CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

Differential and difference equations invariant under a given solvable Lie algebra

Research project

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V Praze, dne 30. 6. 2010 Dalibor Karásek

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Název práce: **Diferenciální a diferenční rovnice invariantní vzhledem k dané řešitelné Lieově algebře**
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Abstrakt:

Zkounstruujeme všechny diferenciální a diferenční rovnice, které mají za své infinitesimální symetrie stejnou algebru. Tyto diferenciální rovnice vyřešíme kvadraturami a nalezneme pro ně Lagrangiány. Prezentujeme myšlenku diferenčních schémat.

Klíčová slova: Lieovy algebry, symetrie, diferenční rovnice, diferenční schema, prolongace

Title: **Differential and difference equations invariant under a given solvable Lie algebra**
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Abstract:

We construct all differential and difference equations with specific Lie algebra as its infinitesimal symmetries. We solve these ODEs by quadratures and find the Lagrangian for each of them. We present the idea of difference schema.

Keywords: Lie algebras, symmetries, difference equations, prolongation, difference schema

Contents

Introduction

This research work deals with the ordinary differential equations, the difference equations Lagrangians, actions and their symmetries. Symmetries play significant role in the modern physics and mathematics. It is mainly because of Emmy Noether and her fundamental work which found connection between symmetries and integrals of motion. Symmetries are used during the discovering of physics laws, or creating physicals models. For example the standard model, in these days the most accurate way how to describe our universe, was developed by scrutinising of the symmetries of elementary particles.

We'll start with brief introduction to the mathematical theory. We assume that reader knows the basics of differential geometry and Lie algebras. They can be found for example in [1]. Then we'll link up with the previous work, take specific algebra and find all equations which has this algebra in common as the algebra of symmetries. We'll analyse these equations, for differential ones we'll find Lagrangian.

Finally, in the end, we present an idea of difference schemas, which can be used to produce numerical methods for solving ODEs. These methods can embody interesting attributes and it's the promising field, where is still lot a work to be done.

Chapter 1

Symmetries of ordinary differential equations

1.1 Basic definitions

It's not surprising that symmetries play a significant role in solution of the ODEs. For example when the problem has roots in the physics one can guess the coordinates (e.g. spherical), in which is the equation significantly simpler. But the symmetries of ODE is the much stronger tool. We can even use them to solve the ODE completely. In this section we provide basic definitions and ideas which hopefully enables anybody use this powerful tool. For more detailed information search e.g. in [2].

In this section

$$F(x, y, \ldots, y^{(n)}) = 0 \tag{1.1}$$

will be a given ODE.

Definition 1.1.

$$\mathcal{S}_F := \text{set of solutions of } F(x, y, \ldots, y^{(n)}) = 0 \tag{1.2}$$

Definition 1.2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function such that it transforms one graph of the solution of (1.1) $y = y(x)$ to another graph $T \triangleright y$.

$$T \text{ is the } \textbf{symmetry} \text{ of (1.1) } \overset{def}{\Longleftrightarrow} T \triangleright \mathcal{S}_F = \mathcal{S}_F \tag{1.3}$$

You can see that this definition is quite natural. We just take all maps which maintain the set of solutions. But there is a caveat, because $T \triangleright y$ doesn't have to be a graph of a function. To be correct we should do everything locally, but things then become hazy. Therefore we sacrifice strict mathematical correctness in favour to the understanding of the idea.

1.2 Search for symmetries

Now we know what the symmetries of an ODE are. But how can we found them? Hardly. The problem of finding the symmetries is highly nontrivial. The point is that we can "easily" find something very similar — infinitesimal symmetries. It also removes the problem with haziness of definitions, because this term is naturally local.

Definition 1.3. Prolongation is a map $\mathfrak{X}(\mathbb{R}^2) \to \mathfrak{X}(\mathbb{R}^k)$ for high enough k.

$$\operatorname{pr} X := \operatorname{pr}(\xi(x,y)\partial_x + \eta(x,y)\partial_y) = \xi(x,y)\partial_x + \sum_{i=0}^{k} \eta^i(x,y,y',\dots,y^{(i)})\partial_{y^{(i)}}, \tag{1.4}$$

$$\eta^0 = \eta, \quad \eta^i = D_x \eta^{i-1} - y^{(i)} \cdot D_x \xi,$$

$$D_x = \partial_x + \sum_{i=0}^{\infty} y^{(i+1)} \partial_{y^{(i)}}. \tag{1.5}$$

We take $x, y, y', \dots$ as the independent coordinates and D_x is sometimes called **total derivative** operator.

Definition 1.4. $X \in \mathfrak{X}(\mathbb{R}^2)$ is the **infinitesimal symmetry** of the $F(x,y,y',\dots) = 0$

$$\overset{def}{\iff} \operatorname{pr} X(F) \restriction_{F=0} (x,y,y',\dots) = 0. \tag{1.6}$$

The infinitesimal symmetries enclose to the Lie algebra.

Summary 1.5. The algorithm how to find symmetries of an ODE is following:

1. Take general vector field X.
2. Calculate $\operatorname{pr} X$.
3. Apply $\operatorname{pr} X$ on $F(x,y,y',\dots)$.
4. Substitute $F(x,y,y',\dots) = 0$ and all differential consequences.
5. Solve (1.6) for ξ and η. It is a system of partial linear homogenous differential equations.

1.3 Equations with given symmetries

Sometimes the other-way-round point of view could be interesting. Suppose we have a vector field algebra $\mathfrak{A}$. How does an ODE for which the $\mathfrak{A}$ are the infinitesimal symmetries look like? Finding the answer isn't very hard.

1. Take a basis X_i of the $\mathfrak{A} \subseteq \mathfrak{X}(\mathbb{R}^2)$.

2. Find the invariants of $\operatorname{pr} X_i$, i.e. functions I_k such that

$$\forall i, \quad \operatorname{pr} X_i(I_k) = 0.$$

   The method of characteristics is commonly used for solution of the equations of this type.

3. Any equation $G(I_1, \ldots, I_n) = 0$ has $\mathfrak{A}$ as its infinitesimal symmetries.

$$(\operatorname{pr} X_i)(G(I_1, \ldots, I_n)) = \sum_{j=1}^{n} \partial_j G \cdot (\operatorname{pr} X_i)(I_j) = \sum_{j=1}^{n} \partial_j G \cdot 0 = 0.$$

Chapter 2

ODEs with given sequence of solvable algebras as symmetries

2.1 Former results

In this chapter we will study equations with specific symmetries, which form a Lie algebras presented in the article [3]. All of them are solvable extensions of the n-dimensional nilpotent Lie algebra $\mathfrak{n}_{n,1}$ ($n \geq 4$). It has following commutating relations

$$\mathfrak{n}_{n,1} = \text{span}\{e_1, \ldots, e_n\}, \tag{2.1}$$
$$[e_k, e_n] = e_{k-1} \qquad 2 \leq k \leq n-1. \tag{2.2}$$

Other Lie brackets are vanishing.

We distinguish five classes of solvable extensions. First four contains are $(n+1)$-dimensional (base $e_1, \ldots, e_n, f$), fifth has one more vector in the base $(e_1, \ldots, e_n, f_1, f_2)$.

1. $\mathfrak{s}_{n+1,1}(B)$ is union of cases $\mathfrak{s}_{n+1,1}(1+B), \mathfrak{s}_{n+1,2}, \mathfrak{s}_{n+1,3}$ from the [3].

$$[f, e_k] = (n - B - k)e_k, \tag{2.3}$$
$$[f, e_n] = e_n.$$

2. $\mathfrak{s}_{n+1,4}$

$$[f, e_k] = e_k, \tag{2.4}$$
$$[f, e_n] = 0.$$

3. $\mathfrak{s}_{n+1,5}$ is a bit trickier.

$$[f, e_k] = (n - k)e_k, \tag{2.5}$$
$$[f, e_n] = e_n + e_{n-1}.$$

4. $s_{n+1,6}(a_1, \dots, a_{n-3})$ is very messy.

$$[f, e_k] = e_k + \sum_{i=1}^{k-2} a_{k-i-1} e_i, \qquad (2.6)$$
$$[f, e_n] = 0.$$

5. s_{n+2} is in fact combination of $\mathfrak{s}_{n+1,4}$ and $\mathfrak{s}_{n+1,1}$.

$$[f_1, e_k] = e_k, \qquad [f_2, e_k] = (n-k-1)e_k, \qquad (2.7)$$
$$[f_1, e_n] = 0, \qquad [f_2, e_n] = e_n.$$

We care about equations whose infinitesimal symmetries forms one of these algebras. Therefore we need realisation (faithful representation) of these algebras on $\mathfrak{X}(\mathbb{R}^2)$. When we know corresponding vector fields we can construct equation by algorithm presented in section 1.3.

2.2 Realisation of algebras

Firstly just recall that two realisations are equivalent if there exist a coordinate change that transforms the first realisation to the second one. When we are finding the realisation we can improve its form by changing the coordinates to the more convenient ones.

2.2.1 Realisation of nilpotent $\mathfrak{n}_{n,1}$

As first step we construct the realisation of the nilpotent algebra (2.2). We do it vector by vector. Look at subalgebra span$\{e_1, e_n\}$. It is Abelian algebra and that means that there exists two its nonequivalent realisations:

$$\text{linearly independent: } E_1 = \partial_y,\ E_n = -\partial_x \qquad (2.8)$$
$$\text{linearly dependent: } E_1 = \partial_y,\ E_n = x\partial_y \qquad (2.9)$$

Firstly we take linearly dependent case and show that this realisation is bad starting point. We extend this realisation successively to realisation of span$\{e_1, e_2, e_3, e_n\}$.

$$E_2 = \xi_2(x, y)\partial_x + \eta_2(x, y)\partial_y, \qquad (2.10)$$
$$E_3 = \xi_3(x, y)\partial_x + \eta_3(x, y)\partial_y. \qquad (2.11)$$

We add E_2 by determining ξ_2, η_2. And then the same for E_3.

$$[E_1, E_2] = 0 \implies E_2 = \xi_2(x)\partial_x + \eta_2(x)\partial_y, \qquad (2.12)$$
$$[E_2, E_n] = E_1 \implies E_2 := \partial_x + \eta_2(x)\partial_y, \qquad (2.13)$$
$$[E_1, E_3] = 0 \implies E_3 = \xi_3(x)\partial_x + \eta_3(x)\partial_y. \qquad (2.14)$$
$$(2.15)$$

That is enough to produce contradiction:

$$[E_3, E_n] = E_2 \tag{2.16}$$
$$\xi_3(x)\partial_y = \partial_x + \eta_2(x)\partial_y \tag{2.17}$$
$$\xi_3 = \eta_2 \quad 0 \cdot \partial_x \stackrel{!}{=} 1 \cdot \partial_x \tag{2.18}$$

Secondly we choose the linearly independent case. We compute rather straightforward, that realisation of $\mathfrak{n}_{n,1}$ has to be in this form:

$$E_n = -\partial_x \tag{2.19}$$
$$E_k = \left(\frac{x^{k-1}}{(k-1)!} + \sum_{i=0}^{k-3} \beta_{k-3-i} \frac{x^i}{i!} \right) \partial_y \tag{2.20}$$

Then we change base of realisation to smoothen the formulae. (We are lucky — it won't change the commutation relations.) This change is defined by relations.

$$E_n =: X, \tag{2.21}$$
$$E_k =: Y_{k-1} + \sum_{i=0}^{k-3} \beta_{k-3-i} Y_i, \quad k = 1, \ldots, n-1. \tag{2.22}$$

That means we get following realisation ρ:

$$\rho(e_n) = X = -\partial_x, \tag{2.23}$$
$$\rho(e_{k+1}) = Y_k = \frac{x^k}{k!}\partial_y, \qquad k = 0, \ldots, n-2. \tag{2.24}$$

It is the unique way how to realise this algebra (except changes of coordinates).

2.2.2 Realisation of solvable extensions of $\mathfrak{n}_{n,1}$

We continue by adding the vector field S (S_1, S_2 respectively) to the basis in the same manner. The structure of the nilpotent subalgebra $\mathfrak{n}_{n,1}$ constrains form of the added vector fields such that there is at most one way how to realise them.

- $s_{n+1,1}(B)$

$$S = -x\partial_x + (1 - n - B)y\partial_y \tag{2.25}$$

- $s_{n+1,4}$

$$S = -y\partial_y \tag{2.26}$$

- $s_{n+1,5}$

$$S = -x\partial_x + \left((1-n)y - \frac{x^{n-1}}{(n-1)!}\right)\partial_y \tag{2.27}$$

- $s_{n+1,6}(a_1, ..., a_{n-3})$

  A realisation of $s_{n+1,6}$ on $\mathbb{R}^2$ doesn't exist. We have to consider at least $\mathbb{R}^3$.

- s_{n+2}

$$S_1 = -y\partial_y \qquad\qquad S_2 = -x\partial_x + (2-n)y\partial_y \tag{2.28}$$

Remark 2.1. The realisation $s_{n+1,6}$ on $\mathbb{R}^3$ is e.g.

$$\begin{aligned}
S &= \partial_x, \\
E_n &= -\partial_z, \\
E_i &= \sum_{k,l\in\mathbb{N}_0} \frac{z^k x^l}{k!l!} \left(\sum_{\substack{|J|=l \\ [J]+k+l=i-1}} \frac{l!}{J!} a^J \right) e^x \partial_y, \\
[J] &= \sum_{s=1}^{\infty} s \cdot J_s,
\end{aligned} \tag{2.29}$$

where J is multiindex.

2.3 Invariants of solvable extensions of $\mathfrak{n}_{n,1}$

Before we can find invariants using the method of characteristics we have to prolong the realisations first. We simply use the formulae in the definition 1.3. It is impossible to get lost during prolongation and using the method of characteristics is also straightforward, so we only present the final results with a few notices. All vector fields can be easily prolongated to any order. This is marked by pr^{∞}.

The prolongated vector fields for the nilradical are

$$\mathrm{pr}^{(\infty)} X = -\partial_x \tag{2.30}$$

$$\mathrm{pr}^{(\infty)} Y_k = \sum_{i=0}^{k} \frac{x^{k-i}}{(k-i)!} \partial_{y^{(i)}}. \tag{2.31}$$

A general invariant ODE looks like

$$I(y^{(n-1)}, y^{(n)}, \dots, y^{(n+j)}, \dots) = 0. \tag{2.32}$$

Let us add one or two vector fields S_a. Because any invariant must be of the form (2.32), the added vector fields effectively act only on $y^{(n-1)}$ and higher jet coordinates.

- $s_{n+1,1}(B)$

$$\mathrm{pr}^{(\infty)} S = -x\partial_x + \sum_{i=0}^{\infty}(1-n-B+i)y^{(i)}\partial_{y^{(i)}}. \tag{2.33}$$

$\mathrm{pr}^{(\infty)} S$ acts effectively as

$$\mathrm{pr}^{(\infty)} S_{eff} = \sum_{i=0}^{\infty}(-B+i)y^{(n-1+i)}\partial_{y^{(n-1+i)}}, \tag{2.34}$$

and the solution is

  – for $B \neq 0$

$$I(y^{(n)}y^{(n-1)^{\frac{1-B}{B}}}, y^{(n+1)}y^{(n-1)^{\frac{2-B}{B}}}, \ldots, y^{(n+j)}y^{(n-1)^{\frac{1+j-B}{B}}}, \ldots) = 0 \tag{2.35}$$

  – for $B = 0$

$$I(y^{(n-1)}, y^{(n+1)}y^{(n)^{-2}}, y^{(n+2)}y^{(n)^{-3}}, \ldots, y^{(n+j)}y^{(n)^{-j-1}}, \ldots) = 0 \tag{2.36}$$

- $s_{n+1,4}$

$$\mathrm{pr}^{(\infty)} S = -\sum_{i=0}^{\infty} y^{(i)}\partial_{y^{(i)}} \tag{2.37}$$

and ODE is in form

$$I\left(\frac{y^{(n)}}{y^{(n-1)}}, \frac{y^{(n+1)}}{y^{(n-1)}}, \ldots, \frac{y^{(n+j)}}{y^{(n-1)}}, \ldots\right) = 0. \tag{2.38}$$

- $s_{n+1,5}$

$$\mathrm{pr}^{(\infty)} S = -x\partial_x + \sum_{i=0}^{n-1}\left(\frac{x^{n-1-i}}{(n-1-i)!} + (1-n+i)y^{(i)}\right)\partial_{y^{(i)}} + \sum_{i=n}^{\infty}(1-n+i)y^{(i)}\partial_{y^{(i)}}. \tag{2.39}$$

Effectively acts $\mathrm{pr}^{(\infty)} S$ as

$$\mathrm{pr}^{(\infty)} S_{eff} = \partial_{y^{(n-1)}} + \sum_{i=1}^{\infty} i \cdot y^{(n-1+i)}\partial_{y^{(n-1+i)}}, \tag{2.40}$$

and

$$I(y^{(n)}e^{-y^{(n-1)}}, y^{(n+1)}e^{-2y^{(n-1)}}, \ldots, y^{(n+j)}e^{-(j+1)y^{(n-1)}}, \ldots) = 0. \tag{2.41}$$

- s_{n+2} Prolongation of the vector fields S_1, S_2 are same as in the first case for $B = 0$ and as in the second case. We can also change the basis of the algebra such that S_a are in the form $x\partial_x$ and $y\partial_y$. Invariant ODE is in the form

$$I\left(\frac{y^{(n+1)}y^{(n-1)}}{y^{(n)^2}}, \frac{y^{(n+2)}y^{(n-1)^2}}{y^{(n)^3}}, \ldots, \frac{y^{(n+j)}y^{(n-1)^j}}{y^{(n)^{j+1}}}, \ldots\right) = 0. \tag{2.42}$$

Chapter 3

Analysis of symmetric ODEs

3.1 Solution by quadratures

When the symmetries form solvable Lie algebra we can lower the order of ODE by the dimension of the algebra, and eventually solve it by quadratures (that is in fact also the reason why these algebras are called solvable). To be more precise when an ODE has a symmetries we can choose one randomly and decrement the order of the equation, but nobody can guarantee that the remaining symmetries prevail. On the other hand if the Lie algebra is solvable, we can take symmetry from the last member of the derived serie. It is then guaranteed that all other symmetries remain.

Whole process of lowering the order is performed by changing the coordinates in the way that a chosen symmetry changes to the form ∂_y. Then the ODE doesn't contain y and we can substitute $u(x) = y'(x)$.

We take the most simple but still interesting equations and uses the theorem about implicit functions to separate the term with the highest derivation. Everyone can see that we can immediately substitute $u = y^{(n-1)}$ and rename it back to y. In fact we have just used symmetries corresponding to $Y_0, \ldots, Y_{n-2}$ to lower the order of the equation. We present the list of the obtained ODEs in the table 3.1.

Original algebra	Remaining symmetries	ODE
$\mathfrak{h}_1(B) := \mathfrak{s}_{n+1,1}(B),\ B \neq 0$	$\partial_x,\ x\partial_x + By\partial_y$	$y'' \cdot y^{\frac{2-B}{B}} = F\left(y' \cdot y^{\frac{1-B}{B}}\right)$
$\mathfrak{h}_2 := \mathfrak{s}_{n+1,1}(0)$	$\partial_x,\ x\partial_x$	$y'' = y'^2 F(y)$
$\mathfrak{h}_4 := \mathfrak{s}_{n+1,4}$	$\partial_x,\ y\partial_y$	$y'' = y' F\left(\frac{y'}{y}\right)$
$\mathfrak{h}_5 := \mathfrak{s}_{n+1,5}$	$\partial_x,\ x\partial_x - \partial_y$	$y'' = e^{2y} F\left(y' e^{-y}\right)$
$h_6 := \mathfrak{s}_{n+2}$	$\partial_x,\ x\partial_x,\ y\partial_y$	$y''' y^2 = y'^3 F\left(\frac{y'' y}{y'^2}\right)$

Table 3.1: List of invariant ODEs

By glimpsing at the Table 3.1 one can easily notice that by proper change of the coordinates (in Table 3.2) we can simplify the symmetries and the equations even more. Furthermore this change of the coordinates is easily reversible. Therefore we can constrain ourselves only to cases in Table 3.3.

Former ODE	Coordinate change	New ODE	Class
$\mathfrak{h}_1(B)$: $y'' \cdot y^{\frac{2-B}{B}} = F\left(y' \cdot y^{\frac{1-B}{B}}\right)$	$t = x$ $z = By^{\frac{1}{B}}$	$\partial_t,\, t\partial_t + z\partial_z$ $z\ddot{z} = (1-B)\dot{z}^2 + BF(\dot{z})$	$\mathfrak{h}_1(1)$
$\mathfrak{h}_4$: $y'' = y'F\left(\frac{y'}{y}\right)$	$t = x$ $z = \ln y$	$\partial_t,\, \partial_z$ $\ddot{z} = \dot{z}F(\dot{z}) - \dot{z}$	Abelian
$\mathfrak{h}_5$: $y'' = e^{2y}F\left(y'e^{-y}\right)$	$t = x$ $z = e^{-y}$	$\partial_t,\, t\partial_t + z\partial_z$ $z\ddot{z} = \dot{z}^2 - F(\dot{z})$	$\mathfrak{h}_1(1)$
$\mathfrak{h}_6$: $y'''y^2 = y'^3F\left(\frac{y''y}{y'^2}\right)$	$t = x$ $z = \ln y$	$\partial_t,\, \partial_z,\, t\partial_t$ $\frac{\dddot{z}}{\dot{z}^3} = F(\frac{\ddot{z}}{\dot{z}^2} - 1) - 3\frac{\ddot{z}}{\dot{z}^2} - 1$	

Table 3.2: Coordinate changes

Class name (definition)	Symmetries	ODE
$\mathfrak{a}$	$\partial_x,\, \partial_y$	$y'' = G\left(y'\right)$
$\mathfrak{b}$	$\partial_x,\, x\partial_x$	$y'' = y'^2G(y)$
$\mathfrak{c}$	$\partial_x,\, x\partial_x + y\partial_y$	$y'' = y^{-1}G\left(y'\right)$
$\mathfrak{d}$	$\partial_x,\, x\partial_x,\, \partial_y$	$y''' = y'^3G\left(\frac{y''}{y'^2}\right)$

Table 3.3: Simplified list of invariant ODEs

We present the solution by quadratures for cases $\mathfrak{a}$–$\mathfrak{d}$ in table 3.4 . We choose a symmetry, transform the equations and lower the order such that we get separable equation of the first order, which is then "easily" solvable. Although we have solved all equations we don't have solutions in the explicit form. It is common flaw by solution with the quadratures that we get only the implicit form (moreover we still need to integrate the implicit form for u to get $y = y(x)$)

	$\mathfrak{a}: \quad y'' = F(y')$	
$\partial_y \to$	$t := x$ $u(t) := y'(t)$	$\Rightarrow \dot{u} = F(u)$
	$\mathfrak{b}: \quad y'' = y'^2 G(y)$	
$x\partial_x \to$	$t := y$ $u := \ln x$ $w(t) := \dot{u}(t)$	$\Rightarrow \dot{w} + w^2 = wG(t)$
$\partial_x \to$	$t := y$ $u := x, u = u(t)$ $w(t) := \dot{u}(t)$ $r := \ln w$	$\Rightarrow \dot{r} = G(t)$
	$\mathfrak{c}: \quad y'' \cdot y = G(y')$	
$x\partial_x + y\partial_y \to$	$t := y^{-1}x$ $\Rightarrow u := \ln y$ $w(t) := \dot{u}(t)$	$\Rightarrow \dot{w} = G(w)$
$\partial_x \to$	$t := y$ $u := x, u = u(t)$ $w(t) := \dot{u}(t)$	$\Rightarrow \ddot{u} = -\frac{\dot{u}^3}{t} G\left(\frac{1}{\dot{u}}\right)$
	$\mathfrak{d}: \quad y''' = y'^3 G\left(\frac{y''}{y'^2}\right)$	
$\partial_y \to$	$t := x$ $u(t) := y'(t)$	$\Rightarrow \ddot{u} = u^3 G\left(\frac{\dot{u}}{u^2}\right)$
	$z = -\frac{1}{u}$	$\Rightarrow zz'' = 2z' + G(z')$ Continue as in the case $\mathfrak{c}$.

Table 3.4: Solving of invariant ODEs with symmetries

3.2 Lagrangian ODEs

3.2.1 Symmetry for Lagrangian

The Lagrangian approach can be sometimes useful during solving ODEs of the second order. When we find the Lagrangian of the ODE, we can analyse it and get benefits e.g. from the first integrals. Let us briefly recall the Lagrangian formalism first (see e.g.[4]).

Definition 3.1. Given that a function $L = L(x, y, y')$ and an interval I is prescribed, we define the functional S on the space of the differentiable curves by the formula

$$S[\gamma] = \int_I L(x, \gamma(x), \dot{\gamma}(x))\mathrm{d}x. \tag{3.1}$$

We call the L **Lagrangian** and the functional S **action**.

Definition 3.2. Suppose we have an action given by formula (3.1). **Euler-Lagrange equation** for action S and Lagrangian $L = L(x, y, y')$ is the equation equivalent to searching the stationary value of S.

$$\delta S = 0 \quad \Longleftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0 \qquad \Longleftrightarrow \quad (D_x\partial_{y'} - \partial_y)L = 0 \tag{3.2}$$

An ODE is **Lagrangian** (or Euler-Lagrangian) if there exists such S, that the ODE is the E-L equation for this action.

The Lagrangian respective the action can have symmetries too. The definition of the symmetry for the action is easy. But we are interested in the infinitesimal symmetries of the Lagrangian, because they are easier to find. Whole snag is that the action is defined by an integral. That implies the presence of degrees of freedom, because the action doesn't change when we take $\tilde{L} = L + D_xV(x, y)$, where V is an arbitrary function, instead of L. We have also keep on mind that when we change x the measure also changes.

Definition 3.3. Action S has T as its **symmetry** if

$$\delta S[\gamma] = 0 \quad \Longrightarrow \quad \delta S[T \triangleright \gamma] = 0 \tag{3.3}$$

holds for all curves γ.

Definition 3.4. Vector field $X = \xi\partial_x + \eta\partial_y$ is the **infinitesimal symmetry** of Lagrangian L if there exists function $V = V(x, y)$ such that

$$\mathrm{pr}\, X(L) + L \cdot D_x(\xi) = D_x(V), \tag{3.4}$$

$$\xi\partial_x + \eta^0\partial_y + \eta^1\partial_{y'} + L \cdot D_x(\xi) = D_x(V). \tag{3.5}$$

Finding symmetries is now harder but it is still quasilinear equation (inhomogeneous now), so we can use method of characteristics again. The important thing is that the E-L equation inherits all symmetries of the Lagrangian.

Finally we present the fundamental Noether's theorem.

Theorem 3.5. Suppose $X = \xi\partial_x + \eta\partial_y$ is the symmetry for the Lagrangian L. Then there exists first integral K.

$$\xi L + (\eta - \xi y')\partial_{y'} L - V = K = const \tag{3.6}$$

3.2.2 Lagrangian for $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$

After the primary analysis of symmetric ODEs in section 3.1 another question arises. Are the second order ODEs in the table 3.3 (cases $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$) Lagrangian for an arbitrary function G? The answer is "Yes".

How did we proceed? We have found Lagrangians with same symmetries and then determine the equation which fixes the form of Lagrangian for given E-L equation.

The symmetry ∂_x determined that Lagrangian must be x-independent.

$$\partial_x L(x, y) + L \cdot D_x(1) = 0 + 0 = D_x(0). \tag{3.7}$$
$$\tag{3.8}$$

Case $\mathfrak{a}$: $y'' = G(y')$

Symmetry ∂_y restricts Lagrangian in the following way:

$$L(x, y, y') = y + H(y'), \quad V(x, y) = x, \tag{3.9}$$
$$\partial_y L(x, y, y') + L \cdot D_x(0) = \partial_y(y + H(y')) + 0 = 1 = D_x(x). \tag{3.10}$$

After substituting into E-L equation (3.2) we get equation

$$y''\ddot{H}(y') = 1. \tag{3.11}$$

That fixes H because equation

$$\ddot{H}(t) = \frac{1}{F(t)} \tag{3.12}$$

must hold.

case $\mathfrak{b}$: $y'' = y'^2 G(y)$

Symmetry $x\partial_x$ restricts Lagrangian this way:

$$L(x, y, y') = -y' \ln y' \dot{A}(y), \quad V(x, y) = A(y), \tag{3.13}$$
$$(x\partial_x - y'\partial'_y)L(x, y, y') + L \cdot D_x(x) = -y'(-\ln y' - 1)\dot{A}(y) - y' \ln y' \dot{A}(y)$$
$$= y'\dot{A}(y) = D_x(A(y)). \tag{3.14}$$

If we fix $A(y) = e^{-\int G(y)}$ we get exactly what we want.

$$(D_x\partial_{y'} - \partial_y)y' \ln y' e^{-\int G(y)} = 0, \tag{3.15}$$

$$D_x\left((\ln y' + 1)e^{-\int G(y)}\right) = -G(y)y' \ln y' e^{-\int G(y)}, \tag{3.16}$$

$$\frac{y''}{y'}e^{-\int G(y)} + -y'G(y)(\ln y' + 1)e^{-\int G(y)} = -G(y)y' \ln y' e^{-\int G(y)}, \tag{3.17}$$

$$\frac{y''}{y'} = y'G(y). \tag{3.18}$$

Case $\mathfrak{c}$: $yy'' = G(y')$

The Lagrangian has to be in the form

$$L(x, y, y') = y^{-1}H(y') \tag{3.19}$$

$$(x\partial_x + y\partial_y)L(x, y, y') + L \cdot D_x(x) = -y^{-1}H(y') + y^{-1}H(y') = 0 \tag{3.20}$$

And H has to fulfil the condition given by the equation

$$\ddot{H}(t)G(t) = t\dot{H}(t) - H(t). \tag{3.21}$$

We summarise results in the table 3.5.

Case: Symmetries	ODE	Lagrangian	
$\mathfrak{a}$: ∂_x, ∂_y	$y'' = G(y')$	$L = y + H(y')$	$\ddot{H}(t) = \frac{1}{G(t)}$
$\mathfrak{b}$: ∂_x, $x\partial_x$	$y'' = y'^2G(y)$	$L = -y' \ln(y')e^{-\int G(y)}$	
$\mathfrak{c}$: ∂_x, $x\partial_x + y\partial_y$	$y'' \cdot y = G(y')$	$L = y^{-1}H(y')$	$\ddot{H}(t)G(t) = t\dot{H}(t) - H(t)$
$\mathfrak{d}$: ∂_x, $x\partial_x$, ∂_y	$y''' = y'^3G\left(\frac{y''}{y'^2}\right)$	—	

Table 3.5: List of Lagrangians for ODEs

Chapter 4

Difference equations with specific symmetries

4.1 Symmetries of difference equations

The definitions of symmetries of difference equations is base on the same concept as the symmetries of differential equation. The only deviation is that we use different prolongation.

In this section

$$F(x_1, \ldots, x_N, y_1, \ldots, y_N) = 0 \tag{4.1}$$

will be a given difference equation of the (N-1)$^{\text{th}}$ order.

The prolongation is based on the idea that all transformations should act on every (x_i, y_i) identically.

Definition 4.1. Prolongation is a map $\mathfrak{X}(\mathbb{R}^2) \to \mathfrak{X}(\mathbb{R}^{2N})$.

$$\operatorname{pr} \mathfrak{X} = \operatorname{pr}(\xi(x,y)\partial_x + \eta(x,y)\partial_y) = \sum_{i=1}^{N} \xi(x_i, y_i)\partial_{x_i} + \eta(x_i, y_i)\partial_{y_i} \tag{4.2}$$

We use the same notation as for ODEs, because it is obvious which prolongation we mean.

And the definition of the infinitesimal symmetry is exactly the same as in continuous case.

Definition 4.2. $X \in \mathfrak{X}(\mathbb{R}^2)$ is the **infinitesimal symmetry** of the $F(x_1, \ldots, x_N, y_1, \ldots, y_N) = 0$

$$\overset{def}{\Longleftrightarrow} \operatorname{pr} X(F) \restriction_{F=0} (x_1, \ldots, x_N, y_1, \ldots, y_N) = 0. \tag{4.3}$$

4.2 Discrete invariants of solvable extensions of $\mathfrak{n}_{n,1}$

We define some symbols to simplify following expressions.

$$h_j := x_{j+1} - x_j \qquad j = 1, \ldots, N-1, \tag{4.4}$$

$$P_j^{(0)} := y_j \qquad j = 1, \ldots, N, \tag{4.5}$$

$$P_j^{(k)} := k \cdot \frac{P_{j+1}^{(k-1)} - P_j^{(k-1)}}{x_{j+k} - x_j} \qquad k = 1, \ldots, N-1, \quad j = 1, \ldots, N-k. \tag{4.6}$$

We want to find the difference equations which admit same symmetries as their differential relatives in chapter 2. In order to do it following identities become handy.

$$(\operatorname{pr} Y_k) P_j^{(k)} = 1, \tag{4.7}$$

$$\sum_{i=1}^{N} x_i \partial_{x_i} = \sum_{i=1}^{N-1} h_i \partial_{h_i} - \sum_{j=0}^{N-1} j \cdot P_1^{(j)} \partial_{P_1^{(j)}}, \tag{4.8}$$

$$\sum_{i=1}^{N} y_i \partial_{y_i} = \sum_{j=0}^{N-1} P_1^{(j)} \partial_{P_1^{(j)}}. \tag{4.9}$$

Identity (4.7) will be proven at the end of this section. The others could be easily proven by mathematical induction.

In the first step we find invariants of the nilpotent subalgebra, using the identity (4.7). The result is

$$I = I(h_1, \ldots, h_{N-1}, P_1^{(n-1)}, \ldots, P_{N-n+1}^{(n-1)}). \tag{4.10}$$

We may also choose a different functionally independent set of invariants

$$I = I(h_1, \ldots, h_{N-1}, P_1^{(n-1)}, \ldots, P_1^{(N-1)}). \tag{4.11}$$

We change the coordinates and express the vector fields in new variables using identities (4.7), (4.8) and (4.9).

Case $\mathfrak{s}_{n+1,1}(B)$:

$$\begin{aligned} \operatorname{pr} S &= -\sum_{i=1}^{N} x_i \partial_{x_i} + (1-n-B) \sum_{i=1}^{N} y_i \partial_{y_i} \\ &= -\sum_{j=1}^{N-1} h_j \partial_{h_j} + \sum_{k=0}^{N-1} k \cdot P_1^{(k)} \partial_{P_1^{(k)}} + (1-n-B) \sum_{k=0}^{N-1} P_1^{(k)} \partial_{P_1^{(k)}} \\ &= -\sum_{j=1}^{N-1} h_j \partial_{h_j} + \sum_{k=0}^{N-1} (1-n+k-B) \cdot P_1^{(k)} \partial_{P_1^{(k)}}. \end{aligned} \tag{4.12}$$

The invariant is therefore a function of the expressions

- $B \neq 0$

$$h_1{}^{-B} \cdot y_1, \ldots, h_{N-1}{}^{-B} \cdot y_1, P_1^{(1)} \cdot y_1{}^{\frac{1-B}{B}}, \ldots, P_1^{(N-n)} \cdot y_1{}^{\frac{N-n-B}{B}} \tag{4.13}$$

- $B = 0$

$$h_1 \cdot P_1^{(1)}, \ldots, h_{N-1} \cdot P_1^{(1)}, y_1, \frac{P_1^{(2)}}{{P_1^{(1)}}^2}, \ldots, \frac{P_1^{(N-n)}}{{P_1^{(1)}}^{N-n}} \tag{4.14}$$

respectively.

Case $\mathfrak{s}_{n+1,4}$: The prolongation in the new coordinates is

$$\operatorname{pr} S = -\sum_{i=1}^{N} y_i \partial_{y_i} = -\sum_{k=0}^{N-1} P_1^{(k)} \partial_{P_1^{(k)}}. \tag{4.15}$$

After a quick computation we get that the invariant is function of

$$h_1, \ldots, h_{N-1}, \frac{P_1^{(1)}}{y_1}, \ldots, \frac{P_1^{(N-n)}}{y_1}. \tag{4.16}$$

Case $\mathfrak{s}_{n+1,5}$: After the change of coordinates prolongated field transforms to

$$\begin{aligned} \operatorname{pr} S &= \sum_{i=1}^{N} -x_i \partial_{x_i} + \left((1-n) y_i + \frac{x_i^{n-1}}{(n-1)!} \right) \partial_{y_i} \\ &= -\sum_{i=1}^{N-1} h_i \partial_{h_i} + \sum_{j=0}^{N-1} j \cdot P_1^{(j)} + (1-n) P_1^{(j)} \partial_{P_1^{(j)}} + \partial_{P_1^{(n-1)}} \\ &= -\sum_{i=1}^{N-1} h_i \partial_{h_i} + \sum_{j=0}^{N-1} (j+1-n) P_1^{(j)} \partial_{P_1^{(j)}} + \partial_{P_1^{(n-1)}}. \end{aligned} \tag{4.17}$$

And solutions are

$$h_1 \cdot e^{y_1}, \ldots, h_{N-1} \cdot e^{y_1}, P_1^{(1)} \cdot e^{-y_1}, \ldots, P_1^{(N-n)} \cdot e^{-(N-n) y_1}. \tag{4.18}$$

Case $\mathfrak{s}_{n+2}$:

$$\operatorname{pr} S_1 = -\sum_{k=0}^{N-1} P_1^{(k)} \partial_{P_1^{(k)}} \tag{4.19}$$

$$\operatorname{pr} S_2 = -\sum_{i=1}^{N-1} h_i \partial_{h_i} + \sum_{k=0}^{N-1} (1-n+k) \cdot P_1^{(k)} \partial_{P_1^{(k)}} \tag{4.20}$$

After the solution of equation for $\mathrm{pr}\, S_2$ is invariant function of expressions (4.14). Only the equation for $\mathrm{pr}\, S_1$ remains and it implies that the invariant depends on

$$h_1 \frac{P_1^{(1)}}{y_1}, \ldots, h_{N-1} \frac{P_1^{(1)}}{y_1}, \frac{P_1^{(2)} \cdot y_1}{{P_1^{(1)}}^2}, \ldots, \frac{P_1^{(N-n)} \cdot {y_1}^{N-n-1}}{{P_1^{(1)}}^{N-n}}. \tag{4.21}$$

We present table 4.2 which summarises the invariants and divide them into two group according to the attribute presented in the next section. The vanishing invariants goes to the zero in the continuous limit.The non-vanishing invariants goes to the continuous invariants presented in the table 3.1.

#	Remaining symmetries	Vanishing invariants	Non-vanishing invariants
1	∂_x	$h_1, \ldots, h_{N-1}$	$P_1^{(0)}, \ldots, P_1^{(N-n)}$
2	∂_x, $x\partial_x + By\partial_y$, $B \neq 0$	${h_1}^{-B} \cdot y_1, \ldots, {h_{N-1}}^{-B} \cdot y_1$	$P_1^{(1)} {y_1}^{\frac{1-B}{B}}, \ldots, P_1^{(N-n)} {y_1}^{\frac{N-n-B}{B}}$
3	∂_x, $x\partial_x$	$h_1 \cdot P_1^{(1)}, \ldots, h_{N-1} \cdot P_1^{(1)}$	$y_1, \frac{P_1^{(2)}}{{P_1^{(1)}}^2}, \ldots, \frac{P_1^{(N-n)}}{{P_1^{(1)}}^{N-n}}$
4	∂_x, $y\partial_y$	$h_1, \ldots, h_{N-1}$	$\frac{P_1^{(1)}}{y_1}, \ldots, \frac{P_1^{(N-n)}}{y_1}$
5	∂_x, $x\partial_x - \partial_y$	$h_1 \cdot e^{y_1}, \ldots, h_{N-1} \cdot e^{y_1}$	$P_1^{(1)} e^{-y_1}, \ldots, P_1^{(N-n)} e^{-(N-n)y_1}$
6	∂_x, $x\partial_x$, $y\partial_y$	$h_1 \frac{P_1^{(1)}}{y_1}, \ldots, h_{N-1} \frac{P_1^{(1)}}{y_1}$	$\frac{P_1^{(2)} \cdot y_1}{{P_1^{(1)}}^2}, \ldots, \frac{P_1^{(N-n)} \cdot {y_1}^{N-n-1}}{{P_1^{(1)}}^{N-n}}$

Table 4.1: Discrete invariants

Theorem 4.3.

$$(\operatorname{pr} Y_k) P_j^{(k)} = 1.$$

Proof. We prove stronger proposition by an incomplete induction.

$$\forall k, j, s \qquad (\operatorname{pr} Y_k) P_s^{(j)} = \left(\sum_{i=1}^{N} \frac{x_i^k}{k!} \partial_{y_i} \right) P_s^{(j)} = \frac{j!}{k!} \sum_{\substack{i_0, \dots, i_j \in \mathbb{N}_0 \\ i_0 + \dots + i_j = k-j}} \prod_{m=0}^{j} x_{s+m}^{i_m} \tag{4.22}$$

$j = 0$

$$\sum_{i=1}^{N} \frac{x_i^k}{k!} \partial_{y_i} P_s^{(0)} = \sum_{i=1}^{N} \frac{x_i^k}{k!} \partial_{y_i} y_s = \sum_{i=1}^{N} \frac{x_i^k}{k!} \delta_{is} = \frac{x_s^k}{k!}$$

$j \to j+1$

$$\begin{aligned}
\frac{k!}{(j+1)!} (\operatorname{pr} Y_k) P_s^{(j+1)} &= \frac{k!}{(j+1)!} \frac{j+1}{x_{s+j+1} - x_s} (pr^N Y_k)(P_{s+1}^{(j)} - P_s^{(j)}) \\
&= \frac{k!}{j!} \frac{1}{x_{s+j+1} - x_s} \sum_{\substack{i_0, \dots, i_j \in \mathbb{N}_0 \\ i_0 + \dots + i_j = k-j}} \frac{j!}{k!} \left(\prod_{m=0}^{j} x_{s+1+m}^{i_m} - \prod_{r=0}^{j} x_{s+r}^{i_r} \right) \\
&= \frac{1}{x_{s+j+1} - x_s} \sum_{\substack{i_0, \dots, i_j \in \mathbb{N}_0 \\ i_0 + \dots + i_j = k-j}} \left(x_{s+j+1}^{i_j} \prod_{m=0}^{j-1} x_{s+1+m}^{i_m} - x_s^{i_0} \prod_{r=1}^{j} x_{s+r}^{i_r} \right) \\
&= \frac{1}{x_{s+j+1} - x_s} \sum_{l=0}^{k-j} (x_{s+j+1}^l - x_s^l) \sum_{\substack{i_1, \dots, i_j \in \mathbb{N}_0 \\ i_1 + \dots + i_j = k-j-l}} \prod_{m=1}^{j} x_{s+m}^{i_m} \\
&= \sum_{l=1}^{k-j} \sum_{p=0}^{l-1} x_s^p x_{s+j+1}^{l-1-p} \sum_{\substack{i_1, \dots, i_j \in \mathbb{N}_0 \\ i_1 + \dots + i_j = k-j-l}} \prod_{m=1}^{j} x_{s+m}^{i_m} \\
&= \sum_{p=0}^{k-j-1} \sum_{l=p+1}^{k-j} x_s^p x_{s+j+1}^{l-1-p} \sum_{\substack{i_1, \dots, i_j \in \mathbb{N}_0 \\ i_1 + \dots + i_j = k-j-l}} \prod_{m=1}^{j} x_{s+m}^{i_m} \\
&= \sum_{p=0}^{k-j-1} \sum_{l=0}^{k-j-1-p} x_s^p x_{s+j+1}^{l} \sum_{\substack{i_1, \dots, i_j \in \mathbb{N}_0 \\ i_1 + \dots + i_j = k-j-l-p-1}} \prod_{m=1}^{j} x_{s+m}^{i_m} \\
&= \sum_{\substack{p, i_1, \dots, i_j, l \in \mathbb{N}_0 \\ p + i_1 + \dots + i_j + l = k-(j+1)}} x_s^l x_{s+j+1}^l \prod_{m=1}^{j} x_{s+m}^{i_m}
\end{aligned}$$

□

4.3 Continuous limit of difference equation

Difference equations can be employed to approximate differential equations. This approximation is called "continuous limit" and it's the key element during constructing so-called difference schema. The idea is following:

1. Take the given difference equation
$$F(x_1, \ldots, x_N, y_1, \ldots, y_N) = 0. \tag{4.23}$$
2. Think of y_i as the function $y = y(x)$ evaluated at the point x_i.
3. Formally Taylor-expand around chosen point (e.g. x_1).
4. Substitute it in the difference equation and expand it again.
5. **Continuous limit** is the term $\propto h^0$.

More precisely is that formal Taylor-expanding only transformation from x_i, y_i to new variables $x = x_1, h_i, y, y', y'', \ldots$.

Example 4.4. Consider the difference equation

$$\frac{y_1 - y_0}{x_1 - x_0} - \sin(x_1) = 0$$

and find its continuous limit.

$$\begin{aligned} \frac{y_0 + y'h_1 + O(h^2) - y_0}{h_1} - \sin(x_0 + h_1) = 0. \\ \frac{y'h_1 + O(h^2)}{h_1} - \sin(x_0) - \cos(x_0)h_1 + O(h^2) = 0. \\ y' - \sin(x) + O(h) = 0. \end{aligned}$$

Continuous limit is

$$y' - \sin(x) = 0.$$

4.4 Ordinary difference scheme

Through this research work we have solved some analytically ODEs (table 3.4). The point is that we don't get an explicit solution, so we are usually forced to use the numerical methods. Usually is used Runge-Kutta or some difference methods which are using the Lagrange interpolating polynomial. But the problem is that even if they could be very accurate, they are very CPU-demanding (Runge-Kutta typically). Furthermore even if the ODE has some symmetries, not only these methods don't use them at all, they break them almost every time.

The idea is to create an algorithm which uses that extra information we have about the ODE—symmetry. In addition we require that this algorithm has to be quick. So the idea of difference schema was created.

Firstly, why every method has the uniform lattice? Isn't be more convenient to adapt the lattice to the symmetries? But how to do it? Take notice of that fact that the lattice can be fixed by difference equation (e.g. $h_1 = h_2$ for uniform lattice). And we can approximate the ODE with the difference equation too. And if we add requirement that both these difference equation must have exactly same symmetries as the approximated ODE, we create method which takes advantage of the symmetries and also is quite fast.

Definition 4.5. Suppose the ODE $F(x, y, y', \dots, y^{(n)}) = 0$ is given and its symmetries are known. **Ordinary difference scheme** is system of two difference equations

$$E_1(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) = 0, \tag{4.24}$$
$$E_2(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) = 0, \tag{4.25}$$

which have the same symmetries and have specific continuous limits. First one has to go in limit to the original ODE (it approximates the ODE) and second one to the zero (it fixes the lattice). The lattice is no longer the "lattice" in the sense of the additive subgroup of $\mathbb{R}$.

The algorithm works this way:

1. Find symmetries and construct difference schema.
2. Get solution of the difference schema in the form

$$x_{n+1} = K_x(x_1, \dots, x_n, y_1, \dots, y_n) \tag{4.26}$$
$$y_{n+1} = K_y(x_1, \dots, x_n, y_1, \dots, y_n). \tag{4.27}$$

3. Set initial condition $x_1, \dots, x_n, y_1, \dots, y_n$.
4. Put initial condition into solution K_x, K_y.
5. Repeat for $x_2, \dots, x_{n+1}, y_2, \dots, y_{n+1}$ as new initial condition.

Summary and conclusions

We have recapitulated the mathematical background related to the symmetries of the differential equations, difference equations and action respective Lagrangian.

We have found realisation of all series of solvable algebras from article [3] on $\mathbb{R}^2$ except one for which $\mathbb{R}^2$ is not enough. It need $\mathbb{R}^3$ at least. Then we have computed the invariants of the prolongated symmetries and with this knowledge we have constructed the general ODE with given symmetries.

During the analysis we have transformed all ODEs on one of four cases and solved them by quadratures. All these ordinary differential equations of the second order were Euler-Lagrangian for some action S, which admits same symmetries as the associated equation.

We have presented concept of the continuous limit of the difference equation and used it to create a numerical method for solution of ODEs. This method is based on the difference equation with same symmetries as the original ODE and on adapted lattice.

The advantages of this method are quite amazing. After pre-solving the difference schema we only insert known data into functions. It also sometimes embodied amazing attributes. Such as tracking the exact solution (absolute precision), tunnelling through singularity or following the graph of the solution even if it is not a function. For more information see into [5, 6, 7]. There are also many disadvantages. Firstly we don't have the explicit recipe how to set difference schema — it's not unique. Secondly it is hard to solve the schema. It could cancel the speed benefit ahead of Runge-Kutta. There are some hints how to choose schema. For example when $P_1^{(1)} \to \infty$, h_n should vanish.

Among questions which naturally arise are for example: "Are all difference schema approximating ODEs in table 3.3 also Lagrangian?", "How to choose the best difference schema?" or "Could be these difference schemas somehow related to the quantisation?".

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