Czech Technical University in Prague
Faculty of Nuclear Science and Physical Engineering

BACHELOR THESIS

QUANTUM WAVEGUIDES WITH ROBIN BOUNDARY CONDITIONS

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I would like to thank my supervisor Mgr. David Krejčiřík, Ph.D. for valuable advices, explanation of notions, and careful reading a previous version of this thesis.

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Název práce:
Kvantové vlnovody s Robinovými okrajovými podmínkami

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Abstrakt: Studujeme spektrum Laplaciánu v zakřiveném rovinném pásku podél nekonečné křivky s Robinovými okrajovými podmínkámi. Dokazujeme, že esenciální spektrum je stabilní, pokud má křivost referenční křivky kompaktní nosič a že vždy, když pásek není rovný, existují vlastní hodnoty pod prahem esenciálního spektra.

Klíčová slova: kvantové vlnovody, Robinovy hraniční podmínky, vázané stavy, Robinův Laplacián.

Title: **Quantum waveguides with Robin boundary conditions**

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Abstract: We study the spectrum of the Laplacian in a curved planar strip built along an infinite curve, subject to Robin boundary conditions. We prove that the essential spectrum is stable under any compactly supported curvature of the reference curve and that there always exist eigenvalues below the bottom of the essential spectrum unless the strip is straight.

Key words: quantum waveguides, Robin boundary conditions, bound states, Robin Laplacian.

Contents

1 Introduction

Modern experimental techniques make it possible to fabricate tiny semiconductor structures which are small enough to exhibit quantum effects. These systems are sometimes called *nanostructures* because of their typical size in a direction and they are expected to become the building elements of the next-generation electronics. Since the used materials are very pure and of crystallic structure, the particle motion inside a nanostructure can be modeled by a free particle with an effective mass m^* living in a spatial planar region Ω. That is, the quantum Hamiltonian can be identified with the operator

$$H = -\frac{\hbar^2}{2m^*}\Delta \tag{1}$$

in the Hilbert space $L^2(\Omega)$, where $\hbar$ denotes the Planck constant. We refer to [11, 5] for more information on the physical background.

An important category of nanostructures is represented by *quantum waveguides*, which are modeled by Ω being an infinitely stretched tubular region in $\mathbb{R}^2$ or $\mathbb{R}^3$. In principle, one can consider various conditions on the boundary of Ω in order to model the fact that the particle is confined to Ω. However, since the particle wavefunctions ψ are observed to be suppressed near the interface between two different semiconductor materials, one usually imposes Dirichlet boundary conditions, *i.e.*

$$\psi = 0 \quad \text{on} \quad \partial\Omega. \tag{2}$$

Such a model was successfully used by P. Exner and P. Šeba in 1989 to describe the binding effect in curved quantum waveguides [7]. We refer to [8, 5] for subsequent generalizations and references on the Dirichlet model.

A fresh impetus to the theoretical studies of curved quantum waveguides has been given by the recent letter [4] of Dittrich and Kříž from 2002 who demonstrated that an alteration of the boundary conditions changes the physical picture completely. More specifically, the authors introduced a waveguide with combined Dirichlet and Neumann boundary conditions and showed that the presence of the latter may lead to the absence of bound states which always exist in the Dirichlet waveguides. We refer to [10] for further studies of the Dirichlet-Neumann model.

Summing up, both the geometry and the choice of boundary conditions are important for the physical interpretation of a quantum-waveguide model. In this thesis, we introduce and study a new model by imposing Robin boundary conditions on the boundary of a quantum waveguide. These conditions can be considered as a generalization of the Dirichlet and Neumann boundary conditions mentioned above, however, our model is closer to the Dirichlet one since we consider a *uniform* coupling constant $\alpha > 0$. More specifically, putting the physical constant in (1) equal to one, we are interested in the solutions ψ of the following stationary Schrödinger equation for a free particle in Ω having energy λ and satisfying Robin boundary conditions on $\partial\Omega$:

$$-\Delta\psi = \lambda\psi \text{ in } \Omega\,, \tag{3}$$

$$\frac{\partial\psi}{\partial n} + \alpha\,\psi = 0 \quad \text{on} \quad \partial\Omega\,. \tag{4}$$

Here Ω is an unbounded curved planar strip, *i.e.* a neighborhood of constant width of an infinite curve in $\mathbb{R}^2$, α is a given positive number and n is the outward unit normal vector to $\partial\Omega$. Later on, we interpret the boundary-value problem (3), (4) rigorously as the spectral problem for a self-adjoint operator in the Hilbert space $L^2(\Omega)$.

While to impose the Dirichlet boundary conditions means to require the vanishing of wavefunction on the boundary of Ω, see (2), the Robin conditions (4) correspond to the weaker requirement of vanishing of the probability current, in the sense that its normal component vanishes on the boundary, *i.e.*

$$j \cdot n = 0 \quad \text{on} \quad \partial\Omega,$$

where the probability current j is defined by

$$j := \frac{i\hbar}{2m^*}\left[\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi\right].$$

This less restrictive requirement may in principle model different types of interface in materials.

The Laplacian subject to Robin boundary conditions can be also used in the problem to find the electro-magnetic field outside the object consisting of a conducting core covered by a dielectric layer. If the thickness of the layer is too small compared to the dimension of the conductive core, the numerical methods of solving this problem fail because of instabilities that then arise. In this case, the problem can by solved by approximation of the dielectric layer by appropriate boundary conditions of the Robin type. We refer to [2, 6] for more information.

The very interesting phenomenon in the curved Dirichlet (and in certain Dirichlet-Neumann) quantum waveguides is the existence of *bound states*, *i.e.*, the eigenvalues under essential spectrum, which classically do not exist. The main aim of the thesis is to show that the geometrically induced discrete spectrum exists also in the Robin waveguides, and this under exactly the same conditions as in the Dirichlet waveguides.

The thesis is organised as follows. In Section 2, we present some notions and tools from spectral theory used in the thesis. Section 3 is devoted to the precise definition of the Robin Laplacian. In Section 4, we find spectrum of straight waveguides, treating separately the longitudinal and transversal motions. The main results are presented in Section 5; firstly, we show that the essential spectrum on a curved strip does not change if the strip is curved only locally; secondly, we prove the existence of discrete spectrum in a non-trivially curved strip.

2 Review of some abstract results in spectral theory

In this section we will summarize some basic results of spectral theory, which we will use in the thesis.

Let $\mathcal{H}$ be a separable complex Hilbert space with the *scalar product* $(\cdot,\cdot)$ and the *norm* $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. A linear *operator* H on $\mathcal{H}$ is, by definition, a linear mapping of a subspace $D(H) \subseteq \mathcal{H}$ into $\mathcal{H}$; $D(H)$ is called the *domain* of H. H is said to be *densely defined* if $D(H)$ is dense in $\mathcal{H}$. $\tilde{H}$ is called an *extension* of H (or H is a *restriction* of $\tilde{H}$) if we have

$$D(H) \subset D(\tilde{H}) \qquad \text{and} \qquad \forall \psi \in D(H), \quad \tilde{H}\psi = H\psi\,.$$

An operator H on $\mathcal{H}$ is said to be *symmetric* if it is densely defined and

$$\forall \phi, \psi \in D(H), \quad (\phi, H\psi) = (H\phi, \psi)\,.$$

If H is a densely defined operator on $\mathcal{H}$ then the *adjoint* operator H^* is uniquely determined by the condition that

$$\forall \phi \in D(H^*), \psi \in D(H), \quad (\phi, H\psi) = (H^*\phi, \psi)\,.$$

We have

$$D(H^*) = \left\{\phi \in \mathcal{H} \mid \exists \eta \in \mathcal{H},\ \forall \psi \in D(H), \quad (\phi, H\psi) = (\eta, \psi)\right\}.$$

If H is symmetric then it is easy to see that the adjoint H^* is an extension of H, *i.e.* $D(H) \subseteq D(H^*)$. We say that H is *self-adjoint* if

$$H \text{ is symmetric} \qquad \text{and} \qquad D(H) = D(H^*)\,.$$

The *spectrum* of a self-adjoint operator H on $\mathcal{H}$, denoted by $\sigma(H)$, is defined as the set of points $\lambda \in \mathbb{C}$ such that either $H - \lambda I$, where I is the identity operator, is not invertible or it is invertible but has range smaller than $\mathcal{H}$. It is easy to see that $\sigma(H) \subseteq \mathbb{R}$. The set $\sigma_{\mathrm{p}}(H)$ of all eigenvalues of H, *i.e.*

$$\sigma_{\mathrm{p}}(H) := \big\{\lambda \in \mathbb{R} \mid \exists \psi \in D(H), \quad \|\psi\| = 1, \quad H\psi = \lambda\psi\big\},$$

is called the *point spectrum* of H, which is obviously contained in $\sigma(H)$. If λ is an eigenvalue of H then the dimension of the kernel of $H - \lambda I$ is called the *multiplicity* of λ.

The *discrete spectrum* of a self-adjoint operator H, denoted by $\sigma_{\mathrm{disc}}(H)$, consists of those eigenvalues of H which are isolated points of $\sigma(H)$ and have finite multiplicity. The set

$$\sigma_{\mathrm{ess}}(H) := \sigma(H) \setminus \sigma_{\mathrm{disc}}(H)$$

is called the *essential spectrum* of H and, by definition, it contains either accumulation points of $\sigma(H)$ or isolated eigenvalues of infinite multiplicity.

The *spectral theorem* provides the following characterization of the spectrum of self-adjoint operators, which we will often use for showing that some value lies in the spectrum (*cf* [3, Lemma 4.1.2]):

Theorem 2.1 (Weyl criterion). *Let H be a self-adjoint operator on $\mathcal{H}$. A point $\lambda \in \mathbb{R}$ belongs to $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset D(H)$ such that*

1. $\forall n \in \mathbb{N}, \quad ||\psi_n|| = 1$,
2. $||H\psi_n - \lambda\psi_n|| \to 0 \quad as \quad n \to \infty.$

We say that a self-adjoint operator H on $\mathcal{H}$ is *semi-bounded* if there exists a real number c such that

$$\forall \psi \in D(H), \quad (\psi, H\psi) \geq c\,\|\psi\|^2\,;$$

in this case we simply write $H \geq c$. We have the following variational formula for the spectral threshold (*cf* [3, Sec. 4.5]):

Theorem 2.2 (Rayleigh-Ritz). *Let H be a semi-bounded self-adjoint operator on $\mathcal{H}$. Then*

$$\inf \sigma(H) = \inf_{\psi \in D(H)\setminus\{0\}} \frac{(\psi, H\psi)}{\|\psi\|^2}\,. \tag{5}$$

As a consequence of Theorem 2.2, $H \geq c$ implies $\sigma(H) \subseteq [c, \infty)$. On the other hand,

$$\inf \sigma(H) \leq \frac{(\psi, H\psi)}{\|\psi\|^2}$$

for any *test function* $\psi \in D(H) \setminus \{0\}$.

Theorem 2.2 is a special case of the following theorem, which is a highly useful tool for analysing the eigenvalues below the threshold of the essential spectrum (*cf* [3, Sec. 4.5]):

Theorem 2.3 (minimax principle). *Let H be a semi-bonded self-adjoint operator on $\mathcal{H}$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a non-decreasing sequence of numbers defined by*

$$\lambda_k := \inf \left\{ \sup_{\psi \in \mathcal{P}} \frac{(\psi, H\psi)}{||\psi||^2} \;\middle|\; \mathcal{P} \subseteq D(H) \;\&\; \dim(\mathcal{P}) = k \right\}. \tag{6}$$

Then one of the following cases occurs:

1. *There exists $\lambda_\infty \in \mathbb{R} \cup \{+\infty\}$ such that*

$$\forall k, \quad \lambda_k < \lambda_\infty \qquad \& \qquad \lim_{k \to \infty} \lambda_k = \lambda_\infty.$$

   Then $\lambda_\infty = \inf \sigma_{ess}(H)$ (with the convention that the essential spectrum is empty if $\lambda_\infty = +\infty$), and the part of the spectrum of H in $(-\infty, \lambda_\infty)$ consists of the eigenvalues λ_k each repeated a number of times equal to its multiplicity.

2. *There exists $\lambda_\infty \in \mathbb{R}$ and $N \in \mathbb{N}$ such that*

$$\forall k \leq N, \quad \lambda_k < \lambda_\infty \qquad \& \qquad \forall k > N, \quad \lambda_k = \lambda_\infty.$$

   Then $\lambda_\infty = \inf \sigma_{ess}(H)$, and the part of the spectrum of H in $(-\infty, \lambda_\infty)$ consists of the eigenvalues $\lambda_1, \ldots, \lambda_N$ each repeated a number of times equal to its multiplicity.

Remark 2.4. *$D(H)$ in (5) and (6) can be replaced by the form domain of H (*cf *[3, Sec. 4.5]).*

3 The Robin Laplacian

In this section we introduce the Laplacian in a general domain with Robin boundary conditions, in a proper way as an operator associated with a closed symmetric form.

Let Ω be an open connected set in $\mathbb{R}^N$, with $N \geq 1$. We assume that the boundary is *nice*, in particular, we require that there exists a unique continuous outward unit normal vector $n : \partial\Omega \to \mathbb{R}^N$. We are interested in the solutions ψ of the following stationary Schrödinger equation for a free particle in Ω having energy λ and satisfying Robin boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in } \Omega\,, \\ \dfrac{\partial\psi}{\partial n} + \alpha\,\psi = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{7}$$

where α is a given positive number.

This problem can be reconsidered in a mathematically rigorous way as the spectral problem for a self-adjoint operator, denoted by $-\Delta_\alpha^\Omega$ and called the Robin Laplacian here, in the Hilbert space $L^2(\Omega)$. It turns out that the correct Hamiltonian $-\Delta_\alpha^\Omega$ is defined by

$$\forall\psi \in D(-\Delta_\alpha^\Omega) := \left\{\psi \in W^{2,2}(\Omega) \;\middle|\; \frac{\partial\psi}{\partial n} + \alpha\,\psi = 0 \text{ on } \partial\Omega\right\}, \quad -\Delta_\alpha^\Omega\psi := -\Delta\psi\,, \tag{8}$$

but several remarks have to be added:

1. The Sobolev space $W^{2,2}(\Omega)$ is, by definition, the space of functions (or rather equivalence classes of functions) with square-integrable *weak* derivatives up to the second order, hence the act of the Laplacian should be understood in the *distributional* sense.

2. The values of ψ and $\nabla\psi$ on the boundary $\partial\Omega$ should be understood in the *trace* sense; this requires additional hypotheses about the boundary regularity.

3. While it is easy to see that $-\Delta_\alpha^\Omega$ is symmetric [1, Thm. 2.19], it is quite difficult to prove that it is self-adjoint.

In order to avoid the above difficulties, we use the quadratic form approach to define the Hamiltonian (*cf* [3, Sec. 4.4]). Let $\dot{Q}_\alpha^\Omega$ be the quadratic form defined in $L^2(\Omega)$ by

$$\forall\psi \in D(\dot{Q}_\alpha^\Omega) := \mathfrak{D}(\Omega)\,, \qquad \dot{Q}_\alpha^\Omega[\psi] := \int_\Omega |\nabla\psi|^2 + \alpha \int_{\partial\Omega} |\psi|^2\,, \tag{9}$$

where the domain

$$\mathfrak{D}(\Omega) := \left\{\psi \in L^2(\Omega) \mid \exists\Psi \in C_0^\infty(\mathbb{R}^N), \quad \psi = \Psi \restriction \Omega\right\} \tag{10}$$

is chosen in such a way that the integrals in (9) are well defined. $\dot{Q}_\alpha^\Omega$ is clearly densely defined, non-negative and the corresponding sesquilinear form is symmetric.

Let us denote by Q_α^Ω the *closure* (*i.e.* the smallest closed extension) of $\dot{Q}_\alpha^\Omega$. By definition, $D(Q_\alpha^\Omega)$ is the completion of $\mathfrak{D}(\Omega)$ with respect to the norm

$$\|\cdot\|_\alpha := \sqrt{\dot{Q}_\alpha^\Omega[\cdot] + \|\cdot\|^2}$$

and

$$\forall\psi \in D(Q_\alpha^\Omega), \qquad Q_\alpha^\Omega[\psi] := \lim_{n\to\infty} \dot{Q}_\alpha^\Omega[\psi_n]\,,$$

where $\{\psi_n\}_{n=1}^\infty$ is any sequence of $D(\dot{Q}_\alpha^\Omega)$ converging to ψ in $L^2(\Omega)$ and such that $\dot{Q}_\alpha^\Omega[\psi_n - \psi_m] \to 0$ for $n, m \to \infty$.

Proposition 3.1. *One has:*

1. *$D(Q_\alpha^\Omega)$ is independent of α,*
2. *$D(Q_\alpha^\Omega) \subseteq W^{1,2}(\Omega)$.*

Proof. The first property follows from the algebraic inequalities

$$\forall\alpha_1 < \alpha_2, \qquad \|\cdot\|_{\alpha_1} \le \|\cdot\|_{\alpha_2} \le \frac{\alpha_2}{\alpha_1}\,\|\cdot\|_{\alpha_1}\,.$$

Given any $\psi \in D(Q_\alpha^\Omega)$, let $\{\psi_n\}_{n=1}^\infty$ be as above and denote by η_ψ the limit of the sequence $\{\nabla\psi_n\}_{n=1}^\infty$ in $L^2(\Omega)$. Then

$$\forall\phi \in C_0^\infty(\Omega), \qquad (\nabla\phi, \psi) = \lim_{n\to\infty} (\nabla\phi, \psi_n) = -\lim_{n\to\infty} (\phi, \nabla\psi_n) = -(\phi, \eta_\psi)\,,$$

which tells us that ψ possesses weak derivatives ($\eta_\psi \in L^2(\Omega)$ is the distributional gradient of ψ), *i.e.* $\psi \in W^{1,2}(\Omega)$. □

As a consequence of the second property, we can write

$$\forall\psi \in D(Q_\alpha^\Omega), \qquad Q_\alpha^\Omega[\psi] = \int_\Omega |\nabla\psi|^2 + \alpha \lim_{n\to\infty} \int_{\partial\Omega} |\psi_n|^2\,.$$

Now, let H_α^Ω be the non-negative self-adjoint operator associated with Q_α^Ω, *i.e.*,

$$D(H_\alpha^\Omega) = \left\{\psi \in D(Q_\alpha^\Omega) \mid \exists\eta \in L^2(\Omega),\ \forall\phi \in D(Q_\alpha^\Omega),\ Q_\alpha^\Omega(\phi,\psi) = (\phi,\eta)\right\}\,,$$
$$\forall\phi \in D(Q_\alpha^\Omega),\ \psi \in D(H_\alpha^\Omega), \quad Q_\alpha^\Omega(\phi,\psi) = (\phi, H_\alpha^\Omega\psi)\,.$$

Using the Green theorem, it is easy to see that H_α^Ω is an extension of an operator which acts as the Laplacian on sufficiently regular functions satisfying the Robin boundary conditions:

Proposition 3.2. *One has*

$$\mathfrak{L}_\alpha(\Omega) \subset D(H_\alpha^\Omega) \qquad \text{and} \qquad \forall \psi \in \mathfrak{L}_\alpha(\Omega), \quad H_\alpha^\Omega \psi = -\Delta\psi\,,$$

where

$$\mathfrak{L}_\alpha(\Omega) := \left\{ \psi \in \mathfrak{D}(\Omega) \,\middle|\, \frac{\partial\psi}{\partial n} + \alpha\,\psi = 0 \ \text{ on } \partial\Omega \right\}.$$

Remark 3.3. *Assuming certain regularity of $\partial\Omega$ and using advanced methods of the theory of elliptic partial differential equations, it is possible to show that*

$$H_\alpha^\Omega = -\Delta_\alpha^\Omega\,,$$

but this proof is out of the scope of the present thesis. Nevertheless, we shall demonstrate the ideas of the proof in the case of Ω being a bounded one-dimensional interval, cf *Proposition 4.1.*

4 Straight waveguide

In this section, we are interested in a straight waveguide of width d, *i.e.*, we consider the Cartesian product $\Omega_0 := \mathbb{R} \times (0, d)$, where d is a given positive number. The problem can be formally simplified as follows. Assuming that the solution of (7) has a form

$$\psi(x, y) = X(x)Y(y).$$

and putting this Ansatz to (7), we get

$$-\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = \lambda \quad \text{in } \Omega_0.$$

This formula has a good sense only for such $(x, y) \in \Omega_0$ that $X(x) \neq 0$ and $Y(y) \neq 0$. It follows that, necessarily,

$$\lambda + \frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = C,$$

where C is a constant. From the boundary conditions in (7), we get the following boundary conditions for Y:

$$\begin{aligned} -Y'(0) + \alpha Y(0) &= 0, \\ Y'(d) + \alpha Y(d) &= 0. \end{aligned}$$

This sort of separation leads us to the study of two independent one-dimensional spectral problems, the transversal and the longitudinal one.

4.1 The transversal Hamiltonian

If $\Omega = (0, d)$, with d being a positive number, then it is easy to see that

$$D(Q_\alpha^{(0,d)}) = W^{1,2}((0, d))$$

because $W^{1,2}((0, d))$ is embedded in the space of uniformly continuous functions on $[0, d]$ due to the Sobolev embedding theorem [1, Thm. 5.4, part II]. By the same argument, the boundary conditions in (8) are defined without problems because $W^{2,2}((0, d))$ is embedded in the space of uniformly smooth functions $C^1([0, d])$.

The fact that $-\Delta_\alpha^{(0,d)}$ is self-adjoint can be proved directly using von Neumann's theory of deficiency indices. Here we prove it by means of the definition of the self-adjoint $H_\alpha^{(0,d)}$ via the associated quadratic form:

Proposition 4.1. $H_\alpha^{(0,d)} = -\Delta_\alpha^{(0,d)}$

Proof. Let $\psi \in D(-\Delta_\alpha^{(0,d)})$. An integration by parts shows that $Q_\alpha^{(0,d)}(\phi, \psi) = (\phi, \eta)$ for every $\phi \in D(Q_\alpha^{(0,d)})$ with $\eta := -\psi'' \in L^2((0, d))$. This proves that $H_\alpha^{(0,d)}$ is an extension of $-\Delta_\alpha^{(0,d)}$.

We are inspired by [9, Example VI. 2.16] to prove the converse inclusion. Let $\psi \in D(H_\alpha^{(0,d)})$ and $H_\alpha^{(0,d)}\psi =: \eta \in L^2((0, d))$. The relation $(\phi, \eta) = (\phi, H_\alpha^{(0,d)}\psi) = (\phi, \eta)$ for every $\phi \in D(Q_\alpha^{(0,d)})$ means

$$\int_0^d \overline{\phi}\eta = \int_0^d \overline{\phi'}\psi' + \alpha\overline{\phi(d)}\psi(d) + \alpha\overline{\phi(0)}\psi(0) \tag{11}$$

for every $\phi \in W^{1,2}((0, d))$. Let z be an indefinite integral of η (which is integrable): $z' = \eta$. Then

$$\int_0^d \overline{\phi}\eta = \int_0^d \overline{\phi}z' = -\int_0^d \overline{\phi'}z + \overline{\phi(d)}z(d) - \overline{\phi(0)}z(0) \tag{12}$$

for every $\phi \in W^{1,2}((0,d))$. Combining (11) with (12), we get the central identity

$$\int_0^d \overline{\phi'}(\psi' + z) + \overline{\phi(d)}[\alpha\psi(d) - z(d)] + \overline{\phi(0)}[\alpha\psi(0) + z(0)] = 0 \tag{13}$$

for every $\phi \in W^{1,2}((0,d))$. For any $\phi' \in L^2((0,d))$ such that $\int_0^d \phi' = 0$, $\phi(x) := \int_0^x \phi'$ satisfies the conditions $\phi \in W^{1,2}((0,d))$ and $\phi(0) = \phi(d) = 0$, so that $\psi' + z$ is orthogonal to ϕ' by (13). Thus $\psi' + z$ must be equal to a constant c, being orthogonal to all functions orthogonal to 1. Substituting this into (13) with arbitrary ϕ, we obtain

$$\overline{\phi(d)}[c + \alpha\psi(d) - z(d)] + \overline{\phi(0)}[-c + \alpha\psi(0) + z(0)] = 0 \tag{14}$$

for every $\phi \in W^{1,2}((0,d))$. Since $\phi(0)$ and $\phi(d)$ vary over all complex numbers when ϕ varies over $W^{1,2}((0,d))$, their coefficients in (14) must vanish. Noting that $c = \psi'(d) + z(d) = \psi'(0) + z(0)$, we thus obtain

$$\psi'(d) + \alpha\psi(d) = 0 \qquad \text{and} \qquad -\psi'(0) + \alpha\psi(0) = 0\,. \tag{15}$$

From $\psi' + z = c$ it follows (*cf* [3, Lem. 7.1.1]) that

$$\psi' \in W^{1,2}((0,d)) \qquad \text{and} \qquad \psi'' = -z' = -\eta\,. \tag{16}$$

In this way we have proved that each $\psi \in D(H_\alpha^{(0,d)})$ has the properties (15) and (16), proving therefore $H_\alpha^{(0,d)} \subset -\Delta_\alpha^{(0,d)}$. □

We divide the study of the spectral problem of $H_\alpha^{(0,d)}$ into several steps. Firstly, we find its point spectrum. Secondly, we establish some properties of the corresponding eigenfunctions we shall need later. Thirdly, we mention two limit situations, the case of Dirichlet and Neumann boundary conditions, respectively. Fourthly, we prove that the essential spectrum of $H_\alpha^{(0,d)}$ is empty. Finally, we present some numerical results to visualise the features of the Robin boundary conditions.

4.1.1 The point spectrum

For finding the point spectrum of $H_\alpha^{(0,d)}$, we search such $\lambda \in \mathbb{R}$ for which there exists a function $\chi \in W^{2,2}((0,d))$ normalised to 1 in $L^2((0,d))$ satisfying

$$-\ddot{\chi} = \lambda\chi \quad \text{in } (0,d) \tag{17}$$

together with the Robin boundary conditions

$$-\dot{\chi}(0) + \alpha\chi(0) = 0 \quad \wedge \quad \dot{\chi}(d) + \alpha\chi(d) = 0. \tag{18}$$

We will demonstrate that, necessarily, $\lambda > 0$. This can be shown by multiplying (17) by $\bar{\chi}$, integrating over $(0, d)$ and by using an integration by parts:

$$\begin{aligned}\lambda &= -\int_0^d \ddot{\chi}\bar{\chi} = -[\dot{\chi}\bar{\chi}]_0^d + \int_0^d |\dot{\chi}|^2 \\ &= \int_0^d |\dot{\chi}|^2 + \alpha|\chi(0)|^2 + \alpha|\chi(d)|^2 > 0.\end{aligned}$$

The inequality is strict, because the equality gives the trivial solution. Since

$$\int_0^d |\dot{\chi}|^2 + \alpha|\chi(0)|^2 + \alpha|\chi(d)|^2$$

is, in fact, the quadratic form corresponding to $H_\alpha^{(0,d)}$, the inequality implies that the operator $H_\alpha^{(0,d)}$ is positive.

The general solution of (17) has the form

$$\chi(y) = A\sin(\sqrt{\lambda}\, y) + B\cos(\sqrt{\lambda}\, y), \tag{19}$$

where $A, B \in \mathbb{C}$ are to be determined by the boundary conditions (18) and the normalisation condition. Restricting the general solution (19) by the boundary conditions (18), we get the system

$$m \begin{pmatrix} A \\ B \end{pmatrix} = 0, \tag{20}$$

where

$$m := \begin{pmatrix} -\sqrt{\lambda} & \alpha \\ \sqrt{\lambda}\cos(\sqrt{\lambda}\, d) + \alpha\sin(\sqrt{\lambda}\, d) & -\sqrt{\lambda}\sin(\sqrt{\lambda}\, d) + \alpha\cos(\sqrt{\lambda}\, d) \end{pmatrix}. \tag{21}$$

Because we are looking for non-trivial solutions of (20), we require that $\det m = 0$. This leads to the following implicit equation for the spectral parameter λ:

$$2\alpha\sqrt{\lambda}\cos(\sqrt{\lambda}\, d) + (\alpha^2 - \lambda)\sin(\sqrt{\lambda}\, d) = 0. \tag{22}$$

4.1.2 The eigenfunctions

In order to get the corresponding eigenfunctions, we solve the system (20), where λ is determined by implicit equation (22). The system (20) is equivalent to the equation

$$-\sqrt{\lambda}A + \alpha B = 0,$$

whence we can express A in terms of B:

$$A = \frac{\alpha}{\sqrt{\lambda}} B.$$

The solution (19) can be then written like

$$\chi(y) = B\left(\frac{\alpha}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\,y) + \cos(\sqrt{\lambda}\,y)\right). \tag{23}$$

The normalisation of χ yields the following condition on B:

$$1 = |B|^2\left[\left(1+\frac{\alpha^2}{\lambda}\right)\frac{d}{2} + \left(1-\frac{\alpha^2}{\lambda}\right)\frac{1}{2\sqrt{\lambda}}\sin(\sqrt{\lambda}\,d)\cos(\sqrt{\lambda}\,d) + \frac{\alpha}{\lambda}\sin^2(\sqrt{\lambda}\,d)\right] \tag{24}$$

We can choose B real and positive.

We show some properties of eigenfunctions χ_n^α, which we will need later. The physical interpretation of the first two lemmas is that the Robin boundary conditions are for the particle "strictly more repulsive" than the Neumann ones.

Lemma 4.2. *If $\alpha > 0$, then E_1^α lies in the open interval $(0, \pi^2/d^2)$.*

Proof. Since we have already shown that E_1^α is non-negative, it is enough to prove that the equation (22) has a solution in the interval $(0, \pi^2/d^2)$. Define

$$f(\sqrt{\lambda}) := 2\alpha\sqrt{\lambda}\cos(\sqrt{\lambda}\,d) + (\alpha^2 - \lambda)\sin(\sqrt{\lambda}\,d).$$

The first derivative of f is

$$f'(\sqrt{\lambda}) = 2\alpha\cos(\sqrt{\lambda}\,d) - 2\alpha\sqrt{\lambda}d\sin(\sqrt{\lambda}\,d) - 2\sqrt{\lambda}\sin(\sqrt{\lambda}\,d) + (\alpha^2-\lambda)d\cos(\sqrt{\lambda}\,d).$$

One has

$$f(0) = 0$$
$$f'(0) = 2\alpha + \alpha^2 > 0$$
$$f(\pi/d) = -2\pi\alpha/d < 0.$$

This and the fact that f is a continuous function follows that the implicit equation (22) has a solution for $\lambda \in (0, \pi^2/d^2)$. □

Lemma 4.3. *If $\alpha > 0$, then $\chi_1^\alpha(0)^2 < \chi_1^N(0)^2 = 1/d$.*

Proof. From (22) we can express α in terms of λ:

$$\alpha = \frac{\sqrt{\lambda}(\pm 1 - \cos(\sqrt{\lambda}\, d))}{\sin(\sqrt{\lambda}\, d)},$$

where the sign is to be chosen in such a way that $\alpha > 0$. In our case $\lambda \in (0, \pi^2/d^2)$ we therefore have

$$\alpha = \frac{\sqrt{\lambda}(1 - \cos(\sqrt{\lambda}\, d))}{\sin(\sqrt{\lambda}\, d)}. \tag{25}$$

If we put (25) to (24), we get

$$B^2 = \frac{\sqrt{\lambda}(1 + \cos(\sqrt{\lambda}\, d))}{(\sqrt{\lambda}\, d) + \sin(\sqrt{\lambda}\, d)}.$$

Since $\sqrt{\lambda} d$ lies in $(0, \pi)$ and $\sin x > x \cos x$ for all $x \in (0, \pi)$, the following inequality holds:

$$\chi_1^\alpha(0)^2 = B^2 = \frac{\sqrt{\lambda}(1 + \cos(\sqrt{\lambda}\, d))}{\sqrt{\lambda} d + \sin(\sqrt{\lambda}\, d)} < \frac{\sqrt{\lambda}(1 + \cos(\sqrt{\lambda}\, d))}{\sqrt{\lambda} d + \sqrt{\lambda} d \cos(\sqrt{\lambda}\, d)} = 1/d.$$

□

In the following lemma we establish certain symmetry of the first eigenfunction:

Lemma 4.4. $\forall y \in [0, d], \quad \chi_1^\alpha(y) = \chi_1^\alpha(d - y)$.

Proof. Since by Lemma 4.2 the eigenvalue $E_1^\alpha \in (0, \pi^2/d^2)$, the parameter α satisfies (25) and we can write

$$\frac{\alpha}{\sqrt{\lambda}} = \frac{1 - \cos(\sqrt{\lambda}\, d)}{\sin(\sqrt{\lambda}\, d)}.$$

Then following equalities hold:

$$\begin{aligned}
\frac{\chi_1^\alpha(d-y)}{B} &= \frac{\alpha}{\sqrt{\lambda}}\sin(\sqrt{\lambda}(d-y)) + \cos(\sqrt{\lambda}(d-y)) = \\
&= \frac{1-\cos(\sqrt{\lambda}\,d)}{\sin(\sqrt{\lambda}\,d)}\sin(\sqrt{\lambda}\,d)\cos(\sqrt{\lambda}\,y) - \\
&- \frac{1-\cos(\sqrt{\lambda}\,d)}{\sin(\sqrt{\lambda}\,d)}\cos(\sqrt{\lambda}\,d)\sin(\sqrt{\lambda}\,y) + \\
&+ \cos(\sqrt{\lambda}\,d)\cos(\sqrt{\lambda}\,y) + \sin(\sqrt{\lambda}\,d)\sin(\sqrt{\lambda}\,y) = \\
&= \cos(\sqrt{\lambda}\,y) - \cos(\sqrt{\lambda}\,d)\cos(\sqrt{\lambda}\,y) + \cos(\sqrt{\lambda}\,d)\cos(\sqrt{\lambda}\,y) + \\
&+ \frac{-\cos(\sqrt{\lambda}\,d) + \cos^2(\sqrt{\lambda}\,d) + \sin^2(\sqrt{\lambda}\,d)}{\sin(\sqrt{\lambda}\,d)}\sin(\sqrt{\lambda}\,y) = \\
&= \frac{\alpha}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\,y) + \cos(\sqrt{\lambda}\,y) = \\
&= \frac{\chi_1^\alpha(y)}{B}
\end{aligned}$$

□

4.1.3 Neumann case

Now, let us explicitly express the eigenvalues and corresponding eigenfunctions of the Laplacian on $(0,d)$ with Neumann boundary conditions. Let us denote

$$H_N^{(0,d)} := H_0^{(0,d)} \quad \text{and} \quad Q_N^{(0,d)} := Q_0^{(0,d)},$$

where $H_0^{(0,d)}$ and $Q_0^{(0,d)}$ can be defined in the same way as $H_\alpha^{(0,d)}$ and $Q_\alpha^{(0,d)}$, respectively, with $\alpha > 0$ (*cf* Section 3). Notice that Proposition 4.1 holds also for $\alpha = 0$. From (22) we get the equation for eigenvalues of $H_N^{(0,d)}$:

$$\lambda \sin(\sqrt{\lambda}\,d) = 0.$$

Here λ can be equal to zero, because it does not lead to the trivial solution, but to the constant one. If follows that the eigenvalues of $H_N^{(0,d)}$ arranged in increasing order are given by

$$E_n^N = (n-1)^2\pi^2/d^2, \quad \text{where } n \in \mathbb{N} \setminus \{0\}.$$

From (23) we get the corresponding family of eigenfunctions normalised to 1:

$$\chi_n^N(y) = \begin{cases} \sqrt{\frac{1}{d}} & \text{if } n = 1, \\ \sqrt{\frac{2}{d}} \cos((n-1)\pi y/d) & \text{if } n \geq 2. \end{cases}$$

$\{\chi_n^N\}_{n=1}^{\infty}$ forms a complete orthonormal family (*cf* [12, Kap. 16.2, Pozn. 13]). This and the fact that $E_n^N \to \infty$ as $n \to \infty$ imply that the essential spectrum of $H_N^{(0,d)}$ is empty (*cf* [3, Thm. 4.1.5]). Hence

$$\sigma(H_N^{(0,d)}) = \sigma_{disc}(H_N^{(0,d)}) = \{E_n^N\}_{n=1}^{\infty}.$$

4.1.4 Dirichlet case

Another limit case of Robin boundary conditions (corresponding formally to $\alpha \to +\infty$) is the Dirichlet Laplacian, denoted by $-\Delta_D^{(0,d)}$ and defined by

$$-\Delta_D^{(0,d)}\chi := -\chi'', \quad \chi \in D(-\Delta_D^{(0,d)}) := \big\{\chi \in W^{2,2}((0,d)) \big| \chi(0) = \chi(d) = 0\big\}.$$

The general solution of the equation (17) is of the form (19). The boundary conditions lead to the system of equations

$$\begin{pmatrix} 0 & 1 \\ \sin(\sqrt{\lambda}\, d) & \cos(\sqrt{\lambda}\, d) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \tag{26}$$

The requirement

$$\begin{vmatrix} 0 & 1 \\ \sin(\sqrt{\lambda}\, d) & \cos(\sqrt{\lambda}\, d) \end{vmatrix} = 0$$

gives the equation for λ:

$$\sin(\sqrt{\lambda}\, d) = 0 \tag{27}$$

Hence the eigenvalues are given by

$$E_n^D = n^2\pi^2/d^2, \quad \forall n \in \mathbb{N} \setminus \{0\}.$$

The case $\lambda = 0$ (which also solves (27)) corresponds to the trivial solution, because the system (26) implies that $B = 0$. The eigenfunctions are then given by

$$\chi_n^D(y) = \sqrt{\frac{2}{d}} \sin(n\pi y/d), \quad \forall n \in \mathbb{N} \setminus \{0\}.$$

By the same reason as in the Neumann case, the essential spectrum of $-\Delta_D^{(0,d)}$ is empty, *i.e.*

$$\sigma(-\Delta_D^{(0,d)}) = \sigma_{disc}(-\Delta_D^{(0,d)}) = \{E_n^D\}_{n=1}^{\infty}.$$

4.1.5 The essential spectrum

We demonstrate, that the essential spectrum of $H_\alpha^{(0,d)}$ is empty.

Proposition 4.5. *If $\alpha \geq 0$, then $\sigma_{ess}(H_\alpha^{(0,d)}) = \varnothing$.*

Proof. Let $\{\lambda_n^N\}_{n=1}^{\infty}$ and $\{\lambda_n^\alpha\}_{n=1}^{\infty}$ be the sequence of numbers associated with $Q_N^{(0,d)}$ and $Q_\alpha^{(0,d)}$, respectively, by the minimax principle, *i.e.*

$$\lambda_n^N := \inf_{\substack{\mathcal{P}\subseteq D(Q_N^{(0,d)}) \\ \dim(\mathcal{P})=n}} \sup_{\psi\in\mathcal{P}} \frac{\int_0^d |\dot{\psi}|^2}{\int_0^d |\psi|^2},$$

$$\lambda_n^\alpha := \inf_{\substack{\mathcal{P}\subseteq D(Q_\alpha^{(0,d)}) \\ \dim(\mathcal{P})=n}} \sup_{\psi\in\mathcal{P}} \frac{\int_0^d |\dot{\psi}|^2 + |\psi(0)|^2 + |\psi(d)|^2}{\int_0^d |\psi|^2}.$$

Let λ_∞^N and λ_∞^α be the threshold of essential spectrum of $H_N^{(0,d)}$ and $H_\alpha^{(0,d)}$, respectively, *i.e.*

$$\lambda_\infty^N := \lim_{n\to\infty} \lambda_n^N, \qquad \lambda_\infty^\alpha := \lim_{n\to\infty} \lambda_n^\alpha.$$

Since $D(Q_N^{(0,d)}) = D(Q_\alpha^{(0,d)})$, the following inequality holds:

$$\forall n \in \mathbb{N}\setminus\{0\}, \quad \lambda_n^N \leq \lambda_n^\alpha.$$

Hence

$$\lambda_\infty^N \leq \lambda_\infty^\alpha.$$

From this and the fact that $\sigma_{ess}(H_N^{(0,d)}) = \varnothing$ (*cf* Section 4.1.3) we get the desired claim. □

Thus, the spectrum of the transversal Hamiltonian can be written like

$$\sigma(H_\alpha^{(0,d)}) = \sigma_{disc}(H_\alpha^{(0,d)}) = \{E_n^\alpha\}_{n=1}^{\infty}.$$

4.1.6 Numerical results

In this section, we show some numerical results of solving the problem of the transversal Hamiltonian. Rewriting the implicit equation (22) into the form

$$\sqrt{\lambda}\cot(\sqrt{\lambda}\,d) = \frac{\lambda - \alpha^2}{2\alpha}, \tag{28}$$

we can visualise its solutions as the intersections of the graph of the function at the l.h.s. and at the r.h.s. in (28) (see Figure 1). The solution of (22) can be also drawn as a dependence of energies E_n^α on the parameter α, see Figure 2. Figure 3 shows how the parameter α influences the first eigenfunction. The quadrate of eigenfunctions corresponding to higher energies are presented in Figure 4.

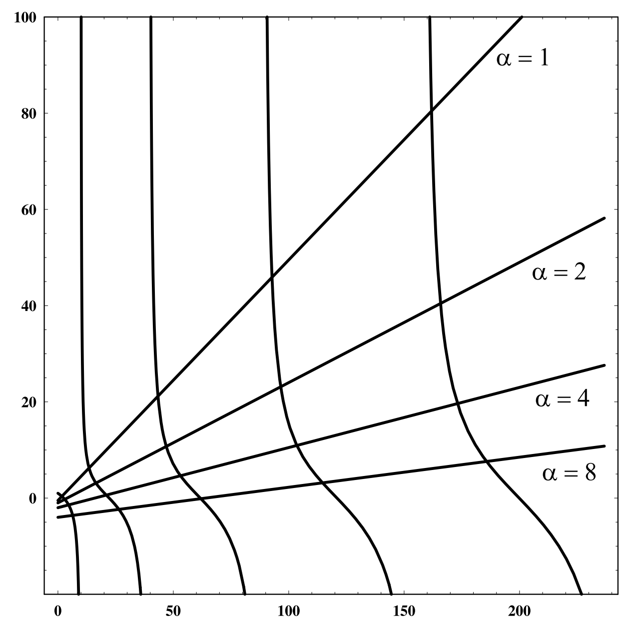

Figure 1: Graphical solution of implicit equation for various α in the case $d = 1$.

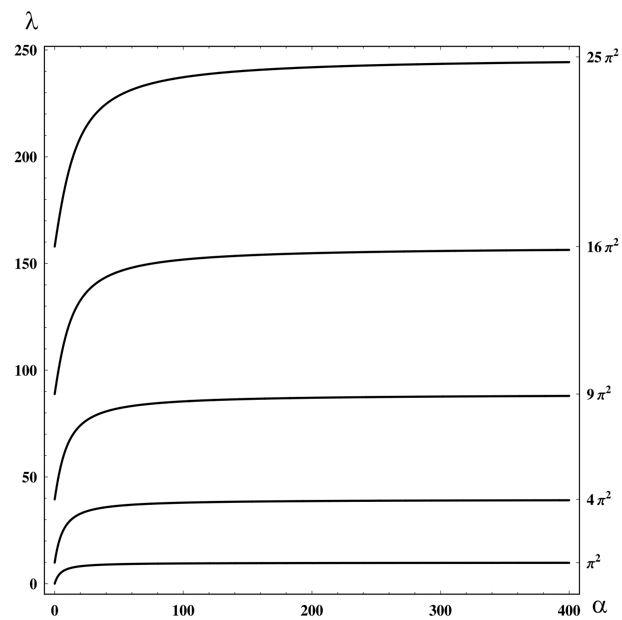

Figure 2: Dependence of the excited energies on α in the case $d = 1$.

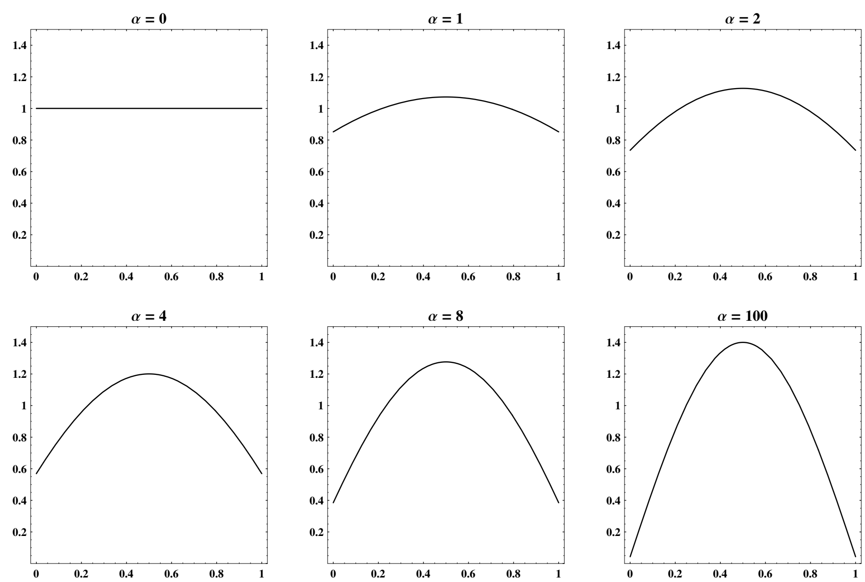

Figure 3: First eigenfunctions of $H_\alpha^{(0,d)}$ for various α in the case $d = 1$.

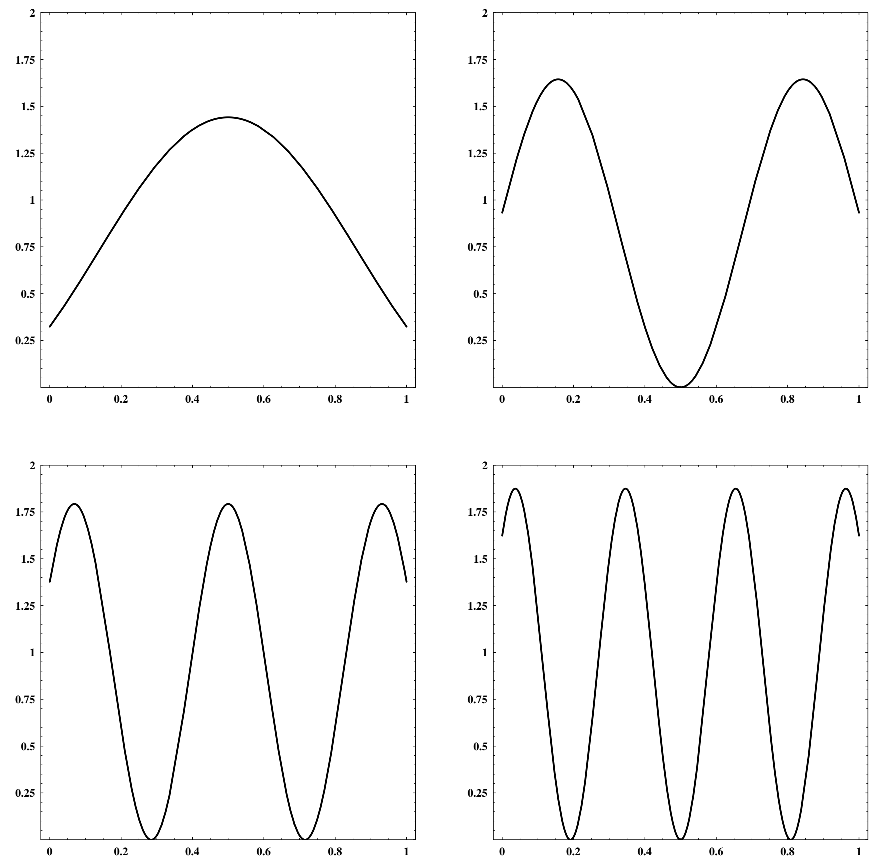

Figure 4: The quadrate of first four eigenfunctions of $H_\alpha^{(0,d)}$ for $\alpha = 4$, $d = 1$.

4.2 The longitudinal Hamiltonian

If $\Omega = \mathbb{R}$, the boundary is empty and we have

$$D(-\Delta^{\mathbb{R}}) = W^{2,2}(\mathbb{R})$$

(we omit the superfluous index α here). On the other hand, since

$$\mathfrak{D}(\mathbb{R}) = \mathfrak{L}_\alpha(\mathbb{R}) = C_0^\infty(\mathbb{R})$$

and $\|\cdot\|_0$ is just the $W^{1,2}$-norm, we have (by the definition of the space $W_0^{1,2}$)

$$D(Q^{\mathbb{R}}) = W_0^{1,2}(\mathbb{R})\,.$$

Again, we shall omit the index α in the notation for the quadratic form $Q^{\mathbb{R}}$ and the associated operator $H^{\mathbb{R}}$.

Proposition 4.6. *One has* $\sigma(H^{\mathbb{R}}) = [0,\infty)$.

Proof. Since the form $Q^{\mathbb{R}}$ is non-negative, it follows that $\sigma(H^{\mathbb{R}}) \subseteq [0,\infty)$.

The proof of the converse inclusion depends upon the use of Theorem 2.1. We need show that for every $E \in [0,\infty)$ there exists a sequence of functions $\psi_n \in D(H^{\mathbb{R}})$ with $\|\psi_n\|_{L^2(\mathbb{R})} = 1$ such that $\lim_{n\to\infty} \|H^{\mathbb{R}}\psi_n - E\psi_n\|_{L^2(\mathbb{R})} = 0$. We define this sequence as follows. Let $\varphi \in C_0^\infty(\mathbb{R})$ has the support in [-1,1] and let $\|\varphi\|_{L^2(\mathbb{R})} = 1$. We define the sequence of functions $\psi_n \in C_0^\infty(\mathbb{R}) = \mathfrak{D}(\mathbb{R}) = \mathfrak{L}(\mathbb{R}) \subset D(H^{\mathbb{R}})$ by

$$\psi_n(x) := \varphi_n(x)e^{i\sqrt{E}x}, \quad \text{where } \varphi_n(x) := n^{-1/2}\varphi(x/n). \tag{29}$$

Here $\mathfrak{D}$ and $\mathfrak{L}$ were defined in (10) and Proposition 3.2, respectively. Notice that $H^{\mathbb{R}}$ acts on $\mathfrak{L}(\mathbb{R})$ as the Laplacian (*cf* Proposition 3.2).

We show that ψ_n is normalised to 1:

$$\|\psi_n\|_{L^2(\mathbb{R})} = \|\varphi_n\|\_L^2(\mathbb{R}) = n^{-1}\int_{\mathbb{R}} |\varphi(x/n)|\mathrm{d}x = \int_{\mathbb{R}} \varphi(u)\mathrm{d}u = \|\varphi\|_{L^2(\mathbb{R})} = 1.$$

One has

$$\begin{aligned} H^{\mathbb{R}}\psi_n(x) - E\psi_n(x) &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[(i\sqrt{E}\varphi_n(x) + \dot{\varphi}_n(x))e^{i\sqrt{E}x}\right] - Ee^{i\sqrt{E}x}\varphi_n(x) \\ &= (-\ddot{\varphi}_n(x) - 2i\sqrt{E}\dot{\varphi}_n(x))e^{i\sqrt{E}x} \end{aligned} \tag{30}$$

and for the norm:

$$\|H^{\mathbb{R}}\psi_n - E\psi_n\|_{L^2(\mathbb{R})} = \|\ddot{\varphi}_n + 2i\sqrt{E}\dot{\varphi}_n\|_{L^2(\mathbb{R})} \le \|\ddot{\varphi}_n\|_{L^2(\mathbb{R})} + 2\sqrt{E}\|\dot{\varphi}_n\|_{L^2(\mathbb{R})}.$$

Since

$$\|\dot{\varphi}_n\|_{L^2(\mathbb{R})} = n^{-1}\|\dot{\varphi}\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|\ddot{\varphi}_n\|_{L^2(\mathbb{R})} = n^{-2}\|\ddot{\varphi}\|_{L^2(\mathbb{R})}, \tag{31}$$

it follows that $\|H^{\mathbb{R}}\psi_n - E\psi_n\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$. Thus, $[0,\infty) \subseteq \sigma(H^{\mathbb{R}})$. □

4.3 The Hamiltonian of a straight planar strip

The spectrum of $H_\alpha^{\Omega_0}$ is equal to the sum of the spectrum of longitudinal Hamiltonian and the transversal one:

Theorem 4.7. *For $\alpha \geq 0$, one has $\sigma(H_\alpha^{\Omega_0}) = [E_1^\alpha, \infty)$.*

Proof. We prove this claim in two steps. Firstly, we show by using Theorem 2.1 that $[E_1^\alpha, \infty) \subseteq \sigma(H_\alpha^{\Omega_0})$. Secondly, we demonstrate that $H_\alpha^{\Omega_0}$ is bounded from below by E_1^α.

Let $\lambda \geq E_1^\alpha$ and $\{\psi_n\}_{n=1}^\infty \subset D(H_\alpha^{\Omega_0})$ be a sequence of functions defined by

$$\psi_n(x,y) := \varphi_n(x) e^{i\sqrt{\lambda - E_1^\alpha}x} \chi_1^\alpha(y),$$

where φ_n are given by (29). The functions ψ_n indeed lie in $D(H_\alpha^{\Omega_0})$, because they are contained in $\mathfrak{L}_\alpha(\Omega_0)$. Notice, that $H_\alpha^{\Omega_0}$ acts on $\mathfrak{L}_\alpha(\Omega_0)$ as the Laplacian (*cf* Proposition 3.2). Since χ_1^α fulfils $-\ddot{\chi}_1^\alpha(y) = E_1^\alpha \chi_1^\alpha(y)$, one get

$$\begin{aligned} H_\alpha^{\Omega_0}\psi_n(x,y) - \lambda\psi_n(x,y) &= -(\ddot{\varphi}_n(x) + 2i\sqrt{\lambda - E_1^\alpha}\dot{\varphi}_n(x))e^{i\sqrt{\lambda - E_1^\alpha}x} \\ \|H_\alpha^{\Omega_0}\psi_n(x,y) - \lambda\psi_n(x,y)\| &= \|\ddot{\varphi}_n + 2i\sqrt{\lambda - E_1^\alpha}\dot{\varphi}_n\|_{L^2(\mathbb{R})} \\ &\leq \|\ddot{\varphi}_n\|_{L^2(\mathbb{R})} + 2\sqrt{\lambda - E_1^\alpha}\|\dot{\varphi}_n\|_{L^2(\mathbb{R})} \to 0 \end{aligned}$$

It follows that $[E_1^\alpha, \infty) \subseteq \sigma(H_\alpha^{\Omega_0})$ by Theorem 2.1.

For proving the inverse inclusion it is enough to show that $Q_\alpha^{\Omega_0}[\psi] \geq E_1^\alpha ||\psi||^2$ for all $\psi \in \mathfrak{D}(\Omega_0)$, a dense subspace of $D(Q_\alpha^{\Omega_0})$.Since the spectrum of $H_\alpha^{(0,d)}$ starts by E_1^α, we can write by Theorem 2.2 and Remark 2.4 for all $\chi \in D(H_\alpha^{(0,d)})$:

$$Q_\alpha^{(0,d)}[\chi] = \|\dot{\chi}\|^2_{L^2(0,d)} + \alpha|\chi(0)|^2 + \alpha|\chi(d)|^2 \geq E_1\|\chi\|^2_{L^2(0,d)}.$$

Using this inequality, together with Fubini's theorem, we get for all $\psi \in \mathfrak{D}(\Omega_0)$

$$\begin{aligned} Q_\alpha^{\Omega_0}[\psi] &= \int_{\Omega_0}|\partial_x\psi|^2 + \int_{\Omega_0}|\partial_y\psi|^2 + \alpha\int_{\partial\Omega_0}|\psi|^2 \\ &\geq \int_{\Omega_0}|\partial_y\psi(x,y)|^2\mathrm{d}x\mathrm{d}y + \alpha\int_{\mathbb{R}}|\psi(x,0)|^2\mathrm{d}x + \alpha\int_{\mathbb{R}}|\psi(x,d)|^2\mathrm{d}x \\ &= \int_{\mathbb{R}}\mathrm{d}x\left[\int_0^d|\partial_y\psi(y)|^2\mathrm{d}y + \alpha|\psi(x,0)|^2 + \alpha|\psi(x,d)|^2\right] \\ &\geq \int_{\mathbb{R}}\mathrm{d}x E_1^\alpha\int_0^d|\psi|^2\mathrm{d}y = E_1^\alpha||\psi||^2. \end{aligned}$$

We have used the fact that $y \mapsto \psi(x,y)$ with $x \in \mathbb{R}$ fixed belongs to $\mathfrak{D}((0,d)) \subset D(Q_\alpha^{(0,d)})$. It follows that $\sigma(H_\alpha) \subseteq [E_1^\alpha, \infty)$. □

5 Curved waveguide

5.1 The Geometry

Let $\Gamma \equiv (\Gamma^1, \Gamma^2)$ be an infinite unit-speed plane curve, *i.e.* a C^2-smooth mapping $\Gamma : \mathbb{R} \to \mathbb{R}^2$ satisfying $|\dot{\Gamma}(s)| = 1$ for all $s \in \mathbb{R}$. Define a unit normal vector field $N := (-\dot{\Gamma}^2, \dot{\Gamma}^1)$ and a tangent field $T := (\dot{\Gamma}^1, \dot{\Gamma}^2)$. The couple (T, N) is a Frenet Frame of Γ. The curvature of Γ defined by $k := \det(\dot{\Gamma}, \ddot{\Gamma})$ is a continuous function of the arc-length parameter s. We define a curved strip of the width d as $\Omega := \mathcal{L}(\Omega_0)$, where

$$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2 : (s, u) \mapsto \Gamma(s) + uN(s). \tag{32}$$

Actually, the image Ω has indeed a geometrical meaning of a non-self-intersecting strip only if we impose some additional hypotheses on d and on the global geometry of Γ. Namely, we want to find sufficient conditions ensuring that the mapping $\mathcal{L} : \Omega_0 \to \Omega$ is a diffeomorphism. By means of the inverse function theorem, it is enough to assume that the restriction $\mathcal{L} \restriction \Omega_0$ is injective and that the Jacobian of $\mathcal{L}$ is non-zero on Ω_0. Using the Frenet formulae

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix}. \tag{33}$$

we find

$$\partial_1 \mathcal{L} = \dot{\Gamma} + u\dot{N} = (1 - uk)\dot{\Gamma}.$$

At the same time, $\partial_2 \mathcal{L} = N$ and it follows that the Jacobian of $\mathcal{L}$ is given by

$$\det(\partial_1 \mathcal{L}, \partial_2 \mathcal{L}) = \begin{vmatrix} (1 - uk)\dot{\Gamma}^1 & N^1 \\ (1 - uk)\dot{\Gamma}^2 & N^2 \end{vmatrix} = (1 - uk)|\dot{\Gamma}|^2 = 1 - uk. \tag{34}$$

The Jacobian is non-zero if we assume that $k(s)u < 1$ for all $s \in \mathbb{R}$ and all $u \in (0, d)$. This follows from the condition that $d \sup_{s \in \mathbb{R}} k(s) < 1$.

More restrictively, we will always assume that

(r1) $\mathcal{L} \restriction \overline{\Omega_0}$ is injective,

(r2) $||k_+||_\infty d < 1$,

where $k_+ := \max\{0, k\}$. The first condition ensures that the outward unit normal vector $n : \partial\Omega \to \mathbb{R}^2$ is defined uniquely.

5.2 The Hamiltonian

Our main strategy is to replace the simple operator H_α^Ω on a complicated space $L^2(\Omega, \mathrm{d}x\mathrm{d}y)$ by a more complicated operator $\tilde{H}_\alpha$ on the simpler Hilbert space $L^2(\Omega_0, (1-uk)\mathrm{d}s\mathrm{d}u)$ by using the diffeomorphism $\mathcal{L} : \Omega_0 \to \Omega$. In other terms, we express the quadratic form Q_α^Ω associated with H_α^Ω in the curvilinear coordinates (s, u) defined by $\mathcal{L}$. That is, for any given $\psi \in L^2(\Omega)$ there is a unique $\phi \in L^2(\Omega_0, (1-uk)\mathrm{d}s\mathrm{d}u)$ defined by $\phi := \psi \circ \mathcal{L}$, and *vice versa*, and the transformed quadratic form $\tilde{Q}_\alpha$ in $L^2(\Omega_0, (1-uk)\mathrm{d}s\mathrm{d}u)$ is defined by

$$\tilde{Q}_\alpha[\phi] := Q_\alpha^\Omega[\phi \circ \mathcal{L}^{-1}]\,, \qquad D(\tilde{Q}_\alpha[\phi]) = \{\phi \in L^2(\Omega_0, (1-uk)\mathrm{d}s\mathrm{d}u) \mid \phi \circ \mathcal{L}^{-1} \in L^2(\Omega)\}\,.$$

The gradient in the curvilinear coordinates is expressed in the following way:

$$\begin{aligned} |\nabla\psi|^2 &= |\partial_x\psi|^2 + |\partial_y\psi|^2 \\ &= \frac{1}{(1-uk)^2}\left\{\left|N^2\partial_1\phi - (1-uk)T^2\partial_2\phi\right|^2 + \left|-N^1\partial_1\phi + (1-uk)T^1\partial_2\phi\right|^2\right\} \\ &= \frac{|\partial_1\phi|^2}{(1-uk)^2} + |\partial_2\phi|^2. \end{aligned} \tag{35}$$

The first part of Q_α^Ω has in the curvilinear coordinates the form

$$\int_\Omega |\nabla\psi|^2 \mathrm{d}x\mathrm{d}y = \int_{\Omega_0} \frac{|\partial_1\phi|^2}{1-uk}\mathrm{d}s\mathrm{d}u + \int_{\Omega_0} |\partial_2\phi|^2(1-uk)\mathrm{d}s\mathrm{d}u.$$

The boundary part of Q_α^Ω is transformed as

$$\int_{\partial\Omega} |\psi|^2 = \int_{\mathbb{R}\times\{0\}} |\phi|^2 + \int_{\mathbb{R}\times\{d\}} |\phi|^2(1-dk).$$

Summing up, the quadratic form in curvilinear coordinates has the form

$$\begin{aligned} \tilde{Q}_\alpha[\phi] = &\int_{\Omega_0} \frac{|\partial_1\phi|^2}{1-uk}\mathrm{d}s\mathrm{d}u + \int_{\Omega_0} |\partial_2\phi|^2(1-uk)\mathrm{d}s\mathrm{d}u + \\ &+ \alpha\int_{\mathbb{R}\times\{0\}} |\phi|^2 + \alpha\int_{\mathbb{R}\times\{d\}} |\phi|^2(1-dk). \end{aligned} \tag{36}$$

The operator $\tilde{H}_\alpha$ is the unique self-adjoint operator associated with the form (36). Notice that following equality holds on $\mathfrak{L}_\alpha(\Omega_0)$:

$$\tilde{H}_\alpha = -\frac{1}{(1-uk(s))^2}\partial_s^2 - \frac{u\dot{k}(s)}{(1-uk(s))^3}\partial_s - \partial_u^2 + \frac{k(s)}{1-uk(s)}\partial_u. \tag{37}$$

5.3 The stability of essential spectrum

As shown in Theorem 4.7, the essential spectrum of the Robin Laplacian on the straight strip Ω_0 is $[E_1^\alpha, \infty)$. In this section, we prove that the essential spectrum of a curved strip Ω coincides with the straight one if we assume that

(k) the curvature k has a compact support.

Theorem 5.1. *Suppose (r1), (r2), and (k). If $\alpha > 0$, then $\sigma_{ess}(H_\alpha^\Omega) = [E_1^\alpha, \infty)$.*

The proof of this theorem is achieved in two steps. Firstly, in Lemma 5.2, we prove that all values above E_1^α belongs to the essential spectrum. Secondly, in Lemma 5.3, we show that the threshold of essential spectrum does not descent below the energy E_1^α.

Lemma 5.2. *Suppose (r1), (r2), and (k). If $\alpha > 0$, then $[E_1^\alpha, \infty) \subseteq \sigma_{ess}(H_\alpha^\Omega)$.*

Proof. First of all, notice that showing that $[E_1^\alpha, \infty) \subseteq \sigma(H_\alpha^\Omega)$, we actually prove that the set $[E_1^\alpha, \infty)$ belongs to the essential spectrum of H_α^Ω, because intervals do not contain isolated points. Let $\lambda \in [E_1^\alpha, \infty)$. Define

$$\psi_n(s,u) := \varphi_n(s)\chi_1^\alpha(u)e^{i\sqrt{\lambda - E_1^\alpha}s}, \quad \varphi_n(s) := \frac{1}{\sqrt{n}}\varphi(\frac{s}{n} - n),$$

where φ is defined in the proof of Theorem 4.6. We show that the sequence $\{\psi_n\}$ satisfies the conditions of Theorem 2.1. It is easy to see that $||\psi_n||_{L^2(\Omega_0,(1-uk)\mathrm{d}s\mathrm{d}u)} = 1$ for sufficiently large n. Note that supp $\varphi_n \subseteq [n^2 - n, n^2 + n]$.

One has

$$\int_{\Omega_0} |\psi_n(s,u)|^2(1-uk(s))\mathrm{d}s\mathrm{d}u = \int_{\Omega_0} |\psi_n(s,u)|^2\mathrm{d}s\mathrm{d}u - \int_0^d |\chi_1^\alpha(u)|^2\mathrm{d}u \int_{\mathbb{R}} |\varphi_n(s)|^2 k(s)\mathrm{d}s.$$

For sufficiently large n the second term at the r.h.s. is equal to zero, because φ_n and k have a disjoint supports. It follows that

$$||\psi_n||_{L^2(\Omega_0,(1-uk)\mathrm{d}s\mathrm{d}u)} = ||\psi_n||_{L^2(\Omega_0)},$$

if n is large enough. The derivatives of the functions ψ_n are

$$\begin{aligned}
\partial_1\psi_n(s,u) &= e^{i\sqrt{\lambda - E_1^\alpha}s}(\dot{\varphi}_n(s)\chi_1^\alpha(u) + \varphi_n(s)\chi_1^\alpha(u)),\\
\partial_1^2\psi_n(s,u) &= e^{i\sqrt{\lambda - E_1^\alpha}s}\chi_1^\alpha(u)[\ddot{\varphi}_n(s) + 2i\sqrt{\lambda - E_1^\alpha}\dot{\varphi}_n(s) - (\lambda - E_1^\alpha)\varphi_n(s)],\\
\partial_2\psi_n(s,u) &= \varphi_n(s)\dot{\chi}_1^\alpha(u)e^{i\sqrt{\lambda - E_1^\alpha}s},\\
\partial_2^2\psi_n(s,u) &= \varphi_n(s)\ddot{\chi}_1^\alpha(u)e^{i\sqrt{\lambda - E_1^\alpha}s}.
\end{aligned}$$

Since the operator $\tilde{H}_\alpha$ acts as (37), we get

$$\begin{aligned}(\tilde{H}_\alpha\psi_n - \lambda\psi_n)(s,u) = &- \frac{1}{(1-uk(s))^2}(\ddot{\varphi}_n(s) + 2i\sqrt{\lambda - E_1^\alpha}\dot{\varphi}_n(s))\chi_1^\alpha(u)e^{i\sqrt{\lambda-E_1^\alpha}s} \\ &- \frac{u\dot{k}(s)}{(1-uk(s))^3}(\dot{\varphi}_n(s) + \sqrt{\lambda - E_1^\alpha}\varphi_n(s))\chi_1^\alpha(u)e^{i\sqrt{\lambda-E_1^\alpha}s} \\ &+ \frac{(2-uk(s))uk(s)}{(1-uk(s))^2}E_1^\alpha\varphi_n(s)\chi_1^\alpha(u)e^{i\sqrt{\lambda-E_1^\alpha}s} \\ &+ \frac{k(s)}{1-uk(s)}\varphi_n(s)\dot{\chi}_1^\alpha(u)e^{i\sqrt{\lambda-E_1^\alpha}s} \\ &+ \frac{(2-uk(s))uk(s)}{(1-uk(s))^2}\lambda\varphi_n(s)\chi_1^\alpha(u)e^{i\sqrt{\lambda-E_1^\alpha}s}\end{aligned}$$

All terms at the r.h.s. except of the first one are zero for sufficiently large n. The equalities (31) imply that the norm

$$\begin{aligned}&\|\tilde{H}_\alpha\psi_n - \lambda\psi_n\|_{L^2(\Omega_0)} = \\ &= \int_{\mathbb{R}}\left(\left|\ddot{\varphi}_n(s) + 2i\sqrt{\lambda - E_1^\alpha}\dot{\varphi}_n(s)\right|^2\int_0^d\left|\chi_1^\alpha(u)\right|^2\frac{1}{(1-uk(s))^4}\mathrm{d}u\right)\mathrm{d}s \leq \\ &\leq \frac{d}{(1-d\|k\|_+)^4}\left(\|\ddot{\varphi}_n\|_{L^2(\mathbb{R})} + 2\sqrt{\lambda - E_1^\alpha}\,\|\dot{\varphi}_n\|_{L^2(\mathbb{R})}\right)\end{aligned}$$

tends to zero as $n \to \infty$. □

Lemma 5.3. *Suppose (r1), (r2), and (k). If* $\alpha > 0$*, then* $\inf\sigma_{ess}(H_\alpha^\Omega) \geq E_1^\alpha$*.*

Proof. For proving this theorem we will use an argument called Neumann bracketing. Let $\{\lambda_n^\alpha(\Omega)\}_{n=1}^\infty$ be a sequence of numbers associated with Q_α^Ω by the minimax principle, *i.e.*

$$\lambda_n^\alpha := \inf_{\substack{\mathcal{P}\subseteq D(Q_\alpha^\Omega)\\ \dim\mathcal{P}=n}}\sup_{\psi\in\mathcal{P}}\frac{\int_\Omega|\nabla\psi|^2 + \alpha\int_{\partial\Omega}|\psi|^2}{\int_\Omega|\psi|^2}. \tag{38}$$

Let $\lambda_\infty^\alpha(\Omega)$ be the spectral threshold of the essential spectrum of H_α^Ω, *i.e.*

$$\lambda_\infty^\alpha(\Omega) := \lim_{n\to\infty}\lambda_n^\alpha(\Omega).$$

Since k has a compact support, there exists a number $s_0 > 0$ such that

$$\forall|s| \geq s_0, \quad k(s) = 0.$$

Let Γ_+ and Γ_- be curves defined by

$$\Gamma_\pm := \mathcal{L}(\{\pm s_0\} \times (0,d)).$$

They divide Ω into three disjoint domains $\Omega_-, \Omega_c, \Omega_+$, *i.e.*

$$\Omega = \Omega_- \cup \Gamma_- \cup \Omega_c \cup \Gamma_+ \cup \Omega_+ \quad \text{and} \quad \Omega_- \cap \Omega_c = \Omega_c \cap \Omega_+ = \Omega_+ \cap \Omega_- = \varnothing,$$

where

$$\begin{aligned} \Omega_- &:= \mathcal{L}((-\infty, -s_0) \times (0,d)) \\ \Omega_c &:= \mathcal{L}((-s_0, s_0) \times (0,d)) \\ \Omega_+ &:= \mathcal{L}((s_0, \infty) \times (0,d)) \end{aligned}$$

We impose an additional Neumann boundary condition on Γ_+ and Γ_- by introducing the quadratic form

$$Q^\Omega_{\alpha N}[\psi] := \int_\Omega |\nabla\psi|^2 + \alpha \int_{\partial\Omega} |\psi|, \quad \psi \in D(Q^\Omega_{\alpha N}) := T_- \oplus T_c \oplus T_+,$$

where

$$\begin{aligned} T_- &:= \left\{\psi \in L^2(\Omega) \mid \psi \upharpoonright \Omega_- \in W^{1,2}(\Omega_-) \ \wedge \ \psi = 0 \text{ a.e. in } \Omega \setminus \Omega_-\right\}, \\ T_c &:= \left\{\psi \in L^2(\Omega) \mid \psi \upharpoonright \Omega_c \in W^{1,2}(\Omega_c) \ \wedge \ \psi = 0 \text{ a.e. in } \Omega \setminus \Omega_c\right\}, \\ T_+ &:= \left\{\psi \in L^2(\Omega) \mid \psi \upharpoonright \Omega_+ \in W^{1,2}(\Omega_+) \ \wedge \ \psi = 0 \text{ a.e. in } \Omega \setminus \Omega_+\right\}. \end{aligned}$$

Let $\{\lambda^{\alpha N}_n(\Omega)\}_{n=1}^\infty$ be the sequence of numbers associated with $Q^\Omega_{\alpha N}$ by the minimax principle, *i.e.*

$$\begin{aligned} \lambda^{\alpha N}_n(\Omega) &:= \inf_{\substack{\mathcal{P} \subseteq T_- \oplus T_c \oplus T_+ \\ \dim(\mathcal{P}) = n}} \sup_{\psi \in \mathcal{P}} \frac{\int_\Omega |\nabla\psi|^2 + \alpha \int_{\partial\Omega} |\psi|^2}{\int_\Omega |\psi|^2} \\ &= \inf_{\substack{\mathcal{P}_- \subseteq T_-,\, \mathcal{P}_c \subseteq T_c,\, \mathcal{P}_+ \subseteq T_+ \\ \dim(\mathcal{P}_-) + \dim(\mathcal{P}_c) + \dim(\mathcal{P}_+) = n}} \\ &\sup_{\psi_- \in \mathcal{P}_-,\, \psi_c \in \mathcal{P}_c,\, \psi_+ \in \mathcal{P}_+} \frac{\int_{\Omega_-} |\nabla\psi_-|^2 + \int_{\Omega_c} |\nabla\psi_c|^2 + \int_{\Omega_+} |\nabla\psi_+|^2 + \alpha \int_{\partial\Omega} |\psi|^2}{\int_{\Omega_-} |\psi_-|^2 + \int_{\Omega_c} |\psi_c|^2 + \int_{\Omega_+} |\psi_+|^2}. \end{aligned}$$

Since $D(Q^\Omega_\alpha) \subset D(Q^\Omega_{\alpha N})$, we clearly have

$$\forall n \in \mathbb{N} \setminus \{0\}, \qquad \lambda^\alpha_n(\Omega) \geq \lambda^{\alpha N}_n(\Omega)\,. \tag{39}$$

In particular,

$$\lambda_\infty^\alpha(\Omega) \geq \lambda_\infty^{\alpha N}(\Omega) := \lim_{n\to\infty} \lambda_n^{\alpha N}(\Omega)\,.$$

It follows from (38) that

$$\forall n \in \mathbb{N}\setminus\{0\}, \quad \forall \mathcal{P} \subseteq D(Q_\alpha^\Omega), \dim(\mathcal{P}) = n, \quad \lambda_n^\alpha(\Omega) \leq \sup_{\psi\in\mathcal{P}} \frac{\int_\Omega |\nabla\psi|^2 + \alpha\int_{\partial\Omega}|\psi|^2}{\int_\Omega |\psi|^2},$$

and similarly for $\lambda_n^\alpha(\Omega_-)$, $\lambda_n^\alpha(\Omega_c)$, and $\lambda_n^\alpha(\Omega_+)$. Consequently,

$$\begin{aligned}&\sup_{\psi_-\in\mathcal{P}_-,\,\psi_c\in\mathcal{P}_c,\,\psi_+\in\mathcal{P}_+} \frac{\int_{\Omega_-}|\nabla\psi_-|^2 + \int_{\Omega_c}|\nabla\psi_c|^2 + \int_{\Omega_+}|\nabla\psi_+|^2 + \alpha\int_{\partial\Omega}|\psi|^2}{\int_{\Omega_-}|\psi_-|^2 + \int_{\Omega_c}|\psi_c|^2 + \int_{\Omega_+}|\psi_+|^2}\\ &\geq \max\left\{\lambda_{n_-}^\alpha(\Omega_-), \lambda_{n_c}^\alpha(\Omega_c), \lambda_{n_+}^\alpha(\Omega_+)\right\},\end{aligned}$$

where $n_- := \dim(\mathcal{P}_-)$, $n_c := \dim(\mathcal{P}_c)$, and $n_+ := \dim(\mathcal{P}_+)$, with the convention that $\lambda_{n_-}^\alpha(\Omega_-) = 0$ or $\lambda_{n_c}^\alpha(\Omega_c) = 0$ or $\lambda_{n_+}^\alpha(\Omega_+) = 0$ if $n_- = 0$ or $n_c = 0$ or $n_+ = 0$, respectively. Hence

$$\forall n \in \mathbb{N}\setminus\{0\}, \quad \lambda_n^{\alpha N}(\Omega) \geq \min_{n_-+n_c+n_+=n} \max\left\{\lambda_{n_-}^\alpha(\Omega_-), \lambda_{n_c}^\alpha(\Omega_c), \lambda_{n_+}^\alpha(\Omega_+)\right\}$$

Taking the limit $n \to \infty$, we get

$$\lambda_\infty^{\alpha N}(\Omega) \geq \min\left\{\lambda_\infty^\alpha(\Omega_-), \lambda_\infty^\alpha(\Omega_c), \lambda_\infty^\alpha(\Omega_+)\right\}.$$

In [3, Thm. 7.2.2], it is shown that the essential spectrum of the Neumann Hamiltonian in a bounded region with smooth boundary is empty. The condition to the boundary can be generalized to our case, *i.e.*, to the piecewise smooth boundary, such that the angle at each vertex lies in an open interval $(0, 2\pi)$ (*cf* [3, Exercise 7.3]). The Robin boundary conditions are bounded from bellow by Neumann ones, in the sense that $\forall\psi \in D(Q_\alpha^{\Omega_c})$, $\quad Q_\alpha^{\Omega_c}[\psi] \geq Q_N^{\Omega_c}[\psi]$. This inequality holds also for the lower bound of the spectrum of operators associated with this forms. Therefore the operator $H_\alpha^{\Omega_c}$ has empty essential spectrum.

Since it can be established by the same way as Theorem 4.7 that $\inf\sigma_{ess}(H_\alpha^{\Omega_-}) = \inf\sigma_{ess}(H_\alpha^{\Omega_+}) = E_1^\alpha$, we get the desired claim. □

5.4 The existence of bound states

In this section we show that the non-trivial curvature of the strip pushes the spectral threshold down the value E_1^α.

Theorem 5.4. *Suppose (r1), (r2). If* $k \not\equiv 0$ *and* $\alpha > 0$*, then* $\inf \sigma(\tilde{H}_\alpha) < E_1^\alpha$.

Proof. By the Rayleigh-Ritz principle (Theorem 2.2), it is enough to find a function $\psi^\alpha \in D(\tilde{Q}_\alpha)$ such that

$$S_\alpha[\psi^\alpha] := \tilde{Q}_\alpha[\psi^\alpha] - E_1^\alpha ||\psi^\alpha||^2_{L^2(\Omega_0,(1-uk)\mathrm{d}s\mathrm{d}u)} < 0.$$

We define

$$\psi_n^\alpha(s,u) := \varphi_n(s)\chi_1^\alpha(u),$$

where $\varphi_n(s)$ is defined by

$$\varphi_n(s) := \begin{cases} 1 & \text{if} |s| \in [0,n), \\ 2 - |s|/n & \text{if} |s| \in [n,2n), \\ 0 & \text{if} |s| \in [2n,\infty). \end{cases} \tag{40}$$

Then

$$\begin{aligned} S_\alpha[\psi_n^\alpha] = & \int_{\Omega_0} |\dot{\varphi_n}(s)|^2 \chi_1^\alpha(u)^2 \frac{\mathrm{d}s\mathrm{d}u}{1-uk(s)} + \\ & + \int_{\Omega_0} |\varphi_n(s)|^2 \dot{\chi_1^\alpha}(u)^2 (1-uk(s))\mathrm{d}s\mathrm{d}u + \\ & + \alpha \int_{\mathbb{R}} |\varphi_n(s)|^2 \chi_1^\alpha(0)^2 \mathrm{d}s + \\ & + \alpha \int_{\mathbb{R}} |\varphi_n(s)|^2 \chi_1^\alpha(d)^2 (1-dk(s))\mathrm{d}s - \\ & - E_1^\alpha \int_{\Omega_0} |\varphi_n(s)|^2 \dot{\chi_1^\alpha}(u)^2 (1-uk(s))\mathrm{d}s\mathrm{d}u. \end{aligned} \tag{41}$$

The second term at the r.h.s. can be written by integration by parts like

$$\begin{aligned} \int_{\Omega_0} |\varphi_n(s)|^2 \dot{\chi_1^\alpha}(u)^2 (1-uk(s))\mathrm{d}s\mathrm{d}u = & \int_{\mathbb{R}} |\varphi_n(s)|^2 \dot{\chi_1^\alpha}(d)\chi_1^\alpha(d)(1-dk(s))\mathrm{d}s - \\ - \int_{\mathbb{R}} |\varphi_n(s)|^2 \dot{\chi_1^\alpha}(0)\chi_1^\alpha(0)\mathrm{d}s - & \int_{\Omega_0} |\varphi_n(s)|^2 \ddot{\chi_1^\alpha}(u)\chi_1^\alpha(u)(1-uk(s))\mathrm{d}s\mathrm{d}u + \\ & + \int_{\Omega_0} |\varphi_n(s)|^2 \dot{\chi_1^\alpha}(u)\chi_1^\alpha(u)k(s)\mathrm{d}s\mathrm{d}u \end{aligned} \tag{42}$$

Since $\dot{\chi}\chi = \frac{1}{2}(\chi^2)^{\cdot}$, the last term in (42) is

$$\int_{\Omega_0} |\varphi_n(s)|^2 \dot{\chi}_1^\alpha(u)\chi_1^\alpha(u)k(s)\mathrm{d}s\mathrm{d}u = \frac{1}{2}(\chi_1^\alpha(d)^2 - \chi_1^\alpha(0)^2)\int_{\mathbb{R}} |\varphi_n(s)|^2 k(s)\mathrm{d}s$$

and it is equal to zero by Lemma 4.4. Substituting the equalities

$$\begin{aligned} \ddot{\chi}_1^\alpha(u) &= -E_1\chi_1^\alpha(u) \\ \dot{\chi}_1^\alpha(0) &= \alpha\chi_1^\alpha(0) \\ \dot{\chi}_1^\alpha(d) &= -\alpha\chi_1^\alpha(d). \end{aligned} \tag{43}$$

to (42), we get

$$S_\alpha[\psi_n^\alpha] = \int_{\Omega_0} |\dot{\varphi}_n(s)|^2 \chi_1^\alpha(u)^2 \frac{\mathrm{d}s\mathrm{d}u}{1-uk(s)}.$$

Since

$$\left| \int_0^d \frac{\chi_1^\alpha(u)^2}{1-uk(s)}\mathrm{d}u \right| \leq \int_0^d \left| \frac{\chi_1^\alpha(u)^2}{1-uk(s)} \right| \mathrm{d}u \leq \frac{1}{1-d||k_+||_\infty},$$

the function

$$f(s) = \int_0^d \frac{\chi_1^\alpha(u)^2}{1-uk(s)}\mathrm{d}u$$

is bounded. Moreover $||\dot{\varphi}_n||_{L^2(\mathbb{R})} \xrightarrow[n\to\infty]{} 0$ and we have the limit

$$S_\alpha[\psi_n^\alpha] \xrightarrow[n\to\infty]{} 0.$$

Now we modify the function ψ_n in a curved part of the waveguide by defining

$$\psi_{n,\varepsilon}^\alpha(s,u) := \psi_n^\alpha(s,u) + \varepsilon\rho(s,u),$$

where $\varepsilon \in \mathbb{R}$ and $\rho(s,u) := \phi(s)u\chi_1^\alpha(u)$. Here $\phi \in C_0^\infty(\mathbb{R})$ is a real, non-negative, non-zero function with compact support contained in a bounded interval, where $k \not\equiv 0$ and does not change sign. We have

$$S_\alpha[\psi_{n,\varepsilon}^\alpha] = S_\alpha[\psi_n^\alpha] + 2\varepsilon S_\alpha(\rho,\psi_n^\alpha) + \varepsilon^2 S_\alpha[\rho].$$

The first term at the r.h.s. tends to zero as $n \to \infty$ and the last one does not depend on n. So we need to calculate the central term:

$$
\begin{aligned}
S_\alpha(\rho, \psi_n^\alpha) = & \int_{\Omega_0} \dot{\varphi}_n(s) u \chi_1^\alpha(u)^2 \dot{\phi}(s) \frac{\mathrm{d}s\mathrm{d}u}{1-uk(s)} + \\
& + \int_{\Omega_0} \varphi_n(s) \dot{\chi_1^\alpha}(u) \phi(s) \chi_1^\alpha(u)(1-uk(s))\mathrm{d}s\mathrm{d}u + \\
& + \int_{\Omega_0} \varphi_n(s) u \dot{\chi_1^\alpha}(u)^2 \phi(s)(1-uk(s))\mathrm{d}s\mathrm{d}u + \\
& + \alpha \int_{\mathbb{R}} \varphi_n(s) d \chi_1^\alpha(d)^2 \phi(s)(1-dk(s))\mathrm{d}s - \\
& - E_1 \int_{\Omega_0} \varphi_n(s) u \chi_1^\alpha(u)^2 \phi(s)(1-uk(s))\mathrm{d}s\mathrm{d}u. \qquad (44)
\end{aligned}
$$

We can write the third term at the r.h.s by integration by parts like

$$
\begin{aligned}
& \int_{\Omega_0} \varphi_n(s) u \dot{\chi_1^\alpha}(u)^2 \phi(s)(1-uk(s))\mathrm{d}s\mathrm{d}u = \\
& = \int_{\mathbb{R}} \varphi_n(s) d \dot{\chi_1^\alpha}(d) \chi_1^\alpha(d) \phi(s)(1-uk(s))\mathrm{d}s - \\
& - \int_{\Omega_0} \varphi_n(s) \phi(s) u \ddot{\chi_1^\alpha}(u) \chi_1^\alpha(u)(1-uk(s))\mathrm{d}s\mathrm{d}u - \\
& - \int_{\Omega_0} \varphi_n(s) \phi(s) \chi_1^\alpha(u) \dot{\chi_1^\alpha}(u)(1-uk(s))\mathrm{d}s\mathrm{d}u + \\
& + \int_{\Omega_0} \varphi_n(s) \phi(s) u \dot{\chi_1^\alpha}(u) \chi_1^\alpha(u) k(s)\mathrm{d}s\mathrm{d}u. \qquad (45)
\end{aligned}
$$

Putting (43) to (44) we finally get

$$
\begin{aligned}
S_\alpha(\rho, \psi_n^\alpha) = & \int_{\Omega_0} \dot{\varphi}_n(s) u \chi_1^\alpha(u)^2 \dot{\phi}(s) \frac{\mathrm{d}s\mathrm{d}u}{1-uk(s)} + \\
& + \int_{\Omega_0} \varphi_n(s) \phi(s) u \dot{\chi_1^\alpha}(u) \chi_1^\alpha(u) k(s)\mathrm{d}s\mathrm{d}u = \\
& = \int_{\Omega_0} \dot{\varphi}_n(s) u \chi_1^\alpha(u)^2 \dot{\phi}(s) \frac{\mathrm{d}s\mathrm{d}u}{1-uk(s)} + \\
& + \frac{1}{2} \left(d \chi_1^\alpha(d)^2 - 1 \right) \int_{\mathbb{R}} \varphi_n(s) \phi(s) k(s)\mathrm{d}s. \qquad (46)
\end{aligned}
$$

Summing up,

$$S_\alpha[\psi^\alpha_{n,\varepsilon}] \xrightarrow[n\to\infty]{} \varepsilon\left(d\chi^\alpha_1(d)^2 - 1\right)\int_{\mathbb{R}} \phi(s)k(s)\mathrm{d}s + \varepsilon^2 Q^\alpha_1[\rho]. \tag{47}$$

Since the integral in 47 is non-zero by the construction of ϕ and the term in brackets is negative by Lemma 4.4 and Lemma 4.3, we can take ε sufficiently small so that the term at the r.h.s. is negative. Hence we choose n sufficiently large so that $S_\alpha[\psi^\alpha_{n,\varepsilon}] < 0$. □

As a consequence of Theorem 5.4 and Theorem 5.1 we get following corollary:

Corollary 5.5. *Suppose (r1),(r2), and $\alpha > 0$. If the strip is not straight but the assumption (k) holds, then $\tilde{H}_\alpha$ has at least one eigenvalue of finite multiplicity below its essential spectrum $[E^\alpha_1, \infty)$, i.e., $\sigma_{disc}(\tilde{H}_\alpha) \neq \varnothing$.*

Remark 5.6. *Notice that for Neumann boundary conditions, i.e. $\alpha = 0$, the spectrum is purely essential and there are no bound states. Indeed, in the same way as in Theorem 5.1, one can prove that $\sigma_{ess}(\tilde{H}_0) = [E^0_1, \infty) = [0, \infty)$.*

6 Conclusion

We were interested in spectral properties of a curved planar waveguide, subject to Robin boundary conditions. We demonstrated the stability of the essential spectrum under the condition that the waveguide is curved only locally (*cf* Theorem 5.1). As the main result, we proved that there always exist quantum bound states bellow the bottom of the essential spectrum whenever the waveguide is not straight (*cf* Theorem 5.4 and Corollary 5.5).

The present thesis was motivated by the theory of quantum waveguides where the bound states are known to exist in waveguides with Dirichlet (and certain Dirichlet-Neumann) boundary coditions. The principal objective of the thesis was to demonstrate that the bound states exist also for other kinds of boundary conditions. The Robin boundary conditions may in principle model different types of interface in materials or approximate very thin layer structures.

There are several possible directions to which the results of the present thesis could be extended. For instance, one can consider the case where the boundary conditions are not uniform, letting the parameter α depend on the longitudinal coordinate s. It is also possible to consider the Robin Laplacian in higher-dimensional tubes. Finally, a detailed numerical study of the bound states would help to understand the interplay between the geometry and spectrum of quantum waveguides with Robin boundary conditions.

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