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DIPLOMA THESIS

Quantum Waveguides under Mild Regularity Assumptions

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Prohlášení

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V Praze dne Helena Šediváková

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Abstract: The Dirichlet Laplacian in a curved three-dimensional tube built along a spatial curve is investigated in the limit when the uniform cross-section of the tube diminishes. Both deformations due to bending and twisting of the tube are considered. We show that the Laplacian converges in a norm resolvent sense to the well known one-dimensional Schrödinger operator whose potential is expressed in terms of the curvature of the reference curve, the twisting angle and a constant measuring the asymmetry of the cross-section. Contrary to previous results, we allow the curves to have non-continuous and possibly vanishing curvature. For such non-smooth curves, the distinguished Frenet frame need not exist, hence an alternative frame defined by the parallel transport is used. The main idea how to deal with the discontinuities is based on a refined smoothing of the curvature via the Steklov approximation.

Key words: curved quantum waveguide, twisting, bending, effective Hamiltonian, constrained systems, relatively parallel adapted frame, Steklov approximation, norm resolvent convergence.

Název: **Kvantové vlnovody za slabých podmínek na regularitu**

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Abstrakt: V této práci zkoumáme dirichletovský laplacián na trubicových okolích prostorových křivek s konstantním průřezem, a to v limitě, kdy plocha průřezu trubice klesá k nule. Je uvažován jak efekt ohýbání trubice, tak i efekt jejího zkroucení a dokazuje se, že laplacián konverguje vzhledem k norm-resolventní konvergenci ke známénu jednodimenzionálnímu schrödingerovskému operátoru, jehož potenciál je vyjádřen pomocí křivostí křivky, úhlu kroucení a konstanty vyjadřující míru asymetrie průřezu trubice. Oproti dřívějším výsledkům uvažujeme i křivky s nespojitou křivostí, která navíc může být v některých bodech nulová. Pro takové křivky neexistuje Frenetův repér, proto je použit repér definovaný paralelním transportem. Hlavním nástojem, který pomáhá řešit problém nespojitosti je tzv. Steklovova aproximace křivosti.

Klíčová slova: kvantový vlnovod, deformace křivostí a kroucením, efektivní Hamiltonián, vázané systémy, repér definovaný paralelním transportem, Steklovova aproximace, norm-resolventní konvergence.

Contents

Chapter 1

Introduction

1.1 Thin quantum waveguides and constrained systems

The notion *quantum waveguide* was introduced for a thin tubular neighborhood of a curve where particles can move effectively freely. In two dimensions this means the thin strip built along a planar curve, in three dimensions, the quantum waveguide is a tube built along a spatial curve. Here we would like to give the motivation why to study such objects.

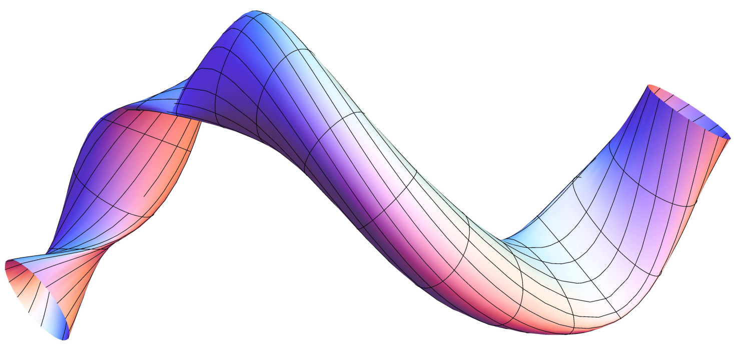

Figure 1.1: The three-dimensional quantum waveguide with the elliptic cross-section. On the right-hand side see the effect of bending, on the left-hand side the effect of twisting.

The limit when the cross-section of the quantum waveguide tends to zero is usually studied. In this sense, the quantum waveguide is a special case of quantum system with constraint which has been studied for a long time and the works on this topic go back to [16]. Let $\mathcal{A}$ be an ambient space and let $\mathcal{C}$ be a submanifold of $\mathcal{A}$ to that the particle is constrained. In the classical mechanics, the d'Alembert principle would be used and we would compute the dynamics on the submanifold $\mathcal{C}$ (where the particle is localized) from the dynamics on $\mathcal{A}$ using only the intrinsic properties of $\mathcal{C}$. However, in quantum mechanics, the uncertainty principle does not allow us to consider the particle as localized on the submanifold and different approach has to be used. The potential that pushes the particle to the submanifold is introduced and the limit when this potential grows to infinity is studied. In this limit, the kinetic energy of the movement transverse to $\mathcal{C}$ is growing to infinity, however, it is possible to look only at the effective movement on the submanifold. The surprising result is that this effective dynamics does not depend only on the intrinsic properties of $\mathcal{C}$, but also on the extrinsic curvature of $\mathcal{C}$, on the curvature of the ambient space $\mathcal{A}$ and on the shape of the constraining potential we chose.

In this way, it was discovered in [16] that if $\mathcal{A}$ is $\mathbb{R}^3$ and $\mathcal{C}$ is a curved surfaces, then the effective dynamics on the surface depends on its mean and Gauss curvatures. Similarly in paper [15], the case when $\mathbb{R}^3$ is the ambient space and the particle is constraint to a spatial curve was considered and it was shown that the dynamics depends on the curvature of the curve. In [27] the case $\mathcal{A} = \mathbb{R}^2$ and $\mathcal{C}$ be a planar curve was considered as well as the more general case of d-dimensional submanifold in $\mathbb{R}^n$ for $n > d$ and again the result that the dynamics depends on the extrinsic properties of the submanifold was obtained. A nuber of papers on this topic was written till today, let us mention the paper [23] that handles the problem in very general form, or more rigorous [14] where also interesting comparison with classical physics was given. From more recent papers let us mention [28] or [26].

In this work the two- and three-dimensional quantum waveguides are considered. We set on the boundary of this strip or tube the Dirichlet boundary conditions which plays here in fact the role of the constraining potential (the potential is zero inside the strip and infinite outside). In the limit when the cross-section of the strip or tube tends to zero, the kinetic energy in the transverse direction tends to infinity due to the Dirichlet boundary conditions. However, we again look only on the effective motion on the curve and the Hamiltonian describing this effective motion was formally derived already in papers like [15].

On the other hand, more rigorous research on the convergence of the spectrum of the Hamiltonian for a particle in such waveguides was not made before the paper [11], where the existence of bound states in a curved planar waveguide was discovered. The generalization of these results to three-dimensional waveguides was given in [7]. In this paper only the effect of *bending* of the waveguide was considered (see Figure 1.1), which yields the negative potential expressed in the terms of the curvature κ of the reference curve Γ and in consequence causes the occurrence of bound states. In two-dimensional case this result is complete and the effective Hamiltonian reads

$$h_{\text{eff}}^{\text{2D}} := -\Delta_D^\Gamma - \frac{\kappa^2}{4}.$$

In three-dimensional case, this Hamiltonian describes the effective motion on the curve in case when the cross-section of the tube is circular or when the non-circular cross-section is rotated with respect to the Frenet frame by angle θ satisfying $\dot{\theta} = \tau$ where τ is the Frenet's torsion. This result was also derived in [7]. However, in paper [9] it is proved that for the three-dimensional waveguides, there is also a counter-effect to bending which can then suppress the bound states. We speak about the effect of twisting the waveguide and also of the torsion of the curve. The formula for the positive potential in the effective Hamiltonian that arises from this effect was given in [4]. Finally, the effective Hamiltonian describing the motion on the spatial curve Γ is expressed in terms of the Frenet's curvature κ, torsion τ and twisting angle θ:

$$h_{\text{eff}}^{\text{3D}} := -\Delta_D^\Gamma - \frac{\kappa^2}{4} + C(\omega)\left(\tau + \dot{\theta}\right)^2. \tag{1.1}$$

The non-negative constant $C(\omega)$ depends only on the cross-section ω and is zero for rotationally symmetric ω.

A review on the effects of bending and twisting of the waveguide is given in [19], the asymptotic expansion for eigenvalues of the Laplacian in the curved thin waveguide is given in [3]. There are different directions, how to extend these results. In [12] the d-dimensional quantum waveguides for general $d \geq 2$ are considered, in [24] the problematic of branched quantum waveguide is studied, whereas [22] considers the tubes with varying cross-section.

Finally, we refer to the extended bibliography of [28] for various works on effective Hamiltonian in thin quantum waveguides and more general constrained systems.

1.2 Quantum waveguides under mild regularity conditions

As we mentioned above, a great number of papers has already been written on the subject of quantum waveguides. That's why we would like to justify here that this work brings something new to this problematic.

In papers like [7] the existence of bound states and other results were proved using the perturbation theory and other methods standard in the theory of linear operators. However, then the assumption on the curvature κ of the reference curve

$$\kappa \in C^2$$

was required (and it will be shown in Section 2.2.2 why this was necessary). This assumption excludes e.g. the curve on Figure 1.2 and the question arises, if the statements of paper [7] and others do hold also for such curves or if there is some physical reason why the assumption $\kappa \in C^2$ must be satisfied for these results to hold. We can partly answer this question using the results of paper [4] or [6]. In there the method of Γ-convergence is used and it is proved that the Laplacian in the three-dimensional waveguide built along the curve Γ where for the curvature it is assumed only

$$\kappa \in L^\infty,$$

converges with respect to the strong-resolvent convergence to the effective Hamiltonian (1.1). In [4] the proof is given for bounded waveguides only, the paper [6] generalizes this result to unbounded waveguides under assumption $\kappa \in C^1$ and adds also the proof of the norm resolvent convergence for the bounded waveguides.

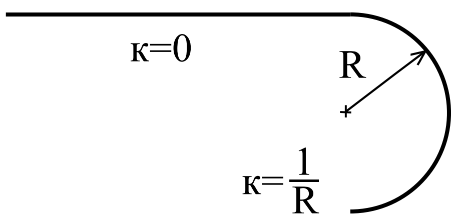

Figure 1.2: A curve with non-continuous curvature.

We stated as a task of this work to prove the norm resolvent convergence of the Laplacian in the tube to the effective Hamiltonian (1.1) under as mild regularity conditions on the curve as possible and we would like also to include the unbounded waveguides. We will proceed in the same way as in paper [7], however, some new ideas have to be used. It is namely the Steklov approximation of the curvature and also working with quadratic forms instead of the operators.

1.3 The results and organization of the text

Our main result is stated in Theorem 3.1 which, roughly said, claims that the Dirichlet Laplacian in the tube built along the spatial curve Γ converges to the effective Hamiltonian (1.1) with respect to the norm resolvent convergence. However, we have to be careful while stating such result, since the effective Hamiltonian acts on the Hilbert space $L^2(I)$ where I is the interval, where the curve Γ is defined, whereas the Dirichlet Laplacian acts on the Hilbert space $L^2(\Omega)$ where Ω is the three-dimensional tube where moreover the cross-section of this tube is diminishing. Hence, certain identification of Hilbert spaces must be done and also the results are not written straightly in terms of the Dirichlet Laplacian, but in terms of a unitarily equivalent operator (which has the same spectrum). This procedure will be described in Sections 2.2 and 3.1.

For Theorem 3.1 to hold, it is assumed that the curvature κ and the derivative of twisting angle $\dot{\theta}$ are bounded. In case when the interval I is bounded, these are all assumptions we need. In case when I is unbounded, we add one more assumption that, roughly speaking, forbids the functions κ and $\dot{\theta}$ to oscillate too quickly in the infinity. This assumption arises from the use of Steklov approximation as is described in Section 2.3, hence it might seem to be only a technical assumption. However, there are curves for that this assumption is not satisfied (the appropriate curvature is found in Section 2.3.4) and we show in Section 4.4 (on the toy-model of two-dimensional quantum waveguide) that for such curve, the statement of our main theorem does not hold. Moreover, we show that the Laplacian on the waveguide built along such curve is not well approximated by the standard effective Hamiltonian.

The proof of Theorem 3.1 is performed in Chapter 3 and it consists of proving a row of auxiliary lemmas. We adopt here an idea from [13] which enables us to switch from comparing the resolvents of the operators to comparing the associated quadratic forms. Also the technique of Hilbert space decomposition is essential in the proof.

In our work we consider the curves that do not fulfil even $\Gamma \in C^2$, i.e. its curvature need not be even continuous (which is the case for curve on Figure 1.2). Moreover, the curvature is allowed to vanish at some parts of the curve. The important consequence is that for such curves, the Frenet frame need not exist. That's why we use so called relatively parallel adapted frame which was studied for C^2 curves in [2] and we generalize the results of this paper on even more general curves (namely we assume that the first and second derivatives of the curve exist only in the weak sense).

The aspects of framing of a spatial curve are considered in Appendix A, the geometry of the three-dimensional quantum waveguide is described in Section 2.1.

Finally, we state also similar result as in Theorem 3.1 for the two-dimensional waveguide. There are two reasons why we included Chapter 4. Firstly since we promised in [8] that we will prove the norm resolvent convergence for the Laplacian in a planar strip. At second, we want to consider the essential spectrum of the waveguide built along the “counterexample” curve (curve that does not fulfill the Assumptions of our main theorem, i.e. is oscillating too quickly in infinity) and this is more illuminating in case of the two-dimensional waveguide (see Section 4.4).

The Conclusion of our results is given in Chapter 5, some mathematical details and theorems stated in classical literature that we need in the text are given in Appendix B.

Chapter 2

Preliminaries

2.1 The three-dimensional quantum waveguide

In the following paragraphs, we construct a thin tubular object along a spatial curve Γ, which will be later called the quantum waveguide. The framing of the curve Γ will be necessary, however, as the scope of this paper is to get the convergence results for a waveguide with the minimal regularity conditions, we won't use the Frenet frame that can be introduced only for C^3 spatial curve (even not for all of them, see Section A.1). Instead we will use the relatively parallel adapted frame (RPAF) which is introduced in Section A.2. Let us note that the construction of RPAF for $W^{2,\infty}_{\mathrm{loc}}(I)$ curves is the generalization of the paper [2] and it is one of the important results of this work although it is for its technicality placed in Appendix.

Let $\Gamma(s)$ be spatial curve, i.e. the (image of the) embedding $\Gamma\colon\ I \to \mathbb{R}^3\colon\ s \longmapsto (\Gamma^1(s), \Gamma^2(s), \Gamma^3(s))$ where the open interval $I \subseteq \mathbb{R}$ is allowed to be either finite, semi-infinite or infinite. For the RPAF to exist we require $\Gamma^i \in W^{2,\infty}_{\mathrm{loc}}(I)$ for $i = 1, 2, 3$ (see Section A.2). Without loss of generality we also assume the curve to be parameterized by arc length.

Then, according to Corollary A.7, there exists a relatively parallel adapted frame $\{T(s), M_1(s), M_2(s)\}$, where the vector fields change continuously with s and their weak derivative exists and is bounded. In fact there exists a whole class of such frames, however, we choose to work with $\{T, M_1, M_2\}$ without loss of generality. The (weak) derivatives of these vector fields satisfy

$$\begin{pmatrix} \dot{T} \\ \dot{M}_1 \\ \dot{M}_2 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ M_1 \\ M_2 \end{pmatrix} \tag{2.1}$$

where $k_1(s)$ and $k_2(s)$ are locally bounded functions. However, they need not to be neither globally bounded, nor continuous. Using the RPAF composed of T, M_1 and M_2 we can introduce a quantum waveguide.

Let ω be an open connected subset of $\mathbb{R}^2$ and let us assume that ω is bounded, i.e.

$$a := \sup_{t\in\omega} |t| < \infty \tag{2.2}$$

where $t = (t_2, t_3)$. Then we can define a curved tube (the waveguide) built along the curve Γ with the uniform cross-section ω as the image of the mapping $\mathcal{L} : I \times \omega =: \Omega_0 \to \mathbb{R}^3$

$$\mathcal{L}(s, t) := \Gamma(s) + \varepsilon t_2 \left(\cos\theta M_1 + \sin\theta M_2\right) + \varepsilon t_3 \left(-\sin\theta M_1 + \cos\theta M_2\right). \tag{2.3}$$

Here $\theta : I \to \mathbb{R}$ has the meaning of the twisting angle, i.e. the rotation of the cross-section with respect to the RPAF. We suppose that $\dot{\theta}$ is bounded:

$$\|\dot{\theta}\|_{L^\infty(I)} =: C_{\dot{\theta}} < \infty. \tag{2.4}$$

The waveguide is then the region in $\mathbb{R}^3$ denoted by

$$\Omega := \mathcal{L}(\Omega_0).$$

We assume the waveguide to be non-self-intersecting, i.e., the mapping $\mathcal{L}$ to be injective. The necessary (but not always sufficient) condition for the injectivity is the non-vanishing determinant of the metric tensor

$$G_{ij} := \partial_i \mathcal{L} \cdot \partial_j \mathcal{L}.$$

($\cdot$ assigns the scalar product in $\mathbb{R}^3$). Here ∂_i denotes the partial derivative with respect to i^{th} variable where the ordered set $(s,\, t_2,\, t_3)$ corresponds to (1,2,3). In our case the matrix $G = (G_{ij})$ reads

$$G = \begin{pmatrix} h^2 + \varepsilon^2(h_2^2 + h_3^2) & -\varepsilon^2 h_3 & -\varepsilon^2 h_2 \\ -\varepsilon^2 h_3 & \varepsilon^2 & 0 \\ -\varepsilon^2 h_2 & 0 & \varepsilon^2 \end{pmatrix} \tag{2.5}$$

where

$$\begin{aligned} h &:= 1 - \varepsilon t_2 \left(k_1 \cos\theta + k_2 \sin\theta\right) - \varepsilon t_3 \left(-k_1 \sin\theta + k_2 \cos\theta\right) \qquad (2.6)\\ h_2 &:= -t_2 \dot{\theta} \\ h_3 &:= t_3 \dot{\theta}. \end{aligned}$$

For the determinant we have

$$|G| = \varepsilon^4 h^2 \tag{2.7}$$

hence the condition on this function being everywhere nonzero reads $h > 0$, i.e.

$$\varepsilon t_2 \left(k_1 \cos\theta + k_2 \sin\theta\right) + \varepsilon t_3 \left(-k_1 \sin\theta + k_2 \cos\theta\right) < 1 \qquad \forall (s,t) \in \Omega_0.$$

This condition can be satisfied only if the functions k_1 and k_2 are bounded. Hence we assume that

$$\|\kappa\|_{L^\infty(I)} =: C_k < \infty, \tag{2.8}$$

where $\kappa = |\ddot{\Gamma}| = \sqrt{k_1^2 + k_2^2}$ (see remark 2.1 below), and consequently

$$k_i(s) \le C_k \qquad \forall s \in I, \quad i = 1, 2. \tag{2.9}$$

Together with the boundedness of ω we can always find ε so small that

$$\varepsilon t_2 \left(k_1 \cos\theta + k_2 \sin\theta\right) + \varepsilon t_3 \left(-k_1 \sin\theta + k_2 \cos\theta\right) \le 4\varepsilon a C_k < 1.$$

Moreover, in the estimates we will often use that ε can be chosen so small that $16\varepsilon a C_k \le 1$ and thus

$$\frac{3}{4} \le 1 - \varepsilon a C_k \le h \le 1 + \varepsilon a C_k \le \frac{5}{4}. \tag{2.10}$$

Remark 2.1. *In other papers on quantum waveguides, the Frenet frame was used in most of cases. Recall therefore that if the Frenet frame exists, then the Frenet's curvature $\kappa(s) = \left|\ddot{\Gamma}(s)\right|$ is connected with the functions k_1 and k_2 by the relation*

$$\kappa = \sqrt{k_1^2 + k_2^2}. \tag{2.11}$$

In our computations (where we use the RPAF), the quantity $k_1^2 + k_2^2$ will occur and we will assign it by κ^2 even if the Frenet frame does not exist (as we already did in (2.8)). Note that the curvatures k_1 and k_2 will occur in our results only in terms proportional to κ, which means that the results are not be dependent on the choice of RPAF.

In Section A.2 we also found that if e_2 is the Frenet's normal then

$$e_2 = \cos\beta M_1 + \sin\beta M_2$$

where $\beta = \arctan\frac{k_2}{k_1}$ (or equivalently $\beta(s) = \arcsin\frac{k_2(s)}{\kappa(s)}$ if $k_1(s) = 0$) and that consequently it holds

$$\dot{\beta} = \tau$$

where τ is the Frenet's torsion. Hence if we consider the twisting angle of a particular waveguide with respect to the relatively parallel adapted frame (θ_{RP}) or the Frenet frame (θ_F), it holds

$$\theta_{RP} = \theta_F + \beta,$$

hence

$$\dot{\theta}_{RP} = \dot{\theta}_F + \tau \tag{2.12}$$

Using these relations we will be able to compare our results with the results of previous papers.

2.2 The Hamiltonian

We assume that the movement of a particle in the quantum waveguide is effectively free and that the particle wavefunction is suppressed on the boundary of the waveguide. Hence the Hamiltonian reads the Laplacian with Dirichlet boundary conditions (we set $\hbar = 2m = 1$)

$$H = -\Delta_D^{\Omega}. \tag{2.13}$$

Since we won't require any regularity of the boundary of the domain Ω (we only suppose that ω is an open set), we start with an operator $\dot{H}$ acting as the Laplace operator (i.e. $-\partial_i\partial_i$) and $\operatorname{Dom}\dot{H}_\varepsilon = C_c^\infty(\Omega)$ (smooth functions with the compact support in Ω). The operator $\dot{H}$ is then symmetric and the Dirichlet boundary conditions are trivially satisfied. The associated quadratic form reads

$$Q[\psi] := \|\nabla\psi\|^2_{L^2(\Omega)} \qquad \operatorname{Dom}(Q) = C_c^\infty(\Omega)$$

and is closable, the domain of the closure being $W_0^{1,2}(\Omega)$ (the closure of $C_c^\infty(\Omega)$ with respect to the norm on the Sobolev space $W^{1,2}$). The operator associated to this closure is then the self-adjoint Friedrichs extension of $\dot{H}$, which will be assigned by H. For the details of this construction see Section B.5.

However, for the description of the waveguide the most suitable coordinates are the curvilinear coordinates $(s,t) \in \Omega_0$ (recall $\Omega = \mathcal{L}(\Omega_0)$) where $\mathcal{L}$ is the mapping introduced by (2.3). Hence we use the unitary transformation $\psi(x) \mapsto (\psi \circ \mathcal{L})(s,t)$, the Hilbert space becomes $\tilde{\mathcal{H}}_\varepsilon := L^2\left(\Omega_0, |G|^{1/2}\, ds\, dt\right)$ and the quadratic form reads

$$\tilde{Q}_\varepsilon[\psi] := \left(\partial_i\psi, G^{ij}\partial_j\psi\right)_{\tilde{\mathcal{H}}_\varepsilon} \qquad \operatorname{Dom}(\tilde{Q}) = W_0^{1,2}(\Omega_0) \tag{2.14}$$

where G^{ij} is the inverse matrix to (2.5) and $|G|$ is the determinant (2.7). The domain of this (closed) quadratic form is the same Sobolev space $W_0^{1,2}(\Omega_0)$ as before, since the metric G^{ij} is bounded (the functions k_i and $\dot{\theta}$ are assumed to be bounded) and uniformly positive (i.e. there exist a constant $c > 0$ such that $G \geq cE$ in the sense of matrices, E is the unit matrix). This fact follows from the construction in Section B.5 and holds even if the coefficients G^{ij} are not to be differentiable. The associated operator acts in the weak sense as the Laplace-Beltrami operator

$$\tilde{H}_\varepsilon = -|G|^{-1/2}\partial_i |G|^{1/2} G^{ij} \partial_j,$$

the operator $\tilde{H}_\varepsilon$ is again understood as the self-adjoint Friedrich's extension where

$$\mathrm{Dom}\, \tilde{H}_\varepsilon^{1/2} = \mathrm{Dom}\, \tilde{Q}_\varepsilon = W_0^{1,2}(\Omega_0).$$

Finally let us summarize the assumptions we made on the curve, the cross-section etc., so that both the waveguide and the Hamiltonian were well defined.

Assumption 1. *Let $\Gamma\colon I \to \mathbb{R}^3$ be a spatial curve where the interval $I \subseteq \mathbb{R}$ is finite, semi-infinite or infinite. Then we assume*

(i) $\Gamma^i \in W^{2,\infty}_{\mathrm{loc}}(I)$ *for* $i = 1, 2, 3$*;*

(ii) $\sup_{s \in I} |\ddot{\Gamma}(s)| < \infty$.

Further, let ω be an open connected subset of $\mathbb{R}^2$ and let $\theta(s)$ be the angle describing the rotation of the cross-section ω with respect to the relatively parallel adapted frame constructed along Γ. We assume

(iii) $\sup_{t \in \omega} |t| < \infty$*;*

(iv) $\sup_{s \in I} |\dot{\theta}| < \infty$.

Finally, let $\mathcal{L} : I \times \omega \to \mathbb{R}^3$ be the mapping introduced by (2.3)*, we assume also that*

(v) the properties of Γ and ω are such that $\mathcal{L}$ is injective for small enough ε.

2.2.1 The asymptotic of the spectrum

In this section we will find the first term in the asymptotic of the spectrum of our Hamiltonian, and using this knowledge we will renormalize the Hamiltonian to get an operator with the finite spectrum even if ε tends to zero.

To get the explicit formula for the form (2.14), we use that

$$G^{-1} = \frac{1}{h^2}\begin{pmatrix} 1 & h_3 & h_2 \\ h_3 & \frac{h^2}{\varepsilon^2} + h_3^2 & h_2 h_3 \\ h_2 & h_2 h_3 & \frac{h^2}{\varepsilon^2} + h_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\varepsilon^2} & 0 \\ 0 & 0 & \frac{1}{\varepsilon^2} \end{pmatrix} + \frac{1}{h^2}\begin{pmatrix} 1 \\ h_3 \\ h_2 \end{pmatrix}\begin{pmatrix} 1 & h_3 & h_2 \end{pmatrix}.$$

It is convenient to introduce $\partial_\tau := t_3\partial_2 - t_2\partial_3$ since than $\partial_1 + h_3\partial_2 + h_2\partial_3 = \partial_s + \dot{\theta}\partial_\tau$. When assigning in addition the gradient in transverse variables t_2, t_3 as $\nabla' = (\partial_2, \partial_3)$, we get

$$\tilde{Q}[\psi] = \int_{\Omega_0} h|\nabla'\psi|^2\, ds\, dt + \int_{\Omega_0} \frac{\varepsilon^2}{h}|(\partial_s + \dot{\theta}\partial_\tau)\psi|^2\, ds\, dt$$

Let us note that we work in the Hilbert space $\tilde{\mathcal{H}}_\varepsilon = L^2\left(\Omega_0, |G|^{1/2}\, ds\, dt\right) = L^2(\Omega_0, \varepsilon^2 h\, ds\, dt)$, so the first term is in fact $\mathcal{O}(\frac{1}{\varepsilon^2})$.

We will estimate this quadratic from below neglecting the positive second term and rewriting the first term in terms of function $\phi = \sqrt{h}\psi$:

$$\int_{\Omega_0} h|\nabla'\psi|^2\, ds\, dt = \int_{\Omega_0} |\nabla'\phi|^2\, ds\, dt - \int_{\Omega_0} \frac{(\nabla' h)^2}{4h^2}|\phi|^2 \geq E_1 \int_{\Omega_0} |\phi|^2\, ds\, dt - 11C_k^2 \int_{\Omega_0} \varepsilon^2|\phi|^2\, ds\, dt =$$
$$= \left(\frac{E_1}{\varepsilon^2} - 11C_k^2\right) \int_{\Omega_0} h\varepsilon^2|\psi|^2\, ds\, dt = \left(\frac{E_1}{\varepsilon^2} - 11C_k^2\right) \|\psi\|^2_{\tilde{\mathcal{H}}_\varepsilon} \tag{2.15}$$

where E_1 is the first eigenvalue of the transverse Laplacian Δ_D^ω and where we obtained the estimate $|\frac{(\nabla' h)^2}{4h^2}| \leq 11C_k^2$ after straight computation (where we assumed that ε is so small that $16\varepsilon a C_k \leq 1$). Thus we can see that the lower bound on the spectrum is asymptotically $\frac{E_1}{\varepsilon^2} + \mathcal{O}(1)$.

The upper bound on the spectrum can be found using the min-max principle (see Section B.8). We estimate the numbers λ_n (according to Theorem B.16 they are either eigenvalues or they are equal to the bottom of the essential spectrum), that are the defined as the infimum over the subsets L_n ($\dim L_n = n$) of the domain of the quadratic form $W_0^{1,2}(\Omega_0)$, by the infimum over smaller subsets $\tilde{L}_n = \left\{\varphi\chi_1 | \varphi \in l_n,\ l_n \subset W_0^{1,2}(I),\ \dim l_n = n\right\}$ where χ_1 is the first eigenvalue of the transverse Laplacian.

$$\lambda_n = \inf_{L_n}\left(\sup_{\psi \in L_n} \tilde{Q}[\psi]\right) \leq \inf_{\tilde{L}_n}\left(\sup_{\varphi \in l_n} \tilde{Q}[\varphi\chi_1]\right)$$

Then we compute (using the integration by parts and the fact that h is linear in variables t)

$$\begin{aligned}\tilde{Q}[\varphi\chi_1] &= \int_{\Omega_0} h|\nabla'\chi_1|^2|\varphi|^2\, ds\, dt + \int_{\Omega_0} \varepsilon^2 h \left|\partial_s\varphi\chi_1 + \dot{\theta}\varphi\partial_\tau\chi_1\right|^2\, ds\, dt = \\ &= -\int_{\Omega_0} h\chi_1\Delta_D^\omega\chi_1|\varphi|^2\, ds\, dt + \left|\partial_s\varphi\chi_1 + \dot{\theta}\varphi\partial_\tau\chi_1\right|^2\, ds\, dt = \\ &= \frac{E_1}{\varepsilon^2}\|\varphi\chi_1\|^2_{\tilde{\mathcal{H}}_\varepsilon} + \|\partial_s\varphi\chi_1 + \dot{\theta}\varphi\partial_\tau\chi_1\|^2_{\tilde{\mathcal{H}}_\varepsilon}.\end{aligned}$$

Hence it is clear that all the eigenvalues and also the bottom of the essential spectrum can be estimated from above by $\frac{E_1}{\varepsilon^2} + \mathcal{O}(1)$ and we proved that the first term in the asymptotic expansion in ε of the spectrum is $\frac{E_1}{\varepsilon^2}$.

In consequence, it is reasonable to renormalize the Hamiltonian by subtracting $\frac{E_1}{\varepsilon^2}$:

$$\tilde{\tilde{H}}_\varepsilon := \tilde{H}_\varepsilon - \frac{E_1}{\varepsilon^2},$$

similarly

$$\tilde{\tilde{Q}}_\varepsilon[\psi] := \tilde{Q}_\varepsilon[\psi] - \frac{E_1}{\varepsilon^2}\|\psi\|^2_{\tilde{\mathcal{H}}_\varepsilon}.$$

Our task is now to show that the operator H_ε converges to some effective Hamiltonian acting only on the interval I and we will see that looking for the terms $\mathcal{O}(1)$ in the asymptotic will be much more difficult then finding the leading term $\frac{E_1}{\varepsilon^2}$.

2.2.2 The standard unitary transformation

As we are interested in the limit when $\varepsilon \to 0$, we would like to work in ε-independent Hilbert space $\mathcal{H}_0 := L^2(\Omega_0,\, ds\, dt)$, ε-dependent being only the coefficients in the Hamiltonian. For this purpose the following unitary transformation is usually performed.

$$\begin{aligned} U_G : L^2\left(\Omega_0, |G|^{1/2}\, ds\, dt\right) &\longrightarrow L^2(\Omega_0,\, ds\, dt) \\ \psi &\longmapsto U_G\psi = |G|^{1/4}\psi \end{aligned}$$

Consequently the operator reads

$$U_G \tilde{\tilde{H}}_\varepsilon U_G^{-1} = |G|^{1/4} \tilde{\tilde{H}}_\varepsilon |G|^{-1/4}.$$

However, the function $|G|$ contains the functions k_1, k_2 which need not to be in our case differentiable. Together with the fact that $\tilde{H}_\varepsilon$ contains the second derivative, it follows that this unitary transformation can not be used in this form for our purposes and we will have to modify it.

2.2.3 The smoothing of the functions k_1, k_2

The main idea of how to follow the well-known procedure either without the strong smoothness conditions is to smooth the functions k_1 and k_2 in the way that these smoothed functions converge to the original ones as ε goes to zero (which is the limit we are interested in). Together with working with the quadratic forms instead of the operators we will be able to prove the convergence properties of the Hamiltonian also for the waveguide with *non-smooth* curvatures.

The smoothing will be performed using so called *Steklov approximation* (see e.g. [1]), i.e. we introduce

$$k_i^\varepsilon(s) := \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} k_i(\xi) d\xi \qquad i = 1, 2 \tag{2.16}$$

where δ is a function of ε which is monotonically increasing in some neighborhood of zero and both

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$$

and

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0. \tag{2.17}$$

It is clear that this function is differentiable

$$\dot{k}_i^\varepsilon(s) = \frac{k_i(s + \frac{\delta(\varepsilon)}{2}) - k_i(s - \frac{\delta(\varepsilon)}{2})}{\delta(\varepsilon)} \qquad i = 1, 2. \tag{2.18}$$

We will also find, that if some additional conditions on k_i are required than $k_i^\varepsilon \xrightarrow{\varepsilon \to 0} k_i$ in certain sense. We will dedicate to this problematic the Section 2.3, since the convergence properties of k_i^ε are essential for the computations below.

Let us also mention that from (2.9) it follows that

$$|k_i^\varepsilon(s)| \leq C_k \qquad \forall s \in I, \quad i = 1, 2. \tag{2.19}$$

2.2.4 The modified unitary transformation

The unitary transformation similar to the one from Section 2.2.2 will be performed using the smoothed curvature. For this purpose we introduce

$$|\tilde{G}| := \varepsilon^4 h_\varepsilon^2 = \varepsilon^4 \left[1 - \varepsilon t_2 \left(k_1^\varepsilon \cos\theta + k_2^\varepsilon \sin\theta\right) - \varepsilon t_3 \left(-k_1^\varepsilon \sin\theta + k_2^\varepsilon \cos\theta\right)\right]^2$$

(the function h_ε is the function h introduced by (2.6) where k_i is replaced by k_i^ε) and we use $|\tilde{G}|$ in the unitary transformation instead of $|G|$

$$\begin{aligned} U_{\tilde{G}} : L^2\left(\Omega_0, |G|^{1/2}\, ds\, dt\right) &\longrightarrow L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right) \\ \psi &\longmapsto U_{\tilde{G}}\psi = |\tilde{G}|^{1/4}\psi \end{aligned}$$

to get

$$H_\varepsilon := U_{\tilde G}\tilde{\tilde{H}}_\varepsilon U_{\tilde G}^{-1} = |\tilde G|^{1/4}\tilde{\tilde{H}}_\varepsilon|\tilde G|^{-1/4}. \tag{2.20}$$

We denote $\mathcal{H}_\varepsilon := L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde G|^{1/2}}\,ds\,dt\right)$ and the scalar product, resp. the norm on this space is then denoted by $(\cdot,\cdot)_\varepsilon$, resp. $\|\cdot\|_\varepsilon$. By $(\cdot,\cdot)$, resp. $\|\cdot\|$ will be denoted the scalar product, resp. the norm on $\mathcal{H}_0$, let us note that

$$\frac{3}{5}\|\cdot\|^2 \le \|\cdot\|_\varepsilon^2 \le \frac{5}{3}\|\cdot\|^2$$

if ε is so small that $16aC_k\varepsilon \le 1$.

The operator H_ε is an operator unitarily equivalent to the (renormalized) initial Dirichlet Laplacian H that was defined as self-adjoint Friedrichs extension, thus we define its action only in the weak sense. That's why we will work with the quadratic forms only, we can compute, that the quadratic form associated to H_ε reads

$$\begin{aligned} Q_\varepsilon[\psi] \;=\;& \int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|(\partial_s+\dot\theta\partial_\tau)\psi\right|^2 ds\,dt + \frac{1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\nabla'\psi|^2\,ds\,dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\psi|^2\,ds\,dt + & (2.21)\\ & +\frac{1}{2}\int_{\Omega_0}\frac{1}{h_\varepsilon^2}(k_1k_1^\varepsilon+k_2k_2^\varepsilon)|\psi|^2\,ds\,dt - \frac{3}{4}\int_{\Omega_0}\frac{h}{h_\varepsilon^3}\left((k_1^\varepsilon)^2+(k_2^\varepsilon)^2\right)|\psi|^2\,ds\,dt + & (2.22)\\ & +\int_{\Omega_0}\frac{\left((\partial_s+\dot\theta\partial_\tau)h_\varepsilon\right)^2}{4hh_\varepsilon^3}|\psi|^2\,ds\,dt - \int_{\Omega_0}\frac{(\partial_s+\dot\theta\partial_\tau)h_\varepsilon}{hh_\varepsilon^2}\,\mathrm{Re}(\bar\psi(\partial_s+\dot\theta\partial_\tau)\psi)\,ds\,dt & (2.23)\end{aligned}$$

(recall that ∇' is the gradient operator in variables t_2, t_3 and $\partial_\tau = t_3\partial_2 - t_2\partial_3$). Again $\mathrm{Dom}\,Q_\varepsilon = W_0^{1,2}(\Omega_0)$.

2.2.5 Existence and boundedness of $(H_\varepsilon + r)^{-1}$

In Section 2.2.1 we proved that for all $\psi \in W_0^{1,2}(\Omega_0)$, $\tilde Q[\psi] \ge \frac{E_1}{\varepsilon^2} - 11C_k^2\|\psi\|^2_{\tilde{\mathcal{H}}_\varepsilon}$. Thus for the renormalized quadratic form $\tilde{\tilde{Q}}[\psi] \ge -11C_k^2\|\psi\|^2_{\tilde{\mathcal{H}}_\varepsilon}$ and since the associated operator $\tilde{\tilde{H}}_\varepsilon$ is unitarily equivalent to H_ε, we get also

$$Q_\varepsilon[\psi] \ge -11C_k^2\|\psi\|_\varepsilon^2. \tag{2.24}$$

Hence there exists a real constant r such that the operator $H_\varepsilon^r := H_\varepsilon + r$ is positive, in consequence $(H_\varepsilon^r)^{-1}$ exists and we will prove that it is bounded, which will be used in the proof of the norm resolvent convergence.

Lemma 2.2. *Let $r > 11C_k^2$ be a real constant. Then*

$$\|(H_\varepsilon + r)^{-1}\|_{\mathcal{B}(H_\varepsilon)} \le \frac{1}{r - 11C_k^2}. \tag{2.25}$$

Proof. This relation is a simple consequence of (2.24):

$$\begin{aligned}\|(H_\varepsilon^r)^{-1}\|^{1/2}_{\mathcal{B}(\mathcal{H}_\varepsilon)} &= \|(H_\varepsilon^r)^{-1/2}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)} = \sup_{\psi\in\mathcal{H}_\varepsilon}\frac{\sqrt{\left((H_\varepsilon^r)^{-1/2}\psi,(H_\varepsilon^r)^{-1/2}\psi\right)_\varepsilon}}{\|\psi\|_\varepsilon} = \\ &= \sup_{\phi\in H_\varepsilon}\frac{\sqrt{(\phi,\phi)_\varepsilon}}{\sqrt{\left((H_\varepsilon^r)^{1/2}\phi,(H_\varepsilon^r)^{1/2}\phi\right)_\varepsilon}} = \sup_{\phi\in\mathcal{H}_\varepsilon}\frac{\sqrt{(\phi,\phi)_\varepsilon}}{\sqrt{Q_\varepsilon^r[\phi]}} \le \sup_{\phi\in\mathcal{H}_\varepsilon}\frac{\|\phi\|_\varepsilon}{\sqrt{r-11C_k^2}\|\phi\|_\varepsilon} = \\ &= \frac{1}{\sqrt{r-11C_k^2}}\end{aligned}$$

□

2.3 Convergence properties of the Steklov approximation

As we mentioned above, the key point in the proof of the norm resolvent convergence will be convergence of the expression $|k_i - k_i^\varepsilon|$ to zero when ε tends to zero. Since in the following also the function $\dot{\theta}$ will be smoothed using the Steklov approximation and similar convergence will be required, we will examine the behavior of the Steklov approximation for a general function $f \in L^\infty(I)$ in this section. Hence we introduce

$$f^\varepsilon(s) := \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} f(\xi) d\xi \tag{2.26}$$

where $\delta(\varepsilon)$ is some continuous function of ε satisfying

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0.$$

In the following we will derive some estimates on $|f(s) - f^\varepsilon(s)|$. As we will show in Section 2.3.1, for the pointwise convergence, very strong assumptions on f have to be required. However, we want to find some convergence results either for more general f, in this case only the convergence in the integral sense of $|f(s) - f^\varepsilon(s)|$ can be proved. Luckily, in the computations below, the relation

$$\int_I |f - f^\varepsilon|^2 |\varphi|^2 ds \xrightarrow{\varepsilon \to 0} 0 \qquad \forall \varphi \in W_0^{1,2}(I) \tag{2.27}$$

will be sufficient. In Sections 2.3.2 and 2.3.3, we will prove this relation for different classes of functions f.

2.3.1 The pointwise convergence

In order to state a lemma concerning the pointwise convergence we have to introduce some terminology which is adopted from [1].

Definition 2.3. *Let f be the uniformly continuous function on either finite or infinite interval I. Then we introduce the modulus of continuity as*

$$\omega(\delta, f) := \sup_{|\xi_1 - \xi_2| \leq \delta} |f(\xi_1) - f(\xi_2)| \qquad \xi_1, \xi_2 \in I$$

Recall that f is uniformly continuous on I if and only if

$$(\forall \tilde{\varepsilon} > 0)(\exists \delta > 0)(\forall \xi_1, \xi_2 \in I, |\xi_2 - \xi_1| < \delta \Rightarrow |\kappa(\xi_2) - \kappa(\xi_1)| < \tilde{\varepsilon}).$$

Hence it is clear that the modulus of continuity tends to zero when $\delta \to 0$.

Lemma 2.4. *Let $f(s)$ be a uniformly continuous function and $f^\varepsilon(s)$ its Steklov approximation given by* (2.26). *Then*

$$|f(s) - f^\varepsilon(s)| \leq \omega\left(\delta(\varepsilon), f\right) \xrightarrow{\varepsilon \to 0} 0.$$

Proof. The lemma is a simple consequence of the definition of the Steklov approximation and the modulus of continuity:

$$|f(s) - f^\varepsilon(s)| = \left| \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} \left(f(s) - f(\xi)\right) d\xi \right| \leq \sup_{|\eta| \leq \frac{\delta(\varepsilon)}{2}, s \in I} |f(s) - f(s+\eta)| = \omega(\delta(\varepsilon), f).$$

□

2.3.2 The integral convergence for L^2 functions

At first we will examine the integral convergence in the special case when $f \in L^2$. In the next section we will generalize these results, however considering the case of square integrable functions will show us the basic ideas that we will use later. Some of the ideas in this section can be found in [1] and we will again use the terminology of this book.

Definition 2.5. *Let f be a function defined on interval I, let $J \subseteq I$ and let $f \upharpoonright J \in L^2(J)$. Then we introduce the second modulus of continuity as*

$$\omega_2(\delta, f, J) := \sup_{|\eta| \leq \frac{\delta}{2}} \left(\int_J |f(s+\eta) - f(s)|^2 \, ds \right)^{1/2}.$$

This definition may involve function values outside I, it is understood that f is extended by zero to the whole $\mathbb{R}$ in order to give a meaning to the definition.

Lemma 2.6. *Let $f(s) \in L^2(I)$ be a function and $f^\varepsilon(s)$ its Steklov approximation given by (2.26). Let also $\varphi \in W^{1,2}(I)$. Then*

$$\int_I |f - f^\varepsilon|^2 \, |\varphi|^2 ds \leq (\omega_2(\delta(\varepsilon), f, I))^2 \, \|\varphi\|^2_{W^{1,2}(I)}.$$

Proof. At first we estimate the integral as

$$\int_I |f - f^\varepsilon|^2 \, |\varphi|^2 ds \leq \|\varphi^2\|_\infty \int_I |f - f^\varepsilon|^2 \, ds \leq \|\varphi\|^2_{W^{1,2}(I)} \|f - f^\varepsilon\|^2_I. \tag{2.28}$$

Here the second estimate follows from

$$\begin{aligned} |\varphi(s)|^2 &= \int_{s_0}^s \left(|\varphi(\xi)|^2\right)^\cdot d\xi = \int_{s_0}^s 2\mathrm{Re}\left(\bar{\varphi}(\xi)\dot{\varphi}(\xi)\right) d\xi \leq 2\sqrt{\int_{s_0}^s |\varphi(\xi)|^2 d\xi}\sqrt{\int_{s_0}^s |\dot{\varphi}(\xi)|^2 d\xi} \leq \\ &\leq \int_{s_0}^s |\varphi(\xi)|^2 d\xi + \int_{s_0}^s |\dot{\varphi}(\xi)|^2 d\xi \leq \|\varphi\|^2_{W^{1,2}(I)} \end{aligned} \tag{2.29}$$

which holds for all $\varphi \in W_0^{1,2}(I)$ and for all $s \in I$. We assigned $s_0 = \inf I$, we used that since φ has the compact support in I, $\varphi(s_0) = 0$, and we used also the Schwarz and Young inequalities. Now we can adopt the proof of the convergence of $\|f - f^\varepsilon\|_I$ from [1]. At first we use the generalized Minkowski inequality (B.2) to get

$$\|f - f^\varepsilon\|_I = \left(\int_I \left| \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} (f(s) - f(\xi)) \, d\xi \right|^2 ds \right)^{1/2} \leq \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} \left(\int_I |f(s) - f(\xi)|^2 \, ds \right)^{1/2} d\xi.$$

If we recall the definition of the second modulus of continuity, it is clear that

$$\|f - f^\varepsilon\|_I \leq \omega_2(\delta(\varepsilon), f, I)$$

which together with (2.28) proves the lemma. □

Remark 2.7.

(i) In case when I is unbounded, we can use the Corollary B.3 stated in Appendix B.1 for $p = 2$ to get

$$\lim_{\varepsilon \to 0} \omega_2(\delta(\varepsilon), f, I) = 0 \tag{2.30}$$

(if $I \neq \mathbb{R}$, we prolong f by zero on $\mathbb{R} \setminus I$). This relation then proves (2.27) for $f \in L^2(I)$.

(ii) If I is bounded, for every function $f \in L^\infty(I)$ it also holds that $f \in L^2(I)$. If we prolong f by zero on $\mathbb{R} \setminus I$, we can apply the Corollary B.2 stated also in Appendix B.1 to get (2.30). Hence (2.27) is proved also for $f \in L^\infty(I)$ in case I is bounded.

2.3.3 The integral convergence for general functions

To get similar results for even more general functions f, we start with the following auxiliary lemma:

Lemma 2.8. *Let $f \in L^\infty(I)$ and let f^ε be the Steklov approximation of f. Let $\varphi \in W^{1,2}(I)$ and finally let $\{a_n\}_{n=n_-}^{n_+} \subset I$, $n \in \mathbb{N}$ be the strictly increasing sequence of numbers where $a_{n_-} = \inf I$, $a_{n_+} = \sup I$, all the intervals (a_n, a_{n+1}) are finite and $n_\pm$ can be either finite or infinite. Then*

$$\int_I |f - f^\varepsilon|^2 \, |\varphi|^2 ds \leq \left[\sup_{n_- \leq n \leq n_+} \left(\frac{\|f - f^\varepsilon\|^2_{L^2(a_n, a_{n+1})}}{a_{n+1} - a_n} \right) + 2 \sup_{n_- \leq n \leq n_+} \|f - f^\varepsilon\|^2_{L^2(a_n, a_{n+1})} \right] \|\varphi\|^2_{W^{1,2}(I)}. \tag{2.31}$$

Proof. We start the proof by rewriting the integral in the following way.

$$\int_I |\varphi|^2 |f - f^\varepsilon|^2 ds = \sum_{n=n_-}^{n_+ - 1} \int_{a_n}^{a_{n+1}} |\varphi|^2 \dot{g}^n_\varepsilon ds$$

where $\forall s \in I$

$$g^n_\varepsilon(s) := \int_{a_n}^{s} |f(\xi) - f^\varepsilon(\xi)|^2 \zeta^n(\xi) d\xi$$

and $\zeta^n(\xi)$ is the characteristic function of the interval $(a_n, a_(n+1))$

$$\zeta^n(\xi) := \begin{cases} 1 & \text{for } \xi \in [a_n, a_{n+1}] \\ 0 & \text{else.} \end{cases}$$

Then

$$\begin{aligned}
\int_I |\varphi|^2 |f - f^\varepsilon|^2 ds &= \sum_{n=n_-}^{n_+ - 1} \int_{a_n}^{a_{n+1}} |\varphi|^2 \dot{g}^n_\varepsilon ds = \sum_{n=n_-}^{n_+ - 1} \left(\left[|\varphi|^2 g^n_\varepsilon \right]_{a_n}^{a_{n+1}} - \int_{a_n}^{a_{n+1}} 2\mathrm{Re}\left(\bar{\varphi}\dot{\varphi}\right) g^n_\varepsilon ds \right) \leq \\
&\leq \sum_{n=n_-}^{n_+ - 1} \left(|\varphi(a_{n+1})|^2 \int_{a_n}^{a_{n+1}} |f(\xi) - f^\varepsilon(\xi)|^2 d\xi + \int_{a_n}^{a_{n+1}} 2|\varphi||\dot{\varphi}||g^n_\varepsilon| ds \right) \leq \\
&\leq \sum_{n=n_-}^{n_+ - 1} \left(\int_{a_n}^{a_{n+1}} |f(\xi) - f^\varepsilon(\xi)|^2 d\xi \right) \left(|\varphi(a_{n+1})|^2 + \int_{a_n}^{a_{n+1}} 2|\varphi||\dot{\varphi}| ds \right)
\end{aligned}$$

where we repeatedly used that

$$\sup_{s \in I} g^n_\varepsilon(s) = g^n_\varepsilon(a_{n+1}) = \int_{a_n}^{a_{n+1}} |f(\xi) - f^\varepsilon(\xi)|^2 d\xi.$$

Then using similar steps as in (2.29) we estimate

$$\begin{aligned}
|\varphi(a_{n+1})|^2 &= |\varphi(a_{n+1})|^2 \frac{a_{n+1} - a_n}{a_{n+1} - a_n} = \int_{a_n}^{a_{n+1}} \frac{d}{dx} \left(|\varphi(x)|^2 \frac{x - a_n}{a_{n+1} - a_n} \right) dx \leq \\
&\leq \int_{a_n}^{a_{n+1}} |\varphi(x)|^2 \frac{1}{a_{n+1} - a_n} dx + \int_{a_n}^{a_{n+1}} 2\mathrm{Re}\left(\bar{\varphi}(x)\dot{\varphi}(x)\right) dx \leq \\
&\leq \left(\frac{1}{a_{n+1} - a_n} + 1 \right) \|\varphi\|_{W^{1,2}(a_n, a_n + 1)}
\end{aligned}$$

and finally

$$
\begin{aligned}
\int_I |\varphi|^2 |f-f^\varepsilon|^2 ds &\leq \sum_{n=n_-}^{n_+-1} \|f-f^\varepsilon\|^2_{L^2(a_n,a_{n+1})} \left(\frac{1}{a_{n+1}-a_n} + 2 \right) \|\varphi\|_{W^{1,2}(a_n,a_n+1)} \leq \\
&\leq \sup_{n_-\leq n\leq n_+} \left(\frac{\|f-f^\varepsilon\|^2_{L^2(a_n,a_{n+1})}}{a_{n+1}-a_n} \right) \|\varphi\|^2_{W^{1,2}(I)} + \\
&\quad +2 \sup_{n_-\leq n\leq n_+} \|f-f^\varepsilon\|^2_{L^2(a_n,a_{n+1})} \|\varphi\|^2_{W^{1,2}(I)}
\end{aligned}
$$

which completes the proof. □

Let us note that when we use the generalized Minkowski inequality we get

$$
\begin{aligned}
\|f-f^\varepsilon\|_{L^2(a_n,a_{n+1})} &= \left(\int_{a_n}^{a_n+1} \left| \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} (f(s)-f(\xi))\, d\xi \right|^2 ds \right)^{1/2} \leq \\
&\leq \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} \left(\int_{a_n}^{a_n+1} |f(s)-f(\xi)|^2\, ds \right)^{1/2} \leq \\
&\leq \sup_{|\eta|\leq\frac{\delta(\varepsilon)}{2}} \left(\int_{a_n}^{a_n+1} |f(s)-f(s+\eta)|^2\, ds \right)^{1/2}. \qquad (2.32)
\end{aligned}
$$

Since $(a_{n+1}-a_n)$ is a finite interval, Corollary B.2 yields that the last expression converges to zero for $\varepsilon\to 0$ and thus $\forall n\in\mathbb{N}$

$$\|f-f^\varepsilon\|^2_{L^2(a_n,a_{n+1})} \xrightarrow{\varepsilon\to 0} 0.$$

This need not to mean that the right-hand side of (2.31) (i.e. the supremum over such expressions) converges to zero, however, we will now state a simple example, when it holds true.

Example 2.9. *Let $f\in L^\infty(\mathbb{R})$ be a periodic function. Then $\forall\varphi\in W^{1,2}(I)$*

$$\int_I |f-f^\varepsilon|^2\, |\varphi|^2 ds \xrightarrow{\varepsilon\to 0} 0.$$

Proof. Let the period of f be $q>0$. Then we can choose $a_n = nq$ for $n\in\mathbb{Z}$ and this sequence will fulfil the hypothesis of Lemma 2.8. Consequently

$$\int_I |f-f^\varepsilon|^2\, |\varphi|^2 ds \leq \left(\frac{1}{q} + 2 \right) \|f-f^\varepsilon\|^2_{L^2(nq,(n+1)q)}$$

where the term on the right-hand side converges to zero according to remarks above. □

Clearly, it would be possible to give more examples of functions that fulfil (2.27) in consequence of the lemma above. In fact, we need not require anything about the behavior of f on some finite sub-interval, the only point is that the function is controlled somehow when $s\to\pm\infty$. Thus for example let f be a function that is constant on $\mathbb{R}\setminus(a,b)$ where (a,b) is arbitrary finite interval, then f also fulfils (2.27). However, we will try to give some general condition on f to satisfy (2.27).

When considering a particular function f, then taking some special sequence $\{a_n\}$ might be useful. However, while deriving estimates on general function f, it will be more convenient to fix some simple sequence $\{a_n\}$. The reason is also that for general f neither the choice of $\{a_n\}$ where $(a_{n+1}-a_n)$ is small won't give us finer estimate (because of the first term on the right-hand side in (2.31)), nor the choice of $\{a_n\}$ with large $(a_{n+1}-a_n)$ will improve the estimate (because of the second term on the right-hand side in (2.31)). For example when $I = \mathbb{R}$, we can use the sequence $a_n = n$, $n \in \mathbb{Z}$, i.e. $n_- = -\infty$, $n_+ = \infty$. Then the Lemma 2.8 gives us

$$\int_I |f - f^\varepsilon|^2 \, |\varphi|^2 ds \leq 3 \sup_{n \in \mathbb{Z}} \left(\|f - f^\varepsilon\|^2_{L^2(n,n+1)} \right) \|\varphi\|^2_{W^{1,2}(I)}.$$

From the relation (2.32) we get that

$$\|f - f^\varepsilon\|^2_{L^2(n,n+1)} \leq \sup_{|\eta| \leq \frac{\delta(\varepsilon)}{2}} \int_n^{n+1} |f(s) - f(s+\eta)|^2 \, ds = \omega_2 \left(\delta(\varepsilon), f, (n, n+1)\right).$$

Let us note, that the fact, if $\omega_2 (\delta(\varepsilon), f, (n, n+1))$ converges to zero or not, does not depend on the choice of function $\delta(\varepsilon)$, however if $\omega_2 (\delta(\varepsilon), f, (n, n+1))$ converges to zero, we might sometimes improve the convergence properties by some convenient choice of $\delta(\varepsilon)$.

We sum up the ideas above and also the idea of Remark 2.7 (*ii*) in the following theorem which gives us the most general criteria on the function f to fulfil (2.27).

Theorem 2.10. *Let $\varphi \in W_0^{1,2}(I)$, $f \in L^\infty(I)$ and let f^ε be the Steklov approximation of f given by* (2.26)*. Then*

$$\int_I |f - f^\varepsilon|^2 \, |\varphi|^2 ds \leq \sigma_f(\delta(\varepsilon)) \|\varphi\|^2_{W^{1,2}(I)} \tag{2.33}$$

where

(i) for I bounded

$$\sigma_f(\delta(\varepsilon)) = \omega_2 \left(\delta(\varepsilon), f, I\right) \xrightarrow{\varepsilon \to 0} 0,$$

(ii) for I unbounded

$$\sigma_f(\delta(\varepsilon)) = 3 \sup_{n \in \mathbb{Z}} \left(\omega_2 \left(\delta(\varepsilon), f, (n, n+1)\right) \right) = 3 \sup_{n \in \mathbb{Z}} \left(\sup_{|\eta| \leq \frac{\delta(\varepsilon)}{2}} \int_n^{n+1} |f(s) - f(s+\eta)|^2 \, ds \right)$$

(for I semi-bounded we compute the same for f prolonged by zero on $\mathbb{R} \setminus I$).

Remark 2.11. *For the unbounded interval I,*

$$\sigma_f(\delta(\varepsilon)) \xrightarrow{\varepsilon \to 0} 0 \tag{2.34}$$

is fulfilled by functions from all the classes we already considered (i.e. uniformly continuous functions, L^2 functions or periodic functions), we can add for example the class of BV-functions (functions of bounded variation) for that (2.34) *also holds, however, we didn't succeed to find any class $\mathcal{F}$ of function that fulfil* (2.34) *and*

$$f \notin \mathcal{F} \Rightarrow \lim_{\varepsilon \to 0} \sigma_f(\delta(\varepsilon)) \neq 0.$$

Thus our most general result is that f satisfies (2.27) *on I unbounded if for some $\delta(\varepsilon)$ it holds*

$$\sup_{n \in \mathbb{Z}} \left(\omega_2 \left(\delta(\varepsilon), f, (n, n+1)\right) \right) \xrightarrow{\varepsilon \to 0} 0. \tag{2.35}$$

Since in the following we will also need some estimate on $\int_I |f(s) - f^\varepsilon(s)| \, |\varphi|^2 ds$, we state the following Corollary.

Corollary 2.12. *Let $\varphi \in W_0^{1,2}(I)$ and $f \in L^\infty(I)$. Then*

$$\int_I |f(s) - f^\varepsilon(s)| \, |\varphi|^2 ds \leq \sqrt{\sigma_f(\delta(\varepsilon))} \|\varphi\|_{W^{1,2}(I)} \|\varphi\|_{L^2(I)} \tag{2.36}$$

where σ_f is the same function as in Theorem 2.10.

Proof. The relation (2.36) is the simple consequence of the Schwarz inequality in L^2 and the Theorem 2.10:

$$\left|\int_I |f(s) - f^\varepsilon(s)|\,|\varphi|^2 ds\right| \leq \sqrt{\int_I |f(s) - f^\varepsilon(s)|^2\,|\varphi|^2 ds}\sqrt{\int_I |\varphi|^2 ds} \leq \sqrt{\sigma_f(\delta(\varepsilon))}\|\varphi\|_{W^{1,2}(I)}\|\varphi\|_{L^2(I)}.$$

□

2.3.4 Counterexample

In Theorem 2.10 we found the condition on a function f such that the relation (2.27) is satisfied. However, we will now show, that there are also functions, that do not fulfil (2.35).

Let us introduce

$$f_{\mathrm{osc}}(s) := \begin{cases} 1 & \text{if } s \in (n-1+\frac{2k}{2n}, n-1+\frac{2k+1}{2n}) \\ -1 & \text{if } s \in (n-1+\frac{2k+1}{2n}, n-1+\frac{2k+2}{2n}) \end{cases} \qquad n \in \mathbb{N},\ k = 0, 1..., n-1, \tag{2.37}$$

i.e. the function f_{osc} is defined on $I = (0, \infty)$ and it is oscillating between the values 1 and -1 faster and faster when $s \to \infty$ (see Figure 2.1).

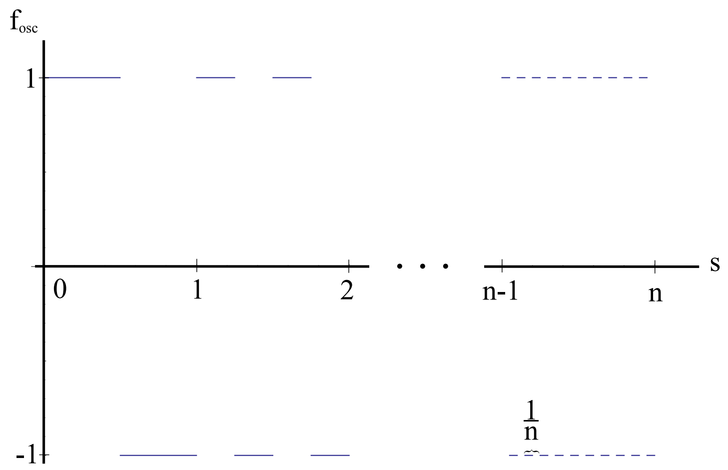

Figure 2.1: The plot describing the function f_{osc}.

Lemma 2.13. *For all $\varepsilon > 0$ there exists $n_0 \in \mathbb{R}$ such that*

$$\sup_{|\eta| \leq \frac{\delta(\varepsilon)}{2}} \int_{n-1}^{n} |f_{\mathrm{osc}}(s) - f_{\mathrm{osc}}(s+\eta)|^2\, ds = 4 \qquad \forall n \in \mathbb{N},\ n \geq n_0.$$

This holds true for any choice of function δ.

Proof. If we choose $n_0 = \frac{1}{\delta(\varepsilon)}$, then $\forall n \geq n_0$ $\frac{1}{2n} \leq \frac{\delta(\varepsilon)}{2}$. Hence

$$\sup_{|\eta| \leq \frac{\delta(\varepsilon)}{2}} \int_{n-1}^{n} |f_{\mathrm{osc}}(s) - f_{\mathrm{osc}}(s+\eta)|^2\, ds = \int_{n-1}^{n} \left|f_{\mathrm{osc}}(s) - f_{\mathrm{osc}}(s - \frac{1}{2n})\right|^2 ds = \int_{n-1}^{n} 2^2 ds = 4$$

where we choose $\eta = -\frac{1}{2n}$ since we know that in the interval $(n-2, n-1)$ the length of the segment where f_{osc} is constant equals $\frac{1}{2(n-1)} \geq \frac{1}{2n}$, hence the difference $\left|f_{\mathrm{osc}}(s) - f_{\mathrm{osc}}(s - \frac{1}{2n})\right|$ is indeed equal to 2 for all $s \in (n-1, n)$. It is clear that this construction works for any function δ as is also stated in the lemma. $\square$

Let us note that if we pose $k_1 = f(s)$, $k_2 = 0$ we can find a curve whose curvatures are k_1, k_2 and it is possible to built a waveguide along such curve. The properties of such waveguide will be examined in Section 4.4 (see Remark 4.9).

Chapter 3

The norm resolvent convergence

3.1 The main result

Our main result states that the Dirichlet Laplacian in the curved three-dimensional waveguide (or more precisely the unitarily equivalent operator H_ε) converges in certain sense to a one-dimensional effective Hamiltonian h_{eff} describing the dynamics on the curve Γ as the cross section of the waveguide diminishes. The effective Hamiltonian reads

$$h_{\text{eff}} := -\Delta_D^I - \frac{\kappa^2}{4} + C(\omega)\dot{\theta}^2. \tag{3.1}$$

where $\kappa = |\ddot{\Gamma}|$ and θ is the twisting angle. The non-negative constant

$$C(\omega) = \int_\omega (\partial_\tau \chi_1)^2 dt \tag{3.2}$$

depends only on the cross section ω, χ_1 is the eigenfunction of the Dirichlet Laplacian on ω corresponding to its first eigenvalue. Let us note that ∂_τ reads in polar coordinates (ρ, φ) the derivative with respect to the angle φ, hence for rotationally symmetric ω, $C(\omega) = 0$ since the eigenfunction χ_1 is independent of φ due to the symmetry. In fact, the constant $C(\omega)$ measures the asymmetry of ω.

Since the operators H_ε and h_{eff} act on different Hilbert spaces (namely $L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right)$ and $L^2(I)$), we will at first describe the way how to compare such operators. Then we will state our main result as a Theorem 3.1. Let us note that in this Chapter, the norms on spaces $\mathcal{H}_0 = L^2(\Omega_0,\, ds\, dt)$, $\mathcal{H}_\varepsilon = L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right)$, $L^2(\omega)$, resp. $L^2(I)$ will be denoted by $\|\cdot\|$, $\|\cdot\|_\varepsilon$, $\|\cdot\|_\omega$, resp. $\|\cdot\|_I$ and similarly for the scalar products on these spaces.

3.1.1 Comparing the operators acting on Hilbert spaces $\mathcal{H}_\varepsilon$ and $L^2(I)$

In the first step, we will show, how to identify the operators and quadratic forms acting on the Hilbert spaces $L^2(\Omega_0)$ and $L^2(I)$.

Let $\chi_1(t)$ be the eigenfunction corresponding to E_1 (the first eigenvalue of the transverse Dirichlet Laplacian) which we can choose real, positive and normalized to 1 (i.e. $\|\chi_1\|_\omega = 1$). Then we introduce the subspace $\mathcal{H}_0^1$ of $\mathcal{H}_0$

$$\mathcal{H}_0^1 := \left\{\psi \in \mathcal{H}_0 \mid \exists \varphi \in L^2(I),\, \psi(s,t) = \varphi(s)\chi_1(t)\right\} \tag{3.3}$$

which is closed, thus

$$\mathcal{H}_0 = \mathcal{H}_0^1 \oplus (\mathcal{H}_0^1)^\perp.$$

Here $(\mathcal{H}_0^1)^\perp$ is the orthogonal complement to $\mathcal{H}_0^1$, and every function $\psi \in \mathcal{H}_0$ can be (due to the projection theorem B.15) uniquely written in the form

$$\psi = \psi_1\chi_1 + \psi^\perp = P_1\psi + (1 - P_1)\psi. \tag{3.4}$$

By $\psi_1\chi_1$ we mean the function $\psi_1 \otimes \chi_1$ from $\mathcal{H}_0^1$ (in every point we compute it as $\psi_1(s)\chi_1(t)$), however, we will use for simplicity (maybe not completely accurate) notation $\psi_1\chi_1$ in the whole text. Further, $\psi^\perp \in (\mathcal{H}_0^1)^\perp$, the projection P_1 acts like

$$(P_1\psi)(s,t) := \left(\int_\omega \chi_1(t)\psi(s,t)dt\right)\chi_1(t) \tag{3.5}$$

and we assigned

$$\psi_1 = \int_\omega \chi_1(t)\psi(s,t)dt.$$

We introduce the identification π of spaces $\mathcal{H}_0^1$ and $L^2(I)$ as

$$\left(\pi\left(\psi_1(s)\chi_1(t)\right)\right)(s) = \psi_1(s). \tag{3.6}$$

The mapping π is an isometric isomorphism since this mapping is bijective and

$$\|\psi_1\chi_1\|^2 = \int_{\Omega_0} |\psi_1|^2\chi_1^2\, ds\, dt = \int_I |\psi_1|^2 ds = \|\psi_1\|_I^2.$$

According to this identification we can identify also the quadratic forms on spaces $\mathcal{H}_0^1$ and $L^2(I)$. We introduce a quadratic form Q_{eff} acting on the domain $W_0^{1,2}(\Omega_0) \cap \mathcal{H}_0^1 \subset \mathcal{H}_0$ and we identify it with the quadratic form q_{eff} associated to h_{eff} acting on $W_0^{1,2}(I)$ in the following way:

$$\begin{aligned} Q_{\text{eff}}[\psi_1\chi] &:= \int_{\Omega_0} |\partial_s\psi_1\chi|^2\, ds\, dt - \frac{1}{4}\int_{\Omega_0} \kappa(s)^2|\psi_1\chi_1|^2\, ds\, dt + C(\omega)\int_{\Omega_0} \dot{\theta}(s)^2|\psi_1\chi_1|^2\, ds\, dt = \\ &= \int_I |\partial_s\psi_1|^2 ds + C(\omega)\int_I \dot{\theta}^2|\psi_1|^2 ds - \frac{1}{4}\int_I \kappa^2|\psi_1|^2 ds =: q_{\text{eff}}[\psi_1]. \end{aligned}$$

Similarly we can identify the operators acting on $\mathcal{H}_0^1$ and $L^2(I)$. Let us note that if $\psi_1\chi_1 \in W_0^{1,2}(\Omega_0) \cap \mathcal{H}_0^1$ then $\psi_1 \in W_0^{1,2}(I)$.

Finally, by $0^\perp$ will be denoted the zero operator on the subspace $(\mathcal{H}_0^1)^\perp$.

As the second step, we have to find the way, how to come from the Hilbert space $\mathcal{H}_\varepsilon = L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right)$ to $\mathcal{H}_0 = L^2(\Omega_0,\, ds\, dt)$, i.e. how to compare the operator H_ε acting on $\mathcal{H}_\varepsilon$ with some operator acting on $\mathcal{H}_0$. For this purpose we introduce a unitary transformation

$$\begin{aligned} U_\varepsilon : L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right) \quad &\longrightarrow \quad L^2(\Omega_0,\, ds\, dt) \\ \psi \quad &\longmapsto \quad U_\varepsilon\psi = \frac{|G|^{1/4}}{|\tilde{G}|^{1/4}}\psi \end{aligned} \tag{3.7}$$

We can not apply this unitary transformation straightly on H_ε, since the function $U_\varepsilon\psi$ need not to be in its domain, however, $(H_\varepsilon + r)^{-1}$ is a bounded operator whose domain consist of whole $\mathcal{H}_\varepsilon$, so also non-differentiable functions are in domain of $(H_\varepsilon + r)^{-1}$. The operator $(H_\varepsilon + r)^{-1}$ becomes after the unitary transformation $U_\varepsilon(H_\varepsilon + r)^{-1}U_\varepsilon^{-1}$, which is the operator acting on $\mathcal{H}_0$. Let us note that $U_\varepsilon(H_\varepsilon + r)^{-1}U_\varepsilon^{-1}$ is not self-adjoint on $\mathcal{H}_0$, however we don't mind this fact since in definition of the norm resolvent convergence only the closedness of the operators is required and the consequences for the spectrum still hold.

3.1.2 The main theorem

Before stating the theorem, we will separately specify the assumptions on the curvatures k_1, k_2 and the twisting angle θ of the waveguide. These assumptions follow from the convergence properties of Steklov approximation described in Section 2.3 and it will be clear from the proof of the auxiliary Lemma 3.7, why they are necessary.

Assumption 2. *We assume that at least one of following conditions is satisfied.*

(i) The interval I is bounded.

(ii) If we set

$$\sigma_f(\delta(\varepsilon)) = 3\sup_{n\in\mathbb{Z}}\left(\sup_{|\eta|\le\frac{\delta(\varepsilon)}{2}}\int_n^{n+1}|f(s)-f(s+\eta)|^2\,ds\right)$$

then both

$$\lim_{\varepsilon\to 0}\sigma_k(\delta(\varepsilon)) := \lim_{\varepsilon\to 0}\left(\max_{i\in\{1,2\}}\sigma_{k_i}(\delta(\varepsilon))\right) = 0, \tag{3.8}$$

$$\lim_{\varepsilon\to 0}\sigma_{\dot\theta}(\tilde\delta(\varepsilon)) = 0 \tag{3.9}$$

for some continuous functions $\delta(\varepsilon)$, $\tilde\delta(\varepsilon)$ satisfying

$$\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0,$$

$$\lim_{\varepsilon\to 0}\tilde\delta(\varepsilon) = 0$$

and in addition

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta(\varepsilon)} = 0. \tag{3.10}$$

Theorem 3.1. *Let H_ε be the operator defined by* (2.20)*, i.e. the operator unitarily equivalent to the renormalized Hamiltonian $-\Delta_D^\Omega$ describing the dynamics on a curved quantum waveguide built along a spatial curve $\Gamma(s)$, $s\in I$, such that the Assumptions 1 and 2 are satisfied. Let h_{eff} be the effective Hamiltonian on the interval I defined by* (3.1) *and let U_ε be the unitary transformation* (3.7)*. Then*

$$\left\|U_\varepsilon(H_\varepsilon+r)^{-1}U_\varepsilon^{-1}-\left((h_{\text{eff}}+r)^{-1}\oplus 0^\perp\right)\right\|_{\mathcal{B}(\mathcal{H}_0)} \le C^{(1)}\varepsilon + C^{(2)}\frac{\varepsilon}{\delta(\varepsilon)} + C^{(3)}\sqrt{\sigma_k(\delta(\varepsilon))} + C^{(4)}\sqrt{\sigma_{\dot\theta}(\tilde\delta(\varepsilon))}$$

for some r satisfying $-r\in\mathbb{C}\setminus(\sigma(H_\varepsilon)\cup\sigma(h_{\text{eff}}))$ and some constants $C^{(i)}$, $i=1,2,3,4$. The right-hand side tends to zero when $\varepsilon\to 0$.

Remark 3.2.

(i) This theorem expresses in fact the norm resolvent convergence of the initial operator H (resp. some unitarily equivalent operator) to the effective Hamiltonian. This implies e.g. the convergence of the spectrum of H to the spectrum of h_{eff} (possible bound states etc.), see Section B.3.

(ii) In Assumption 2, we state some requirements on the functions k_1 and k_2 that are not unique for the curve. However, let us fix some particular RPAF, for that the curvatures are k_1^0 and k_2^0 and recall that the curvatures for different RPAFs are only the linear combination of k_1^0 and k_2^0. When we examine the condition (3.8)*, we easily find that if k_1^0 and k_2^0 satisfy it, then all their linear combinations do satisfy it as well (due to the triangle inequality in L^2), hence there is no ambiguity in the theorem.*

Before coming to proof of this theorem we will give some examples, how this theorem is used.

Example 3.3. *Let the curvatures $k_1(s)$, $k_2(s)$, $s \in \mathbb{R}$ be Lipschitz continuous. Then it holds*

$$|k_i(s_1) - k_i(s_2)| \le L_i |s_1 - s_2| \qquad \forall s_1, s_2 \in I, \quad i = 1, 2$$

and we get

$$\sigma_k(\delta(\varepsilon)) = 3 \max_{i=1,2} \sup_{n\in\mathbb{Z}} \left(\sup_{|\eta| \le \frac{\delta(\varepsilon)}{2}} \int_n^{n+1} |k_i(s) - k_i(s+\eta)|^2 \, ds \right) \le \tag{3.11}$$

$$\le 3 \max_{i=1,2} \left(\sup_{|\eta| \le \frac{\delta(\varepsilon)}{2}} L_i^2 (\eta)^2 \right) = 3 \left(\max_{i=1,2} L_i \right)^2 (\delta(\varepsilon))^2. \tag{3.12}$$

Hence the quantity $\sqrt{\sigma_k(\delta(\varepsilon))} \propto \delta(\varepsilon)$ and since the asymptotic also depends on $\frac{\varepsilon}{\delta(\varepsilon)}$, it is convenient to choose $\delta(\varepsilon) = \varepsilon^{1/2}$. If we assume in addition that for $\dot{\theta}$ we have $\sqrt{\sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon))} \propto F(\tilde{\delta}(\varepsilon))$ where F is some invertible function, then we can choose $\tilde{\delta}(\varepsilon) = F^{-1}(\varepsilon^{1/2})$, then $F(\tilde{\delta}(\varepsilon)) \propto \varepsilon^{1/2}$ and we get

$$\left\| U(H_\varepsilon + r)^{-1} U^{-1} - \left((h_{\text{eff}} + r)^{-1} \oplus 0^\perp \right) \right\|_{\mathcal{B}(\mathcal{H}_0)} \le C \varepsilon^{1/2}.$$

Example 3.4. *Let us now consider one particular curve which has non-continuous curvatures, namely*

$$k_1(s) = \begin{cases} 1 & s \in (2n, 2n+1) \\ -1 & s \in (2n+1, 2n+2) \end{cases} \qquad n \in \mathbb{Z},$$

$k_2(s) = 0$ $\forall s \in \mathbb{R}$. The curve with such curvatures is a curve lying in a plane and formed by arcs of circle with radius 1 which have its center in one half-plane for $s \in (2n, 2n+1)$ and in the other half-plane for $s \in (2n+1, 2n+2)$. Then we compute

$$\sigma_k(\delta(\varepsilon)) = 3 \sup_{n\in\mathbb{Z}} \left(\sup_{|\eta| \le \frac{\delta(\varepsilon)}{2}} \int_n^{n+1} |k_1(s) - k_1(s+\eta)|^2 \, ds \right) = \tag{3.13}$$

$$= 3 \int_n^{n+1} |k_1(s) - k_1(s+\eta)|^2 \, ds \Big|_{\eta = \pm \frac{\delta(\varepsilon)}{2}} = 3 \int_{n+1-\frac{\delta(\varepsilon)}{2}}^{n+1} 2^2 ds = 6\delta(\varepsilon), \tag{3.14}$$

hence $\sqrt{\sigma_k(\delta(\varepsilon))} \propto \sqrt{\delta(\varepsilon)}$. Here it is convenient to choose $\delta(\varepsilon) = \varepsilon^{2/3}$, since than both $\sqrt{\sigma_k} \propto \varepsilon^{1/3}$ and $\frac{\varepsilon}{\delta(\varepsilon)} \propto \varepsilon^{1/3}$. Again with suitable choice of $\tilde{\delta}(\varepsilon)$, we get

$$\left\| U(H_\varepsilon + r)^{-1} U^{-1} - \left((h_{\text{eff}} + r)^{-1} \oplus 0^\perp \right) \right\|_{\mathcal{B}(\mathcal{H}_0)} \le C \varepsilon^{1/3}.$$

3.2 Proof of Theorem 3.1

We will proof this theorem for real $r > 11C_k^2$, since as we showed in Section 2.2.1, for all $\psi \in W_0^{1,2}(\Omega_0)$, $\tilde{\tilde{Q}}_\varepsilon[\psi] \ge -11C_k^2 \|\psi\|$, where $\tilde{\tilde{Q}}_\varepsilon$ is a quadratic form associated with an operator unitarily equivalent to H_ε, thus $-r$ is indeed in the resolvent set of the operator H_ε. In addition $q_{\text{eff}}[\psi_1] \ge -\frac{C_k^2}{4} \|\psi_1\|_I$, hence $r > 11C_k^2$ satisfies the hypothesis of the theorem.

The proof will be divided into several steps, in the first step we state a lemma which gives the connection between operator H_ε defined by (2.20) and

$$H_0 := 1 \otimes \left(-\frac{1}{\varepsilon^2} \Delta_D^\omega - \frac{E_1}{\varepsilon^2} \right) + \left(-\Delta_D^I - \frac{\kappa^2}{4} + C(\omega)\dot{\theta}^2 \right) \otimes 1 \tag{3.15}$$

(recall that $-\Delta_D^\omega$ is the Dirichlet Laplacian in the cross section ω). The associated quadratic form reads

$$\begin{aligned} Q_0[\psi] = \int_{\Omega_0} |\partial_s \psi|^2 \, ds\, dt + \frac{1}{\varepsilon^2} \int_{\Omega_0} |\nabla' \psi|^2 \, ds\, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} |\psi|^2 \, ds\, dt + \\ + C(\omega) \int_{\Omega_0} \dot{\theta}^2 |\psi|^2 \, ds\, dt - \frac{1}{4} \int_{\Omega_0} \kappa^2 |\psi|^2 \, ds\, dt \end{aligned} \tag{3.16}$$

with

$$\operatorname{Dom} Q_0 = W_0^{1,2}(\Omega_0).$$

We can estimate $Q_0[\psi] \geq -\frac{C_k^2}{4} \|\psi\|^2$, hence the operator $H_0 + r$ will be positive for $r > 11C_k^2$ and using similar steps as in Section 2.2.5, we get for its inverse

$$\|(H_0 + r)^{-1}\| \leq \frac{1}{r - \frac{C_k^2}{4}}. \tag{3.17}$$

To compare the operators $(H_0+r)^{-1}$ and $(H_\varepsilon+r)^{-1}$ acting on Hilbert spaces $\mathcal{H}_0 := L^2(\Omega_0,\, ds\, dt)$ resp. $\mathcal{H}_\varepsilon := L^2\left(\Omega_0,\, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right)$, we use the unitary transformation (3.7).

Lemma 3.5. *Let $r > 11C_k^2$ be a real constant and let the assumptions of Theorem 3.1 be satisfied. Then*

$$\|U_\varepsilon (H_\varepsilon + r)^{-1} U_\varepsilon^{-1} - (H_0 + r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} \leq C_1^{(1)} \varepsilon + C^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} + C^{(3)} \sqrt{\sigma_k(\delta(\varepsilon))} + C^{(4)} \sqrt{\sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon))} \tag{3.18}$$

for some constants $C_1^{(1)}$ and $C^{(i)}$, $i = 2, 3, 4$.

The proof of Lemma 3.5 will be subject of Section 3.2.1.

The second part of the proof of Theorem 3.1 lies in proving another auxiliary lemma.

Lemma 3.6. *Let H_0 be the operator defined by* (3.15) *and let h_{eff} be the effective Hamiltonian* (3.1). *Then*

$$\| \, (H_0 + r)^{-1} - \left((h_{\text{eff}} + r)^{-1} \oplus 0^\perp \right) \|_{\mathcal{B}(\mathcal{H}_0)} \leq C_2^{(1)} \varepsilon$$

for some real constants $C_2^{(1)}$ and $r > 11C_k^2$.

Proof. In this proof we will use the ideas of [13] (that were used also in [20]), i.e. we will get the norm resolvent convergence by comparing the associated quadratic forms. The quadratic form associated with H_0 is (3.16), and the important feature of this form is that it acts on the functions $\psi_1 \chi_1 \in W_0^{1,2}(\Omega_0) \cap \mathcal{H}_0^1$ as

$$\begin{aligned} Q_0[\psi_1 \chi_1] &= \int_{\Omega_0} |\partial_s \psi_1|^2 \chi_1^2 \, ds\, dt + \frac{1}{\varepsilon^2} \int_{\Omega_0} |\psi_1|^2 \left(|\nabla' \chi_1|^2 - \frac{E_1}{\varepsilon^2} \right) ds\, dt + \\ &+ C(\omega) \int_{\Omega_0} \dot{\theta}^2 |\psi_1|^2 \chi_1^2 \, ds\, dt - \frac{1}{4} \int_{\Omega_0} \kappa^2 |\psi_1|^2 \chi_1^2 \, ds\, dt = \\ &= \int_I |\partial_s \psi_1|^2 ds + C(\omega) \int_I \dot{\theta}^2 |\psi_1|^2 ds - \frac{1}{4} \int_I \kappa^2 |\psi_1|^2 ds + r \int_I |\psi_1|^2 ds = q_{\text{eff}}[\psi_1] \end{aligned} \tag{3.19}$$

where we integrated the second term by parts with respect to t and we used that $-\Delta_D^\omega \chi_1 = E_1 \chi_1$.

To prove the norm resolvent convergence we use that

$$\| \, (H_0^r)^{-1} - \left((h_{\text{eff}}^r)^{-1} \oplus 0^\perp \right) \|_{\mathcal{B}(\mathcal{H}_0)} \leq \sup_{f, g \in \mathcal{H}_0} \frac{\left| \left(f,\, \left[(H_0^r)^{-1} - \left((h_{\text{eff}}^r)^{-1} \oplus 0^\perp \right) \right] g \right) \right|}{\|f\| \|g\|} \tag{3.20}$$

where we assigned $H_0^r = H_0 + r$ and $h_{\text{eff}}^r = h_{\text{eff}} + r$. Recall that r is from the resolvent set of both H_0 and h_{eff}, thus the operators $(H_0^r)^{-1}$ and $(h_{\text{eff}}^r)^{-1}$ are bounded and thus defined on the whole Hilbert space $\mathcal{H}_0$ and we can compute the supremum over all $f, g \in \mathcal{H}_0$. Then we define the functions $\phi \in \operatorname{Dom} H_0$ and $\psi_1 \in \operatorname{Dom} h_{\text{eff}}$ by the resolvent equations

$$\mathcal{H}_0^r \phi = f,$$
$$(h_{\text{eff}}^r \psi_1)\chi_1 = P_1 g.$$

and we rewrite the numerator from the right-hand side of (3.20) (without the absolute value) as

$$\Big(H_0^r\phi, \Big[(H_0^r)^{-1} - \Big((h_{\text{eff}}^r)^{-1} \oplus 0^\perp\Big)\Big] \left((h_{\text{eff}}^r\psi_1)\chi_1 + g^\perp\right)\Big) = \left(\phi, (h_{\text{eff}}^r\psi_1)\chi_1 + g^\perp\right) - \left(\mathcal{H}_0^r\phi, \psi_1\chi_1\right) =$$
$$= q_{\text{eff}}^r(\phi_1, \psi_1) + \left(\phi^\perp, (h_{\text{eff}}^r\psi_1)\chi_1\right) + \left(\phi, g^\perp\right) - Q_0^r(\phi_1\chi_1, \psi_1\chi_1) - Q_0^r\left(\phi^\perp, \psi_1\chi_1\right).$$

Here we used the representation theorem (Theorem B.10) and we also use the decomposition (3.4) with the notation used therein. By the similar computations as in (3.19) we get $q_{\text{eff}}^r(\phi_1, \psi_1) - Q_0^r(\phi_1\chi_1, \psi_1\chi_1) = 0$. Also $\left(\phi^\perp, (h_{\text{eff}}^r\psi_1)\chi_1\right) = 0$ since $\phi^\perp \perp \chi_1$ in $L^2(\omega)$, and after straightforward computation we get $Q_0^r\left(\phi^\perp, \psi_1\chi_1\right) = 0$ from the same reason (this will be shown in (3.45)). Hence the only nonzero term is $\left(\phi, g^\perp\right)$, and we can make the following estimate on numerator in (3.20) which holds for all $f, g \in \mathcal{H}_0$.

$$\left|\Big(f, \Big[(H_0^r)^{-1} - \Big((h_{\text{eff}}^r)^{-1} \oplus 0^\perp\Big)\Big]\, g\Big)\right| = \left|\left(\phi, g^\perp\right)\right| = \left|\left((H_0^r)^{-1}f, (1-P_1)g\right)\right| = \left|\left(f, (H_0^r)^{-1}(1-P_1)g\right)\right| \le$$
$$\le \|f\|\|g\|\|(H_0^r)^{-1}(1-P_1)\|_{\mathcal{B}(\mathcal{H}_0)}. \tag{3.21}$$

The last task is then to show that $\|(H_0^r)^{-1}(1-P_1)\|_{\mathcal{B}(\mathcal{H}_0)} \propto \varepsilon$. We rewrite this norm as

$$\|(H_0^r)^{-1}(1-P_1)\|_{\mathcal{B}(\mathcal{H}_0)} = \sup_{\psi \in (\mathcal{H}_0^1)^\perp} \frac{\|(H_0^r)^{-1}\psi\|}{\|\psi\|}$$

and we will examine the expression $u = (H_0^r)^{-1}\psi$. We show that if $H_0^r u = \psi \in (\mathcal{H}_0^1)^\perp$ then also $u \in (\mathcal{H}_0^1)^\perp$. Indeed, for all $\eta \in W_0^{1,2}(\Omega_0)$ it holds

$$(\eta, H_0^r u) = Q_0^r(\eta, u).$$

If we set $\eta = P_1 u$ then using $H_0^r u \in (\mathcal{H}_0^1)^\perp$ we get

$$0 = Q_0^r(P_1u, u) = Q_0^r(P_1u, P_1u) \ge \left(r - \frac{C_k^2}{4}\right)\|P_1u\|$$

where we again used that $Q_0^r(P_1u, (1-P_1)u) = 0$. On the right-hand side there is a positive constant $r - \frac{C_k^2}{4}$, hence $\|P_1u\| = 0$, and $u \in (\mathcal{H}_0^1)^\perp$. For every $f \in W_0^{1,2}(\Omega_0)$ it holds

$$\|f^\perp\| \le \varepsilon C_1\sqrt{Q_0^r[f]}$$

(this estimate will be proved in Section 3.3.3 as relation (3.43)), hence

$$\|u\| = \|u^\perp\| \le \varepsilon C_1\sqrt{Q_0^r[u]} = \varepsilon C_1\sqrt{Q_0^r[(H_0^r)^{-1}\psi]} \le \varepsilon C_1\sqrt{((H_0^r)^{-1}\psi, \psi)} \le \varepsilon C_1\sqrt{\|(H_0^r)^{-1}\|}\|\psi\|,$$

and we conclude

$$\|(H_0^r)^{-1}(1-P_1)\|_{\mathcal{B}(\mathcal{H}_0)} \le \varepsilon C_1\sqrt{\|(H_0^r)^{-1}\|}.$$

Altogether it follows from (3.20) and (3.21) that

$$\|\,(H_0^r)^{-1} - \Big((h_{\text{eff}}^r)^{-1} \oplus 0^\perp\Big)\,\|_{\mathcal{B}(\mathcal{H}_0)} \le \varepsilon C_1\sqrt{\|(H_0^r)^{-1}\|} \le \varepsilon C_1\sqrt{\frac{1}{r - \frac{C_k^2}{4}}}.$$

were we used that $\|(H_0^r)^{-1}\|$ is bounded due to (3.17) and this relation completes the proof of Lemma 3.6 with $C_2^{(1)} = C_1\sqrt{\frac{1}{r-\frac{C_k^2}{4}}}$. $\square$

Now it is easy to complete also the proof of Theorem 3.1, since from the triangle inequality for the norm in $\mathcal{B}(\mathcal{H}_0)$ we get

$$\begin{aligned}
&\|U_\varepsilon(H_\varepsilon+r)^{-1}U_\varepsilon^{-1} - \left((h_{\text{eff}}+r)^{-1}\oplus 0^\perp\right)\|_{\mathcal{B}(\mathcal{H}_0)} \leq \\
&\leq \|U_\varepsilon(H_\varepsilon+r)^{-1}U_\varepsilon^{-1} - (H_0+r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} + \|\,(H_0+r)^{-1} - (h_{\text{eff}}+r)^{-1}\oplus 0^\perp\|_{\mathcal{B}(\mathcal{H}_0)} \leq \\
&\leq (C_1^{(1)}+C_2^{(1)})\varepsilon + C^{(2)}\frac{\varepsilon}{\delta(\varepsilon)} + C^{(3)}\sqrt{\sigma_k(\delta(\varepsilon))} + C^{(4)}\sqrt{\sigma_{\dot\theta}(\tilde\delta(\varepsilon))}.
\end{aligned}$$

Since H_ε is unitarily equivalent to H, it is possible to rewrite this relation into the form stated by Theorem 3.1. $\square$

3.2.1 Proof of Lemma 3.5

We start again with an auxiliary lemma and the proof will be again based on the ideas from [13].

Lemma 3.7. *Let Q_ε^r, resp. Q_0^r be quadratic form associated with $H_\varepsilon+r =: H_\varepsilon^r$, resp. $H_0+r =: H_0^r$ where $r > 11C_k^2$ and let the assumption of the Theorem 3.1 be satisfied. Then $\forall\phi,\psi\in W_0^{1,2}(\Omega_0)$*

$$|Q_\varepsilon^r(\phi,\psi) - Q_0^r(\phi,\psi)| \leq \left(\tilde C^{(1)}\varepsilon + \tilde C^{(2)}\frac{\varepsilon}{\delta(\varepsilon)} + \tilde C^{(3)}\sqrt{\sigma_k(\delta(\varepsilon))} + \tilde C^{(4)}\sqrt{\sigma_{\dot\theta}(\tilde\delta(\varepsilon))}\right)\sqrt{Q_0^r[\phi]Q_\varepsilon^r[\psi]}.$$

for some constants $\tilde C^{(i)}$, $i = 1,2,3,4$ and the right-hand side tends to zero when $\varepsilon\to 0$.

The proof of this lemma will be the subject of Section 3.3 and it is the most difficult part of the proof of the main theorem. However, assuming that it holds, we pose $\phi = (H_0^r)^{-1}f$, $\psi = (H_\varepsilon^r)^{-1}g$ for some functions $f,g\in\mathcal{H}_0$ and we also assign $\sigma(\varepsilon) = \tilde C^{(1)}\varepsilon + \tilde C^{(2)}\frac{\varepsilon}{\delta(\varepsilon)} + \tilde C^{(3)}\sqrt{\sigma_k(\delta(\varepsilon))} + \tilde C^{(4)}\sqrt{\sigma_{\dot\theta}(\tilde\delta(\varepsilon))}$ to get

$$\begin{aligned}
&\left|Q_0^r\left((H_0^r)^{-1}f,(H_\varepsilon^r)^{-1}g\right) - Q_\varepsilon^r\left((H_0^r)^{-1}f,(H_\varepsilon^r)^{-1}g\right)\right| = \left|\left(f,(H_\varepsilon^r)^{-1}g\right) - \left((H_0^r)^{-1}f,g\right)_\varepsilon\right| \leq \\
&\leq \sigma(\varepsilon)\left((H_0^r)^{-1}f,f\right)\left((H_\varepsilon^r)^{-1}g,g\right)_\varepsilon \leq 2\sigma(\varepsilon)\sqrt{\|(H_0^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)}\|(H_\varepsilon^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)}}\|f\|\|g\|. \quad (3.22)
\end{aligned}$$

Here we again used the representation theorem and the fact that $\phi = (H_0^r)^{-1}f \in \operatorname{Dom} H_0 \subseteq \operatorname{Dom} Q_\varepsilon$, $\psi = (H_\varepsilon^r)^{-1}g \in \operatorname{Dom} H_\varepsilon \subseteq \operatorname{Dom} Q_0$. We also used the fact that $\|.\|_\varepsilon \leq 2\|.\|$

Next we again use that

$$\|U_\varepsilon(H_\varepsilon+r)^{-1}U_\varepsilon^{-1} - (H_0+r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} \leq \sup_{f,g\in\mathcal{H}_0}\left(\frac{\left|\left(f,U_\varepsilon(H_\varepsilon^r)^{-1}U_\varepsilon^{-1}g\right) - \left(f,(H_0^r)^{-1}g\right)\right|}{\|f\|\|g\|}\right). \quad (3.23)$$

Noticing that the scalar products on $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ are related by

$$(\phi,\psi)_\varepsilon = (U_\varepsilon\phi, U_\varepsilon\psi).$$

we can rewrite the numerator on the right-hand side of (3.23) as

$$\begin{aligned}
&\left|\left(f,U_\varepsilon(H_\varepsilon^r)^{-1}U_\varepsilon^{-1}g\right) - \left(f,(H_0^r)^{-1}g\right)\right| = \left|\left(f,U_\varepsilon(H_\varepsilon^r)^{-1}U_\varepsilon^{-1}g\right) - \left((H_0^r)^{-1}f,g\right)\right| \leq \quad (3.24)\\
\leq\ &\left|\left(f,U_\varepsilon(H_\varepsilon^r)^{-1}U_\varepsilon^{-1}g\right) - \left(f,(H_\varepsilon^r)^{-1}g\right)\right| + \left|\left(U_\varepsilon(H_0^r)^{-1}f,U_\varepsilon g\right) - \left((H_0^r)^{-1}f,g\right)\right| + \\
&+\left|\left(f,(H_\varepsilon^r)^{-1}g\right) - \left(U_\varepsilon(H_0^r)^{-1}f,U_\varepsilon g\right)\right| \leq \quad (3.25)\\
\leq\ &\left|\left(f,(U_\varepsilon-1)(H_\varepsilon^r)^{-1}U_\varepsilon^{-1}g\right)\right| + \left|\left(f,(H_\varepsilon^r)^{-1}(U_\varepsilon^{-1}-1)g\right)\right| + \left|\left((H_0^r)^{-1}f,(U_\varepsilon^2-1)g\right)\right| + \\
&+\left|\left(f,(H_\varepsilon^r)^{-1}g\right) - \left(U_\varepsilon(H_0^r)^{-1}f,U_\varepsilon g\right)\right|.
\end{aligned}$$

For the unitary transformation $U_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_0$ and arbitrary $\psi \in \mathcal{H}_0$ holds

$$\|(U_\varepsilon - 1)\psi\|^2 = \int_{\Omega_0} \left(\frac{|G|^{1/4}}{|\tilde{G}|^{1/4}} - 1 \right)^2 |\psi|^2 \, ds \, dt \le (6\varepsilon a C_k)^2 \|\psi\|_\varepsilon^2, \tag{3.26}$$

similarly

$$\begin{aligned} \|(U_\varepsilon^{-1} - 1)\psi\|_\varepsilon &\le 6\varepsilon a C_k \|\psi\|, \\ \|(U_\varepsilon^2 - 1)\psi\| &\le 12\varepsilon a C_k \|\psi\|, \\ \|U_\varepsilon^{-1}\psi\|_\varepsilon &= \|\psi\|. \end{aligned}$$

Hence using the Schwarz inequality and the relation (3.22) we get

$$\begin{aligned} &\left| \left(f, U_\varepsilon (H_\varepsilon^r)^{-1} U_\varepsilon^{-1} g\right) - \left(f, (H_0^r)^{-1} g\right) \right| \le \\ \le\ & \left[12\varepsilon a C_k \left(\|(H_\varepsilon^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)} + \|(H_0^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} \right) + 2\sigma(\varepsilon) \sqrt{\|(H_0^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} \|(H_\varepsilon^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)}} \right] \|f\| \|g\| \le \\ \le\ & \left(C_3^{(1)}\varepsilon + \tilde{C} \left(\tilde{C}^{(1)}\varepsilon + \tilde{C}^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} + \tilde{C}^{(3)} \sqrt{\sigma_k(\delta(\varepsilon))} + \tilde{C}^{(4)} \sqrt{\sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon))} \right) \right) \|f\| \|g\| \end{aligned}$$

which proves Lemma 3.5. Here we used that $\|(H_\varepsilon^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)}$ and $\|(H_0^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)}$ are bounded by constants independent of ε (see (2.25) and (3.17)). □

3.3 Proof of Lemma 3.7

Let $\phi, \psi \in W_0^{1,2}(\Omega_0)$. Then the proof consists of estimating the difference of appropriate terms in the sesquilinear forms $Q_\varepsilon^r(\phi,\psi)$ resp. $Q_0^r(\phi,\psi)$ which read

$$Q_\varepsilon^r(\phi,\psi) := \int_{\Omega_0} \frac{1}{h h_\varepsilon} (\partial_s + \dot{\theta}\partial_\tau)\bar{\phi} (\partial_s + \dot{\theta}\partial_\tau)\psi \, ds \, dt + \tag{3.27a}$$

$$+ \frac{1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} \nabla'\bar{\phi} \cdot \nabla'\psi \, ds \, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} \bar{\phi}\psi \, ds \, dt + \tag{3.27b}$$

$$+ \frac{1}{2} \int_{\Omega_0} \frac{1}{h_\varepsilon^2} (k_1 k_1^\varepsilon + k_2 k_2^\varepsilon) \bar{\phi}\psi \, ds \, dt - \frac{3}{4} \int_{\Omega_0} \frac{h}{h_\varepsilon^3} \left((k_1^\varepsilon)^2 + (k_2^\varepsilon)^2 \right) \bar{\phi}\psi \, ds \, dt + \tag{3.27c}$$

$$- \int_{\Omega_0} \frac{(\partial_s + \dot{\theta}\partial_\tau) h_\varepsilon}{2 h h_\varepsilon^2} \left(\bar{\phi} (\partial_s + \dot{\theta}\partial_\tau)\psi + (\partial_s + \dot{\theta}\partial_\tau)\bar{\phi}\psi \right) ds \, dt + \tag{3.27d}$$

$$+ \int_{\Omega_0} \frac{\left((\partial_s + \dot{\theta}\partial_\tau) h_\varepsilon \right)^2}{4 h h_\varepsilon^3} \bar{\phi}\psi \, ds \, dt + r \int_{\Omega_0} \frac{h}{h_\varepsilon} \bar{\phi}\psi \, ds \, dt \tag{3.27e}$$

resp.

$$Q_0^r(\phi,\psi) := \int_{\Omega_0} \partial_s\bar{\phi}\partial_s\psi \, ds \, dt + C(\omega) \int_{\Omega_0} \dot{\theta}^2 \bar{\phi}\psi \, ds \, dt + \tag{3.28a}$$

$$+ \frac{1}{\varepsilon^2} \int_{\Omega_0} \nabla'\bar{\phi} \cdot \nabla'\psi \, ds \, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \bar{\phi}\psi \, ds \, dt + \tag{3.28b}$$

$$- \frac{1}{4} \int_{\Omega_0} (k_1^2 + k_2^2) \bar{\phi}\psi \, ds \, dt + r \int_{\Omega_0} \bar{\phi}\psi \, ds \, dt \tag{3.28c}$$

where $C(\omega)$ is given by (3.2). The final formulas for individual estimates that form together the proof of Lemma 3.7, will be denoted by (p1)-(p6) to point them out among all the auxiliary estimates.

3.3.1 Preliminaries

In the following, it will be convenient to estimate individual terms in $|Q^r_\varepsilon(\phi,\psi) - Q^r_0(\phi,\psi)|$ by norms like $\|\psi\|$, $\|\partial_s\psi\|$, $\|\phi\|$ etc., thus as the first step we will find the estimates on such norms by $\sqrt{Q^r_\varepsilon[\psi]}$ resp. $\sqrt{Q^r_0[\phi]}$.

At first we will have to find some finer estimate on the lower bound of Q^r_ε then the one we found in Section 2.2.5.

Lemma 3.8. *Let $\psi \in W^{1,2}_0(\Omega_0)$ and let r be a positive constant. Then*

$$Q^r_\varepsilon[\psi] \geq \frac{1}{2}\int_{\Omega_0} \frac{1}{hh_\varepsilon}\left|(\partial_s + \dot\theta\partial_\tau)\psi\right|^2 ds\,dt + \left(r - 9C_k^2\right)\|\psi\|^2_\varepsilon. \tag{3.29}$$

Proof. The proof will consist of estimating individual terms in $Q_\varepsilon[\psi]$, i.e. the terms in (3.27) with $\phi = \psi$. If we assume that ε is so small that $\varepsilon a C_k \leq \frac{1}{16}$ (we will assume this while doing all the following estimates as well), then (2.10) holds and the same holds when in (2.10) h is replaced by h_ε. Consequently we get for the term corespondent to the term on line (3.27c)

$$\left|\frac{1}{2}\int_{\Omega_0}\frac{1}{h_\varepsilon^2}(k_1k_1^\varepsilon + k_2k_2^\varepsilon)|\psi|^2\,ds\,dt - \frac{3}{4}\int_{\Omega_0}\frac{h}{h_\varepsilon^3}\left((k_1^\varepsilon)^2 + (k_2^\varepsilon)^2\right)|\psi|^2\,ds\,dt\right| \leq 5C_k^2\|\psi\|^2_\varepsilon.$$

The estimate on $\frac{1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\nabla'\psi|^2\,ds\,dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0}|\psi|^2\,ds\,dt$ will be performed in the similar way as in (2.15). We rewrite the integral using the function $\phi := \sqrt{\frac{h}{h_\varepsilon}}\psi$, we use that for the functions $\phi \in W^{1,2}_0(\Omega_0)$

$$\int_\omega |\nabla'\phi|^2 dt - E_1\int_\omega |\phi|^2 dt \geq 0$$

and finally using the Fubini theorem we get

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\nabla'\psi|^2\,ds\,dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\psi|^2\,ds\,dt = \\
= \;& \frac{1}{\varepsilon^2}\int_{\Omega_0}\left(|\partial_2\phi|^2 + |\partial_3\phi|^2 - E_1|\phi|^2\right)ds\,dt + \\
& + \int_{\Omega_0}\left(\frac{3(k_1^2+k_2^2)}{4h^2} - \frac{k_1k_1^\varepsilon + k_2k_2^\varepsilon}{2hh_\varepsilon} - \frac{(k_1^\varepsilon)^2 + (k_2^\varepsilon)^2}{4h_\varepsilon^2}\right)|\phi|^2\,ds\,dt \geq \\
\geq \;& \int_{\Omega_0}\left(\frac{3(k_1^2+k_2^2)}{4h^2} - \frac{k_1k_1^\varepsilon + k_2k_2^\varepsilon}{2hh_\varepsilon} - \frac{(k_1^\varepsilon)^2 + (k_2^\varepsilon)^2}{4h_\varepsilon^2}\right)|\phi|^2\,ds\,dt \geq \\
\geq \;& -3C_k^2\int_{\Omega_0}|\phi|^2\,ds\,dt = -3C_k^2\|\psi\|^2_\varepsilon.
\end{aligned}$$

The second term on line (3.27d) can estimated using the Schwarz inequality and the simple Young's inequality ($2ab \leq a^2 + b^2$)

$$\begin{aligned}
&\left|\int_{\Omega_0}\frac{(\partial_s + \dot\theta\partial_\tau)h_\varepsilon}{hh_\varepsilon^2}\operatorname{Re}(\bar\psi(\partial_s + \dot\theta\partial_\tau)\psi)\,ds\,dt\right| \leq \\
&\leq \frac{1}{2}\left[\int_{\Omega_0}\frac{\left((\partial_s + \dot\theta\partial_\tau)h_\varepsilon\right)^2}{4hh_\varepsilon^3}|\psi|^2\,ds\,dt + \int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|(\partial_s + \dot\theta\partial_\tau)\psi\right|^2 ds\,dt\right].
\end{aligned} \tag{3.30}$$

Using all the estimates above we get

$$Q_\varepsilon[\psi] \geq \frac{1}{2}\int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|(\partial_s + \dot\theta\partial_\tau)\psi\right|^2 ds\,dt - 3C_k^2\|\psi\|^2_\varepsilon - 5C_k^2\|\psi\|^2_\varepsilon - \int_{\Omega_0}\frac{\left((\partial_s + \dot\theta\partial_\tau)h_\varepsilon\right)^2}{4hh_\varepsilon^3}|\psi|^2\,ds\,dt. \tag{3.31}$$

Hence as the last step we have to estimate the last term above. For this purpose we rewrite

$$(\partial_s + \dot\theta\partial_\tau)h_\varepsilon = -\varepsilon\left[t_2(\dot k_1^\varepsilon\cos\theta + \dot k_2^\varepsilon\sin\theta) + t_3(-(\dot k_1^\varepsilon\sin\theta + \dot k_2^\varepsilon\cos\theta)\right]$$

and recalling the relation (2.18) we get

$$|(\partial_s + \dot\theta\partial_\tau)h_\varepsilon| \le 8\frac{\varepsilon}{\delta(\varepsilon)}aC_k. \tag{3.32}$$

In consequence of (3.10) we can assume

$$\left|\int_{\Omega_0}\frac{\left((\partial_s + \dot\theta\partial_\tau)h_\varepsilon\right)^2}{4hh_\varepsilon^3}|\psi|^2\,ds\,dt\right| \le 51\frac{\varepsilon^2}{\delta(\varepsilon)^2}a^2C_k^2\|\psi\|_\varepsilon^2 \le C_k^2\|\psi\|_\varepsilon^2$$

and adding $r\|\psi\|_\varepsilon^2$ to both sides of (3.31) we get the final estimate

$$Q_\varepsilon^r[\psi] \ge \frac{1}{2}\int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|(\partial_s + \dot\theta\partial_\tau)\psi\right|^2\,ds\,dt + (r - 9C_k^2)\|\psi\|_\varepsilon^2.$$

□

Using similar steps, it is easy to prove that also

$$Q_0^r[\phi] \ge \|\partial_s\phi\|^2 + (r - \frac{C_k^2}{4})\|\phi\|^2. \tag{3.33}$$

As a consequence of relations (3.29) and (3.33), we get

$$\begin{aligned}
\|\psi\|^2 &\le 2\|\psi\|_\varepsilon^2 \le \frac{2}{(r-9C_k^2)}Q_\varepsilon^r[\psi] \le \frac{1}{C_k^2}Q_\varepsilon^r[\psi] &\quad (3.34a)\\
\|\phi\|^2 &\le \frac{1}{(r-\frac{C_k^2}{4})}Q_0^r[\phi] \le \frac{1}{C_k^2}Q_0^r[\phi] &\quad (3.34b)\\
\|(\partial_s + \dot\theta\partial_\tau)\psi\|^2 &\le 2\int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|(\partial_s + \dot\theta\partial_\tau)\psi\right|^2\,ds\,dt \le 4Q_\varepsilon^r[\psi] &\quad (3.34c)\\
\|\partial_s\phi\|^2 &\le Q_0^r[\phi]. &\quad (3.34d)
\end{aligned}$$

To get simpler formulas in (3.34a) and (3.34b) we used the assumption that $r > 11C_k^2$.

Similar inequalities for $\|\partial_s\psi\|$ resp. $\|(\partial_s + \dot\theta\partial_\tau)\phi\|^2$ with $Q_\varepsilon^r[\psi]$ resp. $Q_0^r[\phi]$ on the right hand side cannot be derived straightly from the inequalities (3.29), (3.33), to prove them the techniques used in following paragraphs will be necessary. Thus we state them as a lemma which will be proved in Section 3.3.6.

Lemma 3.9. *Let the assumption of the Theorem 3.1 be fulfilled. Then* $\forall\phi,\psi \in W_0^{1,2}$

$$\begin{aligned}
\|(\partial_s + \dot\theta\partial_\tau)\phi\|^2 &\le 2Q_0^r[\phi], &\quad (3.34e)\\
\|\partial_s\psi\|^2 &\le 8Q_\varepsilon^r[\psi]. &\quad (3.34f)
\end{aligned}$$

Finally, let us note that in the estimates below, the techniques that we used in proof of 3.8 will be often used automatically. Namely, we will assume that $\varepsilon aC_k \le \frac{1}{16}$, thus (2.10) holds, and similarly for h_ε we have $\frac{3}{4} \le h_\varepsilon \le \frac{5}{4}$. Also the Schwarz and Young inequalities will be often used as well as the relations (2.9), (2.19) and (2.2).

3.3.2 The easy estimates

Now we start to estimate individual terms in $|Q^r_\varepsilon(\phi,\psi) - Q^r_0(\phi,\psi)|$, in this section we will perform the estimates where the simple ideas will be sufficient.

The terms (3.27d) and the first term on line (3.27e) have to tend to zero since there are no equivalent terms in Q^r_0. To estimate the term (3.27d) we use (3.32), then the Schwarz inequality yields

$$\int_{\Omega_0} \frac{\left((\partial_s + \dot\theta\partial_\tau)h_\varepsilon\right)^2}{4hh_\varepsilon^3}\bar\phi\psi\, ds\, dt \le 51\frac{\varepsilon^2}{\delta(\varepsilon)^2}a^2C_k^2\|\phi\|\|\psi\| \le 51\frac{\varepsilon^2}{\delta(\varepsilon)^2}a^2\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]}$$
$$=: \tilde C^{(5)}\frac{\varepsilon^2}{\delta(\varepsilon)^2}\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]}. \tag{p1}$$

Here $\frac{\varepsilon^2}{\delta(\varepsilon)^2}$ tend to zero since we assumed (3.10). The term (3.27d) can be decomposed using Schwarz inequality similarly as in (3.30). We will estimate the two terms in the bracket individually, for the first of them we have:

$$\left|\int_{\Omega_0}\frac{(\partial_s + \dot\theta\partial_\tau)h_\varepsilon}{2hh_\varepsilon^2}\bar\phi(\partial_s + \dot\theta\partial_\tau)\psi\, ds\, dt\right| \le$$
$$\le \sqrt{\int_{\Omega_0}\frac{\left((\partial_s + \dot\theta\partial_\tau)h_\varepsilon\right)^2}{4hh_\varepsilon^3}|\phi|^2\, ds\, dt}\sqrt{\int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|(\partial_s + \dot\theta\partial_\tau)\psi\right|^2\, ds\, dt} \le$$
$$\le 10\frac{\varepsilon}{\delta(\varepsilon)}aC_k\|\phi\|\|(\partial_s + \dot\theta\partial_\tau)\psi\| \le 20\frac{\varepsilon}{\delta(\varepsilon)}a\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]} =: \frac{\tilde C_1^{(2)}}{2}\frac{\varepsilon}{\delta(\varepsilon)}\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]}. \tag{p2}$$

Similarly for the second one:

$$\left|\int_{\Omega_0}\frac{(\partial_s + \dot\theta\partial_\tau)h_\varepsilon}{2hh_\varepsilon^2}(\partial_s + \dot\theta\partial_\tau)\bar\phi\psi\, ds\, dt\right| \le 20\frac{\varepsilon}{\delta(\varepsilon)}a\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]} =: \frac{\tilde C_1^{(2)}}{2}\frac{\varepsilon}{\delta(\varepsilon)}\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]}. \tag{p2'}$$

Also the difference of the terms with r is easy to estimate:

$$\left|r\int_{\Omega_0}\frac{h}{h_\varepsilon}\bar\phi\psi\, ds\, dt - r\int_{\Omega_0}\bar\phi\psi\, ds\, dt\right| \le r\left|\frac{h}{h_\varepsilon} - 1\right|\|\phi\|\|\psi\| \le 12\varepsilon arC_k\|\phi\|\|\psi\|$$
$$\le \frac{12\varepsilon ar}{C_k}\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]} =: \tilde C_1^{(1)}\varepsilon\sqrt{Q^r_0[\phi]}\sqrt{Q^r_\varepsilon[\psi]}. \tag{p3}$$

Next we estimate the difference of the terms on line (3.27c) and the first term on line (3.28c):

$$|q(\phi,\psi)| := \left|\frac{1}{2}\int_{\Omega_0}\frac{1}{h_\varepsilon^2}(k_1k_1^\varepsilon + k_2k_2^\varepsilon)\bar\phi\psi\, ds\, dt - \frac{3}{4}\int_{\Omega_0}\frac{h}{h_\varepsilon^3}\left((k_1^\varepsilon)^2 + (k_2^\varepsilon)^2\right)\bar\phi\psi\, ds\, dt + \right.$$
$$\left. + \frac{1}{4}\int_{\Omega_0}(k_1^2 + k_2^2)\bar\phi\psi\, ds\, dt\right| = \tag{3.35}$$
$$= \left|\int_{\Omega_0}\left(\frac{(3k_1^\varepsilon + k_1)(k_1 - k_1^\varepsilon) + (3k_2^\varepsilon + k_2)(k_2 - k_2^\varepsilon)}{4h_\varepsilon^3} + \frac{(k_1k_1^\varepsilon + k_2k_2^\varepsilon)(h_\varepsilon - 1)}{2h_\varepsilon^3} + \right.\right.$$
$$\left.\left. - \frac{3\left((k_1^\varepsilon)^2 + (k_2^\varepsilon)^2\right)(h-1)}{4h_\varepsilon^3} + \frac{(k_1^2 + k_2^2)(h_\varepsilon^3 - 1)}{h_\varepsilon^3}\right)\bar\phi\psi\, ds\, dt\right| \le$$
$$\le 3C_k\int_{\Omega_0}\left(|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|\right)|\phi|\,|\psi|\, ds\, dt + \varepsilon aC_k^3\|\phi\|\|\psi\| \tag{3.36}$$

The task is now to estimate somehow the first term on the last line. If we assumed e.g. that the functions k_1 resp. k_2 are uniformly continuous, we could use Lemma 2.4 which says that

$|k_1(s)-k_1^\varepsilon(s)| \leq \omega(\delta(\varepsilon),k_1)$ resp. $|k_2(s)-k_2^\varepsilon(s)| \leq \omega(\delta(\varepsilon),k_2)$ for all $s \in I$ where $\omega(\delta(\varepsilon),k_i) \xrightarrow{\varepsilon\to 0} 0$. Then we could estimate the integral easily by $(\omega(\delta(\varepsilon),k_1)+\omega(\delta(\varepsilon),k_2))\,\|\phi\|\|\psi\|$. However, we don't want to restrict our results to uniformly continuous k_i, thus the Hilbert space decomposition is necessary for this estimate and we will continue with the estimate of (3.36) in the end of the next paragraph. Also while estimating the sesquilinear forms

$$m(\phi,\psi) := \frac{1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\nabla'\bar{\phi}\cdot\nabla'\psi\,ds\,dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\bar{\phi}\psi\,ds\,dt \tag{3.37}$$

(the difference of terms (3.27b) and (3.28b)) resp.

$$l(\phi,\psi) := \int_{\Omega_0}\frac{1}{hh_\varepsilon}(\partial_s+\dot{\theta}\partial_\tau)\bar{\phi}(\partial_s+\dot{\theta}\partial_\tau)\psi\,ds\,dt - \int_{\Omega_0}\partial_s\bar{\phi}\partial_s\psi\,ds\,dt - C(\omega)\int_{\Omega_0}\dot{\theta}^2\bar{\phi}\psi\,ds\,dt \tag{3.38}$$

(the difference of terms (3.27a) and (3.28a)) we will have to use the Hilbert space decomposition and these estimates are subject of Sections 3.3.4 resp. 3.3.5.

3.3.3 The Hilbert space decomposition and some other technical tools

In Section 3.1 we have introduced the subspace $\mathcal{H}_0^1$ (see relation (3.3)) and we know that every function $\psi \in \mathcal{H}_0$ can be uniquely decomposed into two parts, first of which is from $\mathcal{H}_0^1$:

$$\psi = P_1^\varepsilon\psi + (1-P_1^\varepsilon)\psi =: \psi_1\chi_1 + \psi^\perp. \tag{3.39}$$

where the projection P_1 was introduced by (3.5). The convenience of this decomposition lies in the fact that the function $\psi^\perp$ vanishes for small ε as will be proved in the following paragraph.

Similarly as in Lemma 3.8 we can find that for every $\psi \in W_0^{1,2}$

$$\begin{aligned} Q_\varepsilon^r[\psi] \;&\geq\; \frac{1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\nabla'\psi|^2\,ds\,dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\psi|^2\,ds\,dt + (r-6C_k^2)\|\psi\|_\varepsilon^2 \geq \\ &\geq\; \frac{1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\nabla'\psi|^2\,ds\,dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0}\frac{h}{h_\varepsilon}|\psi|^2\,ds\,dt \geq \\ &\geq\; \frac{1}{\varepsilon^2}\frac{1-4\varepsilon aC_k}{1+4\varepsilon aC_k}\int_{\Omega_0}|\nabla'\psi|^2\,ds\,dt - \frac{E_1}{\varepsilon^2}\frac{1+4\varepsilon aC_k}{1-4\varepsilon aC_k}\int_{\Omega_0}|\psi|^2\,ds\,dt. \end{aligned}$$

where we used that $r > 11C_k^2$ and the relation (2.10). Then we can apply this estimate on the function $\psi^\perp$ from (3.39) which is (as a function of t) orthogonal to χ_1, thus we have

$$\int_{\Omega_0}|\nabla'\psi^\perp|^2\,ds\,dt \geq E_2\|\psi^\perp\|^2$$

where E_2 is the second eigenvalue of the transverse Laplacian $-\Delta_D^\omega$. Consequently

$$Q_\varepsilon^r[\psi^\perp] \geq \frac{1}{\varepsilon^2}\frac{1-4\varepsilon aC_k}{1+4\varepsilon aC_k}\int_{\Omega_0}|\nabla'\psi^\perp|^2\,ds\,dt - \frac{E_1}{\varepsilon^2}\frac{1+4\varepsilon aC_k}{1-4\varepsilon aC_k}\int_{\Omega_0}|\psi^\perp|^2\,ds\,dt \geq \tag{3.40a}$$

$$\geq \frac{1}{\varepsilon^2}\left(E_2\frac{1-4\varepsilon aC_k}{1+4\varepsilon aC_k}(1-\beta) - E_1\frac{1+4\varepsilon aC_k}{1-4\varepsilon aC_k}\right)\|\psi^\perp\|^2 + \frac{1}{\varepsilon^2}\beta\frac{1-4\varepsilon aC_k}{1+4\varepsilon aC_k}\|\nabla'\psi^\perp\|^2 \tag{3.40b}$$

where we multiplied the first term in (3.40a) by $(\beta+(1-\beta))$ with β assigning a small positive number. Then we estimated the terms with β and $(1-\beta)$ separately getting (3.40b). The point is that β can be chosen in such way that the coefficient in front of $\|\psi^\perp\|^2$ in (3.40b) is positive. This follows from the fact that for symmetric elliptic operators always $E_2 > E_1$ (see e.g. Theorem 2 in Section 6.5.1. in [10]). It is reasonable to rewrite $\beta := \frac{E_2-E_1}{E_2}\tilde{\beta}$, then

$$\lim_{\varepsilon\to 0} E_2\frac{1-4\varepsilon aC_k}{1+4\varepsilon aC_k}\left(1-\frac{E_2-E_1}{E_2}\tilde{\beta}\right) - E_1\frac{1+4\varepsilon aC_k}{1-4\varepsilon aC_k} = (1-\tilde{\beta})(E_2-E_1) > 0 \qquad \forall\tilde{\beta} < 1.$$

Hence we choose e.g. $\tilde{\beta} = \frac{1}{2}$, and it is possible to show that for all $\varepsilon < \frac{E_2 - E_1}{16aC_k(E_2+3E_1)}$ the coefficient in front of $\|\psi^\perp\|^2$ in (3.40b) is greater then $\frac{E_2-E_1}{4\varepsilon^2}$. Finally

$$Q_\varepsilon^r[\psi^\perp] \geq \frac{E_2 - E_1}{4\varepsilon^2}\|\psi^\perp\|^2 + \frac{E_2 - E_1}{5E_2\varepsilon^2}\|\nabla'\psi^\perp\|^2 =: \frac{1}{C_1^2\varepsilon^2}\|\psi^\perp\|^2 + \frac{1}{C_2^2\varepsilon^2}\|\nabla'\psi^\perp\|^2.$$

The desirable formulas then read

$$\|\psi^\perp\| \leq \varepsilon C_1\sqrt{Q_\varepsilon^r[\psi^\perp]}, \tag{3.41}$$

$$\|\nabla'\psi^\perp\| \leq \varepsilon C_2\sqrt{Q_\varepsilon^r[\psi^\perp]}. \tag{3.42}$$

Similarly we can decompose $\phi = \phi_1\chi_1 + \phi^\perp$ and we can show that

$$Q_0^r[\phi^\perp] \geq \frac{1}{\varepsilon^2}\frac{E_2 - E_1}{2}\|\phi^\perp\|^2 + \frac{1}{\varepsilon^2}\frac{E_2 - E_1}{2E_2}\|\nabla'\phi^\perp\|^2 \geq \frac{1}{C_1^2\varepsilon^2}\|\psi^\perp\|^2 + \frac{1}{C_2^2\varepsilon^2}\|\nabla'\psi^\perp\|^2$$

where we for simplicity used rougher estimate in order to get the same coefficients as in the previous estimate:

$$\|\phi^\perp\| \leq \varepsilon C_1\sqrt{Q_0^r[\phi^\perp]}, \tag{3.43}$$

$$\|\nabla'\phi^\perp\| \leq \varepsilon C_2\sqrt{Q_0^r[\phi^\perp]}. \tag{3.44}$$

Since our goal is to get some estimate where on the right side stands $\sqrt{Q_0^r[\phi]}$ resp. $\sqrt{Q_\varepsilon^r[\psi]}$ we would also like to have some relation between $\sqrt{Q_0^r[\phi^\perp]}$ and $\sqrt{Q_0^r[\phi]}$ resp. $\sqrt{Q_\varepsilon^r[\psi^\perp]}$ and $\sqrt{Q_\varepsilon^r[\psi]}$. In the first case it is simple, since

$$Q_0^r[\phi_1\chi_1 + \phi^\perp] = Q_0^r[\phi_1\chi_1] + 2\mathrm{Re}Q_0^r(\phi_1\chi_1, \phi^\perp) + Q_0^r[\phi^\perp] = Q_0^r[\phi_1\chi_1] + Q_0^r[\phi^\perp]$$

where we used that

$$\begin{aligned} Q_0^r(\phi_1\chi_1, \phi^\perp) = &\int_{\Omega_0} \partial_s\bar{\phi}_1\chi_1\partial_s\phi^\perp\, ds\, dt + C(\omega)\int_{\Omega_0} \dot{\theta}^2\bar{\phi}_1\chi_1\phi^\perp\, ds\, dt - \frac{1}{4}\int_{\Omega_0}(k_1^2 + k_2^2)\bar{\phi}_1\chi_1\phi^\perp\, ds\, dt + \\ &+ r\int_{\Omega_0} \bar{\phi}_1\chi_1\phi^\perp\, ds\, dt + \frac{1}{\varepsilon^2}\int_{\Omega_0} \bar{\phi}_1\nabla'\chi_1\nabla'\phi^\perp\, ds\, dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0} \bar{\phi}_1\chi_1\phi^\perp\, ds\, dt = 0. \end{aligned} \tag{3.45}$$

Here all the individual terms on the first line are equal to zero since

$$f(s) := \int_\omega \chi_1(t)\phi^\perp(s,t)dt = 0 \qquad \forall s \in I \tag{3.46}$$

which follows straightly from the definition of the function $\phi^\perp$. Thus also $\frac{df(s)}{ds} = 0$ which is used in the first term and the two terms on the second line subtract in consequence of relation $-\Delta_D^\omega\chi_1 = E_1\chi_1$. Hence we get

$$Q_0^r[\phi^\perp] \leq Q_0^r[\phi], \tag{3.47a}$$

$$Q_0^r[\phi_1\chi_1] \leq Q_0^r[\phi]. \tag{3.47b}$$

More complicated is the situation for Q_ε, since here $Q_\varepsilon^r(\psi_1\chi_1, \psi^\perp) \neq 0$. That's why we will again state the following estimates as a lemma, which will be proved in Section 3.3.6 where the techniques of the Sections 3.3.4 and 3.3.5 can be used.

Lemma 3.10. $\forall\psi = \psi_1\chi_1 + \psi^\perp \in W_0^{1,2}$

$$Q_\varepsilon^r[\psi^\perp] \leq 2Q_\varepsilon^r[\psi], \tag{3.47c}$$

$$Q_\varepsilon^r[\psi_1\chi_1] \leq 2Q_\varepsilon^r[\psi]. \tag{3.47d}$$

Now we can come back to the estimate of the first term in (3.36). At first we use the Schwarz inequality to get

$$
\begin{aligned}
&\int_{\Omega_0} (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\phi|\, |\psi|\, ds\, dt \leq \\
&\leq \sqrt{\int_{\Omega_0} (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\phi|^2\, ds\, dt} \sqrt{\int_{\Omega_0} (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\psi|^2\, ds\, dt}
\end{aligned}
$$

and then we will for simplicity continue with the estimate of the second multiplicand only.

$$
\begin{aligned}
&\int_{\Omega_0} (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\psi_1\chi_1 + \psi^\perp|^2\, ds\, dt \leq \\
&\leq \int_{\Omega_0} (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\psi_1\chi_1|^2\, ds\, dt + 4C_k \int_{\Omega_0} |\psi_1\chi_1||\psi^\perp|\, ds\, dt + 4C_k \int_{\Omega_0} |\psi^\perp|^2\, ds\, dt \leq \\
&\leq \int_I (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\psi_1|^2 ds + 4C_k\varepsilon \sqrt{Q_\varepsilon^r[\psi_1\chi_1]} \sqrt{Q_\varepsilon^r[\psi^\perp]} + 4C_k\varepsilon^2 Q_\varepsilon^r[\psi^\perp] \leq \\
&\leq \int_I (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\psi_1|^2 ds + 8C_k\varepsilon Q_\varepsilon^r[\psi] + 8C_k\varepsilon^2 Q_\varepsilon^r[\psi]
\end{aligned}
$$

Similarly we would get the estimate of the first multiplicand using the decomposition $\phi = \phi_1\chi_1 + \phi^\perp$:

$$
\begin{aligned}
&\int_{\Omega_0} (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\phi_1\chi_1 + \phi^\perp|^2\, ds\, dt \leq \\
&\leq \int_I (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\phi_1|^2 ds + 4C_k\varepsilon Q_0^r[\phi] + 4C_k\varepsilon^2 Q_0^r[\phi].
\end{aligned}
$$

The first term again remains to be estimated. It looks similarly as $\int_I (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\phi|^2 ds$ we started with, however the important point is that now there stands the function ϕ_1 which is only a function of one variable and lies in $W^{1,2}(I)$, thus the Theorem 2.10 or more precisely its corollary (2.36) can be used. Hence

$$
\int_I |k_i - k_i^\varepsilon||\phi_1|^2 ds \leq \sqrt{\sigma_k(\delta(\varepsilon))} \|\phi_1\|_{W^{1,2}(I)} \|\phi_1\|_I \qquad i = 1, 2
$$

where we for simplicity assigned

$$
\sigma_k(\delta(\varepsilon)) := \max_{i \in \{1,2\}} \sigma_{k_i}(\delta(\varepsilon)) \tag{3.48}
$$

and we assume in Assumption 2 that this quantity tends to zero (if the interval I is finite, this holds automatically). Similar estimate holds for ψ_1. The last step consists of getting some estimate on $\|\partial_s\phi_1\|_I$ which occurs in $\|\phi_1\|^2_{W^{1,2}(I)}$ by $\sqrt{Q_0^r[\phi]}$. Here the key relation reads $\forall \phi = \phi_1\chi_1 + \phi^\perp \in W_0^{1,2}$

$$
\|\partial_s\phi\|^2 = \|\partial_s\phi_1\chi_1\|^2 + 2\mathrm{Re}\left(\partial_s\phi_1\chi_1, \phi^\perp\right) + \|\partial_s\phi^\perp\|^2 = \|\partial_s\phi_1\chi_1\|^2 + \|\partial_s\phi^\perp\|^2
$$

where $\left(\partial_s\phi_1\chi_1, \phi^\perp\right) = 0$ follows from (3.46). Hence

$$
\|\partial_s\phi_1\chi_1\|^2 = \|\partial_s\phi_1\|_I^2 \leq \|\partial_s\phi\|^2, \tag{3.49a}
$$

$$
\|\partial_s\phi^\perp\|^2 \leq \|\partial_s\phi\|^2. \tag{3.49b}
$$

Consequently we can use the estimate (3.34f) or (3.34d) for estimating $\|\phi_1\|^2_{W^{1,2}(I)}$ and altogether we get

$$\int_{\Omega_0} (|k_1 - k_1^\varepsilon| + |k_2 - k_2^\varepsilon|)\, |\phi|\, |\psi|\, ds\, dt \le$$

$$\le \sqrt{\left[\frac{4\sqrt{\sigma_k(\delta(\varepsilon))}}{C_k}\sqrt{4 + \frac{1}{C_k^2}} + 8C_k\varepsilon + 8C_k\varepsilon^2\right] Q_\varepsilon^r[\psi]} \times$$

$$\times \sqrt{\left[\frac{2\sqrt{\sigma_k(\delta(\varepsilon))}}{C_k}\sqrt{1 + \frac{1}{C_k^2}} + 4C_k\varepsilon + 4C_k\varepsilon^2\right] Q_0^r[\phi]} \le$$

$$\le 4\left[\frac{\sqrt{\sigma_k(\delta(\varepsilon))}}{C_k}\sqrt{1 + \frac{1}{C_k^2}} + 2C_k\varepsilon + 2C_k\varepsilon^2\right]\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]}$$

Hence we can finish the estimate (3.36):

$$|q(\phi,\psi)| \le \left(12C_k\left[\frac{\sqrt{\sigma_k(\delta(\varepsilon))}}{C_k}\sqrt{1 + \frac{1}{C_k^2}} + 2\varepsilon C_k + 2\varepsilon^2 C_k\right] + \varepsilon a C_k\right)\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]} = \qquad \text{(p4)}$$

$$=: \left(\tilde{C}_2^{(1)}\varepsilon + \tilde{C}_1^{(3)}\sqrt{\sigma_k(\delta(\varepsilon))} + \tilde{C}_1^{(6)}\varepsilon^2\right)\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]}. \qquad (3.50)$$

3.3.4 The estimate on $m(\phi,\psi)$

The form $m(\phi,\psi)$ was defined by (3.37) and the goal of this section is to prove that this form also tends to zero for $\varepsilon \to 0$. For this purpose we will use the Hilbert space decomposition, thus we rewrite $m(\phi,\psi)$ as

$$m(\phi,\psi) = m(\phi_1\chi_1 + \phi^\perp, \psi_1\chi_1 + \psi^\perp) = m(\phi_1\chi_1, \psi_1\chi_1) + m(\phi_1\chi_1, \psi^\perp) + \overline{m(\psi_1\chi_1, \phi^\perp)} + m(\phi^\perp, \psi^\perp)$$

and we will estimate individual terms in this expression.

The first term will be estimated using integration by parts. Let us note that the integration by parts can be used if we assume $\phi, \psi \in C_c^\infty(\Omega_0)$, for these functions the boundary term vanishes. Then we can generalize the results also on $\phi, \psi \in W_0^{1,2}(\Omega_0)$ since $C_c^\infty(\Omega_0)$ is dense in $W_0^{1,2}(\Omega_0)$. After integrating by parts twice we get

$$\begin{aligned} m(\phi_1\chi_1, \psi_1\chi_1) &= \frac{1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon} - 1\right)\left[\bar{\phi}_1\psi_1(\nabla'\chi_1)^2 - E_1\bar{\phi}_1\psi_1\chi_1^2\right] ds\, dt = \\ &= \frac{1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon} - 1\right)\bar{\phi}_1\psi_1\chi_1\left[-\Delta_D^\omega\chi_1 - E_1\chi_1\right] - \frac{1}{\varepsilon^2}\int_{\Omega_0}\Delta\left(\frac{h}{h_\varepsilon} - 1\right)\bar{\phi}_1\psi_1\chi_1^2\, ds\, dt. \end{aligned}$$

Here the first term is zero since χ_1 is the eigenfunction correspondent to eigenvalue E_1 of the transverse Laplacian $-\Delta_D^\omega$. We compute

$$\Delta\left(\frac{h}{h_\varepsilon} - 1\right) = 2\varepsilon^2\frac{k_1^\varepsilon(k_1 - k_1^\varepsilon) + k_2^\varepsilon(k_2 - k_2^\varepsilon)}{h_\varepsilon^3} +$$

$$+ 2\varepsilon^3(k_2k_1^\varepsilon - k_1k_2^\varepsilon)\frac{t_2(-k_1^\varepsilon\sin\theta + k_2^\varepsilon\cos\theta) + t_3(k_1^\varepsilon\cos\theta + k_2^\varepsilon\sin\theta)}{h_\varepsilon^3}$$

and hence we can estimate

$$\begin{aligned}|m(\phi_1\chi_1,\psi_1\chi_1)| &\le \frac{1}{\varepsilon^2}\int_{\Omega_0}\left|\Delta\left(\frac{h}{h_\varepsilon}-1\right)\right||\phi_1||\psi_1|\chi_1^2\,ds\,dt \le\\ &\le 6C_k\int_I\left(|k_1-k_1^\varepsilon|+|k_2-k_2^\varepsilon|\right)|\phi_1||\psi_1|ds+48\varepsilon C_k^3\int_I|\phi_1||\psi_1|ds\le\\ &\le 6C_k\left(\sqrt{\int_I|k_1-k_1^\varepsilon|^2|\phi_1|^2ds}+\sqrt{\int_I|k_2-k_2^\varepsilon|^2|\phi_1|^2ds}\right)\|\psi_1\|_I+\\ &\quad+48\varepsilon C_k^3\|\phi_1\|_I\|\psi_1\|_I\le\\ &\le 12C_k\sqrt{\sigma_k(\delta(\varepsilon))}\|\phi_1\|_{W^{1,2}(I)}\|\psi_1\|_I+48\varepsilon C_k^3\|\phi_1\|_I\|\psi_1\|_I.\end{aligned}$$

Here we used the relation (2.33) stated in Theorem 2.10 applied on k_i:

$$\int_I|k_i-k_i^\varepsilon|^2|\phi_1|^2ds\le\sigma_k(\delta(\varepsilon))\|\phi_1\|^2_{W^{1,2}(I)}\qquad\forall\phi_1\in W^{1,2}(I)\qquad i=1,2$$

(again with simplification (3.48)).

By integrating by parts we also get

$$\begin{aligned}m(\phi_1\chi_1,\psi^\perp) &= \frac{1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\left[\bar{\phi_1}\nabla'\chi_1\nabla'\psi^\perp-E_1\bar{\phi_1}\chi_1\psi^\perp\right]\,ds\,dt=\\ &=\frac{1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\bar{\phi_1}\left[-\Delta_D^\omega\chi_1-E_1\chi_1\right]\psi^\perp+\\ &\quad-\frac{1}{\varepsilon^2}\int_{\Omega_0}\nabla'\left(\frac{h}{h_\varepsilon}-1\right)\cdot\nabla'\chi_1\bar{\phi_1}\psi^\perp\,ds\,dt.\end{aligned}$$

Again the first term is zero and

$$\begin{aligned}\nabla'\left(\frac{h}{h_\varepsilon}-1\right)\cdot\nabla'\chi_1 &= \varepsilon^2(k_2k_1^\varepsilon-k_1k_2^\varepsilon)\frac{\partial_2\chi_1t_3+\partial_3\chi_1t_2}{h_\varepsilon^3}+\\ &+\varepsilon\frac{\partial_2\chi_1\left((k_1-k_1^\varepsilon)\cos\theta+(k_2-k_2^\varepsilon)\sin\theta\right)+\partial_3\chi_1\left((-k_1+k_1^\varepsilon)\sin\theta+(k_2-k_2^\varepsilon)\cos\theta\right)}{h_\varepsilon^3}.\end{aligned}$$

Hence

$$\begin{aligned}|m(\phi_1\chi_1,\psi^\perp)| &\le \frac{1}{\varepsilon^2}\int_{\Omega_0}\left|\nabla'\left(\frac{h}{h_\varepsilon}-1\right)\cdot\nabla'\chi_1\right||\phi_1||\psi^\perp|\,ds\,dt\le\\ &\le \frac{1}{\varepsilon^2}\int_{\Omega_0}\left(2\varepsilon\left(|k_1-k_1^\varepsilon|+|k_2-k_2^\varepsilon|\right)+4aC_k^2\varepsilon^2\right)(|\partial_2\chi_1|+|\partial_3\chi_1|)|\phi_1||\psi^\perp|\,ds\,dt\le\\ &\le 4\left(\sqrt{\int|k_1-k_1^\varepsilon|^2|\phi_1|^2ds}+\sqrt{\int|k_2-k_2^\varepsilon|^2|\phi_1|^2ds}\right)\|\nabla'\chi_1\|_\omega\frac{\|\psi^\perp\|}{\varepsilon}+\\ &\quad+8\varepsilon aC_k\|\phi_1\|_I\|\nabla'\chi_1\|_\omega\frac{\|\psi^\perp\|}{\varepsilon}\le\\ &\le 8\sqrt{E_1}\left(\sqrt{\sigma_k(\delta(\varepsilon))}\|\phi_1\|_{W^{1,2}(I)}+\varepsilon aC_k\|\phi_1\|_I\right)\frac{\|\psi^\perp\|}{\varepsilon}\end{aligned}\tag{3.51}$$

where we again substituted (2.33). In the last step we used that $-\Delta_D^\omega\chi_1=E_1\chi_1$ and $\|\chi_1\|_\omega=1$ which yields

$$\|\nabla'\chi_1\|_\omega=\sqrt{E_1}.\tag{3.52}$$

Recall also the relations (3.41) or (3.43) that ensure that the expression $\frac{\|\psi^\perp\|}{\varepsilon}$ is bounded.

Very similarly we would get

$$|m(\psi_1\chi_1,\phi^\perp)|\le 8\sqrt{E_1}\left(\sqrt{\sigma_k(\delta(\varepsilon))}\|\psi_1\|_{W^{1,2}(I)}+aC_k\varepsilon\|\psi_1\|_I\right)\frac{\|\phi^\perp\|}{\varepsilon}.$$

Finally we estimate

$$
\begin{aligned}
|m(\phi^\perp,\psi^\perp)| &= \frac{1}{\varepsilon^2}\int_{\Omega_0}\left|\frac{h}{h_\varepsilon}-1\right|\left(|\nabla'\phi^\perp||\nabla'\psi^\perp|+E_1|\phi^\perp||\psi^\perp|\right)\,ds\,dt \le \\
&\le 16\varepsilon a C_k\left(\frac{\|\nabla'\phi^\perp\|}{\varepsilon}\frac{\|\nabla'\psi^\perp\|}{\varepsilon}+E_1\frac{\|\phi^\perp\|}{\varepsilon}\frac{\|\psi^\perp\|}{\varepsilon}\right)
\end{aligned}
$$

where $\frac{\|\nabla'\phi^\perp\|}{\varepsilon}$ and $\frac{\|\nabla'\psi^\perp\|}{\varepsilon}$ are bounded due to (3.42) and (3.44).

Altogether we get

$$
\begin{aligned}
|m(\phi,\psi)| \le &\left[12\sqrt{\sigma_k(\delta(\varepsilon))\left(2+\frac{2}{C_k}\right)}+96\varepsilon C_k^2+32\varepsilon a C_k\left(C_2^2+E_1C_1^2\right)+\right. \\
&\left.+8\sqrt{E_1}C_1\left(\sqrt{\sigma_k(\delta(\varepsilon))}\left(\sqrt{1+\frac{1}{C_k^2}}+\sqrt{8+\frac{2}{C_k^2}}\right)+3\varepsilon a\right)\right]\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]} =: \\
&=: \left(\tilde{C}_3^{(1)}\varepsilon+\tilde{C}_2^{(3)}\sqrt{\sigma_k(\delta(\varepsilon))}\right)\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]} \qquad \text{(p5)}
\end{aligned}
$$

where again know, that the last expression tends to zero, if Assumption 2 is satisfied.

3.3.5 The estimate on $l(\phi,\psi)$

The sesquilinear form l was introduced by (3.38) and in the first step we will rewrite it as

$$
\begin{aligned}
l(\phi,\psi) &= \int_{\Omega_0}\left(\frac{1}{hh_\varepsilon}-1\right)(\partial_s+\dot\theta\partial_\tau)\bar\phi(\partial_s+\dot\theta\partial_\tau)\psi\,ds\,dt+ \\
&+\int_{\Omega_0}(\partial_s+\dot\theta\partial_\tau)\bar\phi(\partial_s+\dot\theta\partial_\tau)\psi\,ds\,dt-\int_{\Omega_0}\partial_s\bar\phi\partial_s\psi\,ds\,dt-C(\omega)\int_{\Omega_0}\dot\theta^2\bar\phi\psi\,ds\,dt.
\end{aligned}
$$

Since

$$
\left|\frac{1}{hh_\varepsilon}-1\right| \le 16\varepsilon a C_k
$$

we can estimate the first term as

$$
\left|\int_{\Omega_0}\left(\frac{1}{hh_\varepsilon}-1\right)(\partial_s+\dot\theta\partial_\tau)\bar\phi(\partial_s+\dot\theta\partial_\tau)\psi\,ds\,dt\right| \le 16\varepsilon a C_k\|(\partial_s+\dot\theta\partial_\tau)\phi\|\|(\partial_s+\dot\theta\partial_\tau)\psi\|. \qquad (3.53)
$$

The remaining terms have to be again estimated using the Hilbert space decomposition, thus we substitute $\psi=\psi_1\chi_1+\psi^\perp$ and $\phi=\phi_1\chi_1+\phi^\perp$. After rewriting also $C(\omega)$ by (3.2) we get

$$
\begin{aligned}
\tilde{l}(\phi,\psi) :=& \int_{\Omega_0}(\partial_s+\dot\theta\partial_\tau)(\bar\phi_1\chi_1+\bar\phi^\perp)(\partial_s+\dot\theta\partial_\tau)(\psi_1\chi_1+\psi^\perp)\,ds\,dt+ \\
&-\int_{\Omega_0}\partial_s(\bar\phi_1\chi_1+\bar\phi^\perp)\partial_s(\psi_1\chi_1+\psi^\perp)\,ds\,dt+ \\
&-\int_\omega(\partial_\tau\chi_1)^2dt\int_{\Omega_0}\dot\theta^2(\bar\phi_1\chi_1+\bar\phi^\perp)(\psi_1\chi_1+\psi^\perp)\,ds\,dt = \\
&=\int_{\Omega_0}\dot\theta\chi_1\left(\partial_s\bar\phi_1\partial_\tau\psi^\perp+\partial_\tau\bar\phi^\perp\partial_s\psi_1\right)+\dot\theta\left(\partial_s\bar\phi^\perp\partial_\tau\psi^\perp+\partial_\tau\bar\phi^\perp\partial_s\psi^\perp\right)+ \qquad (3.54)\\
&+\dot\theta\partial_\tau\chi_1\left(\partial_s\bar\phi^\perp\psi_1+\bar\phi_1\partial_s\psi^\perp\right)+\dot\theta^2\left(\bar\phi_1\partial_\tau\chi_1\partial_\tau\psi^\perp+\partial_\tau\bar\phi^\perp\psi_1\partial_\tau\chi_1+\partial_\tau\bar\phi^\perp\partial_\tau\psi^\perp\right)\,ds\,dt+ \\
&-C(\omega)\int_{\Omega_0}\dot\theta^2\left(\bar\phi_1\chi_1\psi^\perp+\bar\phi^\perp\psi_1\chi_1+\bar\phi^\perp\psi^\perp\right)\,ds\,dt
\end{aligned}
$$

where we have rewritten the expression $\tilde{l}(\phi,\psi)$ without the terms that straightly subtract and also without the terms containing (after applying the Fubiny theorem) the expression $\int_\omega\chi_1\partial_\tau\chi_1 dt$ since

$$
\int_\omega\chi_1(t_2,t_3)\,(t_3\partial_2-t_2\partial_3)\chi_1(t_2,t_3)dt_2\,dt_3=-\int_\omega(t_3\partial_2-t_2\partial_3)\chi_1(t_2,t_3)\,\chi_1(t_2,t_3)dt_2\,dt_3=0. \qquad (3.55)
$$

Here the integration by parts could be done again using some approximation of $\chi_1 \in W_0^{1,2}(\omega)$ by functions from $C_c(\omega)$. For these functions the boundary term is evidently zero and since $C_c(\omega)$ is dense in $W_0^{1,2}(\omega)$ the boundary term will be zero also in our integral above.

Now we have to estimate individual terms in (3.54). The terms where $\psi^\perp$ or $\phi^\perp$ occur will be proportional to ε due to (3.41), (3.43). The same holds for terms with $\partial_\tau\psi^\perp$ or $\partial_\tau\phi^\perp$ since generally in every point $(s,t)\in\Omega_0$ we can use the Schwarz inequality in $\mathbb{C}^2$ in the following way

$$|\partial_\tau\psi| = |(t_3,-t_2)\cdot\nabla'\psi| \le |(t_3,-t_2)||\nabla'\psi| \le a|\nabla'\psi|.$$

In the first equality we mean by "$\cdot$" the scalar product in C^2 and of course $|\nabla'\psi| = \|\nabla'\psi\|_{\mathbb{C}^2}$. Hence

$$\|\partial_\tau\psi\|^2 = \int_{\Omega_0} |\partial_\tau\psi|^2\,ds\,dt \le a^2\int_{\Omega_0}|\nabla'\psi|^2\,ds\,dt = a^2\|\nabla'\psi\|^2 \tag{3.56}$$

and for $\psi=\psi^\perp$ resp. $\psi=\phi^\perp$ the last expression can be estimated by (3.42) or (3.44).

The most problematic term in $l(\phi,\psi)$ is the third term in (3.54). We don't know any convergence properties of $\partial_s\phi^\perp$ or $\partial_s\psi^\perp$, thus we need to integrate by parts with respect to s, then we would get $\phi^\perp$ or $\psi^\perp$, that tend to zero. However the differentiation of $\dot\theta$ would be necessary in this case and as we assumed only that θ is once differentiable, we will use the Steklov approximation of $\dot\theta$ similarly as for the functions k_i. Thus we define

$$\dot\theta^\varepsilon(s) := \frac{1}{\tilde\delta(\varepsilon)}\int_{s-\frac{\tilde\delta(\varepsilon)}{2}}^{s+\frac{\tilde\delta(\varepsilon)}{2}} \dot\theta(\xi)d\xi \tag{3.57}$$

where $\tilde\delta$ is again a continuous function of ε satisfying

$$\lim_{\varepsilon\to 0}\tilde\delta(\varepsilon) = 0.$$

This function is differentiable

$$\left(\dot\theta^\varepsilon\right)^{\boldsymbol{\cdot}}(s) = \frac{\dot\theta(s+\frac{\tilde\delta(\varepsilon)}{2}) - \dot\theta(s-\frac{\tilde\delta(\varepsilon)}{2})}{\tilde\delta(\varepsilon)},$$

in consequence we can rewrite

$$\begin{aligned}\int_{\Omega_0}\dot\theta\partial_\tau\chi_1\partial_s\bar\phi\psi_1\,ds\,dt &= \int_{\Omega_0}(\dot\theta-\dot\theta^\varepsilon)\partial_\tau\chi_1\partial_s\bar\phi^\perp\psi_1\,ds\,dt + \int_{\Omega_0}\dot\theta^\varepsilon\partial_\tau\chi_1\partial_s\bar\phi^\perp\psi_1\,ds\,dt = \\ &= \int_{\Omega_0}(\dot\theta-\dot\theta^\varepsilon)\partial_\tau\chi_1\partial_s\bar\phi^\perp\psi_1\,ds\,dt + \int_{\Omega_0}\left(\dot\theta^\varepsilon\right)^{\boldsymbol{\cdot}}\chi_1\partial_\tau\bar\phi^\perp\psi_1\,ds\,dt + \int_{\Omega_0}\dot\theta^\varepsilon\chi_1\partial_\tau\bar\phi^\perp\partial_s\psi_1\,ds\,dt\end{aligned} \tag{3.58}$$

where we integrated by parts also with respect to τ (this can be done similarly as in (3.55)). Then the second and third term in (3.58) are already proportional to ε due to (3.42), (3.44) and also since

$$\begin{aligned}\left|\left(\dot\theta^\varepsilon\right)^{\boldsymbol{\cdot}}(s)\right| &\le 2C_{\dot\theta} \qquad \forall s\in I\\ \left|\dot\theta^\varepsilon(s)\right| &\le C_{\dot\theta} \qquad \forall s\in I\end{aligned}$$

where $C_{\dot\theta}$ is the upper bound on $|\dot\theta|$ defined by (2.4).

The first term in (3.58) will be estimated using the Schwarz inequality as

$$\left|\int_{\Omega_0}(\dot\theta-\dot\theta^\varepsilon)\partial_\tau\chi_1\partial_s\bar\phi^\perp\psi_1\,ds\,dt\right| \le \|\partial_\tau\chi_1\|_\omega\sqrt{\int_I|\dot\theta-\dot\theta^\varepsilon|^2|\psi_1|^2ds}\,\|\partial_s\bar\phi^\perp\|.$$

Now we can apply the Theorem 2.10 on the function $\dot{\theta}$ to get

$$\int_I |\dot{\theta} - \dot{\theta}^{\varepsilon}|^2 |\psi_1|^2 ds \leq \sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon)) \|\psi_1\|^2_{W^{1,2}(I)}$$

which proves that also the first term in (3.58) tends to zero, if $\dot{\theta}$ satisfies (3.9) from Assumption 2 or I is finite.

Similar estimates will hold for $\int_{\Omega_0} \dot{\theta} \partial_\tau \chi_1 \bar{\phi}_1 \partial_s \psi^\perp \, ds \, dt$. Finally, summing up all the ideas mentioned above and using also the inequality derived from (3.56) and (3.52)

$$\|\partial_\tau \chi_1\|_\omega \leq a \|\nabla' \chi_1\|_\omega = a\sqrt{E_1}$$

we get

$$\begin{aligned}
\left|\tilde{l}(\phi, \psi)\right| \leq{}& \varepsilon C_{\dot{\theta}} \left(\|\partial_s \phi_1\|_I \frac{\|\partial_\tau \psi^\perp\|}{\varepsilon} + \|\partial_s \psi_1\|_I \frac{\|\partial_\tau \phi^\perp\|}{\varepsilon} \right) + \varepsilon C_{\dot{\theta}} \left(\|\partial_s \phi^\perp\| \frac{\|\partial_\tau \psi^\perp\|}{\varepsilon} + \|\partial_s \psi^\perp\| \frac{\|\partial_\tau \phi^\perp\|}{\varepsilon} \right) + \\
&+ \sqrt{\sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon))} a \sqrt{E_1} \left(\|\psi_1\|_{W^{1,2}(I)} \|\partial_s \phi^\perp\| + \|\phi_1\|_{W^{1,2}(I)} \|\partial_s \psi^\perp\| \right) + \\
&+ 2\varepsilon C_{\dot{\theta}} \left(\|\psi_1\|_I \frac{\|\partial_\tau \phi^\perp\|}{\varepsilon} + \|\phi_1\|_I \frac{\|\partial_\tau \psi^\perp\|}{\varepsilon} \right) + \varepsilon C_{\dot{\theta}} \left(\|\partial_s \psi_1\|_I \frac{\|\partial_\tau \phi^\perp\|}{\varepsilon} + \|\partial_s \phi_1\|_I \frac{\|\partial_\tau \psi^\perp\|}{\varepsilon} \right) + \\
&+ \varepsilon C^2_{\dot{\theta}} a \sqrt{E_1} \left(\|\psi_1\|_I \frac{\|\partial_\tau \phi^\perp\|}{\varepsilon} + \|\phi_1\|_I \frac{\|\partial_\tau \psi^\perp\|}{\varepsilon} \right) + \varepsilon^2 C^2_{\dot{\theta}} \frac{\|\partial_\tau \phi^\perp\|}{\varepsilon} \frac{\|\partial_\tau \psi^\perp\|}{\varepsilon} + \\
&+ \varepsilon C^2_{\dot{\theta}} C(\omega) \left(\|\psi_1\|_I \frac{\|\phi^\perp\|}{\varepsilon} + \|\phi_1\|_I \frac{\|\psi^\perp\|}{\varepsilon} \right) + \varepsilon^2 C^2_{\dot{\theta}} C(\omega) \frac{\|\phi^\perp\|}{\varepsilon} \frac{\|\psi^\perp\|}{\varepsilon}.
\end{aligned} \tag{3.59}$$

This expression is slightly long but the point is that all the terms here tend to zero. Finally we use the estimates (3.49), (3.34), (3.41)-(3.44) together with (3.56) and (3.47) to get

$$\begin{aligned}
|l(\phi, \psi)| \leq{}& \Bigg[3\varepsilon C_1 C_{\dot{\theta}} C(\omega) + 3aC_2 \left(13\varepsilon C_{\dot{\theta}} + \varepsilon C^2_{\dot{\theta}} a \sqrt{E_1} \right) + 4a \sqrt{\sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon)) E_1 \left(1 + \frac{1}{C_k^2} \right)} \\
&+ 2\varepsilon^2 C^2_{\dot{\theta}} \left(a^2 C_2^2 + C(\omega) C_1^2 \right) + 64\varepsilon a C_k \Bigg] \sqrt{Q_0^r[\phi]} \sqrt{Q_\varepsilon^r[\psi]} = \\
&=: \left(\tilde{C}_4^{(1)} \varepsilon + \tilde{C}^{(4)} \sqrt{\sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon))} + \tilde{C}_2^{(6)} \varepsilon^2 \right) \sqrt{Q_0^r[\phi]} \sqrt{Q_\varepsilon^r[\psi]}.
\end{aligned} \tag{p6}$$

3.3.6 Proof of the technical Lemmas 3.9 and 3.10.

At the beginning of the Section 3.3 some estimates on the norms of the functions ϕ, ψ or their derivatives were made, however, the relations (3.34e) and (3.34f) stated by Lemma 3.9 were not proved, since the results of Sections 3.3.4 and 3.3.5 would be needed. Also the Lemma 3.10 containing the relations (3.47c) and (3.47d) could not be proved without these new techniques. Hence in this section we will prove the relations

$$\|(\partial_s + \dot{\theta}\partial_\tau)\phi\|^2 \leq 2Q_0^r[\phi] \tag{3.60}$$

$$\|\partial_s \psi\|^2 \leq 8Q_\varepsilon^r[\psi] \tag{3.61}$$

$$Q_\varepsilon^r[\psi^\perp] \leq 2Q_\varepsilon^r[\psi], \tag{3.62}$$

$$Q_\varepsilon^r[\psi_1 \chi_1] \leq 2Q_\varepsilon^r[\psi] \tag{3.63}$$

using some ideas of Sections 3.3.4 and 3.3.5.

At first we prove (3.60). We will use that the form (3.54) as the quadratic form reads in fact

$$\tilde{l}[\psi] = \|(\partial_s + \dot{\theta}\partial_\tau)\psi\|^2 - \|\partial_s \psi\|^2 - C(\omega)\|\dot{\theta}\psi\|^2. \tag{3.64}$$

Using similar steps as while deriving (p6) we get

$$\tilde{l}[\psi] \leq \sigma_1(\varepsilon) Q_0^r[\psi].$$

This relation was derived similarly as (3.59), the estimates by $Q_0^r[\psi]$ and $Q_\varepsilon^r[\psi]$ differ only by constants, thus $\sigma_1(\varepsilon)$ also tends to zero. The important point is that none of the relations (3.61) - (3.63) were not used in these estimates. Hence

$$\|(\partial_s + \dot\theta\partial_\tau)\psi\|^2 \leq \|\partial_s\psi\|^2 + C(\omega)\|\dot\theta\psi\|^2 + \tilde{l}[\psi] \leq (1+\sigma_1(\varepsilon))\, Q_0^r[\psi] \leq 2Q_0^r[\psi]$$

where the last estimate holds for small enough ε and proves (3.60).

Next we will prove (3.62) and (3.63), since it is convenient to use the steps made here in proof of (3.61). In fact we prove the relation

$$\left|Q_\varepsilon^r(\psi_1\chi_1, \psi^\perp)\right| \leq \tilde\sigma(\varepsilon)\sqrt{Q_\varepsilon^r[\psi_1\chi_1]}\sqrt{Q_\varepsilon^r[\psi^\perp]} \leq \frac{\tilde\sigma(\varepsilon)}{2}\left(Q_\varepsilon^r[\psi_1\chi_1] + Q_\varepsilon^r[\psi^\perp]\right) \tag{3.65}$$

where $\tilde\sigma$ is some function of ε that tends to zero for $\varepsilon \to 0$. Hence

$$Q_\varepsilon^r[\psi_1\chi_1 + \psi^\perp] \geq (1-\tilde\sigma(\varepsilon))Q_\varepsilon^r[\psi_1\chi_1] + (1-\tilde\sigma(\varepsilon))Q_\varepsilon^r[\psi^\perp]$$

and for small enough ε we get the relations (3.62) and (3.63).

To get (3.65) we estimate individual terms in $Q_\varepsilon^r(\psi_1\chi_1, \psi^\perp)$, where we often use the estimates made in proceeding paragraphs on $\left|Q_\varepsilon^r(\psi_1\chi_1,\psi^\perp) - Q_0^r(\psi_1\chi_1,\psi^\perp)\right|$, since $Q_0^r(\psi_1\chi_1,\psi^\perp) = 0$. For the terms corresponding to terms (3.27c), (3.27d) and (3.27e) in $Q_\varepsilon^r(\phi,\psi)$ we can straightly say (using the estimates from the beginning of Section 3.3 or in case of the term (3.27c) using the orthogonality of χ_1 and $\psi^\perp$) that they are proportional to $\sqrt{Q_\varepsilon^r[\psi_1\chi_1]}\sqrt{Q_\varepsilon^r[\psi^\perp]}$ and vanishing for small ε. The reason is that in these estimates only the terms with $\|\psi_1\chi_1\|$, $\|\psi^\perp\|$ or $\|(\partial_s + \dot\theta\partial_\tau)(\psi_1\chi_1)\|$, $\|(\partial_s+\dot\theta\partial_\tau)(\psi^\perp)\|$ occur and these expressions can be estimated using relations (3.34a) - (3.34d) that were proved in Section 3.3. Hence

$$\left|Q_\varepsilon^r(\psi_1\chi_1,\psi^\perp) - m(\psi_1\chi_1,\psi^\perp) - l(\psi_1\chi_1,\psi^\perp)\right| \leq \sigma_2(\varepsilon)\sqrt{Q_\varepsilon^r[\psi_1\chi_1]}\sqrt{Q_\varepsilon^r[\psi^\perp]}$$

where $\sigma_2(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$. In the estimate on $m(\psi_1\chi_1,\psi^\perp)$ similar to (3.51) the norm $\|\partial_s\psi_1\|_I$ occurs. The relation (3.61) is not proved, yet, however, it is easy to see that

$$\|\partial_s\psi_1\|_I^2 = \|\partial_s\psi_1\chi_1\|^2 = \|(\partial_s+\dot\theta\partial_\tau)(\psi_1\chi_1)\|^2 - C(\omega)\|\dot\theta\psi_1\chi_1\|^2 \leq \|(\partial_s+\dot\theta\partial_\tau)(\psi_1\chi_1)\|^2 \leq 4Q_\varepsilon^r[\psi_1\chi_1] \tag{3.66}$$

where the last inequality follows from (3.34c) and thus we get

$$\left|m(\psi_1\chi_1,\psi^\perp)\right| \leq \sigma_3(\varepsilon)\sqrt{Q_\varepsilon^r[\psi_1\chi_1]}\sqrt{Q_\varepsilon^r[\psi^\perp]}$$

where $\sigma_3(\varepsilon)$ can be evaluated using (3.51) and tends to zero for $\varepsilon \to 0$. The last term to estimate is $l(\psi_1\chi_1,\psi^\perp)$. Recalling that we can already estimate $\|\partial_s\psi_1\|_I^2$ by $Q_\varepsilon^r[\psi_1\chi_1]$, we can use (3.53) and (3.59) to get the estimate of form

$$\left|l(\psi_1\chi_1,\psi^\perp)\right| \leq \sigma_4(\varepsilon)\sqrt{Q_\varepsilon^r[\psi_1\chi_1]}\sqrt{Q_\varepsilon^r[\psi^\perp]} + 2\sqrt{\sigma_{\dot\theta}(\tilde\delta(\varepsilon))}a\sqrt{E_1}\|\phi_1\|_{W^{1,2}(I)}\|\partial_s\psi^\perp\|. \tag{3.67}$$

where σ_4 tends to zero with ε. Again we have to estimate the term $\|\partial_s\psi^\perp\|$ without using (3.61), for this purpose similar estimate as in the proof of (3.60) can be used. We know that

$$\|\partial_s\psi^\perp\|^2 \leq \|\partial_s\psi^\perp\|^2 + C(\omega)\|\dot\theta\psi^\perp\|^2 \leq \|(\partial_s+\dot\theta\partial_\tau)\psi^\perp\|^2 + \left|\tilde{l}[\psi^\perp]\right|,$$

using (3.59) we get

$$\left|\tilde{l}[\psi^\perp]\right| \leq \frac{\varepsilon}{2} C_{\dot\theta} \left(\|\partial_s \psi^\perp\|^2 + \frac{\|\partial_\tau \psi^\perp\|^2}{\varepsilon^2} \right) + \varepsilon^2 C_{\dot\theta}^2 \frac{\|\partial_\tau \psi^\perp\|^2}{\varepsilon^2} + C_{\dot\theta}^2 C(\omega) \frac{\|\psi^\perp\|^2}{\varepsilon^2},$$

hence

$$\|\partial_s \psi^\perp\|^2 \leq \frac{\|(\partial_s + \dot\theta \partial_\tau)\psi^\perp\|^2 + \frac{\varepsilon}{2} C_{\dot\theta} \frac{\|\partial_\tau \psi^\perp\|^2}{\varepsilon^2} + \varepsilon^2 C_{\dot\theta}^2 \frac{\|\partial_\tau \psi^\perp\|^2}{\varepsilon^2} + C_{\dot\theta}^2 C(\omega) \frac{\|\psi^\perp\|^2}{\varepsilon^2}}{1 - \frac{\varepsilon}{2} C_{\dot\theta}} \leq 6 Q_\varepsilon^r[\psi^\perp] \quad (3.68)$$

for small enough ε. This can be substituted into (3.67) and we again get some $\sigma_5(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$ such that

$$\left| l(\psi_1 \chi_1, \psi^\perp) \right| \leq \sigma_5(\varepsilon) \sqrt{Q_\varepsilon^r[\psi_1 \chi_1]} \sqrt{Q_\varepsilon^r[\psi^\perp]}.$$

In conclusion we get

$$\left| Q_\varepsilon^r(\psi_1 \chi_1, \psi^\perp) \right| \leq (\sigma_3(\varepsilon) + \sigma_3(\varepsilon) + \sigma_5(\varepsilon)) \sqrt{Q_\varepsilon^r[\psi_1 \chi_1]} \sqrt{Q_\varepsilon^r[\psi^\perp]}$$

which together with the arguments mentioned above proves (3.62) and (3.63).

To proof (3.61) we can now use the relations (3.66) and (3.68). Combining them with (3.65) and relations (3.62) and (3.63) we get

$$\begin{aligned} \|\partial_s \psi\|^2 &= \|\partial_s \psi_1 \chi_1\|^2 + \|\partial_s \psi^\perp\|^2 \leq 4 Q_\varepsilon^r[\psi_1 \chi_1] + 6 Q_\varepsilon^r[\psi^\perp] \leq 6 \left(Q_\varepsilon^r[\psi_1 \chi_1] + Q_\varepsilon^r[\psi^\perp] \right) \leq \\ &\leq 6 \left(Q_\varepsilon^r[\psi] + 4\tilde\sigma(\varepsilon) Q_\varepsilon^r[\psi] \right) \leq 8 Q_\varepsilon^r[\psi]. \end{aligned}$$

□

3.3.7 Conclusion

It follows from (p1)-(p6) that

$$\begin{aligned} |Q_\varepsilon^r(\phi, \psi) - Q_0^r(\phi, \psi)| \leq \Bigg[&\left(\tilde C_1^{(1)} + \tilde C_2^{(1)} + \tilde C_3^{(1)} + \tilde C_4^{(1)} + \varepsilon \tilde C_1^{(6)} + \varepsilon \tilde C_1^{(6)} \right) \varepsilon + \left(\tilde C_1^{(2)} + \frac{\varepsilon}{\delta(\varepsilon)} \tilde C_1^{(6)} \right) \frac{\varepsilon}{\delta(\varepsilon)} + \\ &+ \left(\tilde C_1^{(3)} + \tilde C_2^{(3)} \right) \sqrt{\sigma_k(\delta(\varepsilon))} + \tilde C^{(4)} \sqrt{\sigma_{\dot\theta}(\tilde\delta(\varepsilon))} \Bigg] \sqrt{Q_0^r[\phi]} \sqrt{Q_\varepsilon^r[\psi]} \end{aligned}$$

which proves Lemma (3.7) since we can assume $\varepsilon \leq 1$ and $\frac{\varepsilon}{\delta(\varepsilon)} \leq 1$ without loss of generality. Let us note that we did not mention in the statement of the lemma the dependance on ε^2 and $\frac{\varepsilon^2}{\delta(\varepsilon)^2}$ since the leading terms are ε and $\frac{\varepsilon}{\delta(\varepsilon)} \leq 1$ that converge slower. □

Chapter 4

The two-dimensional quantum waveguide

In [8], where we considered the two-dimensional waveguides and where we proved the strong resolvent convergence of the Hamiltonian on these planar strips, we stated as a task for future to prove also the norm resolvent convergence. We would like to fulfill this promise now, however, the proofs will be very similar as those in the three-dimensional case, so we won't write them with all the details and we will often refer to Chapter 3. We will also use the same notation as in case of the three-dimensional waveguide for the equivalent notions. In this Chapter, such notions will always refer to the two-dimensional case.

Let us note that the two-dimensional waveguide (a strip in plane whose width tends to zero) can be understood as a model of infinite "wall" built above the strip where we separated out one variable. It might seem that this object is a special case of the three-dimensional waveguide we considered before, however, this is not true, since we assumed the cross-section of the three-dimensional waveguide to be bounded, which is not fulfilled in case of the infinite wall.

4.1 Preliminaries

4.1.1 Strip in plane

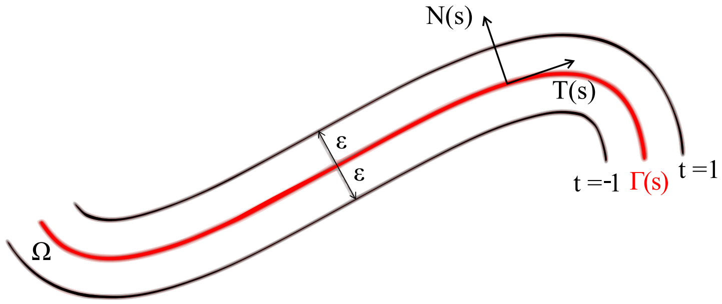

Figure 4.1: The two-dimensional quantum waveguide

Let Γ be a unit-speed plane curve, i.e. the (image of the) embedding $\Gamma\colon\ I \to \mathbb{R}^2\colon\ s \longmapsto (\Gamma^1(s), \Gamma^2(s))$ satisfying $T(s) := |\dot{\Gamma}(s)| = 1$ for all $s \in I$. I is an open interval in $\mathbb{R}$, we allow both finite and semi-infinite or infinite intervals. We assume $\Gamma^i \in W^{2,\infty}_{\mathrm{loc}}(I)$ for $i = 1, 2$, then the tangent vector field T is again continuous and its derivative exists in the weak sense and is locally bounded. The function $N := (-\dot{\Gamma}^2, \dot{\Gamma}^1)$ defines a unit normal vector field with the same properties as the tangent one and the couple (T, N) gives an adapted moving frame for the curve. We introduce a scalar function $\kappa(s)$ called the curvature, that satisfies the equation

$$\begin{pmatrix} \dot{T} \\ \dot{N} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

From $\Gamma^i \in W^{2,\infty}_{\mathrm{loc}}(I)$ it follows that κ (which is defined in the weak sense) is locally bounded, however, we will assume in addition that the curvature is globally bounded, i.e.

$$\|\kappa(s)\|_\infty =: C_\kappa < \infty. \tag{4.1}$$

Let $\Omega_0 := I \times (-1, 1)$ be a straight strip in the plane. We define a curved strip $\Omega := \mathcal{L}(\Omega_0)$ using the mapping

$$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2 : \{(s, t) \longmapsto \Gamma(s) + \varepsilon t N(s)\}. \tag{4.2}$$

If we denote the coordinates (s, t) by $(1, 2)$, we can compute the metric tensor $G = (G_{ij})$ in Ω using $G_{ij} = \partial_i \mathcal{L} \cdot \partial_j \mathcal{L}$, then

$$G = \begin{pmatrix} (1 - \varepsilon t \kappa(s))^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

We denote by $|G|$ the determinant of the matrix G_{ij}:

$$|G| = \varepsilon^2 (1 - \varepsilon t \kappa(s))^2 =: \varepsilon^2 h^2 \tag{4.3}$$

where we denoted

$$h(s, t) := 1 - \varepsilon t \kappa(s).$$

To ensure that $|G| \neq 0$ we state the condition $\varepsilon C_\kappa < 1$. However, this is only the necessary condition for injectivity of the mapping $\mathcal{L}$, we have to require the injectivity of $\mathcal{L}$ as an extra condition.

Finally let us note that for our computations we will often assume $\varepsilon C_\kappa \leq \frac{1}{4}$, so that

$$\frac{3}{4} \leq 1 - \varepsilon C_\kappa \leq h = 1 - \varepsilon t \kappa(s) \leq 1 + \varepsilon C_\kappa \leq \frac{5}{4}. \tag{4.4}$$

Let us summarize the assumptions on the curve Γ that we make in this section.

Assumption 3. *Let $\Gamma\colon I \to \mathbb{R}^2$ be a planar curve where the interval $I \subseteq \mathbb{R}$ is finite, semi-infinite or infinite. Then we assume*

(i) $\Gamma^i \in W^{2,\infty}_{\mathrm{loc}}(I)$ *for* $i = 1, 2$

(ii) $\sup_{s \in I} |\kappa(s)| < \infty$,

(iii) the properties of Γ are such that $\mathcal{L}$ introduced by (4.2) is injective for small enough ε.

4.1.2 The Hamiltonian

If we describe our curved strip Ω with the cartesian coordinates we again consider the Hamiltonian

$$H = -\Delta_D^\Omega,$$

i.e. the Dirichlet Laplacian where $\operatorname{Dom} H = \{\psi \in W^{2,2}(\Omega) | \psi \restriction \partial\Omega = 0\}$, since the boundary $\partial\Omega$ is smooth enough here. However, we will again work rather with curvilinear coordinates (s,t), the Hilbert space becomes then $\tilde{\mathcal{H}}_\varepsilon := L^2(\Omega, |G|^{1/2} ds du)$ and H becomes the Laplace-Beltrami operator

$$\tilde{H}_\varepsilon = -|G|^{-1/2}\partial_i |G|^{1/2} G^{ij} \partial_j$$

which has to be understood in the weak sense, since the elements of matrix G need not to be differentiable (because of the occurrence of κ there). Thus we will again work with the associated quadratic form

$$\tilde{Q}_\varepsilon[\psi] = \left(\partial_i\psi, G^{ij}\partial_j\psi\right)_{\tilde{\mathcal{H}}_\varepsilon} \qquad \operatorname{Dom}(\tilde{Q}) = W_0^{1,2}(\Omega_0).$$

On this domain, the quadratic form $\tilde{Q}_\varepsilon$ is closed (in the construction of the closure, it was again needed that the coefficients in G are bounded, this is ensured by the assumption on boundedness of κ). This form is then associated with the self-adjoint Friedrichs extension of $\tilde{H}_\varepsilon$ which will be denoted by the same symbol.

Using similar steps as in Section 2.2.1, we would show that the first term in the asymptotic in ε of the spectrum of $\tilde{H}_\varepsilon$ is $\frac{E_1}{\varepsilon^2}$ where $E_1 = \frac{\pi^2}{4}$ is the first eigenvalue of the Laplace operator on segment $(-1,1)$ with Dirichlet boundary conditions in points $t = \pm 1$. Hence we will renormalize

$$\tilde{\tilde{H}}_\varepsilon := \tilde{H}_\varepsilon - \frac{E_1}{\varepsilon^2},$$

similarly

$$\tilde{\tilde{Q}}_\varepsilon[\psi] := \tilde{Q}_\varepsilon[\psi] - \frac{E_1}{\varepsilon^2}\|\psi\|^2_{\tilde{\mathcal{H}}_\varepsilon}.$$

The next step will consist of introducing the unitary transformation to "straighten" the strip, however, again the standard unitary transformation (the multiplication by $|G|^{1/4}$) won't work since $|G|$ is not differentiable. That's why we will use the smoothing of the function κ with help of the Steklov approximation.

4.1.3 Smoothing of the curvature κ

We introduce the Steklov approximation of κ as

$$\kappa_\varepsilon(s) := \frac{\int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} \kappa(\xi) d\xi}{\delta(\varepsilon)}$$

where δ is a continuous function of ε satisfying

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$$

and also

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0. \tag{4.5}$$

We consider that the curve is defined on an opened interval I, if I is finite or semi-infinite, we prolong the function by zero on $\mathbb{R} \setminus I$ to give the definition of κ_ε a good sense.

As a consequence of (4.1) we get that

$$\kappa_\varepsilon(s) \le C_\kappa \qquad \forall s \in I.$$

Also in the two-dimensional case we will use the results of Section 2.3 on the convergence properties of the expression $|\kappa - \kappa_\varepsilon|$.

4.1.4 The unitary transformation

Now we can set

$$h_\varepsilon := 1 - \varepsilon t \kappa_\varepsilon(s)$$

and

$$|\tilde{G}|(s,t) := \varepsilon^2 h_\varepsilon^2. \tag{4.6}$$

Using this expression we introduce the unitary transformation

$$\begin{aligned} U_{\tilde{G}} : L^2\left(\Omega_0, |G|^{1/2}\, ds\, dt\right) &\longrightarrow L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right) \\ \psi &\longmapsto U_{\tilde{G}}\psi = |\tilde{G}|^{1/4}\psi \end{aligned}$$

to get

$$H_\varepsilon := U_{\tilde{G}} \tilde{\tilde{H}}_\varepsilon U_{\tilde{G}}^{-1} = |\tilde{G}|^{1/4} \tilde{\tilde{H}}_\varepsilon |\tilde{G}|^{-1/4}. \tag{4.7}$$

We denote $\mathcal{H}_\varepsilon := L^2\left(\Omega_0, \frac{|G|^{1/2}}{|\tilde{G}|^{1/2}}\, ds\, dt\right)$ and the scalar product, resp. the norm on this space is then denoted by $(\cdot,\cdot)_\varepsilon$, resp. $\|\cdot\|_\varepsilon$. By $(\cdot,\cdot)$, resp. $\|\cdot\|$ will be denoted the scalar product, resp. the norm on $\mathcal{H}_0 := L^2(\Omega_0, dsdt)$.

The formula for the associated quadratic form Q_ε with $\operatorname{Dom} Q_\varepsilon = W_0^{1,2}(\Omega)$ reads then

$$\begin{aligned} Q_\varepsilon[\psi] = &\int_{\Omega_0} \frac{1}{h h_\varepsilon} |\partial_s \psi|^2\, ds\, dt + \frac{1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\partial_t \psi|^2\, ds\, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\psi|^2\, ds\, dt + \\ &+ \frac{1}{2} \int_{\Omega_0} \frac{1}{h_\varepsilon^2} \kappa \kappa_\varepsilon |\psi|^2\, ds\, dt - \frac{3}{4} \int_{\Omega_0} \frac{h}{h_\varepsilon^3} \kappa_\varepsilon^2 |\psi|^2\, ds\, dt + \int_{\Omega_0} \frac{|\partial_s h_\varepsilon|^2}{4 h_\varepsilon^3 h} |\psi|^2\, ds\, dt + \\ &+ \int_{\Omega_0} \frac{\partial_s h_\varepsilon}{h_\varepsilon^2 h} \operatorname{Re}(\bar{\psi} \partial_s \psi)\, ds\, dt. \end{aligned} \tag{4.8}$$

4.1.5 Boundedness from below of H_ε

It will be again necessary to find some lower bound on the operator H_ε. This could be done on the level, when we worked with the unitarily equivalent operator $\tilde{\tilde{H}}_\varepsilon$, however, similarly as in the case of three dimensions, some more precise result concerning in the lower bound also $\|\partial_s\psi\|$ will be needed in the following steps, hence we will skip to the finer estimate (analogue of Lemma 3.8).

Lemma 4.1. *Let $\psi \in W_0^{1,2}(\Omega)$ and let r be a positive constant. Then*

$$Q_\varepsilon[\psi] \ge \frac{1}{2} \int_{\Omega_0} \frac{1}{h h_\varepsilon} |\partial_s \psi|^2\, ds\, dt - 6 C_\kappa^2 \|\psi\|_\varepsilon^2. \tag{4.9}$$

Proof. We will proceed similarly as in proof of Lemma 3.8, hence we will not give the details of the estimates any more. At first the second and third term in (4.8) will be estimated using the substitution $\phi := \sqrt{\frac{h}{h_\varepsilon}}\psi$ and using that $\int_{\Omega_0} |\partial_t \phi|^2\, ds\, dt \ge E_1 \int_{\Omega_0} |\phi|^2\, ds\, dt$ to get the formula

$$\frac{1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\partial_t \psi|^2\, ds\, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\psi|^2\, ds\, dt \ge -2 C_\kappa^2 \|\psi\|_\varepsilon^2.$$

Then we estimate the third and fourth term as

$$\left|\frac{1}{2}\int_{\Omega_0}\frac{1}{h_\varepsilon^2}\kappa\kappa_\varepsilon|\psi|^2\,ds\,dt-\frac{3}{4}\int_{\Omega_0}\frac{h}{h_\varepsilon^3}\kappa_\varepsilon^2|\psi|^2\,ds\,dt\right|\geq 3C_\kappa^2$$

and we decompose the seventh term using Schwarz and Young inequalities:

$$\left|\int_{\Omega_0}\frac{\partial_s h_\varepsilon}{h_\varepsilon^2 h}\mathrm{Re}(\bar{\psi}\partial_s\psi)\,ds\,dt\right|\leq\frac{1}{2}\int_{\Omega_0}\frac{1}{hh_\varepsilon}|\partial_s\psi|^2\,ds\,dt+\frac{1}{2}\int_{\Omega_0}\frac{|\partial_s h_\varepsilon|^2}{h_\varepsilon^3 h}|\psi|^2\,ds\,dt.$$

In consequence we get

$$Q_\varepsilon[\psi]\geq\frac{1}{2}\int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|\partial_s\psi\right|^2\,ds\,dt-5C_\kappa^2\|\psi\|_\varepsilon^2-\int_{\Omega_0}\frac{|\partial_s h_\varepsilon|^2}{4h_\varepsilon^3 h}|\psi|^2\,ds\,dt$$

However, for all $(s,t)\in\Omega_0$ it holds

$$|\partial_s h_\varepsilon(s,t)|=\left|-\varepsilon t\frac{\kappa(s+\frac{\delta(\varepsilon)}{2})-\kappa(s+\frac{\delta(\varepsilon)}{2})}{\delta(\varepsilon)}\right|\leq\frac{2\varepsilon C_\kappa}{\delta(\varepsilon)}$$

and since we assumed (4.5), we can estimate that

$$\left|\int_{\Omega_0}\frac{|\partial_s h_\varepsilon|^2}{4h_\varepsilon^3 h}|\psi|^2\,ds\,dt\right|\leq\frac{4\varepsilon^2 C_\kappa^2}{\delta(\varepsilon)^2}\|\psi\|_\varepsilon^2\leq C_\kappa^2\|\psi\|_\varepsilon^2 \tag{4.10}$$

to get the final estimate

$$Q_\varepsilon[\psi]\geq\frac{1}{2}\int_{\Omega_0}\frac{1}{hh_\varepsilon}\left|\partial_s\psi\right|^2\,ds\,dt-6C_\kappa^2\|\psi\|_\varepsilon^2.$$

□

Hence for sure also

$$Q_\varepsilon[\psi]\geq-6C_\kappa^2\|\psi\|_\varepsilon^2$$

and we get similarly as in Lemma 2.2 that if $r>6C_\kappa^2$ is a real constant, then

$$\|(H_\varepsilon+r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)}\leq\frac{1}{r-6C_k^2}. \tag{4.11}$$

4.2 Norm resolvent convergence

4.2.1 The main result

At first, we will again have to find the way how to compare the operator H_ε acting on $\mathcal{H}_\varepsilon$ and the one dimensional effective Hamiltonian

$$h_{\text{eff}}:=-\Delta_D^I-\frac{\kappa^2}{4}.$$

Let χ_1 be the eigenfunction of the transverse Dirichlet Laplacian $-\Delta_D^{(-1,1)}$ associated with the first eigenvalue E_1, normalized to 1:

$$\chi_1(t)=\cos\frac{\pi t}{2}. \tag{4.12}$$

Then we introduce the subspace $\mathcal{H}_0^1$ in the same way as in the three-dimensional case, i.e. by (3.3), the functions $\psi\in\mathcal{H}_0$ are then decomposed as

$$\psi=\psi_1\chi_1+\psi^\perp=P_1\psi+(1-P_1)\psi$$

where

$$(P_1\psi)(s,t) := \left(\int_{-1}^{1} \chi_1(t)\psi(s,t)dt\right)\chi_1(t)$$

in the projection on subspace $\mathcal{H}_0^1$ and

$$\psi_1 = \int_{-1}^{1} \chi_1(t)\psi(s,t)dt.$$

The isomorphism π between the spaces $\mathcal{H}_0^1$ and $L^2(I)$ is defined in the same way as before (see (3.6)) and the quadratic forms on these spaces can be identified in the following way.

$$Q_{\mathrm{eff}}[\psi_1\chi] := \int_{\Omega_0} |\partial_s\psi_1\chi_1|^2\, ds\, dt - \frac{1}{4}\int_{\Omega_0} \kappa(s)^2|\psi_1\chi_1|^2\, ds\, dt = \int_I |\partial_s\psi_1|^2 ds - \frac{1}{4}\int_I \kappa^2|\psi_1|^2 ds =: q_{\mathrm{eff}}[\psi_1]. \tag{4.13}$$

Similarly we identify the associated operators. By $0^\perp$ we will again assign the zero operator on the orthogonal complement of $\mathcal{H}_0^1$ in $\mathcal{H}_0$.

Also the comparison of spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ will be done in the same way as in three dimensions, i.e. we apply the unitary transformation U_ε given by (3.7) on the (bounded) operator $(H_\varepsilon + r)^{-1}$, the only difference is that the terms $|G|$ and $|\tilde{G}|$ are now given by (4.3) and (4.6).

Let us summarize the assumptions of our main theorem in case of two-dimensional waveguide.

Assumption 4. *We assume that at least one of following conditions is satisfied.*

(i) The interval I is bounded.

(ii) If we set

$$\sigma_\kappa(\delta(\varepsilon)) = 3\sup_{n\in\mathbb{Z}}\left(\sup_{|\eta|\le\frac{\delta(\varepsilon)}{2}} \int_n^{n+1} |\kappa(s)-\kappa(s+\eta)|^2\, ds\right)$$

then

$$\lim_{\varepsilon\to 0}\sigma_\kappa(\delta(\varepsilon)) = 0 \tag{4.14}$$

for some continuous functions $\delta(\varepsilon)$ satisfying

$$\begin{aligned} &\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0,\\ &\lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta(\varepsilon)} = 0. \end{aligned} \tag{4.15}$$

Theorem 4.2. *Let H_ε be the operator defined by (4.7), i.e. the operator unitarily equivalent to the renormalized Hamiltonian $-\Delta_D^\Omega$ describing the dynamics on a curved quantum waveguide built along a planar curve $\Gamma(s)$, $s\in I$, such that the Assumptions 3 and 4 are satisfied. Let h_{eff} be the effective Hamiltonian on the interval I defined by (3.1) and let U_ε be the unitary transformation (3.7). Then*

$$\left\|U_\varepsilon(H_\varepsilon+r)^{-1}U_\varepsilon^{-1} - \left((h_{\mathrm{eff}}+r)^{-1}\oplus 0^\perp\right)\right\|_{\mathcal{B}(\mathcal{H}_0)} \le C^{(1)}\varepsilon + C^{(2)}\frac{\varepsilon}{\delta(\varepsilon)} + C^{(3)}\sqrt{\sigma_\kappa(\delta(\varepsilon))}$$

for some r satisfying $-r\in\mathbb{C}\setminus(\sigma(H_\varepsilon)\cup\sigma(h_{\mathrm{eff}}))$ and some constants $C^{(i)}$, $i=1,2,3$. The right-hand side tends to zero when $\varepsilon\to 0$.

4.2.2 Proof of Theorem 4.2

We will prove the theorem for some real $r \geq 8C_\kappa^2$ (since for such r, $(H_\varepsilon + r)^{-1}$ is bounded and since it will be convenient in the proof of the auxiliary Lemma 4.4) and we will again divide the proof into proving two lemmas. The first one compares the operators H_ε and H_0 where in this case

$$H_0 := 1 \otimes \left(-\frac{1}{\varepsilon^2}\Delta_D^{(-1,1)} - \frac{E_1}{\varepsilon^2} \right) + \left(-\Delta_D^I - \frac{\kappa^2}{4} \right) \otimes 1.$$

The associated quadratic form reads

$$Q_0[\psi] = \int_{\Omega_0} |\partial_s \psi|^2 \, ds \, dt + \frac{1}{\varepsilon^2} \int_{\Omega_0} |\partial_t \psi|^2 \, ds \, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} |\psi|^2 \, ds \, dt - \frac{1}{4} \int_{\Omega_0} \kappa^2 |\psi|^2 \, ds \, dt$$

with $\operatorname{Dom} Q_0 = W_0^{1,2}(\Omega_0)$. It again acts on the functions $\psi_1 \chi_1 \in \mathcal{H}_0^1$ in the same way as Q_{eff} given by (4.13). We can estimate $Q_0[\psi] \geq -\frac{C_\kappa^2}{4} \|\psi\|^2$, hence the operator $H_0 + r$ will be positive for $r > \frac{C_k^2}{4}$ and again for its inverse holds

$$\|(H_0 + r)^{-1}\| \leq \frac{1}{r - \frac{C_k^2}{4}}. \tag{4.16}$$

In the following lemma we again use the unitary transformation (3.7) (with appropriate (4.3) and (4.6)).

Lemma 4.3. *Let $r \geq 8C_k^2$ be a real constant and let the assumptions of Theorem 4.2 be satisfied. Then*

$$\|U_\varepsilon (H_\varepsilon + r)^{-1} U_\varepsilon^{-1} - (H_0 + r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} \leq C_1^{(1)} \varepsilon + C^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} + C^{(3)} \sqrt{\sigma_\kappa(\delta(\varepsilon))}$$

for some constants $C_1^{(1)}$, $C^{(2)}$ and $C^{(3)}$.

Proof. The proof will be again based on the auxiliary lemma with the estimate on the difference of quadratic forms Q_ε and Q_0. This lemma will be proved in Section 4.3.

Lemma 4.4. *Let Q_ε^r, resp. Q_0^r be quadratic form associated with $H_\varepsilon + r =: H_\varepsilon^r$, resp. $H_0 + r =: H_0^r$ where $r \geq 8C_k^2$ and let the assumption of the Theorem 4.2 be satisfied. Then $\forall \phi, \psi \in W_0^{1,2}(\Omega_0)$*

$$|Q_\varepsilon^r(\phi, \psi) - Q_0^r(\phi, \psi)| \leq \left(\tilde{C}^{(1)} \varepsilon + \tilde{C}^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} + \tilde{C}^{(3)} \sqrt{\sigma_\kappa(\delta(\varepsilon))} \right) \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]}.$$

for some constants $\tilde{C}^{(i)}$, $i = 1, 2, 3$ and the right-hand side tends to zero when $\varepsilon \to 0$.

Using this lemma, we can make similar estimates as in proof of Lemma 3.5, namely we use the special choice of functions ϕ, ψ as in (3.22) and then we can estimate analogously to (3.24) (we assign $\sigma(\varepsilon) = C_1^{(1)} \varepsilon + C^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} + C^{(3)} \sqrt{\sigma_\kappa(\delta(\varepsilon))}$). For all $f, g \in \mathcal{H}_0$

$$\begin{aligned}
&\left|\left(f, U_\varepsilon (H_\varepsilon^r)^{-1} U_\varepsilon^{-1} g\right) - \left((H_0^r)^{-1} f, g\right)\right| \leq \\
&\leq \left|\left(f, (U_\varepsilon - 1)(H_\varepsilon^r)^{-1} U_\varepsilon^{-1} g\right)\right| + \left|\left(f, (H_\varepsilon^r)^{-1}(U_\varepsilon^{-1} - 1) g\right)\right| + \left|\left((H_0^r)^{-1} f, (U_\varepsilon^2 - 1) g\right)\right| + \\
&+ \left|\left(f, (H_\varepsilon^r)^{-1} g\right) - \left(U_\varepsilon (H_0^r)^{-1} f, U_\varepsilon g\right)\right| \leq \\
&\leq \left[12 \varepsilon a C_\kappa \left(\|(H_\varepsilon^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)} + \|(H_0^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} \right) + 2\sigma(\varepsilon) \sqrt{\|(H_0^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)} \|(H_\varepsilon^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)}} \right] \|f\| \|g\| \leq \\
&\leq \left(C_3^{(1)} \varepsilon + \tilde{C} \left(\tilde{C}^{(1)} \varepsilon + \tilde{C}^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} + \tilde{C}^{(3)} \sqrt{\sigma_\kappa(\delta(\varepsilon))} \right) \right) \|f\| \|g\|
\end{aligned}$$

where we used similar estimates as (3.26) on norms like $\|(U_\varepsilon - 1)\psi\|$ and also the boundedness of $\|(H_\varepsilon^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_\varepsilon)}$ and $\|(H_0^r)^{-1}\|_{\mathcal{B}(\mathcal{H}_0)}$ (see (4.11) and (4.16)). This relation proves the Lemma 4.3 according to relation (3.23). $\square$

The second lemma expresses the connection between the operator H_0 acting on $\mathcal{H}_0$ and the operator h_{eff} acting on $L^2(I)$.

Lemma 4.5. *Let H_0 be the operator defined by* (3.15) *and let h_{eff} be the effective Hamiltonian* (3.1)*. Then*

$$\| (H_0 + r)^{-1} - \left((h_{\text{eff}} + r)^{-1} \oplus 0^\perp \right) \|_{\mathcal{B}(\mathcal{H}_0)} \leq C_2^{(1)} \varepsilon$$

for some real constants $C_2^{(1)}$ and $r \geq 8C_k^2$.

Proof. The proof is completely analogous to proof of Lemma 3.6. If we now assign by C_1 the constant from (4.21a), we get

$$\left\| (H_0^r)^{-1} - \left((h_{\text{eff}}^r)^{-1} \oplus 0^\perp \right) \right\|_{\mathcal{B}(\mathcal{H}_0)} \leq \varepsilon C_1 \sqrt{\|(H_0^r)^{-1}\|} \leq \varepsilon C_1 \sqrt{\frac{1}{r - \frac{C_k^2}{4}}} =: C_2^{(1)} \varepsilon.$$

□

Now we can proof the main theorem using simple estimate

$$\begin{aligned}
&\left\| U_\varepsilon (H_\varepsilon + r)^{-1} U_\varepsilon^{-1} - \left((h_{\text{eff}} + r)^{-1} \oplus 0^\perp \right) \right\|_{\mathcal{B}(\mathcal{H}_0)} \leq \\
&\leq \| U_\varepsilon (H_\varepsilon + r)^{-1} U_\varepsilon^{-1} - (H_0 + r)^{-1} \|_{\mathcal{B}(\mathcal{H}_0)} + \| (H_0 + r)^{-1} - \left((h_{\text{eff}} + r)^{-1} \oplus 0^\perp \right) \|_{\mathcal{B}(\mathcal{H}_0)} \leq \\
&\leq \left(C_1^{(1)} + C_2^{(1)} \right) \varepsilon + C^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} + C^{(3)} \sqrt{\sigma_\kappa(\delta(\varepsilon))}.
\end{aligned}$$

□

4.3 Proof of Lemma 4.4

Our task is to proof the difference of the sesquilinear forms

$$Q_\varepsilon^r(\phi, \psi) = \int_{\Omega_0} \frac{1}{h h_\varepsilon} \partial_s \bar{\phi} \partial_s \psi \, ds \, dt + \frac{1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} \partial_t \bar{\phi} \partial_t \psi \, ds \, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} \bar{\phi} \psi \, ds \, dt + \tag{4.17a}$$

$$+ \frac{1}{2} \int_{\Omega_0} \frac{1}{h_\varepsilon^2} \kappa \kappa_\varepsilon \bar{\phi} \psi \, ds \, dt - \frac{3}{4} \int_{\Omega_0} \frac{h}{h_\varepsilon^3} \kappa_\varepsilon^2 \bar{\phi} \psi \, ds \, dt + \int_{\Omega_0} \frac{|\partial_s h_\varepsilon|^2}{4 h_\varepsilon^3 h} \bar{\phi} \psi \, ds \, dt + \tag{4.17b}$$

$$+ \int_{\Omega_0} \frac{\partial_s h_\varepsilon}{2 h_\varepsilon^2 h} \left(\bar{\phi} \partial_s \psi + \partial_s \bar{\phi} \psi \right) \, ds \, dt + r \int_{\Omega_0} \frac{h}{h_\varepsilon} \bar{\phi} \psi \, ds \, dt \tag{4.17c}$$

and

$$Q_0^r(\phi, \psi) = \int_{\Omega_0} \partial_s \bar{\phi} \partial_s \psi \, ds \, dt + \frac{1}{\varepsilon^2} \int_{\Omega_0} \partial_t \bar{\phi} \partial_t \psi \, ds \, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \bar{\phi} \psi \, ds \, dt + \tag{4.18a}$$

$$- \frac{1}{4} \int_{\Omega_0} \kappa^2 \bar{\phi} \psi \, ds \, dt + r \int_{\Omega_0} \bar{\phi} \psi \, ds \, dt. \tag{4.18b}$$

We will again assign the final formulas by (p1)-(p6).

From (4.9) and the relation

$$Q_0^r[\psi] \geq \|\partial_s \psi\|^2 + \left(r - \frac{C_\kappa^2}{4} \right) \|\psi\|^2$$

it follows that

$$\|\psi\|^2 \leq 2\|\psi\|_\varepsilon^2 \leq \frac{2}{r - 6C_\kappa^2} Q_\varepsilon^r[\psi] \leq \frac{1}{C_\kappa^2} Q_\varepsilon^r[\psi] \tag{4.19a}$$

$$\|\partial_s\psi\|^2 \leq 4Q_\varepsilon^r[\psi] \tag{4.19b}$$

$$\|\psi\|^2 \leq \frac{1}{r - \frac{C_\kappa^2}{4}} Q_0^r[\psi] \leq \frac{1}{C_\kappa^2} Q_0^r[\psi] \tag{4.19c}$$

$$\|\partial_s\psi\|^2 \leq Q_0^r[\psi] \tag{4.19d}$$

where we used the assumption $r \geq 8C_\kappa^2$ in the first an third relation. Now we can perform the easy estimates.

Using the estimate (4.10) from Section 4.1.5, the Schwarz inequality, the relations (4.19) and also the assumption (4.15), we get

$$\int_{\Omega_0} \frac{|\partial_s h_\varepsilon|^2}{4h_\varepsilon^3 h} \bar{\phi}\psi \, ds \, dt \leq 4\frac{\varepsilon^2}{\delta(\varepsilon)^2} \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]} \leq 4\frac{\varepsilon}{\delta(\varepsilon)} \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]} =: \tilde{C}_1^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]}. \tag{p1}$$

Similarly we would get

$$\int_{\Omega_0} \frac{\partial_s h_\varepsilon}{2h_\varepsilon^2 h} \left(\bar{\phi}\partial_s\psi + \partial_s\bar{\phi}\psi\right) \, ds \, dt \leq 8\frac{\varepsilon}{\delta(\varepsilon)} \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]} =: \tilde{C}_2^{(2)} \frac{\varepsilon}{\delta(\varepsilon)} \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]}. \tag{p2}$$

It is easy to estimate the difference of terms with r,

$$r \int_{\Omega_0} \left(\frac{h}{h_\varepsilon} - 1\right) \bar{\phi}\psi \, ds \, dt \leq \frac{3\varepsilon r}{C_\kappa} \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]} =: \tilde{C}_1^{(1)} \varepsilon \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]}. \tag{p3}$$

In the two-dimensional case the difference of first terms on line (4.17a) resp. (4.18a) falls also into easy estimates:

$$\left| \int_{\Omega_0} \left(\frac{1}{hh_\varepsilon} - 1\right) \partial_s\bar{\phi}\partial_s\psi \, ds \, dt \right| \leq 3\varepsilon C_\kappa \|\partial_s\phi\| \|\partial_s\psi\| \leq 6\varepsilon C_\kappa \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]} =: \tilde{C}_2^{(1)} \varepsilon \sqrt{Q_0^r[\phi] Q_\varepsilon^r[\psi]}. \tag{p4}$$

The estimate on the difference of first two terms on line (4.17b) and the first term in (4.18b) will be slightly harder. It is easy to start with

$$\begin{aligned} |q_\varepsilon(\phi,\psi)| &:= \left| \frac{1}{2} \int_{\Omega_0} \frac{1}{h_\varepsilon^2} \kappa\kappa_\varepsilon \bar{\phi}\psi \, ds \, dt - \frac{3}{4} \int_{\Omega_0} \frac{h}{h_\varepsilon^3} \kappa_\varepsilon^2 \bar{\phi}\psi \, ds \, dt + \frac{1}{4} \int_{\Omega_0} \kappa^2 \bar{\phi}\psi \, ds \, dt \right| \leq \\ &\leq 3C_\kappa \int_{\Omega_0} |\kappa_\varepsilon - \kappa| \bar{\phi}\psi \, ds \, dt + 3\varepsilon C_\kappa^3 \|\phi\| \|\psi\|, \end{aligned} \tag{4.20}$$

however, to proceed further, we will have to use the Hilbert space decomposition.

4.3.1 Hilbert space decomposition

It was already mentioned in Section 4.2.1, how we introduce the subspace $\mathcal{H}_0^1$. Let us only recall that we decompose the functions $\psi \in \mathcal{H}_0$ as

$$\psi = P_1\psi + (1 - P_1)\psi = \psi_1\chi_1 + \psi^\perp$$

where the projection P_1 and the function ψ_1 were introduced also in Section 4.2.1. We again show that the function $\psi^\perp$ vanishes for small ε similarly as in (3.41) and (3.42).

From the proof of Lemma 4.1 and from the fact that $r \geq 8C_\kappa^2$ it follows that for all $\psi \in W_0^{1,2}$

$$Q_\varepsilon^r[\psi] \geq \frac{1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\partial_t\psi|^2 \, ds \, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\psi|^2 \, ds \, dt.$$

If we apply this inequality on $\psi^\perp$, if we realize that $\int_{\Omega_0} |\partial_t \psi|^2\, ds\, dt \geq E_2 \|\psi\|^2$ and if we use the same trick with small parameter β as in (3.40), we get

$$\begin{aligned}
Q^r_\varepsilon[\psi] &\geq \frac{1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\partial_t \psi|^2\, ds\, dt - \frac{E_1}{\varepsilon^2} \int_{\Omega_0} \frac{h}{h_\varepsilon} |\psi|^2\, ds\, dt \geq \\
&\geq \frac{1}{\varepsilon^2} \frac{1 - \varepsilon C_\kappa}{1 + \varepsilon C_\kappa} \int_{\Omega_0} |\nabla' \psi^\perp|^2\, ds\, dt - \frac{E_1}{\varepsilon^2} \frac{1 + \varepsilon C_\kappa}{1 - \varepsilon C_\kappa} \int_{\Omega_0} |\psi^\perp|^2\, ds\, dt \geq \\
&\geq \frac{1}{\varepsilon^2} \left(E_2 \frac{1 - \varepsilon C_\kappa}{1 + \varepsilon C_\kappa} (1 - \beta) - E_1 \frac{1 + \varepsilon C_\kappa}{1 - \varepsilon C_\kappa} \right) \|\psi^\perp\|^2 + \frac{1}{\varepsilon^2} \beta \frac{1 - \varepsilon C_\kappa}{1 + \varepsilon C_\kappa} \|\nabla' \psi^\perp\|^2.
\end{aligned}$$

Since here we know E_1 and E_2 explicitly ($E_1 = \frac{\pi^2}{4}$, $E_2 = \pi^2$), we know that if $\varepsilon C_\kappa \leq \frac{1}{4}$, we can put $\beta = \frac{1}{4}$ to get

$$Q^r_\varepsilon[\psi^\perp] \geq \frac{\pi^2}{30\varepsilon^2} \|\partial_t \psi^\perp\|^2 + \frac{3\pi^2}{20\varepsilon^2} \|\psi^\perp\|^2.$$

This yields

$$\|\psi^\perp\|^2 \leq \frac{20\varepsilon^2}{3\pi^2} Q^r_\varepsilon[\psi^\perp] =: \varepsilon^2 C_1^2 Q^r_\varepsilon[\psi^\perp], \tag{4.21a}$$

$$\|\partial_t \psi^\perp\|^2 \leq \frac{30}{\pi^2} \varepsilon^2 Q^r_\varepsilon[\psi^\perp] =: \varepsilon^2 C_2^2 Q^r_\varepsilon[\psi^\perp]. \tag{4.21b}$$

Similarly as while deriving (3.43) and (3.44) we would get for $\phi = \phi_1 \chi_1 + \phi^\perp$ that

$$\|\phi^\perp\|^2 \leq \varepsilon^2 C_1^2 Q^r_0[\phi^\perp], \tag{4.21c}$$

$$\|\partial_t \phi^\perp\|^2 \leq \varepsilon^2 C_2^2 Q^r_0[\phi^\perp]. \tag{4.21d}$$

The estimate

$$Q^r_0[\phi^\perp] \leq Q^r_0[\phi] \tag{4.22a}$$

is obvious, on the other hand, to prove the estimate analogous to (3.47c)

$$Q^r_\varepsilon[\psi^\perp] \leq 2 Q^r_\varepsilon[\psi] \tag{4.22b}$$

again longer computation is made (similarly as we saw in Section 3.3.6), however, we will not give here these details.

Now we can finish the estimate on the term $q_\varepsilon(\phi, \psi)$ started in (4.20). Using the Schwarz inequality we get

$$\int_{\Omega_0} |\kappa_\varepsilon - \kappa| \bar{\phi} \psi\, ds\, dt \leq \sqrt{\int_{\Omega_0} |\kappa_\varepsilon - \kappa| |\phi|^2\, ds\, dt} \sqrt{\int_{\Omega_0} |\kappa_\varepsilon - \kappa| |\psi|^2\, ds\, dt} \tag{4.23}$$

where we can estimate

$$\begin{aligned}
\int_{\Omega_0} |\kappa_\varepsilon - \kappa| |\psi_1 \chi_1 + \psi^\perp|^2\, ds\, dt &\leq \int_I |\kappa_\varepsilon - \kappa| |\psi_1|^2 ds + 2 C_\kappa \|\psi^\perp\|^2 \leq \\
&\leq \sqrt{\sigma_\kappa(\delta(\varepsilon))} \|\psi_1\|_{W^{1,2}(I)} \|\psi\|_{L^2(I)} + 2 C_\kappa \|\psi^\perp\|^2 \leq \\
&\leq \left(\sqrt{\sigma_\kappa(\delta(\varepsilon))} \frac{1}{C_\kappa} \sqrt{2 + \frac{1}{C_\kappa^2}} + 4 C_\kappa C_1 \varepsilon^2 \right) \sqrt{Q^r_\varepsilon[\psi]}.
\end{aligned}$$

Above, we used the relation (2.36) and the relations (4.21a), (4.22b). We also used that the relation $\|\psi\|^2 = \|\psi_1\|^2_{L^2(I)} + \|\psi^\perp\|^2$ yields $\|\psi_1\|^2_{L^2(I)} \leq \|\psi\|^2$ and similarly $\|\partial_s \psi_1\|^2_{L^2(I)} \leq \|\partial_s \psi\|^2$. For the first term in (4.23) we find

$$\int_{\Omega_0} |\kappa_\varepsilon - \kappa| |\phi_1 \chi_1 + \phi^\perp|^2\, ds\, dt \leq \left(\sqrt{\sigma_\kappa(\delta(\varepsilon))} \frac{1}{C_\kappa} \sqrt{1 + \frac{1}{C_\kappa^2}} + 2 C_\kappa C_1 \varepsilon^2 \right) \sqrt{Q^r_0[\phi]}.$$

Altogether we get (similarly as (p4))

$$|q_\varepsilon(\phi,\psi)| \leq \left[6\left(\sqrt{\sigma_\kappa(\delta(\varepsilon))}\sqrt{1+\frac{1}{C_\kappa^2}}+2\varepsilon^2 C_\kappa^2 C_1\right)+3\varepsilon C_\kappa\right]\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]} =$$
$$=: \left(\tilde{C}_3^{(1)}\varepsilon+\tilde{C}_1^{(3)}\sqrt{\sigma_\kappa(\delta(\varepsilon))}\right)\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]}. \tag{p5}$$

In the two-dimensional case, the most problematic term is the difference of second and third term on line (4.17a) and the second and third term on line (4.18a) which reads

$$m(\phi,\psi) := \frac{1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\partial_t\bar{\phi}\partial_t\psi\,ds\,dt - \frac{E_1}{\varepsilon^2}\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\bar{\phi}\psi\,ds\,dt.$$

It is analogous to $m(\phi,\psi)$ in three-dimensional case, hence we will estimate it using the Hilbert space decomposition and the integration by parts. Using the integration by parts twice and using the relation $-\partial_t^2\chi_1 = E_1\chi_1$ we get

$$|m(\phi_1\chi_1,\psi_1\chi_1)| = \frac{1}{2\varepsilon^2}\left|\int_{\Omega_0}\partial_t^2\left(\frac{h}{h_\varepsilon}-1\right)\bar{\phi}_1\psi_1\chi_1^2\right| \leq$$
$$\leq 4C_\kappa\int_I|\kappa_\varepsilon-\kappa||\phi_1||\psi_1|ds + 10\varepsilon C_\kappa^3\|\phi_1\|_{L^2(I)}\|\psi_1\|_{L^2(I)} \leq$$
$$\leq \sqrt{\sigma_\kappa(\delta(\varepsilon))}\sqrt{\|\phi_1\|_{W^{1,2}(I)}\|\phi_1\|_{L^2(I)}}\sqrt{\|\psi_1\|_{W^{1,2}(I)}\|\psi_1\|_{L^2(I)}} + 10\varepsilon C_\kappa^3\|\phi_1\|_{L^2(I)}\|\psi_1\|_{L^2(I)}.$$

The estimate on $|m(\phi_1\chi_1,\psi^\perp)|$ is performed integrating by parts once:

$$|m(\phi_1\chi_1,\psi^\perp)| = \left|-\frac{1}{\varepsilon^2}\int_{\Omega_0}\partial_t\left(\frac{h}{h_\varepsilon}-1\right)\bar{\phi}_1\chi_1\psi^\perp\,ds\,dt\right| \leq$$
$$\leq 2\sqrt{\int_I|\kappa_\varepsilon-\kappa|^2|\phi_1|^2ds}\,\frac{\|\psi^\perp\|}{\varepsilon} + 4\varepsilon C_\kappa^2\|\phi_1\|_{L^2(I)}\frac{\|\psi^\perp\|}{\varepsilon} \leq$$
$$\leq \sqrt{\sigma_\kappa(\delta(\varepsilon))}\|\phi_1\|_{W^{1,2}(I)}\frac{\|\psi^\perp\|}{\varepsilon} + 4\varepsilon C_\kappa^2\|\phi_1\|_{L^2(I)}\frac{\|\psi^\perp\|}{\varepsilon}.$$

Similarly

$$|m(\phi^\perp,\psi_1\chi_1)| \leq \sqrt{\sigma_\kappa(\delta(\varepsilon))}\|\psi_1\|_{W^{1,2}(I)}\frac{\|\phi^\perp\|}{\varepsilon} + 4\varepsilon C_\kappa^2\|\psi_1\|_{L^2(I)}\frac{\|\phi^\perp\|}{\varepsilon}.$$

In the last estimate we don't need any integration by parts:

$$|m(\phi^\perp,\psi^\perp)| \leq \frac{1}{\varepsilon^2}\left|\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\partial_t\bar{\phi^\perp}\partial_t\psi^\perp\,ds\,dt\right| + \frac{E_1}{\varepsilon^2}\left|\int_{\Omega_0}\left(\frac{h}{h_\varepsilon}-1\right)\bar{\phi}^\perp\psi^\perp\right| \leq$$
$$\leq 3\varepsilon C_\kappa\frac{\|\partial_t\phi^\perp\|}{\varepsilon}\frac{\|\partial_t\psi^\perp\|}{\varepsilon} + 3\varepsilon E_1 C_\kappa\frac{\|\phi^\perp\|}{\varepsilon}\frac{\|\psi^\perp\|}{\varepsilon}.$$

Altogether we get (using the relations (4.21), (4.22) and (4.19)), we get

$$|m(\phi,\psi)| \leq \Bigg[\sqrt{\sigma_\kappa(\delta(\varepsilon))}\sqrt{1+\frac{1}{C_\kappa^2}}\left(\frac{2}{C_\kappa}+3C_1\right)+$$
$$+\varepsilon\left(10C_\kappa + C_\kappa\left(12C_1+5C_2^2+5E_1C_1^2\right)\right)\Bigg]\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]} = \tag{4.24}$$
$$=: \left(\tilde{C}_4^{(1)}\varepsilon+\tilde{C}_2^{(3)}\right)\sqrt{Q_0^r[\phi]}\sqrt{Q_\varepsilon^r[\psi]} \tag{p6}$$

4.3.2 Conclusion

According to relations (p1)-(p6), we get that

$$|Q^r_\varepsilon(\phi,\psi) - Q^r_0(\phi,\psi)| \le$$
$$\le \left[\left(\tilde{C}^{(1)}_1 + \tilde{C}^{(1)}_2 + \tilde{C}^{(1)}_3 + \tilde{C}^{(1)}_4\right)\varepsilon + \left(\tilde{C}^{(2)}_1 + \tilde{C}^{(2)}_2\right)\frac{\varepsilon}{\delta(\varepsilon)} + \left(\tilde{C}^{(3)}_1 + \tilde{C}^{(3)}_2\right)\sqrt{\sigma_\kappa(\delta(\varepsilon))}\right]\sqrt{Q^r_0[\phi]Q^r_\varepsilon[\psi]}.$$

The right-hand side converges to zero due to Assumption 4. □

However, in Section 2.3.4 we saw that the Assumption 4 does not hold for all curves and in next section we show that then the Theorem 4.2 need not to hold.

4.4 The essential spectrum for the counterexample waveguide

In this section we will find the lower bound on the essential spectrum of the Dirichlet Laplacian on the strip Ω built along the curve $\Gamma_{\mathrm{osc}} : I = \mathbb{R}^+ \to \mathbb{R}^2$ (see Figure 4.3), whose curvature is oscillating more and more when $s \to \infty$ (see Figure 4.2). The function describing the curvature of Γ_{osc} was already introduced in Section 2.3.4, we slightly modified it so that the picture of the curve was nicer, however, its properties (namely that κ does not satisfy the assumption $\sigma_\kappa(\delta(\varepsilon)) \to 0$ for any $\delta(\varepsilon)$) are preserved. When describing Ω, we will again use the coordinates (s,t), recall $\Omega = \mathcal{L}(\Omega_0)$ where $\Omega_0 = R^+ \times (-1,1)$ and the mapping $\mathcal{L}$ was introduced by (4.2). On the line $\{0\} \times (-1,1)$ we will pose the Dirichlet boundary condition. Recall that we introduce the Dirichlet Laplacian $H := -\Delta^\Omega_D$ as the self-adjoint Friedrich's extension and for the associated quadratic form Q it holds

$$Q[\psi] = \left(\partial_i\psi, G^{ij}\partial_j\psi\right)_{L^2(\Omega_0,|G|^{1/2}ds dt)} \qquad \mathrm{Dom}\, Q = W^{1,2}_0(\Omega_0).$$

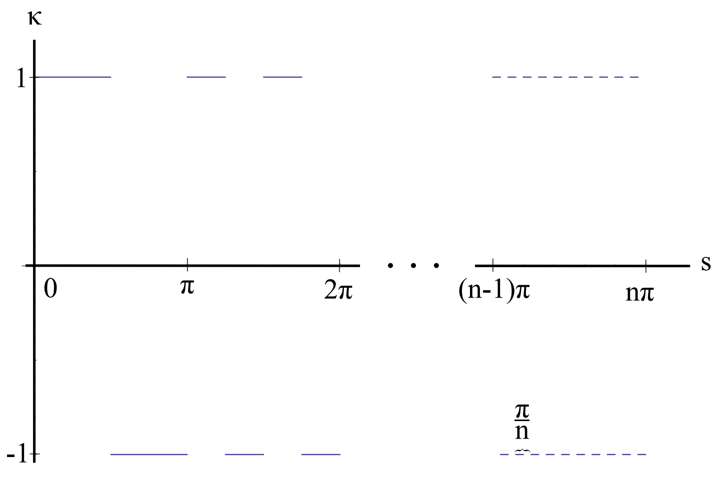

Figure 4.2: The plot describing the curvature $\kappa(s)$ of the curve Γ_{osc}.

If the curve Γ_{osc} satisfied the assumptions of Theorem 4.2, according to our main result, the spectrum of the (renormalized) Dirichlet Laplacian on this strip $-\Delta^\Omega_D$ would converge to the

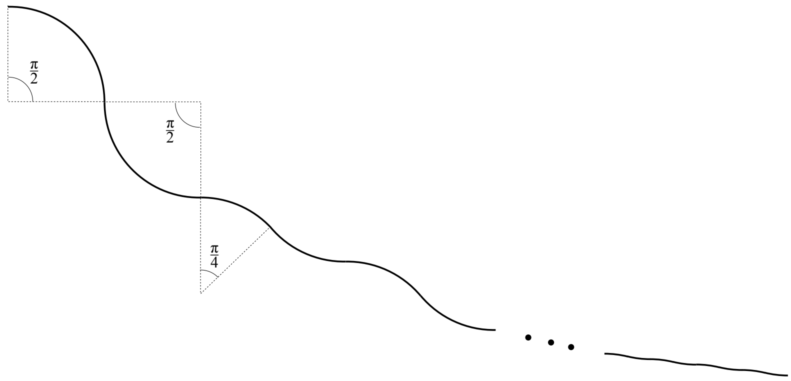

Figure 4.3: The curve Γ_{osc}.

spectrum of the 1D operator acting on $L^2(\mathbb{R}^+)$, $h_{\text{eff}} = -\Delta_D - \frac{\kappa^2}{4} = -\Delta_D - \frac{1}{4}$. Recall that the renormalization of the Hamiltonian consisted of subtracting $\frac{E_1}{\varepsilon^2}$, that's why the threshold of the essential spectrum of $-\Delta_D^\Omega$ would be shifted by this quantity, thus equal to $\frac{E_1}{\varepsilon^2} - \frac{1}{4}$.

However, as we already mentioned above, the curve Γ_{osc} does not satisfy the assumptions of the Theorem 4.2, hence we did not prove the norm resolvent convergence of the renormalized Hamiltonian to h_{eff}. In this section we will find that the essential spectrum of $-\Delta_D^\Omega$ for the strip Ω built along the curve Γ_{osc} does not start below $\frac{E_1}{\varepsilon^2}$ thus it can not be $\frac{E_1}{\varepsilon^2} - \frac{1}{4}$. Therefore the spectrum of $-\Delta_D^\Omega$ is indeed not well approximated by h_{eff} in the limit as $\varepsilon \to 0$.

Theorem 4.6. *Let $H = -\Delta_D^\Omega$ be the Dirichlet Laplacian in the strip built along the curve Γ_{osc}. Then*

$$\inf \sigma_{\text{ess}}(H) \geq \frac{E_1}{\varepsilon^2}$$

where 2ε is the width of the strip.

Remark 4.7. *In this section we consider the width of the strip 2ε as fixed, the only requirement on ε is to be so small that the strip is non-self-intersecting which is for our curve satisfied in case $\varepsilon < 1$ (since the curve consists of arcs, whose radius is 1).*

Proof. In the proof of this theorem we will proceed similarly as in the paper [21]. We will use repeatedly the so called Neumann bracketing (see Section B.8.1), in the first step we will divide the waveguide into two parts. Let $I^{\text{int}} = (0, n_0\pi)$ ($I^{\text{ext}} = \mathbb{R}^+ \setminus \overline{I^{\text{int}}}$), then $\Omega^{\text{int}} := \left(I^{\text{int}} \times (-1,1)\right)$ and $\Omega^{\text{ext}} := \Omega_0 \setminus \overline{\Omega^{\text{int}}}$ where $n_0 \in \mathbb{N}$ will be specified later. On the segment $\{n_0\pi\} \times (-1,1)$ we will pose the Neumann boundary condition, the Laplacian with this extra Neumann boundary condition will be assigned $H_N = H_N^{\text{int}} \oplus H_N^{\text{ext}}$, the associate quadratic form reads $Q_N := Q_N^{\text{int}} \oplus Q_N^{\text{ext}}$ where

$$Q_N^\iota[\psi] = \left(\partial_i \psi, G^{ij} \partial_j \psi\right)_{L^2(\Omega^\iota, |G|^{1/2} ds dt)},$$
$$\operatorname{Dom} Q_N^\iota = \left\{\psi \restriction I^\iota \times (-1,1) \middle| \psi \in \operatorname{Dom} Q\right\},$$

$\iota \in \{\text{int}, \text{ext}\}$.

Here we can use the Proposition B.24 (or more precisely its modification in the sense of Remark B.25 point (i)) to get $H \geq H_N$ and together with Lemma B.22 we get

$$\lambda_j(H) \geq \lambda_j(H_N) \qquad \forall j \geq 1.$$

Due to the Theorem B.16, the threshold of the essential spectrum for the operator H is $\lim_{j\to\infty}\lambda_j(H)$ and similarly for H_N. From the inequality above it follows that also $\lim_{j\to\infty}\lambda_j(H) \geq \lim_{j\to\infty}\lambda_j(H_N)$, hence the threshold of the essential spectrum of the operator H is greater or equal to the threshold of the essential spectrum of H_N. Furthermore since Ω^{int} is finite and regular enough (i.e. it has the extension property, see [5]), the spectrum of H_N^{int} is purely discrete and we get

$$\inf \sigma_{\text{ess}}(H) \geq \inf \sigma_{\text{ess}}(H_N^{\text{ext}}) \geq \inf \sigma(H_N^{\text{ext}}). \tag{4.25}$$

Now we will examine the spectrum of H_N^{ext} and we will again use the Neumann bracketing. On interval $((n-1)\pi, n\pi)$ for each $n \in \mathbb{N}$ there are subintervals $\left((n-1+\frac{2k}{2n})\pi, (n-1+\frac{2k+1}{2n})\pi\right)$ where the curvature is 1 and subintervals $\left((n-1+\frac{2k+1}{2n})\pi, (n-1+\frac{2k+2}{2n})\pi\right)$ where the curvature is -1, $k = 0, 1, ..n-1$. We will divide $\mathbb{R}^+$ into subintervals

$$S_n^k := \left((n-1+\frac{2k}{2n})\pi, (n-1+\frac{2k+2}{2n})\pi\right),$$

and the strip Ω^{ext} will be divided into segments

$$\Omega_n^k := S_n^k \times (-1, 1)$$

(one of those segments is depicted on Figure 4.4). Let us note that the segments Ω_n^k for fixed n differ only in position, the shape is the same. Between two such segments there is again placed a Neumann curve and we get the operator $\tilde{H}_N^{\text{ext}}$ associated with the quadratic form

$$\tilde{Q}_N^{\text{ext}} := \bigoplus_{n=n_0}^{\infty} \bigoplus_{k=0}^{n-1} Q_n^k$$

where

$$Q_n^k[\psi] = \left(\partial_i\psi, G^{ij}\partial_j\psi\right)_{L^2(\Omega_n^k, |G|^{1/2}dsdt)},$$
$$\operatorname{Dom} Q_n^k = \left\{\psi \restriction \Omega_n^k \middle| \psi \in \operatorname{Dom} Q_N^{\text{ext}}\right\}.$$

Using similar arguments as before we get $H_N^{\text{ext}} \geq \tilde{H}_N^{\text{ext}}$ and thus $\lambda_1(H_N^{\text{ext}}) \geq \lambda_1(\tilde{H}_N^{\text{ext}})$ yields

$$\inf \sigma(H_N^{\text{ext}}) \geq \inf \sigma(\tilde{H}_N^{\text{ext}}) \geq \inf_{n\geq n_0} \left(\inf \sigma(H_n^k)\right). \tag{4.26}$$

where H_n^k is the operator associated with Q_n^k. The last inequality follows again from the min-max principle, since $\forall m \geq n_0$, $\forall l = 0, 1..m-1$

$$\lambda_j(\tilde{H}_N^{\text{ext}}) = \inf_{L_j \subseteq \bigoplus_{n,k} \operatorname{Dom} Q_n^k} \sup_{\psi \in L_j} \tilde{Q}_N^{\text{ext}}[\psi] \geq \inf_{L_j^{m,l} \subseteq \operatorname{Dom} Q_m^l} \sup_{\phi \in L_j^{m,l}} Q_m^l[\phi] = \lambda_j(H_m^l)$$

where $\dim L_j = \dim L_j^{m,l} = j$ and $\phi \in \operatorname{Dom} Q_m^l$ can be considered as the function from $\bigoplus_{n,k} \operatorname{Dom} Q_n^k$ when we prolong it by zero on all the segments Ω_n^k where either $n = m \wedge k \neq l$ or $n \neq m$. Now the last task is to evaluate the term on the right-hand side of (4.26) (in there we didn't write the supremum over k since all the H_n^k for fixed n have the same spectrum due to the fact that the shape of Ω_n^k is the same for all k).

Our last task is to prove following lemma which gives us the estimate on the spectrum of the operators H_n^k.

Lemma 4.8.

$$\inf \sigma(H_n^k) \geq \frac{\pi^2}{4\left(\varepsilon + 2\sin^2\frac{\pi}{4n}\right)}.$$

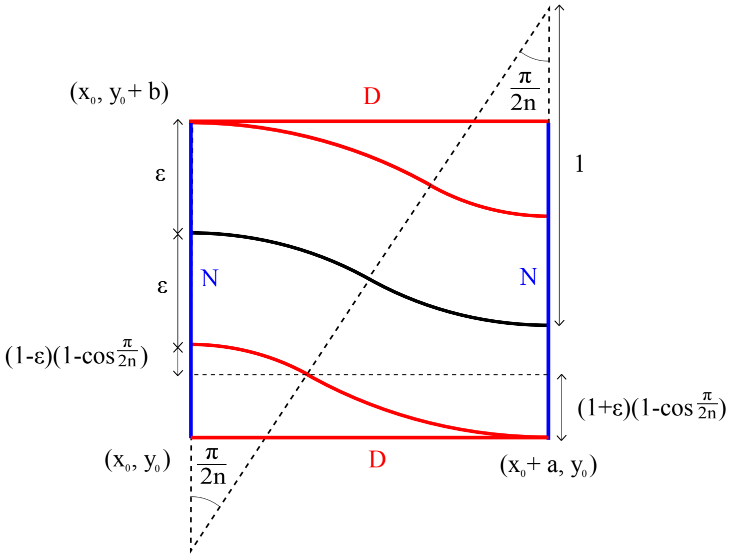

Figure 4.4: The segment Ω_N^k for some $k \in \{0, 1, .., N-1\}$.

Proof. To find the estimate on spectrum of the Laplacian on the segment Ω_n^k where there is Neumann boundary condition on the lines dividing the strip from the neighbor segments and the Dirichlet boundary condition on $\partial\Omega_n^k \cap \partial\Omega$ (see Figure 4.4), we will use the (slightly modified) idea of the Proposition B.23. We confine the segment Ω_n^k by a rectangle of height $b := 2\varepsilon + 2\left(1 - \cos\frac{\pi}{2n}\right) = 2\varepsilon + 4\sin^2\frac{\pi}{4n}$ as depicted on Figure 4.4 and we will pose on the vertical lines the Neumann boundary condition and on the horizontal ones the Dirichlet boundary conditions. The Laplacian on this rectangle will be assigned $H_\square$, the associated quadratic form $Q_\square$. The rectangle is denoted by $\Omega_\square \subset \Omega_0$, however we will come back to the cartesian coordinates (x, y) when describing it. If the lower left corner of the rectangle has the coordinates (x_0, y_0) and we denote the width of the rectangle a, then $(x_0, x_0 + a) \times (y_0, y_0 + b) = \mathcal{L}(\Omega_\square)$. Consequently

$$Q_\square[\psi] = \left(\partial_i\psi, G^{ij}\partial_j\psi\right)_{L^2(\Omega_\square, |G|^{1/2}dsdt)},$$
$$\operatorname{Dom} Q_\square = \left\{\psi \in W^{1,2}(\Omega_\square) \middle| \psi\left(\mathcal{L}^{-1}(x, y_0)\right) = \psi\left(\mathcal{L}^{-1}(x, y_0 + b)\right) = 0 \text{ for a.e. } x \in (x_0, x_0 + a)\right\}.$$

Note that $\psi(x, \cdot)$ denotes the trace of ψ on the boundary part of the rectangle. Our task is now to show that

$$\operatorname{Dom} Q_n^k \subset \operatorname{Dom} Q_\square.$$

To prove this it is enough to realize that if we prolong the function $\psi \in \operatorname{Dom} Q_n^k$ (i.e. the function which is in fact the restriction of some $\Psi \in W_0^{1,2}(\Omega_0)$ on Ω_n^k) by zero on $\Omega_\square \setminus \Omega_n^k$, we will get the function that is for sure in $W^{1,2}(\Omega_\square)$, and also this function is zero on the horizontal lines (the only points where this need not to be satisfied are the corners $(x_0 + a, y_0)$ and $(x_0, y_0 + b)$, however, this is the set of zero measure). It is also clear the forms act in the same way on the functions from $\operatorname{Dom} Q_n^k$. Hence

$$H_n^k \geq H_\square. \tag{4.27}$$

It is easy to show that

$$\sigma(H_\square) = \left\{\frac{((j-1)\pi)^2}{a^2} + \frac{(l\pi)^2}{b^2} \middle| j, l = 1, 2...\right\}.$$

Using (4.27) and rewriting $b = 2\varepsilon + 4\sin^2\frac{\pi}{4n}$ we get

$$\inf\sigma(H_n^k) \geq \inf\sigma(H_\square) = \frac{\pi^2}{4\left(\varepsilon + 2\sin^2\frac{\pi}{4n}\right)}$$

which proves the lemma. $\square$

Using this result we get

$$\inf_{n\geq n_0}\left(\inf\sigma(H_n^k)\right) \geq \inf_{n\geq n_0}\frac{\pi^2}{4\left(\varepsilon + 2\sin^2\frac{\pi}{4n}\right)^2} = \frac{\pi^2}{4\left(\varepsilon + 2\sin^2\frac{\pi}{4n_0}\right)^2}. \tag{4.28}$$

Combining the estimates (4.25), (4.26), and (4.28) we get

$$\inf\sigma_{\mathrm{ess}}(H) \geq \frac{\pi^2}{4\left(\varepsilon + 2\sin^2\frac{\pi}{4n_0}\right)^2}$$

which yields the statement of Theorem 4.6 since n_0 can be chosen arbitrarily big. $\square$

Remark 4.9. *In Section 2.3.4 we mentioned that along the curve Γ_{osc} we can built also a three-dimensional wave-guide. If we chose e.g. the waveguide with the uniform square cross-section without any twisting, the proof of the fact that the spectrum of the Hamiltonian on such tube is not well approximated by the spectrum of the effective Hamiltonian, would be very similar to one we performed here.*

Chapter 5

Conclusion

It is known that the Hamiltonian in quantum waveguides built along a spatial curve $\Gamma(s)$, $s \in I$, converges, under rather restrictive conditions on the curve Γ, in certain way to the effective Hamiltonian

$$h_{\text{eff}}^{\text{FF}} = -\Delta_D^{\Gamma} - \frac{\kappa^2}{4} + C(\omega)\left(\tau + \dot{\theta}_{\text{FF}}\right)^2 \tag{5.1}$$

where κ and τ are Frenet's curvature and torsion and $\dot{\theta}$ is the twisting angle of the uniform cross-section with respect to the Frenet's frame (FF). In this work we generalized this well known result to curves that are more general in two aspects. At first we considered less regular curves, we assumed only

$$\Gamma \in W_{\text{loc}}^{2,\infty}(I)$$

in comparison with C^4-curves considered in papers like [7]. At second, we generalized also the results of papers like [4] or [6] since we proved the norm resolvent convergence of the initial Hamiltonian to (5.1) for wide range of $W_{\text{loc}}^{2,\infty}(I)$ curves also in case when the interval I is *unbounded*.

To introduce the quantum waveguide, firstly, we had to find an adapted moving frame for a general curve $\Gamma \in W_{\text{loc}}^{2,\infty}(I)$. Here we could not use the well known Frenet frame, since it exists only for C^3 curves. That's why we generalized the results of [2] on $W_{\text{loc}}^{2,\infty}(I)$ curves and we built the waveguide using the relatively parallel adapted frame (RPAF). Using this frame we also included in our consideration the curves for which the curvature is allowed to vanish at some parts of the curve. Such curves were excluded in other works, since for them the Frenet frame does not exist.

While using RPAF, the curvatures are assigned by k_1, k_2 and they are related to Frenet's curvature κ as $\kappa^2 = k_1^2 + k_2^2$. Also the Frenet's torsion can be expressed in terms of k_1, k_2, however, for us the following relation concerning the torsion will be sufficient. The quantum waveguide Ω was built using the mapping (2.3), i.e. we rotated the cross-section ω (an open subset of $\mathbb{R}^2$) in every point $s \in I$ by an angle $\theta_{\text{RPAF}}(s)$ with respect to the normal vectors in RPAF. From the definition of the RPAF it follows that θ_{RPAF} is then related to θ_{FF} by

$$\dot{\theta}_{\text{RPAF}} = \tau + \dot{\theta}_{\text{FF}}. \tag{5.2}$$

Then we studied the Dirichlet Laplacian on Ω, $-\Delta_D^{\Omega}$. At first we renormalized the Hamiltonian by subtracting the divergent term in the asymptotic of the spectrum $\frac{E_1}{\varepsilon^2}$ where the E_1 is the first eigenvalue of the transverse Dirichlet Laplacian $-\Delta_D^{\omega}$. Then we followed the procedure which was used e.g. in [7] and where the unitary transformation "straightening" the tube was performed. However, we had to modify slightly this procedure since the curvatures need not to be differentiable in our case. This was done using the so called Steklov approximation which is described e.g. in [1] and reads

$$k_i^\varepsilon(s) = \frac{1}{\delta(\varepsilon)} \int_{s-\frac{\delta(\varepsilon)}{2}}^{s+\frac{\delta(\varepsilon)}{2}} k_i(\xi) d\xi \qquad i = 1, 2.$$

Of course, other approximation of the curvature could be used, we let as an open question if the results would be better in this case. We used the Steklov approximation for we can work with it easily in explicit form.

After the unitary transformation, the operator $-\Delta_D^\Omega$ became an operator that we assigned by H_ε. Unfortunately, when the modified unitary transformation is used, the Hilbert space the Hamiltonian H_ε is acting on is dependent on ε. We solved this problem using another unitary transformation which was applied on the resolvent of H_ε (a bounded operator) and which transforms the resolvents of H_ε to operators that act on fixed Hilbert space $\mathcal{H}_0$. After the identification of subspace $\mathcal{H}_0^1$ of $\mathcal{H}_0$ with the Hilbert space $L^2(I)$, we found in Theorem 3.1 that the operators unitarily equivalent to the resolvents of H_ε converge in norm to the resolvent of effective Hamiltonian

$$h_{\text{eff}}^{\text{RPAF}} = -\Delta_D^\Gamma - \frac{\kappa^2}{4} + C(\omega)\dot{\theta}_{\text{RPAF}}^2.$$

This is in accordance with previous results due to the relation (5.2).

The assumptions on the waveguide we required consisted of boundedness of the cross-section ω and boundedness of the functions $\kappa = k_1^2 + k_2^2$ (curvature) and $\dot{\theta}$ (derivative of the twisting angle). For unbounded intervals we had to assume in addition that the functions $\sigma_{k_i}(\delta(\varepsilon))$, $i = 1, 2$ and $\sigma_{\dot{\theta}}(\tilde{\delta}(\varepsilon))$ tend to zero when ε tends to zero. Recall that in general

$$\sigma_f(\delta(\varepsilon)) = 3 \sup_{n\in\mathbb{Z}} \left(\sup_{|\eta|\le\frac{\delta(\varepsilon)}{2}} \int_n^{n+1} |f(s) - f(s+\eta)|^2 \, ds \right),$$

i.e. $\sigma_f(\delta(\varepsilon))$ expresses how much the function f is oscillating. Recall also that for all "well-behaved" functions f as uniformly continuous functions, L^2- functions or periodic functions, $\sigma_f(\delta(\varepsilon))$ converges to zero with ε. Let us note that we include also the curvatures that do not vanish in infinity which were excluded in [7].

On the other hand, we saw that in case of unbounded intervals, there must be some additional assumption on the curvatures beyond the boundedness, since we have found in Section 4.4 the curve Γ_{osc} such that the spectrum of the Hamiltonian on the waveguide built along Γ_{osc} is not well approximated by the effective Hamiltonian. This was done on the two-dimensional model, however it could be easily extended on the three-dimensional case.

As an open question remains if the class of curves we include into our considerations could be described by some nicer condition. In future we would like also to explore in more detail the spectrum of the Hamiltonian on the waveguide built along Γ_{osc} and what effective Hamiltonian corresponds to this curve.

We gave only a brief overview of the consequences of the norm resolvent convergence, therefore it could be interesting to find the particular consequences of the norm resolvent convergence in our case, i.e. for example state a theorem on the convergence of the eigenvalues with the "speed" of convergence. Also the spectral results such as the existence of Hardy inequality or the existence of bound-states for waveguides with finite (not infinitely small) cross-section could be extended to less regular curves in comparison with other works, if we used our techniques. This all are suggestions for the future work.

Appendix A

Framing of a 3D curve

Let Γ be a spatial curve, i.e. the (image of the) mapping $\Gamma : I \to \mathbb{R}^3 : s \mapsto (\Gamma^1(s), \Gamma^2(s), \Gamma^3(s))$. We assume that this curve is regular, i.e. $|\dot{\Gamma}(s)| \neq 0 \quad \forall s \in I$.

We define an *adapted moving 3-frame* using the terminology of [18] and [2].

Definition A.1. *Let $\Gamma : I \to \mathbb{R}^3$ be a curve.*

(i) A moving 3-frame along Γ is a collection of three differential mappings

$$e_i : \ I \to \mathbb{R}^3 \qquad i = 1, 2, 3$$

such that for all $s \in I$, $e_i \cdot e_j = \delta_{ij}$. Each $e_i(s)$ is then a vector field along Γ and in some particular point s_0 the vector $e_i(s_0)$ is thought of as lying in the copy of $\mathbb{R}^3$ identified with the tangent space $\mathbb{T}_{\Gamma(s_0)}\mathbb{R}^3$.

(ii) We say that a moving frame is adapted to the curve if the members of the frame are either tangent of perpendicular to the curve.

Our goal is now to find the adapted moving frame for as general curves as possible.

A.1 The Frenet frame

The commonly used adapted moving frame is the Frenet frame. We adopt the following definition from [18]:

Definition A.2. *A moving 3-frame is called a Frenet-3-frame, or simply Frenet frame, if for all $k = 1, 2, 3$, the k-th derivative $\Gamma^{(k)}(s)$ of $\Gamma(s)$ lies in the span of the vectors $e_1(s), ..., e_k(s)$.*

In particular, it follows from this definition that the vector field $e_1(s)$ is the normalized tangent vector field of Γ (i.e. the vector field $\frac{\dot{\Gamma}(s)}{|\dot{\Gamma}(s)|}$) and thus the Frenet frame is adapted to the curve Γ.

It is also clear that the Frenet frame can be defined for at least C^3 curves, however this condition is not sufficient for Frenet frame to exist. From the following proposition (adopted from [18]) it follows that the existence of the Frenet frame is determined also by the linear independence of the derivatives of the curve. We will mention here also the proof of this proposition since it gives us the recipe how to construct the Frenet frame. The proposition is the special case of the general proposition for $\mathbb{R}^n$ and we will prove it in the same way as it is done in $\mathbb{R}^n$ even if for $\mathbb{R}^3$ there are also other possibilities how to find the Frenet frame.

Proposition A.3. *Let $\Gamma: I \to \mathbb{R}^3$ be a C^3 curve such that for all $s \in I$, the vectors $\dot{\Gamma}(s)$ and $\ddot{\Gamma}(s)$ are linearly independent. Then there exists a unique Frenet frame with following properties:*

i $\{\dot{\Gamma}(s), \ddot{\Gamma}(s)\}$ *and* $\{e_1(s), e_2(s)\}$ *have the same orientation,*

ii $\{e_1(s), e_2(s), e_3(s)\}$ *has the positive orientation.*

Note that this frame is called the *distinguished Frenet frame.*

Remark A.4. *Recall that two basis for a real vector space have the same orientation provided the linear transformation taking one basis into the other has positive determinant. Basis of $\mathbb{R}^3$ is positively oriented if it has the same orientation as the canonical basis of $\mathbb{R}^3$.*

Proof. We will use the Gram-Schmidt orthogonalization process on $\dot{\Gamma}(s)$ and $\ddot{\Gamma}(s)$. Assumption that $\dot{\Gamma}(s)$ and $\ddot{\Gamma}(s)$ are linearly independent implies that $\dot{\Gamma}(s) \neq 0$ and we can set $e_1(s) := \frac{\dot{\Gamma}(s)}{|\dot{\Gamma}(s)|}$. Then we define

$$\tilde{e}_2(s) := \ddot{\Gamma}(s) - \left(\ddot{\Gamma}(s) \cdot \dot{\Gamma}(s)\right) \dot{\Gamma}(s) \tag{A.1}$$

and we let

$$e_2(s) = \frac{\tilde{e}_2}{|\tilde{e}_2|}. \tag{A.2}$$

Clearly, then e_1 and e_2 satisfy the assertion (i) of the Theorem. The third vector is added in such way that the basis $e_1(s)$, $e_2(s)$, $e_3(s)$ has the positive orientation, this can be done e.g. by setting

$$e_3(s) := e_1(s) \times e_2(s). \tag{A.3}$$

The differentiability of e_1 and e_2 is clear from their definition and from the fact that $\Gamma(s) \in C^3$. The differentiability of e_3 is due to all the components e_3^i can be expressed from (A.3) using the components of e_1 and e_2 which are differentiable. □

Let us note that in the case of $\mathbb{R}^3$ which was considered here, the triplet $\{e_1, e_2, e_3\}$ is sometimes denoted by $\{T, N, B\}$ where the vector fields are called *tangent*, *normal* and *binormal*.

In particular the Frenet frame does not exist for curves where $\ddot{\Gamma}(s) = 0$ for some $s \in I$, i.e. for example the curves where there is inflection in some point or where in some part the straight line occurs. We refer to such curves as *degenerate* curves. In some cases it is possible to patch together the frames for different parts of the curve where $\ddot{\Gamma}(s) \neq 0$ and we still get a differentiable Frenet frame. However, let us take the curve

$$\Gamma(s) = \begin{cases} \left(s, \exp\left(-\frac{1}{s^2}\right), 0\right) & s < 0, \\ (0,0,0) & s = 0, \\ \left(s, 0, \exp\left(-\frac{1}{s^2}\right)\right) & s > 0, \end{cases}$$

which turns from the (x,y)-plane to the (x,z)-plane in the point $s = 0$, where $\ddot{\Gamma}(s) = 0$, but all the derivatives of the curve in this point are continuous. For this curve, it is not possible to find a continuous Frenet frame, even if this curve is C^∞ (see Figure A.1). That's why we will look for the frame that can be built even for such degenerate curve, in the next section. However, now we will mention some other properties of the Frenet frame.

We will look for some relations for the derivatives of the vector fields in the Frenet frame, more precisely we will look for the so called *Cartan* matrix $\mathbb{M}$ satisfying $\dot{e}_i(s) = \mathbb{M}_{ij}(s) e_j(s)$. Let us note that the last relation yields $\mathbb{M}_{ij}(s) = \dot{e}_i(s) \cdot e_j(s)$. Differentiating the relation $e_i(s) \cdot e_j(s) = \delta_{ij}$ we

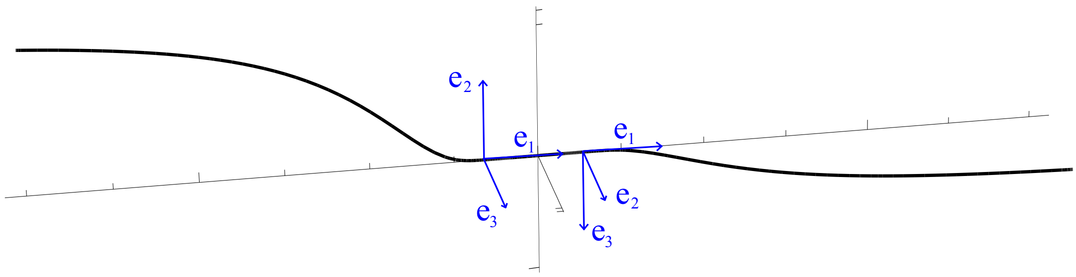

Figure A.1: Example of a C^∞ curve for that the Frenet frame does not exist.

get that $\dot{e}_i(s) \cdot e_j(s) + e_i(s)\dot{e}_j(s) = 0 \; \forall s \in I$, hence the matrix $\mathbb{M}$ is antisymmetric (for any moving frame).

From the fact that $e_1(s)$ is the multiple of $\dot{\Gamma}(s)$ and e_2 is the linear combination of $\dot{\Gamma}(s)$ and $\ddot{\Gamma}(s)$, it follows that $\dot{e}_1(s)$ must be the linear combination of $e_1(s)$ and $e_2(s)$. Furthermore due to the argument above, $\dot{e}_1(s)$ must be the multiple of $e_2(s)$. Summing up all these ideas we get

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{A.4}$$

where the function $\kappa(s)$ is usually called the *curvature* and $\tau(s)$ the *torsion*.

Let us note that from (A.4) follows $\dot{e}_1(s) = \kappa(s)e_2(s)$ and thus the curvature can be computed as

$$\kappa(s) = |\dot{e}_1(s)| = \left|\ddot{\Gamma}(s)\right|.$$

If we now introduce the notion of the *unit-speed* curve (or also the curve *parameterized by arc length*), which assigns the curve where $|\dot{\Gamma}(s)| = 1 \quad \forall s \in I$, we get $e_1 = \dot{\Gamma}$, hence

$$\kappa(s) = \left|\ddot{\Gamma}(s)\right|.$$

for unit-speed curves. Let us note that for a regular curve we can also make the reparameterization such that the reparameterized curve is unit-speed, so we can assume the curve to be parameterized by arc-length without loss of generality. For a unit speed curve we have also simple formula for the torsion:

$$\tau(s) = \frac{\left(\dot{\Gamma}(s) \times \ddot{\Gamma}(s)\right) \cdot \dddot{\Gamma}(s)}{\kappa(s)^2}.$$

From the uniqueness of the Frenet frame, it follows also the uniqueness of the functions $\kappa(s)$ and $\tau(s)$ for the given curve. On the other hand, in [18] it is proved that if the differentiable functions $\kappa(s)$ and $\tau(s)$ are given, there exists a unit-speed curve satisfying the assumptions of Proposition A.3, with curvature $\kappa(s)$ and torsion $\tau(s)$, and two such curves differ only by Euclidian motion (i.e. by translation and rotation of the curve).

A.2 The relatively parallel adapted frame

Using the ideas of the paper [2] we will find another adapted moving frame for more general (unit-speed) curves. Namely, we will require $\Gamma^i \in W^{2,\infty}_{\text{loc}}(I)$ for $i = 1, 2, 3$, where $W^{2,\infty}_{\text{loc}}$ is the Sobolev space of functions ϕ for that (in case of functions of one variable) $\dot{\phi}$ and $\ddot{\phi}$ exist in the weak sense and $\|\phi\|_{L^\infty(J)} + \|\dot{\phi}\|_{L^\infty(J)} + \|\ddot{\phi}\|_{L^\infty(J)} < \infty$ for every compact subinterval $J \subset I$. In fact, we will generalize the results of [2], since in there it is assumed $\Gamma \in C^2$. The assumption $\Gamma^i \in W^{2,\infty}_{\text{loc}}(I)$ yields that $\dot{\Gamma}^i$ is locally Lipschitz continuous (see [10], chapter 5) which allows us to introduce a continuous tangent vector field

$$T(s) := \dot{\Gamma}(s)$$

for every $s \in I$. T has then the bounded weak derivative and it is identical to Frenet's e_1 if the Frenet frame exists. In fact we will slightly modify the Definition A.1 of the moving frame, since we will require the differentiability of the individual vector fields only in the weak sense.

The frame will then consist of the tangent vector field T and two *relatively parallel* normal vector fields.

Definition A.5. *We say that a normal vector field $M(s)$ along the curve $\Gamma : I \to \mathbb{R}^3$ is parallel if there exists a scalar function $f(s)$ such that*

$$\dot{M}(s) = f(s)T(s).$$

Let us note that the relatively parallel normal field has the constant length, since its derivative is perpendicular to it. Another important feature of these vector fields is stated in the following proposition.

Proposition A.6. *Let Γ be a unit-speed spatial curve with $\Gamma^i \in W^{2,\infty}_{\text{loc}}(I)$ for $i = 1, 2, 3$ and let M_0 be a normal vector to this curve at the point $\Gamma(s_0)$. Then there exists a unique relatively parallel normal vector field M such that $M(s_0) = M_0$. This vector field is continuous, its weak derivative exists and is locally bounded.*

Proof. To prove the uniqueness it's enough to realize that if there is another vector field $\tilde{M}$ satisfying the conditions above, the difference $M - \tilde{M}$ is again the relatively parallel vector field. Since the vector field $M - \tilde{M}$ should be zero at s_0 and it should have the constant length, the fields M and $\tilde{M}$ must coincide.

To prove the existence we will at first have to find *some* differentiable normal vector field of length $|M_0|$ and then we will modify this field to get the relatively parallel one. Thus we look for a continuous and weakly differentiable vector field N such that $N(s) \cdot T(s) = 0$ and $|N(s)| = |M_0|$ $\forall s \in I$. Without loss of generality we will assume $|M_0| = 1$. Hence we have two equations for the three components of N:

$$N_1T_1 + N_2T_2 + N_3T_3 = 0 \tag{A.5}$$

$$N_1^2 + N_2^2 + N_3^2 = 1. \tag{A.6}$$

Since we assume $|T(s)| = 1$, there must always be one nonzero component of T, let us assume that for some subinterval $J \subset I$, $s_0 \in J$, it holds $T_1 \geq c > 0$. Then we can write

$$N_1 = -\frac{N_2T_2 + N_3T_3}{T_1}.$$

Since we have still some freedom in choice of the vector N, we pose $N_2 = 0$ which yields

$$N_1 = \frac{-T_3}{\sqrt{T_1^2 + T_3^2}}$$
$$N_3 = \frac{T_1}{\sqrt{T_1^2 + T_3^2}}.$$

Thus we have the components of N expressed in terms of the components of T, the components N_i are all bounded ($|N_1| \leq \frac{1}{c}$, $|N_3| \leq \frac{1}{c}$), and they are continuous because of the continuity of T. Also the weak derivative of N_i exists, and we can compute

$$\left|\dot{N}_1\right| = \left|\frac{T_1(T_3\dot{T}_1 - T_1\dot{T}_3)}{(T_1^2 + T_3^2)^{3/2}}\right| \leq \frac{|\dot{T}_1| + |\dot{T}_3|}{c} = \frac{|\ddot{\Gamma}^3| + |\ddot{\Gamma}^1|}{c} \tag{A.7}$$

$$\left|\dot{N}_3\right| = \left|\frac{T_3(T_3\dot{T}_1 - T_1\dot{T}_3)}{(T_1^2 + T_3^2)^{3/2}}\right| \leq \frac{|\dot{T}_1| + |\dot{T}_3|}{c} = \frac{|\ddot{\Gamma}^1| + |\ddot{\Gamma}^3|}{c} \tag{A.8}$$

and from $\Gamma^i \in W^{2,\infty}_{\text{loc}}(I)$ it follows that these quantities have an upper bound on the interval J. The same properties as N will have the vector field $\tilde{N} := T \times N$ since its components are just products of the components of T and N.

Then the triplet $\{T, N, \tilde{N}\}$ is the adapted frame and similarly as we derived the fact that the Cartan matrix $\mathbb{M}$ is antisymmetric we get that for the derivatives of N and $\tilde{N}$ that there exist functions of s p_{01}, p_{02} and p_{12} such that

$$\begin{aligned}
\dot{T} &= p_{01}N + p_{02}\tilde{N}, \\
\dot{N} &= -p_{01}T + p_{12}\tilde{N}, \\
\dot{\tilde{N}} &= -p_{02}T - p_{12}N.
\end{aligned}$$

Now we can construct the vector field M

$$M := N\cos\vartheta + \tilde{N}\sin\vartheta \tag{A.9}$$

where $\vartheta(s)$ is a differentiable (in the weak sense) angle function which will be chosen in such way that M is relatively parallel. Differentiating this relation we get

$$\dot{M} = (\dot{\vartheta} + p_{12})\left(-\sin\vartheta N + \cos\vartheta\tilde{N}\right) - (p_{01}\cos\vartheta + p_{02}\sin\vartheta)\, T$$

thus the vector field M is relatively parallel if and only if

$$\dot{\vartheta}(s) = -p_{12}(s) \qquad \forall s \in I. \tag{A.10}$$

We derived above that $\dot{N}$ (which we mean in the weak sense) is locally bounded, thus the function p_{12} (defined in the weak sense) is also locally bounded. Hence the function

$$\vartheta(s) := -\int_{s_0}^{s} p_{12}(\xi)d\xi + \vartheta(s_0)$$

is well defined and satisfies (A.10). Furthermore if we set $\vartheta(s_0)$ such that

$$M_0 = N(s_0)\cos\vartheta(s_0) + \tilde{N(s_0)}\sin\vartheta(s_0)$$

then M defined by (A.9) is the relatively parallel vector field we sought for.

The construction above was local, so the last step lies in finding some global vector field M. This is possible patching together the fields on individual subintervals where we construct M as above. If in s_0 the component T_1 is nonzero, we will construct $M^{(0)}$ on some neighborhood J_0 of s_0 where $T_1 \geq c$, then we find $M^{(0)}(s_1)$ for $s_1 \in \partial_J$, here another component of T is nonzero and we can continue with the construction of the vector field $M^{(1)}$ on some neighborhood J_1 of s_1 satisfying $M^{(1)}(s_1) = M^{(0)}(s_1)$. Patching together $M^{(i)}(s)$ we get the field M on the whole I. $\square$

This proposition allows us to introduce the *relatively parallel adapted frame* (RPAF), i.e. the frame where the vectors fields are relatively parallel. The tangential vector field is said to be relatively parallel if it is the constant multiple of the unit tangent field T.

Corollary A.7. *Let $\Gamma : I \to \mathbb{R}^3$ be a unit-speed spatial curve with $\Gamma^i \in W^{2,\infty}_{\mathrm{loc}}(I)$ for $i = 1, 2, 3$ and let M_1^0 and M_2^0 be two normal vectors in the point $\Gamma(s_0)$ such that they form together with the tangent vector $T(s_0)$ the orthonormal basis of the tangent space $\mathbb{T}_{\Gamma(s_0)}\mathbb{R}^3$. Then there exists a unique relatively parallel adapted frame $\{T, M_1, M_2\}$, such that $M_1(s_0) = M_1^0$ and $M_2(s_0) = M_2^0$. The vector fields in this frame change continuously with s and their weak derivative exists and is locally bounded.*

Proof. The existence and uniqueness of the unit vector fields M_1 and M_2 follows from the Proposition A.6 and also clearly $T(s) = \dot{\Gamma}(s)$ fulfils the statement of the corollary. The regularity properties of the vector fields T, M_1 and M_2 were found in the proof of the Proposition A.6, too. The only thing to check is then if the vectors $M_1(s)$ and $M_2(s)$ remain perpendicular. However, knowing that there exist the functions f, g such that $\dot{M}_1(s) = f(s)T(s)$ and $\dot{M}_2(s) = g(s)T(s)$, we can differentiate the scalar product (M_1, M_2) getting $(fT, M_2) + (M_1, gT) = 0$. Thus the angle between M_1 and M_2 remains the same and this holds generally for every pair of two relatively parallel vector fields. □

Remark A.8.

(i) If there is some preferred orientation in R^3, then also the normal space of the curve has some preferred orientation and we may refer to a properly oriented RPAF.

(ii) It is clear that we can get a unique RPAF for all the initial normal vectors in form $M_1^\alpha = \cos\alpha M_1^0 + \sin\alpha M_2^0$, $M_2^\alpha = -\sin\alpha M_1^0 + \cos\alpha M_2^0$. Therefore for a given curve there exists whole one-parametric set of RPAF's consisting of frames $\{T, \cos\alpha M_1 + \sin\alpha M_2, -\sin\alpha M_1 + \cos\alpha M_2\}$ where $\alpha \in [0, 2\pi)$ is a constant.

Let $\{T, M_1, M_2\}$ be a RPAF. Let us have a look on the derivatives of the vector fields in this frame. We already know that there must exist functions $k_1(s)$ and $k_2(s)$ such that $\dot{M}_1(s) = -k_1(s)T(s)$ and $\dot{M}_2(s) = -k_2(s)T(s)$. Since again the Cartan matrix should be antisymmetric, we get

$$\begin{pmatrix} \dot{T} \\ \dot{M}_1 \\ \dot{M}_2 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ M_1 \\ M_2 \end{pmatrix} \tag{A.11}$$

Recalling remark A.8, point (*ii*), certainly the Cartan matrix for a given curve is not unique, however, all the matrices can be written in the form

$$\begin{pmatrix} 0 & k_1\cos\alpha + k_2\sin\alpha & -k_1\sin\alpha + k_2\cos\alpha \\ -k_1\cos\alpha - k_2\sin\alpha & 0 & 0 \\ k_1\sin\alpha - k_2\cos\alpha & 0 & 0 \end{pmatrix} \tag{A.12}$$

where $\alpha \in [0, 2\pi)$ is constant. Thus for a given curve, the vector $(k_1(s), k_2(s)) \in \mathbb{R}^2$ can differ only by an orthogonal transformation, if only properly oriented RPAFs are considered, just the transformations with matrix with determinant $+1$ can occur.

Let us now find the connection between the functions k_1, k_2 (from the relation (A.11)) and κ, τ, in case the Frenet frame exists. We can easily find that

$$\kappa = \left|\ddot{\Gamma}(s)\right| = \left|\dot{T}(s)\right| = |k_1 M_1 + k_2 M_2| = \sqrt{k_1^2 + k_2^2}.$$

Let us note that since $\Gamma^i \in W^{2,\infty}_{\mathrm{loc}}(I)$ for $i = 1, 2, 3$, $\left|\ddot{\Gamma}(s)\right|$ is locally bounded and the same then holds for the functions k_1, k_2. We will assign $\kappa = \left|\ddot{\Gamma}(s)\right|$ even if the Frenet frame does not exist, the important feature of κ is that it is not dependent on choice of the RPAF.

The Frenet's e_2 was defined by (A.1) and (A.2). In case when the curve is parameterized by arc length, we have $\dot{\Gamma} \cdot \ddot{\Gamma} = \frac{1}{2}\left(\left(\dot{\Gamma}\right)^2\right)^{\bullet} = 0$, hence

$$e_2 = \frac{\ddot{\Gamma}}{|\ddot{\Gamma}|} = \frac{k_1}{\kappa}M_1 + \frac{k_2}{\kappa}M_2 =: \cos\beta M_1 + \sin\beta M_2$$

where we denoted $\beta(s) = \arctan\frac{k_2(s)}{k_1(s)}$ (or equivalently $\beta(s) = \arcsin\frac{k_2(s)}{\kappa(s)}$ if $k_1(s) = 0$). Differentiating this relation we get

$$\dot{e}_2 = -\kappa T + \dot{\beta}\left(-\sin\beta M_1 + \cos\beta M_2\right).$$

If our RPAF is properly oriented, then Frenet's $e_3 = -\sin\beta M_1 + \cos\beta M_2$ and comparing the relation above with

$$\dot{e}_2 = -\kappa e_1 + \tau e_3$$

we can identify

$$\tau(s) = \dot{\beta}(s).$$

We observe that the κ and the indefinite integral $\int \tau ds$ are the polar coordinates for the curve (k_1, k_2).

Finally let us mention some uniqueness properties of the functions k_1, k_2. We already mentioned that for a given curve, the vector (k_1, k_2) is given up to rotations in a plane, on the other hand we would like to know, if for a given curvatures, there exists a unique curve. This won't be so straightforward as in case of the Frenet's curvatures, however, the notion of *normal development* will help us solve this problem. We don't use this part of theory in our text, hence we won't generalize it on $W^{2,\infty}_{\mathrm{loc}}(I)$ curves and we will adopt following ideas straightly from [2], i.e. we will consider only the C^2 curves.

Definition A.9. *Let Γ be a C^2 unit-speed curve. Let (k_1, k_2) be a curve parameterized by the arc-length of Γ and lying in the centro-euclidian plane (i.e. the plane having a distinguished point). Then (k_1, k_2) is called the normal development of Γ.*

As we mentioned above, the orthogonal transformations of the vector $(k_1(s), k_2(s))$ in a plane describes exactly the ambiguity of the RPAF, which is expressed by following Theorem.

Theorem A.10. *Two C^2 regular curves in Euclidean space are congruent if and only if they have the same normal development. For any parameterized continuous curve in a centro-euclidian plane there is a C^2 regular curve in euclidian space having the given curve as its normal development.*

Appendix B

Selected topics from the functional analysis

B.1 Weierstrass theorem for L^p functions

In this section we cite the Theorem adopted from [1] saying that it is possible to approximate the functions $f \in L^p(a,b)$ by the polynomials and we yield some consequences of this result (which can be also partly found in [1]).

Theorem B.1. *Let $f(x) \in L^p(a,b)$, where (a,b) is a finite interval and $p \geq 1$. Then for arbitrary $\varepsilon > 0$ there exists a polynomial $P_\varepsilon(x)$ such that*

$$\|f - P_\varepsilon\|_{L^p(a,b)} \leq \varepsilon.$$

This Theorem results from the theory of Lebesque integral, in particular from the fact that the set of continuous functions is dense in L^p. Then it is possible to prove Theorem B.1 using the Weierstrass Theorem in its usual form, i.e. for the continuous functions.

Corollary B.2. *Let $f(x) \in L^\infty(\mathbb{R})$ and let (a,b) be a finite interval. Then*

$$\lim_{h\to 0} \int_a^b |f(x+h) - f(x)|^p \, dx = 0.$$

Proof. We will show that for arbitrary $\varepsilon > 0$ there exists h such that

$$\int_a^b |f(x+h) - f(x)|^p \, dx \leq \varepsilon^p. \tag{B.1}$$

It is clear that $f(x) \restriction (a-1, b+1) \in L^p(a-1, b+1)$ for all $p \geq 1$, thus as a consequence of Theorem B.1 there exists a polynomial P such that

$$\int_{a-1}^{b+1} |f(x) - P(x)|^p dx \leq \left(\frac{\varepsilon}{4}\right)^p.$$

Hence

$$\int_a^b |f(x) - P(x)|^p dx \leq \left(\frac{\varepsilon}{4}\right)^p$$

and if we assume $|h| \leq 1$ then also

$$\int_a^b |f(x+h) - P(x+h)|^p dx \leq \left(\frac{\varepsilon}{4}\right)^p.$$

Thus using the triangle inequality in $L^p(a,b)$ we get

$$\left(\int_a^b |f(x+h)-f(x)|^p\,dx\right)^{1/p} \leq$$
$$\leq \left(\int_a^b |f(x+h)-P(x+h)|^p\,dx\right)^{1/p} + \left(\int_a^b |P(x+h)-P(x)|^p\,dx\right)^{1/p} +$$
$$+ \left(\int_a^b |P(x)-f(x)|^p\,dx\right)^{1/p} \leq$$
$$\leq \frac{\varepsilon}{4} + \left(\int_a^b |P(x+h)-P(x)|^p\,dx\right)^{1/p} + \frac{\varepsilon}{4}.$$

Now we use that every polynomial is a uniformly continuous function on $[a-1,b+1]$, thus for every $\varepsilon>0$ there exists h such that

$$|P(x+h)-P(x)| \leq \frac{\varepsilon}{2(b-a)^{1/p}} \qquad \forall x \in [a,b].$$

Then

$$\left(\int_a^b |P(x+h)-P(x)|^p\,dx\right)^{1/p} \leq \frac{\varepsilon}{2}$$

and

$$\left(\int_a^b |f(x+h)-f(x)|^p\,dx\right)^{1/p} \leq \varepsilon$$

which proves (B.1). □

Corollary B.3. *Let $f(x) \in L^p(\mathbb{R})$. Then*

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h)-f(x)|^p\,dx = 0.$$

Proof. We will prove that for all $\varepsilon>0$ there exists h such that

$$\int_{\mathbb{R}} |f(x+h)-f(x)|^p\,dx \leq \varepsilon.$$

Since $f(x) \in L^p(\mathbb{R})$, there exists $N>0$ such that

$$\int_{-\infty}^{-N} |f(x+h)-f(x)|^p\,dx + \int_N^{\infty} |f(x+h)-f(x)|^p\,dx \leq \frac{\varepsilon}{2}.$$

Then since the interval $(-N,N)$ is finite, we can use the statement from the proof of previous corollary saying that there exists h such that

$$\int_{-N}^{N} |f(x+h)-f(x)|^p\,dx \leq \frac{\varepsilon}{2}.$$

Combining these to estimates we get

$$\int_{\mathbb{R}} |f(x+h)-f(x)|^p\,dx \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which completes the proof. □

B.2 Generalized Minkowski inequality

Here we will mention an inequality adopted from [1], which we also use in the text. Let (a, b) and (c, d) be intervals and $p \geq 1$. Then

$$\left(\int_a^b \left| \int_c^d |f(x,y) dy \right|^p dx \right)^{1/p} \geq \int_c^d \left(\int_a^b |f(x,y)|^p dx \right)^{1/p} dy \tag{B.2}$$

if $f(x, y)$ as a function of x lies in $L^p(a, b)$ and $\left(\int_a^b |f(x,y)|^p dx \right)^{1/p}$ is an integrable function. Let us note that this relation for $p = 1$ is the consequence of the Fubini Theorem.

B.3 The Norm Resolvent Convergence and its consequences

In this section we give a brief insight into the consequences of the fact that the sequence of operators T_n converges to an operator T with respect to norm resolvent convergence. By $\mathcal{C}(\mathcal{H})$ we assign the set of closed operator on Hilbert space $\mathcal{H}$, $\rho(T)$ denotes the resolvent set of the operator T.

Definition B.4. *Let $\{T_n\}_{n=1}^{\infty} \subset \mathcal{C}(\mathcal{H})$ be a sequence of operators and let $T \in \mathcal{C}(\mathcal{H})$ be an operator satisfying*

$$\lim_{n \to \infty} \|(T_n - \lambda)^{-1} - (T - \lambda)^{-1}\|_{\mathcal{B}(\mathcal{H})} = 0$$

for some $\lambda \in \rho(T)$. Then we say that T_n converges to T with respect to the norm resolvent convergence.

Sometimes (e.g. in [5]) the norm resolvent convergence is defined for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ or even $\lambda = -i$. However, the following theorem (Theorem IV.2.25 in [17]) shows that these definitions are equivalent. Originally, this theorem refers to so called *generalized convergence*, which is in [17] defined using the notion of gap between closed operators. However, the notions of generalized and norm resolvent convergence coincide and we won't give the original definition of the first of them here.

Theorem B.5. *Let $T \in \mathcal{C}(\mathcal{H})$ have a non-empty resolvent set $\rho(T)$. In order that a sequence $T_n \in \mathcal{C}(\mathcal{H})$ converge to T with respect to norm resolvent convergence, it is necessary that each $\lambda \in \rho(T)$ belong to $\rho(T_n)$ for sufficiently large n and*

$$\|(T_n - \lambda)^{-1} - (T - \lambda)^{-1}\|_{\mathcal{B}(\mathcal{H})}$$

while it is sufficient that this be true for some $\lambda \in \rho(T)$.

In [17], there can be found number of consequences of the norm resolvent convergence for the convergence of spectrum of operators T_n. We mention here as an example the results of Section IV.3.5. on the finite system of eigenvalues (i.e. a set of finite number of points of spectrum $\sigma(T)$ that are eigenvalues with finite multiplicity). Briefly said, it is claimed there that the finite system of eigenvalues changes with T continuously in the sense that the change of these eigenvalues is small when T is subject to small perturbation in the sense of norm resolvent convergence.

Let T_n be the sequence of closed operators that converge to T with respect to norm resolvent convergence. The important point in proof of the statement above is that the finite system of eigenvalues $\sigma'(T)$ can be separated from the rest of the spectrum $\sigma''(T)$ by a closed curve Γ in complex plane that lies in the resolvent set of T. According to Theorem IV.3.6 in [17], then also the spectrum of the operators T_n is for large enough n separated by Γ into two parts $\sigma'(T_n)$ and

$\sigma''(T_n)$ where in $\sigma'(T_n)$ there are again only eigenvalues and their total multiplicity is the same as total multiplicity of eigenvalues in $\sigma'(T)$. If we realize, that we can place into $\sigma'(T)$ right one eigenvalue λ of T, we find that in arbitrary small neighborhood of λ, there must be for large enough n, also the eigenvalue λ_n of operators T_n with the same multiplicity.

Finally, let us note that we can think of the relation $\lim_{n\to\infty} \|(T_n+\lambda)^{-1} - (T+\lambda)^{-1}\|_{\mathcal{B}(\mathcal{H})} = 0$ for self-adjoint operators T_n and T and for some $\lambda \in \mathbb{C} \setminus (\sigma(T_n) \cup \sigma(T))$ also in another way. We can say that the sequence of bounded operators $(T_n + \lambda)^{-1}$ converges in norm to the operator $(T+\lambda)^{-1}$, hence the spectrum of $(T_n+\lambda)^{-1}$ converges to the spectrum of $(T+\lambda)^{-1}$. Then we can use one consequence of spectral mapping Theorem which is formulated in [17]. Let us note that by $\tilde{\sigma}(T)$ we assign the extended spectrum of T where the point $\lambda = \infty$ is added if the operator is unbounded.

Theorem B.6. *Let T be a closed invertible operator in $\mathcal{H}$. $\tilde{\sigma}(T)$ and $\tilde{\sigma}(T^{-1})$ are mapped onto each other by the mapping $\lambda \to \lambda^{-1}$ of the extended complex plane.*

This approach is convenient in our case, since in fact, we prove the relation

$$\lim_{n\to\infty} \|U(T_n+\lambda)^{-1}U^{-1} - (T+\lambda)^{-1}\|_{\mathcal{B}(\mathcal{H})} = 0$$

where U is some unitary transformation. The operator $\big(U(T_n+\lambda)^{-1}U^{-1}\big)^{-1} = U(T_n+\lambda)U^{-1}$ does not have good sense in our case, hence the consequences of the norm resolvent convergence can not be in fact used in the original form, however, the spectrum of $U(T_n+\lambda)^{-1}U^{-1}$ is the same as the spectrum of $(T_n+\lambda)^{-1}$ and we can use the ideas above.

B.4 Quadratic forms

Definition B.7. *Let H be a non-negative self-adjoint operator acting on Hilbert space $\mathcal{H}$. For $\phi, \psi \in \mathrm{Dom}\,(H^{1/2}) =: \mathcal{D} \subset \mathcal{H}$ we define sesquilinear form $Q : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$:*

$$Q(\phi, \psi) := \Big(H^{1/2}\phi, H^{1/2}\psi\Big)_{\mathcal{H}}$$

and the quadratic form $Q : \mathcal{D} \to [0, +\infty)$ associated with Q':

$$Q[\psi] := Q(\psi, \psi).$$

It might seem confusing that we assign the sesquilinear and the quadratic form by the same letter, however, it is always clear from the number of arguments and the shape of the parenthesis which of these two is intended.

In the following lemma, we will introduce the term of closed quadratic form (for proof see Theorem 4.4.2 in [5]).

Lemma B.8. *The following conditions are equivalent:*

(i) Q is the form arising form a non-negative self-adjoint operator H.

(ii) The domain $\mathcal{D}$ of Q is complete for the norm defined by

$$\|f\|_Q := (Q[f] + \|f\|^2)^{1/2}. \tag{B.3}$$

Definition B.9. *We say that the quadratic form fulfilling the conditions stated in Lemma B.8 is closed.*

A form Q_2 is said to be an extension of Q_1 if it has a larger domain but coincides with Q_1 on the domain of Q_1. A form Q is said to be closable of it has a closed extension, the smallest closed extension is called its closure $\bar{Q}$.

Finally let us introduce the notion of the *core* of the closed sesquilinear form Q. Let Q' be a sesquilinear form for that $\overline{Q'} = Q$. Then the linear submanifold $\operatorname{Dom} Q'$ of $\operatorname{Dom} Q$ is called the core of Q.

The connection between the quadratic form and the associated operator is expressed by the representation theorem that we adopt from [17]:

Theorem B.10. *Let $Q(\phi, \psi)$ be a densely defined, closed sesquilinear form in $\mathcal{H}$. Then there exists an operator H such that*

(i) $\operatorname{Dom} H \subset \operatorname{Dom} Q$ *and for every* $\phi \in \operatorname{Dom} H$, $\psi \in \operatorname{Dom} Q$ *it holds*

$$Q(\phi, \psi) = (H\phi, \psi)_{\mathcal{H}},$$

(ii) $\operatorname{Dom} H$ *is a core of* Q,

(iii) if $\varphi \in \operatorname{Dom} Q$, $\mu \in \mathcal{H}$ *and*

$$Q(\phi, \psi) = (\varphi, \psi)_{\mathcal{H}}$$

holds for $\psi \in \operatorname{Dom} Q$, *then* $\phi \in \operatorname{Dom} H$ *and* $H\phi = \varphi$. *The operator H is uniquely determined by the condition* (i).

B.5 Dirichlet boundary conditions and Sobolev spaces

In this text, we work with differential operators with specified boundary conditions. This restricts the domain of the operator and the domain of the associated quadratic form as well. In this section, we introduce in few steps the Sobolev space $W_0^{1,2}(\Omega)$ and we show, that it is suitable domain for the operators considered in this work. The details can be found in Section 6.1 of [5].

The operators in the form

$$Hf := -b(x)^{-1} \frac{\partial}{\partial x^i} \left(a^{ij}(x) \frac{\partial f}{\partial x^j} \right). \tag{B.4}$$

acting on $L^2(\Omega, b(x)d^N x)$ will be studied. We will present here the most general result from [5], where no smoothness conditions on coefficients $a_{i,j}(x)$ an $b(x)$ are required and we also assume that Ω is any open connected subset of $\mathbb{R}^N$. Namely, we assume that $a(x) := \{a_{i,j}(x)\}$ is a real symmetric matrix depending measurably upon the variable $x \in \Omega$ and that the matrices $a(x)$ are uniformly positive and bounded in the sense that there exists a constant $c \geq 1$ such that

$$c^{-1} E \leq a(x) \leq cE \qquad \forall x \in \Omega \tag{B.5}$$

in the sense of matrices (E is the unit matrix). In addition, we suppose that $b(x)$ is a positive (thus real) measurable function on Ω satisfying

$$c^{-1} \leq b(x) \leq c \qquad \forall x \in \Omega. \tag{B.6}$$

Usually, the construction is started with the initial domain $C_0^\infty(\bar{\Omega})$ which is the space of smooth functions on Ω all of whose partial derivatives can be extended continuously to $\bar{\Omega}$ and which fulfil the Dirichlet boundary conditions $\psi(x) = 0$ for $x \in \partial\Omega$. However, in case of non-smooth coefficients, it might happen that even $C_0^\infty(\bar{\Omega}) \nsubseteq \operatorname{Dom} H$, hence we will skip to a more general construction. In fact, in case we don't assume the differentiability of $a(x)$ and $b(x)$, the operator (B.4) is no more well defined in the classical sense and we will have to work with the quadratic forms only. The quadratic form associated with (B.4) reads

$$Q(f,g) := \int_\Omega a^{ij}(x) \frac{\partial \bar{f}}{\partial x^i} \frac{\partial g}{\partial x^j} d^N x. \tag{B.7}$$

The initial domain of such quadratic form is $C_c^\infty(\Omega)$, the space of smooth functions with compact supports contained in Ω. On this domain the operator (understood in the weak sense) is symmetric and the Dirichlet boundary conditions in the original sense are fulfilled. Since the matrices $a(x)$ are positive, also the form (B.7) is positive and together with the symmetricity of H, we get that this form is closable. To find the closure of this quadratic form, we will have to introduce the notion of weak derivative and the Sobolev spaces.

At first we define the distribution as a linear functional ϕ: $C_c^\infty(\Omega) \to \mathbb{C}$. If g is a function on Ω which is integrable when restricted to every compact subset of Ω, then g determines a distribution ϕ_g by means of the formula

$$\phi_g(f) := \int_\Omega f(x) g(x) d^N x$$

If α is any multi-index, the weak derivative $D^\alpha \phi$ of the distribution ϕ is defined by

$$(D^\alpha \phi)(f) := (-1)^{|\alpha|} \phi(D^\alpha f).$$

If h is a smooth function on Ω, then we define the product $h\phi$ to be the distribution $(h\phi)(f) := \phi(hf)$. Now we can define the Sobolev space $W^{1,2}(\Omega)$ (the definition of Sobolev spaces is more general, but in this text this special case is sufficient).

Definition B.11. *Let Ω be an open connected subset of $\mathbb{R}^N$ and $f \in L^2(\Omega)$. We say that f lies in the Sobolev space $W^{1,2}(\mathbb{R}^N)$, if the weak partial derivatives $\partial_i f := \frac{\partial f}{\partial x_i}$ lie in $L^2(\Omega)$. For these functions we define the Sobolev norm*

$$\|f\|_{1,2}^2 := \int_\Omega \left(|f|^2 + |\nabla f|^2\right) d^N x.$$

In [5] it is e.g. shown that for any choice of $\Omega \subseteq \mathbb{R}^N$, the space $W^{1,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(f,g)_{1,2} = \int_\Omega \left(\overline{f(x)} g(x) + \overline{\nabla f(x)} \nabla g(x)\right) d^N x.$$

Finally, we define the subspace $W_0^{1,2}(\Omega)$ of $W^{1,2}(\Omega)$ to be the closure of the subspace $C_c^\infty(\Omega)$ for the norm $\|\cdot\|_{1,2}$. It can be shown, that since the coefficients $a(x)$ and $b(x)$ are bounded, the norms $\|\cdot\|_{1,2}$ and $\|\cdot\|_Q$ given by (B.3) are equivalent, thus the closures with respect to the norm $\|\cdot\|_{1,2}$ and $\|\cdot\|_Q$ are equal, hence the space $W_0^{1,2}(\Omega)$ is precisely the space where the form (B.7) is closed. According to the Lemma B.8, we know that the closure $\bar{Q}$ is associated with a non-negative self-adjoint operator. All these facts are summarized in the following theorem.

Theorem B.12. *Under the conditions above stated, the quadratic form Q defined by (B.7) is closed on the domain $W_0^{1,2}(\Omega)$ in the Hilbert space $\mathcal{H} = L^2(\Omega, b(x)d^N x)$. There exists a non-negative self-adjoint operator H_D on $L^2(\Omega, b(x)d^N x)$ associated to the form, in such a way that*

$$\left(H_D^{1/2} f, H_D^{1/2} g\right)_{\mathcal{H}} = Q(f,g)$$

for all $f, g \in \operatorname{Dom}(H_D^{1/2}) = W_0^{1,2}(\Omega)$.

The operator H_D is then called the Friedrichs extension of H. The crucial point is that the domain of the form (B.7) is $W_0^{1,2}(\Omega)$ independently on the coefficients $a(x)$ and $b(x)$. Hence this construction can be used for introducing the basic Laplace operator with Dirichlet conditions where $a(x) = E$ and $b(x) = 1$, but also the Laplace-Beltrami operator which is the Laplace operator in the curvilinear coordinates, can be introduced in this way and the domain of these two operators is the same.

B.6 Neumann boundary conditions

In this section we will study an operator given by the same formula as the operator considered in the previous section, i.e. by (B.4), however the initial domain $\mathcal{D}$ of this operator will be the set of functions $f \in C^\infty(\bar{\Omega})$ satisfying

$$a^{ij}(x)\frac{\partial f}{\partial x^j}n_i(x) = 0 \qquad \forall x \in \partial\Omega \tag{B.8}$$

where $n(x)$ is the unit normal vector in the point $x \in \partial\Omega$. If we assume that the matrices $a(x)$ are symmetric and satisfy (B.5), the operator H is symmetric on $\mathcal{D}$ and the associated quadratic form is positive on this domain, hence closable. Let us note that if $a_{i,j}(x) = \alpha(x)\delta_{i,j}$, then the condition (B.8) says that the normal derivatives of f vanish on the boundary, which is the usual way how the Neumann boundary conditions are introduced. We will again cite the most general results from [5], hence we again assume only that the matrices $a(x)$ are measurable, for $b(x)$ we assume (B.6) and that it is also a measurable function.

If the boundary of Ω is smooth, then (as it is stated in [5]) the associated form (B.7) with the initial domain $\mathcal{D}$ is closable and its closure is defined on the Sobolev space $W^{1,2}(\Omega)$ that we introduced in previous section. However, for arbitrary Ω, the quadratic form (B.7) is closed on $W^{1,2}$ (this follows straightly from the equivalence of norms $\|\cdot\|_{W^{1,2}}$ and $\|\cdot\|_Q$) and the associated self-adjoint operator that we denote by H_N is said to satisfy Neumann boundary conditions, even though it is not possible to identify its domain $\mathcal{D}$ in general, and even though a normal direction may not be definable at any point of the boundary $\partial\Omega$.

For another result to hold, we have to assume that Ω satisfies so called *extension property.*

Definition B.13. *We say that the bounded open connected subset Ω of $\mathbb{R}^N$ has the extension property if there exists a bounded linear extension operator E: $W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^N)$ such that $(Ef)(x) = f(x)$ for all $f \in W^{1,2}(\Omega)$ and all $x \in \Omega$.*

This condition holds e.g. for Ω with piecewise smooth or Lipshitz boundary (see [5]). Now we can cite the following theorem from [5].

Theorem B.14. *If Ω is a bounded region with extension property, then the Friedrichs extension H_N of the operator defined on the domain $\mathcal{D}$ by* (B.4) *has a compact resolvent. If $\{\mu_n\}_{n=1}^\infty$ (resp. $\{\lambda_n\}_{n=1}^\infty$) are the eigenvalues of H_N (resp. the operator H_D satisfying Dirichlet boundary conditions), then*

$$0 \le \mu_{n-1} \le \lambda_n$$

for all $n > 1$.

Finally, let us note that in the similar way we can introduce also an operator with mixed Dirichlet and Neumann conditions. Let Ω be a bounded region with the extension property and let S be a closed subset of $\partial\Omega$. Then we can introduce an operator H_S given by (B.4) with Dirichlet boundary conditions on S and the Neumann one on $\partial\Omega \setminus S$. It is again sufficient to assume the coefficients $a(x)$, $b(x)$ to be measurable. As the domain of this operator we take the closure in $W^{1,2}(\Omega)$ of the set of all smooth functions on $\bar{\Omega}$ which vanish in a neighborhood of S. It again holds that the operator H_S has then the compact resolvent (see [5]).

B.7 The projection theorem

Since we use the Hilbert space decomposition in the proof of norm resolvent convergence we recall here the projection theorem. Notice that for the subset M of the Hilbert space $\mathcal{H}$ we assign $M^\perp$ the set of all vectors in $\mathcal{H}$ that are orthogonal to all vectors of M.

Theorem B.15. *Let $\mathcal{G}$ be a closed subspace in the Hilbert space $\mathcal{H}$. Then for all $x \in \mathcal{H}$ there exist unique vectors $y \in \mathcal{G}$ and $z \in \mathcal{G}^\perp$ such that $x = y + z$.*

B.8 The min-max principle

When estimating the eigenvalues of self-adjoint operators, the variational formulae or the so called min-max principle is very useful. Following theorems can be found e.g. in [5] and are proved therein.

Let H be a non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$ and let L be any finite-dimensional subspace of the domain of H. We define

$$\lambda(L) := \sup\{(Hf, f) : f \in L, \|f\| = 1\} \tag{B.9}$$

We'll use these numbers to define a non-decreasing sequence of numbers λ_n:

$$\lambda_n := \inf\{\lambda(L) : L \subseteq \mathrm{Dom}\,(H), \dim\,(L) = n\}. \tag{B.10}$$

Theorem B.16. *Let H be a non-negative self-adjoint operator on $\mathcal{H}$, and let λ_n be defined by (B.10). If H has non-empty essential spectrum then one of the following cases occurs.*

(1) There exists $a < \infty$ such that $\lambda_m < a$ for all m and $\lim_{m\to\infty} \lambda_m = a$. Then a is the smallest number in the essential spectrum, and the part of the spectrum of H in $[0, a)$ consists of the eigenvalues λ_m each repeated a number of times equal to its multiplicity.

(2) There exists $a < \infty$ and $N < \infty$ such that $\lambda_N < a$ but $\lambda_m = a$ for all $m > N$. Then a is the smallest number in the essential spectrum, and the part of the spectrum of H in $[0, a)$ consists of the eigenvalues $\lambda_1, ..., \lambda_N$ each repeated a number of times equal to its multiplicity.

When working with quadratic forms, some alternative to the definition (B.10) is needed. Let Q be a closed quadratic form and let $\mathcal{D}$ be a core for Q, that is a subspace of $\mathrm{Dom}\,(H^{1/2})$ which is dense in it for the norm $\|\cdot\|_Q = \left(Q[f] + \|f\|^2\right)^{1/2}$. If L is a finite-dimensional subspace of $\mathrm{Dom}\,(H^{1/2})$, then we define the modified $\lambda(L)$ as:

$$\tilde{\lambda}(L) := \sup\{Q[f] : f \in L, \|f\| = 1\}.$$

Following theorem shows the equivalent definition of λ_n and is of great importance, since it enables us to compare two operators with different domains if the domains of the associated quadratic forms are identical (which is the case e.g. for wide range of elliptic operators with Dirichlet boundary conditions).

Theorem B.17. *If we put*

$$\lambda'_n := \inf\{\tilde{\lambda}(L) : L \subseteq \mathcal{D}, \dim(L) = n\} \tag{B.11}$$

$$\lambda''_n := \inf\{\tilde{\lambda}(L) : L \subseteq \mathrm{Dom}\,(H^{1/2}), \dim(L) = n\}, \tag{B.12}$$

then $\lambda_n = \lambda'_n = \lambda''_n$ *for all* $n \geq 1$.

In the following we will in most cases use the definition of λ_n with help of $\tilde{\lambda}(L)$, i.e. using the quadratic forms associated to a self-adjoint operator H. Sometimes we will point out the operator H for that λ_n is computed, by assigning $\lambda_n(H)$.

Remark B.18. *The simple consequence of these theorems is e.g. the fact that if* $\mathrm{Dom}\,Q_1 \subset \mathrm{Dom}\,Q_2$ *and the action of the two forms is identical then, since in definition of* λ_n *we make the infimum over smaller set, it holds for the associated operators* H_1 *and* H_2, $\lambda_n(H_1) \leq \lambda_n(H_2)$.

B.8.1 The Dirichlet-Neumann Bracketing

In this section we will present some results of [25] that will help us to get some estimates on the spectrum of the Dirichlet Laplacian when the domain (the strip or the tube) we are working on is divided into more parts. The Dirichlet Laplacian on a domain Ω will be assigned $-\Delta_D^\Omega$ and we introduce it in the way we described in Section B.5. Similarly $-\Delta_N^\Omega$ assigns the Neumann Laplacian, which is also introduced as the Friedrichs extension as is described in Section B.6. We will usually work with the associate quadratic forms instead of these operators, let us recall that for the (closed) quadratic form associated to the Dirichlet Laplacian we have $\mathrm{Dom}\,Q_D^\Omega = W_0^{1,2}(\Omega)$ and similarly $\mathrm{Dom}\,Q_N^\Omega = W^{1,2}(\Omega)$ for the Neumann Laplacian.

The following formalism will be used.

Definition B.19. *Let* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. *Let* A_1 *resp.* A_2 *be a self-adjoint operator on* $\mathcal{H}_1$, *resp.* $\mathcal{H}_2$. *Then we define the direct sum* $A = A_1 \oplus A_2$ *with domain* $\mathrm{Dom}\,A = \{(\phi,\psi) | \phi \in \mathrm{Dom}\,A_1, \psi \in \mathrm{Dom}\,A_2\}$ *and with the action* $A(\phi,\psi) = (A_1\phi, A_2\psi)$.

It is easy to prove that A is then also a self-adjoint operator. It also follows from this definition that for the associated quadratic forms we get $\mathrm{Dom}\,Q_A = \mathrm{Dom}\,Q_{A_1} \oplus \mathrm{Dom}\,Q_{A_2}$.

Proposition B.20. *Let* Ω_1 *and* Ω_2 *be disjoint open sets such that* $L^2(\Omega) = L^2(\Omega_1) \oplus L^2(\Omega_2)$. *Under this decomposition*

$$-\Delta_D^{\Omega_1 \cup \Omega_2} = -\Delta_D^{\Omega_1} \oplus -\Delta_D^{\Omega_1}$$
$$-\Delta_N^{\Omega_1 \cup \Omega_2} = -\Delta_N^{\Omega_1} \oplus -\Delta_N^{\Omega_1}.$$

The proof of this proposition can be found in [25] and it is very easy due to the fact that Ω_1 and Ω_2 are disjoint.

Next we define the order in the set of non-negative self-adjoint operators:

Definition B.21. *Let* A *and* B *be self-adjoint non-negative operators with associated quadratic forms* Q_A *and* Q_B. *We write* $A \leq B$ *if and only if*

(i) $\mathrm{Dom}\,Q_B \subset \mathrm{Dom}\,Q_A$

(ii) For any $\psi \in \mathrm{Dom}\,Q_B$

$$0 \leq Q_A[\psi] \leq Q_B[\psi].$$

This definition is convenient since:

Lemma B.22. *If $0 \leq A \leq B$ then $\lambda_n(A) \leq \lambda_n(B)$ for all n where λ_n is given by* (B.10).

This lemma follows straightly from Theorem B.17 which gives the equivalent definition of λ_n using the quadratic forms. Now we will state two propositions that both compare two Laplacians on different domains.

Proposition B.23. *If $\Omega \subset \Omega'$ then*

$$0 \leq -\Delta_D^{\Omega'} \leq -\Delta_D^{\Omega}.$$

Proof. The first step consists of showing that $W_0^{1,2}(\Omega) \subset W_0^{1,2}(\Omega')$. This will hold true if we adopt that $L^2(\Omega) \subset L^2(\Omega')$ in the sense that the function from $L^2(\Omega)$ can be extended by zero on $\Omega' \setminus \Omega$ and such extension then lies in $L^2(\Omega')$. Hence in this sense also $C_0^\infty(\Omega) \subset C_0^\infty(\Omega')$ and the same will hold for the closures. Finally for $\psi \in W_0^{1,2}(\Omega)$, $Q_D^{\Omega}[\psi] = Q_D^{\Omega'}[\psi]$ where on the right-hand side we mean the prolonged function ψ. □

Before we state the second proposition let us note that by M^{int} we mean the interior of the set M and by $\overline{M}$ is assigned the closure of M.

Proposition B.24. *Let Ω_1 and Ω_2 be disjoint open subsets of an open set Ω so that $\overline{(\Omega_1 \cup \Omega_2)}^{\text{int}} = \Omega$ and $\Omega \setminus (\Omega_1 \cup \Omega_2)$ has zero measure (see Figure B.1). Then*

$$0 \leq -\Delta_N^{\Omega_1 \cup \Omega_2} \leq -\Delta_N^{\Omega}.$$

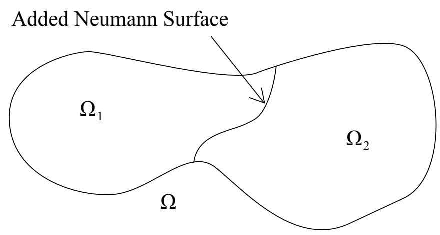

Figure B.1: The Neumann bracketing.

Proof. It is clear that if $\psi \in W^{1,2}(\Omega)$ then the restriction of ψ to $\Omega_1 \cup \Omega_2$ lies in $W^{1,2}(\Omega_1) \oplus W^{1,2}(\Omega_2)$. In proving the point (*ii*) of the definition B.21 we use that $\Omega \setminus (\Omega_1 \cup \Omega_2)$ has zero measure, and therefore $\forall \psi \in W^{1,2}(\Omega)$

$$\int_\Omega |\nabla\psi|^2 dx = \int_{\Omega_1 \cup \Omega_2} |\nabla\psi|^2 dx.$$

□

Remark B.25.

(i) *The proposition would hold also in case when we start with the Dirichlet Laplacian on Ω and then we pose the extra Neumann condition on the surface dividing Ω_1 and Ω_2. To prove this the argumentation would be very similar to the proof above.*

(ii) In the proposition above we performed so called Neumann bracketing, the Dirichlet bracketing refers to the case where we add the extra Dirichlet boundary. We don't use the Dirichlet bracketing in our text, however, let's note that the Dirichlet bracketing has the opposite effect than the Neumann one, i.e. it holds $-\Delta_D^{\Omega} \leq -\Delta_D^{\Omega_1 \cup \Omega_2}$ *which is in fact the corollary of the Proposition B.23.*

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