CZECH TECHNICAL UNIVERSITY IN PRAGUE
FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING

DIPLOMA THESIS

Analysis of Two-Dimensional Quantum Models with Singular Potentials

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May 10, 2006 in Prague

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Podpis

Název práce:	**Analýza dvourozměrných kvantových modelů se singulárními potenciály**
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Abstrakt:	Uvažujeme singulární poruchu $-\Delta$ v $L^2(\mathbb{R}^2)$ nesenou bodem nebo uzavřenou křivkou. Nalezneme explicitní výrazy pro rezolventu a amplitudu rozptylu. Dále zavedeme model popisující dvourozměrnou rotačně symetrickou kvantovou tečku (izotropní harmonický oscilátor) s nečistotou krátkého dosahu (bodová interakce). Nalezneme Kreinovu Q-funkci systému ve formě rychle konvergující sumy. Prozkoumáme strukturu spektra a její závislost na poloze nečistoty.
Klíčová slova:	Matematická fyzika, bodová interakce, singulární potenciál, kvantová tečka.

Title:	**Analysis of Two-Dimensional Quantum Models with Singular Potentials**
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Abstract:	A singular perturbation of $-\Delta$ in $L^2(\mathbb{R}^2)$ supported by a point or by a closed curve is considered. Explicit expressions for the resolvent and the scattering amplitude are given. Next a model describing a two-dimensional rotationally symmetric quantum dot (the isotropic harmonic oscillator) with a short-range impurity (the point interaction) is introduced. The Krein Q-function of the system is obtained in the form of a fast converging infinite sum. The structure of the spectrum and its dependence on the impurity position is analyzed.
Key-words:	Mathematical physics, point interaction, singular potential, quantum dot.

Contents

Notation

We replace the ordinary equal sign by $:=$ in defining equalities.

$\bar{A}$ denotes a complex conjugation, if A is a number or a function, and a closure, if A is a set or an operator. To label the set of all complex conjugated elements to the elements of a set A we use a star: A^*.

By operator we mean a linear mapping with a domain of definition $Dom(.)$ being a subspace of some Hilbert space $\mathscr{H}$. The standard inner product on the space of all quadratically integrable (with a non-negative measure $\mathrm{d}\mu$) functions defined on M: $\mathscr{H} = L^2(M, \mathrm{d}\mu)$ and the associated norm we denote by $\langle .,. \rangle$ and $\|.\|$.

$$\langle f, g \rangle := \int_M \bar{f} g \; \mathrm{d}\mu, \quad \forall f, g \in \mathscr{H}$$

To prevent from a misunderstanding, we will sometimes add a lower index of a Hilbert space to the inner product and the norm signs: $\langle .,. \rangle_{\mathscr{H}}$, $\|.\|_{\mathscr{H}}$. By inner product of vector functions we mean the sum of inner products of partial components.

$\mathbb{C}$ ($\mathbb{R}$) denotes the complex (real) numbers, the inner product on $\mathbb{C}^n$ ($\mathbb{R}^n$) we define as usual: $\langle a, b \rangle := \sum_{i=1}^{n} \bar{a}_i b_i$.

We denote a quadratic form associated to a sesquilinear form $u(.,.)$ by the same letter $u[.]$.

Let $A \subset \mathbb{R}^n$, then $D(A)$ stands for the vector space of smooth functions with a bounded support lying in A. Next we denote the space of all continuous linear functionals on $D(A)$ by $D'(A)$. The elements of $D(A)$ we call test functions, the elements of $D'(A)$ we call generalized functions. $(f, \varphi) := f(\varphi)$ for arbitrary $f \in D'(A)$ and $\varphi \in D(A)$. If a function f is locally integrable on J (regular generalized function),we set

$$(f, \varphi) := \int_A \bar{f}(x) \varphi(x) \mathrm{d}^n x$$

For an arbitrary $c \in \mathbb{C}$ we consider $\Im(c) \geq 0$.

For the convenience of a reader the list of symbols is submitted.

Symbol	**Meaning**
$\bar{\ }$	complex conjugation or closure
*	set of complex conjugated elements
$'$	derivative with respect to the argument
$\dagger$	hermitian conjugation
$\upharpoonright$	restriction
$:=$	defining equality
$\langle .,. \rangle_{\mathscr{H}}$	inner product on $\mathscr{H}$
$\lVert . \rVert_{\mathscr{H}}$	norm on $\mathscr{H}$
$[.,.]_x$	Lagrange bracket (see (A.3))
$AC^n(J)$	space of functions that their derivatives up to n^{th} order are absolutely continuous on a set J, $AC \equiv AC^0$
arc	mapping $\mathbb{R}^2 \to \langle 0, 2\pi),\ x = r(\cos\varphi, \sin\varphi) \mapsto \varphi$
$\mathscr{B}(\mathscr{H})$	space of bounded operators on $\mathscr{H}$
$\mathbb{C}$	the complex numbers
C^∞	space of all functions having all the derivatives continuous
δ^n	Dirac delta function on $\mathbb{R}^n$
$D(A)$	space of all test functions with a support in A
$D'(A)$	space of all generalized functions (continuous functionals on $D(A)$)
$Dom(.)$	domain of definition
$H_n^{\substack{(1)\\(2)}}$	Hankel functions (see appendix A.2)
H^m	m^{th} Sobolev space (see appendix A.5)
$\mathscr{F}$	Fourier transform or Fourier-Plancherel operator, for $f \in L^1(\mathbb{R}^n, \mathrm{d}^n x)$ we set $(\mathscr{F}f)(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{i\langle x,\xi \rangle} f(x) \mathrm{d}^n x, \quad \xi \in \mathbb{R}^n$
$f(a_\pm)$	limit of the function f in the point a from the right (from the left respectively)
$\Im$	imaginary part
Id_A	identity operator on A
$Ker(.)$	kernel of an operator
$L^{1(2)}(M, \mathrm{d}\mu)$	space of (quadratically) integrable functions on the set M with the measure μ
$L^{1(2)}_{loc}(M, \mathrm{d}\mu)$	space of locally (quadratically) integrable functions on the set M with the measure μ
$M_{a,b}$	Whittaker functions (see appendix A.3.2)
$M(a,b)$	Kummer functions (see appendix A.3.1)
max	maximum
min	minimum
$\mathbb{N}$	the natural numbers
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
Ψ	digamma function
$\varrho(.)$	resolvent set of an operator: $\varrho(.) = \mathbb{C} \setminus \sigma(.)$
$r_{>,<}$	the smaller and the greater of r, r' respectively

Symbol	**Meaning**
$\Re$	real part
$\mathbb{R}$	the real numbers
$\mathbb{R}^{\pm}$	positive (negative) half-axis: $\mathbb{R}^{+} := (0, \infty),\ \mathbb{R}^{-} := (-\infty, 0)$
$R_H(z)$	resolvent of H in z
$Ran(.)$	rank
$\sigma_{ac,c,ess,p,pp,r}$	absolutely continuous, continuous, essencial, point, pure point and residual spectrum
S^1	unit circle
$\mathscr{S}(\mathbb{R}^n)$	Schwartz space
$\mathscr{S}'(\mathbb{R}^n)$	space of all tempered distributions (continuous functionals on $\mathscr{S}(\mathbb{R}^n)$)
$span\,\{.\}$	vector space span
sup	supremum
supp	support
$Tr(.)$	trace of an operator: $Tr(A) := \sum_n \langle\varphi_n, A\varphi_n\rangle$, where $\{\varphi_n\}$ is an orthonormal base of a seperable Hilbert space
$W_{a,b}$	Whittaker functions (see appendix A.3.2)
$W(a, b)$	Kummer functions (see appendix A.3.1)
$\mathbb{Z}$	the integers

Introduction

Schrödinger operators with potentials supported by sets of zero Lebesgue measure are of great interest from both physical and mathematical viewpoint.

Such operators provide a reasonable description in the physical situations when the real interaction is of a very small range comparing to the other characteristic lengths. For example studying the interaction of a quantum particle with a polymer, the Schrödinger operator with a potential supported by a curve may be employed. For some references see [1], [2] or [3].

It should be mentioned that the operator $-\Delta$ in $L^2(\mathbb{R}^n)$, $n = 1, 2, 3$, with point interactions, i.e. interactions supported by points, represents one of rare solvable models [1].

The main goal of the thesis is to review two different rigorous methods introducing those Schrödinger operators and then to apply them to some concrete two-dimensional models.

Namely, in the first chapter we will study the hamiltonian formally given by $-\Delta_x + \alpha\delta(x)$ in detail. We will use a selfadjoint extension method and then a quadratic form technique to define this hamiltonian properly. At last all scattering quantities are calculated.

In the second chapter we will introduce a quadratic form, which gives a mathematical meaning to the formal hamiltonian $-\Delta_x + \alpha(y(s))\delta(x - y(s))$, where $\{y(s)|\ s \in I\} =: \Gamma$ is a closed curve and α is a real continuous non-zero function. We will make use of the techniques developed in [4] and [3]. For Γ being a circle and $\alpha = const.$, the partial wave decomposition of the scattering amplitude is given.

The third chapter is devoted to the isotropic harmonic oscillator hamiltonian with the one-center point interaction. This hamiltonian can be used to approximate so-called quantum dot with a short-range impurity [5]. We will analyze the hamiltonian spectrum and we will show that, breaking the rotational symmetry, its structure changes.

The one-center point interaction in two dimensions

At first we introduce the one-center point interaction by the method of selfadjoint extensions [1], then we show, that finding a selfadjoint operator associated to a appropriately chosen quadratic form we come to the same result [4]. Both of the methods can be generalized to the case of finitely many centers. In the case of the one-center point interaction we solve the scattering problem.

1 Method of selfadjoint extensions

Let's start with a densely defined operator H on $\mathscr{H} = L^2(\mathbb{R}^2, \mathrm{d}^2x)$

$$H := -\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}, \quad Dom(H) := D(\mathbb{R}^2 \setminus \{0\}). \tag{1}$$

At first we transform H to the polar coordinates and we use the partial wave decomposition. Since the following spaces are isometric [6]

$$\begin{aligned} \mathscr{H} = L^2(\mathbb{R}^2, \mathrm{d}^2x) &\overset{U}{\equiv} L^2(\mathbb{R}^+ \times (0, 2\pi), r\mathrm{d}r\mathrm{d}\varphi) \equiv L^2(\mathbb{R}^+, r\mathrm{d}r) \otimes L^2((0, 2\pi), \mathrm{d}\varphi) \\ &\equiv \bigoplus_{n=-\infty}^{\infty} L^2(\mathbb{R}^+, r\mathrm{d}r) \otimes Y_n =: \mathscr{H}', \quad \text{where } Y_n(\varphi) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{in\varphi}, \end{aligned} \tag{2}$$

to each $g \in \mathscr{H}$ we can assign exactly one $f \in \mathscr{H}'$ such that

$$\begin{aligned} (U(g))(r, \varphi) := g(r\cos\varphi, r\sin\varphi) = f(r, \varphi) = \sum_{n=-\infty}^{\infty} f_n(r) Y_n(\varphi), \\ \text{where } f_n(r) = \langle Y_n, f \rangle_{L^2((0,2\pi), \mathrm{d}\varphi)}, \end{aligned}$$

whereas $\|g\|_{\mathscr{H}} = \|f\|_{\mathscr{H}'}$.

The following operator decomposition corresponds to isometry (2)

$$H = U^{-1} H_{polar} U = U^{-1} \left(\sum_{n=-\infty}^{\infty} H_n \otimes \mathrm{Id}_{span\{Y_n\}} \right) U, \quad \text{where}$$

$$H_{polar} = -\frac{\partial}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}, \quad Dom(H_{polar}) = D(\mathbb{R}^+ \times (0, 2\pi))$$

$$H_n = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{n^2}{r^2}, \quad Dom(H_n) = D(\mathbb{R}^+)$$

The differential expression for the action of H_n can be simplified by another isometry $V:\ L^2(\mathbb{R}^+, r\mathrm{d}r) \to L^2(\mathbb{R}^+, \mathrm{d}r),\ f(r) \overset{V}{\mapsto} \sqrt{r} f(r)$:

$$h_n := V H_n V^{-1} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{n^2 - \frac{1}{4}}{r^2}, \quad Dom(h_n) = D(\mathbb{R}^+)$$

Now the problem of finding all selfadjoit extensions of H can be formulated as follows.

Proposition 1 *All rotationally symmetric[1] selfadjoint extensions of the operator H can be constructed using the family of all selfadjoint extensions of the operators $h_n,\ n \in \mathbb{Z}$, in the following way. Let $\tilde{h}_n$ be an arbitrary fixed selfadjoint extension of h_n for all $n \in \mathbb{Z}$, then*

$$\tilde{H} := U^{-1} \left(\bigoplus_{n=-\infty}^{\infty} V^{-1} \tilde{h}_n V \otimes \mathrm{Id}_{span\{Y_n\}} \right) U$$

is a selfadjoint extension of H.

Proof Unitary equivalent operators are simultaneously selfadjoint. So the proposition is a consequence of the treatment above.

■

Proposition 2 *The operator $h_n^\dagger$ hermitian conjugated to the operator h_n is of the form*

$$h_n^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{n^2 - \frac{1}{4}}{r^2}$$

$$Dom(h_n^\dagger) = \left\{ f \in L^2(\mathbb{R}^+, \mathrm{d}r) |\ f \in AC^1(\mathbb{R}^+),\ \left(-f'' + \frac{n^2 - \frac{1}{4}}{r^2} f \right) \in L^2(\mathbb{R}^+, \mathrm{d}r) \right\}.$$

Thus h_n is symmetric for all $n \in \mathbb{Z}$.

Proof The operator h_n is densely defined. The definition of a selfadjoint operator says that

$$f \in Dom(h_n^\dagger) \Leftrightarrow \exists g \in L^2(\mathbb{R}^+, \mathrm{d}r) |\ \forall \varphi \in Dom(h_n) \equiv D(\mathbb{R}^+)\ \langle f, h_n \varphi \rangle = \langle g, \varphi \rangle, \qquad (3)$$

[1] i.e. the extensions commuting with the projection operator $\mathrm{Id}_{L^2(\mathbb{R}^+, r\mathrm{d}r)} \otimes P_n$, where $P_n := \langle Y_n, . \rangle Y_n,\ n \in \mathbb{Z}$

which is equivalent to the condition

$$f \in Dom(h_n^\dagger) \Leftrightarrow \exists g \in L^2(\mathbb{R}^+, \mathrm{d}r)|\ \forall \varphi \in D(\mathbb{R}^+)\ (f, h_n\varphi) = \left(-f'' + \frac{n^2 - \frac{1}{4}}{r^2} f, \varphi\right) = (g, \varphi)$$

$$\Leftrightarrow \exists g \in L^2(\mathbb{R}^+, \mathrm{d}r)|\ -f'' + \frac{n^2 - \frac{1}{4}}{r^2} f = g \text{ v } D'(\mathbb{R}^+),$$

since $f, g \in L^2(\mathbb{R}^+, \mathrm{d}r) \subset L^1_{loc}(\mathbb{R}^+, \mathrm{d}r)$. So if $f \in Dom(h_n^\dagger)$, then by corollary A.4, f must be in $AC^1(\mathbb{R}^+)$. All in all we have the following implication

$$f \in Dom(h_n^\dagger) \Rightarrow$$
$$f \in M_n := \left\{ f \in L^2(\mathbb{R}^+, \mathrm{d}r)|\ f \in AC^1(\mathbb{R}^+),\ \left(-f'' + \frac{n^2 - \frac{1}{4}}{r^2} f\right) \in L^2(\mathbb{R}^+, \mathrm{d}r) \right\}.$$

Using the Lagrange formula (A.4) for an arbitrary fixed $f \in M_n$ and each $\varphi \in D(\mathbb{R}^+)$ we obtain [2]

$$\langle f, h_n\varphi\rangle - \langle -f'' + \frac{n^2 - \frac{1}{4}}{r^2} f, \varphi\rangle = [f, \varphi]_\infty - [f, \varphi]_0 = 0 - 0 = 0.$$

Thus we have found $g := -f'' + \frac{n^2 - \frac{1}{4}}{r^2} f \in L^2(\mathbb{R}^+, \mathrm{d}r)$ such that the rhs of (3) is fulfilled. $Dom(h_n^\dagger) = M_n$, $h_n \subset h_n^\dagger$ for $n \in \mathbb{Z}$.

■

Proposition 3 *For $n \in \mathbb{Z} \setminus \{0\}$, h_n is essentially selfadjoint $\bar{h}_n = h_n^\dagger$. h_0 has deficiency indices (1,1), its closure is of the form*

$$\bar{h}_0 = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{4r^2}$$
$$Dom\left(\bar{h}_0\right) = \Big\{ f \in L^2(\mathbb{R}^+, \mathrm{d}r)|\ f \in AC^1(\mathbb{R}^+),\ \left(-f'' - \frac{1}{4r^2} f\right) \in L^2(\mathbb{R}^+, \mathrm{d}r),$$
$$[g, f]_0 = [\bar{g}, f]_0 = 0 \Big\},$$

where $g(r) := \sqrt{r} H_0^{(1)}(\sqrt{i}r)$.

Proof At first we will find the defect subspace $Ker(h_n^\dagger - i)$, i.e. we will solve the following differential equation on $Dom(h_n^\dagger)$

$$-g'' + \frac{n^2 - \frac{1}{4}}{r^2} g = ig.$$

A pair of linearly independent solutions is $\sqrt{r} H_n^{(1)}(\sqrt{i}r)$ and $\sqrt{r} H_n^{(2)}(\sqrt{i}r)$ (see appendix A.2). Moreover a solution g has to be in $Dom(h_n^\dagger)$, then in accordance with corollary

[2] alternatively we integrate twice by parts

A.9 $\lim_{r\to\infty} g(r) = 0$ necessarily. Due to (A.12) and (A.13) we consider the solution $g = \sqrt{r}H_n^{(1)}(\sqrt{i}r)$, however it is quadratically integrable only for $|n| < 1$ [7], which means

$$Ker(h_n^\dagger - i) = \begin{cases} \{0\} & \text{for } n \in \mathbb{Z}\setminus\{0\} \\ span\left\{\sqrt{r}H_0^{(1)}(\sqrt{i}r)\right\} & \text{for } n = 0 \end{cases}$$

The operator $h_n^\dagger$ is real and so $Ker(h_n^\dagger + i) = \left(Ker(h_n^\dagger - i)\right)^*$.

The deficiency indices of h_0 are really (1,1). For $n \in \mathbb{Z}\setminus\{0\}$ we have $Ker(h_n^\dagger \pm i) = \{0\}$, hence h_n is essentially selfadjoint $\bar{h}_n = h_n^\dagger$ by the selfadjointness criterion [6].

At last we find the closure of h_0. Substituting the Lagrange formula (A.4) to lemma A.10 we have

$$Dom(\bar{h}_0) = \left\{ f \in Dom(h_0^\dagger) |\ [g,f]_\infty - [g,f]_0 = [\bar{g},f]_\infty - [\bar{g},f]_0 = 0 \right\}.$$

Moreover with regard to corollary A.9, $[h,f]_\infty = 0$ for all $f, h \in Dom(h_0^\dagger)$ (and so for g and $\bar{g}$ too).

■

Proposition 4 *Every selfadjoint extension of the operator $\bar{h}_0$ can be described by the parameter $\alpha \in (-\infty,\infty\rangle$:*

$$h_{0,\alpha} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{4r^2}$$
$$Dom(h_{0,\alpha}) = \left\{ f \in L^2(\mathbb{R}^+, \mathrm{d}r) \cap AC^1(\mathbb{R}^+) |\ \left(-f'' - \frac{1}{r^2}f\right) \in L^2(\mathbb{R}^+,\mathrm{d}r),\ 2\pi\alpha f_0 + f_1 = 0 \right\},$$

where

$$f_0 := \lim_{r\to 0+} (\sqrt{r}\ln(r))^{-1} f(r), \quad f_1 := \lim_{r\to 0+} \frac{1}{\sqrt{r}}\left(f(r) - f_0\sqrt{r}\ln r\right).$$

Proof Regarding the second von Neumann formula [6], all selfadjoint extensions of $\bar{h}_0$ are of the form

$$h_{0,\beta} f = \bar{h}_0 \tilde{f} + i\xi\left(g + \mathrm{e}^{i\beta}\bar{g}\right), \quad \beta \in \langle 0, 2\pi)$$
$$Dom(h_{0,\beta}) = \left\{ f \in L^2(\mathbb{R}^+,\mathrm{d}r) |\ f = \tilde{f} + \xi\left(g - \mathrm{e}^{i\beta}\bar{g}\right),\ \tilde{f} \in Dom(\bar{h}_0),\ \xi \in \mathbb{C} \right\},$$

where $g(r) := \sqrt{r}H_0^{(1)}(\sqrt{i}r)$.

At first we will show that the limits $\tilde{f}_0, \tilde{f}_1$ are zero for all $\tilde{f} \in Dom(\bar{h}_0)$. The conditions $[g,\tilde{f}] = [\bar{g},\tilde{f}] = 0$ imply

$$\Re g'(0+)\tilde{f}(0+) - \Re g(0+)\tilde{f}'(0+) = 0$$
$$\Im g'(0+)\tilde{f}(0+) - \Im g(0+)\tilde{f}'(0+) = 0.$$

For $g(0+)$ and $g'(0+)$ we substitute their asymptotic expansions (see note A.11) and after some elementary rearrangement we have

$$\tilde{f}(0+)\left(\frac{1}{\sqrt{r}}+O(\sqrt{r})\right)\overset{r\to 0+}{\sim} 0$$
$$\tilde{f}'(0+)\left(\sqrt{r}+O(r^{\frac{5}{2}}\ln r)\right)\overset{r\to 0+}{\sim} 0,$$

which means

$$\lim_{r\to 0+}\frac{\tilde{f}(r)}{\sqrt{r}}=0,\quad \lim_{r\to 0+}\tilde{f}'(r)\sqrt{r}=0.$$

Hence $\tilde{f}_0=\tilde{f}_1=0$.

The remaining part of $f\in Dom(h_{0,\beta})$ may be expanded:

$$\xi\left(g(r)-\mathrm{e}^{i\beta}\bar{g}(r)\right)\overset{r\to 0+}{\sim}\xi\frac{2i}{\pi}\left(1+\mathrm{e}^{i\beta}\right)\sqrt{r}\ln r+\xi\left(A-\mathrm{e}^{i\beta}\bar{A}\right)\sqrt{r}=f_0\sqrt{r}\ln r+f_1\sqrt{r},$$

where $A:=\frac{1}{2}-\frac{2i}{\pi}(\ln 2+\Psi(1))$. For the ratio $\frac{f_1}{f_0}$ we obtain a relation

$$\frac{f_1}{f_0}=\frac{\pi}{2i}\left(i\Im A+\Re A\frac{1-\mathrm{e}^{i\beta}}{1+\mathrm{e}^{i\beta}}\right)=-\ln 2-\Psi(1)-\frac{\pi}{4}\tan\frac{\beta}{2},$$

which introducing a notion

$$\alpha:=\frac{1}{2\pi}\left(\ln 2+\Psi(1)+\frac{\pi}{4}\tan\frac{\beta}{2}\right)\tag{4}$$

takes the form $2\pi\alpha f_0+f_1=0,\ \alpha\in(-\infty,\infty\rangle$. We have verified the inclusion $h_{0,\beta}\subset h_{0,\alpha}$, where α is related to β through formula (4).

Now let $f\in Dom(h_{0,\alpha})\subset Dom(h_0^\dagger)$. Decomposing f by the first von Neumann formula (A.8), the only thing left is to show that there is exactly the same linear combination of g and $\bar{g}$ in $Dom(h_{0,\alpha})$ that lies in $Dom(h_{0,\beta})$.

Consider a general element (up to a multiple) of the subspace $Ker(h_0^\dagger-i)\dotplus Ker(h_0^\dagger-i)$: $g+\gamma\bar{g},\ \gamma\in\mathbb{C}$. Its asymptotic expansion gives

$$(g+\gamma\bar{g})_0=\frac{2i}{\pi}(1-\gamma),\quad (g+\gamma\bar{g})_1=\frac{1}{2}(1+\gamma)-\frac{2i}{\pi}(\ln 2+\Psi(1))(1-\gamma),$$

whereas we require that

$$-2\pi\alpha=\frac{(g+\gamma\bar{g})_1}{(g+\gamma\bar{g})_0}=-\ln 2-\Psi(1)+\frac{\pi}{4i}\frac{1+\gamma}{1-\gamma},$$

which regarding (4) means

$$i\frac{1+\gamma}{1-\gamma}=\tan\frac{\beta}{2}=i\frac{1-\mathrm{e}^{i\beta}}{1+\mathrm{e}^{i\beta}},$$

and so $\gamma=-\mathrm{e}^{i\beta}$.

■

Note 5 *Let us denote the operator introduced in proposition 1 by a lower index α in accordance with the partial hamiltonian $h_{0,\alpha}$, which is the only partial hamiltonian that differs in dependence on a selfadjoint extension of (1).*

For $\alpha = \infty$ we obtain the free particle hamiltonian with the mass $m = \frac{1}{2}$

$$H_\infty = -\Delta, \quad Dom(H_\infty) = H^2(\mathbb{R}^2). \tag{5}$$

Proposition 6 *Let $z \in \mathbb{C} \setminus \mathbb{R}$. The Green function of the operator H_α is of the form*

$$\begin{aligned}
&\mathcal{G}_z^\alpha(x,y) = \mathcal{G}_z(x-y) + \lambda_\alpha(z)\mathcal{G}_z(x)\mathcal{G}_z(y), \text{ where}\\
&\mathcal{G}_z(x-y) = \frac{i}{4}H_0^{(1)}(\sqrt{z}|x-y|)\\
&\lambda_\alpha(z) = 2\pi\left(2\pi\alpha - \Psi(1) + \ln\frac{\sqrt{z}}{2i}\right)^{-1}
\end{aligned} \tag{6}$$

Proof Selfadjoint extensions of H differ only on the subspace $\mathscr{H}_0 := L^2(\mathbb{R}^+, r\mathrm{d}r) \otimes span\{Y_0\}$ of decomposition (2), so searching the subspaces $Ker(H^\dagger - z)$ and $Ker(H^\dagger - \bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$, we can restrict ourselves to it.

The operator $H^\dagger$ acts on the non-trivial part of $\mathscr{H}_0$ as follows

$$\begin{aligned}
&H_0^\dagger = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\\
&Dom(H_0^\dagger) = \left\{ f \in L^2(\mathbb{R}^+, r\mathrm{d}r) |\ f \in AC^1(\mathbb{R}^+),\ \left(-\frac{\partial f^2}{\partial r^2} - \frac{1}{r}\frac{\partial f}{\partial r}\right) \in L^2(\mathbb{R}^+, r\mathrm{d}r) \right\},
\end{aligned}$$

hence

$$Ker(H^\dagger - z) = span\left\{H_0^{(1)}(\sqrt{z}r)\right\}, \quad Ker(H^\dagger - \bar{z}) = span\left\{\overline{H_0^{(1)}(\sqrt{z}r)}\right\}.$$

Using the Krein formula (see theorem A.12), for the resolvent of H_α we conclude

$$(H_\alpha - z)^{-1} = (H_\infty - z)^{-1} + \lambda_\alpha(z)\langle\overline{\mathcal{G}_z}, .\rangle\mathcal{G}_z, \quad \lambda_\alpha(z) \in \mathbb{C} \tag{7}$$

and for the corresponding Green function we have

$$\mathcal{G}_z^\alpha(x,y) = \mathcal{G}_z(x-y) + \lambda_\alpha(z)\mathcal{G}_z(x)\mathcal{G}_z(y),$$

because the Green function of H_∞ is just $\mathcal{G}_z$ (see appendix A.7).

The Green function $\mathcal{G}_z^\alpha(x,y)$ obeys the boundary condition of the selfadjoint extension H_α for an arbitrary fixed $y \neq 0$.

$$\begin{aligned}
&2\pi\alpha f_0 + f_1 = 0, \text{ where}\\
&f_0 = \lim_{r\to 0+}(\ln r)^{-1}f(r), \quad f_1 = \lim_{r\to 0+}(f(r) - f_0\ln r).
\end{aligned} \tag{8}$$

Regarding expansion (A.15) the following relation holds

$$\mathcal{G}_z^\alpha(x,y) \overset{x\to 0}{\sim} \mathcal{G}_z(y) + \frac{i}{4}\lambda_\alpha(z)\left(1 + \frac{2i}{\pi}\left(\ln\frac{\sqrt{z}|x|}{2} - \Psi(1)\right)\right)\mathcal{G}_z(y),$$

which according to condition (8) gives

$$\lambda_\alpha(z) = 2\pi\left(2\pi\alpha - \Psi(1) + \ln\frac{\sqrt{z}}{2i}\right)^{-1}.$$

■

Proposition 7 *Every $f \in Dom(H_\alpha)$ can be uniquely decomposed as follows*

$$f(x) = \tilde{f}_z(x) + \lambda_\alpha(z)\tilde{f}_z(0)\mathcal{G}_z(x), \quad x \neq 0, \tag{9}$$

where $z \in \mathbb{C}\setminus\mathbb{R}$ and $\tilde{f}_z \in Dom(H_\infty) \equiv H^2(\mathbb{R}^2)$.

$$(H_\alpha - z)f = (H_\infty - z)\tilde{f}_z. \tag{10}$$

Proof Let as consider an arbitrary fixed $z \in \mathbb{C}\setminus\mathbb{R}$. Using relation (7) we obtain

$$\begin{aligned} Dom(H_\alpha) &= (H_\alpha - z)^{-1}\mathscr{H} = (H_\alpha - z)^{-1}(H_\infty - z)Dom(H_\infty) \\ &= \left[(H_\infty - z)^{-1} + \lambda_\alpha(z)\langle\overline{\mathcal{G}_z}, .\rangle\mathcal{G}_z\right](H_\infty - z)Dom(H_\infty). \end{aligned} \tag{11}$$

For an arbitrary $\tilde{f}_z \in Dom(H_\infty)$ we have

$$\langle\overline{\mathcal{G}_z}, (H_\infty - z)\tilde{f}_z\rangle = \{\text{int. by parts}\} = \int\limits_{\mathbb{R}^2}\tilde{f}_z(x)(-\Delta - z)\mathcal{G}_z(x)\mathrm{d}x^2 = \int\limits_{\mathbb{R}^2}\tilde{f}_z(x)\delta^2(x)\mathrm{d}x^2 = \tilde{f}_z(0),$$

which together with (11) give decomposition (9). Moreover

$$f = (H_\alpha - z)^{-1}(H_\infty - z)\tilde{f}_z \in Dom(H_\alpha),$$

and so relation (10) holds too.

At last we verify the uniqueness of the decomposition. Let $f_1, f_2 \in Dom(H_\infty)$ such that

$$f(x) = f_{1,2}(x) + \lambda_\alpha(z)f_{1,2}(0)\mathcal{G}_z(x), \quad x \neq 0,$$

thus

$$(f_1 - f_2)(x) = \lambda_\alpha(z)(f_2 - f_1)(0)\mathcal{G}_z(x), \quad x \neq 0.$$

Since the function $(f_1 - f_2)$ is continuous on $\mathbb{R}^2$, whereas the function $\mathcal{G}_z$ is discontinuous in the point $x = 0$, $f_1(0) = f_2(0)$ and consequently $f_1 = f_2$.

■

Proposition 8 *The spectrum of H_α is of the form*

$$\begin{aligned} &\sigma_{ess}(H_\alpha) = \sigma_{ac}(H_\alpha) = \langle 0, \infty) && \textit{for } \alpha \in (-\infty, \infty\rangle \\ &\sigma_p(H_\alpha) = \left\{-4\mathrm{e}^{2[-2\pi\alpha + \Psi(1)]}\right\} && \textit{for } \alpha \in \mathbb{R} \\ &\sigma_p(H_\infty) = \emptyset. \end{aligned}$$

Proof The energy levels of the point spectrum corresponds to the singularities of the resolvent. The equation

$$\frac{1}{\lambda_\alpha(z)} = 0$$

has exactly one solution in $\mathbb{R}:\ z = -4\mathrm{e}^{2[-2\pi\alpha+\Psi(1)]} < 0$.

It is well known that $\sigma(H_\infty) = \sigma_{ess}(H_\infty) = \sigma_{ac}(H_\infty) = \langle 0, \infty)$. According to relation (7), the difference of the resolvents $R_{H_\infty}(z)$ and $R_{H_\alpha}(z)$ is an onedimensional (and so necessarily trace class[3]) operator. Absolutely continuous and essential parts of a spectrum are invariant under a trace class perturbation [8][4], i.e. $\sigma_{ac,ess}(R_{H_\alpha}(z)) = \sigma_{ac,ess}(R_{H_\infty}(z))$.

Since the resolvent of an arbitrary selfadjoint operator H can be written in the following form [6],

$$R_H(z) = \int\limits_{\sigma(H)} \frac{1}{t-z}\mathrm{d}E_H(t) \quad \text{for } z \in \varrho(H)$$

where $E_H(.)$ is the projection-valued measure of the operator H, and thus

$$\sigma_{\substack{ac \\ ess}}(H) = \left\{ \lambda \in \mathbb{R} |\ \frac{1}{\lambda - z} \in \sigma_{\substack{ac \\ ess}}(R_H(z)) \right\},$$

$\sigma_{ess}(H_\alpha) = \sigma_{ac}(H_\alpha) = \langle 0, \infty)$.

The spectrum of an arbitrary closed operator can be decomposed to the not necessarily disjoint parts

$$\sigma = \sigma_{ess} \cup \sigma_p \cup \sigma_r$$

Moreover for selfadjoint (and so necessarily normal) operators $\sigma_r = \emptyset$. Thus we have found the whole spectrum of the operator H_α.

■

Note 9 *The point interaction in an arbitrary point $y \in \mathbb{R}^2$ can be introduced in the same way as in the origin, but at first we have to translate the reference frame $y \mapsto 0$.*

2 Quadratic form for the one-centered point interaction

Motivated by the results of section 1, we introduce a sesquilinear form F_α on $L^2(\mathbb{R}^2, \mathrm{d}^2x)$ as follows:

$$\begin{aligned} &Dom(F_\alpha) := \left\{ u \in L^2(\mathbb{R}^2, \mathrm{d}^2x) |\ \exists Q_u \in \mathbb{C}:\ (u - Q_u\mathcal{G}_z) \in H^1(\mathbb{R}^2) \right\} \\ &F_\alpha(u,v) := \mathcal{F}^z(u,v) + \Phi^z_\alpha(Q_u, Q_v) \end{aligned} \tag{12}$$

$$\mathcal{F}^z(u,v) := \langle \nabla(u - Q_u\mathcal{G}_z), \nabla(v - Q_v\mathcal{G}_z) \rangle - z\langle u - Q_u\mathcal{G}_z, v - Q_v\mathcal{G}_z \rangle + z\langle u, v\rangle$$

$$\Phi^z_\alpha(Q_u, Q_v) := \frac{1}{2\pi}\left(2\pi\alpha - \Psi(1) + \ln\frac{\sqrt{z}}{2i} \right) \bar{Q}_u Q_v =: \Gamma_\alpha(z)\bar{Q}_u Q_v,$$

[3]An operator A is called trace class iff $Tr(A) < \infty$.

[4]Essential spectrum is invariant even under a compact perturbation.

where $z<0$, $\alpha\in\mathbb{R}$ and $\mathcal{G}_z$ denotes the Green function of the operator $H_\infty=-\Delta,\ Dom(H_\infty)=H^2(\mathbb{R}^2)$ (the free hamiltonian Green function-see appendix A.7):

$$\mathcal{G}_z(x-y)=\frac{i}{4}H_0^{(1)}(\sqrt{z}|x-y|)\in\mathbb{R}\quad\text{for } z<0. \tag{13}$$

Note 10 *The form F_α is independent of the parameter $z<0$.*

Proof Since for arbitrary $z,z'<0$: $\mathcal{G}_z-\mathcal{G}_{z'}\in H^1(\mathbb{R}^2)$, $Dom(F_\alpha)$ is z independent. Moreover for an arbitrary $u\in Dom(F_\alpha)$ we have

$$\begin{aligned}
\mathcal{F}^z[u]-\mathcal{F}^{z'}[u]&=\int_{\mathbb{R}^2}\left\{\nabla(\mathcal{G}_z-\mathcal{G}_{z'})\nabla\left[|Q_u|^2\mathcal{G}_z+|Q_u|^2\mathcal{G}_{z'}-(Q_u\bar{u}+\bar{Q}_uu)\right]\right\}\\
&+\int_{\mathbb{R}^2}\left\{z(Q_u\bar{u}+\bar{Q}_uu)\mathcal{G}_z-z|Q_u|^2\mathcal{G}_z^2-z'(Q_u\bar{u}+\bar{Q}_uu)\mathcal{G}_{z'}+z'|Q_u|^2\mathcal{G}_{z'}^2\right\}\\
&=\{\text{int. by parts}\}=-\int_{\mathbb{R}^2}\left\{\Delta(\mathcal{G}_z-\mathcal{G}_{z'})\left[|Q_u|^2\mathcal{G}_z+|Q_u|^2\mathcal{G}_{z'}-(Q_u\bar{u}+\bar{Q}_uu)\right]\right\}\\
&+\int_{\mathbb{R}^2}\left\{z(Q_u\bar{u}+\bar{Q}_uu)\mathcal{G}_z-z|Q_u|^2\mathcal{G}_z^2-z'(Q_u\bar{u}+\bar{Q}_uu)\mathcal{G}_{z'}+z'|Q_u|^2\mathcal{G}_{z'}^2\right\}\\
&=\{-\Delta\mathcal{G}_z=z\mathcal{G}_z+\delta\}=(z-z')|Q_u|^2\int_{\mathbb{R}^2}\mathcal{G}_z\mathcal{G}_{z'}\\
&=\left\{\text{the resolvent formula: } R(z)-R(z')=(z-z')R(z)R(z')\right\}\\
&=|Q_u|^2\lim_{x\to 0}(\mathcal{G}_z-\mathcal{G}_{z'})(x)=\{(A.14)\}=-\frac{1}{2\pi}(\ln\sqrt{z}-\ln\sqrt{z'})|Q_u|^2=\\
&=[\Gamma_\alpha(z')-\Gamma_\alpha(z)]|Q_u|^2=\Phi_\alpha^{z'}[Q_u]-\Phi_\alpha^z[Q_u].
\end{aligned}$$

■

Proposition 11 *The quadratic form F_α is symmetric, closed and bounded from below.*

Proof For an arbitrary fixed $\alpha\in\mathbb{R}$ we can choose $z(\alpha)<0$ such that $\Gamma_\alpha(z(\alpha))>0$. Estimating

$$F_\alpha[u]=\|\nabla(u-Q_u\mathcal{G}_{z(\alpha)})\|^2-z(\alpha)\|u-Q_u\mathcal{G}_{z(\alpha)}\|^2+z(\alpha)\|u\|^2+\Gamma_\alpha(z(\alpha))|Q_u|^2\geq z(\alpha)\|u\|^2 \tag{14}$$

we verify, that F_α is bounded from below.

F_α is closed iff the form

$$F_\alpha^{z(\alpha)}:=F_\alpha-z(\alpha)\|u\|^2,\ Dom(F_\alpha^{z(\alpha)}):=Dom(F_\alpha)$$

is closed. Let $\{u_n\}\subset Dom(F_\alpha^{z(\alpha)})$ be a sequence such that

$$\lim_{m,n\to\infty}F_\alpha^{z(\alpha)}[u_n-u_m]=0 \tag{15}$$

Set $w_n := u_n - Q_{u_n}\mathcal{G}_{z(\alpha)}$. Assumption (15) implies

$$\begin{aligned}\lim_{m,n\to\infty}|Q_{u_n-u_m}| &= \lim_{m,n\to\infty}|Q_{u_n}-Q_{u_m}| = 0\\ \lim_{m,n\to\infty}\|\nabla(w_n-w_m)\| &= 0, \lim_{m,n\to\infty}\|w_n-w_m\| = 0.\end{aligned} \tag{16}$$

So $\lim_{m,n\to\infty}\|w_n - w_m\|_{H^1(\mathbb{R}^2)} = 0$ by (16). $\mathbb{C}$ and $H^1(\mathbb{R}^2)$ are complete (see note A.16), hence $Q \in \mathbb{C}$ and $w \in H^1(\mathbb{R}^2)$ exist such that

$$\begin{aligned}&\lim_{n\to\infty}|Q-Q_{u_n}| = 0\\ &\lim_{n\to\infty}\|\nabla(w-w_n)\| = 0, \lim_{m,n\to\infty}\|w-w_n\| = 0.\end{aligned}$$

For $u := w + Q\mathcal{G}_{z(\alpha)}$ we have $u \in Dom(F_\alpha^{z(\alpha)})$ and $\lim_{n\to\infty} F_\alpha^{z(\alpha)}[u-u_n] = 0$, hence $F_\alpha^{z(\alpha)}$ is closed.

The symmetry of the form is a direct consequence of the inner product symmetry.

■

Proposition 12 *The quadratic form F_α associates a selfadjoint operator $-\Delta_\alpha$:*

$$\begin{aligned}&Dom(-\Delta_\alpha) = \left\{u \in L^2(\mathbb{R}^2) \mid \exists Q_u \in \mathbb{C}: \ (u - Q_u\mathcal{G}_z) \in H^2(\mathbb{R}^2),\ (u-Q_u\mathcal{G}_z)(0) = \Gamma_\alpha(z)Q_u\right\}\\ &(-\Delta_\alpha - z)u = (-\Delta - z)(u - Q_u\mathcal{G}_z), \quad z < 0.\end{aligned}$$

Proof As has been shown, the form F_α is symmetric, closed and bounded from below. So we can apply the representation theorem A.20, which states the existence of a selfadjoint operator A_α such that for an arbitrary fixed $v \in Dom(A_\alpha) \subset Dom(F_\alpha)$, $g \in L^2(\mathbb{R}^2, \mathrm{d}^2x)$ with the following property exists

$$F_\alpha(u,v) = \langle u, g\rangle \quad \text{for } \forall\ u \in Dom(F_\alpha) \tag{17}$$

We set $A_\alpha v = g$.

Now let $u \in H^1(\mathbb{R}^2)$, then $Q_u = 0$ and

$$F_\alpha(u,v) = \langle \nabla u, \nabla(v - Q_v\mathcal{G}_z)\rangle - z\langle u, v - Q_v\mathcal{G}_z\rangle + z\langle u, v\rangle = \langle u, g\rangle, \tag{18}$$

which means

$$\langle \nabla u, \nabla(v - Q_v\mathcal{G}_z)\rangle = \langle u, g - zQ_v\mathcal{G}_z\rangle \quad \forall u \in H^1(\mathbb{R}^2).$$

Lemma A.18 then implies $(v - Q_v\mathcal{G}_z) \in H^2(\mathbb{R}^2)$. Thus in (18) we may integrate by parts:

$$\langle u, (-\Delta - z)(v - Q_v\mathcal{G}_z)\rangle = \langle u, g - zv\rangle \quad \text{pro } \forall u \in H^1(\mathbb{R}^2),$$

which regarding that $H^1(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2, \mathrm{d}^2x)$, means

$$(-\Delta - z)(v - Q_v\mathcal{G}_z) = g - zv = (A_\alpha - z)v.$$

Next we consider an arbitrary $u \in D(F_\alpha)$, then $(u - Q_u\mathcal{G}_z) \in H^1(\mathbb{R}^2)$ and condition (18) takes form

$$\langle \nabla(u - Q_u\mathcal{G}_z), \nabla(v - Q_v\mathcal{G}_z)\rangle - z\langle u - Q_u\mathcal{G}_z, v - Q_v\mathcal{G}_z\rangle = \langle u - Q_u\mathcal{G}_z, g - zv\rangle,$$

which can be further rearranged with the help of (17):

$$\langle u, g\rangle - z\langle u, v\rangle - \Phi^z_\alpha(Q_u, Q_v) = \langle u - Q_u\mathcal{G}_z, g - zv\rangle,$$

and so

$$\Phi^z_\alpha(Q_u, Q_v) = \langle Q_u\mathcal{G}_z, g - zv\rangle = \bar{Q}_u \int\limits_{\mathbb{R}^2} \mathcal{G}_z(-\Delta - z)(v - Q_v\mathcal{G}_z) = \bar{Q}_u(v - Q_v\mathcal{G}_z)(0),$$

since $\mathcal{G}_z$ is the Green function of H_∞. Hence $(v - Q_v\mathcal{G}_z)(0) = \Gamma_\alpha(z)Q_v$.

All in all we have shown that $A_\alpha \subset -\Delta_\alpha$.

The reverse inclusion, i.e. for an arbitrary $v \in Dom(-\Delta_\alpha)$ an element $g \in L^2(\mathbb{R}^2)$ exists such that (17) holds and

$$(-\Delta - z)(v - Q_v\mathcal{G}_z) = (-\Delta_\alpha - z)v,$$

can be proved following the same line of reasoning, considering $u \in H^1(\mathbb{R}^2)$ at first.

■

Note 13 *We have come to the same family of operators as by applying the method of selfadjoint extensions (see proposition 7). The corresponding operators are related by the equation* $\Gamma_\alpha(z) = \frac{1}{\lambda_\alpha(z)}$.

3 Scattering

Proposition 14 *The generalized eigenfunctions of the operator H_α with the eigenvalue k^2, $k \in \mathbb{R}^2$ are of the form*

$$F^\pm_k(x) = \mathrm{e}^{il\langle\omega,x\rangle} + \frac{i\pi}{2}\left(2\pi\alpha - \Psi(1) + \ln\frac{\pm l}{2i}\right)^{-1} H^{(1)}_0(\pm lr)^5,$$

where k is decomposed as $k = l\omega,\ l = |k|$ and $r = |x|$.

Proof We are looking for solutions to the equation

$$-\Delta F^\pm_k = k^2 F^\pm_k,$$

for which boundary condition (8) together with asymptotic condition (A.34) are fulfilled.

[5]We consider the Hankel function $H^{(1)}_0$ of a negative argument in the sense of definition (A.29).

The asymptotic condition gives

$$F_k^{\pm}(x) = \mathrm{e}^{il\langle\omega,x\rangle} + L^{\pm}(l)H_0^{(1)}(\pm lr).$$

Using the boundary condition we find the constants $L^{\pm}$:

$$L^{\pm}(l) = \frac{i\pi}{2}\left(2\pi\alpha - \Psi(1) + \ln\frac{\pm l}{2i}\right)^{-1}.$$

■

Note 15 *The functions $F_k^{\pm}(x) \equiv F_k^{\pm}(r\hat{\Omega}) \equiv F^{\pm}(l,\omega;x)$ are in fact functions only of those variables: $l, r, |\Omega - \operatorname{arc}\omega|$.*

$$F^{\pm}(l,\omega;r,\Omega) = \mathrm{e}^{ilr\cos(\Omega-\operatorname{arc}\omega)} + \frac{i\pi}{2}\left(2\pi\alpha - \Psi(1) + \ln\frac{\pm l}{2i}\right)^{-1} H_0^{(1)}(\pm lr).$$

Note 16 *Since*

$$F_k^{\pm}(x) \overset{r\to\infty}{\sim} \mathrm{e}^{il\langle\omega,x\rangle} + \mathrm{e}^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{\sqrt{\pm 2l}}\left(2\pi\alpha - \Psi(1) + \ln\frac{\pm l}{2i}\right)^{-1}\frac{\mathrm{e}^{\pm ilr}}{\sqrt{r}} = \mathrm{e}^{il\langle\omega,x\rangle} + f^{\pm}(l)\frac{\mathrm{e}^{\pm ilr}}{\sqrt{r}}[6],$$

the scattering amplitude (see formula (A.34)) is independent of the angular variables (the directions of an incoming and an outgoing particle) and the relation

$$f(l) \equiv f^{+}(l) = \mathrm{e}^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{2l}}\left(2\pi\alpha - \Psi(1) + \ln\frac{l}{2i}\right)^{-1}$$

holds.

Proposition 17 *For g of the form*

$$g(x) := \int_0^{\infty}\int_{S^1} \mathrm{e}^{il\langle\omega,x\rangle}\varphi(l,\omega)\mathrm{d}\omega l\mathrm{d}l,$$

the scattering operator S_α of the point interaction is given be the following prescription

$$(S_\alpha g)(x) = \int_0^{\infty}\int_{S^1} \mathrm{e}^{il\langle\omega',x\rangle}\left(\int_{S^1}\mathcal{S}_\alpha^{(l)}(\omega,\omega')\varphi(l,\omega)\mathrm{d}\omega\right)\mathrm{d}\omega' l\mathrm{d}l,$$

where $\mathcal{S}_\alpha^{(l)}(\omega,\omega') = \delta_{S^1}(\omega,\omega') + \frac{i}{2}\left(2\pi\alpha - \Psi(1) + \ln\frac{l}{2i}\right)^{-1}$

[6] Evidently $f^{+}(l) = \overline{f^{-}(l)}$.

Proof To ease the notation we will omit the index α in this proof.

The generalized eigenfunctions $F^{\pm}$ of H_α fulfil the assumptions of lemma A.23 [7]. If we verify (A.40) (we even compute $\mathcal{S}^{(l)}$), the proposition is a direct consequence of lemma A.25

Let's find the integral kernel $\mathcal{S}^{(l)}(\omega,\omega')$. Considering the rotational symmetry of the model, we may choose it of the form

$$\mathcal{S}^{(l)}(\omega,\omega') = \sum_{m=-\infty}^{\infty} \mathcal{S}_m^{(l)} \mathrm{e}^{im(\mathrm{arc}\,\omega - \mathrm{arc}\,\omega')}.$$

Using (A.36) we write down defining equality (A.40) for $r \to \infty$

$$\sum_{m=-\infty}^{m=\infty} \left[\frac{(-)^m \mathrm{e}^{i\frac{\pi}{4}}}{\sqrt{2\pi l r}} \mathrm{e}^{-ilr} + \left(\frac{\mathrm{e}^{-i\frac{\pi}{4}}}{\sqrt{2\pi l r}} + \frac{f_m^+(l)}{\sqrt{r}} \right) \mathrm{e}^{ilr} \right] \mathrm{e}^{im(\Omega - \mathrm{arc}\,\omega)} =$$
$$2\pi \sum_{m=-\infty}^{m=\infty} \left[\left(\frac{(-)^m \mathrm{e}^{i\frac{\pi}{4}}}{\sqrt{2\pi l r}} + \frac{f_m^-(l)}{\sqrt{r}} \right) \mathrm{e}^{-ilr} + \frac{\mathrm{e}^{-i\frac{\pi}{4}}}{\sqrt{2\pi l r}} \mathrm{e}^{ilr} \right] S_m^{(l)} \mathrm{e}^{im(\Omega - \mathrm{arc}\,\omega)},$$

where $f_m^{\pm}(l)$ denotes the m^{th} Fourier coefficient of $f^{\pm}(l)$ (regarding the angular variable independence, $f_m^{\pm}(l) = 0$ for $m \neq 0$ and $f_0^{\pm}(l) = f^{\pm}(l)$.).

One can easily verify that for all $m \in \mathbb{Z}$:

$$\frac{\frac{(-)^m \mathrm{e}^{i\frac{\pi}{4}}}{\sqrt{2\pi l}} + f_m^-(l)}{\frac{\mathrm{e}^{-i\frac{\pi}{4}}}{\sqrt{2\pi l}}} = \frac{\frac{(-)^m \mathrm{e}^{i\frac{\pi}{4}}}{\sqrt{2\pi l}}}{\frac{\mathrm{e}^{-i\frac{\pi}{4}}}{\sqrt{2\pi l}} + f_m^+(l)},$$

and thus for an arbitrary $m \in \mathbb{Z}$ we have

$$\mathcal{S}_m^{(l)} = \frac{1}{2\pi} \left(1 + \mathrm{e}^{i\frac{\pi}{4}} \sqrt{2\pi l} f_m^+(l) \right).$$

For $\mathcal{S}^{(l)}(\omega,\omega')$ we may conclude

$$\mathcal{S}^{(l)}(\omega,\omega') = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}\setminus\{0\}} \mathrm{e}^{im(\mathrm{arc}\,\omega - \mathrm{arc}\,\omega')} + \frac{1}{2\pi} \left[1 + i\pi \left(2\pi\alpha - \Psi(1) + \ln \frac{l}{2i} \right)^{-1} \right]$$
$$= \delta_{S^1}(\omega,\omega') + \frac{i}{2} \left(2\pi\alpha - \Psi(1) + \ln \frac{l}{2i} \right)^{-1}.$$

■

Note 18 *The result is in agreement with general formula (A.41). Alternatively the proposition can be directly proved verifying (A.40) for $S_\alpha^{(l)}$.*

[7] Verifying the conditions, we make use of the relation $\frac{\partial}{\partial l} H_0^{(1)}(lr) = -r H_1^{(1)}(lr)$.

Perturbation supported by a curve

At the beginning let us summarize some basic facts about the restriction to a submanifold problem. The results will be formulated for our two-dimensional case.

Let $y :< 0, L >\equiv I \to \mathbb{R}^2$ be a regular loop (a closed path), that is $y'(s) \neq 0$ for all $s \in I$ and $y(0) = y(L)$, additionally we suppose y' to be continuous. We call the image $\Gamma := \{y(s)|\ s \in I\}$ a curve. We introduce a natural length measure on Γ and we denote the space of square integrable functions with this measure by $L^2(\Gamma)$:

$$f \in L^2(\Gamma) \Leftrightarrow \int\limits_\Gamma |f|^2 := \int\limits_0^L |f(y(s))|^2 |y'(s)| \mathrm{d}s < \infty.$$

Finally let $T_\Gamma f$ be the restriction of f to Γ for every $f \in \mathscr{S}(\mathbb{R}^2)$.

Theorem 19 *There is a constant C such that $\|T_\Gamma f\|_{L^2(\Gamma)} \le C\|f\|_{H^1(\mathbb{R}^2)}$ for all $f \in \mathscr{S}(\mathbb{R}^2)$. Since $\mathscr{S}(\mathbb{R}^2)$ is a dense subspace of $H^1(\mathbb{R}^2)$, T_Γ can be uniquely extended to a bounded map of $H^1(\mathbb{R}^2)$ into $L^2(\Gamma)$.*

Proof A proof even for more general case can be found for example in [9].

Let y be a loop with the properties mentioned above such that the free hamiltonian Green function $\mathcal{G}_z(. - .)$ (13) belongs to $L^2(\Gamma) \otimes L^2(\Gamma)$, and α be a real continuous nonzero function defined on Γ (and so α is bounded). For each pair y and α we introduce the following quadratic form:

$$\begin{aligned} &Dom(F_{\alpha,\Gamma}) = H^1(\mathbb{R}^2) \\ &F_{\alpha,\Gamma}[u] = \int\limits_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}^2 x + \int\limits_\Gamma \alpha |u_\Gamma|^2. \end{aligned} \tag{19}$$

For $u \in H^1(\mathbb{R}^2)$ we denote its restriction to Γ by $u_\Gamma := T_\Gamma u$ and the restriction's $-\alpha-$multiple by $\sigma_u := -\alpha u_\Gamma$. With this notation in hand we define

$$[G^z_\Gamma u](x) := \int\limits_\Gamma u \mathcal{G}_z(x - .) \equiv \int\limits_0^L u(y(s)) \mathcal{G}_z(x - y(s)) |y'(s)| \mathrm{d}s, \quad u \in L^2(\Gamma),\ x \in \mathbb{R}^2,\ z < 0.$$

Proposition 20 *The mapping $G^z_\Gamma:\ L^2(\Gamma)\to H^1(\mathbb{R}^2)$ given by prescription (19) is bounded.*

Proof Since $\|f\|^2_{H^1(\mathbb{R}^2)}=\int_{\mathbb{R}^2}(1+|\xi|^2)|(\mathscr{F}f)(\xi)|^2\mathrm{d}^2\xi=\|f\|^2_{L^2(\mathbb{R}^2)}+\|\nabla f\|^2_{L^2(\mathbb{R}^2)}$, it is enough to find constants $K_1,\ K_2$ such that

$$\text{a) } \|G^z_\Gamma u\|_{L^2(\mathbb{R}^2)}\le K_1\|u\|_{L^2(\Gamma)}$$
$$\text{b) } \|\nabla G^z_\Gamma u\|_{L^2(\mathbb{R}^2)}\le K_2\|u\|_{L^2(\Gamma)}$$

a) With the help of the Green function integral representation we make the following estimates

$$\begin{aligned}\int\limits_{\mathbb{R}^2}|G^z_\Gamma u|^2\mathrm{d}^2x=&\int\limits_{\mathbb{R}^2}\int\limits_{I\times I}\int\limits_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{(2\pi)^4}u(y(s))\overline{u(y(s'))}\frac{1}{k^2-z}\mathrm{e}^{i\langle k,x-y(s)\rangle}\frac{1}{l^2-z}\mathrm{e}^{i\langle l,y(s')-x\rangle}\\ &|y'(s)||y'(s')|\mathrm{d}^2k\mathrm{d}^2l\mathrm{d}s\mathrm{d}s'\mathrm{d}^2x=\frac{1}{(2\pi)^2}\int\limits_{I\times I}\int\limits_{\mathbb{R}^2}u(y(s))\overline{u(y(s'))}\frac{1}{(k^2-z)^2}\mathrm{e}^{i\langle k,y(s')-y(s)\rangle}\\ &|y'(s)||y'(s')|\mathrm{d}^2k\mathrm{d}s\mathrm{d}s'\le\frac{1}{(2\pi)^2}\int\limits_{\mathbb{R}^2}\frac{1}{(k^2-z)^2}\mathrm{d}^2k\int\limits_{I\times I}|u(y(s))u(y(s'))||y'(s)||y'(s')|\\ &\mathrm{d}s\mathrm{d}s'\le \mathit{const.}\|u\otimes\mathrm{Id}\|_{L^2(\Gamma)\otimes L^2(\Gamma)}\|\mathrm{Id}\otimes u\|_{L^2(\Gamma)\otimes L^2(\Gamma)}=\mathit{const.}\Big(\int\limits_\Gamma\mathrm{Id}\Big)^2\|u\|^2_{L^2(\Gamma)}\\ &\le K_1\|u\|^2_{L^2(\Gamma)}.\end{aligned}$$

b)Using the integration by parts we come to the following estimates

$$\begin{aligned}\int\limits_{\mathbb{R}^2}|\nabla G^z_\Gamma u|^2\mathrm{d}^2x=&\int\limits_{I\times I}u(y(s))\overline{u(y(s'))}\left(\int\limits_{\mathbb{R}^2}\nabla_x\mathcal{G}_z(x-y(s))\nabla_x\overline{\mathcal{G}_z(x-y(s'))}\mathrm{d}^2x\right)|y'(s)||y'(s')|\\ &\mathrm{d}s\mathrm{d}s'=\int\limits_{I\times I}u(y(s))\overline{u(y(s'))}\left(-\int\limits_{\mathbb{R}^2}\mathcal{G}_z(x-y(s))\Delta_x\overline{\mathcal{G}_z(x-y(s'))}\mathrm{d}^2x\right)\\ &|y'(s)||y'(s')|\mathrm{d}s\mathrm{d}s'=\{-\Delta_x\mathcal{G}_z(x-y)=z\mathcal{G}_z(x-y)+\delta(x-y)\}\\ &=z\int\limits_{\mathbb{R}^2}|G^z_\Gamma u|^2\mathrm{d}^2x+\int\limits_{I\times I}u(y(s))\overline{u(y(s'))}\mathcal{G}_z(y(s)-y(s'))|y'(s)||y'(s')|\mathrm{d}s\mathrm{d}s'\\ &\le|z|K_1\|u\|^2_{L^2(\Gamma)}+\|u\|^2_{L^2(\Gamma)}\|\mathcal{G}_z\|_{L^2(\Gamma)\otimes L^2(\Gamma)}\le K_2\|u\|^2_{L^2(\Gamma)}\end{aligned}$$

■

Proposition 21 *There is an alternative prescription for quadratic form (19):*

$$\begin{aligned}&F_{\alpha,\Gamma}[u]=\mathscr{F}^z_\Gamma[u]+\Phi^z_{\alpha,\Gamma}[\sigma_u]\\ &\mathscr{F}^z_\Gamma[u]=\int\limits_{\mathbb{R}^2}|\nabla(u-G^z_\Gamma\sigma_u)|^2\mathrm{d}^2x-z\int\limits_{\mathbb{R}^2}|u-G^z_\Gamma\sigma_u|^2\mathrm{d}^2x+z\int\limits_{\mathbb{R}^2}|u|^2\mathrm{d}^2x\\ &\Phi^z_{\alpha,\Gamma}[\sigma_u]=-\int\limits_\Gamma\frac{|\sigma_u|^2}{\alpha}-\int\limits_\Gamma\sigma_u(\overline{G^z_\Gamma\sigma_u}).\end{aligned}$$

Proof

$$F_{\alpha,\Gamma}[u] - \int_{\Gamma} \alpha |u_\Gamma|^2 = \int_{\mathbb{R}^2} |\nabla(u - G^z_\Gamma \sigma_u)|^2 - \int_{\mathbb{R}^2} |\nabla G^z_\Gamma \sigma_u|^2 + \int_{\mathbb{R}^2} \nabla u \nabla(\overline{G^z_\Gamma \sigma_u}) + \int_{\mathbb{R}^2} \nabla \overline{u} \nabla (G^z_\Gamma \sigma_u)$$

$$= \{\text{int. by parts}\} = \int_{\mathbb{R}^2} |\nabla(u - G^z_\Gamma \sigma_u)|^2 - z \int_{\mathbb{R}^2} |G^z_\Gamma \sigma_u|^2 - \int_{\Gamma} \sigma_u \overline{G^z_\Gamma \sigma_u} + z \int_{\mathbb{R}^2} u \overline{G^z_\Gamma \sigma_u}$$

$$- 2 \int_{\Gamma} \frac{|\sigma_u|^2}{\alpha} + z \int_{\mathbb{R}^2} \overline{u} G^z_\Gamma \sigma_u = \int_{\mathbb{R}^2} |\nabla(u - G^z_\Gamma \sigma_u)|^2 - z \int_{\mathbb{R}^2} |u - G^z_\Gamma \sigma_u|^2 + z \int_{\mathbb{R}^2} |u|^2 - 2 \int_{\Gamma} \frac{|\sigma_u|^2}{\alpha}$$

$$- \int_{\Gamma} \sigma_u \overline{G^z_\Gamma \sigma_u}$$

■

Proposition 22 *Let $0 \neq u \in C(\Gamma)$, then $G^z_\Gamma u \in H^m(\mathbb{R}^2 \setminus \Gamma), \forall m \in \mathbb{N}$, $G^z_\Gamma u \notin H^2(\mathbb{R}^2)$ and*

$$\left[\frac{\partial G^z_\Gamma u}{\partial n}\right]_\Gamma \equiv \partial_{n_{out}} G^z_\Gamma u - \partial_{n_{in}} G^z_\Gamma u = -u,$$

where ∂_{out} and ∂_{in} stand for the derivatives in the direction of the unit normal vector to the loop Γ from outside and inside respectively.

Proof By proposition 20 $G^z_\Gamma u \in H^1(\mathbb{R}^2)$. Moreover due to the smoothness of $\mathcal{G}_z(. - y(s))$ on $\mathbb{R}^2 \setminus \Gamma$, $G^z_\Gamma u \in C^\infty(\mathbb{R}^2 \setminus \Gamma)$.

Let $\varphi \in D(\mathbb{R}^2 \setminus \Gamma)$. Since $\varphi G^z_\Gamma u \in D(\mathbb{R}^2 \setminus \Gamma) \subset H^m(\mathbb{R}^2)$, $G^z_\Gamma u \in H^m(\mathbb{R}^2 \setminus \Gamma)$, $\forall m \in \mathbb{N}$.

Now let $\varphi \in D(\mathbb{R}^2)$. Then

$$\begin{aligned} ((-\Delta - z)(G^z_\Gamma u), \varphi) &= \int_{\mathbb{R}^2} \int_I \delta(x - y(s)) u(y(s)) |y'(s)| \mathrm{d}s \varphi(x) \mathrm{d}^2 x \\ &= \int_I u(y(s)) \varphi(y(s)) |y'(s)| \mathrm{d}s = (u \delta_\Gamma, \varphi) \end{aligned}$$

and thus $(-\Delta - z)(G^z_\Gamma u) = u\delta_\Gamma$ in $D'(\mathbb{R}^2)$. Alternatively

$$\begin{aligned} (-\Delta - z)(G^z_\Gamma u) &= \{(-\Delta - z)(G^z_\Gamma u)\} - \left[\frac{\partial G^z_\Gamma u}{\partial n}\right]_\Gamma \delta_\Gamma - \frac{\partial}{\partial n}([G^z_\Gamma u]_\Gamma \delta_\Gamma) \\ &= \left\{G^z_\Gamma u \in H^1(\mathbb{R}^2) \subset C(\mathbb{R}^2)\right\} = - \left[\frac{\partial G^z_\Gamma u}{\partial n}\right]_\Gamma \delta_\Gamma. \end{aligned}$$

Comparing those two expressions for $(-\Delta - z)(G^z_\Gamma u)$, we have $\left[\frac{\partial G^z_\Gamma u}{\partial n}\right]_\Gamma = -u$. If $u \neq 0$, $G^z_\Gamma u$ can not be in $H^2(\mathbb{R}^2)$, because every element of this space has absolutely continuous derivatives.

■

Proposition 23 *$\Phi^z_{\alpha,\Gamma}$ is a symmetric bounded quadratic form.*

Proof The symmetry is a direct consequence of the Green function symmetry $\mathcal{G}_z(x,y) = \overline{\mathcal{G}_z(y,x)}$. To prove the boundedness we make the following estimates

$$\begin{aligned}|\Phi^z_{\alpha,\Gamma}[\sigma_u]| &\leq \frac{1}{\min\{|\alpha(y)||\ y \in \Gamma\}}\|\sigma_u\|^2_{L^2(\Gamma)} + \|\sigma_u\|_{L^2(\Gamma)}\|G^z_\Gamma \sigma_u\|_{L^2(\Gamma)} \\ &\leq \left(\frac{1}{\min\{|\alpha|\}} + \|T_\Gamma\|\|G^z_\Gamma\|\right)\|\sigma_u\|^2_{L^2(\Gamma)}.\end{aligned}$$

$\min\{|\alpha(y)||\ y \in \Gamma\}$ is nonzero, since α is a nonzero and continuous function and Γ is a compact set as an image of the compact interval I.

■

Note 24 *A hermitian operator, which we denote $B^z_{\alpha,\Gamma}$, is associated to the form $\Phi^z_{\alpha,\Gamma}$:*

$$Dom(B^z_{\alpha,\Gamma}) = L^2(\Gamma), \quad B^z_{\alpha,\Gamma} = -\frac{1}{\alpha}Id - T_\Gamma G^z_\Gamma.$$

Proposition 25 *Let $(1 + \min\{\alpha(y)|\ y \in \Gamma\}\|T_\Gamma\|^2) \geq 0$, then $F_{\alpha,\Gamma}$ is symmetric, closed and bounded from below.*

Proof We make use of former prescription (19) for the form $F_{\alpha,\Gamma}$

$$F_{\alpha,\Gamma}(u,v) = \langle \nabla u, \nabla v\rangle + \langle \alpha u_\Gamma, v_\Gamma\rangle_{L^2(\Gamma)}.$$

The symmetry of the form then follows from the symmetry of inner products.

For $\alpha > 0$ we immediately find a lower bound: $F_{\alpha,\Gamma}[u] \geq 0$ for all $u \in H^1(\mathbb{R}^2)$, moreover the form is positive. For $\alpha < 0$ satisfying the condition of the proposition we obtain

$$\begin{aligned}F_{\alpha,\Gamma}[u] &\geq \|\nabla u\|^2 + \min\{\alpha\}\|u_\Gamma\|^2_{L^2(\Gamma)} \geq \|\nabla u\|^2 + \min\{\alpha\}\|T_\Gamma\|^2\left(\|u\|^2 + \|\nabla u\|^2\right) \\ &\geq (1 + \min\{\alpha\}\|T_\Gamma\|^2)\|\nabla u\|^2 + \min\{\alpha\}\|T_\Gamma\|^2\|u\|^2 \geq \min\{\alpha\}\|T_\Gamma\|^2\|u\|^2.\end{aligned}$$

Finally we show that the form $F_{\alpha,\Gamma}$ is closed. Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^2)$ be a sequence converging in $L^2(\mathbb{R}^2)$ to u such that $\lim_{m,n\to\infty} F_{\alpha,\Gamma}[u_n - u_m] = 0$. The sequence u_n converges, so it must be Cauchy: $\lim_{m,n\to\infty}\|u_n - u_m\| = 0$. For $\alpha > 0$ we conclude $\lim_{m,n\to\infty}\|u_n - u_m\|_{H^1(\mathbb{R}^2)} = 0$. For $\alpha < 0$ we take use of the estimate above

$$\underbrace{F_{\alpha,\Gamma}[u_n - u_m]}_{\downarrow \atop 0} \geq (1 + \min\{\alpha\}\|T_\Gamma\|^2)\|\nabla(u_n - u_m)\|^2 + \min\{\alpha\}\underbrace{\|T_\Gamma\|^2\|(u_n - u_m\|^2}_{\downarrow \atop 0}$$

as $m,n \to \infty$. Hence $\lim_{m,n\to\infty}\|\nabla(u_n - u_m)\| = 0$ too and consequently $\lim_{m,n\to\infty}\|u_n - u_m\|_{H^1(\mathbb{R}^2)} = 0$. Due to the completeness of the Sobolev space $H^1(\mathbb{R}^2)$, there exists $v \in H^1(\mathbb{R}^2)$ such that $\lim_{n\to\infty}\|u_n - u\|_{H^1(\mathbb{R}^2)} = 0$. The limit uniqueness and the prescription for the $H^1(\mathbb{R}^2)$ norm implies $v = u$.

$\lim_{n\to\infty}\int_\Gamma \alpha|u_n - u|^2 = 0$ iff $\lim_{n\to\infty}\|u_n - u\|_{L^2(\Gamma)} = 0$, so the first limit is really zero, since the restriction mapping T_Γ is continuous and $\lim_{n\to\infty}\|u_n - u\|_{H^1(\mathbb{R}^2)} = 0$. Hence $\lim_{n\to\infty} F_{\alpha,\Gamma}[u_n - u] = 0$ in all.

■

Proposition 26 *A selfadjoint operator $-\Delta_{\alpha,\Gamma}$ is associated to the quadratic form $F_{\alpha,\Gamma}$:*

$$Dom(-\Delta_{\alpha,\Gamma}) = \left\{u \in H^1(\mathbb{R}^2)|\ (u - G^z_\Gamma \sigma_u) \in H^2(\mathbb{R}^2)\right\}$$
$$(-\Delta_{\alpha,\Gamma} - z)u = (-\Delta - z)(u - G^z_\Gamma \sigma_u)$$

Proof With regard to proposition 25, we can apply the form representation theorem A.20, which states the existence of a selfadjoint operator $A_{\alpha,\Gamma}$ such that $\forall v \in Dom(A_{\alpha,\Gamma}) \subset Dom(F_{\alpha,\Gamma})$ fixed, there is g in $L^2(\mathbb{R}^2)$ with the following property:

$$F_{\alpha,\Gamma}(u, v) = \langle u, g\rangle \quad \forall u \in Dom(F_{\alpha,\Gamma}). \tag{20}$$

We set $A_{\alpha,\Gamma}v = g$.

Let us take $u \in H^1(\mathbb{R}^2)$ such that $u_\Gamma \equiv 0$. Then

$$F_{\alpha,\Gamma}(u, v) = \langle \nabla u, \nabla(v - G^z_\Gamma \sigma_v)\rangle - z\langle u, v - G^z_\Gamma \sigma_v\rangle + z\langle u, v\rangle = \langle u, g\rangle \quad \text{for } \forall u \in H^1(\mathbb{R}^2), \tag{21}$$

which means

$$\langle \nabla u, \nabla(v - G^z_\Gamma \sigma_v)\rangle = \langle u, g - zG^z_\Gamma \sigma_v\rangle \quad \text{for } \forall u \in H^1(\mathbb{R}^2).$$

Lemma A.18 then implies $(v - G^z_\Gamma \sigma_v) \in H^2(\mathbb{R}^2)$. Hence in (21) we may integrate by parts :

$$\langle u, (-\Delta - z)(v - G^z_\Gamma \sigma_v)\rangle = \langle u, g - zv\rangle$$

which considering that $H^1(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$ implies

$$(-\Delta - z)(v - G^z_\Gamma \sigma_v) = g - zv = (A_{\alpha,\Gamma} - z)v.$$

We have proved $A_{\alpha,\Gamma} \subset -\Delta_{\alpha,\Gamma}$.

The inverse inclusion, i.e. for arbitrary $v \in Dom(-\Delta_{\alpha,\Gamma})$ such $g \in L^2(\mathbb{R}^2)$ exists that (20) is fulfilled and $(-\Delta - z)(v - G^z_\Gamma \sigma_v) = (-\Delta_{\alpha,\Gamma} - z)v$, can be immediately obtained considering $u \in H^1(\mathbb{R}^2)$ such that $u_\Gamma \equiv 0$ and integrating by parts.

■

Corollary 27 *For $u \in H^1(\mathbb{R}^2)$, $\sigma_u \in C(\Gamma)$, and hence by proposition 22, $G^z_\Gamma \sigma_u \notin H^2(\mathbb{R}^2)$. Thus each $u \in Dom(-\Delta_{\alpha,\Gamma})$ can be decomposed as follows: $u = v + G^z_\Gamma \sigma_u$, where $v \in H^2(\mathbb{R}^2)$. In accordance with proposition 22 we have*

$$\left[\frac{\partial G^z_\Gamma \sigma_u}{\partial n}\right] = -u_\Gamma = \alpha u_\Gamma. \tag{22}$$

Proposition 28 *For $z < 0$ and $g \in L^2(\mathbb{R}^2)$ we have*

$$(-\Delta_{\alpha,\Gamma} - z)^{-1}g = (-\Delta - z)^{-1}g + G^z_\Gamma (B^z_{\alpha,\Gamma})^{-1} T_\Gamma (-\Delta - z)^{-1}g$$

Proof Let $g \in L^2(\mathbb{R}^2)$, then $(-\Delta_{\alpha,\Gamma} - z)^{-1} g \equiv u \in Dom(-\Delta_{\alpha,\Gamma})$ and we have

$$g = (-\Delta_{\alpha,\Gamma} - z)u = (-\Delta - z)(u - G_\Gamma^z \sigma_u) = (-\Delta - z)\left[(-\Delta_{\alpha,\Gamma} - z)^{-1} g - G_\Gamma^z \sigma_u\right]$$

and hence $(-\Delta - z)^{-1} g = (-\Delta_{\alpha,\Gamma} - z)^{-1} g - G_\Gamma^z \sigma_u$.
At last we show that $\sigma_u = (B_{\alpha,\Gamma}^z)^{-1} T_\Gamma (-\Delta - z)^{-1} g$:

$$B_{\alpha,\Gamma}^z \sigma_u = T_\Gamma (u - G_\Gamma^z \sigma_u) = T_\Gamma (-\Delta - z)^{-1} (-\Delta_{\alpha,\Gamma} - z) u = T_\Gamma (-\Delta - z)^{-1} g.$$

■

Note 29 *For the Green function of the operator $-\Delta_{\alpha,\Gamma}$ we have:*

$$\mathcal{G}_{\alpha,\Gamma}^z(x,y) = \mathcal{G}_z(x,y) + \left[G_\Gamma^z (B_{\alpha,\Gamma}^z)^{-1} T_\Gamma \mathcal{G}_z(.-y)\right](x),$$

where $\mathcal{G}_z$ is the free hamiltonian Green function (13).

4 Time-independent scattering theory

It is convenient to express the spectral parameter z by the impulse $k \in \mathbb{R}^2$: $z = k^2 = l^2$, where $l := |k|,\ k = l\omega$.

Proposition 30 *The generalized eigenfunctions of the operator $-\Delta_{\alpha,\Gamma}$ are*

$$\psi_k(x) = \mathrm{e}^{il\langle x,\omega\rangle} + G_\Gamma^z[(B_{\alpha,\Gamma}^z)^{-1} \mathrm{e}^{il\langle .,\omega\rangle}](x). \tag{23}$$

Proof
At first we show that $(-\Delta_{\alpha,\Gamma} - k^2)\psi_k = 0$ in the distributional sense. With the help of the asymptotic expansions of $\mathcal{G}_{(l+i\varepsilon)^2}(x,y)$ and $|x-y|$ for fixed x and $|y| \to \infty$:

$$\mathcal{G}_{(l+i\varepsilon)^2}(x,y) \overset{|y|\to\infty}{\sim} \frac{1}{2\sqrt{2\pi}} \mathrm{e}^{i\frac{\pi}{4}} \mathrm{e}^{i(l+i\varepsilon)|y|} \mathrm{e}^{i(l+i\varepsilon)\langle x,\omega\rangle} \frac{1}{\sqrt{(l+i\varepsilon)|y|}}, \quad \text{where } \omega = -\frac{y}{|y|},$$

one can easily verify that

$$\psi_k(x) \equiv \psi_{l\omega}(x) = \lim_{\varepsilon\to 0+} \lim_{|y|\to\infty} 2\sqrt{2\pi} \mathrm{e}^{-i\frac{\pi}{4}} \sqrt{(l+i\varepsilon)|y|} \mathrm{e}^{-i(l+i\varepsilon)|y|} \mathcal{G}_{\alpha,\Gamma}^{(l+i\varepsilon)^2}(x,y).$$

Since we assume fixed x and $|y| \to \infty$, we may set $\delta(x-y) = 0$ on $D(\mathbb{R}^2)$, so

$$(-\Delta_{\alpha,\Gamma} - (l+i\varepsilon)^2) \mathcal{G}_{\alpha,\Gamma}^{(l+i\varepsilon)^2}(x,y) = 0$$

and hence $(-\Delta_{\alpha,\Gamma} - k^2)\psi_k = 0$ in $D'(\mathbb{R}^2)$.
$(B_{\alpha,\Gamma}^z)^{-1} \mathrm{e}^{il\langle .,\omega\rangle} \in C(\Gamma)$, so boundary condition (22) of $Dom(-\Delta_{\alpha,\Gamma})$ is fulfilled.

■

Proposition 31 *For the scattering amplitude we have*

$$f(k,k') = \frac{\mathrm{e}^{i\frac{\pi}{4}}}{2\sqrt{2\pi l}} \int\limits_I \mathrm{e}^{-i\langle k', y(s)\rangle} \left[(B^z_{\alpha,\Gamma})^{-1}\mathrm{e}^{i\langle .,k\rangle}\right](y(s))|y'(s)|\mathrm{d}s, \quad \textit{where } |x| = r,\ k' = l\frac{x}{r} \quad (24)$$

Proof Knowing generalized eigenfunctions (23) we extract the scattering amplitude from the general asymptotic expansion

$$\psi_k(x) \overset{r\to\infty}{\sim} \mathrm{e}^{il\langle x,\omega\rangle} + f(k,k')\frac{\mathrm{e}^{ilr}}{\sqrt{r}}, \quad \text{where } k' = l\frac{x}{r}$$

Hence

$$f(k,k') = \lim_{r\to\infty} \sqrt{r}\mathrm{e}^{-ilr} \int\limits_I \mathcal{G}_z(x - y(s)) \left[(B^z_{\alpha,\Gamma})^{-1}\mathrm{e}^{i\langle .,k\rangle}\right](y(s))|y'(s)|\mathrm{d}s,$$

which gives (24) after substituting the asymptotic expansion

$$\mathcal{G}_z(x-y(s)) \overset{r\to\infty}{\sim} \frac{1}{2\sqrt{2\pi}}\mathrm{e}^{i\frac{\pi}{4}}\mathrm{e}^{ilr}\mathrm{e}^{-il\langle\frac{x}{r},y(s)\rangle}\frac{1}{\sqrt{lr}}.$$

■

5 Constant δ-like interaction supported by a circle

We will discuss the special case of Γ being a circle $y(\omega) = R(\cos\omega, \sin\omega),\ \omega \in< 0, 2\pi)$ and of α being a positive constant. Due to the rotational symmetry it is convenient to pass to the polar coordinates: $x = r(\cos\varphi, \sin\varphi)$. Using the Fourier expansion, $\sigma_u|\ u \in Dom(-\Delta_{\alpha,\Gamma})$ can be written as follows

$$\sigma_u = \sum_{n=-\infty}^{\infty} u_n(R)\mathrm{e}^{in\varphi}, \quad \text{where } u_n(r) = -\frac{\alpha}{2\pi}\int\limits_0^{2\pi} u(r,\varphi)\mathrm{e}^{-in\varphi}\mathrm{d}\varphi.$$

Relevant operators are then diagonalized

$$\begin{aligned}
[G^z_\Gamma\sigma_u](r,\varphi) &= \int\limits_0^{2\pi}\sum_{n=-\infty}^{\infty} u_n(R)\mathrm{e}^{in\omega}\mathcal{G}_z(\sqrt{r^2+R^2-2rR\cos(\varphi-\omega)})R\mathrm{d}\omega \\
&= 2\pi R\sum_{n=-\infty}^{\infty}\mathcal{G}^z_n(r)u_n(R)\mathrm{e}^{in\varphi} \\
[T_\Gamma G^z_\Gamma\sigma_u](\varphi) &= 2\pi R\sum_{n=-\infty}^{\infty}\mathcal{G}^z_n(R)u_n(R)\mathrm{e}^{in\varphi} \\
[B^z_{\alpha,\Gamma}\sigma_u](\varphi) &= \sum_{n=-\infty}^{\infty}\left(-2\pi R\mathcal{G}^z_n(R) - \frac{1}{\alpha}\right)u_n(R)\mathrm{e}^{in\varphi} =: \sum_{n=-\infty}^{\infty} B_n u_n(R)\mathrm{e}^{in\varphi},
\end{aligned}$$

where

$$\mathcal{G}_n^z(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{G}_z(\sqrt{r^2+R^2-2rR\cos\alpha})\mathrm{e}^{-in\alpha}\mathrm{d}\alpha$$
$$= \begin{cases} \frac{i}{4}H_n^{(1)}(\sqrt{z}r)J_n(\sqrt{z}R) & \text{for } r>R \\ \frac{i}{4}H_n^{(1)}(\sqrt{z}R)J_n(\sqrt{z}r) & \text{for } r<R \end{cases}$$

by the Graf addition theorem A.20.

The inverse operator $(B_{\alpha,\Gamma}^z)^{-1}$ is of the diagonal form too: $[(B_{\alpha,\Gamma}^z)^{-1}]_{nn} = B_n^{-1}$ for $B_n \neq 0$.

5.1 Stationary scattering theory

Thanks to the rotational symmetry we may restrict ourselves to the case of a particle coming in the x_1 axis direction, i.e. $\omega = (1,0)$. In order to find the Fourier decomposition of the generalized eigenfunctions ψ_l, we expand $\mathrm{e}^{ilR\cos\varphi}$ (see formula (A.35)):

$$\mathrm{e}^{ilR\cos\varphi} = \sum_{n=-\infty}^{\infty} i^n J_n(lR)\mathrm{e}^{in\varphi}.$$

Then we have

$$\psi_l(r,\varphi) = \mathrm{e}^{ilr\cos\varphi} + 2\pi R\sum_{n=-\infty}^{\infty}\frac{i^n J_n(lR)}{B_n}\mathcal{G}_n^z(r)\mathrm{e}^{in\varphi}.$$

The scattering amplitude can be explicitly computed too:

$$f(k,k') \equiv f(l,\varphi) = 2\pi R\lim_{r\to\infty}\sqrt{r}\mathrm{e}^{-ilr}\sum_{n=-\infty}^{\infty}\frac{i^n J_n(lR)}{B_n}\mathcal{G}_n^z(r)\mathrm{e}^{in\varphi} = \frac{\sqrt{\pi}R\mathrm{e}^{i\frac{\pi}{4}}}{\sqrt{2l}}\sum_{n=-\infty}^{\infty}\frac{J_n^2(lR)}{B_n}\mathrm{e}^{in\varphi},$$

where we take use of the asymptotic expansion $\mathcal{G}_n^z(r) \overset{r\to\infty}{\sim} \frac{\mathrm{e}^{i\frac{\pi}{4}}}{4}(-i)^n\sqrt{\frac{2}{\pi l r}}\mathrm{e}^{ilr}J_n(lR)$.

Quantum dot with the point interaction

Some basic facts about the point perturbation of Schrödinger operators are summarized and developed further in [5], especially the case of the threedimensional harmonic oscillator is investigated in detail. Here we at first recapitulate some general results included in the mentioned article and then apply them to examine the spectral properties of the point perturbed two-dimensional harmonic oscillator.

Similarly to the case of the free hamiltonian, we introduce the point interaction in $q \in \mathbb{R}^2$ as a self-adjoint extension of some suitable restriction of the unperturbed hamiltonian. Let us denote this restriction $H(q)$ and define it as follows:

$$Dom(H(q)) := \{f \in Dom(H) |\ f(q) = 0\}\,, \quad H_q := H\upharpoonright_{Dom(H(q))},$$

where H is the hamiltonian of the two-dimensional isotropic harmonic oscillator (A.42). $H(q)$ is a closed symmetric operator with the deficiency indices (1,1).

All self-adjoint extensions of $H(q)$ form a one-parametric family $\{H_\alpha(q) |\ \alpha \in \mathbb{R}\}$. The Green function of $H_\alpha(q)$ is in accordance with Krein formula given by

$$\mathcal{G}_z^{\alpha,q}(x,y) = \mathcal{G}_z^{\mathrm{ho}}(x,y) - [Q(z,q) - \alpha]^{-1}\mathcal{G}_z^{\mathrm{ho}}(x,q)\mathcal{G}_z^{\mathrm{ho}}(q,y), \tag{25}$$

where $\mathcal{G}_z^{\mathrm{ho}}$ is the Green function of H and $Q(z,q) = \mathcal{G}_{z,\mathrm{reg}}^{\mathrm{ho}}(q,q)$ is the regularized Green function of H evaluated in $x = y = q$ (so-called Krein Q-function).

The unperturbed operator H corresponds formally to $\alpha = \infty$.

The function $z \to Q(z,q)$ is analytic in the domain $\mathbb{C} \setminus \sigma(H)$ and $\frac{\partial Q(z,q)}{\partial z} > 0$ for $z \in \mathbb{R} \setminus \sigma(H)$. The set of all poles of the function $z \to Q(z,q)$ coincides with the set $\rho(q)$ defined as follows:

$$\rho(q) := \{\lambda_n \in \sigma(H) :\ \exists f \in L_n |\ f(q) \neq 0\}\,,$$

where L_n denotes the eigenspace associated with the n^{th} eigenvalue λ_n.

6 Spectral properties of $H_\alpha(q)$

$H_\alpha(q)$ is a rank one perturbation[8] of H, so the spectrum of H_α is discrete too. An eigenvalue λ_n of H of the multiplicity k_n is an eigenvalue of H_α of the multiplicity $k_n + 1$, k_n or $k_n - 1$ (if

[8]That is $\dim Ran(R_{H_\alpha}(z) - R_H(z)) = 1$.

$k_n = 1$, then λ_n does not belong to $\sigma(H_\alpha)$). There are additional eigenvalues different from λ_n, which can be found as solutions to the equation

$$Q(z,q) = \alpha. \tag{26}$$

Since $\frac{\partial Q(z,q)}{\partial z} > 0$ equation (26) has exactly one solution on each interval $(-\infty, \epsilon_0(q))$, $(\epsilon_0(q), \epsilon_1(q)), \ldots$, where $\{\epsilon_n(q) |\ n \in \mathbb{N}_0\}$ is the sequence of strictly increasing poles of the function $z \to Q(z,q)$. Denote those solutions $\xi_0(q),\ \xi_1(q), \ldots$.

Theorem 32 ([5]) *The spectrum of $H_\alpha(q)$ is discrete and consists of four nonintersecting parts $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ described as follows:*

i. σ_1 is the set of all solutions ξ_n to equation (26), which do not belong to $\sigma(H)$. The multiplicity of ξ_n in the spectrum of $H_\alpha(q)$ equals 1.

ii. σ_2 is the set of all $\lambda_n \in \rho(q)$ that are multiple eigenvalues of H. The multiplicity of the eigenvalue $\lambda_n \in \sigma_2$ in the spectrum of $H_\alpha(q)$ equals $k_n - 1$.

iii. σ_3 consists of all $\lambda_n \in \sigma(H) \setminus \rho(q)$, that are not solutions to equation (26). The multiplicity of the eigenvalue λ_n in the spectrum of H_α equals k_n.

iv. σ_4 consists of all $\lambda_n \in \sigma(H) \setminus \rho(q)$, that are solutions to equation (26). The multiplicity of the eigenvalue λ_n in the spectrum of H_α equals $k_n + 1$.

Proposition 33 $\rho(q) = \{\lambda_{2n} |\ n \in \mathbb{N}_0\}$, *if* $q = 0$, *and* $\rho(q) = \sigma(H)$ *otherwise.*

Proof At first we investigate the case $q = 0$. $\lambda_n \equiv \lambda_{n_1,n_2}$, $n_1 + n_2 = n$ (see appendix A.9), so if n is odd, then n_1 is odd and n_2 is even or vice versa. Since $H_{n_i}(0) = 0$ for n_i odd and $H_{n_i}(0) \neq 0$ for n_i even, $\psi_{n_1,n_2}(0) = 0$. On the other hand, if n is even, then $\psi_{n,0}(0) \neq 0$.

Now suppose that $q \neq 0$. Since $H_1(q) = 2q = 0$ only for $q = 0$ and $H_0 \equiv 1$, we have $\lambda_0, \lambda_1 \in \rho(q)$. Let $n > 0$. Suppose that $\psi_{n-1,1}(q) = 0$, then according to lemma A.27, $\psi_{n,0}(q) \neq 0$, i.e. $\lambda_n \in \rho(q)$.

■

7 Derivation of the Krein Q-function

Thanks to theorem 32, only a rather computation work is left to fully describe the spectrum of $H_\alpha(q)$ in dependence on α and q.

Consider the Green function of the two-dimensional isotropic harmonic oscillator in the polar coordinates [10]

$$\mathcal{G}_z^{\mathrm{ho}}(r\hat{\varphi}, r'\hat{\varphi}') = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathcal{G}_n^z(r,r')\mathrm{e}^{in(\varphi-\varphi')} = \frac{1}{\pi}\sum_{n=1}^{\infty}\mathcal{G}_n^z(r,r')\cos[n(\varphi-\varphi')] + \frac{1}{2\pi}\mathcal{G}_0^z(r,r') \tag{27}$$

$$\mathcal{G}_n^z(r,r') = \frac{\Gamma\left(\frac{1}{2}(|n|+1-\frac{z}{\omega})\right)}{\omega\Gamma(|n|+1)}\frac{1}{rr'}M_{\frac{z}{2\omega},\frac{|n|}{2}}\left(\frac{\omega}{2}r_<^2\right)W_{\frac{z}{2\omega},\frac{|n|}{2}}\left(\frac{\omega}{2}r_>^2\right)$$

$$(H-z)\mathcal{G}_z^{\mathrm{ho}}(x,y) = \delta(x-y), \quad \text{for } z \in \mathbb{C}\setminus\sigma(H),$$

where Γ denotes the Gamma function, $M_{a,b}$ and $W_{a,b}$ denote the Whittaker functions and $r_<, r_>$ are smaller and the greater of r and r', respectively. The singularities of $\mathcal{G}_z^{\mathrm{ho}}$ come from the singularities of $\Gamma\left(\frac{1}{2}(|n|+1-\frac{z}{\omega})\right)$ in $\frac{z}{\omega} \in \mathbb{N}$.

Proposition 34 *The divergent part of $\mathcal{G}_z^{ho}(x,y)$ as $x, y \to q$ is $-\frac{1}{2\pi}\ln(x-y)$.*

Proof This proof is a little heuristic, however its purpose is to give a guess.

Consider equation (A.25) for fixed a and for b much greater than a, then its solutions are close to $M_{0,b}$ and $W_{0,b}$. Those functions can be expressed in terms of the cylindrical functions (see formulas (A.27)). With the help of the asymptotic expansions

$$J_\nu(z) \overset{\nu\to\infty}{\sim} \frac{1}{\sqrt{2\pi\nu}}\left(\frac{\mathrm{e}z}{2\nu}\right)^{\nu}$$
$$Y_\nu(z) \overset{\nu\to\infty}{\sim} -\sqrt{\frac{2}{\pi\nu}}\left(\frac{\mathrm{e}z}{2\nu}\right)^{-\nu}$$

we obtain

$$M_{0,b}(z) \overset{b\to\infty}{\sim} z^{b+\frac{1}{2}}$$
$$W_{0,b}(z) \overset{b\to\infty}{\sim} \frac{1}{\sqrt{2b}}\left(\frac{4b}{\mathrm{e}}\right)^{b} z^{\frac{1}{2}-b}$$

and so for $\mathcal{G}_n^0$ we have

$$\mathcal{G}_n^0(r,r') \overset{n\to\infty}{\sim} \frac{\Gamma(\frac{n+1}{2})}{2\sqrt{n}\Gamma(n+1)}\left(\frac{2n}{\mathrm{e}}\right)^{\frac{n}{2}}\left(\frac{r_<}{r_>}\right)^{n} \sim \left\{\text{Stirling formula for } \Gamma(\frac{n}{2})\right\}$$
$$\sim \frac{2^{n-2}\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})}\left(\frac{r_<}{r_>}\right)^{n}.$$

Now suppose that $x, y \to q$ in such a way that $\varphi = \varphi'$. For the divergent part of $\mathcal{G}_z^{\mathrm{ho}}(x-y)$ we have

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\mathcal{G}_n^0(r,r') = -\frac{1}{2\pi}\ln\left(1-\frac{r_<}{r_>}\right) = -\frac{1}{2\pi}\ln(r_> - r_<) + \frac{1}{2\pi}\ln r_>.$$

The term $\lim_{r_>\to|q|}\frac{1}{2\pi}\ln r_>$ is finite for $q \neq 0$. For $q = 0$ we should consider the summand $\mathcal{G}_0^z(r,r')$ too. Using the asymptotic expansions $M_{a,0}(z) \overset{z\to 0}{\sim} \sqrt{z}$, $W_{a,0}(z) \overset{z\to 0}{\sim} -\frac{1}{\Gamma(1/2-a)}\sqrt{z}\ln z$ we conclude

$$\frac{1}{2\pi}\mathcal{G}_0^z(r,r') \overset{r,r'\to 0}{\sim} -\frac{1}{2\pi}\ln r_>,$$

i.e. exactly the unwanted part with the opposite sign.

■

The following lemma enables us to simplify the further analysis by setting $\omega = 1$ without the loss of generality.

Lemma 35 *Let $\tilde{\mathcal{G}}^{ho}_z$ be the Green function of $\tilde{H} := -\Delta + \frac{1}{4}x^2$ ($\omega = 1$), then $\tilde{\mathcal{G}}^{ho}_{\frac{z}{\omega}}(\sqrt{\omega}x, \sqrt{\omega}y)$ is the Green function of H.*

Proof

$$(H - z)\tilde{\mathcal{G}}^{\mathrm{ho}}_{\frac{z}{\omega}}(\sqrt{\omega}x, \sqrt{\omega}y) = (-\Delta_x + \frac{1}{4}\omega^2 x^2 - z)\tilde{\mathcal{G}}^{\mathrm{ho}}_{\frac{z}{\omega}}(\sqrt{\omega}x, \sqrt{\omega}y)$$
$$= \omega\left(-\Delta_{\sqrt{\omega}x} + \frac{1}{4}(\sqrt{\omega}x)^2 - \frac{z}{\omega}\right)\tilde{\mathcal{G}}^{\mathrm{ho}}_{\frac{z}{\omega}}(\sqrt{\omega}x, \sqrt{\omega}y) = \omega\delta(\sqrt{\omega}x - \sqrt{\omega}y) = \delta(x - y)$$

■

Consequently $Q(z,q) = \tilde{Q}(\frac{z}{\omega}, \sqrt{\omega}q)$, where $\tilde{Q}$ is the Krein Q-function for $\tilde{H}$, and equation (26) takes the form

$$\tilde{Q}(\frac{z}{\omega}, \sqrt{\omega}q) = \alpha.$$

Hence a change of the frequency ω does not change the numerical values of energy levels in $\sigma(H)$ if $\frac{1}{\sqrt{\omega}}$ is used as the unit of length and ω as the unit of energy.

So we restrict ourselves to the case $\omega = 1$ and we keep notation H for $\tilde{H}$ and Q for $\tilde{Q}$.

Proposition 36 *Let $|x-y| \to 0$ in such a way that $\varphi - \varphi' = 0$, then*

$$-\frac{1}{2\pi}\ln|x-y| = \begin{cases} -\frac{1}{2}\sum\limits_{n=1}^{\infty} Y_n(\sqrt{z}r_>)J_n(\sqrt{z}r_<) - \frac{1}{4}Y_0(\sqrt{z}r_>)J_0(\sqrt{z}r_<) \\ +\frac{1}{2\pi}(\ln\frac{\sqrt{z}}{2} - \Psi(1)) + O(|x-y|^2\ln|x-y|) \quad \text{for } z > 0 \\ \frac{i}{2}\sum\limits_{n=1}^{\infty} H_n^{(1)}(i\sqrt{-z}r_>)J_n(i\sqrt{-z}r_<) + \frac{i}{4}H_0^{(1)}(i\sqrt{-z}r_>)J_0(i\sqrt{-z}r_<) \\ +\frac{1}{2\pi}(\ln\frac{\sqrt{-z}}{2} - \Psi(1)) + O(|x-y|^2\ln|x-y|) \quad \text{for } z < 0. \end{cases}$$

Proof a) Let $\mathcal{G}_z$ be the free hamiltonian Green function $\mathcal{G}_z(x-y) = \frac{i}{4}H_0^{(1)}(\sqrt{z}|x-y|)$ and $z > 0$. For the real part of $\mathcal{G}_z$ we have

$$\Re\mathcal{G}_z(x-y) \overset{|x-y|\to 0}{\sim} -\frac{1}{2\pi}(\ln|x-y| + \ln\frac{\sqrt{z}}{2} - \Psi(1)) + O(|x-y|^2\ln|x-y|).$$

By the Graf formula (A.20):

$$\Re\mathcal{G}_z(r\hat{\varphi} - r'\hat{\varphi}') = -\frac{1}{4}\sum_{n=-\infty}^{\infty} Y_n(\sqrt{z}r_>)J_n(\sqrt{z}r_<)\cos[n(\varphi - \varphi')]$$
$$= -\frac{1}{2}\sum_{n=1}^{\infty} Y_n(\sqrt{z}r_>)J_n(\sqrt{z}r_<)\cos[n(\varphi - \varphi')] - \frac{1}{4}Y_0(\sqrt{z}r_>)J_0(\sqrt{z}r_<).$$

Setting $\varphi - \varphi' = 0$ and comparing those two expressions, we come to the statement of the proposition.

b) For $z < 0$, the Green function $\mathcal{G}_z$ is real and its asymptotic expansion is

$$\mathcal{G}_z(x-y) \overset{|x-y|\to 0}{\sim} -\frac{1}{2\pi}(\ln|x-y| + \ln\frac{\sqrt{-z}}{2} - \Psi(1)) + O(|x-y|^2\ln|x-y|).$$

The Graf formula then states that

$$\begin{aligned}\mathcal{G}_z(x-y) &= \frac{i}{4}\sum_{n=-\infty}^{\infty} H_n^{(1)}(i\sqrt{-z}r_>)J_n(i\sqrt{-z}r_<)\cos[n(\varphi-\varphi')] \\ &= \frac{i}{2}\sum_{n=1}^{\infty} H_n^{(1)}(i\sqrt{-z}r_>)J_n(i\sqrt{-z}r_<)\cos[n(\varphi-\varphi')] + \frac{i}{4}H_0^{(1)}(i\sqrt{-z}r_>)J_0(i\sqrt{-z}r_<).\end{aligned}$$

Again comparing those expressions one obtains the proposition.

■

For the Krein Q-function of the two-dimensional isotropic harmonic oscillator we conclude:

$$Q(z,q) = \begin{cases} \sum\limits_{n=1}^{\infty}\left(\frac{1}{\pi}\mathcal{G}_n^z(q,q)+\frac{1}{2}Y_n(\sqrt{z}q)J_n(\sqrt{z}q)\right)+\frac{1}{2\pi}\mathcal{G}_0^z(q,q) \\ +\frac{1}{4}Y_0(\sqrt{z}q)J_0(\sqrt{z}q)-\frac{1}{2\pi}(\ln\frac{\sqrt{z}}{2}-\Psi(1)) \quad \text{for } z>0 \\ \sum\limits_{n=1}^{\infty}\left(\frac{1}{\pi}\mathcal{G}_n^z(q,q)-\frac{i}{2}H_n^{(1)}(i\sqrt{-z}q)J_n(i\sqrt{-z}q)\right) \\ +\frac{1}{2\pi}\mathcal{G}_0^z(q,q)-\frac{i}{4}H_0^{(1)}(i\sqrt{-z}q)J_0(i\sqrt{-z}q)-\frac{1}{2\pi}(\ln\frac{\sqrt{-z}}{2}-\Psi(1)) \quad \text{for } z<0. \end{cases} \tag{28}$$

where q stands for $|q|$. Thus for fixed z, $Q(z,q)$ is only a function of the perturbation distance from the origin. It is a direct consequence of the rotational symmetry of the hamiltonian H.

7.1 Perturbation in the origin

If $q=0$, we get a rotational symmetric model. In this case the Krein Q-function takes much simpler form than the general one (28). Let $z>0$ and $\Delta_n := \frac{1}{\pi}\mathcal{G}_n^z(q,q)+\frac{1}{2}Y_n(\sqrt{z}q)J_n(\sqrt{z}q)$. Substituting asymptotic expansions of the cylindric and the Whittaker functions as $q\to 0$ we obtain $\Delta_n = O(q^4\ln q)$ for $n\geq 2$, $\Delta_1 = O(q^2)$ (to the order I chose to compute) and finally

$$\frac{\Delta_0}{2} \equiv \frac{1}{2\pi}\mathcal{G}_0^z(q,q)+\frac{1}{4}Y_0(\sqrt{z}q)J_0(\sqrt{z}q) = -\frac{1}{4\pi}\left(\Psi(\frac{1-z}{2})-2\ln\sqrt{z}+\ln 2\right).$$

All in all we have

$$Q(z,0) = \frac{1}{4\pi}\left(-\Psi(\frac{1-z}{2})+\ln 2+2\Psi(1)\right). \tag{29}$$

For $z<0$ the similar steps lead to the same conclusion.

There is another way how to come to (29). The Green function (25) have to satisfy the boundary condition of the point interaction (8) in every $0\neq y\in\mathbb{R}^2$. Let us take such fixed y and define

$$f_{z,\alpha}(x) := \mathcal{G}_z^{\alpha,0}(x,y) = \mathcal{G}_z^{\mathrm{ho}}(x,y) - [Q(z,0)-\alpha]^{-1}\mathcal{G}_z^{\mathrm{ho}}(x,0)\mathcal{G}_z^{\mathrm{ho}}(0,y).$$

Suppose that $f_{z,\alpha}(x) \overset{x\to 0}{\sim} \tilde{\phi}_0(z,\alpha)\ln|x|+\tilde{\phi}_1(z,\alpha)$ and $\mathcal{G}_z^{\mathrm{ho}}(x,0) \overset{x\to 0}{\sim} \phi_0(z)\ln|x|+\phi_1(z)$. The boundary condition takes form $2\pi\alpha\tilde{\phi}_0+\tilde{\phi}_1=0$, that is

$$Q(z,0)-\alpha = 2\pi\alpha\phi_0(z)+\phi_1(z), \tag{30}$$

since $\tilde{\phi}_0(z,\alpha) = -\frac{\mathcal{G}_z^{\mathrm{ho}}(0,y)}{Q(z,0)-\alpha}\phi_0(z)$ and $\tilde{\phi}_1(z,\alpha) = \mathcal{G}_z^{\mathrm{ho}}(0,y) - \frac{\mathcal{G}_z^{\mathrm{ho}}(0,y)}{Q(z,0)-\alpha}\phi_1(z)$.

Now the only thing left is to find $\mathcal{G}_z^{\mathrm{ho}}(x,0) \equiv \mathcal{G}_z^{\mathrm{ho}}(r)$. It is a solution to the differential equation

$$-(\mathcal{G}_z^{\mathrm{ho}})'' - \frac{1}{r}(\mathcal{G}_z^{\mathrm{ho}})' + \frac{1}{4}r^2\mathcal{G}_z^{\mathrm{ho}} - z\mathcal{G}_z^{\mathrm{ho}} = \delta(x) = \frac{1}{r}\delta(r).$$

For $r \neq 0$, the general solution is $\mathcal{G}_z^{\mathrm{ho}}(r) = \frac{C_1}{r}M_{\frac{z}{2},0}(\frac{r^2}{2}) + \frac{C_2}{r}W_{\frac{z}{2},0}(\frac{r^2}{2})$, where C_1, C_2 are complex constants. If we assume $z > 0$, we set $C_1 = 0$, because the first term diverges as $r \to \infty$:

$$\frac{1}{r}M_{\frac{z}{2},0}(\frac{r^2}{2}) \overset{r\to\infty}{\sim} \frac{1}{\sqrt{2}\Gamma(\frac{1-z}{2})}\left(\frac{r^2}{2}\right)^{-\frac{1+z}{2}} \mathrm{e}^{\frac{r^2}{4}}, \quad \frac{1}{r}W_{\frac{z}{2},0}(\frac{r^2}{2}) \overset{r\to\infty}{\sim} \frac{1}{\sqrt{2}}\left(\frac{r^2}{2}\right)^{\frac{z-1}{2}} \mathrm{e}^{-\frac{r^2}{4}},$$

as one can verify using asymptotic expansions (A.22), (A.23) and relations (A.26). The constant C_2 can be for example derived by comparing the expression for $\mathcal{G}_z^{\mathrm{ho}}(r)$ with the general one (27). Finally we have

$$\mathcal{G}_z^{\mathrm{ho}}(r) = \frac{\Gamma(\frac{1-z}{2})}{2\sqrt{2}\pi}\frac{1}{r}W_{\frac{z}{2},0}\left(\frac{r^2}{2}\right).$$

Using (A.24) together with (A.26) we obtain $\phi_0 = -\frac{1}{2\pi}$ and $\phi_1 = \frac{1}{4\pi}\left(-\Psi(\frac{1-z}{2}) + \ln 2 + 2\Psi(1)\right)$, substituting those expressions to (30) we come just to (29).

8 Some concrete results

With theorem 32, proposition 33 and formula (28) in hand we are able to find the spectrum of $H_\alpha(q)$. The eigenvalues, which do not belong to $\sigma(H)$, must be computed numerically by solving (26) with respect to z for fixed q and α. The sum in the Krein Q-function prescription (28) converges very fast especially for q not too large (see figure 1).

Figure 3 shows Q as a function of the spectral parameter q. Note that the graph of $Q(z,0)$ differs from the graphs of $Q(z,q), q \neq 0$ qualitatively. If the perturbation is decentralized and so the rotational symmetry is broken, the structure of the spectrum changes. Equation (26) has exactly one solution on each interval $(\lambda_n, \lambda_{n+1}), n \in \mathbb{N}_0$ instead of the interval $(\lambda_{2n}, \lambda_{2n+2})$ as for $q = 0$.

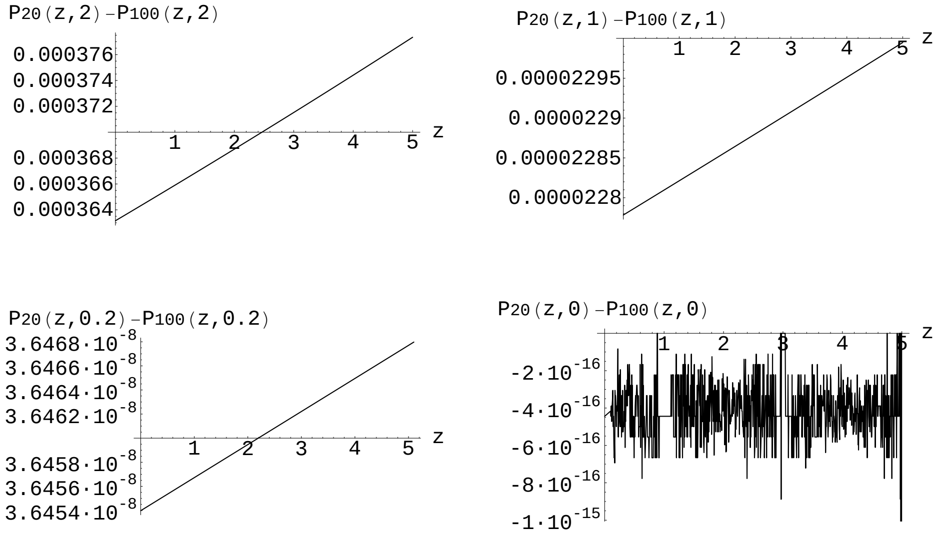
For an arbitrary fixed α let us define

$$E_n(q) := \begin{cases} \xi_n(q) & \text{for } q \neq 0 \\ \text{continuous extension} & \text{for } q = 0. \end{cases}$$

Graphs of several first functions E_n for $\alpha = 2$, $\alpha = 0$ and $\alpha = -2$ are displayed in figure 4. One can see that for $\alpha = 0$: $E_{2n-1}(0) = \lambda_{2n-1}, n \in \mathbb{N}$, however in contrast to the threedimensional case [5] $E_{2n}(0) > \lambda_{2n-1}, n \in \mathbb{N}$.

Finally let us summarize what we know about the structure of $\sigma(H_\alpha(q))$ from the viewpoint of theorem 32. If $q \neq 0$, then: $\sigma_1 = \{E_n(q)|\ n \in \mathbb{N}_0\}$, $\sigma_2 = \{\lambda_n | n \in \mathbb{N}\}$ and $\sigma_3 = \sigma_4 = \emptyset$. The case $q = 0$ is little more complicated.

Figure 1: P_m denotes the Krein Q-function evaluated up to its m^{th} summand.



We define $\mathcal{M} := \{Q(2n,0),\ n \in \mathbb{N}\}$. $\{Q(2n,0)\}_{n\in\mathbb{N}}$ is a negative decreasing sequence (see figure 2), so if $\alpha \in \mathcal{M}$, then there is just one n with the property $Q(2n,0) = \alpha$. Moreover $\lim_{n\to\infty} Q(2n,0) = -\infty$. The spectrum structure differs in dependence on the parameter α. For the convenience of a reader a transparent summary is given (see table 1).

Table 1: The spectrum of $H_\alpha(0)$

	$\alpha>Q(2,0)$	$\alpha\in(Q(2n,0),Q(2n+2,0))$	$\alpha=Q(2n,0)$
σ_1	$\{E_{2m}(0),m\in\mathbb{N}_0\}$	$E_0(0)\cup\{E_{2m-1}(0),m\in\mathbb{N},m\leq n\}$ $\cup\{E_{2m}(0),m\in\mathbb{N},m>n\}$	$E_0(0)\cup\{E_{2m-1}(0),m\in\mathbb{N},m<n\}$ $\cup\{E_{2m}(0),m\in\mathbb{N},m>n\}$
σ_2	$\{\lambda_{2m},m\in\mathbb{N}\}$	$\{\lambda_{2m},m\in\mathbb{N}\}$	$\{\lambda_{2m},m\in\mathbb{N}\}$
σ_3	$\{\lambda_{2m+1},m\in\mathbb{N}_0\}$	$\{\lambda_{2m+1},m\in\mathbb{N}_0\}$	$\{\lambda_{2m+1},m\in\mathbb{N}_0\}\backslash\lambda_{2n-1}$
σ_4	$\emptyset$	$\emptyset$	λ_{2n-1}

Figure 2: Points of the sequence $\{Q(2n,0)\}_{n\in\mathbb{N}}$

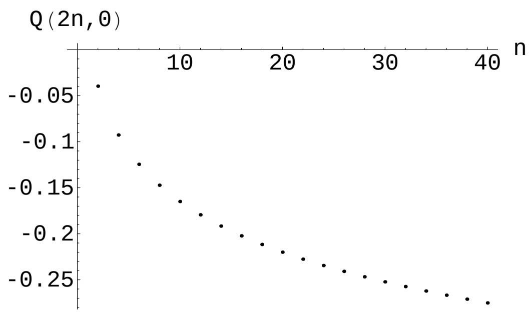


Figure 3: Krein Q-function for several values of q

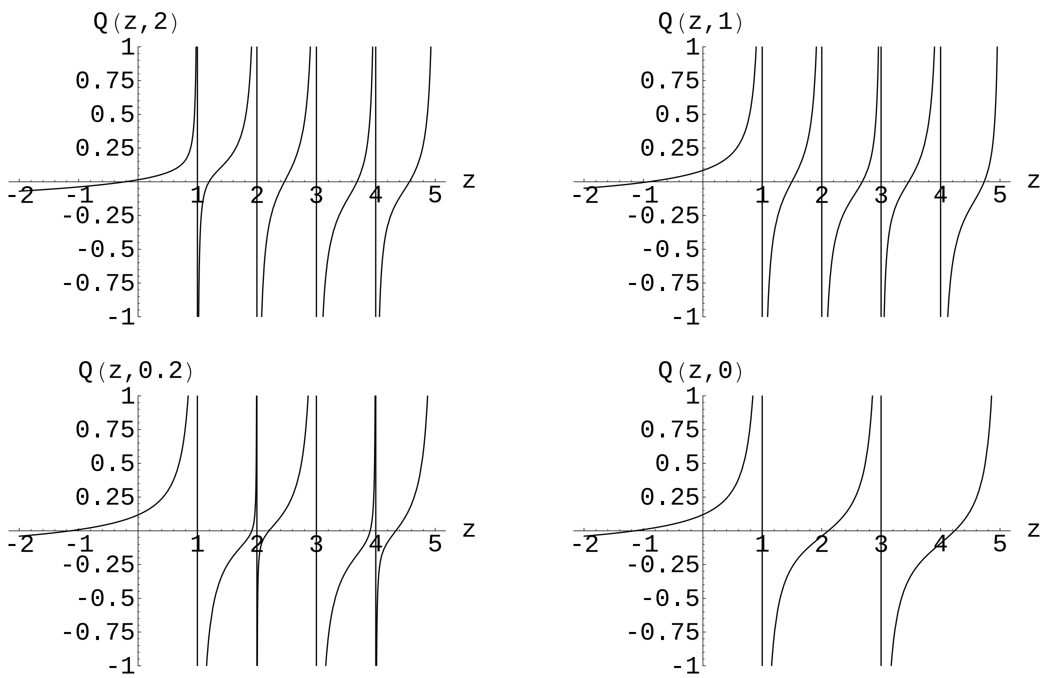
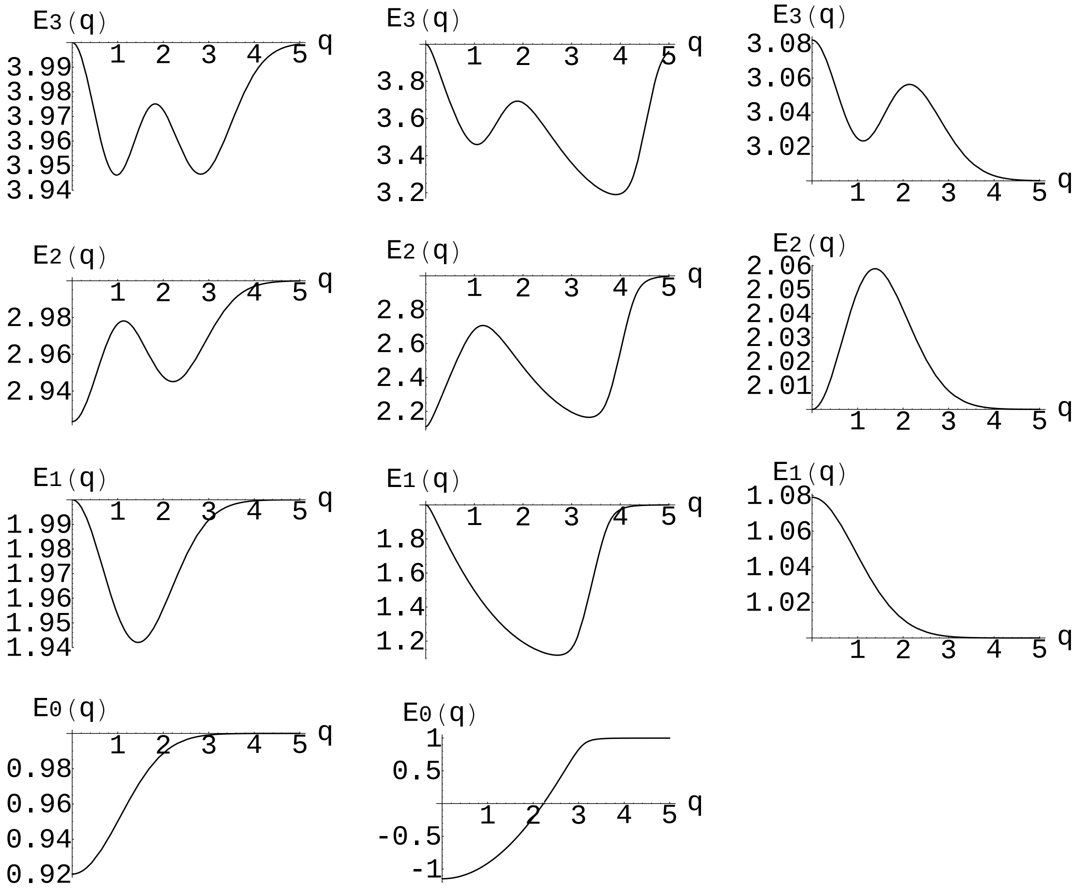


Figure 4: Energy levels E_n for $\alpha = 2,\ \alpha = 0,\ \alpha = -2$ (in the order of columns)



Appendices

A.1 Some useful lemmas

Lemma A.1 (Du Bois-Reymond) [9] *Let $f \in L^1_{loc}(J, \mathrm{d}x)$, where J is an open interval of $\mathbb{R}$, and let $(f, \varphi) = 0$ for all $\varphi \in D(J)$. Then $f = 0$ as an element of $L^1_{loc}(J, \mathrm{d}x)$.*

Proof For the proof see for example [11].

Lemma A.2 *Let $f \in D'(J)$, where J is an interval of $\mathbb{R}$, and let $f' = 0$ in $D'(J)$. Then $f = const.$ in $D'(J)$.*

Proof Let us take $\eta \in D(J)$ such that $(1, \eta) = \int_J \eta = 1$ and $\mathrm{supp}(\eta) \subset (a, b) \subset J$. For an arbitrary $\varphi \in D(J)$ define $\psi := \varphi - (1, \varphi)\eta$. ψ can be expressed as a derivative of the function $\tilde{\psi}$:

$$\tilde{\psi}(x) := \int_{-\infty}^{x} (\varphi(t) - (1, \varphi)\eta(t))\, \mathrm{d}t.$$

$\tilde{\psi}$ is evidently a smooth function and regarding that $\mathrm{supp}(\varphi) \subset (c, d) \subset J$ and the prescription for ψ, the support of $\tilde{\psi}$ is bounded too: $\mathrm{supp}(\tilde{\psi}) \subset (m, M)$, where $m := \min\{a, c\}$ and $M := \max\{b, d\}$. Thus $\tilde{\psi} \in D(J)$.

Since we suppose that $(f', \varphi) = -(f, \varphi') = 0$ for an arbitrary $\varphi \in D(J)$,

$$0 = (f, \tilde{\psi}') = (f, \psi) = (f, \varphi - (1, \varphi)\eta),$$

has to hold. So we have

$$(f, \varphi) = (f, \eta)(1, \varphi) = const.\ (1, \varphi) = (const., \varphi).$$

$\blacksquare$

Lemma A.3 *Let $f, g \in L^1_{loc}(J, \mathrm{d}x)$, where J is an interval of $\mathbb{R}$, and let $f' = g$ in $D'(J)$. Then $f \in AC(J)$ and $f'(x) = g(x)$ a.e. on J.*

[9] The lemma generally holds for $f \in L^1_{loc}(M, \mathrm{d}^n x)$, where M is an open subset of $\mathbb{R}^n$.

Proof Consider an arbitrary fixed point $a \in J$ and for $x \in J$ define

$$h(x) := \int\limits_a^x g(t)\mathrm{d}t. \tag{A.1}$$

The function h is absolutely continuous on J as a function of the integral limit and $h'(x) = g(x)$ a.e. on J.

For all $\varphi \in D(J)$ we have

$$(f', \varphi) = -\int\limits_J \tilde{f}\varphi' = (g, \varphi) = \int\limits_J \tilde{g}\varphi = \int\limits_J \tilde{h}'\varphi = \{\text{int. by parts}\} = -\int\limits_J \tilde{h}\varphi' = (h', \varphi),$$

and so $(f-h)' = 0$ in $D'(J)$. According to lemma A.2, $f = h + const.$ in $D'(J)$. Lemma A.1 then says, that $f(x) = h(x) + const.$ a.e. on J. Thus the function f as an element of $L^1_{loc}(J, \mathrm{d}x)$ can be represented by the element $h + const.$, from which follows the propositions of the lemma.

■

Corollary A.4 *Let $f, V \in L^2_{loc}(J, \mathrm{d}x) \subset L^1_{loc}(J, \mathrm{d}x)$ and $g \in L^1_{loc}(J, \mathrm{d}x)$. If $f'' + Vf = g$ in $D'(J)$, then $f \in AC^1(J)$.*

Proof $(g - Vf) \in L^1_{loc}(J, \mathrm{d}x)$. Following the same line of reasoning as in the proof of lemma A.3, we show, that f' is absolutely continuous on J: $f \in AC^1(J)$.

■

Lemma A.5 (Lagrange formula) *Let $J \equiv (a, b)$ be an interval of $\mathbb{R}$ and l a differential expression of the form*

$$l := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \quad \textit{where } V \in L^1_{loc}(J),\ V(x) \in \mathbb{R}. \tag{A.2}$$

We call the point a (b) a regular end, iff $a > -\infty$ ($b < \infty$) and V is integrable on some right (left) neighbourhood of a (b). J_l denotes the union of J and the set of regular ends. For each $x \in J_l$ we define

$$[f, g]_x := \bar{f}(x)g'(x) - \bar{f}'(x)g(x). \tag{A.3}$$

Then for an arbitrary closed interval $\langle c, d\rangle \subset J_l$ and functions $f, g \in AC^1(J_l)$ we have

$$\int\limits_c^d \big(l(\bar{f})g - \bar{f}l(g)\big)\,\mathrm{d}x = [f, g]_d - [f, g]_c. \tag{A.4}$$

Relation (A.4) is called the Lagrange formula [6].

Proof For $f, g \in AC^1(J_l)$ the mapping $x \mapsto [f,g]_x$ is absolutely continuous on J_l, and so it is differentiable a.e. on J_l:

$$[f,g]' = \bar{f}g'' - \bar{f}''g = l(\bar{f})g - \bar{f}l(g).$$

Integrating from c to d we obtain the Lagrange formula.

■

Note A.6 *An operator H on $L^2(J, \mathrm{d}x)$ with the maximal domain of definition is introduced by differential expression (A.2):*

$$Dom(H) = \left\{ f \in L^2(J, \mathrm{d}x) |\ f \in AC^1(J_l),\ l(f) \in L^2(J, \mathrm{d}x) \right\}.$$

For $f, g \in Dom(H)$, the limits $\lim_{x \to a+}[f,g]_x$ and $\lim_{x \to b-}[f,g]_x$ exist and are finite.

Proof The proof will be carried out for example for the left end point a.

Let us take an arbitrary closed interval $\langle c, d \rangle \subset J_l$ and $f, g \in Dom(H)$. Then $\big(l(\bar{f})g - \bar{f}l(g)\big) \in L^1(J, \mathrm{d}x)$ and so the finite limit

$$\lim_{c \to a+} \int_c^d \big(l(\bar{f})g - \bar{f}l(g)\big)\, \mathrm{d}x$$

exists. By the Lagrange formula (A.4) the integral above equals to $[f,g]_d - \lim_{c \to a+}[f,g]_c$, and since $[f,g]_d$ is finite, the limit $\lim_{x \to a+}[f,g]_x$ is finite too.

■

Lemma A.7 *Let $a \in \mathbb{R}$. Let $f \in L^2((a,\infty), \mathrm{d}r) \cap AC(a,\infty)$ such that $f' \in L^2((a,\infty), \mathrm{d}r)$. Then $\lim_{r \to \infty} f(r) = 0$.*

Proof Define $g := f'\bar{f} + f\bar{f}' \left\{= (|f|^2)'\right\}$. Since $g \in L^1((a,\infty), \mathrm{d}r)$, the following has to hold

$$\forall \varepsilon > 0\ \exists K \in \mathbb{R}|\ \forall r_1, r_2 > K \quad |\int_{r_2}^{r_1} g(r)| < \varepsilon. \tag{A.5}$$

Since $\int_{r_2}^{r_1} g(r)\mathrm{d}r = |f(r_1)|^2 - |f(r_2)|^2$, condition (A.5) says, that the limit $\lim_{r \to \infty} |f(r)|^2$ exists. If we require f to be in $L^2((a,\infty), \mathrm{d}, r)$, this limit is necessarily zero.

■

Lemma A.8 *Let $a \in \mathbb{R}$. Let $f \in L^2((a,\infty), \mathrm{d}r) \cap AC^1(a,\infty)$ such that $f'' \in L^2((a,\infty), \mathrm{d}r)$. Then $f' \in L^2((a,\infty), \mathrm{d}r)$ too.*

Proof Suppose that $f' \notin L^2((a,\infty), \mathrm{d}r)$. For an arbitrary $r \in (a,\infty)$ the following identity holds

$$g(r) = 2\int_a^r |f'|^2 + \int_a^r f''\bar{f} + \int_a^r f\bar{f}'' + g(a), \tag{A.6}$$

where $g := f'\bar{f} + f\bar{f}'$. $g(a)$ is finite as a consequence of the absolute continuity of f and f'. The second and the third integral on the rhs of (A.6) are finite as $r \to \infty$, whereas the first one goes to the infinity as $r \to \infty$. Thus $\lim_{r\to\infty} g(r)$ exists and

$$\lim_{r\to\infty} g(r) = \lim_{r\to\infty} \left(|f|^2\right)' = \infty,$$

which is a contradiction of the assumption $f \in L^2((a,\infty), \mathrm{d}r)$.

■

Corollary A.9 *Let V be bounded on (a,∞) for some $a \in \mathbb{R}^+$, next let $f \in L^2(\mathbb{R}^+, \mathrm{d}r) \cap AC^1(\mathbb{R}^+)$ such that $(f'' + Vf) \in L^2(\mathbb{R}^+, \mathrm{d}r)$. Then* $\lim_{r\to\infty} f(r) = \lim_{r\to\infty} f'(r) = 0$

Proof Using the estimates

$$\int_a^\infty |f'' + Vf|^2 < \int_0^\infty |f'' + Vf|^2 < \infty$$

$$\int_a^\infty |Vf|^2 < const. \int_a^\infty |f|^2 < \infty$$

we conclude $(f'' + Vf), Vf \in L^2((a,\infty), \mathrm{d}r)$ and so $f'' \in L^2((a,\infty), \mathrm{d}r)$ too. Lemmas A.7 and A.8 imply the propositions.

■

Lemma A.10 (about a closure) [10] *Let H be a symmetric operator $H \subset H^\dagger$. Let us define a set*

$$M(H) := \left\{ f \in Dom(H^\dagger) |\ \forall g \in Ker(H^\dagger \pm i)\ \langle H^\dagger g, f\rangle = \langle g, H^\dagger f\rangle \right\}.$$

Then $\bar{H} = H^\dagger \restriction M(H)$.

Proof Having $\bar{H} = H^{\dagger\dagger}$ in mind, the proof is rather simple.

$$f \in Dom(H^{\dagger\dagger}) \Leftrightarrow \exists h \in \mathscr{H} |\ \forall g \in Dom(H^\dagger)\ \langle H^\dagger g, f\rangle = \langle g, h\rangle$$

$H \subset H^\dagger$ implies $H^{\dagger\dagger} \subset H^\dagger$, so we set $h = H^\dagger f$:

$$\langle g, h\rangle = \langle g, H^{\dagger\dagger} f\rangle = \langle g, H^\dagger f\rangle,$$

[10] The name of the lemma is only internal.

which gives

$$Dom(\bar{H}) = \left\{ f \in Dom(H^\dagger) |\ \forall g \in Dom(H^\dagger)\ \langle H^\dagger g, f\rangle = \langle g, H^\dagger f\rangle \right\}. \tag{A.7}$$

Since H is a symmetric operator, the first von Neumann formula [6] can be used. Hence an arbitrary $f \in Dom(H^\dagger)$ can be uniquely decomposed as

$$f = \tilde{f} + g_+ + g_-, \quad \text{where } \tilde{f} \in Dom(\bar{H}),\ g_\pm \in Ker(H^\dagger \pm i). \tag{A.8}$$

For $g \in Dom(\bar{H})$ domain condition (A.7) is always fulfilled and it should be verified only for $g \in Ker(H^\dagger \pm i)$.

■

A.2 Cylindric functions

We call the solutions to the linear second order differential equation

$$u'' + \frac{1}{z}u' + \left(1 - \frac{\nu^2}{z^2}\right) u = 0 \tag{A.9}$$

the cylindric functions. z is a complex variable and ν is a complex parameter. A pair of linearly independent solutions consists for example of the Bessel functions of the first and the second kind: J_ν and Y_ν, which are holomorphic on $\mathbb{C}\backslash\mathbb{R}^-$. The next important pair consists of the Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$, which are the following linear combinations of the Bessel functions

$$H_\nu^{(1)} = J_\nu + iY_\nu, \quad H_\nu^{(2)} = J_\nu - iY_\nu$$

The Hankel functions are related by the formula [12]

$$H_\nu^{(1)}(-z) = -\mathrm{e}^{-\nu\pi i} H_\nu^{(2)}(z). \tag{A.10}$$

The solutions to the slightly modified differential equation (A.9):

$$u'' + \frac{1}{z}u' - \left(1 + \frac{\nu^2}{z^2}\right) u = 0$$

are so-called modified Bessel functions I_ν a K_ν.

A.2.1 Wronskians [12]

$$W\left\{J_\nu(z),\ Y_\nu(z)\right\} = \frac{2}{\pi z}$$
$$W\left\{H_\nu^{(1)}(z),\ H_\nu^{(2)}(z)\right\} = -\frac{4i}{\pi z}$$
$$W\left\{K_\nu(z),\ I_\nu(z)\right\} = \frac{1}{z}$$

A.2.2 Asymptotic expansions for large arguments $|z| \to \infty$ [12]

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + \mathrm{e}^{|\Im z|} O(|z|^{-1}) \right\} \quad \text{for } |\arg z| < \pi \tag{A.11}$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + \mathrm{e}^{|\Im z|} O(|z|^{-1}) \right\} \quad \text{for } |\arg z| < \pi$$

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \mathrm{e}^{i\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)} \quad \text{for } -\pi < \arg z < 2\pi \tag{A.12}$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} \mathrm{e}^{-i\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)} \quad \text{for } -2\pi < \arg z < \pi \tag{A.13}$$

$$I_\nu(z) \sim \frac{\mathrm{e}^z}{\sqrt{2\pi z}} \left\{1 + O(|z|^{-1})\right\} \quad \text{for } |\arg z| < \frac{1}{2}\pi$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \left\{1 + O(|z|^{-1})\right\} \quad \text{for } |\arg z| < \frac{3}{2}\pi$$

A.2.3 Asymptotic expansions for small arguments $z \to 0$ [12]

$$J_\nu(z) \sim \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)} \quad \text{for } \nu \neq -1,\ -2,\ -3, \ldots$$

$$Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim \frac{2}{\pi} \ln z \tag{A.14}$$

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -\frac{1}{\pi}\Gamma(\nu) \left(\frac{1}{2}z\right)^{-\nu} \quad \text{for } \Re\nu > 0$$

$$I_\nu(z) \sim \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)} \quad \text{for } \nu \neq -1,\ -2,\ -3, \ldots$$

$$K_0(z) \sim -\ln z$$

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu) \left(\frac{1}{2}z\right)^{-\nu} \quad \text{for } \Re\nu > 0$$

Note A.11 *Using the software Maple 9 we obtain next terms of the expansions*

$$H_0^{(1)}(z) \overset{z\to 0}{\sim} 1 + \frac{2i}{\pi}\left(\ln\frac{z}{2} - \Psi(1)\right) + O(z^2 \ln z) \tag{A.15}$$

$$H_1^{(1)}(z) \overset{z\to 0}{\sim} -\frac{2i}{\pi z} + \frac{1}{2\pi}\left(\pi - i - 2i\Psi(1) + 2i\ln\frac{z}{2}\right) z + O(z^3 \ln z)$$

where $\Psi(z) := \frac{\mathrm{d}}{\mathrm{d}z} \ln\Gamma(z)$ *denotes the digamma function.* $(-\Psi(1))$ *equals to the Euler constant* $\gamma \doteq 0.5772$*.*

A.2.4 Relations between solutions [12]

$$K_\nu(z) = \frac{1}{2}\pi i \mathrm{e}^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(iz) \quad \text{for } -\pi < \arg z \le \frac{1}{2}\pi \tag{A.16}$$

$$K_\nu(z) = -\frac{1}{2}\pi i \mathrm{e}^{-\frac{1}{2}\nu\pi i} H_\nu^{(2)}(-iz) \quad \text{for } -\frac{1}{2}\pi < \arg z \le \pi \tag{A.17}$$

A.2.5 Integral representations [12]

$$J_m(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\alpha - m\alpha)\mathrm{d}\alpha \quad \text{for } \Re z > 0,\ m \in \mathbb{Z} \tag{A.18}$$

$$K_0(Z|x|) = \int_0^\infty \frac{tJ_0(t|x|)}{t^2 + Z^2}\mathrm{d}t \quad \text{for } \Re Z > 0 \tag{A.19}$$

A.2.6 Graf formula [12]

$$\mathscr{C}_\nu(w) {\cos \atop \sin}(\nu\chi) = \sum_{n=-\infty}^{\infty} \mathscr{C}_{\nu+n}(u) J_n(v) {\cos \atop \sin}(n\alpha), \quad |v\mathrm{e}^{\pm i\alpha}| < |u|, \tag{A.20}$$

where $\mathscr{C}_\nu$ denotes J_ν, Y_ν or any linear combinations of these functions, and where

$$w = \sqrt{u^2 + v^2 - 2uv\cos\alpha},\ u - v\cos\alpha = w\cos\chi,\ v\sin\alpha = w\sin\chi.$$

A.3 Confluent hypergeometric functions

A.3.1 Kummer functions

The Kummer functions $M(a,b,z), W(a,b,z)$ are independent solutions to the equation

$$zw'' + (b-z)w' - aw = 0 :$$

$$M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \ldots + \frac{(a)_n z^n}{(b)_n n!} + \ldots \tag{A.21}$$

$$U(a,b,z) = \frac{\pi}{\sin(\pi b)}\left(\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right),$$

where $(a)_n := a(a+1)(a+2)\ldots(a+n-1), (a)_0 := 1$ are so-called Pochhammer symbols.

Series (A.21) for $M(a,b,z)$ is absolutely convergent for all values of a,b and x, real or complex, excluding $b=-n|\ n\in\mathbb{N}_0$, however $U(a,b,z)$ can be defined as a limit $b\to\pm n$ [13]:

$$\begin{aligned}U(a,1+n,z) &= \frac{(-)^{n-1}}{n!\Gamma(a-n)}\sum_{r=0}^{\infty}\frac{(a)_r z^r}{(1+n)_r r!}\left[\ln z+\Psi(a+r)-\Psi(1+r)-\Psi(1+n+r)\right]\\ &+\frac{(n-1)!}{\Gamma(a)}\sum_{r=0}^{n-1}\frac{(a-n)_r x^{r-n}}{(1-n)_r r!}\\ U(a,1-n,z) &= z^n U(a+n,1+n,z),\end{aligned}$$

where $n\in\mathbb{N}_0$.

Asymptotic expansions for $|z|\to\infty$ [12]

$$M(a,b,z)=\begin{cases}\frac{\Gamma(b)}{\Gamma(a)}\mathrm{e}^z z^{a-b}[1+O(|z|^{-1})] & \text{for } \Re z>0\\ \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}[1+O(|z|^{-1})] & \text{for } \Re z<0\end{cases} \tag{A.22}$$

$$U(a,b,z)=z^{-a}[1+O(|z|^{-1})] \quad \text{for } \Re z\to\infty \tag{A.23}$$

Asymptotic expansions for $|z|\to 0$ [12]

$$U(a,b,z)=\begin{cases}\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}+O(|z|^{\Re b-2}) & \text{for } \Re b\ge 2, b\ne 2\\ \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}+O(|\ln z|) & \text{for } b=2\\ \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}+O(1) & \text{for } 1<\Re b<2\\ -\frac{1}{\Gamma(a)}[\ln z+\Psi(a)]+O(|z\ln z|) & \text{for } b=1\\ \frac{\Gamma(1-b)}{\Gamma(1+a-b)}+O(|z|^{1-\Re b}) & \text{for } 0<\Re b<1\\ \frac{1}{\Gamma(1+a)}+O(|z\ln z|) & \text{for } b=0\\ \frac{\Gamma(1-b)}{\Gamma(1+a-b)}+O(|z|) & \text{for } \Re b\le 0, b\ne 0\end{cases} \tag{A.24}$$

A.3.2 Whittaker functions

The Whittaker functions $M_{a,b}(z), W_{a,b}(z)$ are independent solutions to the equation

$$w''+\left[-\frac{1}{4}+\frac{a}{z}+\frac{1/4-b^2}{z^2}\right]w=0: \tag{A.25}$$

$$\begin{aligned}M_{a,b}(z) &= \mathrm{e}^{-\frac{z}{2}}z^{\frac{1}{2}+b}M(\frac{1}{2}+b-a,1+2b,z)\\ W_{a,b}(z) &= \mathrm{e}^{-\frac{z}{2}}z^{\frac{1}{2}+b}U(\frac{1}{2}+b-a,1+2b,z)\end{aligned} \tag{A.26}$$

for $-\pi<\arg z\le\pi$.

Special cases [12]

$$\begin{aligned} M_{0,b}(z) &= \Gamma(1+b)(4i)^b \sqrt{z} J_b(\frac{z}{2i}) \\ W_{0,b}(z) &= \frac{\sqrt{\pi}}{2}(i)^{-b} \mathrm{e}^{i\pi(b+\frac{1}{2})} \sqrt{z} H_b^{(1)}(i\frac{z}{2}) \end{aligned} \tag{A.27}$$

A.4 Krein formula

Theorem A.12 (Krein formula) *Let H be a closed, symmetric and densely defined operator on a Hilbert space $\mathscr{H}$. For an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$ and a pair of selfadjoint extensions H_α, H_∞ of the operator H, let us denote the isometry from $Ker(H^\dagger - z)$ on $Ker(H^\dagger - \bar{z})$ determining the extension H_α (H_∞) by $V_\alpha(z)$ ($V_\infty(z)$), the orthogonal projection operator onto $Ker(H^\dagger - z)$ by $P(z)$ and the immersion mapping of the subspace $Ker(H^\dagger - z)$ to $\mathscr{H}$ by $P^*(z)$. Then the resolvents of H_α and H_∞ in z are related as follows*

$$(H_\alpha - z)^{-1} = (H_\infty - z)^{-1} + P^*(z) \frac{V_\alpha(\bar{z}) - V_\infty(\bar{z})}{\bar{z} - z} P(\bar{z})$$

Proof We prove the formula on each of the subspaces of the decomposition $\mathscr{H} = Ker(H^\dagger - \bar{z}) \oplus \big(Ker(H^\dagger - \bar{z})\big)^\perp$ separately.

At first let

$$f \in \Big(Ker(H^\dagger - \bar{z})\Big)^\perp = \Big((Ran(H - z))^\perp\Big)^\perp = \overline{Ran(H - z)} = Ran(H - z).$$

The last equality holds, because H is a closed, symmetric operator and $z \in \mathbb{C} \setminus \mathbb{R}$ [6]. Thus $g \in Dom(H)$ exists such that $(H - z)g = f$, and since $H \subset H_{\alpha,\infty}$, we have

$$(H_\alpha - z)g = (H_\infty - z)g = f.$$

Since $z \in \mathbb{C} \setminus \mathbb{R} \subset \varrho(H_{\alpha,\infty})$, we can invert this equality to the Krein formula for $f \in \big(Ker(H^\dagger - \bar{z})\big)^\perp$:

$$(H_\alpha - z)^{-1} f = (H_\infty - z)^{-1} f$$

For $f \in Ker(H^\dagger - \bar{z})$ we make use of the second von Neumann formula [6]

$$H_{\alpha,\infty}(f + V_{\alpha,\infty}(\bar{z})f) = \bar{z} f + z V_{\alpha,\infty}(\bar{z}) f,$$

which can be rearranged to

$$(H_{\alpha,\infty} - z)^{-1} f = \frac{f + V_{\alpha,\infty}(\bar{z}) f}{\bar{z} - z}.$$

Therefore for $f \in Ker(H^\dagger - \bar{z})$ we have:

$$(H_\alpha - z)^{-1} = (H_\infty - z)^{-1} + \frac{V_\alpha(\bar{z}) - V_\infty(\bar{z})}{\bar{z} - z}$$

■

Note A.13 *If* $\dim\left(Ker(H^\dagger - z)\right) = \dim\left(Ker(H^\dagger - \bar{z})\right) = 1$, *the Krein formula may be written as follows*

$$(H_\alpha - z)^{-1} = (H_\infty - z)^{-1} + \lambda(z)\langle\phi(\bar{z}), .\rangle\phi(z),$$

where $0 \neq \phi(z) \in Ker(H^\dagger - z)$ *and* $\lambda(z) \in \mathbb{C}$.

A.5 Sobolev spaces

Let us introduce the standard multiindex notation. By multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ we mean the ordered n-tuple of non-negative integers $\alpha_1, \alpha_2 \dots, \alpha_n$. Symbols $|\alpha|$, x^α a D^α stand for:

$$\begin{aligned} |\alpha| &:= \sum_{i=1}^{n} \alpha_i \\ x^\alpha &:= x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{for } x = (x_1, \dots, x_n) \\ D^\alpha &:= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \end{aligned}$$

Definition A.14 *A tempered distribution* $f \in \mathscr{S}'(\mathbb{R}^n)$ *lies in the* m^{th} *Sobolev space* $H^m(\mathbb{R}^n)$ $(m \in \mathbb{N}_0)$[11], *iff* $\mathscr{F}f$ *is measurable and*

$$\|f\|^2_{H^m(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^m |(\mathscr{F}f)(\xi)|^2 \mathrm{d}^n\xi < \infty.$$

Note A.15 *According to the Plancherel theorem [9](which says that the Fourier transform can be uniquely extended to an unitary mapping from* $L^2(\mathbb{R}^n, \mathrm{d}^n x)$ *on* $L^2(\mathbb{R}^n, \mathrm{d}^n\xi)$*),* $H^0 \equiv L^2(\mathbb{R}^n, \mathrm{d}^n x)$.

Note A.16 *The spaces* $H^m(\mathbb{R}^n)$ *are complete.*

Proof Let us consider an arbitrary Cauchy sequence $\{f_l\} \subset H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, \mathrm{d}x)$. According to the unitarity of the Fourier-Plancherel operator $\{\mathscr{F}f_l\} \subset \mathscr{F}(H^m(\mathbb{R}^n)) = L^2(\mathbb{R}^n, (1 + |\xi|^2)^m \mathrm{d}^n\xi)$ is Cauchy too. $L^2(\mathbb{R}^n, (1 + |\xi|^2)^m \mathrm{d}^n\xi)$ is complete [6], i.e. $f \in H^m(\mathbb{R}^n)$ exists such that

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} |(\mathscr{F}f_l - \mathscr{F}f)(\xi)|^2 (1 + |\xi|^2)^m \mathrm{d}^n\xi = 0.$$

Thus the sequence $\{f_l\}$ converges in $H^m(\mathbb{R}^n)$.

■

Theorem A.17 $f \in H^m(\mathbb{R}^n)$, *iff* $D^\alpha f \in L^2(\mathbb{R}^n, \mathrm{d}^n x)$ *for all multiindices* α, $|\alpha| \leq m$, *where* D^α *denotes a generalized derivative.*

[11] $m \in \mathbb{R}$ in general

Proof Let $f \in H^m(\mathbb{R}^n)$, then by the definition $\xi^\alpha(\mathscr{F} f) \in L^2(\mathbb{R}^n, \mathrm{d}^n\xi)$ for $|\alpha| \leq m$. Since $\mathscr{F}(D^\alpha f) = (-i\xi)^\alpha \mathscr{F} f$ for $f \in \mathscr{S}'(\mathbb{R}^n)$, $D^\alpha f \in L^2(\mathbb{R}^n, \mathrm{d}^n x)$ in accordance with the Plancherel theorem.

Now let $D^\alpha f \in L^2(\mathbb{R}^n, \mathrm{d}^n x)$ for $|\alpha| \leq m$. Inverting the previous part of the proof we show that $\xi^\alpha(\mathscr{F} f) \in L^2(\mathbb{R}^n, \mathrm{d}^n\xi)$, hence $\|f\|_{H^m(\mathbb{R}^n)} < \infty$.

■

Lemma A.18 *Let $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n, \mathrm{d}^n x)$. If*

$$\langle \nabla f, \nabla \varphi \rangle = \langle g, \varphi \rangle \qquad \text{for } \forall \varphi \in H^1(\mathbb{R}^n), \tag{A.28}$$

then $f \in H^2(\mathbb{R}^n)$.

Proof Using the unitarity of the Fourier-Plancherel operator $\mathscr{F}$, condition (A.28) takes form

$$\langle \mathscr{F}(\nabla f), \mathscr{F}(\nabla \varphi) \rangle = \langle \mathscr{F} g, \mathscr{F} \varphi \rangle \qquad \text{for } \forall \varphi \in H^1(\mathbb{R}^n)$$

thus

$$\langle \xi(\mathscr{F} f), \xi(\mathscr{F} \varphi) \rangle = \langle |\xi|^2 \mathscr{F} f, \mathscr{F} \varphi \rangle = \langle \mathscr{F} g, \mathscr{F} \varphi \rangle \qquad \text{for } \forall \varphi \in H^1(\mathbb{R}^n).$$

Since $\mathscr{F}(H^1(\mathbb{R}^n)) = L^2(\mathbb{R}^n, (1 + |\xi|^2)\mathrm{d}^n\xi)$ is a dense subspace of $L^2(\mathbb{R}^n, \mathrm{d}^n\xi)$, $|\xi|^2 \mathscr{F} f = \mathscr{F} g$, thus

$$(1 + |\xi|^2)\mathscr{F} f = \mathscr{F} g + \mathscr{F} f \in L^2(\mathbb{R}^n, \mathrm{d}^n\xi),$$

and so $f \in H^2(\mathbb{R}^n)$.

■

Definition A.19 *Let Ω be an open subset of $\mathbb{R}^n$. The local Sobolev space $H^m(\Omega)$ is the set of tempered distributions $f \in D'(\Omega)$ so that $\varphi f \in H^m(\mathbb{R}^n)$ for all $\varphi \in D(\Omega)$.*

A.6 The form representation theorem

Theorem A.20 (representation theorem) *Let s be a densely defined, closed and symmetric sesquilinear form on $\mathscr{H}$ bounded from below. Then there exists a selfadjoint operator A such that*

i. $Dom(A) \subset Dom(s)$, and for all $f \in Dom(s)$ and $g \in Dom(A)$

$$s(f, g) = \langle f, Ag \rangle$$

ii. $Dom(A)$ is a core of s, i.e. $\overline{s \restriction Dom(A)} = s$

iii. if $f \in Dom(s)$ and $h \in \mathscr{H}$ exist such that for all $g \in Dom(s)$:

$$s(f, g) = \langle h, g \rangle,$$

then $f \in Dom(A)$ and $h = Af$.

The selfadjoint operator A is uniquely determined by condition (i).

Proof For the proof see [6], [8] or [14].

A.7 The free hamiltonian Green function

The hamiltonian of a free particle with the mass $m = \frac{1}{2}$ in the plane is

$$H_\infty = -\Delta, \quad Dom(H_\infty) = H^2(\mathbb{R}^2).$$

The integral kernel of $(H_\infty - z)^{-1}$ (i.e. the Green function of H_∞) can be found using the Fourier transformation $\mathscr{F}$. The Green function has to read a generalized equation

$$(-\Delta - z)\mathcal{G}_z(x-y) = \delta(x-y) \quad \text{for a fixed } y \in \mathbb{R}^2,$$

thus applying the inverse Fourier transform:

$$\mathscr{F}^{-1}(-\Delta - z)\mathscr{F}\mathscr{F}^{-1}(\mathcal{G}_z(x-y)) = (k^2 - z)\mathscr{F}^{-1}(\mathcal{G}_z(x-y)) = \frac{1}{2\pi}.$$

Hence for $\mathcal{G}_z(x-y)$ we have

$$\mathcal{G}_z(x-y) = \frac{1}{(2\pi)^2}\int\limits_{\mathbb{R}^2} \frac{1}{k^2 - z}\mathrm{e}^{i\langle k, x-y\rangle}\mathrm{d}^2k = \left\{sub.: \;\; k = (\sqrt{v}\cos\varphi, \sqrt{v}\sin\varphi)\right\}$$

$$= \frac{1}{2(2\pi)^2}\int\limits_0^\infty \mathrm{d}v \int\limits_0^{2\pi} \mathrm{d}\varphi \frac{1}{v-z}\mathrm{e}^{i\sqrt{v}\langle(\cos\varphi, \sin\varphi), x-y\rangle} = \frac{1}{2(2\pi)^2}\int\limits_0^\infty \mathrm{d}v \int\limits_{-\pi}^{\pi} \mathrm{d}\alpha \frac{1}{v-z}\mathrm{e}^{i\sqrt{v}|x-y|\sin\alpha}$$

$$= \frac{1}{(2\pi)^2}\int\limits_0^\infty \mathrm{d}v \; \frac{1}{v-z}\int\limits_0^\pi \mathrm{d}\alpha \cos(\sqrt{v}|x-y|\sin\alpha) = \{(A.18)\} = \frac{1}{4\pi}\int\limits_0^\infty \frac{1}{v-z}J_0(\sqrt{v}|x-y|)\mathrm{d}v$$

$$= \left\{sub.: \;\; \sqrt{v} = t\right\} = \frac{1}{2\pi}\int\limits_0^\infty \frac{t}{t^2 - z}J_0(t|x-y|)\mathrm{d}t.$$

For $z \in \mathbb{C}\backslash\mathbb{R}$ the last integral can be evaluated using formula (A.19), in which we substitute $Z = -i\sqrt{z}$ ($\Im\sqrt{z} > 0$, and so $\Re Z > 0$):

$$\mathcal{G}_z(x-y) = \frac{i}{4}H_0^{(1)}(\sqrt{z}|x-y|), \quad \Im\sqrt{z} > 0.$$

For $z > 0$ the last integral diverges, $(H_\infty - z)^{-1} \notin \mathscr{B}(\mathscr{H})$, i.e. the resolvent does not exist. However this integral can be regularized adding (or subtracting) a small imaginary number $i\varepsilon|\; \varepsilon > 0$ to the denominator:

$$\mathcal{G}_z(x-y)^{\pm} := \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int\limits_0^\infty \frac{t}{t^2 - z \mp i\varepsilon}J_0(t|x-y|)\mathrm{d}t = \{(A.19)\} = \frac{1}{2\pi}K_0\left(-i\sqrt{z \pm i\varepsilon}|x-y|\right)$$

$$= \{(A.16),\ (A.17)\} = \pm\frac{i}{4}H_0^{\substack{(1)\\(2)}}(\sqrt{z}|x-y|) = \frac{i}{4}H_0^{(1)}(\pm\sqrt{z}|x-y|),$$

where

$$H_0^{(1)}(-z) := \lim_{\varepsilon\to 0+} H_0^{(1)}(-z + i\varepsilon) \quad \text{for } z > 0. \tag{A.29}$$

A.8 Review of scattering theory

In this section we summarize some basic notions of the time-independent and the time-dependent scattering theory. The most of the results are formulated for the two-dimensional case. For more detailed treatment see [15], [16], [17] or [6].

A.8.1 Time-independent scattering theory

Consider a potential V of a finite range, i.e. $V(x) = 0$ for $|x| > a \in \mathbb{R}^+$. The wave function of an incoming particle, which is localized out of the potential range in a fixed time t_0, denote by ψ_0. Long before the particle comes near the region $|x| \leq a$, it is essentially governed by the free hamiltonian H_∞:

$$i\psi_0'(t) = H_\infty \psi_0(t), \quad \psi_0(t_0) = \psi_0, \tag{A.30}$$

where $'$ stands for the time derivative in the strong sense:

$$\lim_{h\to 0} \|\frac{1}{h}\left(\psi_0(t+h) - \psi_0(t)\right) - \psi_0'(t)\| = 0.$$

In this region the solution to (A.30) may be decomposed as follows

$$\psi_0(x,t) = \int_{\mathbb{R}^2} \varphi(k) \mathrm{e}^{i\langle k,x\rangle - ik^2(t-t_0)} \mathrm{d}^2 k, \quad \text{where } \varphi(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{e}^{-i\langle k,x\rangle} \psi_0(x) \mathrm{d}^2 x.$$

Within the range of the potential, the generalized eigenfunctions $\mathrm{e}^{i\langle k,x\rangle}$ of the hamiltonian H_∞ must be replaced by the generalized eigenfunctions F_k of the hamiltonian $-\Delta + V$:

$$(-\Delta + V)F_k^\pm = k^2 F_k^\pm.[12] \tag{A.31}$$

Then the time evolution of our particle is given by

$$\psi_+(x,t) = \int_{\mathbb{R}^2} \varphi_+(k) F_k^+(x) \mathrm{e}^{-ik^2(t-t_0)} \mathrm{d}^2 k$$
$$\psi_-(x,t) = \int_{\mathbb{R}^2} \varphi_-(k) F_k^-(x) \mathrm{e}^{-ik^2(t-t_0)} \mathrm{d}^2 k,$$

where the continuous coefficients $\varphi_\pm$ are determined by the initial condition.

The Schrödinger equation (A.31) can be written down in the integral form (the Lippmann-Schwinger equation)

$$F_k^\pm(x) = \mathrm{e}^{i\langle k,x\rangle} + \int_{\mathbb{R}^2} \mathcal{G}_k^\pm(x-y) V(y) F_k^\pm(y) \mathrm{d}^2 y =: \mathrm{e}^{i\langle k,x\rangle} + G_\pm(k;x), \tag{A.32}$$

[12] The solutions $F^\pm$ have specified asymptotic expansions (A.34), which will be determined below.

where $\mathcal{G}_k^{\pm}$ is so-called retarded (advanced respectively) Green function of the Schrödinger equation.

$$\mathcal{G}_k^{\pm}(x-y) = \frac{i}{4} H_0^{(1)}(\pm|k||x-y|)$$

For a fixed y and $|x| =: r \to \infty$, the functions $\mathcal{G}_z^{\pm}$ can be expanded using (A.12):

$$\mathcal{G}_k^{\pm}(x-y) \overset{r\to\infty}{\sim} \frac{1}{2\sqrt{2\pi}} \mathrm{e}^{i\frac{\pi}{4}} \mathrm{e}^{\pm i|k||x-y|} \frac{1}{\sqrt{\pm|k||x-y|}} + O\left(\frac{1}{(|k||x-y|)^{\frac{3}{2}}}\right), \tag{A.33}$$

and since $|x-y| \overset{r\to\infty}{\sim} r - \langle n, y\rangle$, where $n := \frac{x}{r}$, we have

$$F_k^{\pm}(x) \overset{r\to\infty}{\sim} \mathrm{e}^{i\langle k,x\rangle} + f^{\pm}(k',k) \frac{\mathrm{e}^{\pm i|k|r}}{\sqrt{r}}, \quad \text{where} \tag{A.34}$$

$$f^{\pm}(k',k) = \frac{1}{2\sqrt{\pm 2\pi|k|}} \mathrm{e}^{i\frac{\pi}{4}} \int\limits_{\mathbb{R}^2} \mathrm{e}^{\mp i\langle k',y\rangle} V(y) F_k^{\pm}(y) \mathrm{d}^2 y, \quad k' := |k|\frac{x}{r} = |k|n.$$

Setting $k = l\omega,\ l = |k|$ and $x = r(\cos\Omega, \sin\Omega)$, one can see that $f^{\pm}$ are in fact functions only of those variables: l, ω, Ω. $f^{\pm}(k',k) \equiv f^{\pm}(l,\omega,\Omega)$.

For a rotationally symmetric potential, we may consider $k = (l, 0)$ without loss of generality. Expansion (A.34) is of the form

$$F_k^{\pm}(r,\Omega) \overset{r\to\infty}{\sim} \mathrm{e}^{ilr\cos\Omega} + f^{\pm}(l,\Omega) \frac{\mathrm{e}^{\pm ilr}}{\sqrt{r}}.$$

.

The function $f(k',k) \equiv f^{+}(k',k)$ (so-called scattering amplitude) is of the big importance, since the differential scattering cross-section can be computed as follows

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(k',k)|^2.$$

Note A.21 *In the two-dimensional scattering theory we take use of the following expansion:*

$$\mathrm{e}^{ir\cos\Omega} = \sum_{m=-\infty}^{\infty} i^m J_m(r) \mathrm{e}^{im\Omega} \quad \textit{for } r > 0. \tag{A.35}$$

Proof Using (A.18) we obtain

$$\mathrm{e}^{ir\sin\Omega} = \sum_{m=-\infty}^{\infty} J_m(r) \mathrm{e}^{im\Omega},$$

and hence

$$\mathrm{e}^{ir\cos\Omega} = \mathrm{e}^{ir\sin(\Omega+\frac{\pi}{2})} = \sum_{m=-\infty}^{\infty} i^m J_m(r) \mathrm{e}^{im\Omega}.$$

■

Note A.22 *Using the asymptotic expansions of the Bessel functions (A.11) we have*

$$\mathrm{e}^{ir\cos\Omega} \overset{r\to\infty}{\sim} \frac{1}{\sqrt{2\pi r}} \sum_{m=-\infty}^{\infty} \left(\mathrm{e}^{-i\frac{\pi}{4}} \mathrm{e}^{ir} + (-)^m \mathrm{e}^{i\frac{\pi}{4}} \mathrm{e}^{-ir}\right) \mathrm{e}^{im\Omega}. \tag{A.36}$$

A.8.2 Time-dependent scattering theory

Again consider a potential of a finite range. The time evolution of an incoming particle ψ is governed by a total hamiltonian H:

$$\psi(t) = \mathrm{e}^{-itH}\psi(0).$$

Before reaching the interaction region the particle acts like a free particle, thus there should be a state g_- governed by the free hamiltonian H_∞ satisfying

$$\lim_{t\to-\infty} \|\psi(t) - g_-(t)\| = \lim_{t\to-\infty} \|\mathrm{e}^{-itH}\psi(0) - \mathrm{e}^{-itH_\infty}g_-(0)\| = 0. \tag{A.37}$$

If it exists, we call it the incoming asymptotic state for ψ. Similarly, the outgoing asymptotic state g_+ may be introduced:

$$\lim_{t\to\infty} \|\psi(t) - g_+(t)\| = \lim_{t\to\infty} \|\mathrm{e}^{-itH}\psi(0) - \mathrm{e}^{-itH_\infty}g_+(0)\| = 0. \tag{A.38}$$

If ψ possesses both incoming and outgoing asymptotic states, we call it a scattering state. Conditions (A.37) and (A.38) may be formulated using so-called wave operators $W_\pm$

$$\psi = W_\pm g_\pm, \quad \text{where } W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-itH_\infty}.$$

Asymptotic states $g_\pm$ of a scattering state ψ are related by a scattering operator S:

$$g_+(0) = W_+^{-1}W_-g_-(0) =: Sg_-(0).$$

A.8.3 Relationship between time-independent and time-dependent scattering theory

Lemma A.23 *Let $\varphi \in D(\mathbb{R}^+ \times S^1)$. Let us define a closed ball $B_K := \left\{x \in \mathbb{R}^2 |\ |x| \le K\right\}$. If for the functions $G_\pm$ (see (A.32)) holds $G_\pm, \frac{\partial}{\partial l}G_\pm \in L^1_{loc}(\mathbb{R}^+ \times S^1, \mathrm{d}l\mathrm{d}\omega)$ for all $x \in \mathbb{R}^2$, if $\sqrt{r}G_\pm, \sqrt{r}\frac{\partial}{\partial l}G_\pm$ are continuous as functions of x on $\mathbb{R}^2$ for $k \in B_b \setminus B_a$, and moreover if the functions $\left(f^\pm, \frac{\partial}{\partial l}f^\pm, \frac{\partial^2}{\partial l^2}f^\pm\right)(l,\omega,\Omega)$ are essentially bounded on $(B_b \setminus B_a) \times \langle 0, 2\pi)$, where $\infty > b > a > 0$, then the wave function*

$$g(x) := \int_0^\infty \int_{S^1} \mathrm{e}^{il\langle\omega,x\rangle}\varphi(l,\omega)\mathrm{d}\omega l\mathrm{d}l \tag{A.39}$$

describes the incoming (outgoing) asymptotic state of the state ψ^- (ψ^+, respectively):

$$\psi^\pm(x) = \int_0^\infty \int_{S^1} F_k^\mp(x)\varphi(l,\omega)\mathrm{d}\omega l\mathrm{d}l.$$

Proof We have to prove that

$$\lim_{t\to\pm\infty} \|\mathrm{e}^{-itH_\infty}g - \mathrm{e}^{-itH}\psi^{\pm}\| = 0.$$

The proof will be carried out for the limit in $-\infty$. A proof for the other limit is similar.

On an arbitrary closed ball B_K we estimate the function

$$\Delta(x) := \left(\mathrm{e}^{-itH_\infty}g - \mathrm{e}^{-itH}\psi^{-}\right)(x) = \int_0^\infty \mathrm{e}^{-itl^2} \int_{S^1} \varphi(l,\omega)G_+(l,\omega;x)\mathrm{d}\omega l\mathrm{d}l = \{\text{int. by parts}\}$$

$$= \frac{1}{2it}\int_0^\infty \mathrm{e}^{-itl^2} \int_{S^1} \left(\frac{\partial\varphi}{\partial l}G_+ + \varphi\frac{\partial G_+}{\partial l}\right)(l,\omega;x)\mathrm{d}\omega\mathrm{d}l$$

by a continuous non-negative function C_K:

$$|\Delta(x)| \leq \frac{1}{2|t|}\int_a^b \int_{S^1} \left|\frac{\partial\varphi}{\partial l}G_+ + \varphi\frac{\partial G_+}{\partial l}\right|(l,\omega;x)\mathrm{d}\omega\mathrm{d}l < \frac{1}{|t|}\frac{C_K(x)}{\sqrt{r}},$$

where $\mathrm{supp}\left[\int_{S^1}\varphi(l,\omega)G_+(l,\omega;x)\mathrm{d}\omega\right](l) \subset (a,b)$.

$$\|\Delta\|^2 = I_1 + I_2 = \int_0^{K_0}\int_0^{2\pi} |\Delta(r,\Omega)|^2\mathrm{d}\Omega r\mathrm{d}r + \int_{K_0}^{\infty}\int_0^{2\pi} |\Delta(r,\Omega)|^2\mathrm{d}\Omega r\mathrm{d}r,$$

where $K_0 > 0$ such that $\displaystyle\sup_{x\in\mathbb{R}^2\setminus B(K_0)} \left|G_+(k;x) - f^+(k',k)\frac{\mathrm{e}^{ilr}}{\sqrt{r}}\right| < \frac{const.}{(lr)^{\frac{3}{2}}}$.

The existence of K_0 with the properties above is a consequence of the asymptotic expansion (A.33).

C_{K_0} is non-negative and continuous on the compact set B_{K_0} (and hence it is bounded on it), thus we can estimate:

$$I_1 < \frac{1}{|t|}\int_0^{K_0}\int_0^{2\pi} C_{K_0}(r,\Omega)^2\mathrm{d}\Omega\mathrm{d}r \overset{t\to-\infty}{\longrightarrow} 0.$$

We substitute (A.34) for ψ^- to I_2, latter we will show that the residue of the order $O\left(\frac{1}{(lr)^{\frac{3}{2}}}\right)$ does not contribute to I_2 as $t \to -\infty$.

Let's estimate the function

$$\tilde{\Delta}(r,\Omega) := \left(\mathrm{e}^{-itH_\infty}g - \mathrm{e}^{-itH}\psi^{-}_{(r\to\infty)}\right)(r,\Omega) = \int_0^\infty \mathrm{e}^{-itl^2}\int_{S^1}\varphi(l,\omega)f^+(l,\omega,\Omega)\frac{\mathrm{e}^{ilr}}{\sqrt{r}}\mathrm{d}\omega l\mathrm{d}l.$$

The function $\varphi(l,\omega)f^+(l,\omega,\Omega)$ can be decomposed to the Fourier series

$$\varphi(l,\omega)f^+(l,\omega,\Omega) =: s(l,\omega,\Omega) = \sum_{m=-\infty}^{\infty} s_m(l,\Omega)\mathrm{e}^{im\omega}.$$

Evaluating the angular integral we have

$$\begin{aligned}\tilde{\Delta}(r,\Omega) &= 2\pi \int_0^\infty \mathrm{e}^{-itl^2} s_0(l,\Omega)\frac{\mathrm{e}^{ilr}}{\sqrt{r}} l\mathrm{d}l = \{s(l,\Omega) := 2\pi l s_0(l,\Omega),\ \text{2x int. by parts}\}\\ &= \frac{1}{\sqrt{r}}\int_0^\infty \mathrm{e}^{-itl^2+ilr}\frac{\partial}{\partial l}\left[\frac{1}{i(r-2tl)}\frac{\partial}{\partial l}\left(\frac{1}{i(r-2tl)}s(l,\Omega)\right)\right]\mathrm{d}l\end{aligned}$$

Regarding that $\mathrm{supp}(s) \subset B_b \setminus B_a,\ 0 < a < b < \infty$, and in accordance with the assumed properties of f^+, we make the following estimate (we consider $t < 0$)

$$|\tilde{\Delta}(r,\Omega)| \leq \frac{1}{\sqrt{r}}\int_a^\infty \frac{M}{(r-2tl)^2}\mathrm{d}l = -\frac{1}{\sqrt{r}}\frac{1}{2t}\frac{M}{r-2ta},$$

where $0 < M = const.$ Thus for a contribution to I_2 we have

$$\int_K^\infty\int_0^{2\pi} |\tilde{\Delta}(r,\Omega)|^2 \mathrm{d}\Omega r\mathrm{d}r \leq \frac{\pi M^2}{2t^2}\int_0^\infty \frac{1}{(r-2ta)^2}\mathrm{d}r = -\frac{\pi M^2}{4at^3} \overset{t\to-\infty}{\longrightarrow} 0.$$

Let us denote the residue of Δ by $\tilde{\tilde{\Delta}}$

$$\begin{aligned}\tilde{\tilde{\Delta}}(r) &:= \int_0^\infty \mathrm{e}^{-itl^2}\int_{S^1}\varphi(l,\omega)\frac{1}{(lr)^{\frac{3}{2}}}\mathrm{d}\omega l\mathrm{d}l = \left\{\tilde{\varphi}(l) := \int_{S^1}\varphi(l,\omega)\mathrm{d}\omega \in D(\mathbb{R}^+), \text{int. by parts}\right\}\\ &= \frac{1}{2it}\int_0^\infty \mathrm{e}^{-itl^2}\frac{\partial}{\partial l}\left(\tilde{\varphi}(l)\frac{1}{(lr)^{\frac{3}{2}}}\right)\mathrm{d}l.\end{aligned}$$

Since we consider $r > K_0$ and $\mathrm{supp}(\tilde{\varphi})$ is bounded, we may estimate

$$|\tilde{\tilde{\Delta}}(r)| \leq \frac{const.}{|t|r^{\frac{3}{2}}},$$

and hence for a contribution to I_2 we have

$$\int_{K_0}^\infty\int_0^{2\pi} |\tilde{\tilde{\Delta}}|^2 \mathrm{d}\Omega r\mathrm{d}r \leq \frac{2\pi\ const.}{K_0|t|} \overset{t\to-\infty}{\longrightarrow} 0.$$

All in all $\lim_{t\to-\infty}\|\Delta\| = 0$.

■

Note A.24 *The generalized eigenfunctions $F^{\pm}$ are integral kernels of the wave operators $\tilde{W}_{\pm}$ in the "impulse space":*

$$W_{\pm} = \tilde{W}_{\pm}\mathscr{F}^{-1}.$$

Lemma A.25 *If generalized eigenfunctions $F^{\pm}$ of a total hamiltonian H fulfil the assumptions of lemma A.23, and if they are related by an integral operator on S^1 with a kernel $\mathcal{S}^{(l)}$:*

$$F^{+}(l,\omega;x) = \int\limits_{S^1} \mathcal{S}^{(l)}(\omega,\omega')F^{-}(l,\omega';x)\mathrm{d}\omega', \tag{A.40}$$

then for g of the form (A.39) the scattering operator S is given by the following prescription

$$(Sg)(x) = \int\limits_0^{\infty}\int\limits_{S^1} \mathrm{e}^{il\langle\omega',x\rangle}\left(\int\limits_{S^1}\mathcal{S}^{(l)}(\omega,\omega')\varphi(l,\omega)\mathrm{d}\omega\right)\mathrm{d}\omega' l\mathrm{d}l.$$

Proof Using lemma A.23 we have

$$\begin{aligned}(W_{-}g)(x) = \psi^{-}(x) &= \int\limits_0^{\infty}\int\limits_{S^1} F^{+}(l,\omega;x)\varphi(l,\omega)\mathrm{d}\omega l\mathrm{d}l \\ &= \int\limits_0^{\infty}\int\limits_{S^1} F^{-}(l,\omega';x)\left(\int\limits_{S^1}\mathcal{S}^{(l)}(\omega,\omega')\varphi(l,\omega)\mathrm{d}\omega\right)\mathrm{d}\omega' l\mathrm{d}l \\ &= W_{+}\int\limits_0^{\infty}\int\limits_{S^1} \mathrm{e}^{il\langle\omega',x\rangle}\left(\int\limits_{S^1}\mathcal{S}^{(l)}(\omega,\omega')\varphi(l,\omega)\mathrm{d}\omega\right)\mathrm{d}\omega' l\mathrm{d}l,\end{aligned}$$

which gives the general form (A.25) of S, because $S = W_{+}^{-1}W_{-}$.

■

Note A.26 *In the case of a rotationally symmetric potential, the integral kernel $\mathcal{S}^{(l)}$ is of the form [7]*

$$\mathcal{S}^{(l)}(\omega,\omega') = \delta_{S^1}(\omega,\omega') + \sqrt{\frac{il}{2\pi}}f(l,\operatorname{arc}\omega - \operatorname{arc}\omega'), \tag{A.41}$$

where $\delta_{S^1}(\omega,\omega')$ stands for the identity operator kernel on S^1.

A.9 Two-dimensional isotropic harmonic oscillator

We set the reduced Planck constant $\hbar = 1$ and the particle's mass $m = \frac{1}{2}$ to simplify further computation. In those units the isotropic harmonic oscillator hamiltonian is given as follows

$$\begin{aligned} &H = -\Delta + \frac{1}{4}\omega^2 x^2, \quad \text{where } \omega \geq 0 \\ &Dom(H) = span\left\{ x_1^{n_1} x_2^{n_2} \mathrm{e}^{-\frac{\omega x^2}{4}} \mid n_1, n_2 \in \mathbb{N}_0 \right\} \equiv span\left\{ \psi_{n_1,n_2} \mid n_1, n_2 \in \mathbb{N}_0 \right\}, \end{aligned} \tag{A.42}$$

where the functions ψ_{n_1,n_2} are introduced below. The operator H just defined is essentially selfadjoint.

The harmonic oscillator potential $V(x) = \frac{1}{4}\omega^2 x^2$ fulfils $V \in L^1_{loc}(\mathbb{R}^2)$ and $V(x) \overset{|x|\to\infty}{\to} \infty$, so H is an operator with compact resolvent [18], in particular it has purely discrete spectrum and a complete set of eigenfunctions: $\sigma(H) = \sigma_{pp}(H) = \{n + 1 \mid n \in \mathbb{N}_0\}$

$$\begin{aligned} &\psi_{n_1,n_2}(x) = \psi_{n_1}(x_1)\psi_{n_2}(x_2), \quad n_1, n_2 \in \mathbb{N}_0 \\ &H\psi_{n_1,n_2} = (n_1 + n_2 + 1)\omega\psi_{n_1,n_2}, \end{aligned}$$

where ψ_n stands for the n^{th} eigenfunction of the onedimensional harmonic oscillator $H_{(1)} = -\partial_x^2 + \frac{1}{4}\omega^2 x^2$ [19]:

$$\begin{aligned} &\psi_n(x) = \sqrt[4]{\frac{\omega}{2\pi}}\sqrt{\frac{1}{n!2^n}} H_n\left(\sqrt{\frac{\omega}{2}}x\right)\mathrm{e}^{-\frac{\omega x^2}{4}}, \quad n \in N_0 \\ &H_{(1)}\psi_n = (n + \frac{1}{2})\omega\psi_n. \end{aligned}$$

The functions $H_n,\ n \in \mathbb{N}_0$ are called the Hermite polynomials. The eigenfunctions ψ_n and ψ_{n_1,n_2} are of the unit norm: $\|\psi_n\|_{L^2(\mathbb{R})} = 1,\ \|\psi_{n_1,n_2}\|_{L^2(\mathbb{R}^2)} = 1$. The multiplicity k_n of the n^{th} eigenvalue $\lambda_n = (n_1 + n_2 + 1) \mid n_1 + n_2 = n$ is equal to $n + 1$.

Lemma A.27 *If $H_n(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then $H_{n+1}(x_0) \neq 0$.*

Proof For all $n \in \mathbb{N}_0$ the following relation takes place [12]: $H'_{n+1}(x) = 2(n+1)H_n(x)$, so if $H_n(x_0) = 0$, then $H'_{n+1}(x_0) = 0$. Now assume that $H_{n+1}(x_0) = 0$, then $H_{n+1} \equiv 0$, because $y = H_{n+1}$ is a solution to the second order linear differential equation $y'' - 2xy' + 2(n+1)y = 0$.

$\blacksquare$

Acknowledgements

I thank my supervisor prof. Pavel Šťovíček for his support, help and worthy consultations during our cooperation.

I thank my family, my friends and Eva for their support during writting this thesis.

I thank my colleague Tomáš Kalvoda for saving my time, providing me the LaTeX source-code of the declaration.

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