric bilinear form on $\mathfrak{g}$. Then*

$$(\forall X, Y \in \mathfrak{g})(\forall g \in G)\ (\langle Ad_g(X), Ad_g(Y)\rangle_{\mathfrak{g}} = \langle X, Y\rangle_{\mathfrak{g}}),$$

if and only if

$$(\forall X, Y, Z \in \mathfrak{g})\ (\langle ad_Z(X), Y\rangle_{\mathfrak{g}} + \langle X, ad_Z(Y)\rangle_{\mathfrak{g}} = 0).$$

That is ad-invariant bilinear form is Ad-invariant and vice versa.

Proof. Implication $\Rightarrow$:

We take arbitrary basis (T_k) in $\mathfrak{g}$ and $g = e^{tZ}$ for $Z \in \mathfrak{g}$. We can then rewrite the presumption as

$$X^kY^n((Ad_{e^{tZ}})_{\mathcal{X}})^l{}_k((Ad_{e^{tZ}})_{\mathcal{X}})^m{}_n\langle T_l, T_m\rangle_{\mathfrak{g}} = X^lY^m\langle T_l, T_m\rangle_{\mathfrak{g}},$$

where $X = X^kT_k, Y = Y^nT_n$. We can differentiate this identity with respect to t at $t = 0$ to get

$$X^kY^n((ad_Z)_{\mathcal{X}})^l{}_k\delta^m_n\langle T_l, T_m\rangle_{\mathfrak{g}} + X^kY^n\delta^l_k((ad_Z)_{\mathcal{X}})^m{}_n\langle T_l, T_m\rangle_{\mathfrak{g}} = 0. \tag{4.9}$$

This is the right hand side of the equivalence in the lemma statement.

Implication $\Leftarrow$:

Let $\widetilde{X}, \widetilde{Y}, Z \in \mathfrak{g}$. Starting from (4.9) we have

$$\frac{d}{dt}\bigg|_{t=0}\langle Ad_{e^{tZ}}(\widetilde{X}), Ad_{e^{tZ}}(\widetilde{Y})\rangle_{\mathfrak{g}} = 0.$$

If we set $\widetilde{X} = Ad_{e^{sZ}}(X)$, $\widetilde{Y} = Ad_{e^{sZ}}(Y)$ for $X, Y \in \mathfrak{g}$, we have

$$\frac{d}{dt}\bigg|_{t=0}\langle Ad_{e^{(t+s)Z}}(X), Ad_{e^{(t+s)Z}}(Y)\rangle_{\mathfrak{g}} = 0.$$

and therefore

$$\frac{d}{dt}\langle Ad_{e^{tZ}}(X), Ad_{e^{tZ}}(Y)\rangle_{\mathfrak{g}} = 0.$$

We can integrate this identity to get

$$\langle Ad_{e^{tZ}}(X), Ad_{e^{tZ}}(Y)\rangle_{\mathfrak{g}} = \langle X, Y\rangle_{\mathfrak{g}}.$$

We have thus proven a left hand side of the equivalence of the lemma statement for all $g \in exp(\mathfrak{g})$. On a connected Lie group every element can be written as a finite product of such elements (4.3.3) and for $U \in \mathfrak{g}$ we can write

$$\langle Ad_{e^{tZ}e^{sU}}(X), Ad_{e^{tZ}e^{sU}}(Y)\rangle_{\mathfrak{g}} = \langle Ad_{e^{tZ}}(Ad_{e^{sU}}(X)), Ad_{e^{tZ}}(Ad_{e^{sU}}(Y))\rangle_{\mathfrak{g}} = \cdots = \langle X, Y\rangle_{\mathfrak{g}}.$$

The statement for all $g \in G$ can be then proven by induction.

■

We shall now prove the skew-symmetry of the matrix $a(g)^{-T}b(g)^T = (PAd_gPAd_{g^{-1}}\widetilde{P})_{\mathcal{X}}$. To do this we should remark that from the ad-invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ and preceding lemma we get

$$\langle Ad_g(X), Y\rangle_{\mathfrak{d}} = \langle X, Ad_{g^{-1}}(Y)\rangle_{\mathfrak{d}}, \tag{4.10}$$

for all $X, Y \in \mathfrak{d}$. By simple computation we get also

$$\langle P(X), Y\rangle_{\mathfrak{d}} = \langle X, \widetilde{P}(Y)\rangle_{\mathfrak{d}}, \tag{4.11}$$

for all $X, Y \in \mathfrak{d}$. Thus we can write

$$(a(g)^{-T}b(g)^T)^{ij} = ((PAd_gPAd_{g^{-1}}\widetilde{P})_{\mathcal{X}})^{ij} = \langle \widetilde{T}^i, PAd_gPAd_{g^{-1}}\widetilde{P}(\widetilde{T}^j)\rangle_{\mathfrak{d}} \overset{(4.10)(4.11)}{=}$$

$$\overset{(4.10)(4.11)}{=} \langle PAd_g\widetilde{P}\widetilde{P}Ad_{g^{-1}}\widetilde{P}(\widetilde{T}^i), \widetilde{T}^j\rangle_{\mathfrak{d}} = ((PAd_g\widetilde{P}\widetilde{P}Ad_{g^{-1}}\widetilde{P})_{\mathcal{X}})^{ji} = \otimes.$$

Because we have

$$(PAd_g\widetilde{P})_{\mathcal{X}} = -a(g)^{-T}b(g)^T d(g)^{-T},$$
$$(\widetilde{P}Ad_{g^{-1}}\widetilde{P})_{\mathcal{X}} = d(g)^T,$$

we can write

$$\otimes = ((PAd_g\widetilde{P})_{\mathcal{X}}(\widetilde{P}Ad_{g^{-1}}\widetilde{P})_{\mathcal{X}})^{ji} = -(a(g)^{-T}b(g)^T)^{ji},$$

The matrix $(\Pi(g))_{\mathcal{X}}$ is therefore skew-symmetric and moreover we can write

$$(\Pi(g))_{\mathcal{X}} = -a(g)^{-T}b(g)^T = (PAd_g\widetilde{P}Ad_{g^{-1}}\widetilde{P})_{\mathcal{X}}.$$

So we have just found the coordinate-free definition of the map $\Pi(g)$ (see [15], eq. (13))

$$\Pi(g) = PAd_g\widetilde{P}Ad_{g^{-1}}\widetilde{P}, \tag{4.12}$$

and we know that it is skew-symmetric with respect to the bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$, that is

$$\langle X, \Pi(g)(Y)\rangle_{\mathfrak{d}} = -\langle \Pi(g)(X), Y\rangle_{\mathfrak{d}},$$

for all $X, Y \in \widetilde{\mathfrak{g}}$ and all $g \in G$.

4.5 Using $\Pi(g)$ to define a multiplicative bivector field on G

Next step is to use the linear map $\Pi(g)$ to define a bivector field on G. Because we are looking for a Poisson-Lie group bivector, we would like it to be multiplicative.

Recall that for every Manin triple $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$ and the corresponding Lie bialgebra $(\mathfrak{g}, \delta)$ we have the Lie algebra isomorphism $\mathbf{P} : \widetilde{\mathfrak{g}} \to \mathfrak{g}^*$, such that $\mathbf{P}(\widetilde{T}^i) = T^i$ and

$$(\forall X, Y \in \widetilde{\mathfrak{g}})\ (\delta^*(\mathbf{P}(X), \mathbf{P}(Y)) = \mathbf{P}([X, Y]_{\mathfrak{d}})).$$

Using this isomorphism we can define the bivector field $\mathcal{P}$ on M as

$$\mathcal{P}(g)(\alpha_g, \beta_g) = \langle R_g^*(\beta_g), \Pi(g)\mathbf{P}^{-1}R_g^*(\alpha_g)\rangle, \tag{4.13}$$

for $g \in G$, α_g, β_g are the values of the arbitrary covector fields α, β at g and $R_g^* : T_g^*(M) \to \mathfrak{g}^*$ is the pullback generated by the right translation on G.

For $g, h \in G$ one finds by multiplying the matrices of $Ad_{g^{-1}}$ and $Ad_{h^{-1}}$, that

$$a(gh) = a(g)a(h),$$
$$b(gh) = b(g)a(h) + d(g)b(h),$$

and therefore

$$(\Pi(gh))_{\mathcal{X}} = (\Pi(g))_{\mathcal{X}} + d(g)(\Pi(h))_{\mathcal{X}}a(g)^{-1}. \tag{4.14}$$

Using the conjugation tricks again one can easily show that

$$a(g)^{-1} = (\widetilde{P}Ad_{g^{-1}}\widetilde{P})_{\mathcal{X}},$$
$$d(g) = (PAd_gP)_{\mathcal{X}},$$

and thus we finally get

$$\Pi(gh) = \Pi(g) + Ad_g\Pi(h)\widetilde{P}Ad_{g^{-1}}\widetilde{P}. \tag{4.15}$$

Proposition 4.5.1. *The bivector field defined by (4.13) is multiplicative, i.e. it satisfies the equation (4.2) for every $g, h \in G$.*

Proof. Let us take arbitrary group elements g, h. We can rewrite the left hand side of the equation (4.2) using the definition of $\mathcal{P}$ and (4.15) as

$$\mathcal{P}(gh)(\alpha_{gh}, \beta_{gh}) = \langle R^*_{gh}(\beta_{gh}), \Pi(gh)\mathbf{P}^{-1}R^*_{gh}(\alpha_{gh})\rangle =$$

$$\overset{(4.15)}{=} \underbrace{\langle R^*_{gh}(\beta_{gh}), \Pi(g)\mathbf{P}^{-1}R^*_{gh}(\alpha_{gh})\rangle}_{A} + \underbrace{\langle R^*_{gh}(\beta_{gh}), Ad_g\Pi(h)\widetilde{P}Ad_{g^{-1}}\widetilde{P}\mathbf{P}^{-1}R^*_{gh}(\alpha_{gh})\rangle}_{B}.$$

Now we rewrite the second term (easier one) on the right hand side of (4.2)

$$(R_{h*}(\mathcal{P}(g)))(\alpha_{gh}, \beta_{gh}) = \mathcal{P}(g)(R^*_h(\alpha_{gh}), R^*_h(\beta_{gh})) = \langle R^*_g R^*_h(\beta_{gh}), \Pi(g)\mathbf{P}^{-1}R^*_g R^*_h(\alpha_{gh})\rangle =$$

$$= \langle R^*_{gh}(\beta_{gh}), \Pi(g)\mathbf{P}^{-1}R^*_{gh}(\alpha_{gh})\rangle,$$

which is exactly the term A above. Now we take the first term on the right hand side of (4.2) and write

$$(L_{g*}(\mathcal{P}(h)))(\alpha_{gh}, \beta_{gh}) = \mathcal{P}(h)(L^*_g(\alpha_{gh}), L^*_g(\beta_{gh})) = \langle R^*_h L^*_g(\beta_{gh}), \Pi(h)\mathbf{P}^{-1}R^*_h L^*_g(\alpha_{gh})\rangle =$$

$$= \langle L^*_g R^*_{g^{-1}} R^*_g R^*_h(\beta_{gh}), \Pi(h)\mathbf{P}^{-1}L^*_g R^*_{g^{-1}} R^*_g R^*_h(\alpha_{gh})\rangle =$$

$$= \langle R^*_{gh}(\beta_{gh}), L_{g*}R_{g^{-1}*}\Pi(h)\mathbf{P}^{-1}L^*_g R^*_{g^{-1}} R^*_{gh}(\alpha_{gh})\rangle =$$

$$= \langle R^*_{gh}(\beta_{gh}), Ad_g\Pi(h)\mathbf{P}^{-1}L^*_g R^*_{g^{-1}} R^*_{gh}(\alpha_{gh})\rangle.$$

This has be equal to the term B. To finish the proof it is sufficient to show that maps $A_1, A_2 : \mathfrak{g}^* \to \widetilde{\mathfrak{g}}$ are equal, where

$$A_1 = \widetilde{P}Ad_{g^{-1}}\widetilde{P}\mathbf{P}^{-1},$$

$$A_2 = \mathbf{P}^{-1}L^*_g R^*_{g^{-1}}.$$

Recall that for arbitrary $\widetilde{V} \in \widetilde{\mathfrak{g}}$ and $X \in \mathfrak{g}$

$$\langle X, \mathbf{P}(\widetilde{V})\rangle = \langle X, \widetilde{V}\rangle_{\mathfrak{d}},$$

where $\langle\cdot,\cdot\rangle$ is the canonical pairing on $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ is the bilinear form on $\mathfrak{d}$. Then for arbitrary $\alpha \in \mathfrak{g}^*$ we define $\widetilde{V} = \mathbf{P}^{-1}\alpha$ and we can write

$$\langle X, A_1(\alpha)\rangle_{\mathfrak{d}} = \langle X, \widetilde{P}Ad_{g^{-1}}\mathbf{P}^{-1}(\alpha)\rangle_{\mathfrak{d}} = \langle X, \widetilde{P}Ad_{g^{-1}}(\widetilde{V})\rangle_{\mathfrak{d}} =$$

$$= \langle P(X), Ad_{g^{-1}}(\widetilde{V})\rangle_{\mathfrak{d}} = \langle Ad_g(X), \widetilde{V}\rangle_{\mathfrak{d}}.$$

On the other side

$$\langle X, A_2(\alpha)\rangle_{\mathfrak{d}} = \langle X, \mathbf{P}^{-1}L^*_g R^*_{g^{-1}}(\alpha)\rangle_{\mathfrak{d}} = \langle X, L^*_g R^*_{g^{-1}}(\alpha)\rangle =$$

$$= \langle L_{g*}R_{g^{-1}*}(X), \alpha\rangle = \langle Ad_g(X), \mathbf{P}(\widetilde{V})\rangle = \langle Ad_g(X), \widetilde{V}\rangle_{\mathfrak{d}}.$$

Hence we have

$$\langle X, A_1(\alpha)\rangle_{\mathfrak{d}} = \langle X, A_2(\alpha)\rangle_{\mathfrak{d}},$$

for arbitrary $\alpha \in \mathfrak{g}^*$ and $X \in g$. From the non-degeneracy of the bilinear form we finally obtain $A_1 = A_2$. ■

Remark 4.5.2. The bivector field $\mathcal{P}$ defined by (4.13) can be written in the components with respect to the right-invariant basis as

$$\mathcal{P}(g) = ((\Pi(g))_{\mathcal{X}})^{ji} R_{T_i} \otimes R_{T_j}, \tag{4.16}$$

where $R_{T_j}(g) = R_{g*}(T_j)$. Indeed, the dual basis to $R_{T_j}(g)$ is $R^*_{g^{-1}}(T^j)$ and so

$$\mathcal{P}(g)(R^*_{g^{-1}}(T^i), R^*_{g^{-1}}(T^j)) = \langle T^j, \Pi(g)\mathbf{P}^{-1}(T^i)\rangle = \langle \widetilde{T}^j, \Pi(g)(\widetilde{T}^i)\rangle_{\mathfrak{d}} = ((\Pi(g))_{\mathcal{X}})^{ji}.$$

4.6 The intrinsic derivative of $\mathcal{P}$

For an arbitrary k-times contravariant tensor field K on G satisfying $K(e) = 0$, we can define so called **intrinsic derivative** of K at e, which is a linear map $DK : \mathfrak{g} \to \bigotimes^k \mathfrak{g}$ defined as

$$DK(X) := [\mathcal{L}_{\bar{X}}(K)](e), \tag{4.17}$$

for all $X \in \mathfrak{g}$, where $\bar{X}$ is an *arbitrary* vector field on G such that $\bar{X}(e) = X$.

The map defined by (4.17) does not depend on the choice of $\bar{X}$, which can be seen for example from the coordinate expression of the Lie derivative, where the terms containing the partial derivatives of $\bar{X}$ vanish due to $K(e) = 0$.

We will now derive this intrinsic derivative of $\mathcal{P}$ defined by (4.13). As the extension of $X \in \mathfrak{g}$, denoted as $\bar{X}$, we naturally take the left-invariant field L_X generated by X.

Just for the easier notation we define the n^2 functions Π^{ij} on G as

$$\Pi^{ij}(g) = (\Pi(g))^{ji}_{\mathcal{X}}. \tag{4.18}$$

Now we can write

$$\mathcal{P} = \Pi^{ij} R_{T_i} \otimes R_{T_j} \equiv \frac{1}{2}\Pi^{ij} R_{T_i} \wedge R_{T_j}.$$

We use the derivation property of the Lie derivative and since the left-invariant and right-invariant fields commute, we have

$$D\mathcal{P}(X) = [\mathcal{L}_{L_X}(\mathcal{P})](e) = [\mathcal{L}_{L_X}(\Pi^{ij} R_{T_i} \otimes R_{T_j})](e) = [L_X(\Pi^{ij})](e) T_i \otimes T_j =$$

$$= X(\Pi^{ij}) T_i \otimes T_j = \frac{d}{dt}\bigg|_{t=0} \Pi^{ij}(e^{tX}) T_i \otimes T_j.$$

Therefore we get

$$D\mathcal{P}(X) = \frac{d}{dt}\bigg|_{t=0} \{((\Pi(e^{tX}))_{\mathcal{X}})^{ji}\} T_i \otimes T_j.$$

To calculate $\frac{d}{dt}\big|_{t=0}\{((\Pi(e^{tX}))_{\mathcal{X}})^{ji}\}$ let us consider the map $\Pi(g)$ as the linear operator on $\mathfrak{d}$, we denote it as $\mathbf{K}(g)$. Hence

$$((\Pi(g))_{\mathcal{X}})^{ij} = (\mathbf{K}(g)_{\mathcal{X}})^{i,n+j}.$$

Then we can write $\mathbf{K}(g)_{\mathcal{X}}$ as the product of $2n \times 2n$ matrices

$$\mathbf{K}(g)_{\mathcal{X}} = P_{\mathcal{X}}(Ad_g)_{\mathcal{X}} \widetilde{P}_{\mathcal{X}} (Ad_{g^{-1}})_{\mathcal{X}} \widetilde{P}_{\mathcal{X}},$$

where $P_{\mathcal{X}} = blockdiag(\mathbf{1}_n, \mathbf{0}_n)$ and $\widetilde{P}_{\mathcal{X}} = blockdiag(\mathbf{0}_n, \mathbf{1}_n)$, where $\mathbf{1}_n$ and $\mathbf{0}_n$ are $n \times n$ unit and zero matrices respectively. Therefore $((\Pi(e^{tX}))_{\mathcal{X}})^{ji}$ is just the matrix element of the product of the matrices of Ad and the constant matrices $P_{\mathcal{X}}$ and $\widetilde{P}_{\mathcal{X}}$. It is now justified to say that it is easy to derive

$$\frac{d}{dt}\bigg|_{t=0}\{((\Pi(e^{tX}))_{\mathcal{X}})^{ji}\} = ((Pad_X\widetilde{P})_{\mathcal{X}})^{ji}.$$

We can now directly compute the components of $D\mathcal{P}(T_k)$:

$$D\mathcal{P}(T_k)^{ij} = ((Pad_{T_k}\widetilde{P})_{\mathcal{X}})^{ji} = \langle \widetilde{T}^j, Pad_{T_k}(\widetilde{T}^i)\rangle_{\mathfrak{d}} = \langle \widetilde{T}^j, P(f^{il}{}_k T_l - c_{kl}{}^i\widetilde{T}^l)\rangle_{\mathfrak{d}} = f^{ij}{}_k,$$

where $c_{ij}{}^k$ are the structure constants of $\mathfrak{g}$ with respect to the basis T_i and $f^{ij}{}_k$ to the basis $\widetilde{T}^i$ of $\widetilde{\mathfrak{g}}$ respectively. So we have just found that

$$D\mathcal{P}(T_k) = f^{ij}{}_k T_i \otimes T_j.$$

Hence $D\mathcal{P}$ is exactly the cocommutator of the Lie bialgebra corresponding to the Manin triple $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$.

4.7 Vanishing of the Schouten-Nijenhuis bracket $[\mathcal{P}, \mathcal{P}]$

In this section we will show that in fact every bivector field defined by (4.13) is the Poisson bivector, that is it satisfies $[\mathcal{P}, \mathcal{P}] = 0$. To reach this goal we have to prove several useful lemmas.

Proposition 4.7.1. *Let K be a multiplicative bivector field on G. Let us choose an arbitrary basis $(T_i)_{i=1}^n$ of $\mathfrak{g}$. We denote the components of $DK(T_k)$ in the basis $T_i \otimes T_j$ as $DK(T_k)^{ij} = f^{ij}{}_k$.*

Then for the components of $D[K,K]$ in the basis $T_a \otimes T_b \otimes T_c$ there holds

$$D[K,K](T_k)^{abc} = 2(f^{am}{}_k f^{bc}{}_m) + cyclic\{a,b,c\}. \tag{4.19}$$

Proof. From lemma 4.3.8 we know that $[K,K]$ is again a multiplicative bivector field, thus $D[K,K]$ is well defined. We again use the graded Jacobi identities (3.11) and write

$$D[K,K](T_k) = [\mathcal{L}_{L_{T_k}}([K,K])](e) = 2[\mathcal{L}_{L_{T_k}}(K), K](e).$$

We know from lemma 4.3.4 that $\mathcal{L}_{L_{T_k}}(K)$ is a left-invariant bivector field, hence $[\mathcal{L}_{L_{T_k}}(K)](g) = L_{g*}(DK(T_k))$. If we expand it in the left-invariant basis, the components are just the real numbers, and we can write

$$\mathcal{L}_{L_{T_k}}(K) = \frac{1}{2}DK(T_k)^{ij}L_{T_i} \wedge L_{T_j} = \frac{1}{2}f^{ij}{}_k L_{T_i} \wedge L_{T_j}.$$

We can now continue and use the property of the Schouten-Nijenhuis bracket (3.10) to get

$$2[\mathcal{L}_{L_{T_k}}(K), K] = \frac{1}{2}f^{ij}{}_k[L_{T_i} \wedge L_{T_j}, K^{mn}L_{T_m} \wedge L_{T_n}] =$$

$$= \frac{1}{2}f^{ij}{}_k K^{mn}[L_{T_i} \wedge L_{T_j}, L_{T_m} \wedge L_{T_n}] + \frac{1}{2}f^{ij}{}_k[K^{mn}, L_{T_i} \wedge L_{T_j}] \wedge L_{T_m} \wedge L_{T_n}.$$

Evaluating at the unit e and using $K(e) = 0$ we get

$$D[K,K](T_k) = \frac{1}{2}f^{ij}{}_k[K^{mn}, L_{T_i} \wedge L_{T_j}](e) \wedge T_m \wedge T_n = f^{ij}{}_k T_i \wedge T_j(K^{mn})T_m \wedge T_n.$$

Since we can write

$$DK(T_j) = [\mathcal{L}_{L_{T_j}}(\frac{1}{2}K^{mn}L_{T_m} \wedge L_{T_n})](e) = \frac{1}{2}T_j(K^{mn})T_m \wedge T_n$$

we get

$$D[K,K](T_k) = f^{ij}{}_k DK(T_j)^{mn} T_i \wedge T_m \wedge T_n = (f^{ij}{}_k f^{mn}{}_j) T_i \wedge T_m \wedge T_n.$$

Finally for the *abc*-th component we get

$$D[K,K](T_k)^{abc} = (f^{ij}{}_k f^{mn}{}_j)\delta^{abc}_{imn} = 2(f^{am}{}_k f^{bc}{}_m) + cyclic\{a,b,c\},$$

which was to be proved. ■

After some technical results we have finally got to the main theorem of this chapter. It shows that $\mathcal{P}$ is the desired Poisson bivector on group G. Moreover it is multiplicative and therefore we have found a Poisson-Lie group structure on G, starting only from the bialgebra $(\mathfrak{g}, \delta)$. What is more important, we found the way to directly construct it. Thus we can calculate many useful examples, what would be done at the end of this chapter.

Theorem 4.7.2. *The bivector field $\mathcal{P}$ defined by (4.13) is a Poisson bivector, that is $[\mathcal{P},\mathcal{P}] = 0$.*

Proof. We have already shown that $\mathcal{P}$ is multiplicative. Thus by lemma 4.3.8 $[\mathcal{P},\mathcal{P}]$ is also multiplicative. To prove $[\mathcal{P},\mathcal{P}] = 0$ it is by proposition 4.3.9 sufficient to show that

$$[\mathcal{L}_V([\mathcal{P},\mathcal{P}])](e) = 0,$$

for all left-invariant vector fields V.

This is by definition equal to $D[\mathcal{P},\mathcal{P}] = 0$. From the section 4.6 we know that components of $D\mathcal{P}(T_k)$ in the basis $T_i \otimes T_j$ coincide with the structure constants of Lie algebra $\widetilde{\mathfrak{g}}$ in the basis $(\widetilde{T}^j)$. Therefore the right hand side of (4.19) is equal to zero because of the Jacobi identities in $\widetilde{\mathfrak{g}}$. Thus by the proposition 4.7.1 we have $D[\mathcal{P},\mathcal{P}] = 0$.

So we have proved that $[\mathcal{P},\mathcal{P}] = 0$ and $\mathcal{P}$ is then the Poisson bivector on G. ■

4.8 Relation between Poisson-Lie groups and Lie bialgebras

As we have presumed in the introduction of this chapter, for every Poisson-Lie group G there exists an additional structure on its Lie algebra $\mathfrak{g}$. We can start with the more general situation, Lie group G equipped just by the multiplicative contravariant tensor field. See that in general we *do not* require G to be connected.

Proposition 4.8.1. *Let K be a multiplicative k-times contravariant tensor field on a Lie group G. A map $\delta : \mathfrak{g} \to \bigotimes^k \mathfrak{g}$ defined as $\delta(X) := DK(X)$ is a 1-cocycle with the values in $\bigotimes^k \mathfrak{g}$ with respect to the generalized adjoint representation $ad^{(k)}$, that is*

$$\delta([X,Y]) = ad^{(k)}_X(\delta(Y)) - ad^{(k)}_Y(\delta(X)), \tag{4.20}$$

for all $X, Y \in \mathfrak{g}$.

Proof. We define $\bar{X} = L_X$, $\bar{Y} = L_Y$ and $\overline{[X,Y]} = [L_X, L_Y]$. From the multiplicativity of K we have $K(e) = 0$ and the intrinsic derivative of K is then well defined.

We can write using the classical property of the Lie derivative

$$\delta([X,Y]) = [\mathcal{L}_{[\bar{X},\bar{Y}]}(K)](e) = [\mathcal{L}_{\bar{X}}(\mathcal{L}_{\bar{Y}}(K))](e) - [\mathcal{L}_{\bar{Y}}(\mathcal{L}_{\bar{X}}(K))](e). \tag{4.21}$$

From lemma 4.3.4 we know (no matter whether G is connected) that $\mathcal{L}_{\bar{Y}}(K)$ is a left-invariant tensor field. Hence we have

$$[\mathcal{L}_{\bar{Y}}(K)](g) = L_{g*}(\delta(Y)).$$

We can use this to rewrite the first term of the right hand side of (4.21) as

$$[\mathcal{L}_{\bar{X}}(\mathcal{L}_{\bar{Y}}(K))](e) = \frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}(L_{e^{tX}*}(\delta(Y)) = \frac{d}{dt}\bigg|_{t=0} Ad_{e^{tX}}(\delta(Y)) = ad_X^{(k)}(\delta(Y)).$$

We can do the same with the second term and we get the statement of the proposition. ■

Now we take G to be a Poisson-Lie group $(G,\mathcal{P})$. It turns out that $\delta \equiv D\mathcal{P}$ is not only a 1-cocycle, but a Lie bialgebra cocommutator on $\mathfrak{g}$.

Proposition 4.8.2. *Let $(G,\mathcal{P})$ be a Poisson-Lie group, that is a Lie group G with a Lie algebra $\mathfrak{g}$, equipped with the multiplicative bivector field $\mathcal{P}$ such that $[\mathcal{P},\mathcal{P}] = 0$. Then the map $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ defined as*

$$\delta(X) := D\mathcal{P}(X) \equiv \mathcal{L}_{\bar{X}}(\mathcal{P}), \tag{4.22}$$

for all $X \in \mathfrak{g}$, is the cocommutator of a Lie bialgebra $(\mathfrak{g},\delta)$.

Proof. Let us choose an arbitrary basis $(T_i)_{i=1}^n$ of $\mathfrak{g}$. We denote the components of $DK(T_k)$ in the basis $T_i \otimes T_j$ as $D\mathcal{P}(T_k)^{ij} = f^{ij}{}_k$.

This constants play the role of the structure constants in $\mathfrak{g}^*$. Thus they have to satisfy the Jacobi identities

$$f^{am}{}_k f^{bc}{}_m + cyclic\{a,b,c\} = 0. \tag{4.23}$$

The skew-symmetry $f^{ij}{}_k = -f^{ji}{}_k$ is obvious, because the $D\mathcal{P}$ is a Lie derivative of the bivector field, i.e. also the bivector field.

It is clear that $[\mathcal{P},\mathcal{P}] = 0$ implies $D[\mathcal{P},\mathcal{P}] = 0$. Hence the left hand side of (4.19) vanishes and the right hand side is nothing but the left hand side of (4.23). ■

We can sum up the preceding results in the following theorem.

Theorem 4.8.3. *(i.) Let $(G,\mathcal{P})$ be a Poisson-Lie group, that is a Lie group G with a Lie algebra $\mathfrak{g}$ equipped with the multiplicative bivector field $\mathcal{P}$, such that $[\mathcal{P},\mathcal{P}] = 0$.*

*There exists a unique Lie bialgebra structure δ on $\mathfrak{g}$, such that $\delta = D\mathcal{P}$. Lie bialgebra $(\mathfrak{g},\delta)$ is called a **tangent Lie bialgebra** to $(G,\mathcal{P})$.*

(ii.) Let (g,δ) be a finite-dimensional real Lie bialgebra. There exists a connected and simply connected Lie group G, such that $\mathfrak{g}$ is its Lie algebra, equipped with a unique multiplicative bivector field $\mathcal{P}$ satisfying $[\mathcal{P},\mathcal{P}] = 0$, such that (g,δ) is its tangent Lie bialgebra.

The Poisson bivector $\mathcal{P}$ is the one defined by (4.13).

Proof. The proof of the theorem was in fact given in the previous sections. The only unsolved thing is the uniqueness of the Poisson bivector $\mathcal{P}$ from the second statement of the theorem.

We consider $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ to be the multiplicative bivector fields such that $D\mathcal{P} = D\widetilde{\mathcal{P}}$. Then by definition $[\mathcal{L}_V(\mathcal{P} - \widetilde{\mathcal{P}})](e) = 0$, for all the left-invariant vector fields V. The linear combination of the multiplicative tensor fields is always multiplicative and thus by the proposition 4.3.9 we have $\mathcal{P} \equiv \widetilde{\mathcal{P}}$. ■

Definition 4.8.4. Let $(G,\mathcal{P})$ be a Poisson-Lie group. Let $(\mathfrak{g},\delta)$ be its tangent Lie bialgebra. Let $(\mathfrak{g}^*,\gamma)$ be a Lie bialgebra dual to $(\mathfrak{g},\delta)$. The connected and simply connected Poisson-Lie group $(\widetilde{G},\widetilde{\mathcal{P}})$ corresponding to $(\mathfrak{g}^*,\gamma)$, that is $D\widetilde{\mathcal{P}} = \gamma$, is called a **dual of a Poisson-Lie group**.

Remark 4.8.5. Note that in general G is not dual of $\widetilde{G}$, because G does not have to be connected and simply connected.

4.9 Sklyanin bracket

We have found the corresponding Poisson bivector $\mathcal{P}$ for every given Lie bialgebra $(\mathfrak{g}, \delta)$. But in the chapter 2 we have found that there exists a class of Lie bialgebras given by particular $r \in \mathfrak{g} \otimes \mathfrak{g}$, that is $\delta = \Delta(r)$. For this class (coboundary Lie bialgebras) there exists a very simple and elegant way to find a corresponding Poisson bivector.

Proposition 4.9.1. *Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Let r be an arbitrary element of $\bigotimes^k \mathfrak{g}$. $Q \in \mathcal{T}_0^k(G)$ is defined as*

$$Q(g) := L_{g*}(r) - R_{g*}(r), \tag{4.24}$$

for $g \in G$. Then Q is a multiplicative tensor field on G. Moreover for its intrinsic derivative we have

$$DQ = \Delta(r).$$

Proof. Let $g, h \in G$. Then

$$Q(gh) = L_{gh*}(r) - R_{gh*}(r) = L_{g*}(L_{h*}(r)) - R_{h*}(R_{g*}(r)) =$$

$$L_{g*}(L_{h*}(r)) - L_{g*}(R_{h*}(r)) + L_{g*}(R_{h*}(r)) - R_{h*}(R_{g*}(r)) =$$

$$= L_{g*}(Q(h)) + R_{h*}(Q(g)).$$

Hence Q is a multiplicative tensor field. To calculate DQ let us choose $\bar{X} := L_X$ for $X \in \mathfrak{g}$. Then

$$DQ(X) \equiv [\mathcal{L}_{\bar{X}}(Q)](e) = [\mathcal{L}_{L_X}(Q)](e) = \frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}(Q(e^{tX})) =$$

$$= \frac{d}{dt}\bigg|_{t=0} R_{e^{-tX}*}(L_{e^{tX}*}(r) - R_{e^{tX}*}(r)) = \frac{d}{dt}\bigg|_{t=0} \{Ad_{e^{tX}}(r) - r\} = ad_X^{(k)}(r) \equiv \Delta(r)(X).$$

■

Q is therefore a pretty good candidate (if $k = 2$) for a Poisson bivector of the Poisson-Lie group corresponding to a coboundary Lie bialgebra. First we have to show the important equivalence of ad-invariance and Ad-invariance on connected Lie groups.

Lemma 4.9.2. *Let G be a connected Lie group. $q \in \bigotimes^k \mathfrak{g}$. Then*

$$(\forall g \in G)\ (Ad_g(q) = q) \Longleftrightarrow (\forall X \in \mathfrak{g})\ (ad_X^{(k)}(q) = 0).$$

Proof. The left to right direction is just the definition of $ad^{(k)}$, therefore trivial. Starting from the right hand side, again using the definition, we have

$$0 = ad_X^{(k)}(q) = \frac{d}{dt}\bigg|_{t=0} Ad_{e^{tX}}(q),$$

for every $X \in \mathfrak{g}$. Then we find the derivative for every value of t

$$\frac{d}{dt} Ad_{e^{tX}}(q) = Ad_{e^{tX}}(\frac{d}{ds}\bigg|_{s=0} Ad_{e^{sX}}(q)) = 0.$$

We can integrate this identity to get

$$Ad_{e^{tX}}(q) = q, \tag{4.25}$$

for every $X \in \mathfrak{g}$. We know from the connectedness of G and proposition 4.3.3, that every element of G can be written as a finite product of exponential images. So we can extend (4.25) by induction to whole G, because

$$Ad_{e^{tX}e^{sY}}(q) = Ad_{e^{tX}}(Ad_{e^{sY}}(q)) = \cdots = q.$$

■

Theorem 4.9.3. *Let G be a connected Lie group, $\mathfrak{g}$ is its Lie algebra with a coboundary Lie bialgebra cocommutator $\delta = \Delta(r)$, where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is its classical r-matrix. Then $\mathcal{P} \in L_2(G)$, defined as*

$$\mathcal{P}(g) = L_{g*}(r) - R_{g*}(r), \tag{4.26}$$

for $g \in G$, is a Poisson bivector on G, making G into a Poisson-Lie group. Moreover $(\mathfrak{g}, \Delta(r))$ is its tangent Lie bialgebra.

Proof. We can write $r = a + s$, where $a \in \bigwedge^2 \mathfrak{g}$ and $s \in S^2\mathfrak{g}$. From 2.5.11 we know that (because r is a classical r-matrix)

$$ad_X^{(2)}(s) \equiv \Delta(s)(X) = 0$$

and

$$ad_X^{(3)}([\![a,a]\!]) \equiv \Delta([\![a,a]\!])(X) = 0,$$

for every $X \in \mathfrak{g}$. From the lemma 4.9.2 we then get

$$Ad_g(s) = s, \;\; Ad_g([\![a,a]\!]) = [\![a,a]\!],$$

for all $g \in G$. Using this we can show, that $P(g) = L_{g*}(a) - R_{g*}(a)$. Indeed, we have

$$s = Ad_g(s) = L_{g*}(R_{g^{-1}*}(s)) \Rightarrow L_{g*}(s) - R_{g*}(s) = 0.$$

Hence we can write

$$\mathcal{P}(g) = L_{g*}(a+s) - R_{g*}(a+s) = L_{g*}(a) + L_{g*}(s) - R_{g*}(s) - R_{g*}(a) = L_{g*}(a) - R_{g*}(a).$$

Now it is clear that $\mathcal{P}$ defined as 4.26 is indeed a bivector field, $\mathcal{P} \in L_2(G)$.

We have remarked in 2.5.8, that $[L_a, L_a](g) = L_{g*}([\![a,a]\!])$ for $g \in G$ and $L_a(g) \equiv L_{g*}(a)$. It is easy to see that for $R_a(g) := R_{g*}(a)$ we get $[R_a, R_a](g) = -R_{g*}([\![a,a]\!])$. And finally, there also holds $[L_a, R_a] = 0$. Therefore

$$R_{g^{-1}*}([\mathcal{P},\mathcal{P}](g)) = R_{g^{-1}*}([L_a - R_a, L_a - R_a](g)) = R_{g^{-1}*}([L_a, L_a](g) + [R_a, R_a](g)) =$$

$$= R_{g^{-1}*}(L_{g*}([\![a,a]\!]) - R_{g*}([\![a,a]\!])) = Ad_g([\![a,a]\!]) - [\![a,a]\!] = 0.$$

We have found that $[\mathcal{P},\mathcal{P}] = 0$ and hence $\mathcal{P}$ is a Poisson bivector on G.

From 4.9.1 we know that $\mathcal{P}$ is multiplicative and moreover $D\mathcal{P} = \Delta(r)$, which was to be proven. ■

Example 4.9.4. As we have promised, we can now proceed with example 2.7.8. We have just learned a simple way of the Poisson bivector construction for coboundary Lie bialgebras. This allows us to construct a Poisson structure on Drinfel'd double D.

We choose the usual basis in Lie algebra $\mathfrak{d}$, that is $(X_i)_{i=1}^{2n} = (T_i, \widetilde{T}^j)_{i,j=1}^{n,n}$. Then we can find the matrices $R_{\mathcal{X}}$ of the map R and $\mathcal{D}_{\mathcal{X}}^{-1}$ of the bilinear form $\mathcal{D}^{-1}$ on $\mathfrak{d}^*$ as:

$$R_{\mathcal{X}} = \frac{1}{2}\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}, \; \mathcal{D}_{\mathcal{X}}^{-1} = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix},$$

where $\mathbf{1}_n$ is $n \times n$ unit matrix. If we calculate the matrix $a_{\mathcal{X}}$ of the skew-symmetric part a of the corresponding r-matrix, we get

$$a_{\mathcal{X}} = R_{\mathcal{X}} \mathcal{D}_{\mathcal{X}}^{-1} = \frac{1}{2} \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

Using the Sklyanin bracket (4.26), we can define the Poisson bivector $\mathcal{P}_D$ on the Drinfel'd double D as

$$\mathcal{P}_D(d) := L_{d*}(a) - R_{d*}(a),$$

for all $d \in D$. We can write it in the right-invariant basis as

$$\mathcal{P}_D(d) = R_{d*}\big(Ad_d(a) - a\big) = \big(Ad_d(a) - a\big)^{ij}_{\mathcal{X}} R_{X_i}(d) \otimes R_{X_j}(d).$$

Thus for the matrix $\mathbf{P}_D$ of $\mathcal{P}_D$ in the right-invariant basis (on D) we have

$$\mathbf{P}_D(d) = \big(Ad_d(a) - a\big)_{\mathcal{X}} = (Ad_d)_{\mathcal{X}} a_{\mathcal{X}} (Ad_d)^T_X - a_{\mathcal{X}}. \tag{4.27}$$

We can now define the bivector $\mathcal{P}_G$ on the subgroup G as

$$\mathcal{P}_G(g) := \mathcal{P}_D(g)|_G\,,$$

for $g \in G$. If we calculate the matrix $\mathbf{P}_D(g)$ for $g \in G$, we get, using the notation from section 4.4:

$$\mathbf{P}_D(g) = \begin{pmatrix} -b(g)a(g)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{pmatrix},$$

where $\mathbf{0}_n$ is $n \times n$ zero matrix. One has to use the identities

$$a(g)^T = d(g)^{-1}, \tag{4.28}$$

$$b(g)^T = -a(g)^T b(g) a(g)^{-1}, \tag{4.29}$$

which can be obtained from (4.7) using the conjugation tricks introduced in section 4.4.

Therefore

$$\mathcal{P}_G(g) = (-b(g)a(g)^{-1})^{ij} R_{T_i}(g) \otimes R_{T_j}(g).$$

We have just discovered, that $\mathcal{P}_G$ is nothing but the Poisson bivector $\mathcal{P}$ on G, defined by (4.13)!!

Moreover, because $\mathcal{P}_D(g) \in T_g(G) \otimes T_g(G)$, we have found that G is a Poisson-Lie subgroup of D. This gives us another proof of important properties of $\mathcal{P}$, the multiplicativity and the vanishing of Schouten-Nijenhuis bracket, because all properties of $\mathcal{P}_G$ are just inherited from $\mathcal{P}_D$.

Furthermore, we can calculate $\mathcal{P}_{\widetilde{G}}(\widetilde{g}) := \mathcal{P}_D(\widetilde{g})|_{\widetilde{G}}$ for $\widetilde{g} \in \widetilde{G}$. We discover that $\mathcal{P}_{\widetilde{G}}(\widetilde{g}) = -\widetilde{\mathcal{P}}$, where $\widetilde{\mathcal{P}}$ is the Poisson bivector of Poisson-Lie group dual to $\mathcal{P} = \mathcal{P}_G$. Dual Poisson-Lie group $(\widetilde{G}, \widetilde{\mathcal{P}})$ is thus an anti-Poisson-Lie subgroup (its inclusion is anti-Poisson map) of $(D, \mathcal{P}_D)$.

We have found a very interesting conclusion - for every Lie bialgebra $(\mathfrak{g}, \delta)$ we can construct a unique Poisson-Lie structure $\mathcal{P}_D$ on Drinfel'd Double D, such that its subgroups G and $\widetilde{G}$ are mutually dual Poisson-Lie groups and moreover G is Poisson-Lie subgroup of $(D, \mathcal{P}_D)$, whereas $\widetilde{G}$ is anti-Poisson-Lie subgroup of $(D, \mathcal{P}_D)$.

4.10 Examples

In this section we will give two examples. First we will find the Poisson brackets for a given coboundary Lie bialgebra. We will use two different approaches, the construction using the map $\Pi(g)$ and a Sklyanin bracket, and show that they give the same results.

4.10.1 Poisson-Lie group corresponding to Manin triple $(\mathbf{5}|\mathbf{2.i})$

Manin triple $(\mathbf{5}|\mathbf{2.i})$ is a six-dimensional Manin triple $(\mathfrak{d}, \mathfrak{g}, \widetilde{\mathfrak{g}})$, where the commutation relations of $\mathfrak{g}$ are given in basis (T_1, T_2, T_3) as

$$[T_1, T_2] = -T_2,\ [T_2, T_3] = 0,\ [T_3, T_1] = T_3.$$

The commutation relations of $\widetilde{\mathfrak{g}}$ are given in the basis $(\widetilde{T}^1, \widetilde{T}^2, \widetilde{T}^3)$ (it satisfies 4.1), as

$$[\widetilde{T}^1, \widetilde{T}^2] = 0,\ [\widetilde{T}^2, \widetilde{T}^3] = \widetilde{T}^1,\ [\widetilde{T}^3, \widetilde{T}^1] = 0.$$

The mixed commutation relations of $\mathfrak{d}$, like $[T_1, \widetilde{T}^2], \ldots$, are given by (2.25) and there is no need to explicitly write them.

As we have said in the section 4.3, we only have to know the values of a multiplicative Poisson bivector in some neighbourhood of the unit of G. In a sufficiently small neighbourhood of the unit of every Lie group G we can use the special coordinates, defined by relation

$$(\alpha_1, \alpha_2, \alpha_3) \leftrightarrow e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}.$$

We denote the corresponding coordinate functions as y^i, that is

$$y^i(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) := \alpha_i.$$

In this coordinates we can calculate the matrices of the adjoint representation of G in an amazingly simple way. We can write the ad matrices right from the commutation relations

$$(ad_{T_1})_{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(ad_{T_2})_{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(ad_{T_3})_{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the definition of the adjoint representation of $\mathfrak{g}$ we have

$$(Ad_{e^{tX}})_{\mathcal{X}} = e^{(t(ad_X)_{\mathcal{X}})}, \tag{4.30}$$

for every $X \in \mathfrak{g}$. On the right hand side we have the ordinary matrix exponential. Using this we calculate

$$(Ad_{e^{\alpha_1 T_1}})_{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & e^{-\alpha_1} & 0 & 0 & 0 & -\sinh(\alpha_1) \\ 0 & 0 & e^{-\alpha_1} & 0 & \sinh(\alpha_1) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\alpha_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\alpha_1} \end{pmatrix},$$

$$(Ad_{e^{\alpha_2 T_2}})_{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(Ad_{e^{\alpha_3 T_3}})_{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\alpha_3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

The matrix of adjoint representation for a general Lie group element $e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}$ can be found by multiplying the three matrices given above

$$(Ad_{e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}})_{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 e^{-\alpha_1} & e^{-\alpha_1} & 0 & 0 & 0 & -\sinh(\alpha_1) \\ \alpha_3 e^{-\alpha_1} & 0 & -e^{\alpha_1} & 0 & \sinh\alpha_1 & 0 \\ 0 & 0 & 0 & 1 & -\alpha_2 & -\alpha_3 \\ 0 & 0 & 0 & 0 & e^{\alpha_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\alpha_1} \end{pmatrix} \tag{4.31}$$

The matrix of Ad corresponding to the inverse element $e^{-\alpha_3 T_3} e^{-\alpha_2 T_2} e^{-\alpha_1 T_1}$ is obtained by multiplying the same three matrices in reverse order and by changing the signs of parameters α_i

$$(Ad_{(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3})^{-1}})_{\mathcal{X}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_2 & e^{\alpha_1} & 0 & 0 & 0 & \sinh(\alpha_1) \\ -\alpha_3 & 0 & e^{\alpha_1} & 0 & -\sinh\alpha_1 & 0 \\ 0 & 0 & 0 & 1 & \alpha_2 e^{-\alpha_1} & \alpha_3 e^{-\alpha_1} \\ 0 & 0 & 0 & 0 & e^{-\alpha_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\alpha_1} \end{pmatrix}$$

We know that $b(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = (P Ad_{e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}} \widetilde{P})_{\mathcal{X}}$, that is the upper-right 3×3 submatrix of (4.31)

$$b(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sinh(\alpha_1) \\ 0 & \sinh(\alpha_1) & 0 \end{pmatrix}. \tag{4.32}$$

In the same way for $a^{-1}(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = (\widetilde{P} Ad_{(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3})^{-1}} \widetilde{P})_{\mathcal{X}}$ we get

$$a^{-1}(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = \begin{pmatrix} 1 & \alpha_2 e^{-\alpha_1} & \alpha_3 e^{-\alpha_1} \\ 0 & e^{-\alpha_1} & 0 \\ 0 & 0 & e^{-\alpha_1} \end{pmatrix}. \tag{4.33}$$

Multiplying (4.32) and (4.33) we finally obtain

$$(\Pi(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}))_{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(e^{-2\alpha_1} - 1) \\ 0 & -\frac{1}{2}(e^{-2\alpha_1} - 1) & 0 \end{pmatrix}. \tag{4.34}$$

We now have a Poisson bivector $\mathcal{P}$ written in the right-invariant basis R_{T_i}, because $\mathcal{P}(g) = ((\Pi(g))_{\mathcal{X}})^{ji} R_{T_i} \otimes R_{T_j}$. This form is sometimes inconvenient for real use, we would like to have it written in the coordinate basis $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$. This seems to be an impossible task, because in general we do not know the multiplication laws on the group G. Fortunately in this particular coordinates this can be done using one interesting trick.

We will show how to calculate the transformation matrices between the coordinate frame fields $\frac{\partial}{\partial y^i}$ and the left-invariant frame fields generated by $T_i \in \mathfrak{g}$, i.e. $L_{T_i}(g) = L_{g*}(T_i)$. We are looking for the matrices $e^L(g)$, such that

$$\left.\frac{\partial}{\partial y^i}\right|_g = (e^L(g))^k{}_i \, L_{g*}(T_k). \tag{4.35}$$

Recall that for arbitrary smooth map $f : G \to G$ and the point $p \in G$ the linear map $f_* : T_pG \to T_{f(p)}G$ is defined as

$$f_*(V) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma^V(t)), \tag{4.36}$$

where $V \in T_pG$ and γ^V is its defining curve (i.e. V is tangent to γ^V at p).

It is easy to find the integral curves $\gamma^i(t)$ of the vector $\left.\frac{\partial}{\partial y^i}\right|_g$ starting from g. For $g = e^{\alpha_1 T_1} \cdots e^{\alpha_n T_n}$ we get

$$\gamma^i(t) = e^{\alpha_1 T_1} \cdots e^{(t+\alpha_i)T_i} \cdots e^{\alpha_n T_n}. \tag{4.37}$$

Now it is easy to calculate $L_{g^{-1}*}(\left.\frac{\partial}{\partial y^i}\right|_g)$ using (4.36) and (4.37)

$$L_{g^{-1}*}(\left.\frac{\partial}{\partial y^i}\right|_g) = \left.\frac{d}{dt}\right|_{t=0} L_{g^{-1}}(\gamma^i(t)) = \left.\frac{d}{dt}\right|_{t=0} \{e^{-\alpha_n T_n} \cdots e^{-\alpha_1 T_1} e^{\alpha_1 T_1} \cdots e^{(t+\alpha_i)T_i} \cdots e^{\alpha_n T_n}\} =$$

$$= \left.\frac{d}{dt}\right|_{t=0} \{e^{-\alpha_n T_n} \cdots e^{-\alpha_{i+1} T_{i+1}} e^{tT_i} e^{\alpha_{i+1} T_{i+1}} \cdots e^{\alpha_n T_n}\}.$$

$$= \left.\frac{d}{dt}\right|_{t=0} I_{(e^{-\alpha_n T_n} \cdots e^{-\alpha_{i+1} T_{i+1}})}(e^{tT_i}) = Ad_{(e^{-\alpha_n T_n} \cdots e^{-\alpha_{i+1} T_{i+1}})}(T_i) =$$

$$= \langle T^k, Ad_{(e^{-\alpha_n T_n} \cdots e^{-\alpha_{i+1} T_{i+1}})}(T_i) \rangle T_k,$$

where I_g denotes the conjugation by the element g. If we define the matrices $e^L(g)$ as

$$e^L(e^{\alpha_1 T_1 \cdots \alpha_n T_n})^k{}_i = \langle T^k, Ad_{(e^{-\alpha_n T_n} \cdots e^{-\alpha_{i+1} T_{i+1}})}(T_i) \rangle,$$

we get the result, because we now have

$$L_{g^{-1}*}(\left.\frac{\partial}{\partial y^i}\right|_g) = e^L(g)^k{}_i T_k$$

and therefore

$$\left.\frac{\partial}{\partial y^i}\right|_g = e^L(g)^k{}_i \, L_{g*}(T_k).$$

We can define the inverse transformations as

$$L_{g*}(T_k) = f^L(g)^i{}_k \left.\frac{\partial}{\partial y^i}\right|_g,$$

we of course have $f^L(g) = e^L(g)^{-1}$. It turns out that we can simply calculate the transformation matrices between the coordinate and right-invariant basis

$$\left.\frac{\partial}{\partial y^i}\right|_g = e^R(g)^k{}_i \; R_{g*}(T_k).$$

We can use the result for the left-invariant basis

$$\left.\frac{\partial}{\partial y^i}\right|_g = e^L(g)^k{}_i \; L_{g*}(T_k) = e^L(g)^k{}_i \; R_{g*}(R_{g^{-1}*}(L_{g*}(T_k)))) =$$

$$= e^L(g)^k{}_i R_{g*}(Ad_g(T_k)) = e^L(g)^k{}_i((Ad_g)_{\mathcal{X}})^m{}_k R_{g*}(T_m) =$$

$$= [(Ad_g)_{\mathcal{X}} e^L(g)]^m{}_i R_{g*}(T_m),$$

therefore

$$e^R(g) = (Ad_g)_{\mathcal{X}} e^L(g),$$

and

$$f^R(g) = f^L(g)(Ad_{g^{-1}})_{\mathcal{X}}.$$

Using the ordinary rules for a tensor transformation we find that for

$$\mathcal{P}(g) = \mathcal{P}^{ij}(g) \left.\frac{\partial}{\partial y^i}\right|_g \otimes \left.\frac{\partial}{\partial y^j}\right|_g$$

we find the components $\mathcal{P}^{ij}(g)$ as

$$\mathcal{P}^{ij}(g) = \left[f^R(g)((\Pi(g))_{\mathcal{X}})^T f^R(g)^T\right]^{ij}. \tag{4.38}$$

For our particular example the matrices e^L, f^L, e^R and f^R are

$$e^L(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_2 & 1 & 0 \\ -\alpha_3 & 0 & 1 \end{pmatrix}, \; f^L(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & 0 \\ \alpha_3 & 0 & 1 \end{pmatrix},$$

$$e^R(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\alpha_1} & 0 \\ 0 & 0 & e^{-\alpha_1} \end{pmatrix}, \; f^R(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\alpha_1} & 0 \\ 0 & 0 & e^{\alpha_1} \end{pmatrix}.$$

Using 4.38 we can finally calculate the components of $\mathcal{P}$ with respect to the coordinate basis $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$. We get

$$\mathcal{P}^{ij}(e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(e^{2\alpha_1} - 1) \\ 0 & -\frac{1}{2}(e^{2\alpha_1} - 1) & 0 \end{pmatrix}. \tag{4.39}$$

We can then express $\mathcal{P}^{ij}$ as a matrix of functions on G using the coordinate functions y^i

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(e^{2y^1} - 1) \\ 0 & -\frac{1}{2}(e^{2y^1} - 1) & 0 \end{pmatrix}. \tag{4.40}$$

The Poisson brackets defined by $\mathcal{P}$ then read

$$\{y^1, y^2\} = 0, \; \{y^1, y^3\} = 0, \; \{y^2, y^3\} = \frac{1}{2}(e^{2y^1} - 1). \tag{4.41}$$

We will now try to reproduce this result using the Sklyanin bracket (4.26). First we have to find the r-matrix corresponding to the Lie bialgebra defined by Manin triple (**5|2.i**) (if there exists one). One can show that this bialgebra is triangular, and if we express its r-matrix as $r = r^{ij} T_i \otimes T_j$, we have

$$r^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}. \tag{4.42}$$

To calculate the Sklyanin bracket, all we have to do is to calculate $L_{g*}(r)$ and $R_{g*}(r)$. It is easy because we have prepared the matrices f^L and f^R. We can write

$$L_{g*}(r) = r^{ij} L_{g*}(T_i) \otimes L_{g*}(T_j) = r^{ij} f^L(g)^k{}_i \left. \frac{\partial}{\partial y^k} \right|_g \otimes f^L(g)^l{}_j \left. \frac{\partial}{\partial y^l} \right|_g .$$

Hence we get for the coordinates of $L_{g*}(r)$ in the basis $\left. \frac{\partial}{\partial y^k} \right|_g \otimes \left. \frac{\partial}{\partial y^l} \right|_g$

$$L_{g*}(r)^{kl} = r^{ij} f^L(g)^k{}_i f^L(g)^l{}_j .$$

In the same way for the right translation

$$R_{g*}(r)^{kl} = r^{ij} f^R(g)^k{}_i f^R(g)^l{}_j$$

and therefore

$$\mathcal{P}^{kl}(g) = r^{ij} f^L(g)^k{}_i f^L(g)^l{}_j - r^{ij} f^R(g)^k{}_i f^R(g)^l{}_j .$$

If we define the matrix $\mathbf{r}$ as $(\mathbf{r})^{ij} = r^{ij}$, we get the matrix relation

$$\mathcal{P}^{kl}(g) = \left[f^L(g) \mathbf{r} (f^L(g))^T - f^R(g) \mathbf{r} (f^R(g))^T \right]^{kl} .$$

Using the matrices calculated above and (4.42) it is easy to verify that for $g = e^{\alpha_1 T_1} e^{\alpha_2 T_2} e^{\alpha_3 T_3}$ we indeed get the same result as (4.39).

The method using the Sklyanin bracket can be slightly faster and easier then the method using the map Π. Although the computation of the matrices of adjoint representation cannot be avoided (we need them in the calculation of f^L and f^R matrices), we have to calculate only the submatrices corresponding to the Lie group G ($n \times n$ matrix instead of $2n \times 2n$ matrix). On the other hand, for the computation of $\mathcal{P}$ using the Sklyanin bracket we have to find the r-matrix, that is to solve an additional system of linear algebraic equations ($\Delta r(X) = \delta(X)$).

But what is more important, it is completely useless for non-coboundary Lie bialgebras. The method using the map Π works for an arbitrary given Lie bialgebra.

4.10.2 Lie bialgebra corresponding to the Lie-Poisson structure

Now the example of the opposite direction will be shown. We are going to find a Lie bialgebra corresponding to the Poisson-Lie group defined in examples 3.3.4 and 3.8.4. For arbitrary finite-dimensional real Lie algebra $\mathfrak{g}$, there is a Poisson bivector $\mathcal{P}$ defined on $\mathfrak{g}^*$ in coordinates $(X_1, \ldots, X_n)$ as

$$\mathcal{P}_{ij}(\xi) = \langle \xi, [X_i, X_j]_{\mathfrak{g}} \rangle, \tag{4.43}$$

where $\xi \in \mathfrak{g}^*$. $\mathfrak{g}^*$ is considered as Abelian Lie group $(\mathfrak{g}^*, +)$ with coordinates $(X_1, \ldots, X_n)$ given by an arbitrary chosen basis $(X_i)_{i=1}^n$ of $\mathfrak{g}$.

Let us denote the Lie algebra of $\mathfrak{g}^*$ as $\mathscr{G}$. $\mathscr{G}$ is of course an Abelian Lie algebra, if we choose its basis $(T^i)_{i=1}^n$ in $\mathscr{G}$ as $T^i = \left. \frac{\partial}{\partial X_i} \right|_0$, we have

$$[T^i, T^j] = 0. \tag{4.44}$$

In this particular basis we get

$$L_{\xi *}(T^i) \equiv L_{\xi *}(\frac{\partial}{\partial X_i}\Big|_0) = \frac{\partial}{\partial X_i}\Big|_\xi, \tag{4.45}$$

for every $\xi \in \mathfrak{g}^*$. Hence it is easy to find the integral curves corresponding to the left-invariant fields generated by the basis vectors T^k, starting from the point $0 \in \mathfrak{g}^*$:

$$exp(tT^k) = tX^k, \tag{4.46}$$

for all $k \in \{1, \dots, n\}$. Moreover, we can write

$$\mathcal{P}(\xi) = \mathcal{P}_{ij}(\xi) L_{\xi *}(T^i) \otimes L_{\xi *}(T^j),$$

for $\xi \in \mathfrak{g}^*$. Then for the calculation of $D\mathcal{P}$ we choose the extension of T^k as R_{T^k}, and

$$DP(T^k) \equiv [\mathcal{L}_{\bar{T^k}}(\mathcal{P})](0) = [\mathcal{L}_{R_{T^k}}(\mathcal{P})](0) = \frac{d}{dt}\Big|_{t=0} L_{e^{-tT^k}*}(\mathcal{P}(e^{tT^k})) =$$

$$= \frac{d}{dt}\Big|_{t=0} L_{e^{-tT^k}*}(\mathcal{P}_{ij}(e^{tT^k}) L_{e^{tT^k}*}(T^i) \otimes L_{e^{tT^k}*}(T^j)) = \frac{d}{dt}\Big|_{t=0} \mathcal{P}_{ij}(e^{tT^k}) T^i \otimes T^j =$$

$$= \frac{d}{dt}\Big|_{t=0} \langle tX^k, [X_i, X_j]\rangle T^i \otimes T^j = c_{ij}{}^k T^i \otimes T^j.$$

So we have found that the Lie algebra on $\mathscr{G}^*$ defined by $D\mathcal{P}$ is isomorphic to the original Lie algebra $\mathfrak{g}$. This is not a surprising fact.

Recall that there exists a linear isomorphism $\mathbf{A_0} : \mathfrak{g}^* \to T_0(\mathfrak{g}^*) \equiv \mathscr{G}$, such that $\mathbf{A}_0(X^i) = \frac{\partial}{\partial X_i}\Big|_0 \equiv T^i$. Let $(T_i)_{i=1}^n$ be a basis of $\mathscr{G}^*$ dual to (T^i). It's easy to see that the dual map $\mathbf{A}_0^*$ sends T_i to $X_i \in \mathfrak{g}$. $D\mathcal{P}$ defines a commutator on $\mathscr{G}^*$ as

$$[T_i, T_j]_{\mathscr{G}^*} = c_{ij}{}^k T_k.$$

Finally we get

$$\mathbf{A}_0^*([T_i, T_j]) = c_{ij}{}^k \mathbf{A}_0^*(T_k) = c_{ij}{}^k X_k = [X_i, X_j]_{\mathfrak{g}} = [\mathbf{A}_0^*(T_i), \mathbf{A}_0^*(T_j)].$$

Therefore A_0^* is the isomorphism between $\mathscr{G}^*$ and $\mathfrak{g}$.

See that $(\mathfrak{g}^*, +, \mathcal{P})$ is connected and simply connected (a vector space). The preceding computations have shown that it is a dual Poisson-Lie group to any Poisson-Lie group (G, Θ), such that $\mathfrak{g}$ is the Lie algebra of G, G equipped with a trivial Poisson structure Θ.

Chapter 5

Poisson sigma models

5.1 Introduction

A theory of Poisson manifolds gives us a possibility to construct an interesting field theory (classical). It was first introduced by Peter Schaller and Thomas Strobl in 1994 in [16]. It turned out that this theory included some of the well-known models, such as two-dimensional Yang-Mills model or $\mathcal{R}^2$ gravity model. The basic idea is to use a Poisson manifold as a target, instead of a Riemannian manifold.

We will in detail discuss the definition of Poisson sigma model fields, using the formalism of fibre bundles. After the necessary introduction of involved objects, we will proceed and give a definition of a globally defined action integral of the model.

We will show that local solutions of (general) Poisson sigma model exist, using the symplectic foliation of a Poisson manifold. We give an example of the solution for the simplest existing Poisson sigma model, using this method.

In articles [17], [16] or [18] the equations of motion "follow immediately". However, the process of their derivation was never presented there. In section 5.4 we will derive the equations of motion using the variational principle, after giving a geometric sense to the variation of the fields.

One of the most proclaimed properties of Poisson sigma models is an invariance of their action with respect to a class of infinitesimal transformations, called for certain reasons "gauge" transformations. However, this transformations depend on the coordinates in the manifold M and are thus suitable only for the particular Poisson-Sigma models. Moreover, we have not found their geometric interpretation yet.

These are two main reasons why we leave the gauge transformations for the future investigation and do not mention them in the following. One exception will be made in the case of linear Poisson sigma model, where we have no problem with local coordinates.

In section 5.5 we will deal with the most simple non-trivial example - linear Poisson sigma model, and find a coordinate-free form of its equations of motion

As the preceding chapters clue, we will be most interested in Poisson sigma models constructed on Poisson-Lie groups, with Lie group as a target. Even for this (quite wide) class of Poisson sigma models we are able to work out an amazingly elegant form of the equations of motion.

5.2 A brief introduction to fibre bundles

In this section we have to bring in a few definitions essential for the proper setting of Poisson sigma models. We do not intend to go in details, there exists a plenty of classical literature on this topic, see for example [19] or, for more detailed explanation, [20].

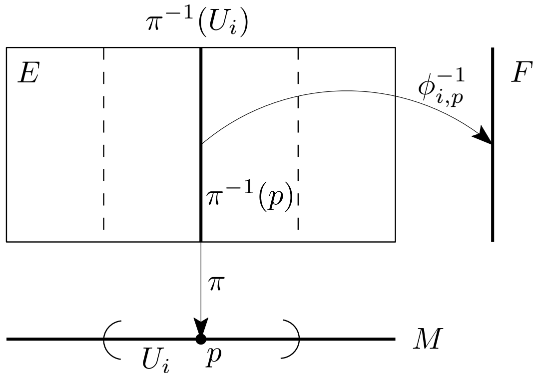

Figure 5.1: Fibre bundle (E, π, M, F)

Definition 5.2.1. A differentiable **fibre bundle** (E, π, M, F) consists of four elements:

(i.) A differentiable manifold E, called the **total space**.

(ii.) A differentiable manifold M, called the **base manifold**.

(iii.) A smooth surjective map $\pi : E \to M$, called the **projection**.

(iv.) A differentiable manifold F, called the **typical fibre**.

Moreover, fibre bundle has to obey the **local trivialization property**:

There exists an open covering $U = \{U_i\}$ of M together with a set of maps $\{\phi_i\}$, such that every map $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ is a diffeomorphism and $\pi \circ \phi_i = \pi_i$, where π_i denotes the natural projection of $U_i \times F$ on U_i.

Furthermore, for $p \in U_i$ the map $\phi_{i,p} := \phi_i(p, \cdot) : F \to \pi^{-1}(p)$ has to be a diffeomorphism.

For every $p \in M$, $F_p := \pi^{-1}(p)$ is called the **fibre** at p. Every fibre is a diffeomorphic image of the typical fibre F, $F_p \cong F$. For illustration see Fig. 5.1.

A shorthand notation $E \xrightarrow{\pi} M$ is often used.

Remark 5.2.2. Let U_i and U_j be two open sets from the covering of M, $U_i \cap U_j \neq \emptyset$. Let $p \in U_i \cap U_j$. From the definition of fibre bundle we have two trivialization diffeomorphisms ϕ_i and ϕ_j available. Let $u \in \pi^{-1}(p)$. Then we can define $f_i, f_j \in F$ as

$$f_i := \phi_{i,p}^{-1}(u), \ f_j := \phi_{j,p}^{-1}(u).$$

We can define $t_{ij}(p) := \phi_{i,p}^{-1} \circ \phi_{j,p}$. Clearly $t_{ij}(p) : F \to F$ is a diffeomorphism of F. Then $f_i = t_{ij}(p) f_j$. $t_{ij}(p)$ are called the transition functions at $p \in M$ and they in fact carry all the information about the fibre bundle.

The transition functions $t_{ij}(p)$ have to satisfy following consistency conditions:

(i.) $(\forall p \in U_i)\ (t_{ii}(p) = Id_F)$.

(ii.) $(\forall p \in U_i \cap U_j)\ (t_{ji}(p) = t_{ij}(p)^{-1})$.

(iii.) $(\forall p \in U_i \cap U_j \cap U_k)\ (t_{ik}(p) = t_{ij}(p) \circ t_{jk}(p))$.

For a given base manifold M with covering $\{U_i\}$, fibre F and a set of diffeomorphisms $t_{ij}(p)$ for all $p \in M$ satisfying the conditions above, there exists a unique (up to diffeomorphism) differentiable fibre bundle (E, π, M, F).

If all diffeomorphisms $t_{ij}(p)$ form a group G, we call G a structure group of the fibre bundle.

Definition 5.2.3. Fibre bundle $E \xrightarrow{\pi} M$ is called the **vector bundle**, if:

(i.) Typical fibre F is a vector space.

(ii.) Every fibre F_p, $p \in M$ is a vector space.

(iii.) Every diffeomorphism $\phi_{i,p} : F \to F_p$ is a linear isomorphism.

Example 5.2.4. Cotangent bundle (see example 3.5.7) is the vector bundle.

We have shown that its total space $T^*M = \bigcup_{x \in M} T^*_x(M)$ is a differentiable manifold. In this case the typical fibre $F = \mathbb{R}^n$. A fibre F_x at $x \in M$ is the cotangent space $T^*_x(M)$.

Let $\mathcal{O} \subset M$ be an open set, where we have the coordinates $(x^1, \ldots, x^n)$. Then on $\pi^{-1}(\mathcal{O})$ we have the canonical coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$. For every $[x, (a_1, \ldots, a_n)] \in \mathcal{O} \times \mathbb{R}^n$ we set the map $\phi_{\mathcal{O}} : \mathcal{O} \times \mathbb{R}^n \to \pi^{-1}(\mathcal{O})$ as:

$$\phi_{\mathcal{O}}([x, (a_1, \ldots, a_n)]) := a_i \, dx^i\big|_x .$$

This map is clearly a diffeomorphism. Moreover, $\phi_{\mathcal{O},x} : (a_1, \ldots, a_n) \mapsto a_i \, dx^i\big|_x$ is a linear isomorphism of vector spaces.

T^*M is thus a vector bundle.

Definition 5.2.5. A global smooth section of the fibre bundle $E \xrightarrow{\pi} M$ is a smooth map $\sigma : M \to E$, such that $\pi \circ \sigma = Id_M$. In the other words, σ always maps $p \in M$ only to "its own" fibre $\pi^{-1}(p)$.

A local smooth section is a smooth map $\sigma : U \to E$ of the same properties as a global smooth section, for $U \subset M$.

We denote the set of all smooth sections as $\Gamma(M, E)$. The subset of global smooth sections is denoted as $\Gamma_G(M, E)$, the subset of local smooth sections on $U \subset M$ as $\Gamma(U, E)$.

Definition 5.2.6. Let $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} N$ be two fibre bundles. A pair $(f, g) : E \to E'$ is called the **bundle map**, if:

(i.) $f : M \to N$ is a smooth map of the base manifolds.

(ii.) $g : E \to E'$ is a smooth map of the total spaces.

(iii.) Following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ \downarrow_{\pi} & & \downarrow_{\pi'} \\ M & \xrightarrow{f} & N \end{array}$$

In the other words, for $u \in \pi^{-1}(p)$ the image $g(u)$ lies above $f(p)$, that is $\pi'(g(u)) = f(p)$.

If E and E' are vector bundles, we call (f, g) the **vector bundle map**, if $(f, g) : E \to E'$ is a bundle map and $g : E \to E'$ is linear "in the fibres", i.e.:

$$(\forall p \in M)(\forall u, v \in \pi^{-1}(p))(\forall \alpha \in \mathbb{R}) \ \big(g(\alpha u + v) = \alpha g(u) + g(v)\big). \tag{5.1}$$

Definition 5.2.7. Let N and M be differentiable manifolds. M is equipped with a fibre bundle structure (E, π, M, F). Let $\varphi : N \to M$ be a smooth map. Then we can induce a fibre bundle structure above N, called the **pullback bundle**, as follows:

Its total space $\varphi^*(E)$ is defined as

$$\varphi^*(E) := \{(p, u) \in N \times E \mid \varphi(p) = \pi(u)\}.$$

Its base manifold is N, projection is $\pi' : (p, u) \in \varphi^*(E) \mapsto p \in N$ and typical fibre F remains the same. It is clear that the fibre F_p at $p \in N$ is just the same, as the fibre $F_{\varphi(p)}$ of the original bundle.

Let $\{U_i\}$ be an open covering of M. Then $\{\varphi^{-1}(U_i)\}$ is an open covering of N (φ is continuous map). Let $p \in \varphi^{-1}(U_i)$. Then $\varphi(p)$ lies in U_i, where we have a local trivialization map ϕ_i of the original fibre bundle.

We define a local trivialization map ψ_i corresponding to $\varphi^{-1}(U_i)$ as follows:

$$(\forall p \in \varphi^{-1}(U_i))(\forall f \in F) \ \big(\psi_i(p, f) := (p, \phi_i(\varphi(p), f))\big).$$

The transition functions $t^*_{ij}(p)$ of the pullback bundle can be found as

$$t^*_{ij}(p) = t_{ij}(\varphi(p)),$$

where t_{ij} are the transition functions of the original bundle.

Thus $(\varphi^*(E), \pi', N, F)$ forms a well-defined differentiable fibre bundle.

5.3 Fields, action

Let us suppose we have a 2-dimensional differentiable orientable manifold Σ, called usually the **worldsheet**. Let us remark that in general we do not demand a (pseudo-)metric structure on Σ. We usually take $\partial\Sigma = \emptyset$ (empty boundary) and we want Σ to be such manifold, where integration and the Stokes' theorem have a good sense. Otherwise we have to impose some (boundary) conditions on the fields.

Next suppose an n-dimensional Poisson manifold $(M, \mathcal{P})$. Again we do not demand M to be a (pseudo-)Riemannian manifold. The manifold M is called a **target manifold**.

Definition 5.3.1. Dynamical field of Poisson sigma model is a vector bundle map $(X, A) : T\Sigma \to T^*M$, where $T\Sigma$ is the tangent bundle of Σ and T^*M is the cotangent bundle of M.

$X : \Sigma \to M$ is a smooth map of the base manifolds, whereas $A : T\Sigma \to T^*M$ is a smooth map of the total spaces.

Remark 5.3.2. Let $E \overset{\pi}{\to} M$ be a vector bundle. Its set of global smooth sections $\Gamma_G(M, E)$ has a natural $C^\infty(M)$-linear structure:

$$\big(\forall \sigma, \tau \in \Gamma_G(M, E)\big)\big(\forall f \in C^\infty(M)\big)\big(\forall p \in M\big) \ \big((f\sigma + \tau)(p) := f(p)\sigma(p) + \tau(p)\big).$$

In particular, we will use this property for the pullback bundle $X^*(T^*M)$ and the set $\Gamma_G(\Sigma, X^*(T^*M))$.

Definition 5.3.3. A 1-form α on Σ with values in the set of global smooth sections $\Gamma_G(\Sigma, X^*(T^*M))$ is a smooth assignment $p \in \Sigma \mapsto \alpha(p)$, where $\alpha(p) : T_p(\Sigma) \to \pi'^{-1}(p) \equiv T^*_{X(p)}(M)$ is a linear map. π' denotes the projection of the pullback bundle $X^*(T^*M)$.

We define the action of α on a smooth vector field $V \in \mathfrak{X}(\Sigma)$ as

$$\langle \alpha, V\rangle(p) := \langle \alpha(p), V_p\rangle \in T^*_{X(p)}(M), \tag{5.2}$$

for every $p \in \Sigma$, where V_p denotes the value of V at p. $\langle \alpha, V\rangle$ can be thus interpreted as the global section of the pullback bundle $X^*(T^*M)$.

The requirement of the smooth assignment can be then more precisely stated as the smoothness of the section $\langle \alpha, V\rangle$.

Therefore α defines a $C^\infty(\Sigma)$-linear map from $\mathfrak{X}(\Sigma)$ to $\Gamma_G(\Sigma, X^*(T^*M))$.

In the same way we can define a k-form on Σ with values in $\Gamma_G(\Sigma, X^*(T^*M))$ and a k-form on Σ with values in $\Gamma_G(\Sigma, X^*(TM))$, where $X^*(TM)$ is the pullback bundle of TM by X.

Remark 5.3.4. Previous definition can be equivalently stated as follows: 1-form α on Σ with values in $\Gamma_G(\Sigma, X^*(T^*M))$ is a $C^\infty(\Sigma)$-linear map from $\mathfrak{X}(\Sigma)$ to $\Gamma_G(\Sigma, X^*(T^*M))$.

The reason and proof is similar to the same statement for ordinary 1-forms $\Omega^1(\Sigma)$.

Definition 5.3.5. The space of k-forms on Σ ($k \in \{0, 1, 2\}$) with values in $\Gamma_G(\Sigma, X^*(T^*M))$ is denoted as $\Omega^k(\Sigma, X^*(T^*M))$. In the same way, $\Omega^k(\Sigma, X^*(TM))$ denotes the space of k-forms on Σ with values in $\Gamma_G(\Sigma, X^*(TM))$.

Lemma 5.3.6. *Let (X, A) be a vector bundle map $(X, A) : T\Sigma \to T^*M$.*

The map A of the total spaces can be considered as the element of $\Omega^1(\Sigma, X^*(T^*M))$.

Proof. Let A be the total space map from the vector bundle map (X, A). We can define $\alpha_A \in \Omega^1(\Sigma, X^*(T^*M))$ as

$$\langle \alpha_A, V\rangle(p) \equiv \langle \alpha_A(p), V_p\rangle := A(V_p) \in T^*_{X(p)}(M) \equiv \pi'^{-1}(p), \tag{5.3}$$

where $V \in \mathfrak{X}(\Sigma)$ is a smooth vector field. $\alpha_A(p)$ thus indeed maps linearly vectors from $T_p(\Sigma)$ to $\pi'^{-1}(p)$. From the smoothness of V and of the map A follows, that the resulting section $\langle \alpha_A, V\rangle$ is smooth. Hence $\alpha_A \in \Omega^1(\Sigma, X^*(T^*M))$.

Conversely, let $\alpha \in \Omega^1(\Sigma, X^*(T^*M))$. Then for $u \in T\Sigma$ we define the map A_α as

$$A_\alpha(u) := \langle \alpha(\pi(u)), u\rangle, \tag{5.4}$$

where π is the projection of the tangent bundle $T\Sigma$. The map A_α clearly satisfies the condition

$$\widetilde{\pi} \circ A_\alpha = X \circ \pi,$$

where $\widetilde{\pi}$ denotes the projection of the cotangent bundle T^*M and it is linear in the fibres. The smoothness of A_α follows from the smoothness of α and the projection π. Hence (X, A_α) is a vector bundle map. ■

Remark 5.3.7. Since now we will not distinguish between A and α_A, and we will use the notation $A(p) \equiv \alpha_A(p)$ for $p \in \Sigma$.

Remark 5.3.8. Let $(y^1, \ldots, y^n)$ be a set of local coordinates on M. A can be locally expanded as

$$A(p) = A_i(p)\, dy^i\big|_{X(p)}, \tag{5.5}$$

for $p \in \Sigma$. $A_i \in \Omega^1(\Sigma)$ are uniquely determined 1-forms on Σ, called the **component 1-forms** of A. We will use the notation

$$A = A_i dy^i. \tag{5.6}$$

Remark 5.3.9. In fact, one has to be very careful, because expansion (5.5) has the good sense only for such $p \in \Sigma$, where $X(p)$ stays in the area of M, where coordinates $(y^1, \ldots, y^n)$ are defined. Hence A_i are, strictly speaking, not uniquely determined in the "evil points" of Σ.

However, taking Poisson-Lie groups as target manifolds, we have global frame fields on M - left-(right-)invariant vector fields. This solves our problem in such cases.

Alternatively, we can impose on the map X to not "come out" of the chosen coordinate patch.

Let us examine for a moment the map X_* tangent to the map $X : \Sigma \to M$. Constructed at given point $p \in \Sigma$, it maps linearly $T_p(\Sigma) \to T_{X(p)}(M)$. Therefore we can define a 1-form dX on Σ with values in $\Gamma_G\big(\Sigma, X^*(TM)\big)$, as:

$$\langle dX, Y\rangle(p) := X_*(Y_p) \in T_{X(p)}(M),$$

for $Y \in \mathfrak{X}(\Sigma)$. If we write it in the local coordinates $(y^1, \ldots, y^n)$ on M and local coordinates (σ^1, σ^2) on Σ, we get

$$\langle dX, Y\rangle(p) := Y^\mu(p) X_*(\frac{\partial}{\partial \sigma^\mu}\Big|_p) = Y^\mu(p)\, \frac{\partial X^i}{\partial \sigma^\mu}\Big|_p \frac{\partial}{\partial y^i}\Big|_{X(p)}.$$

Hence

$$dX(p) = dX^i(p)\, \frac{\partial}{\partial y^i}\Big|_{X(p)},$$

where $dX^i := X^*(dy^i)$. Let us remark that dX can be viewed as a total space map of the bundle map $(X, dX) : T\Sigma \to TM$.

Canonical pairing on M allows us to define the induced pairing of k-forms on Σ with values in the global sections of pullback bundles $X^*(TM)$ and $X^*(T^*M)$ respectively.

Definition 5.3.10. Let $A \in \Omega^k(\Sigma, X^*(T^*M))$, $B \in \Omega^l(\Sigma, X^*(TM))$. $A = A_i dy^i$, $B = B^j \frac{\partial}{\partial y^j}$. Then we define a pairing of A with B as

$$\langle A, B\rangle(p) := A_i(p) \wedge B^j(p)\, \langle dy^i\big|_{X(p)}, \frac{\partial}{\partial y^j}\Big|_{X(p)}\rangle = A_i(p) \wedge B^i(p), \tag{5.7}$$

for all $p \in \Sigma$. Hence $\langle A, B\rangle \in \Omega^{k+l}(\Sigma)$. This definition does not depend on the particular choice of coordinates in M.

Let $p \in M$ and $\mathcal{P}$ be a Poisson bivector on M, Let $A \in \Omega^1(\Sigma, X^*(T^*M))$. We define

$$i_A(\mathcal{P})(X(p)) := A_i(p)\mathcal{P}(X(p))(dy^i|_{X(p)}, \cdot) = A_i(p)\mathcal{P}^{ij}(X(p))\, \frac{\partial}{\partial y^j}\Big|_{X(p)}.$$

This is an expansion of some $V \in \Omega^1(\Sigma, X^*(TM))$. We set $i_A(\mathcal{P})(X) := V$.

Definition 5.3.11. Let Σ be a 2-dimensional orientable differentiable manifold, let $(M, \mathcal{P})$ be an n-dimensional Poisson manifold. A **Poisson sigma model** is a field model defined by the action integral

$$S[X, A] := \int_{\Sigma} \langle A, dX \rangle - \frac{1}{2} \langle A, i_A(\mathcal{P})(X) \rangle. \tag{5.8}$$

Suppose we have the local coordinates $(y^1, \ldots, y^n)$ in some neighbourhood $U \subset M$, $\mathcal{P}^{ij} \equiv \mathcal{P}(dy^i, dy^j)$. If $X(\Sigma) \subset U$, we can write the action as

$$S[X, A] := \int_{\Sigma} A_i \wedge dX^i + \frac{1}{2} \mathcal{P}^{jk}(X) A_j \wedge A_k, \tag{5.9}$$

where $A = A_i dy^i$ and $dX^i = X^*(dy^i)$.

Remark 5.3.12. Poisson sigma models are in many articles ([18], [16]) defined by the action (5.9), with the map X not restricted to U. Strictly speaking, this is not correct, because there exist points of Σ, where the integrand has no sense at all. However, we can always locally write the Lagrangian:

$$L[X, A](p) = A_i(p) \wedge dX^i(p) + \frac{1}{2} \mathcal{P}^{jk}(X(p)) A_j(p) \wedge A_k(p), \tag{5.10}$$

for such $p \in \Sigma$, where $X(p) \in U$.

5.4 Variational principle, equations of motion

In this section we will in detail derive the equations of motion of the Poisson sigma model, using the action (5.8) and a variational principle.

It is quite simple to get the equations of motion from (5.9), putting $\widetilde{X}^i = X^i + \epsilon Y^i$, $\widetilde{A}_i = A_i + \widetilde{\epsilon} B_i$ and using the ordinary per partes trick in the calculation of $S[\widetilde{X}, \widetilde{A}] - S[X, A]$. However, this approach is heavily coordinate-dependent, especially in the case, where the form of action 5.9 has no real sense. This led us to the following idea.

We will parametrize each variation of the fields (X, A) by infinitesimal constants $\epsilon, \widetilde{\epsilon} \in \mathbb{R}$, $|\epsilon|, |\widetilde{\epsilon}| \ll 1$, 1-form $B \in \Omega^1(\Sigma, X^*(T^*M))$ and by a smooth vector field $Y \in \mathfrak{X}(M)$ on M, such that its local flow $\phi^Y_\epsilon(X(p))$ is defined for all $p \in \Sigma$. If $\partial\Sigma \neq \emptyset$, we have to impose $(\forall q \in \partial\Sigma)\ (Y(X(q)) = 0)$.

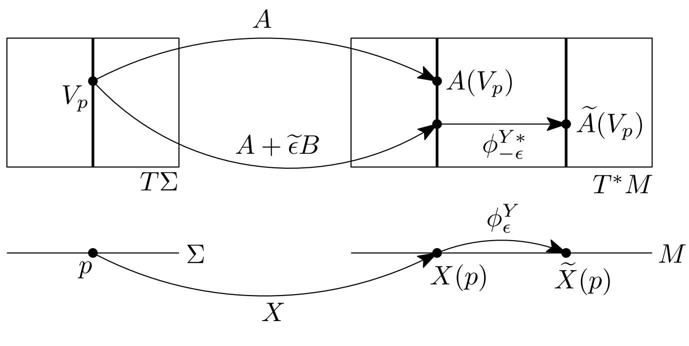

Figure 5.2: Variation of fields (X, A)

Shifted fields $\widetilde{X}$ and $\widetilde{A}$ are then set as

$$\widetilde{X}(p) := \phi^Y_\epsilon(X(p)), \tag{5.11}$$

$$\widetilde{A}(V_p) := \phi_{-\epsilon}^{Y*}\big((A + \widetilde{\epsilon} B)(V_p)\big), \tag{5.12}$$

for all $p \in \Sigma$ and $V_p \in T_p(\Sigma)$. For illustration, see the figure 5.2. It is obvious that $(\widetilde{X}, \widetilde{A})$ is again a vector bundle map $T\Sigma \to T^*M$.

Let $(y^1, \dots, y^n)$ be a set of local coordinates on $U \subset M$. Vector field Y can be for $x \in U$ expanded as

$$Y(x) = Y^i(x) \left.\frac{\partial}{\partial y^i}\right|_x . \tag{5.13}$$

We define $\widetilde{Y} \in \Omega^0(\Sigma, X^*(TM))$ as

$$\widetilde{Y}(p) := Y(X(p)), \tag{5.14}$$

for $p \in \Sigma$. Its local expansion is $\widetilde{Y}(p) = Y^i(X(p)) \left.\frac{\partial}{\partial y^i}\right|_{X(p)} \equiv \widetilde{Y}^i(p) \left.\frac{\partial}{\partial y^i}\right|_{X(p)}$, where we denote $\widetilde{Y}^i(p) := Y^i(X(p))$.

Let us emphasize, that we have no troubles using local coordinate expansions of objects in the Lagrangian - it is a well defined 2-form on Σ, which can be analysed point-wise. Problem arises only when we write the local expansion under the integral.

$\widetilde{A}$ can be locally expanded as

$$\widetilde{A}(p) = (A_i + \widetilde{\epsilon} B_i)(p)\ \phi_{-\epsilon}^{Y*}(dy^i\big|_{X(p)}). \tag{5.15}$$

For $d\widetilde{X}$, $p \in \Sigma$ and $V_p \in T_p(\Sigma)$ we have

$$\langle d\widetilde{X}(p), V_p\rangle \equiv \widetilde{X}_*(V_p) = \phi_{\epsilon *}^{Y}(X_*(V_p)),$$

and thus locally

$$d\widetilde{X}(p) = dX^i(p)\ \phi_{\epsilon *}^{Y}(\left.\frac{\partial}{\partial y^i}\right|_{X(p)}). \tag{5.16}$$

We are now ready to proceed with the computation of the first order (in ϵ and $\widetilde{\epsilon}$) term in the difference of the Lagrangians $L[\widetilde{X}, \widetilde{A}] - L[X, A]$, where

$$L[X, A] := \langle A, dX\rangle - \frac{1}{2}\langle A, i_A(\mathcal{P})(X)\rangle. \tag{5.17}$$

From the definition of pairing (5.7) and the local expansions (5.15) and (5.16) of $\widetilde{A}$ and $d\widetilde{X}$ respectively it is clear, that

$$\langle \widetilde{A}, d\widetilde{X}\rangle = \langle A, dX\rangle + \widetilde{\epsilon}\langle B, dX\rangle. \tag{5.18}$$

To deal with the second term of $L[\widetilde{X}, \widetilde{A}]$, we should remind that ϕ_ϵ^Y is a diffeomorphism of M. Therefore we can for $x \in U$ use $\phi_{\epsilon *}^{Y}(\left.\frac{\partial}{\partial y^i}\right|_x)$ as the basis of the tangent space $T_{\phi_\epsilon^Y(x)}(M)$.

Poisson bivector $\mathcal{P}$ can be thus at $\widetilde{X}(p) \equiv \phi_\epsilon^Y(X(p))$ expanded as

$$\mathcal{P}(\widetilde{X}(p)) = \mathcal{P}_*^{ij}(\widetilde{X}(p))\ \phi_{\epsilon *}^{Y}(\left.\frac{\partial}{\partial y^i}\right|_{X(p)}) \otimes \phi_{\epsilon *}^{Y}(\left.\frac{\partial}{\partial y^j}\right|_{X(p)}), \tag{5.19}$$

where

$$\begin{aligned}\mathcal{P}_*^{ij}(\widetilde{X}(p) = \mathcal{P}(\widetilde{X}(p))\big(\phi_{-\epsilon}^{Y*}(dy^i\big|_{X(p)}), \phi_{-\epsilon}^{Y*}(dy^j\big|_{X(p)})\big) \equiv \\ \equiv \phi_{-\epsilon *}^{Y}\big(\mathcal{P}(\widetilde{X}(p))\big)\Big(\ dy^i\big|_{X(p)}, dy^j\big|_{X(p)}\Big).\end{aligned} \tag{5.20}$$

In the following we will omit the explicit writing of (p) in every term, but we still mean everything written at the particular point $p \in \Sigma$. For infinitesimal ϵ we can rewrite $\mathcal{P}^{ij}_*(\widetilde{X})$ as

$$\mathcal{P}_*(\widetilde{X})^{ij} = \phi^Y_{-\epsilon *}\big(\mathcal{P}(\widetilde{X})\big)\Big(\, dy^i\big|_X \, , dy^j\big|_X \Big) =$$
$$= \big(\mathcal{P}(X) + \epsilon[\mathcal{L}_Y(\mathcal{P})](X)\big)\Big(\, dy^i\big|_X \, , dy^j\big|_X \Big) =$$
$$= \mathcal{P}^{ij}(X) + \epsilon[\mathcal{L}_Y(\mathcal{P})]^{ij}(X).$$

Then we can write

$$i_{\widetilde{A}}(\mathcal{P})(\widetilde{X}) = \mathcal{P}(\widetilde{X})(\widetilde{A}, \cdot) = (A_i + \widetilde{\epsilon} B_i) \; \mathcal{P}(\widetilde{X})\big(\phi^{Y*}_{-\epsilon}(\, dy^i\big|_X), \cdot\big) =$$
$$= (A_i + \widetilde{\epsilon} B_i)\mathcal{P}^{ij}_*(\widetilde{X}) \; \phi^Y_{\epsilon *}\Big(\frac{\partial}{\partial y^j}\Big|_X \Big).$$

Hence

$$-\frac{1}{2}\langle \widetilde{A}, i_{\widetilde{A}}(\mathcal{P})(\widetilde{X})\rangle = -\frac{1}{2}(A_k + \widetilde{\epsilon} B_k) \wedge (A_i + \widetilde{\epsilon} B_i)\mathcal{P}^{ik}_*(\widetilde{X}) =$$
$$= -\frac{1}{2}A_k \wedge A_i \mathcal{P}^{ik}(X) - \widetilde{\epsilon} B_k \wedge A_i \mathcal{P}^{ik} - \epsilon\frac{1}{2}A_k \wedge A_i[\mathcal{L}_Y(\mathcal{P})]^{ik},$$

where we omitted the second order terms in the infinitesimal parameters ϵ and $\widetilde{\epsilon}$. Therefore finally

$$-\frac{1}{2}\langle \widetilde{A}, i_{\widetilde{A}}(\mathcal{P})(\widetilde{X})\rangle = -\frac{1}{2}\langle A, i_A(\mathcal{P})(X)\rangle - \widetilde{\epsilon}\langle B, i_A(\mathcal{P})(X)\rangle - \epsilon\frac{1}{2}\langle A, i_A[\mathcal{L}_Y(\mathcal{P})](X)\rangle. \tag{5.21}$$

Putting together (5.18) and (5.21) we get

$$L[\widetilde{X}, \widetilde{A}] - L[X, A] = \widetilde{\epsilon}\langle B, dX - i_A(\mathcal{P})(X)\rangle - \epsilon\frac{1}{2}\langle A, i_A[\mathcal{L}_Y(\mathcal{P})](X)\rangle \tag{5.22}$$

We will state and prove a following lemma to proceed:

Lemma 5.4.1. $\widetilde{Y} \in \Omega^0(\Sigma, X^*(TM))$ *defined in (5.14) satisfies (in the first order in* ϵ*)*

$$d\widetilde{Y}^i = Y^i{}_{,m}(X)dX^m. \tag{5.23}$$

Proof. Expansion of $\widetilde{X} = \phi^Y_\epsilon(X)$ in local coordinates on M reads

$$\widetilde{X}^i = X^i + \epsilon Y^i(X) = X^i + \epsilon\widetilde{Y}^i.$$

Hence

$$d\widetilde{X}^i = dX^i + \epsilon d\widetilde{Y}^i. \tag{5.24}$$

On the other side, as we know, $d\widetilde{X}^i$ constitute the component 1-forms of $d\widetilde{X} \in \Omega^1(\Sigma, \widetilde{X}^*(TM))$. We can proceed from (5.16):

$$d\widetilde{X} = dX^i \; \phi^Y_{\epsilon *}\Big(\frac{\partial}{\partial y^i}\Big|_X \Big) = dX^i\Big(\frac{\partial}{\partial y^i}\Big|_{\widetilde{X}} - \epsilon[\mathcal{L}_Y(\frac{\partial}{\partial y^i})](\widetilde{X})\Big) =$$
$$= dX^i \; \frac{\partial}{\partial y^i}\Big|_{\widetilde{X}} + \epsilon dX^i \; Y^m{}_{,i}(\widetilde{X}) \; \frac{\partial}{\partial y^m}\Big|_{\widetilde{X}} = dX^i \; \frac{\partial}{\partial y^i}\Big|_{\widetilde{X}} + \epsilon dX^i \; Y^m{}_{,i}(X) \; \frac{\partial}{\partial y^m}\Big|_{\widetilde{X}} =$$
$$= (dX^i + \epsilon Y^i{}_{,m}(X)dX^m) \; \frac{\partial}{\partial y^i}\Big|_{\widetilde{X}}.$$

Comparison with (5.24) gives us (5.23). ■

We can now step to the derivation of the equations of motion. We impose the condition of extremality for (X, A), that is

$$S[\widetilde{X}, \widetilde{A}] - S[X, A] = 0. \tag{5.25}$$

From (5.22) this is equivalent to

$$\int_\Sigma \widetilde{\epsilon}\langle B, dX - i_A(\mathcal{P})(X)\rangle - \epsilon\frac{1}{2}\langle A, i_A[\mathcal{L}_Y(\mathcal{P})](X)\rangle = 0.$$

By putting $\epsilon = 0$ (variation of the 1-form A only), we get the first equation of the motion in the form

$$dX = i_A(\mathcal{P})(X), \tag{5.26}$$

or in the local coordinates as

$$dX^i + \mathcal{P}^{ij}(X)A_j = 0. \tag{5.27}$$

For the analysis of the second term we can (and have to) use the equation 5.27:

$$-\frac{1}{2}\langle A, i_A(\mathcal{L}_Y(P))(X)\rangle = \frac{1}{2}A_i \wedge A_k\ [\mathcal{L}_Y(\mathcal{P})]^{ik}(X) =$$

$$= \frac{1}{2}A_i \wedge A_k\Big(Y(\mathcal{P}^{ik})(X) - Y^i{}_{,m}(X)\mathcal{P}^{mk}(X) - Y^k{}_{,m}(X)\mathcal{P}^{im}(X)\Big) =$$

$$= \frac{1}{2}A_i \wedge A_k Y(\mathcal{P}^{ik})(X) + \mathcal{P}^{mi}(X)A_i \wedge A_k Y^k{}_{,m}(X) \overset{(5.27)}{=}$$

$$\overset{(5.27)}{=} \frac{1}{2}A_i \wedge A_k\widetilde{Y}^m\mathcal{P}^{ik}{}_{,m}(X) - dX^m Y^k{}_{,m}(X) \wedge A_k \overset{(5.23)}{=}$$

$$\overset{(5.23)}{=} \frac{1}{2}A_i \wedge A_k\widetilde{Y}^m\mathcal{P}^{ik}{}_{,m}(X) - d\widetilde{Y}^k \wedge A_k =$$

$$= \widetilde{Y}^m\Big(dA_m + \frac{1}{2}\mathcal{P}^{ik}{}_{,m}(X)A_i \wedge A_k\Big) - d(\widetilde{Y}^k A_k).$$

The boundary term is clearly coordinate invariant and it vanishes under integration. The first term is not coordinate dependent for (X, A) satisfying (5.27), which is enough for the derivation of the extremal equation. Both terms can be thus with no problems integrated. Hence we get the second equation (we can choose the vector field Y (almost) arbitrarily):

$$dA_k + \frac{1}{2}\mathcal{P}(X)^{ij}{}_{,k}A_i \wedge A_j = 0. \tag{5.28}$$

This equation cannot be written globally in general, but it transforms itself well for (X, A) solving the first equation. We will again find various possibilities for linear Poisson sigma models or Poisson-Lie sigma models, where we can use more global structures. We can sum up the preceding text in the following proposition [16]:

Proposition 5.4.2. *The extremal fields (X, A) of the Poisson sigma model given by the action (5.8) have to satisfy the equations written locally as*

$$dX^i + \mathcal{P}^{ij}(X)A_j = 0, \tag{5.29}$$

$$dA_k + \frac{1}{2}\mathcal{P}^{ij}{}_{,k}(X)A_i \wedge A_j = 0. \tag{5.30}$$

Remark 5.4.3. In local coordinates (σ^1, σ^2) on Σ, we can find the expansion $A_i = A_{i\mu} d\sigma^\mu$. In fact then, (5.29) is a system of $2n$ partial differential equations, where (5.30) is a system of n partial differential equations. Together we have $3n$ partial differential equations for $3n$ unknown functions $X^1, \ldots, X^n, A_{11}, A_{12}, \ldots, A_{n1}, A_{n2}$ on $\mathbb{R}^2$ (variables are σ^1 and σ^2).

Until now, there was no reason to use such special bivector field - Poisson bivector. It turns out ([21]) that Jacobi identities ensure the consistency of the equations of motion. If we do the exterior derivative of the equation (5.29), we get

$$\mathcal{P}^{ij}{}_{,k}(X) dX^k \wedge A_j + \mathcal{P}^{ij}(X) dA_j = 0.$$

Using (5.29), we then have

$$\mathcal{P}^{ij}(X) dA_j = \mathcal{P}^{ij}{}_{,k}(X)\mathcal{P}^{kl}(X) A_l \wedge A_j = \frac{1}{2}\big(\mathcal{P}^{ij}{}_{,k}(X)\mathcal{P}^{kl}(X) - \mathcal{P}^{il}{}_{,k}(X)\mathcal{P}^{kj}(X)\big) A_l \wedge A_j.$$

Left hand side of this equation can be rewritten using (5.30) as

$$\mathcal{P}^{ij}(X) dA_j = -\frac{1}{2}\mathcal{P}^{ij}(X)\mathcal{P}^{kl}{}_{,j}(X) A_k \wedge A_l.$$

Putting both sides together and changing of the indices leads us into

$$\frac{1}{2}\big(\mathcal{P}^{ij}{}_{,k}(X)\mathcal{P}^{kl}(X) + cyclic\{i,j,l\}\big) A_l \wedge A_j = 0.$$

This has to be satisfied for every possible solution of the equations of motion. For $\mathcal{P}$ a Poisson bivector there hold the Jacobi identities and the term in the brackets vanishes. In the opposite case that means that equation (5.29) contradicts (5.30) and we get more conditions imposed on the field A.

We can ask if there exist any solutions of a Poisson sigma model. Locally, the answer is positive, thanks to the local canonical coordinates, introduced in theorem 3.7.12.

Proposition 5.4.4. *Let Σ and $(M, \mathcal{P})$ form a Poisson sigma model defined by the action (5.8). Let $\mathcal{F}_\mathcal{P}$ be a symplectic foliation of $(M, \mathcal{P})$.*

For a given point $x \in M$ we can find its neighbourhood U_x, such that (X, A) are the fields of Poisson sigma model (5.8) solving the equations of motion (5.29) and (5.30), and $X(\Sigma) \subset U_x$.

Moreover $X(\Sigma) \subset \mathcal{F}_x$, where $\mathcal{F}_x \in \mathcal{F}_\mathcal{P}$ is a symplectic leaf containing x.

Proof. Denote $2s$ the rank of $\mathcal{P}$ at x, i.e. $\rho_\mathcal{P}(x) = 2s$. From theorem 3.7.12 we know, that there exists a neighbourhood V of x with the local canonical coordinate system $(x^1, \ldots, x^s, p_1, \ldots, p_s, z^1, \ldots, z^{n-2s})$, such that $z^i(V \cap \mathcal{F}_x) = 0$ and the Poisson bivector matrix has the form

$$\mathcal{P}^{ij}(V \cap \mathcal{F}_x) = \left(\begin{array}{cc|c} 0 & -\mathbf{1}_s & 0 \\ \mathbf{1}_s & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) \equiv \left(\begin{array}{c|c} \mathcal{P}_{(s)} & 0 \\ \hline 0 & 0 \end{array}\right), \tag{5.31}$$

where $\mathbf{1}_s$ denotes an $s \times s$ unit matrix. Thus we define $U_x := V \cap \mathcal{F}_x$.

We will denote indices corresponding to (x^i, p_i) coordinates with Greek letters, indices corresponding to (z^i) coordinates with capital Latin letters. For $X(\Sigma) \subset U_x$ the equations of motion take the form

$$dX^\alpha + \mathcal{P}^{\alpha\beta}_{(s)} A_\beta = 0, \tag{5.32}$$

$$dX^I = 0, \tag{5.33}$$

$$dA_\alpha = 0, \tag{5.34}$$

$$dA_I = 0. \tag{5.35}$$

From the condition $X(\Sigma) \subset U_x$ and (5.33) we get $X^I = 0$. We can find the symplectic form $\omega^{(s)}$ corresponding to $\mathcal{P}_{(s)}$, that is $\omega^{(s)}_{\gamma\alpha}\mathcal{P}^{\alpha\beta}_{(s)} = -\delta^\beta_\gamma$. Together with (5.32) we get $A_\beta = \omega^{(s)}_{\beta\alpha} dX^\alpha$. Thus A_β are fully determined by X^α. Condition (5.34) is then satisfied automatically ($\omega^{(s)}_{\beta\alpha}$ are constant on Σ). Finally, using (5.35) A_I can be chosen as arbitrary closed 1-forms on Σ. The $2s$ functions X^α are not restricted by the equations of motion at all. We can choose them as arbitrary smooth functions, such that $X(\Sigma) \subset U_x$. ■

This proposition gives us a one possible way to find the solutions of Poisson sigma models. It is not that difficult to find the symplectic leaves. What is really non-trivial (and in general almost impossible task) is to find the local canonical coordinates (Darboux coordinates on the leaves). For a Poisson bivector given in (some) coordinates, one has to find a Jacobi matrix of the transformation, such that $\mathcal{P}$ in new coordinates gets the form (5.31). Unluckily, this usually leads to a complicated system of partial differential equations, which is impossible to solve, although the solution exists for sure in some neighbourhood of a chosen point.

Example 5.4.5. Let $\mathfrak{g} = span\{X_1, X_2\}$ be a 2-dimensional Lie algebra with basis (X_1, X_2) and a Lie bracket given as

$$[X_1, X_2] = X_2. \tag{5.36}$$

Then we can construct a Poisson structure on $\mathfrak{g}^*$, as was shown in 3.3.4. A Poisson bivector matrix in the coordinates (X_1, X_2) is

$$\mathcal{P}(\xi) = \begin{pmatrix} 0 & \xi_2 \\ -\xi_2 & 0 \end{pmatrix}. \tag{5.37}$$

If we calculate the Hamiltonian fields corresponding to the coordinate functions, we get

$$\zeta_{X_1} = \mathcal{P}^{12}\frac{\partial}{\partial X_2} = X_2\frac{\partial}{\partial X_2}, \tag{5.38}$$

$$\zeta_{X_2} = \mathcal{P}^{21}\frac{\partial}{\partial X_1} = -X_2\frac{\partial}{\partial X_1}. \tag{5.39}$$

From this we can immediately see that we get the following symplectic leaves:

$$\mathcal{F}_I = \{\xi \in \mathfrak{g}^* | \xi_2 > 0\}, \tag{5.40}$$

$$\mathcal{F}_{II} = \{\xi \in \mathfrak{g}^* | \xi_2 < 0\}, \tag{5.41}$$

$$\mathcal{F}_C = \{\xi \in \mathfrak{g}^* | \xi_1 = C, \xi_2 = 0\}, \tag{5.42}$$

where $C \in \mathbb{R}$ is a real constant, determining the leaf $\mathcal{F}_C$. In the coordinates (X_1, X_2), the symplectic leaves $\mathcal{F}_I$ and $\mathcal{F}_{II}$ form two half-planes ($X_2 > 0$ and $X_2 < 0$ respectively) with non-degenerate Poisson structure, and $\mathcal{F}_C$ leaves are the single points (such that $X_1(\mathcal{F}_C) = C$) along the X_1 axis, with completely degenerate Poisson structure $\mathcal{P}(\mathcal{F}_C) = 0$.

We can (for example) try to find the local canonical coordinates for points in $\mathcal{F}_I$. It will turn out that we can find coordinates suitable for the whole leaf $\mathcal{F}_I$ (but not for whole $\mathfrak{g}^*$!). So let us suppose that $\xi_2 > 0$. We denote the Darboux coordinates as (P, Q). We are looking for the transformation

$$P = P(X_1, X_2),\ Q = Q(X_1, X_2),$$

such that it changes $\mathcal{P}$ into the form (5.31), that is

$$\mathcal{P}^{PQ} \equiv \frac{\partial P}{\partial X_1}\frac{\partial Q}{\partial X_2}\mathcal{P}^{12} + \frac{\partial P}{\partial X_2}\frac{\partial Q}{\partial X_1}\mathcal{P}^{21} = \frac{\partial P}{\partial X_1}\frac{\partial Q}{\partial X_2}X_2 - \frac{\partial P}{\partial X_2}\frac{\partial Q}{\partial X_1}X_2 \stackrel{!}{=} -1,$$

where $\mathcal{P}^{PQ}$ denotes $\mathcal{P}(dP, dQ)$. This can be done (for example) by choosing

$$P = \sqrt{X_2},\ Q = \frac{2}{\sqrt{X_2}} X_1. \tag{5.43}$$

Indeed, we get the Jacobi matrix of the transformation:

$$J(X_1, X_2) = \begin{pmatrix} 0 & \frac{1}{2\sqrt{X_2}} \\ \frac{2}{\sqrt{X_2}} & -\frac{X_1}{\sqrt{X_2}^3} \end{pmatrix},$$

with the determinant equal to $\frac{1}{X_2} \neq 0$, and

$$\mathcal{P}^{PQ} = 0 - \frac{1}{2\sqrt{X_2}} \cdot \frac{2}{\sqrt{X_2}} \cdot X_2 = -1.$$

Thus we have found the local canonical coordinates for the whole leaf $\mathcal{F}_I$.

We can examine this coordinates a little bit to give them a geometric meaning. Looking on (5.43), we see that P is always positive, whereas Q can take any real value. Clearly P is just the rescaling of X_2 coordinate for its positive values ($X_2 > 0$). The curves of constant non-zero Q are branches of parabolae with one excluded point. Indeed, for example for $Q = C > 0$, we have

$$C = \frac{2}{\sqrt{X_2}} X_1.$$

For $X_1 > 0$, we can express X_2 via X_1 as

$$X_2 = \frac{4}{C^2} X_1^2. \tag{5.44}$$

Therefore we get the right branch of the parabola, where point $(X_1, X_2) = (0, 0)$ is excluded. For $Q < 0$ we would get left branches and for $Q = 0$ we get the positive part of X_2 axis $(0, X_2 > 0)$.

As a worldsheet we can take for example the square $\Sigma := (0,1) \times (0,1) \subset \mathbb{R}^2$. As (global coordinates) (σ^1, σ^2) we take the ordinary Euclidean coordinates, that is $\sigma^1 = x$, $\sigma^2 = y$.

This manifold has no boundary, $\partial\Sigma = \emptyset$, and we do not have to impose any boundary conditions.

From the proof of 5.4.4 we get, that we can choose $X^P(\sigma^\mu) := P(X(\sigma^\mu))$ and $X^Q(\sigma^\mu) := Q(X(\sigma^\mu))$ as arbitrary smooth functions, such that X stays in $\mathcal{F}_I$, that is

$$X^P(\sigma^\mu) := f(\sigma^\mu) : \Sigma \to (0, +\infty),$$

$$X^Q(\sigma^\mu) := g(\sigma^\mu) : \Sigma \to \mathbb{R},$$

but otherwise arbitrary. The 1-forms A_P and A_Q from the expansion $A = A_P dX^P + A_Q dX^Q$ are then uniquely determined as

$$A_P = -dX^Q = -\frac{\partial g}{\partial \sigma^1} d\sigma^1 - \frac{\partial g}{\partial \sigma^2} d\sigma^2,$$

$$A_Q = dX^P = \frac{\partial f}{\partial \sigma^1} d\sigma^1 + \frac{\partial f}{\partial \sigma^2} d\sigma^2.$$

It is then easy to use the inverse transformation $X_2 = P^2$ and $X_1 = \frac{1}{2}PQ$ to write the solution in the original coordinates (X_1, X_2).

5.5 Linear Poisson sigma model

In this section we will introduce a notation and the form of equations (5.29) and (5.30) for a linear Poisson sigma model. For a target manifold we choose the Lie-Poisson structure introduced in example (3.3.4). That is $M = \mathfrak{g}^*$. In the previous example we have shown an explicit solution of such model, using the method of finding the Darboux coordinates on a chosen symplectic leaf.

First we would prove a useful lemma for the Killing form K of semisimple Lie algebras.

Lemma 5.5.1. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. Then the inverse of the Killing form K^{-1} is ad^*-invariant, that is*

$$K^{-1}(ad^*_X(\xi), \eta) + K^{-1}(\xi, ad^*_X(\eta)) = 0, \tag{5.45}$$

$\forall \xi, \eta \in \mathfrak{g}^*$ *and* $\forall X \in \mathfrak{g}$.

Proof. The proof is very straightforward, one starts with writing (5.45) in the coordinates with respect to an arbitrary basis, multiplies the relation twice by the matrix of K to get rid of K^{-1} and finally, one discovers the condition for the ad-invariance of K, which is satisfied by every Killing form K. ■

Let $(T_1, \ldots, T_n)$ be an arbitrary chosen basis of $\mathfrak{g}$. We will use it as global coordinates on $\mathfrak{g}^*$. Let $A = A^i dX_i$, $A \in \Omega^1(\Sigma, X^*(T^*\mathfrak{g}^*))$, where $dX_i = X^*(dT_i)$. We will see that equations of motion can be in this case written in a simple, coordinate-free way.

We define a 1-form $\widetilde{A}$ on Σ with values in Lie algebra $\mathfrak{g}$:

$$\widetilde{A} := A^i T_i. \tag{5.46}$$

It is clear that $\widetilde{A}$ does not depend on the choice of the basis $(T_i)_{i=1}^n$.

We can consider X as the 0-form on Σ with values in $\mathfrak{g}^*$, that is

$$X(p) = T_i(X(p))T^i \equiv X_i(p)T^i, \tag{5.47}$$

for all $p \in \Sigma$.

To avoid confusion: dX in the following proposition stays for 1-form with values in $\mathfrak{g}^*$, obtained as the exterior derivative of 0-form X defined by (5.47).

This point of view allows us to state the following proposition:

Proposition 5.5.2. *Suppose we have a linear Poisson sigma model and notation described above. The equations of motion (5.29) and (5.30) can be then written in the form*

$$dX + ad^*_{\widetilde{A}}(X) = 0, \tag{5.48}$$

$$d\widetilde{A} + \frac{1}{2}[\widetilde{A} \wedge \widetilde{A}]_{\mathfrak{g}} = 0, \tag{5.49}$$

where ad^* is an ordinary coadjoint representation of $\mathfrak{g}$ (see 2.3.5).

Moreover, if $\mathfrak{g}$ is a semisimple Lie algebra, we can rewrite (5.48) as

$$d\widetilde{X} + [\widetilde{A}, \widetilde{X}]_{\mathfrak{g}} = 0, \tag{5.50}$$

where $\widetilde{X} := K^{-1}(X, \cdot)$ is a 0-form on Σ with values in $\mathfrak{g}$ and K^{-1} is the inverse of the Killing form.

Proof. For the linear Poisson structure, we have $\mathcal{P}_{ij}(\xi) = c_{ij}{}^k \xi_k$ and $\frac{\partial \mathcal{P}_{ij}}{\partial T_k}\Big|_\xi = c_{ij}{}^k$, where $[T_i, T_j] = c_{ij}{}^k T_k$. Therefore the equations (5.29) take the form

$$dX_i + X_k c_{ij}{}^k A^j = 0. \tag{5.51}$$

Now we will compute $\langle T_i, dX + ad^*_{\widetilde{A}}(X)\rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical pairing on $\mathfrak{g}$:

$$0 = \langle T_i, dX + ad^*_{\widetilde{A}}(X)\rangle = dX_i + \langle T_i, ad^*_{\widetilde{A}}(X)\rangle = dX_i - \langle ad_{\widetilde{A}}(T_i), X\rangle =$$

$$= dX_i - A^j X_k \langle ad_{T_j}(T_i), T^k\rangle = dX_i + A^j X_k c_{ij}{}^k.$$

Thus the equation (5.48) is equal to the set of equations (5.29). Let us proceed to the second set of equations. If we again use the properties of $\mathcal{P}$ of the linear Poisson structure, we obtain (5.30) in the form

$$dA^i + \frac{1}{2} c_{jk}{}^i A^j \wedge A^k = 0. \tag{5.52}$$

But $c_{jk}{}^i A^j \wedge A^k = \langle T^i, (A^j \wedge A^k)\ [T_j, T_k]_\mathfrak{g}\rangle \equiv \langle T^i, [\widetilde{A} \wedge \widetilde{A}]_\mathfrak{g}\rangle$.

To obtain the last part of the statement, we take arbitrary $\xi \in \mathfrak{g}^*$ and using the equation (5.48) we write

$$0 = K^{-1}(dX + ad^*_{\widetilde{A}}(X), \xi) = K^{-1}(dX, \xi) + K^{-1}(ad^*_{\widetilde{A}}(X), \xi).$$

For the first term we have straight from the definition $K^{-1}(dX, \xi) \equiv \langle \xi, d\widetilde{X}\rangle$. We apply lemma 5.45 to rewrite the second term:

$$K^{-1}(ad^*_{\widetilde{A}}(X), \xi) = -K^{-1}(X, ad^*_{\widetilde{A}}(\xi)) \equiv -\langle ad^*_{\widetilde{A}}(\xi), \widetilde{X}\rangle = \langle \xi, ad_{\widetilde{A}}(\widetilde{X})\rangle = \langle \xi, [\widetilde{A}, \widetilde{X}]_\mathfrak{g}\rangle.$$

Hence $\langle \xi, d\widetilde{X} + [\widetilde{A}, \widetilde{X}]_\mathfrak{g}\rangle = 0$ for all $\xi \in \mathfrak{g}^*$, and we get $d\widetilde{X} + [\widetilde{A}, \widetilde{X}]_\mathfrak{g} = 0$. ■

Remark 5.5.3. Notice the resemblance of (5.49) to the zero curvature condition for the local gauge potential $\widetilde{A}$ on the principal bundle. This is not a coincidence. Linear Poisson sigma model, where $\mathfrak{g} = su(2)$, originated as the limit case of two-dimensional Yang-Mills theory.

Remark 5.5.4. In fact, we can rewrite the action (5.8) using the 1-form $\widetilde{A}$ as

$$S[X, \widetilde{A}] = \int_\Sigma \langle X, (d\widetilde{A} + \frac{1}{2}[\widetilde{A} \wedge \widetilde{A}])\rangle_\mathfrak{g}. \tag{5.53}$$

Equation (5.49) then follows immediately.

An interesting fact to observe is that equation (5.49) does not involve the field X at all.

As we have promised, for linear model we will give an example of "gauge" transformations of Poisson sigma model. For linear Poisson sigma model we are able to write them using auxiliary 1-forms $\widetilde{A}$ and ϵ.

However, they are still coordinate dependent. Until now, we have wisely chosen global coordinates on M, defined by some basis in $\mathfrak{g}$. But of course, we can choose every possible kind of local coordinates on M ($\mathbb{R}^n$, in fact). Especially the second equation (5.55) could then get into a very "wild" form in these coordinates.

Proposition 5.5.5. *Suppose we have a linear Poisson sigma model with fields X and $\widetilde{A}$ (using the previous notation).*

We define a 0-form ϵ on Σ with values in $\mathfrak{g}$ as $\epsilon := \epsilon^i T_i$, such that $|\epsilon^i| \ll 1$. If $\partial\Sigma \neq \emptyset$, we set $\epsilon(p) = 0$ for $p \in \partial\Sigma$.

The action of linear Poisson sigma model (5.8) is then invariant with respect to infinitesimal transformations

$$X' = X + \delta_\epsilon X := X - ad^*_\epsilon(X), \tag{5.54}$$

$$\widetilde{A}' = \widetilde{A} + \delta_\epsilon \widetilde{A} := \widetilde{A} + d\epsilon + [\widetilde{A}, \epsilon]. \tag{5.55}$$

For semisimple Lie algebra $\mathfrak{g}$, (5.54) can be rewritten as

$$\widetilde{X}' = \widetilde{X} + [\widetilde{X}, \epsilon], \tag{5.56}$$

where $\widetilde{X}' := K^{-1}(X', \cdot)$, $\widetilde{X} := K^{-1}(X, \cdot)$.

Proof. Using the easy algebra in the first order of ϵ_k, we find that

$$L[X', \widetilde{A}'] - L[X, \widetilde{A}] = d\langle \epsilon, dX \rangle.$$

This boundary term vanishes under the integral due to the Stokes theorem. ■

Using this form of gauge transformations, one can easily derive the commutator of such transformations: Let $X_\epsilon := X - ad^*_\epsilon(X)$ and $\widetilde{A}_\epsilon := \widetilde{A} + d\epsilon + [\widetilde{A}, \epsilon]$. Then

$$[\delta_\epsilon, \delta_{\epsilon'}]X := (X_{\epsilon'})_\epsilon - (X_\epsilon)_{\epsilon'}.$$

We calculate the first term

$$(X_{\epsilon'})_\epsilon = X_{\epsilon'} - ad^*_\epsilon(X_{\epsilon'}) = X - ad^*_{\epsilon'}(X) - ad^*_\epsilon(X - ad^*_{\epsilon'}(X)) = X - ad^*_{\epsilon'}(X) - ad^*_\epsilon(X) + ad^*_\epsilon(ad^*_{\epsilon'}(X)).$$

Hence

$$[\delta_\epsilon, \delta_{\epsilon'}]X \equiv (X_{\epsilon'})_\epsilon - (X_\epsilon)_{\epsilon'} = ad^*_\epsilon(ad^*_{\epsilon'}(X)) - ad^*_{\epsilon'}(ad^*_\epsilon(X)) = [ad^*_\epsilon, ad^*_{\epsilon'}](X).$$

And finally, because ad^* is a representation of $\mathfrak{g}$ on $\mathfrak{g}^*$, we get

$$[\delta_\epsilon, \delta_{\epsilon'}]X = ad^*_{[\epsilon, \epsilon']}(X).$$

Therefore the commutator of two gauge transformations is again a gauge transformation, denoted as $\delta_{[\epsilon,\epsilon']^*}X$, where $[\epsilon, \epsilon']^* = -[\epsilon, \epsilon']_{\mathfrak{g}}$.

Now we can examine the same for the gauge transformation of 1-form $\widetilde{A}$. Using the same technique, we obtain

$$[\delta_\epsilon, \delta_{\epsilon'}]\widetilde{A} = [d\epsilon', \epsilon] - [d\epsilon, \epsilon'] + [[\widetilde{A}, \epsilon'], \epsilon] - [[\widetilde{A}, \epsilon], \epsilon'] = d[\epsilon', \epsilon] + [\widetilde{A}, [\epsilon', \epsilon]] \equiv \delta_{[\epsilon,\epsilon']^*}A.$$

This means that infinitesimal gauge transformations of linear Poisson structure form a Lie algebra.

5.6 Poisson-Lie sigma model

We will be now concerned with Poisson manifolds constructed in chapter 4. Our target manifold M will be now a Poisson-Lie group G with Poisson bivector (4.13). We will call such model a **Poisson-Lie sigma model**. From (4.16) it is clear, that it would be convenient to rewrite the equations of motion (5.29) and (5.30) in components with respect to non-coordinate right-invariant frame fields $R_{T_i}(g) \equiv R_{g*}(T_i)$, where $(T_i)_{i=1}^n$ is a chosen basis of $\mathfrak{g}$, Lie algebra of group G.

We would expand the 1-form A not as $A = A_\alpha dy^\alpha$ (let us denote the coordinate indices in Greek letters for now), but rather as $A = A_k R_{T^k}$, where R_{T^k} are the right-invariant 1-forms on G dual to R_{T_m} frame fields. We denote $T^i_X(p) := X^*\big(R_{T^i}(X(p))\big) \in T^*_p(\Sigma)$.

Lemma 5.6.1. *The equations of motion rewritten in the components with respect to the right-invariant frame fields take the form*

$$T^i_X + \Pi^{ij}(X)A_j = 0, \tag{5.57}$$

$$dA_k + \frac{1}{2}R_{T_k}(\Pi^{ij})(X)A_i \wedge A_j + c_{kj}{}^i A_i \wedge T^j_X = 0, \tag{5.58}$$

where $\mathcal{P} = \frac{1}{2}\Pi^{ij} R_{T_i} \wedge R_{T_j}$ *and* $[T_i, T_j] = c_{ij}{}^k T_k$.

Proof. We start from 5.29 and 5.30 written in the local coordinates $(y^1, \dots, y^n)$. We denote the coordinate indices with Greek letters and right-invariant basis indices with Latin letters.

$$dX^\alpha + \mathcal{P}^{\beta\gamma}(X)A_\gamma = 0, \tag{5.59}$$

$$dA_\alpha + \frac{1}{2}P^{\beta\gamma}{}_{,\alpha}(X)A_\beta \wedge A_\gamma = 0. \tag{5.60}$$

For the convenience let us denote $e^k{}_\alpha := (e^R(X))^k{}_\alpha$ and $f^\alpha{}_k := (f^R(X))^\alpha{}_k$. For the definition of matrices, see the example in 4.10.1. If A_k denotes the component 1-forms from expansion $A = A_k R_{T^k}$, we get

$$A_\alpha = e^k{}_\alpha A_k,$$

and for other involved objects

$$dX^\alpha = f^\alpha{}_k T^k_X,\ \mathcal{P}^{\beta\gamma}(X) = f^\beta{}_m f^\gamma{}_n \Pi^{mn}(X).$$

Equation (5.59) can be thus written in the form

$$T^i_X + \Pi^{ij}(X)A_j = 0.$$

We can now deal with the second equation. First we rewrite the term dA_α:

$$dA_\alpha = d\big(e^k{}_\alpha A_k\big) = de^k{}_\alpha \wedge A_k + e^k{}_\alpha dA_k =$$

$$= \frac{\partial(e^k{}_\alpha)}{\partial y^\beta} dX^\beta \wedge A_k + e^k{}_\alpha dA_k, \tag{5.61}$$

where by $\frac{\partial}{\partial y^\beta}$ we always mean $\frac{\partial}{\partial y^\beta}\Big|_X$. The second term in (5.60) reads

$$\frac{1}{2}\mathcal{P}^{\beta\gamma}{}_{,\alpha}(X)A_\beta \wedge A_\gamma = \frac{1}{2}\frac{\partial}{\partial y^\alpha}\Big(f^\beta{}_m f^\gamma{}_n \Pi^{mn}(X)\Big) e^a{}_\beta e^b{}_\gamma A_a \wedge A_b =$$

$$= \frac{1}{2} e^k{}_\alpha R_{T_k}\Big(f^\beta{}_m f^\gamma{}_n \Pi^{mn}(X)\Big) e^a{}_\beta e^b{}_\gamma A_a \wedge A_b =$$

$$= \frac{1}{2} e^k{}_\alpha f^\beta{}_m f^\gamma{}_n R_{T_k}(\Pi^{mn})(X) e^a{}_\beta e^b{}_\gamma A_a \wedge A_b +$$

$$+ e^k{}_\alpha R_{T_k}(f^\beta{}_m)(X) f^\gamma{}_n \Pi^{mn}(X) e^a{}_\beta e^b{}_\gamma A_a \wedge A_b =$$

$$= \frac{1}{2} e^k{}_\alpha R_{T_k}(\Pi^{mn})(X) A_m \wedge A_n + e^k{}_\alpha e^a{}_\beta R_{T_k}(f^\beta{}_m)(X)\Pi^{mb} A_a \wedge A_b = \otimes.$$

Using the equation (5.57), we can write

$$\otimes = \frac{1}{2} e^k{}_\alpha R_{T_k}(\Pi^{mn})(X) A_m \wedge A_n - e^k{}_\alpha e^a{}_\beta R_{T_k}(f^\beta{}_m)(X) A_a \wedge T^m_X. \tag{5.62}$$

To continue, we have to use the following trick

$$R_{T_k}(f^\beta{}_m)(X) = R_{T_k}(R_{T_m}(y^\beta))(X) = R_{T_m}(R_{T_k}(y^\beta))(X) + [R_{T_k}, R_{T_m}](y^\beta)(X) =$$

$$= R_{T_m}(R_{T_k}(y^\beta))(X) - c^l_{km} R_{T_l}(y^\beta)(X) = R_{T_m}(f^\beta{}_k)(X) - c_{km}{}^l f^\beta{}_l.$$

Using this, we can rewrite the second term in (5.62) as

$$-e^k{}_\alpha e^a{}_\beta R_{T_k}(f^\beta{}_m)(X) A_a \wedge T^m_X =$$

$$-e^k{}_\alpha e^a{}_\beta R_{T_m}(f^\beta{}_k)(X) A_a \wedge T^m_X + e^k{}_\alpha e^a{}_\beta c_{km}{}^l f^\beta{}_l A_a \wedge T^m_X =$$

$$-e^k{}_\alpha e^a{}_\beta R_{T_m}(f^\beta{}_k)(X) A_a \wedge T^m_X + e^k{}_\alpha c_{km}{}^a A_a \wedge T^m_X = \boxtimes.$$

We can write

$$e^a{}_\beta R_{T_m}(f^\beta{}_k)(X) = -R_{T_m}(e^a{}_\beta)(X) f^\beta{}_k.$$

Hence

$$\boxtimes = e^k{}_\alpha R_{T_m}(e^a{}_\beta)(X) f^\beta{}_k A_a \wedge T^m_X + e^k{}_\alpha c_{km}{}^a A_a \wedge T^m_X =$$

$$= R_{T_m}(e^a{}_\alpha)(X) A_a \wedge T^m_X + e^k{}_\alpha c_{km}{}^a A_a \wedge T^m_X =$$

$$= f^\beta{}_m \frac{\partial(e^a{}_\alpha)}{\partial y^\beta} A_a \wedge T^m_X + e^k{}_\alpha c_{km}{}^a A_a \wedge T^m_X =$$

$$= -\frac{\partial(e^k{}_\alpha)}{\partial y^\beta} dX^\beta \wedge A_k + e^k{}_\alpha c_{km}{}^a A_a \wedge T^m_X.$$

Putting this back into (5.62), we obtain

$$\frac{1}{2}\mathcal{P}^{\beta\gamma}{}_{,\alpha}(X) A_\beta \wedge A_\gamma =$$

$$= e^k{}_\alpha \Big(\frac{1}{2} R_{T_k}(\Pi^{mn})(X) A_m \wedge A_n + c_{km}{}^a A_a \wedge T^m_X\Big) - \frac{\partial(e^k{}_\alpha)}{\partial y^\beta} dX^\beta \wedge A_k.$$

Together with (5.61), we get

$$dA_\alpha + \frac{1}{2}\mathcal{P}^{\beta\gamma}{}_{,\alpha}(X) A_\beta \wedge A_\gamma =$$

$$= e^k{}_\alpha \Big(dA_k + \frac{1}{2} R_{T_k}(\Pi^{mn})(X) A_m \wedge A_n + c_{km}{}^a A_a \wedge T^m_X\Big).$$

Therefore from (5.60), finally:

$$dA_k + \frac{1}{2} R_{T_k}(\Pi^{mn})(X) A_m \wedge A_n + c_{km}{}^a A_a \wedge T^m_X = 0,$$

which was to be proved. ■

Up to now, this rewriting does not seem to be that much useful. Moreover, instead of an ordinary partial derivative, we have the action of the right-invariant field in (5.58). Fortunately, as we we will show in the following lemma, this is no obstacle at all. We will use the notation introduced in chapter 4.

Lemma 5.6.2. *Let* $(T_k)^n_{k=1}$ *be a chosen (but arbitrary) basis of* $\mathfrak{g}$. $\Pi^{ij}(g) \equiv (\Pi(g))^{ji}_{\mathcal{X}} = \langle \widetilde{T}^j, \Pi(g)(\widetilde{T}^i)\rangle_{\mathfrak{d}}$ *for* $g \in G$. *Denote* $[T_i, T_j] = c_{ij}{}^k T_k$ *and* $[\widetilde{T}^i, \widetilde{T}^j] = f^{ij}{}_k \widetilde{T}^k$. *Then*

$$R_{T_k}(\Pi^{ij}) = c_{kl}{}^i \Pi^{lj} - c_{kl}{}^j \Pi^{li} + f^{ij}{}_k. \tag{5.63}$$

Proof. Let $g \in G$. We use the definitions to write

$$R_{T_k}(\Pi^{ij})(g) = R_{g*}(T_k)(\Pi^{ij}) = T_k(\Pi^{ij} \circ R_g) = \frac{d}{dt}\Big|_{t=0} \Pi^{ij}(e^{tT_k}g) =$$

$$= \frac{d}{dt}\Big|_{t=0} \langle \widetilde{T}^j, \Pi(e^{tT_k}g)(\widetilde{T}^i)\rangle_{\mathfrak{d}} = \otimes.$$

To proceed, we have to calculate (for $X \in \mathfrak{g}$)

$$\frac{d}{dt}\Big|_{t=0} \Pi(e^{tX}g) = \frac{d}{dt}\Big|_{t=0} PAd_{e^{tX}}Ad_g\widetilde{P}Ad_{g^{-1}}Ad_{e^{-tX}}\widetilde{P} =$$

$$= Pad_X Ad_g \widetilde{P} Ad_{g^{-1}}\widetilde{P} - PAd_g\widetilde{P}Ad_{g^{-1}}\widetilde{P}ad_X\widetilde{P} =$$

$$= Pad_X(P+\widetilde{P})Ad_g\widetilde{P}Ad_{g^{-1}}\widetilde{P} - PAd_g\widetilde{P}Ad_{g^{-1}}(P+\widetilde{P})ad_X\widetilde{P} =$$

$$= Pad_X \underbrace{PAd_g\widetilde{P}Ad_{g^{-1}}\widetilde{P}}_{\Pi(g)} + Pad_X \underbrace{\widetilde{P}Ad_g\widetilde{P}\widetilde{P}Ad_{g^{-1}}\widetilde{P}}_{Id_{\widetilde{g}}} -$$

$$- \underbrace{PAd_g\widetilde{P}Ad_{g^{-1}}\widetilde{P}}_{\Pi(g)} ad_X\widetilde{P} - PAd_g \underbrace{\widetilde{P}Ad_{g^{-1}}P}_{0} ad_X\widetilde{P} =$$

$$= Pad_X\Pi(g) + Pad_X\widetilde{P} - \Pi(g)ad_X\widetilde{P}.$$

Hence we can write

$$\otimes = \langle \widetilde{T}^j, Pad_{T_k}\Pi(g)(\widetilde{T}^i)\rangle_{\mathfrak{d}} + \langle \widetilde{T}^j, Pad_{T_k}(\widetilde{T}^i)\rangle_{\mathfrak{d}} - \langle \widetilde{T}^j, \Pi(g)ad_{T_k}(\widetilde{T}^i)\rangle_{\mathfrak{d}} =$$

$$= c_{kl}{}^j\Pi(g)^{il} + f^{ij}{}_k + c_{kl}{}^i\Pi(g)^{lj}.$$

■

Let us observe that if $A = A_k R_{T^k}$, we may consider A_k as the component 1-forms of 1-form $\widetilde{A}$ on Σ with values in Lie algebra $\widetilde{\mathfrak{g}}$. That is

$$\widetilde{A}(p) := A_k(p)\widetilde{T}^k. \tag{5.64}$$

One can easily verify that this definition does not depend on the choice of the basis $(T_i)_{i=1}^n$ in $\mathfrak{g}$.

Right-invariant Maurer-Cartan 1-form Θ_R can be expanded as

$$\Theta_R(g) = R_{T^k}(g)T_k, \tag{5.65}$$

for all $g \in G$. We define Θ_R^X as its pullback by X to Σ:

$$\Theta_R^X := X^*(\Theta_R) \equiv T_X^k T_k. \tag{5.66}$$

Using this definitions, we can rewrite the equations of motion in a very elegant way:

Proposition 5.6.3. *Equations of motion of a Poisson-Lie sigma model can be written in the coordinate-free form*

$$\Theta_R^X = \Pi(X)(\widetilde{A}), \tag{5.67}$$

$$d\widetilde{A} + \frac{1}{2}[\widetilde{A} \wedge \widetilde{A}]_{\widetilde{\mathfrak{g}}} = 0, \tag{5.68}$$

where $\Pi(X)$ stands for the map $\Pi(g) : \widetilde{\mathfrak{g}} \to \mathfrak{g}$, defined by (4.8), taking $g = X(p)$.

Proof. From (5.57) and (4.18), we have

$$\langle T^i, \Theta_R^X \rangle = A_j \Pi^{ji}(X) = A_j \langle T^i, \Pi(X)(\widetilde{T}^j) \rangle = \langle T^i, \Pi(X)(\widetilde{A}) \rangle.$$

To derive the second equation, we just put (5.63) into (5.58) and use (5.57):

$$0 = dA_k + \frac{1}{2} R_{T_k}(\Pi^{ij})(X) A_i \wedge A_j + c_{kj}{}^i A_i \wedge T_X^j =$$

$$= dA_k + \frac{1}{2}\big(c_{kl}{}^i \Pi^{lj}(X) - c_{kl}{}^j \Pi^{li}(X)\big) A_i \wedge A_j +$$

$$+ \frac{1}{2} f^{ij}{}_k A_i \wedge A_j + c_{kj}{}^i A_i \wedge T_X^j =$$

$$= dA_k + \frac{1}{2} f^{ij}{}_k A_i \wedge A_j + c_{kl}{}^i A_i \wedge \Pi^{lj}(X) A_j + c_{kj}{}^i A_i \wedge T_X^j \stackrel{(5.57)}{=}$$

$$\stackrel{(5.57)}{=} dA_k + \frac{1}{2} f^{ij}{}_k A_i \wedge A_j - c_{kl}{}^i A_i \wedge T_X^l + c_{kj}{}^i A_i \wedge T_X^j =$$

$$= dA_k + \frac{1}{2} f^{ij}{}_k A_i \wedge A_j.$$

■

We have just found a very interesting result. For general Poisson-Lie sigma model, the second equation of motion (5.30) takes the form of "zero curvature" equation for $\widetilde{\mathfrak{g}}$-valued 1-form $\widetilde{A}$. The most important fact is that the field X is not anyhow present in the equation (5.68). It is the equation for 2-forms on Σ only.

This generalizes the result brought in [17], where $\mathfrak{g}$ is supposed to be semisimple with coboundary Lie bialgebra $(\mathfrak{g}, \delta = \Delta(r))$. We can quickly derive the form of equations presented there, as is done in the following corollary of the proposition 5.6.3:

Corollary 5.6.4. *Let G be a Poisson-Lie group with semisimple coboundary tangent Lie bialgebra $(\mathfrak{g}, \delta = \Delta(r))$.*

Using the notation introduced in (2.7), we set $R := \underline{a} \circ \underline{K}$, where K is the Killing form on $\mathfrak{g}$ and $a \in \bigwedge^2 \mathfrak{g}$ is the skew-symmetric part of r.

Moreover, we define a 1-form B on Σ with values in $\mathfrak{g}$ as

$$B := \underline{K}^{-1}(\widetilde{A}) \equiv K^{-1}(\widetilde{A}, \cdot),$$

where K^{-1} is the inverse of the Killing form K. Equations (5.67) and (5.68) can be then written in the form

$$\Theta_R^X + \big(R - Ad_X R Ad_X^{-1}\big)(B) = 0, \tag{5.69}$$

$$dB + \frac{1}{2}[B \wedge B]_R = 0. \tag{5.70}$$

Proof. If we expand K and a as $K = K_{ij} T^i \otimes T^j$ and $a = a^{ij} T_i \otimes T_j$ respectively, we get the matrix of R:

$$(R_{\mathcal{X}})^i{}_j = K_{jl} a^{li}.$$

For 1-form B we can write from definition

$$B \equiv B^i T_i = K^{ij} A_j T_i,$$

where K^{ij} are the components of K^{-1} in the basis $T_i \otimes T_j$. Poisson bivector $\mathcal{P}$ on G is given by Sklyanin bracket (4.26) as

$$\mathcal{P}(g) = L_{g*}(a) - R_{g*}(a) = R_{g*}\big(Ad_g(a) - a\big) = \big(Ad_g(a)^{ij} - a^{ij}\big) R_{T_i}(g) \otimes R_{T_j}(g).$$

Thus for the components Π^{ij} in right-invariant basis

$$\Pi^{ij}(g) = Ad_g(a)^{ij} - a^{ij}.$$

If we denote $\mathbf{P}^i{}_j := \langle T^i, Ad_X(T_j)\rangle$, we get

$$\Pi^{ij}(X) = \mathbf{P}^i{}_k \mathbf{P}^j{}_l a^{kl} - a^{ij}.$$

Equation (5.57) then reads

$$0 = T^i_X + \Pi^{ij}(X) A_j = T^i_X + \big(\mathbf{P}^i{}_k \mathbf{P}^j{}_l a^{kl} - a^{ij}\big) A_j =$$

$$= T^i_X + \big(\mathbf{P}^i{}_k \mathbf{P}^j{}_l a^{kl} - a^{ij}\big) K_{jm} B^m = T^i_X + (R_{\mathcal{X}})^i{}_m B^m + K_{jm} \mathbf{P}^j{}_l \mathbf{P}^i{}_k a^{kl} B^m = \otimes.$$

Using the Ad-invariance of K, we have $K_{jm}\mathbf{P}^j{}_l = K_{lj}(\mathbf{P}^{-1})^j{}_m$. Hence

$$\otimes = T^i_X + (R_{\mathcal{X}})^i{}_m B^m + K_{lj} a^{kl} (\mathbf{P}^{-1})^j{}_m \mathbf{P}^i{}_k B^m =$$

$$= T^i_X + (R_{\mathcal{X}})^i{}_m B^m - \mathbf{P}^i{}_k (R_{\mathcal{X}})^k{}_j (\mathbf{P}^{-1})^j{}_m B^m =$$

$$= \langle T^i, \Theta^X_R + \big(R - Ad_X R Ad_X^{-1}\big)(B)\rangle.$$

This finishes the proof of (5.69). Second equation (5.70) follows from (5.68) and (2.57). Indeed, if we apply $\underline{K}^{-1}$ on both sides of (5.68), we get

$$0 = \underline{K}^{-1}\big(d\widetilde{A} + \frac{1}{2}[\widetilde{A} \wedge \widetilde{A}]_{\tilde{\mathfrak{g}}}\big) = dB + \frac{1}{2}\underline{K}^{-1}[\widetilde{A} \wedge \widetilde{A}]_{\tilde{\mathfrak{g}}} \stackrel{(2.57)}{=}$$

$$\stackrel{(2.57)}{=} dB + \frac{1}{2}[B \wedge B]_R.$$

■

Chapter 6

Conclusion

In this chapter we should summarize accomplishments and failures of this work.

We have tried to understand the basics of the theory of Lie bialgebras and Poisson manifolds. It turned out that especially the Poisson geometry nowadays consists of huge parts, ranging from topology to group theory. There exist many books concerning with various parts of the theory ([9], [11]) and countless articles. In the chapters 2 and 3 we have brought a very brief introduction to this wide area of the differential geometry.

An intended key part of this work was the comparison of the map Π, found in the work of Ctirad Klimčík and Pavol Ševera [13], with general Poisson bivector of Poisson-Lie groups. This has been done in the chapter 4. We have found a way how to construct a multiplicative bivector field $\mathcal{P}$ on a Lie group G (4.13) and we have successfully shown, using the theory of multiplicative tensor fields, that such defined bivector field is nothing but the Poisson bivector of the Poisson-Lie group $(G, \mathcal{P})$.

This construction of the Poisson bivector works for every possible Lie bialgebra, unlike the (widely known) Sklyanin bracket (4.26), which can be used only for coboundary Lie bialgebras. We thought at the moment that we were first who brought such method. However, later it turned out that it has been already found before [11]. Despite that, the sequence of lemmas and proofs is kept as I have found it independently.

At the end of the chapter 4 we have given the example, where the Poisson-Lie group bivector is fully reconstructed from the abstract form of the Lie bialgebra (set just by the commutation relations) without the knowledge of the Lie group multiplication. This can be very useful for the construction of simple examples.

In the chapter 5 we have made a proper discussion of the structures involved in the definition of Poisson sigma model. In main articles about Poisson sigma models ([18],[16],[21] or [17]) there is a little attention paid to the definition of the fields, making the understanding difficult for readers (particularly for the author of this work).

We have also written down the derivation of the equations of motion, using the "geometric" method of vector field flows. We avoided the problems with local coordinates this way.

Since we have found a general Poisson-Lie bivector in the chapter 4, we were able to examine the Poisson sigma models with general Poisson-Lie group target. It turned out that the equations of motion can be rewritten in the components with respect to the right-invariant frame fields (5.57) and (5.58). This was in fact rather non-trivial, because the second equation of motion (5.30) does not behave as a tensor equation.

Using the knowledge from the chapter 4, we were able to find the action of the right-invariant vector fields on the components of the Poisson bivector (see lemma 5.6.2) and then we could rewrite the equations of the motion in the elegant form (5.67) and (5.68), what is a generalization of the equations given in [17] only for Poisson-Lie groups with a coboundary semisimple tangent Lie bialgebra.

However, there still remains a lot of unsolved questions. For instance we do not understand at all, where did the gauge transformations of Poisson sigma models come from. We are able to explicitly show that they retain the action of the model unchanged, compute their commutator, but we fail in the attempts of giving them a proper geometric meaning. This is one of our efforts for the future, crucial for the full understanding of Poisson sigma models.

We would also like to discover some kind of duality of Poisson sigma models. Connected and simply connected Poisson-Lie groups come in the mutually dual pairs, moreover both of them are subgroups of one Lie group, Drinfel'd double. It could be very useful to find some relation between the solutions of corresponding Poisson sigma models, as can be done e.g. for Yang-Baxter sigma models [15]. It is completely unclear at the moment, whether such kind of duality could be discovered.

There exist many interesting tools for the description of Poisson manifolds, like the Poisson cohomology, Poisson calculus and the theory of symplectic grupoids (see e.g. [9]). Getting familiar with these could help us with the understanding the structures underlying Poisson sigma models.

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