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Faculty of Nuclear Sciences and Physical Engineering

Differential and difference equations invariant with respect to given solvable Lie groups

Diploma thesis

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V Praze, dne 6. 5. 2011 Dalibor Karásek

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Název práce: **Diferenciální a diferenční rovnice invariantní vzhledem k daným řešitelným Lieovým grupám**
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Obor: Matematické inženýrství
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Abstrakt:

Nalezneme všechny realizace vektorovými poli na $\mathbb{R}^2$ a $\mathbb{R}^3$ jistých sérií nilpotentních Lieových algeber a jejich řešitelných rozšířeních. Znalost těchto realizací využijeme k nalezení diferenciálních rovnic, které mají tyto realizace jako algebru infinitesimálních symetrií. Současně zkonstruujeme operátory invariantní derivace pro tyto realizace a diskutujeme možnost řešení těchto rovnic v kvadraturách za využití známých symmetrií.

Klíčová slova: Lieovy algebry, symetrie, prolongace, operátor invariantní derivace, diferenční rovnice, diferenční schema,

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Abstract:

Certain series of nilpotent algebras and their solvable extensions are realised by vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$. All such realisations are found. Next, the obtained realisations are used to describe all ordinary differential equations whose algebras of the infinitesimal symmetries coincide with these realisations. The operators of invariant differentiations for realisations are also found. Solution of the constructed equations by quadratures using known symmetries is discussed.

Keywords: Lie algebras, symmetries, operator of invariant differentiation, prolongation, difference equations, difference schema

Contents

Introduction

Lie algebras are very useful structures appearing in many fields of modern physics. They appear as essential components of the Standard model and other gauge field theories, as an algebra of functions on any symplectic manifold with the Poisson bracket as a Lie bracket, as sets of observables in quantum mechanics or as algebras of Killing fields in general relativity and string theory.

One of the most important applications of the theory of Lie algebras are symmetries of differential equation. In fact, this is how Lie algebras were discovered. In the end of the 19$^{\text{th}}$ century Marius Sophus Lie published work [1] which made him famous. Like Évariste Galois who discovered groups when he was investigating algebraic equations, M.S. Lie revealed concept of Lie algebras (he called them infinitesimal transformations) when he analysed ordinary differential equations. Lie algebras got their name in the 1930s from Hermann Weyl.

Lie algebras have not been classified yet. Some partial but important results were obtained mainly by Élie Joseph Cartan [2] and Eugenio Elia Levi [3]. Cartan fully classified semisimple Lie algebras and Levi proved that any Lie algebra can be decomposed into a semidirect product of semisimple and solvable one. However the classification of the solvable algebras is a brainteaser.

Despite this, it is useful to have some algebras of sufficiently high dimension, on which we can test our conjectures. That is why Pavel Winternitz et al. have begun to construct series of solvable algebras with a given nilradical and to study their properties in [4–8]. Some secondary outcomes also appeared e.g. in [9].

This diploma thesis builds on these previous works and returns to the origin of Lie algebras — differential equations. When a differential equation is given, we can determine its symmetries in an algorithmic manner. But the converse is also possible. It is theoretically feasible to construct all differential equations with given algebra of infinitesimal symmetries. For this task it is important to classify all vector field realisations of the given Lie algebra. Once we have all such differential equations we can use the symmetries to lower their order. Interesting question is also whether such an equation can be Euler-Lagrange equation for some Lagrangian.

Another intriguing area are difference equations. New numeric methods of solving differential equations have been recently suggested. These methods respect symmetries of the equations (opposed to e.g. Runge-Kutta methods in which symmetries are totally neglected and typically broken in the computation). The scheme is to approximate a differential equation with a system of difference equation which fixes both "lattice" and the original equation for more detailed information look e.g. into [10] or [11]. These methods can have peculiar properties. For example they have low CPU-demand or the computed points corresponds precisely with the solution. This thesis unfortunately does not deal with this topic as was originally intended. It turned out that the task of complete classification of realisations of the given classes of algebras was more demanding than it appeared in the beginning.

However we have been able to finish and refine the classification of realisation on $\mathbb{R}^2$ and $\mathbb{R}^3$. We have been also able to find some differential equations with these realisations as the algebra of infinitesimal symmetries. Unfortunately it appears that we have not enough symmetries to solve the equations by quadratures. Moreover the systems of differential equations, which we have constructed in this manner up to now, are in a form which is easily separated and therefore trivially reduced to one ordinary differential equation.

Still, we have found and learned new interesting methods — particularly the idea of the operator of invariant differentiation is inspiring.

The thesis is structured as follows: We begin with a chapter where we introduce the theory. We define our notation and elementary facts later used for topics with which we are dealing.

The second chapter is the essential part of this work. We present the complete construction of realisations of the algebras $\mathfrak{n}_{n,1}, \mathfrak{n}_{n,2}$ and $\mathfrak{n}_{n,3}$ from [7–9] by vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$ including the realisations of their solvable extensions.

In the third chapter we use these realisations to construct some systems of differential equations with these realisations as algebras of their infinitesimal symmetries. In the end we summarise our results and suggest some interesting areas which deserve further investigation.

Chapter 1

Theoretical introduction

1.1 Lie algebras and their representations

This thesis deals with concepts of symmetries, Lie algebras and vector fields. It is therefore convenient to start with recalling some basic definitions and notation. These definitions can be found in every book about Lie algebras. Among others let us mention [12–14] or if the reader is interested in differential geometry point of view he can look in [15, 16].

Definition 1.1. Lie algebra L is a vector space equipped with a bilinear operation $[\cdot,\cdot] : L \times L \to L$ called Lie bracket (or commutator) which satisfies the following two conditions.

Antisymmetry

$$[x, y] = -[y, x]. \tag{1.1}$$

Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{1.2}$$

Due to the fact that we can multiply vectors we can introduce a new operation on subspaces.

Definition 1.2. Let I, J be subspaces of the Lie algebra L. Then

$$[I, J] := \mathrm{span}\{[x, y] | x \in I, y \in J\}. \tag{1.3}$$

With this operation we can formulate definitions of subalgebras, ideals and several series.

Definition 1.3. Let L be a Lie algebra and K its vector subspace. Then

- K is a **subalgebra** $\overset{def}{\Longleftrightarrow}$ $[K, K] \subseteq K$.
- K is a **Abelian subalgebra** $\overset{def}{\Longleftrightarrow}$ $[K, K] = \{0\}$.

- K is an **ideal** $\overset{def}{\iff} [K,L] \subseteq K$.
- **Derived series** is defined $L^{(0)} := L$, $L^{(i)} := [L^{(i-1)}, L^{(i-1)}]$.
- The algebra is **solvable** if the derived series converges to the zero vector.
- **Lower central series** is defined $L^1 := L$, $L^{i+1} := [L^i, L]$.
- The algebra is **nilpotent** if the lower central series converges to the zero vector.
- The algebra is **semisimple** if it doesn't have any non-zero Abelian ideal.

Eminent roles in every algebra are played by two types of linear maps — derivations and homomorphisms.

Definition 1.4. Let L be a Lie algebra and $A : L \to L$ a linear map. A is called:

Homomorphism if

$$A[x,y] = [Ax, Ay] \tag{1.4}$$

is satisfied for all $x, y \in L$.

Derivation if relation

$$A[x,y] = [Ax, y] + [x, Ay] \tag{1.5}$$

is satisfied for all $x, y \in L$.

Now we know the abstract definition of the Lie algebra. Can we represent a Lie algebra by some concrete objects and their relations? The answer is affirmative and the theory dealing with this topic is the theory of representations or L-modules.

Theorem 1.5. Let V be a vector space. $End(V)$ (set of all linear maps from V to V) with a bracket defined as $[A,B] := AB - BA$ forms a Lie algebra. We call this algebra $gl(V)$.

Proof. The proof is straightforward. Antisymmetry is obviously fulfilled and proof of the Jacobi identity is just a matter of direct calculation. □

We can proceed to the definition.

Definition 1.6. Let L be a Lie algebra and V a vector space. Any homomorphism $\rho : L \to gl(V)$ is **representation** of L. If ρ is injective we call this representation **faithful**.

The representations where V is an associative or Lie algebra and $\rho(L)$ is contained in the algebra of derivations of V are especially interesting for our purposes. See examples 1.7 and 1.11.

Example 1.7. Let L be a Lie algebra. Then

$$\mathrm{ad} : L \to \mathfrak{Der}(L) : x \mapsto [x, \cdot]$$

is a representation of L in the algebra $\mathfrak{Der}(L)$ of derivations of L.

Definition 1.8. Let L be a Lie algebra and $(\rho, V), (\tau, W)$ two representations of L. Representations ρ, τ are **equivalent**

$$\begin{aligned}(\rho, V) \cong (\tau, W) \overset{def}{\Longleftrightarrow} \exists A \in \mathcal{L}(V, W), L \text{ bijection}, \\ A \circ \rho(x) = \tau(x) \circ A, \quad \forall x \in L.\end{aligned} \tag{1.6}$$

1.2 Vector fields and realisations of Lie algebra

Vector field is a fundamental concept both in physics and in differential geometry. For example, in Hamiltonian mechanics vector fields and especially their flows are related to symmetries and solutions of equations of motion.

Definition 1.9. Let M be a manifold. A **vector field** X is a smooth section of the tangent bundle TM. In other words it is a map which assigns to every point m in M a tangent vector X_m in that point in a smooth way. The set of all vector fields is denoted $\mathfrak{X}(M)$.

There is a close relation between $\mathfrak{X}(M)$ and $\mathcal{F}(M)$ — the algebra of real functions on M.

Theorem 1.10. Let M be a manifold. Then $\mathfrak{X}(M)$ equals to an algebra of derivations of $\mathcal{F}(M)$.

Proof. Vector fields act on functions as differential operators of first order and therefore can be written in the form $X^i(m)\frac{\partial}{\partial x_i}$. The defining property of derivations ($X(fg) = (Xf)g + f(Xg)$) can be proven directly.

On the contrary, the opposite direction is nontrivial and advanced parts of the theory are needed. Proof can be found e.g. in [17, chap. 4]. □

Vector fields have a naturally defined Lie bracket (commutator of vector fields). They form a Lie algebra and therefore it makes sense to consider vector field-valued representations on $\mathcal{F}(M)$.

Definition 1.11. Let L be a Lie algebra and M a manifold. A faithful representation ρ of L on $\mathcal{F}(M)$ such that $\rho(L) \in \mathfrak{X}(M)$ is called a **realisation** of the Lie algebra L on M.

Example 1.12. Let $L = \text{span}\{e_1, e_2\}$ be a Lie algebra with the commutation relation

$$[e_1, e_2] = e_1.$$

Define

$$\begin{aligned}\rho(e_1) &:= \partial_x, \\ \rho(e_2) &:= x\partial_x + \partial_y.\end{aligned} \tag{1.7}$$

Then ρ is a realisation of L.

Remark 1.13. Let us investigate what equivalence means for realisation on same manifold M. Any transformation of coordinates $m' = F(m)$ generates an invertible linear transformation on functions on manifold called *pull-back*:

$$F^* : \mathcal{F}(M) \to \mathcal{F}(M) : \quad F^* f(m) := (f \circ F)(m). \tag{1.8}$$

Vector field $X = X^i(m)\partial_{x_i}$, when expressed in the new coordinates, takes form $X' = X'^i(m')\partial_{x'_i}$. It is connected with the original one via relation

$$F^* \circ X' = X \circ F^*. \tag{1.9}$$

Therefore X and X' are equivalent realisations in the sense of definition 1.8, where $A = F^*$.

1.3 Flow of vector field, invariants and method of moving frames

A vector field defines a flow on the manifold. It is the congruence of curves whose tangent vectors coincides with the given vector field.

Definition 1.14. Let M be a manifold and $X \in \mathfrak{X}(M)$. Then $\Phi^X(t, m)$ satisfying the differential equation

$$\frac{\partial}{\partial t}\Phi^X(t, m) = X(\Phi^X(t, m)), \tag{1.10}$$

with the boundary condition

$$\Phi^X(0, m) = m, \tag{1.11}$$

is called **flow** of vector field X.

(1.10) is a system of autonomous linear equations and it is known that solution satisfying the given initial conditions always locally exists. The flow can be used to find integrals or invariants of the vector field X.

Definition 1.15. Let M be a manifold and $X \in \mathfrak{X}(M)$. **Invariant** of X is any function $f \in \mathcal{F}(M)$ such that $Xf = 0$.

Remark 1.16. We shall also consider functions invariant with respect to a given subalgebra of $\mathfrak{X}(M)$, i.e. with respect of an integrable distribution on M. It is obvious that the invariants are determined by the distribution alone and do not depend an a particular choice of vector fields generating it.

$Xf = 0$ is a homogeneous first order PDE. The standard way of solving it is to employ the method of characteristics or method of moving frames. The idea of the latter is to find the flow Φ^X_t because the equation is equivalent to finding such function that $f \circ \Phi^X_t = f$ for all t. Choose a suitable hypersurface H transversal to the X. The integral curves of the X intersect H (at least in some neighbourhood). To the every point on the integral curve we can assign n numbers — n coordinates of the intersection point. This creates n functions ξ_i which are constant along integral curves.

To be more formali: We find a coefficient $\tau(m)$ which takes the point m to H by vector flow.

$$\Phi^X_{\tau(m)}(m) = (\xi_1, \dots, \xi_n) \in H. \tag{1.12}$$

Then $\xi_1, \dots, \xi_n$ viewed as functions of coordinates of point m are the invariant we are looking for. The preliminary computation and the very verification are

$$\begin{aligned} &\Phi^X_{\tau(m)-t}(\Phi_t(m)) = \Phi^X_{\tau(m)}(m) \in H \Longrightarrow t + \tau(\Phi^X_t(m)) = \tau(m), \\ &\xi_i(\Phi^X_t(m)) = \Phi^X_{\tau(\Phi^X_t(m))}(\Phi^X_t(m))_i = \Phi^X_{t+\tau(\Phi^X_t(m))}(m)_i = \Phi^X_{\tau(m)}(m)_i = \xi_i(m). \end{aligned} \tag{1.13}$$

Last thing we have to do is to choose $n-1$ invariants — $\xi_1, \dots, \xi_n$ are functionally dependent since they lie on hypersurface H.

1.4 Jet space, prolongation and symmetries

Lie algebras were developed when the symmetries of differential equations were studied. The idea arises from symmetries of algebraic equations.

Definition 1.17. Let

$$F_a(x_1, \dots, x_n) = 0, \qquad a = 1, \dots, r \tag{1.14}$$

be a system of algebraic equations. **Infinitesimal symmetry** of this algebraic equation is any vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ fulfilling the condition

$$XF_a|_{F=0} = 0 \tag{1.15}$$

for all a. (Assuming that $dF_a|_{F=0} \neq 0$)

Suppose a system of differential equation (SDE) of order N for k dependent variables

$$F_a(x, y_i, y_i', y_i'', \dots, y_i^{(N)}) = 0 \qquad i = 1, \dots, k \tag{1.16}$$

is given.

Similarity with the algebraic equation is obvious. We can employ almost identical approach. The main difference between (1.14) and (1.16) is that there are additional constrains. The variables $y_l^{(j)}$ are clearly dependent on x, y_l because they are derivations. The vector field from definition 1.17 should be somehow determined by action on x, y_j. We introduce the proper mathematical formalism immediately. It can be also found in [18].

We want to interpret (1.16) as an algebraic equation. We construct a manifold J^N with coordinates $x, y_i, y_i', y_i'', \dots, y_i^{(N)}$ in such a way that $F_a \in \mathcal{F}(J^N)$.

Definition 1.18. Let $J^0 := M$ be a manifold with coordinates x, y_i. **Jet space** J^n of (1.16) is $k(n+1)+1$ dimensional manifold with coordinates $x, y_i, y_i', y_i'', \dots, y_i^{(n)}$. We sometimes use the notation J^∞ for the jet space of arbitrarily large, but finite, order.

The J^0 is naturally embedded in J^n. Moreover any set of points $\{(x, y_i)\}$ which is a graph of a function $y_i(x)$ can be extended to J^N by including derivations of this function. When a transformation $(x, y_i) \to (\tilde{x}, \tilde{y}_i)$ is introduced it generates a change of the coordinates on the whole J^N. The formula can be found from the condition that if $\{(x, y_i, y_i', \dots)\}$ was a graph of the function $y(x)$ with all its derivations, $\{(\tilde{x}, \tilde{y}_i, \tilde{y}_i', \dots)\}$ has to be a graph of the function $\tilde{y}(\tilde{x})$ with all its derivations. The formula for prolongation is just an infinitesimal version.

There is a very important vector field. It encodes the behaviour of differentiation.

Definition 1.19. The **total derivative operator** is vector field on J^∞

$$D_x := \partial_x + \sum_{j=1}^{k} y_j' \partial_{y_j} + \dots + \sum_{i=1}^{k} y_i^{(s+1)} \partial_{y_i^{(s)}} + \cdots . \tag{1.17}$$

It is called a total derivative because it a is total derivative when acting on graphs of functions

$$\frac{d}{dx} F(x, y_j, y_j', \dots) = (D_x F)(x, y_j, y_j', \dots), \tag{1.18}$$

where $\{(x, y_j, y_j', \dots)\}$ are from a graph of a function $y(x)$.

Secondly every vector field on $\mathbb{R}^{k+1} = J^0 \subset J^n$ can be uniquely extended (prolonged) on whole J^n.

Definition 1.20. Let

$$X = \xi(x, y_j)\partial_x + \sum_{l=0}^{k} \eta_l^0(x, y_j)\partial_{y_l}$$

be an arbitrary vector field on J^0. Let us define functions

$$\eta_l^i := D_x \eta_l^{i-1} - y_l^{(i)}(D_x \xi). \tag{1.19}$$

Notice that η_l^i depends on x, y_l and derivations of y_l of at most i^{th} order. The $\mathbf{n^{th}}$ **prolongation** of vector field X is $\operatorname{pr}^n X \in \mathfrak{X}(J^n)$

$$\operatorname{pr}^n X := \xi \partial_x + \sum_{i=0}^{n} \sum_{j=1}^{k} \eta_j^i \partial_{y_j^{(i)}}. \tag{1.20}$$

The last step is the definition of infinitesimal symmetries of ODE. It is straightforward, compare with definition 1.17.

Definition 1.21. Infinitesimal symmetry of (1.16) is any vector field on $X \in \mathfrak{X}(J^0) = \mathfrak{X}(\mathbb{R}^{k+1})$ satisfying the condition

$$(\operatorname{pr}^N X) F_a \big|_{F=0} = 0, \tag{1.21}$$

for all a. (Assuming that $dF_a|_{F=0} \neq 0$)

Infinitesimal symmetries close into a Lie algebra.

1.5 ODEs with given algebra of symmetries

So far we can find the symmetries of a given ODE. However the inverted problem is the main subject of this thesis. Suppose that a Lie algebra L is given. The question is how the most general ODE with symmetries forming a realisation of the given Lie algebra looks like.

The procedure of answering this question has two steps. After deciding how many dependent variables will occur in ODE we have to find all realisations of L on accordant $\mathbb{R}^{k+1}$. Two realisations are equivalent if they are related by a point transformation (see remark 1.13).

Definition 1.22. Let a realisation $\mathfrak{L}$ of a Lie algebra L be given on J^0.

Strong invariant is any function $I \in \mathcal{F}(J^n)$ such that

$$(\operatorname{pr} X)I = 0 \tag{1.22}$$

for all $X \in \mathfrak{L}$.

Weak invariant is any function $I \in \mathcal{F}(J^n)$ such that

$$\forall X \in \mathfrak{L}, \exists f \in \mathcal{F}(J^n), \, (\operatorname{pr} X)I = fI. \tag{1.23}$$

Any differential equation with $\mathfrak{L}$ as its algebra of infinitesimal symmetries is, at least locally, equivalent to an equation

$$I(x, y_j, y_j', \dots) = 0, \tag{1.24}$$

where I is a weak invariant of $\mathfrak{L}$. However the computation of strong invariants is significantly easier and it has been proven in [19] that if at least one nontrivial strong invariant exists then every ODE invariant with respect to the same realisation has the form

$$e^{f(x,y,z,y',z',\dots)} \cdot F(I_1, \dots, I_k) = 0, \tag{1.25}$$

where $I_1, \dots, I_k$ are functionally independent strong invariants. In this thesis it is therefore sufficient for our purposes to find all strong invariants.

1.6 Operator of invariant differentiation

In the previous section the fact that it is enough to find only the strong invariants was mentioned. Theoretically speaking it is actually sufficient to find all strong invariants of the lowest possible order. The others can be generated with the help of operators of invariant differentiation. This issue is elaborated on e.g. in [20] or more in detail in [21].

Definition 1.23. Let realisation $\mathcal{L}$ of a Lie algebra is given. A vector field δ on the infinitely prolonged jet space J^∞ is called **operator of invariant differentiation** if for any strong differential invariant I the expression δI is also a strong differential invariant.

The operators of invariant differentiation form themselves an operator Lie algebra.

Theorem 1.24. Any operator commuting with the whole prolonged realisation $\mathcal{L}$ on J^∞ is an operator of invariant differentiation.

Proof. Let $\delta, X \in \operatorname{pr}^\infty \mathcal{L}$ and I a strong invariant.

$$X\delta I = \delta X I = \delta(0) = 0. \tag{1.26}$$

□

Suppose now that $\delta = \lambda D_x$, where $\lambda \in \mathcal{F}(J^\infty)$. How does the sufficient condition from previous theorem look like?

$$\begin{aligned}
0 = [\operatorname{pr} X, \lambda D_x] &= (\operatorname{pr} X\lambda)D_x + \lambda \cdot [\operatorname{pr} X, D_x] = \\
&= (\operatorname{pr} X\lambda)D_x + \lambda \cdot \sum_{j,l=0}^{(\infty,k)} (\eta_l^{j+1} - D_x\eta_l^j)\partial_{y_l^{(j)}} - \lambda \cdot (D_x\xi)\partial_x = \\
&= (\operatorname{pr} X\lambda)D_x + \lambda \cdot \sum_{j,l=0}^{(\infty,k)} (\eta_l^{j+1} - (\eta_l^{j+1} + y_l^{(j+1)}(D_x\xi)))\partial_{y_l^{(j)}} - \lambda \cdot (D_x\xi)\partial_x \\
&= (\operatorname{pr} X\lambda)D_x - \lambda \cdot \sum_{j,l=0}^{(\infty,k)} y_l^{(j+1)}(D_x\xi)\partial_{y_l^{(j)}} - \lambda \cdot (D_x\xi)\partial_x \\
&= (\operatorname{pr} X\lambda)D_x - \lambda \cdot (D_x\xi)\left(\sum_{j,l=0}^{(\infty,k)} y_l^{(j+1)}\partial_{y_l^{(j)}} + \partial_x\right) \\
&= (\operatorname{pr} X\lambda - \lambda \cdot (D_x\xi))D_x.
\end{aligned} \tag{1.27}$$

The determining formula for λ is therefore

$$\operatorname{pr} X\lambda = \lambda \cdot (D_x\xi). \tag{1.28}$$

If λ is sought implicitly by equation $\Omega(x, y_j, y'_j, \ldots, \lambda) = 0$ this equation takes the form

$$(\operatorname{pr} X + \lambda \cdot (D_x\xi)\partial_\lambda)\Omega = 0, \tag{1.29}$$

which can be seen from

$$(\operatorname{pr} X)\lambda = -\frac{(\operatorname{pr} X)\Omega}{\partial_\lambda\Omega} \overset{(1.29)}{=} -\frac{-\lambda \cdot (D_x\xi)\partial_\lambda\Omega}{\partial_\lambda\Omega} = \lambda \cdot (D_x\xi). \tag{1.30}$$

Using the equation in this form not only we can employ the method of moving frames, but if we do it cautiously we can get invariants of $\mathcal{L}$ as a by-product.

Example 1.25. Let $\mathcal{L}$ be the realisation from example 1.12. The equations for Ω are:

$$\begin{aligned} \partial_x \Omega &= 0, \\ (x\partial_x + \partial_y - y'\partial_{y'} + \lambda\partial_\lambda)\Omega &= 0. \end{aligned} \tag{1.31}$$

First equation eliminates the dependence on x. Now we employ the method of mowing frames. The flow is

$$\begin{aligned} x(t) &= xe^t, \\ y(t) &= t + y, \\ y'(t) &= y'e^{-t}, \\ \lambda(t) &= \lambda e^t. \end{aligned} \tag{1.32}$$

We choose a hypersurface given by equation $y = 0$. t can be expressed from the second equation and substituted to the others. The solution is

$$\Omega = \Omega(xe^{-y}, y'e^y, \lambda e^{-y}). \tag{1.33}$$

Now we can conclude that strong invariants of the first order of the chosen realisation $\mathcal{L}$ have the form $I(xe^{-y}, y'e^y)$, and from implicit equation

$$\lambda e^{-y} = 1 \tag{1.34}$$

we deduce that $\delta = e^y D_x$ is an operator of invariant differentiation.

Notice that $\delta(xe^{-y}) = 1 - xy'$. The newly obtained invariant is usually not in the simplest form, we can still subtract or multiply by an already known invariant. In this case we can subtract 1 and change a sign.

Why are operators of invariant differentiation so interesting? We can use them to create all strong invariants.

Theorem 1.26. Let $\mathcal{L}$ be a realisation of a Lie algebra and $\mathcal{I}$ a set of functionally independent strong invariants of the lowest order possible.

The set of all strong invariants is then finitely generated by the operators of invariant derivations and the finite number of functional operations from basal set $\mathcal{I}$.

Proof. The proof can be found for example in [22]. □

Back to our example. This theorem implies that the strong invariant of the n^{th} order could be written in the form $I = I(xe^{-y}, \delta xe^{-y}, \delta^2 xe^{-y}, \ldots, \delta^n xe^{-y})$.

Chapter 2

Realisations of given nilpotent and solvable Lie algebras

2.1 Realisation of $\mathfrak{n}_{4,1}$ by vector fields on $\mathbb{R}^3$

All algebras which appear in this thesis have $\mathfrak{n}_{4,1}$ from [7] as their subalgebra. Therefore it is useful to find realisations of $\mathfrak{n}_{4,1}$ in advance.

The non-vanishing commutation relations can be found in [7, eq.(4)]. We change our notation slightly in order to emphasise the significant role of d_1 (originally denoted by e_n).

$$\begin{aligned}\mathfrak{n}_{4,1} &= \operatorname{span}\{e_0, e_1, e_2, d_1\},\\ [e_2, d_1] &= e_1,\\ [e_1, d_1] &= e_0.\end{aligned} \tag{2.1}$$

Two realisations are equivalent if they are related by a point transformation, i.e. a transformation of the type

$$\begin{aligned}\tilde{x} &= \Lambda(x, y, z),\\ \tilde{y} &= \Omega(x, y, z),\\ \tilde{z} &= \Theta(x, y, z).\end{aligned} \tag{2.2}$$

There are 6 classes of 3-dimensional non-equivalent realisations.

$\mathfrak{N}^{\mathbf{1}}_{4,a}(\tilde{\alpha})$

$$\begin{aligned}D_1 &= -\partial_x,\\ E_0 &= \partial_y,\\ E_1 &= x\partial_y + \partial_z,\\ E_2 &= \left(\frac{x^2}{2} + \tilde{\alpha}\right)\partial_y + x\partial_z.\end{aligned} \tag{2.3}$$

$\mathfrak{N}^{\mathbf{1}}_{4,b}(\alpha)$ $\alpha \neq 0$

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_1 &= x\partial_y + \partial_z, \\ E_2 &= \alpha\partial_x + \left(\frac{x^2}{2} + \alpha z\right)\partial_y + x\partial_z. \end{aligned} \tag{2.4}$$

$\mathfrak{N}^{\mathbf{1}}_{4,c}$

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_1 &= z\partial_x + x\partial_y, \\ E_2 &= xz\partial_x + \frac{x^2}{2}\partial_y + z^2\partial_z. \end{aligned} \tag{2.5}$$

$\mathfrak{N}^{\mathbf{1}}_{4,d}$

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_1 &= x\partial_y, \\ E_2 &= \frac{x^2}{2}\partial_y + \partial_z. \end{aligned} \tag{2.6}$$

$\mathfrak{N}^{\mathbf{1}}_{4,e}$

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_1 &= x\partial_y, \\ E_2 &= \left(\frac{x^2}{2} + z\right)\partial_y. \end{aligned} \tag{2.7}$$

$\mathfrak{N}^{\mathbf{1}}_{4,f}(\tilde{\alpha})$

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_1 &= x\partial_y, \\ E_2 &= \left(\frac{x^2}{2} + \tilde{\alpha}\right)\partial_y. \end{aligned} \tag{2.8}$$

Notice that the parameters with a tilde (e.g. $\tilde{\alpha}$) could be fixed arbitrarily by a change of basis in the realisation, but not by an equivalence transformation (2.2) of the realisation.

Although it is usually useful to undertake such transformation we cannot in general use the results for the realisation of solvable algebras afterwards.

If we perform such change of the basis of the realisation it induces an automorphism on Lie algebra, because our realisation is injective and the Lie brackets are unchanged. The basis d_1, e_0, e_N is not the extraordinary basis — we can use any basis with same commutation relations. But if we have this realisation as the subrealisation of the solvable extension the map generated by a change of the basis is no longer an automorphism and therefore it is not allowed to perform such change of the basis.

The result presented above was derived in the following way. Vectors d_1 and e_0 generate a 2-dimensional abelian subalgebra. Such algebra can be realised either by fields $-\partial_x, \partial_y$ or by fields $x\partial_y, \partial_y$. Let us consider the latter one first.

1.
$$\begin{aligned} D_1 &= x\partial_y, \\ E_0 &= \partial_y. \end{aligned} \tag{2.9}$$

E_1 has to be a vector field commuting with E_0 and satisfying $[E_1, D_1] = E_0$. Hence E_1 is in the form
$$E_1 = \partial_x + \eta(x,z)\partial_y + \rho(x,z)\partial_z. \tag{2.10}$$

We can use the transformation

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + G(x,z), \\ \tilde{z} &= H(x,z), \end{aligned} \tag{2.11}$$

which for suitable $H(x,z)$ and $G(x,z)$ transforms E_1 to the $\partial_{\tilde{x}}$ and keeps the form of D_1 and E_0 unchanged at the same time. H and G have to satisfy the equations

$$\begin{aligned} \eta + \partial_x G + \rho \cdot \partial_z G &= 0, \\ \partial_x H + \rho \cdot \partial_z H &= 0, \end{aligned} \tag{2.12}$$

which always have a solution.

We prove that the realisation
$$\begin{aligned} D_1 &= x\partial_y, \\ E_0 &= \partial_y, \\ E_1 &= \partial_x \end{aligned} \tag{2.13}$$

cannot be further extended and hence is the dead end.

Field E_2 would have to commute with E_0, E_1. That restrict the E_2 to

$$E_2 = \xi(z)\partial_x + \eta(z)\partial_y + \rho(z)\partial_z. \tag{2.14}$$

Let's find out what the last condition says.

$$\begin{aligned} E_1 &= [E_2, D_1], \\ \text{i.e. } \partial_x &= [\xi(z)\partial_x + \eta(z)\partial_y + \rho(z)\partial_z, x\partial_y], \\ &= \xi(z)\partial_y. \end{aligned} \tag{2.15}$$

That is a contradiction because ∂_x and ∂_y are linearly independent.

2. Let's focus on the second case

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y. \end{aligned} \tag{2.16}$$

The field E_1 satisfying the commutation relations with E_0 and D_1 has to be in the form

$$E_1 = \xi_1(z)\partial_x + (x + \eta_1(z))\partial_y + \rho_1(z)\partial_z, \tag{2.17}$$

where ξ_1, η_1 and ρ_1 are arbitrary functions. The point transformation

$$\begin{aligned} \tilde{x} &= x + F(z), \\ \tilde{y} &= y + G(z), \\ \tilde{z} &= H(z), \end{aligned} \tag{2.18}$$

does not change the form of D_1, E_0 and therefore can be used to simplify E_1.

$$\begin{aligned} E_1 &= (\xi_1 + \rho_1\dot{F})\partial_{\tilde{x}} + (\tilde{x} - F + \eta_1 + \rho_1\dot{G})\partial_{\tilde{y}} + \rho_1\dot{H}\partial_{\tilde{z}}, \\ &= \tilde{\xi}_1(z)\partial_{\tilde{x}} + (x + \tilde{\eta}_1(z))\partial_{\tilde{y}} + \tilde{\rho}_1(z)\partial_{\tilde{z}}, \end{aligned} \tag{2.19}$$

where z is taken as a function of $\tilde{z}$.

We want to choose such functions F, G and H that

$$\begin{aligned} \tilde{\xi}_1 &= \xi_1 + \rho_1\dot{F}, \\ \tilde{\eta}_1 &= \eta_1 - F + \rho_1\dot{G}, \\ \tilde{\rho}_1 &= \rho_1\dot{H}, \end{aligned} \tag{2.20}$$

are as simple as possible.

One can see that if $\rho_1 \neq 0$ there are such functions F, G and H that $\tilde{\xi}_1 = \tilde{\eta}_1 = 0$ and $\tilde{\rho}_1 = 1$. In that case we have the realisation

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_1 &= x\partial_y + \partial_z. \end{aligned} \tag{2.21}$$

If $\rho_1 = 0$ we can use the function $F = \eta_1$ to annul $\tilde{\eta}_1$. In the case where ξ_1 is not a constant, $H = \xi_1$ will transform field E_1 to its final form. To sum up, there are three ways how to realise e_1.

$$\begin{aligned} E_1 &= x\partial_y + \partial_z, \\ \hat{E}_1 &= z\partial_x + x\partial_y, \\ \bar{E}_1(a) &= a\partial_x + x\partial_y. \end{aligned} \tag{2.22}$$

We still have to add the last field E_2. Take the realisation with $E_1 = \partial_z + x\partial_y$ first. The required commutation relations $[E_2, E_0] = 0$ and $[E_2, D_1] = E_1$ fix the form of E_2 in the following way:

$$E_2 = \xi_2(z)\partial_x + \left(\frac{x^2}{2} + \eta_2(z)\right)\partial_y + (x + \rho_2(z))\,\partial_z. \tag{2.23}$$

The relation

$$\begin{aligned} 0 &= [E_1, E_2], \\ &= \dot{\xi}_2\partial_x + (\dot{\eta}_2 - \xi_2)\partial_y + \dot{\rho}_2\partial_z \end{aligned} \tag{2.24}$$

has to be satisfied too. Therefore

$$E_2 = \alpha\partial_x + \left(\frac{x^2}{2} + \alpha z + \beta\right)\partial_y + (x + \gamma)\partial_z. \tag{2.25}$$

The transformation which does not spoil the form of D_1, E_0 and E_1 is

$$\begin{aligned} \tilde{x} &= x + a, \\ \tilde{y} &= y + az + b, \\ \tilde{z} &= z + c. \end{aligned} \tag{2.26}$$

The substitution where $a = \gamma, c = 0$ certainly helps and transform E_2 to

$$E_2 = \alpha\partial_{\tilde{x}} + \left(\frac{\tilde{x}^2}{2} + \alpha\tilde{z} + \tilde{\beta}\right)\partial_{\tilde{y}} + \tilde{x}\partial_{\tilde{z}}, \tag{2.27}$$

where $\tilde{\beta} = \beta + \frac{\gamma^2}{2}$.

The parameter a in (2.26) cannot be used anymore. Nevertheless there are still b, c which could be brought into play. After the application of the transformation is

$$E_2 = \alpha\partial_{\hat{x}} + \left(\frac{\hat{x}^2}{2} + \alpha\hat{z} + (\tilde{\beta} - c\alpha)\right)\partial_{\hat{y}} + \hat{x}\partial_{\hat{z}}. \tag{2.28}$$

For $\alpha = 0$ there is nothing we can do and we get $\mathfrak{N}^{\mathbf{1}}_{4,a}(\beta)$. We can set $\beta = 0$ by a convenient choice of c otherwise. In that case we get $\mathfrak{N}^{\mathbf{1}}_{4,b}(\alpha)$.

Our second realisation of e_1 is

$$\hat{E}_1 = z\partial_x + x\partial_y. \tag{2.29}$$

The required commutation relations $[\hat{E}_2, E_0] = 0$ and $[\hat{E}_2, D_1] = \hat{E}_1$ specify the form of $\hat{E}_2$ in the following way:

$$\hat{E}_2 = (xz + \xi_2(z))\partial_x + \left(\frac{x^2}{2} + \eta_2(z)\right)\partial_y + \rho_2(z)\partial_z. \tag{2.30}$$

The last condition

$$\begin{aligned} 0 &= [\hat{E}_1, \hat{E}_2], \\ &= (z^2 - \rho_2(z))\partial_x - \xi_2(z)\partial_y \end{aligned} \tag{2.31}$$

is satisfied if and only if

$$\hat{E}_2 = xz\partial_x + \left(\frac{x^2}{2} + \eta_2(z)\right)\partial_y + z^2\partial_z. \tag{2.32}$$

Finally we can employ the transformation $\tilde{y} = y + G(z)$ which (for a suitable choice of G) helps us remove η_2 and obtain the realisation $\mathfrak{N}^{\mathbf{1}}_{4,c}$.

The last case is the realisation with

$$\bar{E}_1(a) = a\partial_x + x\partial_y. \tag{2.33}$$

The first two conditions ($[E_0, \bar{E}_2] = 0, [\bar{E}_2, D_1] = \bar{E}_1(a)$) enforce

$$\bar{E}_2(a) = (ax + \xi_2(z))\partial_x + \left(\frac{x^2}{2} + \eta_2(z)\right)\partial_y + \rho_2(z)\partial_z. \tag{2.34}$$

We look closely on the third condition

$$\begin{aligned} 0 &= [\bar{E}_2(a), \bar{E}_1(a)], \\ &= a^2\partial_x + \xi_2(z)\partial_y. \end{aligned} \tag{2.35}$$

This implies that in order to proceed in realising the $\mathfrak{n}_{4,1}$ both a and ξ_2 have to be 0. Hence we have realisation

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ \bar{E}_1 &= x\partial_y \\ \bar{E}_2 &= \left(\frac{x^2}{2} + \eta_2(z)\right)\partial_y + \rho_2(z)\partial_z. \end{aligned} \tag{2.36}$$

The transformation leaving D_1, E_0 and $\bar{E}_1$ invariant is

$$\begin{aligned} \tilde{x} &= x, \\ \tilde{y} &= y + G(z), \\ \tilde{z} &= H(z). \end{aligned} \tag{2.37}$$

And $\bar{E}_2$ in the new coordinates is

$$\bar{E}_2 = \left(\frac{\tilde{x}^2}{2} + \eta_2 + \rho_2 \dot{G}\right) \partial_{\tilde{y}} + \rho_2 \dot{H} \partial_{\tilde{z}}. \tag{2.38}$$

This particular situation is very similar to the (2.20). It splits into three cases.

1. $\rho_2 \neq 0$.

   Then η_2 can be removed and ρ_2 set to 1. We get the realisation $\mathfrak{N}^{\mathbf{1}}_{4,d}$.

2. $\rho_2 = 0$ and η_2 is not a constant.

   The only thing that can be done in this case is the transformation $\tilde{z} = \eta_2(z)$. This leads to the realisation $\mathfrak{N}^{\mathbf{1}}_{4,e}$.

3. In other cases we get $\mathfrak{N}^{\mathbf{1}}_{4,f}(\eta_2)$ directly.

2.2 Realisation of $\mathfrak{n}_{n,1}$ by vector fields on $\mathbb{R}^3$

In the previous section the very important case $\mathfrak{n}_{4,1}$ was realised. Here we classify realisations of $\mathfrak{n}_{n,1}$ for all n. This classification will be presented step by step.

This algebra is the only one that can be realised also on $\mathbb{R}^2$. The $\mathbb{R}^2$-realisation will be obtained as a secondary outcome. It is because the structure of the realisation of $\mathfrak{n}_{n,1}$ on $\mathbb{R}^2$ is so constraining that it is nearly impossible to extend it, and both $\mathfrak{n}_{n,2}$ and $\mathfrak{n}_{n,3}$ have $\mathfrak{n}_{n,1}$ as their subalgebra.

First of all let us recall the non-vanishing commutation relations from [7].

$$\begin{aligned} \mathfrak{n}_{n,1} &= \operatorname{span}\{e_0, \dots, e_{n-2}, d_1\}, \\ [e_k, d_1] &= e_{k-1}, \quad k = 1, \dots, n-2. \end{aligned} \tag{2.39}$$

For this algebra it is useful to define $N := n - 2$.

Any algebra $\mathfrak{n}_{n,1}$ has $\mathfrak{n}_{4,1}$ as its subalgebra. It is spanned by vectors d_1, e_0, e_1 and e_2. This implies that every realisation of $\mathfrak{n}_{n,1}$ can be transformed by a point transformation in such a way that D_1, E_0, E_1, E_2 coincides with one of the realisations of $\mathfrak{n}_{4,1}$.

Several facts arise from the analysis of the realisation of $\mathfrak{n}_{4,1}$.

1. We can choose $D_1 = -\partial_x, E_0 = \partial_y$.

2. It has to be true that $(-\operatorname{ad}_{D_1})^k E_k = E_0$. This follows from the commutation relations.

3. The cases $\mathfrak{N}^{\mathbf{1}}_{4,b}(\alpha)$ and $\mathfrak{N}^{\mathbf{1}}_{4,c}$ cannot be further extended (to any dimension n larger than four). A short calculation verifies this claim:

$\mathfrak{N}^{\mathbf{1}}_{4,b}(\alpha)$ The relations $[E_0, E_3] = 0$ and $E_2 = [E_3, D_1]$ determine the form of E_3

$$E_3 = (\alpha x + \xi_3(z))\partial_x + \left(\frac{x^3}{3!} + \alpha xz + \eta_3(z)\right)\partial_y + \left(\frac{x^2}{2} + \rho_3(z)\right)\partial_z. \tag{2.40}$$

A problem arises when we study the relation

$$\begin{aligned} 0 &= [E_1, E_3], \\ &= (\alpha x + \xi_3(z) - \dot{\eta}_3(z))\partial_y - \dot{\xi}_3\partial_x - \dot{\rho}_3(z)\partial_z. \end{aligned} \tag{2.41}$$

The α ought be zero but this is not allowed in this realisation.

$\mathfrak{N}^{\mathbf{1}}_{4,c}$ The procedure is almost identical. The relations $[E_0, E_3] = 0$ and $E_2 = [E_3, D_1]$ determine the form of E_3

$$E_3 = \left(\frac{x^2}{2}z + \xi_3(z)\right)\partial_x + \left(\frac{x^3}{3!} + \eta_3(z)\right)\partial_y + \left(x\frac{z^2}{2} + \rho_3(z)\right)\partial_z. \tag{2.42}$$

The problem again arises when we study the relation

$$\begin{aligned} 0 &= [E_1, E_3], \\ &= \rho_3(z)\partial_x + \xi_3(z)\partial_y - \frac{z^3}{2}\partial_z. \\ &\Rightarrow \frac{z^3}{2} \stackrel{!}{=} 0, \end{aligned} \tag{2.43}$$

which is a contradiction.

4. Other cases enforce $E_k = \eta_k(x, z)\partial_y + \rho_k(x, z)\partial_z$ and this is our crucial assumption.

   This is easy to calculate when we find all fields F which commutes with E_0, E_1 and E_2 (commutator with E_2 is relevant only in the case of $\mathfrak{N}^{\mathbf{1}}_{4,a}(\alpha)$). Notice that E_k has to be such field.

Suppose we have a realisation $D_1 = -\partial_x, E_0 = \partial_y$. According to our observations we have to find $2N$ functions η_i, ρ_i in order to construct a realisation

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_i &= \eta_i(x, z)\partial_y + \rho_i(x, z)\partial_z. \end{aligned} \tag{2.44}$$

There are two cases which we deal with separately.

1. All $\rho_i = 0$. The realisation is hence

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_i &= \eta_i(x, z)\partial_y \end{aligned} \tag{2.45}$$

The condition $(-\operatorname{ad}_{D_1})^k E_k = E_0$ further constrains this form to

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_i &= \left(\frac{x^i}{i!} + \sum_{k=1}^{i} \xi_k(z) \frac{x^{i-k}}{(i-k)!} \right) \partial_y, \end{aligned} \tag{2.46}$$

where $\xi_1(z), \dots, \xi_{n-2}(z)$ represent remaining freedom. The last thing we can do is to use a certain change of coordinates. The transformation

$$\begin{aligned} \tilde{x} &:= x + \xi_1(z), \\ \tilde{y} &:= y, \\ \tilde{z} &:= z \end{aligned} \tag{2.47}$$

allows us to eliminate ξ_1, i.e. we have

$$\begin{aligned} D_1 &= -\partial_{\tilde{x}}, \\ E_0 &= \partial_{\tilde{y}}, \\ E_i &= \left(\frac{\tilde{x}^i}{i!} + \sum_{k=2}^{i} \xi_k(\tilde{z}) \frac{\tilde{x}^{i-k}}{(i-k)!} \right) \partial_{\tilde{y}}, \end{aligned} \tag{2.48}$$

after the change of coordinates (2.47).

We divide the obtained realisations into several cases:

- We put aside the realisation where all ξ_i are constants:
  $\mathfrak{N}^{\mathbf{1}}_{n,A}(\tilde{\alpha}_2, \dots, \tilde{\alpha}_N)$

  $$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_i &= \left(\frac{x^i}{i!} + \sum_{k=0}^{i-2} \tilde{\alpha}_{i-k} \frac{x^k}{k!} \right) \partial_y. \end{aligned} \tag{2.49}$$

  All alphas can be annulled by a change of basis in realisation (this is not the equivalent operation in the sense of our definition 1.8). We would also like to emphasise that this is the only possible realisation of $\mathfrak{n}_{n,1}$ by vector fields on $\mathbb{R}^2$.
- Otherwise, choose the lowest index j such that $\xi_j(z)$ is non-constant. The last thing we can do is to apply the transformation $\tilde{z} = \xi_j(z)$. This does not affect constants $\xi_2, \dots, \xi_{j-1}$ but gives rise to new functions $\tilde{\xi}_j(\tilde{z}), \dots, \tilde{\xi}_N(\tilde{z})$ such that $\tilde{\xi}_j(\tilde{z}) = \tilde{z}$. We point out this j-dependance in the notation of these classes of realisations.

$\mathfrak{N}^{\mathbf{1}}_{n,B,j}(\tilde{\alpha}_2,\ldots,\tilde{\alpha}_{j-1},\xi_{j+1},\ldots,\xi_N)$

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_i &= \left(\frac{x^i}{i!} + \sum_{k=2}^{i} \xi_k(z)\frac{x^{i-k}}{(i-k!)}\right)\partial_y, \end{aligned} \tag{2.50}$$

where $\xi_2 = \tilde{\alpha}_2,\ldots,\xi_{j-1} = \tilde{\alpha}_{j-1}$ and $\xi_j(z) = z$. By a change of the basis in the realisation we could annul all $\tilde{\alpha}_2,\ldots,\tilde{\alpha}_{j-1}$ and on top of that choose functions $\xi_i(z)$ such that $\xi_i(0) = 0$.

2. There exists m such that $\rho_m \neq 0$. We choose the lowest m possible. It is obvious that all $\rho_k, k > m$ are then nonzero too because

$$[E_k, D_1] = E_{k-1}$$

implies

$$\rho_m = \partial_x \rho_{m+1} = \partial_{xx}\rho_{m+2}$$

and so on.

We have to ensure that $[E_i, E_j] = 0$. This condition gives

$$0 = [E_i, E_j] = (\eta_{j,z}\rho_i - \eta_{i,z}\rho_j)\partial_y + (\rho_{j,z}\rho_i - \rho_{i,z}\rho_j)\partial_z,$$

i.e. we have two equations:

$$0 = \eta_{j,z}\rho_i - \eta_{i,z}\rho_j, \tag{2.51}$$

$$0 = \rho_{j,z}\rho_i - \rho_{i,z}\rho_j. \tag{2.52}$$

The equation (2.52) is satisfied automatically whenever i or j is not in M. Suppose $k \in M$ and solve the equation (2.52) for $i = k$ and $j = m = \min M$.

$$\begin{aligned} \rho_{k,z}\rho_m &= \rho_{m,z}\rho_k, \\ \frac{\rho_{k,z}}{\rho_k} &= \frac{\rho_{m,z}}{\rho_m}, \\ \partial_z \ln \rho_k &= \partial_z \ln \rho_m, \\ \rho_k(x,z) &= C_k(x)\cdot\rho_m(x,z). \end{aligned} \tag{2.53}$$

And consecutively (2.51).

$$\begin{aligned} \eta_{k,z}\rho_m &= \eta_{m,z}\rho_k, \\ \partial_z\eta_k(x,z) &= C_k(x)\cdot\partial_z\eta_m(x,z), \\ \eta_k(x,z) &= C_k(x)\cdot\eta_m(x,z) + B_k(x). \end{aligned} \tag{2.54}$$

Let us make the next step. We know that $m-1 \notin M$ and at the same time

$$E_{m-1} = [E_m, D_1] = (\partial_x \eta_m(x,z))\partial_y + (\partial_x \rho_m(x,z))\partial_z. \tag{2.55}$$

Therefore

$$\partial_x \rho_m(x,z) = 0, \tag{2.56}$$

which implies $\rho_m = \rho_m(z)$. Our realisation becomes

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_j &= \eta_j(x,z)\partial_y, \quad j < m, \\ E_m &= \eta_m(x,z)\partial_y + \rho_m(z)\partial_z, \\ E_k &= \left(C_k(x)\eta_m(x,z) + B_k(x)\right)\partial_y + C_k(x)\rho_m(z)\partial_z, \quad k > m. \end{aligned} \tag{2.57}$$

Return to the equation (2.51). Choose $i = m$ and $j = m-1$.

$$\begin{aligned} \eta_{m-1,z}\rho_m &= \eta_{m,z}\rho_j, \\ \eta_{m-1,z}\rho_m &= 0, \\ \partial_z \eta_{m-1}(x,z) &= 0. \end{aligned} \tag{2.58}$$

This fact together with (2.55) gives

$$\partial_{xz}\eta_m(x,z) = 0. \tag{2.59}$$

The solution is $\eta_m(x,z) = \mu(x) + \nu(z), \mu(0) = 0$. We can remove $\nu(z)$ and transform $\rho_m(z)$ to 1 using transformation

$$\begin{aligned} \tilde{x} &:= x, \\ \tilde{y} &:= y + G(z), \\ \tilde{z} &:= z + H(z), \\ \dot{G}(z) = -\frac{\nu(z)}{\rho_m(z)}&, \dot{H}(z) = \frac{1}{\rho_m(z)}. \end{aligned} \tag{2.60}$$

We keep using x, y, z rather than $\tilde{x}, \tilde{y}, \tilde{z}$. The realisation is now in the following form

$$\begin{aligned} D_1 &= -\partial_x, \\ E_0 &= \partial_y, \\ E_j &= \eta_j(x)\partial_y, \quad j < m, \\ E_m &= \mu(x)\partial_y + \partial_z, \\ E_k &= \left(C_k(x)\mu(x) + B_k(x)\right)\partial_y + C_k(x)\partial_z, \quad k > m. \end{aligned} \tag{2.61}$$

The term $C_k(x)\mu(x)$ can be absorbed into $B_k(x)$. The realisation is now specified completely by $B_N(x)$ and $C_N(x)$. The condition $(-\operatorname{ad}_{D_1})^k E_k = E_0$ together with

$\mu(0) = 0$ implies that they have to be polynomials of the order N and $N - m$ respectively, with the specified highest order coefficient and B_N has to have 0 as coefficient for $\frac{x^{N-m}}{(N-m)!}$. There still remains a transformation $\tilde{x} = x + \alpha$ available which can be used to annul N-1[th] coefficient of the first polynomial. To sum up, we have nonequivalent realisations with nonequivalence characterised by two polynomials, one of the order $n-2$ and second one of any lower order (because m can be any number from 1 to $n-2$).

$\mathfrak{N}^{\mathbf{1}}_{n,C}(P_{n-2}, Q_j)$ P_N and Q_j are arbitrary polynomials fulfilling the following conditions.

$$\deg P_N = N = n-2, \qquad \deg Q_j = j,\ 0 \le j < N,$$
$$P_N = \sum_{i=0}^{N} a_i \frac{x^i}{i!}, a_N = 1, a_{N-1} = a_j = 0 \qquad Q_j = \frac{x^j}{j!} + \cdots . \tag{2.62}$$

The realisation has the form

$$\begin{aligned} D_1 &= -\partial_x, \\ E_N &= P_N(x)\partial_y + Q_j(x)\partial_z, \\ E_{N-k} &= \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_N(x)\partial_y + \frac{\mathrm{d}^k}{\mathrm{d}x^k} Q_j(x)\partial_z. \end{aligned} \tag{2.63}$$

This implies that $E_{N-j} = \frac{\mathrm{d}^j}{\mathrm{d}x^j} P_N(x)\partial_y + \partial_z$ and $E_0 = \partial_y$. Furthermore by a change of the basis we can change our realisation to a form where $P_N(x) = \frac{x^N}{N!}$.

2.2.1 Realisation of solvable extension of $\mathfrak{n}_{n,1}$

Before we begin to realise the solvable extensions of $\mathfrak{n}_{n,1}$ it is appropriate to introduce commutation relation, because they are slightly different than in [7]. We recall that non-vanishing commutation relations in the nilradical spanned by $e_0, \ldots, e_N$ and d_1 ($N = n-2$) are

$$[e_k, d_1] = e_{k-1}, \quad k = 1, \ldots, N. \tag{2.64}$$

It has been revealed in [7] that the dimension of the solvable extension could be only one or two greater. We add vector s (s_1 and s_2 respectively) and specify commutation relations.

- $\mathfrak{s}_{n+1,\iota\nu}(\alpha, \beta)$

  This class absorbs cases

$$\mathfrak{s}_{n+1,\iota\nu}(\alpha,\beta) = \begin{cases} \mathfrak{s}_{n+1,1}(\beta), & \alpha = 1, \beta \notin \{0, -N\}, \\ \mathfrak{s}_{n+1,2}, & \alpha = 1, \beta = 0, \\ \mathfrak{s}_{n+1,3}, & \alpha = 1, \beta = -N \\ \mathfrak{s}_{n+1,4}, & \alpha = 0, \beta = 1, \end{cases}$$

and its commutation relations are

$$\begin{aligned}[d_1, s] &= \alpha d_1,\\ [e_k, s] &= [(N-k)\alpha + \beta]e_k.\end{aligned} \tag{2.65}$$

- $\mathfrak{s}_{n+1,5}$

  Its commutation relations are

$$\begin{aligned}[d_1, s] &= d_1 + e_N,\\ [e_k, s] &= (N+1-k)e_k.\end{aligned} \tag{2.66}$$

- $\mathfrak{s}_{n+1,6}(a_2, \ldots, a_N)$

  The parameters $a_2, \ldots, a_N$ are either from $\mathbb{R}$ or $\mathbb{C}$ and at least one of them is nonzero. There are more condition but they are not important for our purposes. Commutation relations are

$$\begin{aligned}[d_1, s] &= 0,\\ [e_k, s] &= e_k + \sum_{i=2}^{k} a_i e_{k-i}.\end{aligned} \tag{2.67}$$

- s_{n+2} This is the unique solvable $n+2$-dimensional extension. Commutation relations are

$$\begin{aligned}[d_1, s_1] &= d_1, & [d_1, s_2] &= 0,\\ [e_k, s_1] &= (N-k)e_k, & [e_k, s_2] &= e_k.\end{aligned} \tag{2.68}$$

The computation of realisation of solvable extensions of $\mathfrak{N}^{\mathbf{1}}_{n,A}$ is rather straightforward because they can be obtained from two-dimensional realisations simply by determining $\rho(z)\partial_z$ in S. All parameters α_i are forced to be zero except for the algebra $\mathfrak{s}_{n+1,4}$. The realisation can be found in the Table 2.1.

Realisation with nilpotent part $\mathfrak{N}^{\mathbf{1}}_{n,C}(P_N, Q_j)$

As far as extending of the realisation $\mathfrak{N}^{\mathbf{1}}_{n,C}(P_N, Q_j)$ is concerned, reader can easily notice that, roughly speaking, this realisation is just two realisations ($\mathfrak{N}^{\mathbf{1}}_{N+2,A}$ for variables x, y and $\mathfrak{N}^{\mathbf{1}}_{j+2,A}$ for x, z) joined together by common variable x. It is not surprising that the results corresponds to this observation.

We begin with determination of the basic form of S for algebras $\mathfrak{s}_{n+1,\iota\nu}(\alpha, \beta)$ and $\mathfrak{s}_{n+1,5}$. The relation $[E_0, S] = (N\alpha + \beta)E_0$ implies

$$S = \xi(x,z)\partial_x + (N\alpha + \beta)(y + \eta(x,z))\partial_y + \rho(x,z)\partial_z. \tag{2.69}$$

Table 2.1: Realisation of solvable extensions of $n_{n,1}$ extended from $\mathfrak{N}^{\mathbf{1}}_{n,A}$

$\mathfrak{N}^{\mathbf{3}}_{N+4,a}(0,\dots,0)$	
$\mathfrak{s}_{n+1,\iota\nu}(\alpha,\beta)$	$S = \alpha x\partial_x + (\alpha N+\beta)y\partial_y + \rho(z)\partial_z$
$\mathfrak{s}_{n+1,5}$	$S = x\partial_x + \left((N+1)y - \frac{x^{N+1}}{(N+1)!}\right)\partial_y + \rho(z)\partial_z$
$\mathfrak{s}_{n+1,6}(a_2,\dots,a_N)$	does not exist
$\mathfrak{s}_{n+2}$	$S_1 = x\partial_x + (N-k)y\partial_y + \rho(z)\partial_z$ $S_2 = y\partial_y + \tilde{\rho}(z;\delta)\partial_z$

$\rho(z)$ could be either z or 0 and those realisations are nonequivalent.

If $\rho_1(z) = z$, then $\rho_2(z,\delta) = \delta\cdot z$ and for various δ we get nonequivalent realisation.

If $\rho_1(z) = 0$, then $\rho_2(z,\delta)$ could be z or 0, and in the latter case

is our realisation purely two-dimensional and nonequivalent to the first one.

E_1 has the form $E_1 = x\partial_y + \epsilon\partial_z$, where $\epsilon = 0, 1$. Let us look at the condition

$$\begin{aligned}[(N-1)\alpha+\beta]E_1 &= [E_1, S],\\ &= (N\alpha+\beta)x\partial_y + \epsilon\rho_z(x,z)\partial_z - \xi(x,z)\partial_y.\end{aligned} \tag{2.70}$$

When we compare ∂_y-terms we find that

$$\xi(x,z) = \alpha x. \tag{2.71}$$

The next relevant relation is $[E_{N-j}, S] = (j\alpha+\beta)E_{N-j}$ because term ∂_z appears in the E_{N-j}. We are interested in the ∂_z-part only.

$$\begin{aligned}(j\alpha+\beta)E_{N-j} = [E_{N-j}, S] &= \dots\partial_y + \rho_z(x,z)\partial_z,\\ \dots\partial_y + (j\alpha+\beta)\partial_z &= \dots\partial_y + \rho_z(x,z)\partial_z.\end{aligned} \tag{2.72}$$

This implies that S has to be in the form

$$S = \alpha x\partial_x + (N\alpha+\beta)(y+\eta(x,z))\partial_y + (j\alpha+\beta)(z+\lambda(x))\partial_z. \tag{2.73}$$

Suppose that $\alpha \neq 0$ and set $\alpha = 1$ by rescaling of S. The relation $(k+\beta)E_{N-k} = [E_{N-k}, S]$ gives

$$
\begin{aligned}
(k+\beta)E_{N-k} &= [P_N^{(k)}(x)\partial_y + Q_j^{(k)}(x)\partial_z, S],\\
\beta P_N^{(k)}\partial_y + \beta Q_j^{(k)}\partial_z &= (N+\beta)P_n^{(k)}\partial_y + (j+\beta)Q_j^{(k)}\partial_z - \eta_z Q_j^{(k)}\partial_y - x(P_N^{(k+1)}\partial_y + Q_j^{(k+1)}\partial_z),\\
0 &= \left\{\left[N \cdot P_N^{(k)} - xP_N^{(k+1)}\right] - \eta_z Q_j^{(k)}\right\}\partial_y + \left[jQ_j^{(k)} - xQ_j^{(k+1)}\right]\partial_z.
\end{aligned}
\tag{2.74}
$$

Choose $k = 0$. The solution of the differential equation $jQ_j - x\dot{Q}_j$ is $Q_j(x) = C\dfrac{x^j}{j!}$ and from properties of Q_j (2.62) we get $C = 1$. Then consider $k = 0, 1$. Then the ∂_y-term gives us two equations

$$
N \cdot P_N - x\dot{P}_N = \eta_z \frac{x^j}{j!}, \tag{2.75}
$$

$$
(N-1) \cdot \dot{P}_N - x\ddot{P}_N = \eta_z \frac{x^{j-1}}{(j-1)!}. \tag{2.76}
$$

This system is equivalent to

$$
N \cdot P_N - x\dot{P}_N = \eta_z \frac{x^j}{j!}, \tag{2.77}
$$

$$
\eta_{xz} = 0. \tag{2.78}
$$

The second equation implies that $\eta(x,z) = \mu(x) + \nu(z), \nu(0) = 0$.

That leaves us with

$$
N \cdot P_N(x) - x\dot{P}_N(x) = \dot{\nu}(z)\frac{x^j}{j!}. \tag{2.79}
$$

From the dependence of the two sides of the equation on different variables it follows that $\nu(z) = \gamma z$ and solution of the differential equation together with (2.62) is $P_N(x) = \dfrac{x^N}{N!} + \dfrac{\gamma}{N-j}\dfrac{x^j}{j!}$. We recall that the coefficient at $\dfrac{x^j}{j!}$ has to be zero, therefore $\gamma = 0$. Altogether we have

$$
\begin{aligned}
&S = x\partial_x + [(N+\beta)y + \mu(x)]\,\partial_y + [(j+\beta)z + \lambda(x)]\,\partial_z,\\
&P_N = \frac{x^N}{N!},\ Q_j = \frac{x^j}{j!}.
\end{aligned}
\tag{2.80}
$$

We can proceed and finish the realisation.

- $s_{n+1,\iota\nu}(1,\beta)$

  We use prepared S and employ the last condition $[D_1, S] = D_1$ together with the transformation $\tilde{y} = y + b, \tilde{z} = z + c$ which is still available and get

$$
S = x\partial_x + (N+\beta)y\partial_y + (j+\beta)z\partial_z. \tag{2.81}
$$

This realisation extends $\mathfrak{N}^{\mathbf{1}}_{n,C}\left(\frac{x^N}{N!}, \frac{x^j}{j!}\right)$ and for various j we get nonequivalent realisations.

- $s_{n+1,\iota\nu}(0,1)$

  We return to the point where we have assumed that $\alpha \neq 0$ which is not true in this case. We have $S = (y + \eta(x,z))\partial_y + (z + \lambda(x))\partial_z$.

$$\begin{aligned} E_N &= [E_N, S] \\ P_N\partial_y + Q_j\partial_z &= P_N\partial_y + Q_j\partial_z + Q_j\eta_z\partial_y. \\ 0 &= [D_1, S], \\ 0 &= -\eta_x\partial_y - \lambda_x\partial_z. \end{aligned} \tag{2.82}$$

  These equations and the available transformation $\tilde{y} = y + b, \tilde{z} = z + c$ gives us the final form

$$S = y\partial_y + z\partial_z. \tag{2.83}$$

  This algebra does not constrain the form of the polynomials P_N and Q_j. We get different realisations for various choice of these polynomial (satisfying conditions (2.62)).

- $\mathfrak{s}_{n+1,5}$

  We can utilise our prepared $S = x\partial_x + [(N+1)y + \mu(x)]\,\partial_y + [(j+1)z + \lambda(x)]\,\partial_z$. The remaining condition is

$$\begin{aligned} [D_1, S] &= D_1 + E_N, \\ -\partial_x - \mu_x\partial_y - \lambda_x\partial_z &= -\partial_x + \frac{x^N}{N!}\partial_y + \frac{x^j}{j!}\partial_z. \end{aligned} \tag{2.84}$$

  This gives us

$$S = x\partial_x + \left[(N+1)y - \frac{x^{N+1}}{(N+1)!}\right]\partial_y + \left[(j+1)z - \frac{x^{j+1}}{(j+1)!}\right]\partial_z. \tag{2.85}$$

  Again we have realisation parametrised by number j.

- $\mathfrak{s}_{n+1,6}(a_2, \ldots, a_N)$

  Can be realised only if $2j < N$.

  We want to extend $\mathfrak{N}^{\mathbf{1}}_{n,C}(P_N, Q_j)$. From relations

$$\begin{aligned} [D_1, S] &= 0, \\ [E_0, S] &= E_0, \\ [E_1, S] &= E_1. \end{aligned} \tag{2.86}$$

it follows that

$$S = (y + \eta(z))\partial_y + \rho(z)\partial_z. \tag{2.87}$$

The parameters $a_2, \ldots, a_{N-j-1}$ have to be zero in order for realisation to exist. This can be seen if we expand the formula

$$\begin{aligned}
[E_{N-j-1}, S] &= E_{N-j-1} + \sum_{i=2}^{N-j-1} a_i E_{N-j-1-i}, \\
\text{i.e. } [P_N^{(j+1)}(x)\partial_y, S] &= P_N^{(j+1)}(x)\partial_y + \sum_{i=2}^{N-j-1} a_i P_N^{(j+1+i)}(x)\partial_y, \\
P_N^{(j+1)}(x)\partial_y &= P_N^{(j+1)}(x)\partial_y + \sum_{i=2}^{N-j-1} a_i P_N^{(j+1+i)}(x)\partial_y, \\
0 &= \sum_{i=2}^{N-j-1} a_i P_N^{(j+1+i)}.
\end{aligned} \tag{2.88}$$

On the other way a_{N-j} has to be nonzero. This can be seen from the following computations. Consider

$$\begin{aligned}
[E_{N-j}, S] &= E_{N-j} + \sum_{i=2}^{N-j} a_i E_{N-j-i}, \\
\text{i.e. } [P_N^{(j)}(x)\partial_y + \partial_z, S] &= P_N^{(j)}(x)\partial_y + \partial_z + a_{N-j}\partial_y, \\
P_N^{(j)}(x)\partial_y + \dot{\eta}(z)\partial_y + \dot{\rho}(z)\partial_z &= P_N^{(j)}(x)\partial_y + \partial_z + a_{N-j}\partial_y,
\end{aligned} \tag{2.89}$$

and therefore

$$\begin{aligned}
\eta(z) &= a_{N-j}z + \beta, \\
\rho(z) &= z + \gamma.
\end{aligned} \tag{2.90}$$

Both β and γ could be removed by transformation $\tilde{y} = y + \beta, \tilde{z} = z + \gamma$. The vector field S have the form

$$S = (y + a_{N-j}z)\partial_y + z\partial_y \tag{2.91}$$

now, but the reason why a_{N-j} has to be nonzero can be seen either from the relation concerning E_N or, in the case $j = 0$, from the fact that at least one a_i most be non-zero. Back to the relation with E_N:

$$
\begin{gathered}
[E_N, S] = E_N + \sum_{i=N-j}^{N} a_i E_{N-i}, \\
[P_N(x)\partial_y + Q_j(x)\partial_z, S] = P_N(x)\partial_y + Q_j(x)\partial_z + \sum_{i=N-j}^{N} a_i \left(P_N^{(i)}(x)\partial_y + Q_j^{(i)}(x)\partial_z\right), \\
a_{N-j} \cdot Q_j(x)\partial_y = \sum_{i=N-j}^{N} a_i \left(P_N^{(i)}(x)\partial_y + Q_j^{(i)}(x)\partial_z\right).
\end{gathered} \tag{2.92}
$$

If a_{N-j} is zero then also all a_i have to be zero and this contradicts the definition of $\mathfrak{s}_{n+1,6}(\dots)$. Therefore we have proved our assertion $a_{N-j} \neq 0$.

Our realisation will be in the nicer form if we rescale $\tilde{z} = a_{N-j} \cdot z$. The highest coefficient in Q_j is no longer $\frac{1}{j!}$ but $\frac{a_{N-j}}{j!}$. The obtained equations are then:

$$
Q_j(x) = \sum_{i=N-j}^{N} a_i \cdot P_N^{(i)}(x), \tag{2.93}
$$

$$
0 = \sum_{i=N-j}^{N} a_i \cdot Q_j^{(i)}(x). \tag{2.94}
$$

We use the (2.93) to define Q_j. The defined function is indeed a polynomial of the required order — j. Now move on to the second condition

$$
0 = \sum_{i=N-j}^{N} a_i \cdot Q_j^{(i)}(x) = \sum_{i=N-j}^{j} a_i \cdot Q_j^{(i)}(x). \tag{2.95}
$$

If $2j < N$ it is satisfied automatically. If $2j \geq N$ let us substitute from (2.93).

$$
\begin{aligned}
0 &= \sum_{i=N-j}^{j} \sum_{k=N-j}^{N-i} a_i \cdot a_k \cdot P_N^{(i+k)} = \sum_{s=2N-2j}^{N} \sum_{i=N-j}^{s-N+j} a_i \cdot a_{s-i} \cdot P_N^{(s)}, \\
&= \sum_{r=N-j}^{j} \sum_{i=N-j}^{r} a_i \cdot a_{N-j+r-i} \cdot P_N^{(N-j+r)}.
\end{aligned} \tag{2.96}
$$

This is the polynomial equation. Consider the coefficient of the highest power term. It is contained only in the least differentiated polynomial i.e. for $r = N - j$. That coefficient is

$$
0 = \sum_{i=N-j}^{N-j} a_i \cdot a_{N-j+N-j-i} = a_{N-j} \cdot a_{N-j} \tag{2.97}
$$

which is contradiction with $a_{N-j} \neq 0$.

To sum up, the algebra $\mathfrak{s}_{n+1,6}(0,\dots,0,a_k,\dots,a_N)$, where $a_k \neq 0$ can be realised as an extension of $\mathfrak{N}^{\mathbf{1}}_{n,C}(P_N, Q_j)$ if and only if $k > \frac{N}{2}$. The realisation is then characterised by an polynomial P_N fulfilling the condition (2.62). Define Q_{N-k} with formula (2.93). The vector fields are

$$\begin{aligned} D_1 &= -\partial_x, \\ E_{N-i} &= P_N^{(i)}(x)\partial_y + Q_{N-k}^{(i)}(x)\partial_z, \\ S &= (y+z)\partial_y + z\partial_z. \end{aligned} \tag{2.98}$$

- s_{n+2}

  In contrast to the previous case this one is very simple. According to our already performed calculations

$$\begin{aligned} S_1 &= x\partial_x + Ny\partial_y + jz\partial_z \\ S_2 &= (y+\beta)\partial_y + (z+\gamma)\partial_z. \end{aligned} \tag{2.99}$$

  Since S_1 and S_2 has to commute both β and γ has to be 0. This realisation extends the case $\mathfrak{N}^{\mathbf{1}}_{n,C}\left(\frac{x^N}{N!}, \frac{x^j}{j!}\right)$, that is why we get N different realisations ($j = 0, \dots, N-1$).

To summarise the main results we present the Table 2.2

Table 2.2: Realisations of solvable extension of $\mathfrak{N}^{\mathbf{1}}_{n,C}(P_N, Q_j)$

Algebra	Realisation	Constrains
$\mathfrak{s}_{n+1,\iota\nu}(1,\beta)$	$x\partial_x + (N+\beta)y\partial_y + (j+\beta)z\partial_z$	$P_N = \dfrac{x^N}{N!}, Q_j = \dfrac{x^j}{j!}$
$\mathfrak{s}_{n+1,\iota\nu}(0,1)$	$y\partial_y + z\partial_z$	—
$\mathfrak{s}_{n+1,5}$	$x\partial_x + \left(y - \dfrac{x^{N+1}}{(N+1)!}\right)\partial_y + \left(z - \dfrac{x^{j+1}}{(j+1)!}\right)\partial_z$	$P_N = \dfrac{x^N}{N!}, Q_j = \dfrac{x^j}{j!}$
$\mathfrak{s}_{n+1,6}$, $k > \frac{N}{2}, a_2 = \dots = a_{k-1} = 0, a_k \neq 0$	$(y+z)\partial_y + z\partial_z$	$Q_{N-k} = \sum_{i=k}^{N} a_i P_N^{(i)}$
$\mathfrak{s}_{n+2}$	$S_1 = x\partial_x + Ny\partial_y + jz\partial_z$ $S_2 = y\partial_y + z\partial_z$	$P_N = \dfrac{x^N}{N!}$ $Q_j = \dfrac{x^j}{j!}$

Realisation with nilpotent part $\mathfrak{N}^{\mathbf{1}}_{n,B,j}$

There still remains $\mathfrak{N}^{\mathbf{1}}_{n,B,j}(\tilde{\alpha}_2, \dots, \tilde{\alpha}_{j-1}, \xi_{j+1}, \dots, \xi_N)$ to be extended. We can again perform some preparatory calculation. For all solvable extensions has to be true that

$$\begin{aligned} [E_0, S] &= (N\alpha + \beta)E_0, \\ [E_1, S] &= [(N-1)\alpha + \beta]E_1. \end{aligned} \tag{2.100}$$

This means that

$$S = \alpha x \partial_x + [(N\alpha + \beta)y + \eta(x,z)]\,\partial_y + \rho(x,z)\partial_z. \tag{2.101}$$

For the case $\mathfrak{s}_{n+1,\iota\nu}(1,\beta)$ is $\alpha = 1$ and $[D_1, S] = D_1$. This causes that both η and ρ can no longer depend on x. As a next step we consider the relation $[E_j, S] = (N - j + \beta)E_j$.

$$\begin{aligned} (N-j+\beta)E_j = [E_j, S] &= \left[\left(\frac{x^j}{j!} + \sum_{k=2}^{j-1} \tilde{\alpha}_k \frac{x^{j-k}}{(j-k)!} + z\right)\partial_y, S\right], \\ &= (N+\beta)E_j - j\frac{x^j}{j!}\partial_y - \sum_{k=2}^{j-1}(j-k)\tilde{\alpha}_k \frac{x^{j-k}}{(j-k)!}\partial_y - \rho(z)\partial_y \\ &= (N-j+\beta)E_j + \sum_{k=2}^{j-1} k\cdot\tilde{\alpha}_k \frac{x^{j-k}}{(j-k)!}\partial_y + (jz - \rho(z))\partial_y. \end{aligned} \tag{2.102}$$

This implies that $\tilde{\alpha}_2 = \dots = \tilde{\alpha}_{j-1} = 0$ and $\rho(z) = jz$.
Finally the relation $[E_N, S] = \beta E_N$ fixes $\xi_k(z)$.

$$\begin{aligned} \beta E_N = [E_N, S] &= \left[\left(\frac{x^N}{N!} + \sum_{k=j}^{N} \xi_k(z) \frac{x^{N-k}}{(N-k)!}\right)\partial_y, S\right], \\ &= (N+\beta)E_N - N\cdot\frac{x^N}{N!}\partial_y - \sum_{k=j}^{N}\Big((N-k)\xi_k(z) + jz\dot{\xi}_k(z)\Big)\frac{x^{N-k}}{(N-k)!}\partial_y, \\ &= \beta E_N + \sum_{k=j}^{N}\Big(k\xi_k(z) - jz\dot{\xi}_k(z)\Big)\frac{x^{N-k}}{(N-k)!}\partial_y. \end{aligned} \tag{2.103}$$

This gives us $N-j$ differential equations, because the equation for $k = j$ is satisfied automatically (we recall that $\xi_j(z) = z$). The realisation extends $\mathfrak{N}^{\mathbf{1}}_{n,B,j}(0, \dots, 0, \xi_{j+1}, \dots, \xi_N)$, where $\xi_k(z) = \beta_k z^{\frac{k}{j}}$. The parameters β_k are arbitrary and

$$S = x\partial_x + [(N+\beta)y + \eta(z)]\partial_y + jz\partial_z. \tag{2.104}$$

We can use the available transformation $\tilde{y} = y + G(z)$ to remove the function $\eta(z)$.

For case $\mathfrak{s}_{n+1,\iota\nu}(0,1)$ the conditions $[D_1, S] = 0$ and $[E_j, S] = E_j$ constrain the form of the solvable element to

$$S = y\partial_y. \tag{2.105}$$

No additional conditions on $a_k, \xi_l(z)$ are required.

The solvable algebra $\mathfrak{s}_{n+1,5}$ is very similar to the $\mathfrak{s}_{n+1,\iota\nu}(1,1)$. We use the prepared form (2.101) and employ the relation $[D_1, S] = D_1 + E_N$ and find out that

$$S = x\partial_x + \left[(N+1)y - \sum_{k=2}^{N} \xi_k(z)\frac{x^{N-k}}{(N-k)!} + \tau(z)\right]\partial_y + \rho(z)\partial_z. \tag{2.106}$$

Our consequent proceeding will be same as in the case $\mathfrak{s}_{n+1,\iota\nu}(1,\beta)$ — we determine $\xi_k(z)$ and $\rho(z)$, transform the function $\tau(z)$ away, so the final form is

$$S = x\partial_x + \left[(N+1)y - zx - \sum_{k=j+1}^{N} \beta_k z^{\frac{k}{j}}\frac{x^{N-k}}{(N-k)!}\right]\partial_y + \rho(z)\partial_z. \tag{2.107}$$

The algebra $\mathfrak{s}_{n+1,6}(a_2, \ldots, a_N)$ with the most cumbersome commutation relation is in order. Firsts steps are easy. We take into account the relations

$$\begin{aligned} [D_1, S] &= 0 \\ [E_0, S] &= E_0 \\ [E_1, S] &= E_1 \end{aligned} \tag{2.108}$$

These conditions imply that

$$S = y\partial_y + \rho(z)\partial_z. \tag{2.109}$$

Consider the relation $[E_j, S] = E_j + \sum_{k=2}^{j} a_k E_{j-k}$. This relation forces $a_2 = \cdots = a_{j-1} = 0$ and $\rho(z) = -a_j$ because when we actually calculate $[E_j, S]$ we get

$$[E_j, S] = E_j - \rho(z)\partial_y. \tag{2.110}$$

From this moment things are getting complicated. To make formulae simple we set $\xi_i(z) = \tilde{\alpha}_i$ for $i < j$ and $\xi_j(z) = z$. As soon as we prove that $a_j \neq 0$ we will make a substitution $a_j\tilde{z} := z$ which introduces a_j into the E_N but on the other hand it removes it from S and the overall structure of the realisation will be more clear.

Further conditions can be obtained from relation of S with E_N:

$$
\begin{aligned}
[E_N, S] &= E_N + \sum_{k=2}^{N} a_k E_{N-k}, \\
\text{i.e.: } E_N + a_j \sum_{k=2}^{N} \dot{\xi}_k(z) \frac{x^{N-k}}{(N-k)!} \partial_y &= E_N + \sum_{k=2}^{N} a_k E_{N-k}, \qquad (2.111) \\
a_j \sum_{k=2}^{N} \dot{\xi}_k(z) \frac{x^{N-k}}{(N-k)!} \partial_y &= \sum_{k=2}^{N} a_k E_{N-k}
\end{aligned}
$$

If $a_j = 0$ then $a_{j+1} = \cdots = a_N = 0$ and this cannot be, therefore $a_j \neq 0$ and we can employ transformation $a_j \tilde{z} := z$ which change the condition to

$$
\sum_{k=2}^{N} \dot{\xi}_k(z) \frac{x^{N-k}}{(N-k)!} \partial_y = \sum_{k=2}^{N} a_k E_{N-k}. \qquad (2.112)
$$

When we separate terms proportional to the monomials $\frac{x^i}{i!}$ we get a system of first order linear differential equations for $\xi_i(z)$ with constant parameters. From the theory of differential equation this system has always solution which depends on $N-1$ constants $c_2, \ldots, c_N$.

We found out whether the solution is in agreement with the original form of $\xi_2, \ldots, \xi_j$.

- Let's find out which equation matches to $\dfrac{x^{N-k}}{(N-k)!}$ for $k < j$.

$$
\dot{\xi}_k(z) \frac{x^{N-k}}{(N-k)!} \partial_y + \cdots = \sum_{m=2}^{k} a_m E_{N-m} + \cdots , \qquad (2.113)
$$

where "$\cdots$" stands for the other terms which do not contain $\dfrac{x^{N-k}}{k!} \partial_y$. Our previous computation implies

$$
m < j \Longrightarrow a_m = 0. \qquad (2.114)
$$

Thus the equation corresponding to $\dfrac{x^{N-k}}{(N-k)!}$ is

$$
\begin{aligned}
&\dot{\xi}_k(z) = 0, \\
&\Rightarrow \xi_k(z) = c_k = \alpha_k.
\end{aligned} \qquad (2.115)
$$

- Similarly for $k = j$ we derive

$$\begin{aligned}
\dot{\xi}_j(z)\frac{x^{N-j}}{(N-j)!}\partial_y + \cdots &= \sum_{m=2}^{j} a_m E_{N-m} + \cdots\ , \\
\dot{\xi}_j(z)\frac{x^{N-j}}{(N-j)!}\partial_y + \cdots &= a_j E_{N-j} + \cdots\ , \\
\Rightarrow \quad & \dot{\xi}_j(z) = a_j, \\
\Rightarrow \quad & \xi_j(z) = a_j z + c_j.
\end{aligned} \tag{2.116}$$

  Which is in agreement when we set $c_j = 0$ because the transformation $a_j\tilde{z} = z$ have made $\xi_j(z) = a_j$.

The unique solution of (2.112) depends on $N-2$ constants of integration (we have already fixed one) $\tilde{\alpha}_2, \ldots, \tilde{\alpha}_{j-1}, \tilde{\alpha}_{j+1}, \ldots, \tilde{\alpha}_N$.

These constants can be theoretically fixed by a change of basis in the realisation in such way that $\xi_i(0) = 0$ for all i. The explicit form is very difficult to find, nevertheless we were able to find that solution for which it is true that $\xi_i(0) = 0$. The realisation is then in the form

$$\begin{aligned}
D_1 &= -\partial_x, \\
E_i &= \sum_{k,l\in\mathbb{N}_0} \frac{z^k x^l}{k!l!}\left(\sum_{\substack{|J|=k \\ \langle J\rangle = i-l}} \frac{k!}{J!}a^J\right)\partial_y, \\
S &= y\partial_y - \partial_z,
\end{aligned} \tag{2.117}$$

where J is a multiindex such that $J_1 = 0$ and $\langle J\rangle := \sum_{s=1}^{\infty} s\cdot J_s$.

As far as the $\mathfrak{s}_{n+2}$ is concerned we can realise the extending vectors by

$$\begin{aligned}
S_1 &= x\partial_x + Ny\partial_y + jz\partial_z, \\
S_2 &= (y+\eta(z))\partial_y.
\end{aligned} \tag{2.118}$$

We also want $[S_1, S_2] = 0$ and from that condition we have $\eta(z) = \beta z^{\frac{N}{j}}$. If $\beta \neq 0$ the transformation $\tilde{z} = \beta^{\frac{1}{N}} z^{\frac{1}{j}}$ simplifies S_2 and moves the parameter β to the nilpotent realisation.

We sum up the results on the following list:

- $\mathfrak{s}_{n+1,\iota\nu}(1,\beta)$

  Choose $j \in \mathbb{N}$. Only the extension of the $\mathfrak{N}^{\mathbf{1}}_{n,B,j}(0,\ldots,0,\beta_{j+1}z^{\frac{j+1}{j}},\ldots,\beta_N z^{\frac{N}{j}})$, where $\beta_{j+1},\ldots,\beta_N$ are arbitrary constants is possible. The extending vector field is

$$S = x\partial_x + [(N+\beta)y + \eta(z)]\partial_y + jz\partial_z. \tag{2.119}$$

- $\mathfrak{s}_{n+1,\iota\nu}(0,1)$

  Choose $j \in \mathbb{N}$. For this realisation the extending vector field is

$$S = y\partial_y \tag{2.120}$$

  And the parameters of the extended realisation $\mathfrak{N}^{\mathbf{1}}_{n,B,j}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_{j-1}, \xi_{j+1}, \ldots, \xi_N)$ are not constrained.

- $\mathfrak{s}_{n+1,5}$

  Choose $j \in \mathbb{N}$. Only the realisation $\mathfrak{N}^{\mathbf{1}}_{n,B,j}(0, \ldots, 0, \beta_{j+1} z^{\frac{j+1}{j}}, \ldots, \beta_N z^{\frac{N}{j}})$ could be extended. The additional vector field is

$$S = x\partial_x + \left[(N+1)y - zx - \sum_{k=j+1}^{N} \beta_k z^{\frac{k}{j}} \frac{x^{N-k}}{(N-k)!}\right] \partial_y + jz\partial_z. \tag{2.121}$$

- $\mathfrak{s}_{n+1,6}(a_2, \ldots, a_N)$

  Let a_k be the first nonzero parameter. We choose $\mathfrak{N}^{\mathbf{1}}_{n+1,B,N-k}(0, \ldots, 0, \xi_j, \ldots, \xi_N)$ for ξ_i solution of (2.112) and $S = y\partial_y - \partial_z$. Using a change of basis in realisation, we can obtain the realisation (2.117).

- $\mathfrak{s}_{n+2}$

  Choose $j \in \mathbb{N}$. We have two cases

  1. First one is just simply "joining" realisation of $\mathfrak{s}_{n+1,\iota\nu}(1,0)$ and $\mathfrak{s}_{n+1,\iota\nu}(1,0)$. It extends $\mathfrak{N}^{\mathbf{1}}_{n,B,j}(0, \ldots, 0, \beta_{j+1} z^{\frac{j+1}{j}}, \ldots, \beta_N z^{\frac{N}{j}})$

$$\begin{aligned} S_1 &= x\partial_x + Ny\partial_y + jz\partial_z, \\ S_2 &= y\partial_y. \end{aligned} \tag{2.122}$$

  2. The second one is more complicated:

$$\begin{aligned} D_1 &= -\partial_x, \\ E_k &= \left(\frac{x^{N-k}}{(N-k)!} + \sum_{m=j}^{k} \beta_m z^m \frac{x^{N-k-m}}{(N-k-m)!} \right) \partial_y, \\ S_1 &= x\partial_x + Ny\partial_y + z\partial_z, \\ S_2 &= (y + z^N)\partial_y, \end{aligned} \tag{2.123}$$

     where $\beta_j, \ldots, \beta_N$ are arbitrary constants.

We have finished a detailed description of the classification of the realisations of $\mathfrak{n}_{n+1}$ and its solvable extension. We suppose that the reader is in this moment familiar with the method and procedure. In the following sections we shall describe the computations in general terms only and focus on the results.

This task is also easier because we can put to use already obtained results.

2.3 Realisation of $\mathfrak{n}_{n,2}$ by vector fields on $\mathbb{R}^3$

The algebras $\mathfrak{n}_{n,2}$ were considered in [8]. Their extension to a solvable Lie algebra is unique. The non-vanishing commutation relations are

$$\begin{aligned} \mathfrak{n}_{n,2} &= \operatorname{span}\{e_0, \ldots, e_N, d_1, d_2\}, & N &= n-3, \\ [e_k, d_1] &= e_{k-1}, & 1 &\le k \le N, \\ [e_l, d_2] &= e_{l-2}, & 2 &\le l \le N, \\ [d_2, d_1] &= e_N. \end{aligned} \tag{2.124}$$

The unique solvable extension is created when we add a vector s with the commutation relations

$$\begin{aligned} [s, e_k] &= (N+3-k)e_k, \\ [s, d_2] &= 2d_2, \\ [s, d_1] &= d_1. \end{aligned} \tag{2.125}$$

The subalgebra spanned by e_0, e_1, e_2, d_1 is isomorphic to $\mathfrak{n}_{4,1}$. We use the results from section 2.1 and extend them by adding vector fields corresponding to e_k and d_2. This is impossible on $\mathbb{R}^2$, and on $\mathbb{R}^3$ it is possible only for two cases.

1. Algebras extending $\mathfrak{N}^{\mathbf{1}}_{4,d}$.

   We can only realise the five- and six-dimensional algebras ($\mathfrak{n}_{5,2}$ and $\mathfrak{n}_{6,2}$).

   $\mathfrak{N}^{\mathbf{2}}_{5,a}(\alpha)$

$$\begin{aligned} E_0 &= \partial_y, \\ E_1 &= x\partial_y, \\ E_2 &= \frac{x^2}{2}\partial_y + \partial_z, \\ D_1 &= -\partial_x, \\ D_2 &= \left(\frac{x^3}{3!} + z\right)\partial_y + (x+\alpha)\partial_z. \end{aligned} \tag{2.126}$$

   The extension to a realisation of the solvable algebra is possible if and only if $\alpha = 0$. The additional vector field is

$$S = -x\partial_x - 5y\partial_y - 3z\partial_z. \tag{2.127}$$

$\mathfrak{N}^{\mathbf{2}}_{6,a}(\alpha, \tilde{\gamma})$

$$
\begin{aligned}
E_0 &= \partial_y, \\
E_1 &= x\partial_y, \\
E_2 &= \frac{x^2}{2}\partial_y + \partial_z, \\
E_3 &= \left(\frac{x^3}{3!} + \tilde{\gamma}\right)\partial_y + x\partial_z, \\
D_1 &= -\partial_x, \\
D_2 &= \left(\frac{x^4}{4!} + \tilde{\gamma}x + z\right)\partial_y + \left(\frac{x^2}{2} + \alpha\right)\partial_z.
\end{aligned}
\tag{2.128}
$$

The extension to a realisation of the solvable algebra is possible if and only if $\alpha = \tilde{\gamma} = 0$. The additional vector field is

$$
S = -x\partial_x - 6y\partial_y - 4z\partial_z. \tag{2.129}
$$

2. Algebras extending $\mathfrak{N}^{\mathbf{1}}_{4,e}$.

We can realise $\mathfrak{n}_{n,2}$ for any n and there is large freedom in choosing the realisation.

Indeed realisations are characterised by two polynomials P_N,P_{N-1}, where $N := n-3$. These polynomials have to fulfil the following conditions:

$$
\begin{aligned}
\deg P_N &= \left\lfloor \frac{N}{2} \right\rfloor =: k, \\
\deg P_{N-1} &= \left\lfloor \frac{N-1}{2} \right\rfloor =: l. \\
P_N(t) &= \frac{t^k}{k!} + 0 \cdot t^{k-1} + \text{lower order terms}, \\
P_{N-1}(t) &= \frac{t^l}{l!} + \text{lower order terms}.
\end{aligned}
\tag{2.130}
$$

The polynomials generate a flag of polynomials P_j satisfying $P_{j-2}(t) := \frac{\mathrm{d}}{\mathrm{d}t}P_j(t)$. Using this flag we get the realisation

$\mathfrak{N}^{\mathbf{2}}_{n,b}(P_N, P_{N-1})$

$$
\begin{aligned}
E_i &= \sum_{j=0}^{i} P_{i-j}(z)\frac{x^j}{j!}\partial_y, \\
D_1 &= -\partial_x, \\
D_2 &= \sum_{j=0}^{N} P_{N-j}(z)\frac{x^{j+1}}{(j+1)!}\partial_y - \partial_z.
\end{aligned}
\tag{2.131}
$$

The situation changes when we add a vector field corresponding to the solvable extension. The polynomials P_N and P_{N-1} can no longer be arbitrary. Their form is prescribed completely. Hence we have a unique realisation of the solvable extension which we present transformed to a nicer form.

$\mathfrak{N}^{\mathbf{2s}}_{n,b}$

$$\begin{aligned} E_k &= \sum_{\substack{i,j\in\mathbb{N}_0\\ i+2j=k}} \frac{x^i}{i!}\frac{z^j}{j!}\partial_y, \quad k=0,\dots,N,\\ D_1 &= -\partial_x,\\ D_2 &= \sum_{\substack{i,j\in\mathbb{N}_0\\ i+2j=N}} \frac{x^{i+1}}{(i+1)!}\frac{z^j}{j!}\partial_y - \partial_z,\\ S &= x\partial_x + (N+3)y\partial_y + 2z\partial_z. \end{aligned} \tag{2.132}$$

2.4 Realisation of $\mathfrak{n}_{n,3}$ by vector fields on $\mathbb{R}^3$

In [9] we dealt with another class of nilpotent algebras $\mathfrak{n}_{n,3}$ which is very similar to $\mathfrak{n}_{n,1}$ (in fact it contains $\mathfrak{n}_{n-2,1}$ as a subalgebra). This similarity was used to simplify calculations in [9]. We choose $n \geq 8$ to avoid anomalies occurring in low dimensions. The non-vanishing commutation relations are

$$\begin{aligned} \mathfrak{n}_{N+4,3} &= \operatorname{span}\{e_0,\dots,e_N,f,d_f,d_1\},\\ [e_k,d_1] &= e_{k-1}, \quad k=1,\dots,N,\\ [f,d_f] &= e_0,\\ [d_f,d_1] &= f. \end{aligned} \tag{2.133}$$

We shall see the similarity between the algebras $\mathfrak{n}_{n,3}$ and $\mathfrak{n}_{n-2,1}$ also on the level of vector fields. The realisation is just the $\mathbb{R}^2$ realisation of $\mathfrak{n}_{N+2,1} = \operatorname{span}\{d_1,e_0,\dots,e_N\}$ extended to $\mathbb{R}^3$ by two additional vector fields.

We again utilise the subalgebra isomorphic to $\mathfrak{n}_{4,1}$. There are two classes of realisations of $\mathfrak{n}_{n,3}$. Both of them share the same form for the vector fields representing $e_0,\dots,e_N$.

$\mathfrak{N}^{\mathbf{3}}_{n,a}(\tilde\alpha_1,\dots,\tilde\alpha_N)$

$$\begin{aligned} E_k &= \sum_{i=0}^{k} \tilde\alpha_{k-i}\frac{x^i}{i!}\partial_y, \quad \tilde\alpha_0 = 1,\\ D_1 &= -\partial_x,\\ F &= \partial_z,\\ D_f &= x\partial_z + z\partial_y. \end{aligned} \tag{2.134}$$

All parameters can be annulled by a change of basis.

$\mathfrak{N}^{\mathbf{3}}_{n,b}(\tilde{\alpha}_2,\ldots,\tilde{\alpha}_N)$

$$\begin{aligned} E_k &= \sum_{i=0}^{k} \tilde{\alpha}_{k-i} \frac{x^i}{i!} \partial_y, \quad \tilde{\alpha}_0 = 1, \tilde{\alpha}_1 = 0, \\ D_1 &= -\partial_x, \\ F &= z\partial_y, \\ D_f &= xz\partial_y - \partial_z. \end{aligned} \tag{2.135}$$

And as in the previous case all parameters with a tilde can be removed by a change of basis.

2.4.1 Realisation of solvable extensions of $\mathfrak{n}_{n,3}$

As far as a realisation of solvable extensions of $\mathfrak{n}_{n,3}$ are concerned, the parameters $\tilde{\alpha}_i$ have to be zero (except the cases $\mathfrak{s}_{n+1,5}$ and $\mathfrak{s}_{n+1,8}$). Except for $\mathfrak{s}_{n+1,8}$ all realisations can be constructed without problems. Notation and commutation relations can be found in [9, pp 10–12]. It is a bit unpractical that the notation in [7] collides with [9] (both uses $s_{n+1,k}$ for different algebras), although it has good reason.

The possible solvable extensions has the dimension either $n+1$ or $n+2$. For the dimension $n \geq 7$ we have five types of the solvable Lie algebras with dimension $n+1$ and the unique algebra with the dimension $n+2$. They are represented by the following (we recall $N = n-4$ for this algebra):

- $\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$ This class absorbs the cases

$$\mathfrak{s}_{n+1,\nu}(\alpha,\beta) \begin{cases} \mathfrak{s}_{n+1,1}(\beta), & \alpha = 1, \beta \notin \{-\frac{1}{2}, 0, \frac{N-1}{2}\}, \\ \mathfrak{s}_{n+1,2}, & \alpha = 1, \beta = \frac{N-1}{2}, \\ \mathfrak{s}_{n+1,3}, & \alpha = 1, \beta = 0, \\ \mathfrak{s}_{n+1,4}, & \alpha = 1, \beta = -\frac{1}{2}, \\ \mathfrak{s}_{n+1,5}, & \alpha = 0, \beta = 1, \end{cases}$$

and its commutation relations are

$$\begin{aligned} [d_1, s] &= \alpha d_1, \\ [d_f, s] &= \beta d_f, \\ [f, s] &= (\alpha+\beta) f, \\ [e_k, s] &= ((1-k)\alpha + 2\beta)\, e_k. \end{aligned} \tag{2.136}$$

- $\mathfrak{s}_{n+1,6}(\varepsilon)$

Its commutation relations are

$$\begin{aligned} [d_1, s] &= d_1 + \varepsilon e_N, \\ [d_f, s] &= \frac{N}{2} d_f, \\ [f, s] &= \frac{N+2}{2} f, \\ [e_k, s] &= (N + 1 - k)\, e_k. \end{aligned} \tag{2.137}$$

- $\mathfrak{s}_{n+1,7}$

  The commutation relation of this algebra are

$$\begin{aligned} [d_1, s] &= d_1, \\ [d_f, s] &= e_1, \\ [f, s] &= f + e_0, \\ [e_k, s] &= (1 - k)\, e_k. \end{aligned} \tag{2.138}$$

- $\mathfrak{s}_{n+1,8}(a_1, \ldots, a_{n-4})$

  Similar to the solvable extensions of $\mathfrak{n}_{n,1}$ the parameters $a_1, \ldots, a_N$ are from field and at least one of them is nonzero. The other conditions are not important for our purposes. The commutation relations are

$$\begin{aligned} [d_1, s] &= 0, \\ [d_f, s] &= d_f + a_1 f, \\ [f, s] &= f, \\ [e_k, s] &= 2e_k + \sum_{i=2}^{k} a_i e_{k-i}. \end{aligned} \tag{2.139}$$

- $\mathfrak{s}_{n+1,9}$

  The commutation relations of this last $(n-1)$-dimensional case are

$$\begin{aligned} [d_1, s] &= d_1, \\ [d_f, s] &= d_f - e_2, \\ [f, s] &= 2f - e_1, \\ [e_k, s] &= (3 - k)\, e_k. \end{aligned} \tag{2.140}$$

- $\mathfrak{s}_{n+2}$

  The unique solvable $(n+2)$-dimensional extension has commutation relations

$$\begin{aligned} [d_1, s_1] &= d_1, & [d_1, s_2] &= 0, \\ [d_f, s_1] &= 0, & [d_f, s_2] &= d_f \\ [f, s_1] &= f, & [f, s_2] &= f, \\ [e_k, s_1] &= (1-k)\, e_k, & [e_k, s_2] &= 2e_k, \\ [s_1, s_2] &= 0. \end{aligned} \tag{2.141}$$

We sum up the two realisations of $\mathfrak{n}_{n,3}$ in Tables 2.3 and 2.4, respectively.

Table 2.3: First realisation of solvable extensions of $n_{n,3}$

$\mathfrak{N}^{\mathbf{3}}_{N+4,a}(0,\ldots,0)$	
$\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$	$S = \alpha x\partial_x + (\alpha+2\beta)y\partial_y + (\alpha+\beta)z\partial_z$
$\mathfrak{s}_{n+1,6}(\varepsilon)$	$S = x\partial_x + \left((N+1)y - \varepsilon\frac{x^{N+1}}{(N+1)!}\right)\partial_y + \frac{N+2}{2}z\partial_z$
$\mathfrak{s}_{n+1,7}$	$S = x\partial_x + (z+y)\partial_y + z\partial_z$
$\mathfrak{s}_{n+1,8}(a_2,\ldots,a_{n-3})$	—
$\mathfrak{s}_{n+1,9}$	$S = x\partial_x + (3y - xz)\partial_y + \left(2z - \dfrac{x^2}{2}\right)\partial_z$
$\mathfrak{s}_{n+2}$	$S_1 = x\partial_x + y\partial_y + z\partial_z$ $S_2 = 2y\partial_y + z\partial_z$

Table 2.4: Second realisation of solvable extensions of $n_{n,3}$

$\mathfrak{N}^{\mathbf{3}}_{N+4,b}(0,\dots,0)$	
$\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$	$S=\alpha x\partial_x+(2\beta+\alpha)y\partial_y+\beta z\partial_z$
$\mathfrak{s}_{n+1,6}(\varepsilon)$	$S=x\partial_x+\left((N+1)y-\varepsilon\frac{x^{N+1}}{(N+1)!}\right)\partial_y+\dfrac{N}{2}z\partial_z$
$\mathfrak{s}_{n+1,7}$	$S=x\partial_x+y\partial_y-\partial_z$
$\mathfrak{s}_{n+1,8}(1,0,0,\dots)$	$S=\left(2y-\dfrac{z^2}{2}\right)\partial_y+z\partial_z$
$\mathfrak{s}_{n+1,9}$	$S=x\partial_x+\left(3y-\dfrac{x^2}{2}z\right)\partial_y+(x+z)\partial_z$
$\mathfrak{s}_{n+2}$	$S_1=x\partial_x+y\partial_y$ $S_2=2y\partial_y+z\partial_z$

Chapter 3

Differential equations with given infinitesimal symmetries

3.1 ODEs with $\mathfrak{n}_{n,1}$ as algebra of infinitesimal symmetries

3.1.1 Realisation $\mathfrak{N}^{1}_{n,A}$

Let us focus on $\mathbb{R}^2$-realisation of $\mathfrak{n}_{n,1}$. The vector fields realising the nilpotent algebra are

$$\begin{aligned} D_1 &= -\partial_x, \\ E_k &= \frac{x^k}{k!}\partial_y, \quad 0 \leq k \leq N = n-2. \end{aligned} \tag{3.1}$$

We have already changed the basis, because for searching of invariants is the basis of the realisation not important.

First thing which is needed is the prolongation of realisation via the definition 1.20. We prolongate them formally to the infinity — that means we can restrain ourselves to any order of prolongation.

$$\begin{aligned} \operatorname{pr} D_1 &= -\partial_x, \\ \operatorname{pr} E_k &= \sum_{i=0}^{k} \frac{x^{k-i}}{(k-i)!}\partial_{y^{(i)}}, \quad 0 \leq k \leq N = n-2. \end{aligned} \tag{3.2}$$

It is obvious that these fields eliminate the variables $x, y, y', \ldots, y^{(N)}$. When we search for differential equations with $\mathfrak{n}_{n,1}$ as the algebra of the infinitesimal symmetries. The theory implies, that if we know at least one strong invariant, then all symmetric ODEs are in the form (1.25):

$$e^{f(x,y,z,y',z',\ldots)} \cdot F(I_1, \ldots, I_k) = 0, \tag{3.3}$$

where $I_1, \ldots, I_k$ are the strong invariants (in general this ODE can have more symmetries — it depends on the function F). Therefore it is sufficient to know just the functionally independent strong invariants and all symmetric ODEs (and therefore also all systems of differential equations) can be obtained from (1.25). Hence we will present only the invariants.

The realisation and the invariants for the solvable extensions are in the Table 3.1.

Table 3.1: $\mathbb{R}^2$-realisations and invariants of solvable extension of $\mathfrak{N}^{\mathbf{1}}_{n,A}$

Algebra	Realisation	Invariants ($k \in \mathbb{N}$)	Op. of inv. diff.
$\mathfrak{s}_{n+1,\iota\nu}(1, 1+B)$, $B \neq 0$	$x\partial_x + (n-1+B)y\partial_y$	$y^{(n-1+k)}\left(y^{(n-1)}\right)^{\frac{k-B}{B}}$	$(y^{(n-1)})^{\frac{1}{B}} \cdot D_x$
$\mathfrak{s}_{n+1,\iota\nu}(1, 1)$	$x\partial_x + (n-1)y\partial_y$	$y^{(n-1)}, \dfrac{y^{(n+k)}}{(y^{(n)})^{1+k}}$	$\dfrac{1}{y^{(n)}} \cdot D_x$
$\mathfrak{s}_{n+1,\iota\nu}(0, 1)$	$y\partial_y$	$\dfrac{y^{(n-1+k)}}{y^{(n-1)}}$	D_x
$\mathfrak{s}_{n+1,5}$	$x\partial_x + \left(y - \dfrac{x^{n-1}}{(n-1)!}\right)\partial_y$	$y^{(n-1+k)}e^{-ky^{(n-1)}}$	$\dfrac{1}{y^{(n)}} \cdot D_x$
$\mathfrak{s}_{n+1,6}(a_2, \ldots, a_N)$	This algebra does not have any realisation on $\mathbb{R}^2$.		
$\mathfrak{s}_{n+2}$	$S_1 = x\partial_x + (n-1)y\partial_y$, $S_2 = y\partial_y$	$\dfrac{y^{(n+k)}(y^{(n-1)})^k}{(y^{(n)})^{k+1}}$	$\dfrac{y^{(n-1)}}{y^{(n)}} \cdot D_x$

3.1.2 Realisation $\mathfrak{N}^{\mathbf{1}}_{n,B,j}(\tilde{\alpha}_2, \ldots, \tilde{\alpha}_{j-1}, \xi_{j+1}, \ldots, \xi_N)$

Our efforts to find invariants of this algebra fail on the fact that the prolongations of vector fields cannot be gained explicitly. The problem is in the term $D_x^k \xi_i(z)$ which can be theoretically expressed by means of Bell's polynomials but it seems to be useless for our purposes.

3.1.3 Realisation $\mathfrak{N}^{\mathbf{1}}_{n,C}(P_N, Q_j)$

The computation is very straightforward and the result is dependent only on the degrees of polynomials P_N and Q_j. In the first step equation with D_1 assure that the invariant does not depend on x. $E_0, \ldots, E_{N-j-1}$ eliminate dependance on $y, \ldots, y^{(N-j-1)}$ and remaining $E_{N-j}, \ldots, E_N$ force $y^{(N-j)} - z, \ldots, y^{(N)} - z^{(j)}$ be invariants. The invariants are therefore

$$\begin{aligned} y^{(N-j)} - z, \ldots, y^{(N-j+k)} - z^{(k)}, \ldots \\ z^{(j+1)}, z^{(j+2)}, \ldots \end{aligned} \tag{3.4}$$

The operator of invariant differentiation can be deduced as D_x.

3.2 ODEs with $\mathfrak{n}_{n,2}$ as algebra of infinitesimal symmetries

Let's deal with the two exceptional cases $\mathfrak{N}^{\mathbf{2}}_{5,a}$ and $\mathfrak{N}^{\mathbf{2}}_{6,a}$ first. The advantage of these cases is that they have fixed finite dimension (not an arbitrary dimension as in the other cases). That means we have to prolong only finitely many vector fields and afterwards solve only finitely many pseudolinear differential equations.

3.2.1 Realisation $\mathfrak{N}^{\mathbf{2}}_{5,a}(\alpha)$

Prolongation of the vector fields are

$$\begin{aligned}
&\operatorname{pr} E_0 = \partial_y,\\
&\operatorname{pr} E_1 = x\partial_y + \partial_{y'},\\
&\operatorname{pr} E_2 = \frac{x^2}{2}\partial_y + x\partial_{y'} + \partial_{y''} + \partial_z,\\
&\operatorname{pr} D_1 = -\partial_x,\\
&\operatorname{pr} D_2 = (x+\alpha)\partial_z + \partial_{z'} + \frac{x^3}{3!}\partial_y + \frac{x^2}{2}\partial_{y'} + x\partial_{y''} + \partial_{y'''} + \sum_{i=0}^{\infty} z^{(i)}\partial_{y^{(i)}}.
\end{aligned} \tag{3.5}$$

When $\alpha = 0$ we can add field S and prolong it.

$$\operatorname{pr} S = -x\partial_x - \sum_{j=0}^{\infty}(5-j)y^{(j)}\partial_{y^{(j)}} - \sum_{k=0}^{\infty}(3-k)z^{(k)}\partial_{z^{(k)}}.$$

Functionally independent invariants for this realisation are

$$\begin{aligned}
&z'', z''', \dots\\
&y'' - z - (z''-\alpha)z', y''' - (z'''+1)z', y^{(4)} - z^{(4)}z', \dots, y^{(4+k)} - z^{(4+k)}z', \dots
\end{aligned} \tag{3.6}$$

and the operator of the invariant differentiation is D_x.

And for the realisation of solvable extensions, functionally independent invariants are

$$\begin{aligned}
&z''', z^{(4)}z'', \dots, z^{(k)}(z'')^{k-3}, \dots\\
&\frac{y''-z-z'z''}{(z'')^3}, \frac{y'''-z'(1+z''')}{(z'')^2}, \left(y^{(k)} - z^{(k)}z'\right)(z'')^{k-5}, k \geq 4.
\end{aligned} \tag{3.7}$$

The operator of invariant differentiation is $z''D_x$.

3.2.2 Realisation $\mathfrak{N}^{\mathbf{2}}_{6,a}(\alpha, \tilde{\gamma})$

The parameter $\tilde{\gamma}$ expresses only different choices of basis in the $\mathfrak{N}^{\mathbf{2}}_{6,a}(\alpha, 0)$. Therefore it is not important for the computation of invariants and can be set $\tilde{\gamma} = 0$. Prolongations of the vector fields are

$$
\begin{aligned}
\operatorname{pr} E_0 &= \partial_y, \\
\operatorname{pr} E_1 &= x\partial_y + \partial_{y'}, \\
\operatorname{pr} E_2 &= \frac{x^2}{2}\partial_y + x\partial_{y'} + \partial_{y''} + \partial_z, \\
\operatorname{pr} E_3 &= \frac{x^3}{3!}\partial_y + \frac{x^2}{2}\partial_{y'} + x\partial_{y''} + \partial_{y'''} + x\partial_z + \partial_{z'} \\
\operatorname{pr} D_1 &= -\partial_x, \\
\operatorname{pr} D_2 &= \left(\frac{x^2}{2} + \alpha\right)\partial_z + x\partial_{z'} + \partial_{z''} + \sum_{j=0}^{4} \frac{x^{4-j}}{(4-j)!}\partial_{y^{(j)}} + \sum_{i=0}^{\infty} z^{(i)}\partial_{y^{(i)}}.
\end{aligned} \tag{3.8}
$$

When $\alpha = \tilde{\gamma} = 0$ we can add field S and prolong it.

$$
\operatorname{pr} S = -x\partial_x - \sum_{j=0}^{\infty}(6-j)y^{(j)}\partial_{y^{(j)}} - \sum_{k=0}^{\infty}(4-k)z^{(k)}\partial_{z^{(k)}}.
$$

Invariants for this realisation are

$$
\begin{aligned}
& y'' - z - (\frac{z''}{2} - \alpha)z'', y''' - z' - z''z''', y^{(4)} - z'' - z''z^{(4)}, \\
& z^{(j)}, y^{(k)} - z^{(k)}z'', j \geq 3, k \geq 5,
\end{aligned} \tag{3.9}
$$

and the operator of the invariant differentiation is D_x.

And for the realisation of solvable extensions, invariants are

$$
\begin{aligned}
& \frac{y'' - \frac{(z'')^2}{2} - z}{(z''')^4}, \frac{y''' - z' - z''z'''}{(z''')^3}, \frac{y^{(4)} - z'' - z''z^{(4)}}{(z''')^2}, \\
& \left(y^{(k)} - z^{(k)}z''\right)(z''')^{k-6}, k \geq 5, \\
& z^{(4)}, z^{(j)}(z''')^{j-4}, j \geq 5.
\end{aligned} \tag{3.10}
$$

The operator of invariant differentiation is clearly $z'''D_x$.

3.2.3 Realisation $\mathfrak{N}^{\mathbf{2}}_{n,b}(P_N, P_{N-1})$

Finding invariants for this realisation in an arbitrary dimension is very complicated. The complexity of prolonged fields increases rapidly due to presence of terms like $D_x^i \frac{z^j}{j!}$. Despite this we were able to successfully reduce the problem.

It can be shown that for any choice of the polynomials P_N, P_{N-1}, the vector fields E_k, D_1, D_2 always spans the same distribution on TM, i.e. the invariants do not depend on the choice of P_N, P_{N-1}. Therefore we can choose one representative, e.g. nilpotent part E_k, D_1, D_2 from (2.132).

Reader can easily see that

$$z', z'', \dots, z^{(k)}, \dots \tag{3.11}$$

are undoubtedly invariants of realisation of nilpotent algebra and

$$z'', z'''z', \dots, z^{(k)}(z')^{k-2}, \dots \tag{3.12}$$

of the solvable extension. The problem is that this set is not complete — the invariants containing derivatives of y are missing.

Regardless of our inability to find tho whole prolongation of vector fields we can find an operator of invariant differentiation. It is sufficient that we know prolongation "in the ∂_z part".

The operator of invariant differentiation is D_x for the realisation of the nilpotent part and $z' \cdot D_x$ for the solvable extension.

If we knew a y-invariant of the least order, we could construct all invariants from this information.

3.3 ODEs with $\mathfrak{n}_{n,3}$ as algebra of infinitesimal symmetries

3.3.1 Realisation of $\mathfrak{N}^3_{n,a}$ and its invariants

We can simplify our work during the realisation, because various choices of the parameters $\tilde{\alpha}_1, \dots, \tilde{\alpha}_N$ represent different choices of basis in same subalgebra of vector fields. Therefore it is sufficient to choose one set of parameters (lets say $\tilde{\alpha}_i = 0$ for all i).

$$\begin{aligned}
\operatorname{pr} D_1 &= -\partial_x, \\
\operatorname{pr} E_k &= \sum_{i=0}^{k} \frac{x^{k-i}}{(k-i)!}\partial_{y^{(i)}}, \\
\operatorname{pr} F &= \partial_z, \\
\operatorname{pr} D_f &= x\partial_z + \partial_{z'} + \sum_{n=0}^{\infty} z^{(n)}\partial_{y^{(n)}}.
\end{aligned} \tag{3.13}$$

The invariants for these realisations are

$$\begin{aligned}
&z'',\ z''',\ z^{(4)},\ \dots \\
&y^{(N+1)} - z^{(N+1)}z',\ y^{(N+2)} - z^{(N+2)}z', \dots;\ N = n-4.
\end{aligned} \tag{3.14}$$

When the invariants of the realisation of the nilradical are known, we can advance to invariants of realisations of the solvable extensions. Computation is rather straightforward hence we only summarise results in tables 3.2 and 3.3.

Table 3.2: Prolongation of realisation of solvable extension of $\mathfrak{N}^{\mathbf{3}}_{n,a}$

Algebra	Prolongation
$\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$	$\alpha x\partial_x + \sum_{n=0}^{\infty}\left(\left(2\beta+(1-n)\,\alpha\right)y^{(n)}\partial_{y^{(n)}} + (\beta+(1-n)\alpha)\,z^{(n)}\partial_{z^{(n)}}\right)$
$\mathfrak{s}_{n+1,6}(\varepsilon)$	$x\partial_x - \varepsilon\sum_{i=0}^{N+1}\frac{x^{N+1-i}}{(N+1-i)!}\partial_{y^{(i)}} + \sum_{j=0}^{\infty}\left(\left(N+1-j\right)y^{(j)}\partial_{y^{(j)}} + \left(\frac{N+2}{2}-j\right)z^{(j)}\partial_{z^{(j)}}\right)$
$\mathfrak{s}_{n+1,7}$	$x\partial_x + \sum_{i=0}^{\infty}\left(z^{(i)}+(1-i)y^{(i)}\right)\partial_{y^{(i)}} + \sum_{j=0}^{\infty}(1-j)z^{(j)}\partial_{z^{(j)}}$
$\mathfrak{s}_{n+1,9}$	$x\partial_x + \sum_{j=0}^{\infty}\left(\left((3-j)y^{(j)} - \frac{\mathrm{d}^j}{\mathrm{d}^j x}(xz)\right)\partial_{y^{(j)}} + (2-j)z^{(j)}\partial_{z^{(j)}}\right) - \sum_{i=0}^{2}\frac{x^i}{i!}\partial_{z^{(2-i)}}$
$\mathfrak{s}_{n+2}$	$\operatorname{pr} S_1 = x\partial_x + \sum_{j=0}^{\infty}(1-j)(y^{(j)}\partial_{y^{(j)}} + z^{(j)}\partial_{z^{(j)}})$ $\operatorname{pr} S_2 = \sum_{j=0}^{\infty}(2y^{(j)}\partial_{y^{(j)}} + z^{(j)}\partial_{z^{(j)}})$

Table 3.3: Invariants of realisation of solvable extension of $\mathfrak{N}^{\mathbf{3}}_{n,a}$

Algebra	Invariants — $k, j \in \mathbb{N}$
$\mathfrak{s}_{n+1,\nu}(1,1)$	$z'',\ \frac{z^{(3+k)}}{(z''')^{k+1}},\ \frac{y^{(N+j)} - z^{(N+j)}z'}{(z''')^{N+j-3}}$
$\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$, $\alpha\neq\beta$	$\frac{(z^{(2+k)})^{\beta-\alpha}}{(z'')^{\beta-(k+1)\alpha}},\ \frac{(y^{(N+j)} - z^{(N+j)}z')^{\beta-\alpha}}{(z'')^{2\beta-(N+j-1)\alpha}}$
$\mathfrak{s}_{n+1,6}(\varepsilon)$	$z^{(j+1)}e^{-\varepsilon(j-\frac{N}{2})(y^{(N+1)}-z^{(N+1)}z')},\ (y^{(N+1+k)} - z^{(N+1+k)}z')e^{-\varepsilon k(y^{(N+1)}-z^{(N+1)}z')}$
$\mathfrak{s}_{n+1,7}$	$\frac{z^{(2+k)}}{(z'')^{k+1}},\ \frac{y^{(N+j)} - z^{(N+j)}(z' - \ln z'')}{(z'')^{N+j-1}}$
$\mathfrak{s}_{n+1,9}$	$z^{(k+2)}e^{-kz''},\ \left(y^{(N+j)} - z^{(N+j)}z' - (N+j)z^{(N+j-1)}z''\right)e^{-(N+j-3)z''}$
$\mathfrak{s}_{n+2}$	$\frac{z^{(k+3)}(z'')^k}{(z''')^{k+1}},\ \frac{\left(y^{(N+j)} - z^{N+j}z'\right)(z'')^{N+j-5}}{(z''')^{N+j-3}}$

3.3.2 Realisation $\mathfrak{N}^3_{n,b}$ and its invariants

This case is slightly more complicated than the previous one. The prolongations of the elements of the nilradical are

$$
\begin{aligned}
\operatorname{pr} D_1 &= -\partial_x, \\
\operatorname{pr} E_k &= \sum_{i=0}^{k} \frac{x^{k-i}}{(k-i)!}\partial_{y^{(i)}}, \\
\operatorname{pr} F &= \sum_{j=0}^{\infty} z^{(j)}\partial_{y^{(j)}}, \\
\operatorname{pr} D_f &= -\partial_z + \sum_{n=0}^{\infty}(xz^{(n)} + nz^{(n-1)})\partial_{y^{(n)}}.
\end{aligned}
\tag{3.15}
$$

The invariants for these realisations are

$$
\begin{aligned}
& z',\ z'',\ z''',\ \ldots, \\
& \chi_j := y^{(N+1+j)} - \frac{z^{(N+1+j)}}{z^{(N+1)}}y^{(N+1)} - (N+1)\frac{z^{(N)}z}{z^{(N+1)}}z^{(N+1+j)} + (N+1+j)z^{(N+j)}z, \\
& N = n-4, j \in \mathbb{N}.
\end{aligned}
\tag{3.16}
$$

We express the invariants of the solvable extensions in terms of $z^{(k)}$ and χ_j — otherwise the formulae would be too complicated.

In tables 3.4 and 3.5 we present the results of rather straightforward computations.

3.3.3 Operators of invariant differentiation

Interesting by-products of our computations are operators of invariant differentiation. We do not present detailed calculations because they are straightforward — compare the example 1.25. Let us just mention that for the nilpotent part of the realisation the vector fields do not have non-constant part of ∂_x, i.e. they are in the form $\alpha\partial_x+\eta(x,y,z)\partial_y+\rho(x,y,z)\partial_z$. That is why D_x itself is the operator of invariant differentiation for the realisation of the nilpotent part. The operator of invariant differentiation is not determined uniquely, but any particularly choice is sufficient for our purposes, i.e for construction of functionally independent higher order invariants from the lower order ones. We summarise results in the table 3.6.

Table 3.4: Prolongation of realisation of solvable extension of $\mathfrak{N}^{\mathbf{3}}_{n,b}$

Algebra	Prolongation
$\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$	$\alpha x\partial_x + \sum_{n=0}^{\infty}\left(\left(2\beta+(1-n)\,\alpha\right)y^{(n)}\partial_{y^{(n)}} + (\beta - n\alpha)\,z^{(n)}\partial_{z^{(n)}}\right)$
$\mathfrak{s}_{n+1,6}(\varepsilon)$	$x\partial_x - \varepsilon\sum_{i=0}^{N+1}\frac{x^{N+1-i}}{(N+1-i)!}\partial_{y^{(i)}} + \sum_{j=0}^{\infty}\left((N+1-j)\,y^{(j)}\partial_{y^{(j)}} + \left(\frac{N}{2}-j\right)z^{(j)}\partial_{z^{(j)}}\right)$
$\mathfrak{s}_{n+1,7}$	$x\partial_x + \sum_{i=0}^{\infty}(1-i)y^{(i)}\partial_{y^{(i)}} - \partial_z - \sum_{j=1}^{\infty}jz^{(j)}\partial_{z^{(j)}}$
$\mathfrak{s}_{n+1,8}(1,\vec{0})$	$\left(2y-\frac{z^2}{2}\right)\partial_y + \sum_{k=1}^{\infty}\left(2y^{(k)} - \sum_{l=0}^{k-1}\binom{k-1}{l}z^{(l)}z^{(k-l)}\right)\partial_{y^{(k)}} + \sum_{j=0}^{\infty}z^{(j)}\partial_{z^{(j)}}$
$\mathfrak{s}_{n+1,9}$	$x\partial_x + \sum_{j=0}^{\infty}\left(\left((3-j)y^{(j)} - \frac{\mathrm{d}^j}{\mathrm{d}^j x}(\frac{x^2}{2}z)\right)\partial_{y^{(j)}} + (1-j)z^{(j)}\partial_{z^{(j)}}\right) + x\partial_z + \partial_{z'}$
$\mathfrak{s}_{n+2}$	$\operatorname{pr}S_1 = x\partial_x + \sum_{j=0}^{\infty}\left((1-j)y^{(j)}\partial_{y^{(j)}} - jz^{(j)}\partial_{z^{(j)}}\right)$ $\operatorname{pr}S_2 = \sum_{j=0}^{\infty}(2y^{(j)}\partial_{y^{(j)}} + z^{(j)}\partial_{z^{(j)}})$

Table 3.5: Invariants of realisation of solvable extension of $\mathfrak{N}^{\mathbf{3}}_{n,b}$

Algebra	Invariants — $k, j \in \mathbb{N},\ k \geq 2$
$\mathfrak{s}_{n+1,\nu}(1,1)$	$z',\ \dfrac{z^{(k+1)}}{(z'')^k},\ \dfrac{\chi_j}{(z'')^{N+j-2}}$
$\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$ $\alpha \neq \beta$	$\dfrac{(z^{(k)})^{\beta-\alpha}}{(z')^{\beta-k\alpha}},\ \dfrac{\chi_j^{\beta-\alpha}}{(z')^{2\beta-(N+j)\alpha}}$
$\mathfrak{s}_{n+1,6}(\varepsilon)$	$z^{(k)}\,(z')^{\frac{N-2k}{2-N}},\ \left(\chi_j + \dfrac{2\varepsilon}{2-N}\dfrac{z^{(N+1+j)}}{z^{(N+1)}}\ln z'\right)(z')^{\frac{2j}{N-2}}$
$\mathfrak{s}_{n+1,7}$	$\dfrac{z^{(k)}}{(z')^k},\ \left\{\chi_j + \left[(N+1)\dfrac{z^{(N)}z^{(N+1+j)}}{z^{(N+1)}} - (N+1+j)z^{(N+j)}\right]\ln z'\right\}(z')^{-(N+j)}$
$\mathfrak{s}_{n+1,8}(1,\vec{0})$	$\dfrac{z^{(k)}}{z'},\ \dfrac{\chi_j + \left[\frac{d^{N+j}}{dx^{N+j}}zz' - \frac{z^{(N+1+j)}}{z^{(N+1)}}\frac{d^N}{dx^N}zz'\right]\ln z'}{(z')^2}$
$\mathfrak{s}_{n+1,9}$	$z^{(k)}e^{(k-1)z'},\ \left\{\chi_j + \left[\binom{N+1+j}{2}z^{(N+j-1)} - \binom{N+1}{2}\dfrac{z^{(N-1)}z^{(N+j+1)}}{z^{(N+1)}}\right]z'\right\}e^{(N+j-2)z'}$
$\mathfrak{s}_{n+2}$	$\dfrac{z^{(k+1)}(z')^{k-1}}{(z'')^k},\ \dfrac{\chi_j(z')^{N+j-4}}{(z'')^{N+j-2}}$

Table 3.6: Operators of invariant differentiation for the realisations of $\mathfrak{n}_{n,3}$ and its solvable extensions

	Realisation $\mathfrak{N}^{\mathbf{3}}_{n,a}$	Realisation $\mathfrak{N}^{\mathbf{3}}_{n,b}$
$\mathfrak{n}_{n,3}$	D_x	D_x
$\mathfrak{s}_{n+1,\nu}(1,1)$	$\frac{1}{z'''}D_x$	$\frac{1}{z''}D_x$
$\mathfrak{s}_{n+1,\nu}(\alpha,\beta)$ $\alpha \neq \beta$	$(z'')^{\frac{\alpha}{\beta-\alpha}}D_x$	$(z')^{\frac{\alpha}{\beta-\alpha}}D_x$
$\mathfrak{s}_{n+1,6}(\varepsilon)$	$(z'')^{\frac{2}{2-N}}D_x$	$(z')^{\frac{2}{2-N}}D_x$
$\mathfrak{s}_{n+1,7}$	$\frac{1}{z''}D_x$	$\frac{1}{z'}D_x$
$\mathfrak{s}_{n+1,8}(1,\vec{0})$	—	D_x
$\mathfrak{s}_{n+1,9}$	$e^{-z''}D_x$	$e^{z'}D_x$
$\mathfrak{s}_{n+2}$	$\frac{z''}{z'''}D_x$	$\frac{z'}{z''}D_x$

Chapter 4

Properties of differential equations with given symmetries

4.1 Solution by quadratures

We briefly outline the situation. Details can be found e.g. in [18].

Suppose a system of differential equation

$$F_a(x, y_i, y_i', \ldots, y_i^{(n)}) = 0 \tag{4.1}$$

with an algebra of infinitesimal symmetries $\mathcal{L}$ is given. This system should not be neither undetermined nor overdetermined, i.e. we have as many equations as dependent variables y_i.

We can use any symmetry $g \in \mathcal{L}$ to lower the order of the equation. The procedure is as following: Find point transformation which alter g to $\partial_{\tilde{y}_j}$. Such transformation always exists. The vector field $\partial_{\tilde{y}_j}$ is still the infinitesimal symmetry of the equations (4.1). This means that the variable $\tilde{y}_j$ does not occur in the equations and we can make substitution $u(x) := \tilde{y}_j'(x)$ which lowers the order.

A natural question is whether we could continue this process until we reduce the system of differential equations (4.1) to the system of algebraic equations i.e. implicit function. If this is possible we say that the system is solvable by quadratures. let us leave aside the problem that even if we are able to solve the system by quadratures we are not in general able to perform the backward transformations to get the explicit solutions.

The trivial observation shows that if we have k dependent variables and the order of the equation (i.e. the highest derivative) is n we usually need $k \cdot n$ independent symmetries. Therefore the dimension of the algebra ought to be at least $k \cdot n$.

Another issue is how to proceed — which symmetry choose as the first one in order to do not lost other symmetries. And is this even possible? This was in fact problem which had studied M. S. Lie when he discovered Lie algebras. We know that if the algebra is solvable, such process is possible — just choose vector from the smallest element of the lower central series. This is the reason why such an algebras are called “solvable”.

The last hassle is how to maintain all transformations point-like, we do not want any derivation in our transformations. We can achieve this when we do a small trick (maybe we should say a small cheat). We artificially lower the order of the system (4.1) to one in exchange with increasing number of dependent variables. We get the system

$$\begin{aligned} F_a(x, y_i, y_{i,1}, y_{i,2} \dots, y'_{i,n-1}) &= 0, \\ y'_i &= y_{i,1}, \\ y'_{i,1} &= y_{i,2}, \\ &\dots \\ y'_{i,n-2} &= y_{i,n-1}. \end{aligned} \tag{4.2}$$

Then all transformations all point transformation in the variables $y_{i,1}, \dots, y_{i,n-1}$ and every symmetry will change one equation to an algebraic one.

4.2 Investigation of ODE with $\mathbb{R}^2$-realisation as the algebra of infinitesimal symmetries

Let's take (n+1)-dimensional solvable extension and find which equations could we solve by quadratures. We have $n+1$ symmetries, therefore we are allowed to make an equation from first two invariants. Our symmetries from nilpotent part are in a perfect form, we can use all E_j symmetries and do substitution $u := y^{(n-1)}$. We present our results in Table 4.1

Table 4.1: List of invariant ODEs

Original algebra	Remaining symmetries	ODE
$\mathfrak{h}_1(B) := \mathfrak{s}_{n+1,\iota\nu}(1, 1+B),\ B \neq 0$	$\partial_x,\ x\partial_x + Bu\partial_u$	$u'' \cdot u^{\frac{2-B}{B}} = F\left(u' \cdot u^{\frac{1-B}{B}}\right)$
$\mathfrak{h}_2 := \mathfrak{s}_{n+1,\iota\nu}(1, 1)$	$\partial_x,\ x\partial_x$	$u'' = u'^2 F(u)$
$\mathfrak{h}_3 := \mathfrak{s}_{n+1,\iota\nu}(0, 1)$	$\partial_x,\ u\partial_u$	$u'' = u' F\left(\dfrac{u'}{u}\right)$
$\mathfrak{h}_4 := \mathfrak{s}_{n+1,5}$	$\partial_x,\ x\partial_x - \partial_u$	$u'' = e^{2u} F\left(u' e^{-u}\right)$

We could transform $\mathfrak{h}_4$ and $\mathfrak{h}_1(B)$ to the $\mathfrak{h}_1(1)$ and the equation of type $\mathfrak{h}_3$ into equation with symmetries ∂_x, ∂_v. The transformations are in the Table 4.2.

It is apparently sufficient to study three cases in the Table 4.3.

Both cases $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are solvable by quadratures. We transform them to the separated differential equations which are obviously solvable by quadratures (see Table 4.4).

Furthermore all three cases are Euler-Lagrangian equations. We do not go into details (they can be found for example in [11]) but Lagrangians listed in the Table 4.5 were found

Table 4.2: Coordinate changes

Former ODE	Coordinate change	New ODE
$\mathfrak{h}_1(B)$: $u'' \cdot u^{\frac{2-B}{B}} = F\left(u' \cdot u^{\frac{1-B}{B}}\right)$	$t = x$ $v = By^{\frac{1}{B}}$	$\partial_t,\, t\partial_t + v\partial_v$ $\ddot{v}v = (1-B)\dot{v}^2 + BF(\dot{v})$
$\mathfrak{h}_4$: $u'' = e^{2u}F\left(u'e^{-u}\right)$	$t = x$ $v = e^{-y}$	$\partial_t,\, x\partial_x + v\partial_v$ $\ddot{v}v = \dot{v}^2 - F(\dot{v})$
$\mathfrak{h}_3$: $u'' = u'F\left(\frac{u'}{u}\right)$	$t = x$ $v = \ln y$	$\partial_t,\, \partial_v$ $\ddot{v} = \dot{v}F(\dot{v}) - \dot{v}$

Table 4.3: Simplified list of invariant ODEs

Class name (definition)	Symmetries	ODE
$\mathfrak{a}$	$\partial_t,\, \partial_v$	$v'' = G\left(v'\right)$
$\mathfrak{b}$	$\partial_t,\, t\partial_t$	$v'' = v'^2 G(v)$
$\mathfrak{c}$	$\partial_t,\, t\partial_t + v\partial_v$	$v'' = v^{-1} G\left(v'\right)$

Table 4.4: Solving of invariant ODEs with symmetries

	$\mathfrak{a}$: $v'' = F(v')$	
$\partial_v \to$	$s := t$ $w(t) := v'(t)$	$\Rightarrow \dot{w} = F(w)$
	$\mathfrak{b}$: $v'' = v'^2 G\left(v\right)$	
$\partial_t \to$	$s := v$ $w := t, w = w(s)$ $r(s) := \ln w'(s)$	$\Rightarrow r' = G(s)$
	$\mathfrak{c}$: $v'' \cdot v = G\left(v'\right)$	
$t\partial_t + v\partial_v \to$	$s := v^{-1}t$ $\Rightarrow w := \ln v$ $r(t) := \dot{w}(t)$	$\Rightarrow r' = G(r)$

because we have assumed that the action generated by those Lagrangians admits same symmetries.

Table 4.5: List of Lagrangians for ODEs

Case: Symmetries	ODE	Lagrangian	
$\mathfrak{a}\colon \partial_x,\, \partial_y$	$y'' = G(y')$	$L = y + H(y')$	$\ddot{H}(t) = \dfrac{1}{G(t)}$
$\mathfrak{b}\colon \partial_x,\, x\partial_x$	$y'' = y'^2 G(y)$	$L = -y' \ln(y') e^{-\int G(y)}$	
$\mathfrak{c}\colon \partial_x,\, x\partial_x + y\partial_y$	$y'' \cdot y = G(y')$	$L = y^{-1} H(y')$	$\ddot{H}(t) G(t) = t\dot{H}(t) - H(t)$

4.3 Partial simplification of SDE with the algebra of symmetries $\mathfrak{N}^3_{n,b}$

We shall demonstrate procedure of solving SDE using known symmetries. The simplest SDE which contains both y and z is

$$\begin{aligned}
y^{(N+2)} - \frac{z^{(N+2)}}{z^{(N+1)}} y^{(N+1)} - (N+1)\frac{zz^{(N)}}{z^{(N+1)}} z^{(N+2)} + (N+2)zz^{(N+1)} &= F(z', z'', \dots, z^{(k-1)}),\\
z^{(k)} &= G(z', z'', \dots, z^{(k-1)}),\\
N &= n-4, k < N.
\end{aligned} \tag{4.3}$$

Available symmetries are

$$\begin{aligned}
E_i &= \frac{x^i}{i!}\partial_y, 0 \geq i \geq N,\\
D_1 &= -\partial_x,\\
F &= z\partial_y,\\
D_f &= xz\partial_y - \partial_z.
\end{aligned} \tag{4.4}$$

Using these symmetries we can partially solve this system. As the first step we enlarge the set of equations in exchange for lowering their order. The other reason is to keep all substitutions as point transformations (2.2).

$$u' - \frac{w'}{w}u - (N+1)\frac{zv}{w}w' + (N+2)wz = F(z', z'', \ldots, z^{(k-1)}), \tag{4.5}$$
$$z^{(k)} = G(z', z'', \ldots, z^{(k-1)}), \tag{4.6}$$
$$u = y^{(N+1)}, \tag{4.7}$$
$$w = v', \tag{4.8}$$
$$v = z^{(N)}. \tag{4.9}$$

Owing to this change of SDE our available symmetries change too.

$$\begin{aligned} E_i &= \frac{x^i}{i!}\partial_y, 0 \leq i \leq N, \\ D_1 &= -\partial_x, \\ F &= z\partial_y + w\partial_u, \\ D_f &= -\partial_z + xz\partial_y + (xw + (N+1)v)\partial_u. \end{aligned} \tag{4.10}$$

It is obvious that we can put aside equation (4.7), it can be solved easily afterwards. We focus on remaining equations which in addition do not contain y at all. We have used all the symmetries E_i the and remaining symmetries act effectively as

$$\begin{aligned} D_1 &= -\partial_x, \\ F &= w\partial_u, \\ D_f &= -\partial_z + (xw + (N+1)v)\partial_u. \end{aligned} \tag{4.11}$$

We proceed with substitution

$$\begin{aligned} \tilde{u} &:= \frac{u}{w} + (N+1)\frac{v}{w}z, \\ \tilde{v} &:= (N+1)\frac{v}{w} \\ w &= w \\ z &= z \end{aligned} \tag{4.12}$$

Our equations are now equivalent to the system

$$\begin{aligned} \tilde{u}'w - (\tilde{v}wz)' + (N+2)wz &= F(z', z'', \ldots, z^{(k-1)}), \\ z^{(k)} &= G(z', z'', \ldots, z^{(k-1)}), \\ (N+1)w &= \tilde{v}'w + \tilde{v}w', \\ \tilde{v}w &= (N+1)z^{(N)}, \end{aligned} \tag{4.13}$$

and symmetries of this system are

$$\partial_x, \partial_{\tilde{u}}, -\partial_z + x\partial_{\tilde{u}}. \tag{4.14}$$

By virtue of the symmetry $\partial_{\tilde{u}}$ the variable $\tilde{u}$ is not present in the system (the lower derivative is the first one). Therefore we can simplify our equation again and set $g := \tilde{u}'$. Remnant symmetries are ∂_x and $\partial_g - \partial_z$. Substitution $h := g + z$ and consecutive order lowering $s := z'$ shape our system into the following form

$$\begin{aligned} hw - \tilde{v}ws &= F(s, s', \dots, s^{(k-2)}), \\ s^{(k-1)} &= G(s, s', \dots, s^{(k-2)}), \\ (N+1)w &= \tilde{v}'w + \tilde{v}w', \\ \tilde{v}w &= (N+1)s^{(N-1)}, \end{aligned} \tag{4.15}$$

with the single known symmetry ∂_x.

4.4 Discussion about simplification of SDEs

Unfortunately apart from some exceptional cases the system of differential equations for y, z with algebra of infinitesimal symmetries isomorphic to $\mathfrak{n}_{n,3}$ and it solvable extensions is not solvable by quadratures. We do not even have enough symmetries. The system has two dependent variables and is of at least $N+1$ (respectively $N+2$) order and we have $N+5$ symmetries maximum. Moreover we can always separate system in the following way:

$$\begin{aligned} y^{(k)} &= F_1(x, z, z', \dots), \\ z^{(j)} &= F_2(x, z, z', \dots). \end{aligned} \tag{4.16}$$

We can just solve the ordinary differential equation for z, then substitute into first equation and integrate. Therefore these equations are not very interesting as a system of differential equations.

The reason why is the SDE always equivalent to the (4.16) is that the highest derivative of y in all invariants appears linearly. Suppose we have

$$\begin{aligned} y^{(k)} &= F_1(x, y, z, y', z', \dots, y^{(k-2)}, z^{(k-2)}, z^{(k-1)}, z^{(k)}), \\ y^{(k-1)} &= F_2(x, y, z, y', z', \dots, y^{(k-2)}, z^{(k-2)}, z^{(k-1)}, z^{(k)}). \end{aligned} \tag{4.17}$$

in the beginning. Next, we can differentiate second equation and compare it with the first one:

$$\begin{aligned} y^{(k-1)} &= F_2(x, y, z, y', z', \dots, y^{(k-2)}, z^{(k-2)}, z^{(k-1)}, z^{(k)}), \\ (D_x F_2 - F_1)&(x, y, z, y', z', \dots, y^{(k-1)}, z^{(k-1)}, z^{(k)}, z^{(k+1)}) = 0. \end{aligned} \tag{4.18}$$

We have removed $y^{(k)}$ from the system in exchange with $z^{(k+1)}$. And we can iterate this process — express $y^{(k-1)}$ and $y^{(k-2)}$, differentiate, compare and so on. In the end we get an equation in the form (4.16).

Conclusions

We have thoroughly classified the realisations of $\mathfrak{n}_{n,1}, \mathfrak{n}_{n,2}$ and $\mathfrak{n}_{n,3}$ from [7–9] in this thesis. It was revealed that the only algebra which posses a realisation on $\mathbb{R}^2$ is $\mathfrak{n}_{n,1}$, that realisation is determined uniquely (as an subalgebra of $\mathfrak{X}(\mathbb{R}^2)$) and we can construct equations which could be solved by quadratures using known symmetries. Furthermore these equations (after reducing to the equation of the second order) are all Euler-Lagrangian equation for some Lagrangian L.

Although the realisation of $\mathfrak{n}_{n,1}$ on $\mathbb{R}^2$ was essentially (i.e. up to automorphisms of $\mathfrak{n}_{n,1}$) unique, the situation on $\mathbb{R}^3$ is bit different. The realisations of $\mathfrak{n}_{n,1}$ splits into two distinct cases on whose is there a functional freedom — we have a set of functions which we have to specify in order to obtain a realisation and for different choices of the set we get different realisations. In addition all solvable extensions of $\mathfrak{n}_{n,1}$ can be realised which was not the case of $\mathbb{R}^2$. For the $\mathbb{R}^2$ the realisation of the algebra $\mathfrak{s}_{n+1,6}(a_2, \dots, a_N)$ does not exist. However this algebra makes trouble also in the realisation on $\mathbb{R}^3$ — we know that such realisation exists but the explicit form is very hard to find. Despite that, we managed to find an explicit solution for a special choice of initial conditions.

We were able to realise also the other algebras. Due to the fact that both $\mathfrak{n}_{n,2}$ and $\mathfrak{n}_{n,3}$ have $\mathfrak{n}_{k,1}$ as their subalgebra, the situation was helped by the knowledge of all realisations of $\mathfrak{n}_{n,1}$ in considered dimensions. Partly because we can use our previous result and partly because some classes of equivalence disappear and some classes are getting narrower. Especially the realisations of the algebra $\mathfrak{n}_{n,3}$ are very similar to the $\mathbb{R}^2$-realisations of $\mathfrak{n}_{n,1}$. It is not surprising because these algebras are closely related as was described right in the [9]. We should also mention that there are two exceptional low dimensional realisations of $\mathfrak{n}_{n,2}$ namely $\mathfrak{N}^{\mathbf{2}}_{5,a}$ and $\mathfrak{N}^{\mathbf{2}}_{6,a}$ which look like a part of some series of realisations. We conjecture that on $\mathbb{R}^4$ there ought to be next elements of this sequence.

In the third chapter we have found all differential equations of one independent and two dependent variables for those two exceptional cases of $\mathfrak{n}_{n,2}$ and for all realisations of $\mathfrak{n}_{n,3}$ including their solvable extensions. In other cases the results were not of particular interest or we were not able to calculate them due to the complexity of realisations.

The main direction of future research is to use the already found realisations to obtain difference equations with our realisations as their algebras of the infinitesimal symmetries. There are still many open questions in the symmetry analysis of systems of difference equations and these equations with prescribed symmetry could be very useful for testing various conjectures.

Among the most interesting and challenging questions belong: How should we properly discretise a system of differential equation so that the difference equations admits the same symmetries? Is it true in general for systems of first order ODEs that there always exists such discretisation that the numeric method originated from that discretisation is exact (that means that the points obtained from the algorithm are exactly on the solution)? If not, can it be deduced ar least for some particular form of the system? Could we do this discretisation in an algorithmic manner and which discretisation shall we choose (because this is the biggest issue for this family of numeric methods)?

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