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Quantum Mechanics in Phase Space

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Abstract: Deformation quantization is a logically complete and self-standing formulation of quantum mechanical observables. We propose an introductory description of basic ideas involved, without referring to the conventional quantum mechanical approach as to the start point. Time-dependent solution of harmonic oscillator and time-independent solutions of one dimensional free particle, infinite wall and infinite square well are presented in the framework of deformation quantization according to recent results by Kryukov-Walton.

Keywords: Deformation quantization, one dimensional free particle, infinite wall, infinite square well, harmonic oscillator

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Introduction

There are basically three autonomous paths to quantization. The first one is the ordinary approach that involves operators in Hilbert space, developed by Schrödinger, Heisenberg, Dirac and others. The second way relies on path integrals, was initiated by Feynman and is today widely used in quantum field theory. The last one is the deformation quantization conceived as a phase-space theory. Basic ideas of deformation quantization were already given by the pioneers of quantum mechanics (Wigner, Weyl, von Neumann) but the autonomous statute was acquired in 1970s when Groenewold together with Moyal pulled the entire formulation together. Since then many others have contributed to the topic. Remarkable were in particular Kontsevich's results from 1990s that made it possible to set the whole concept in the framework of Poisson manifolds.

As emphasized by Dirac, crucial point of quantization process lies in noncommutativity of observables. This feature is in conventional quantum mechanical approach established by operators in Hilbert space. A physical system is thus described using completely new formalism introducing a severe conceptual break with classical mechanics. Consequently, precise formulation of correspondence principle that was the guidance principle throughout the whole history of physics still remains unclear. Rather by changing observables representation in order to introduce noncommutativity, deformation quantization modifies the multiplication of observables. More precisely, considering a Poisson manifold with a given Poisson algebra, deformation quantization alters pointwise multiplication of phase space functions. This is done in a continuous way without changing the nature of observables still being described by phase space functions. It turns out that this process called deformation gives the correct formulation of the correspondence principle.

In this work we make an attempt to emphasize the autonomous status of deformation quantization and provide an introductory description of structures and techniques involved without referring to ordinary quantum mechanical treatment. The second aim is to present solutions of simple physical systems like harmonic oscillator, infinite wall and infinite square well as these are not usually mentioned in papers dealing with the subject. As we will explain in the last chapter, quantization of systems with such simple potentials is not straightforward in the framework of deformation quantization and requires further study.

Chapter 1

Motivating Deformation Quantization

In this chapter we begin by presenting some of the basic features of classical mechanics. This part was inspired by [6]. Then we shall try to motivate the deformation of an algebra and sketch the deformation-contraction relationship. Finally we outline the basic idea of deformation quantization.

1.1 Classical Mechanics

There are three principal things that every theory has to deal with: state representation, observables and time evolution of the system. Let us consider a dynamical system with a finite number of degrees of freedom, for example a system consisting of n particles. In Hamiltonian formulation of classical mechanics every particle is described at any time by its position and momentum. The state of whole system of n particles is thus represented by a point in $2n$-dimensional space M. M has a structure of a smooth manifold and a point x in M is written as $x = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ (in canonical coordinates). The observables of the system are smooth realvalued functions on the space M whereas physical quantities at some time are given by evaluating these functions at the point in phase space $x_0 = (q_0, p_0)$ (one particle system). In general the states of the system are positive functionals on the observables. This is in the case of Hamiltonian function mathematically summarized by expression

$$E = \int H(q,p)\delta^{(2)}(q - q_0, p - p_0)dqdp, \tag{1.1}$$

where $\delta^{(2)}$ is the two-dimensional Dirac delta function representing the definite state $x_0 = (q_0, p_0)$.

Apart from being a manifold, M possesses structures of commutative algebra and Lie algebra. Commutative algebra is realized by ordinary pointwise function multiplication

$$(fg)(x) = f(x)g(x)$$

while Poisson bracket

$$\{f,g\}(q,p) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)\Big|_{q,p} \tag{1.2}$$

leads to Lie algebra structure. This means that function space $C^\infty(M)$ of the manifold M is a *Poisson algebra* as we can see from the following definition,

Definition 1.1.1. A *Poisson algebra* is a complex vector space V equipped with a commutative associative algebra structure

$$(f,g) \longrightarrow fg$$

and Lie algebra structure

$$(f,g) \longrightarrow \{f,g\}$$

which satisfy the compatibility condition

$$\{fg,h\} = f\{g,h\} + \{f,h\}g.$$

Finally the time development of the system is given by Hamilton's equations, which are expressed in terms of Poisson brackets as

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}.$$

For a general observable one has

$$\dot{f} = \{f, H\}.$$

In order to simplify notation we provide the derivatives with vector symbols, which indicate the action as follows

$$f\overleftarrow{\partial_{q_i}}g = \frac{\partial f}{\partial q_i}g, \quad f\overrightarrow{\partial_{q_i}}g = f\frac{\partial g}{\partial q_i}.$$

Using this notation Poisson bracket (1.2) can be written as

$$\{f, g\}(q, p) = \sum_i f\left(\overleftarrow{\partial_{q_i}}\overrightarrow{\partial_{p_i}} - \overleftarrow{\partial_{p_i}}\overrightarrow{\partial_{q_i}}\right) g.$$

This notation may be further abbreviated using the *Poisson tensor* α^{ij} and the Einstein convention of summing over repeated indices. In canonical coordinates α^{ij} is represented by matrix

$$\alpha = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ unit matrix. Let $x = (x_1, \ldots, x_{2n})$, then (1.2) has the form

$$\{f, g\}(x) = \alpha^{ij}\partial_i f(x)\partial_j g(x). \tag{1.3}$$

As the name says, Poisson tensor is a tensor and therefore transforms whenever changing coordinate system. In general coordinates components of α depend on the point of the manifold and thus need not to be constant. In Hamiltonian mechanics we assume α to be invertible and in general a manifold equipped with an invertible Poisson tensor is called *symplectic*. We can further generalize this concept by leaving the invertibility condition and focus on the Lie algebra structure only.

Definition 1.1.2. A *Poisson manifold* is a manifold M whose function space $C^\infty(M)$ is a Poisson algebra with respect to the usual pointwise multiplication of functions and a prescribed Lie algebra structure.

It turns out that such manifolds provide a better context for treating dynamical systems with symmetries. From now on we consider Poisson manifolds only.

1.2 Contractions and Deformations

The idea of symmetry of a physical system being expressed by a corresponding Lie group (Lie algebra) brought a branch of study dealing with relations and connections between Lie groups and Lie algebras. If two physical theories are related by a limiting process (for example relativistic and non-relativistic mechanics), then similar relation between associated invariance groups (Poincaré and Galilean groups in this case) was expected. Such a limiting process for Lie algebras was introduced by Segal and studied further by Inönü, Wigner and Evelyn Weimar-Woods [4]. Work on this subject lead to what we today call generalized Inönü-Wigner contraction, and contraction of Poincaré algebra into Galilei algebra is a classic example.

The idea of Lie algebra contraction thus naturally arises while working with symmetry Lie groups (Lie algebras). As there is a connection between contractions and deformations we can get a vivid image what deformations are about by pointing out this connection. This is done in the following text without any further detail.

A contraction is a limiting process that gives rise to a new Lie algebra that is not isomorphic to the original one whereas throughout the whole process the output algebra and original algebra are isomorphic. This means that we do not get a new nonisomorphic algebra until the limit point is reached. In [1] Gerstenhaber introduced an operation called deformation that is in this perspective inverse to contraction as we can schematically see in Figure 1.1. Using deformations that are in a sense inverse to contractions

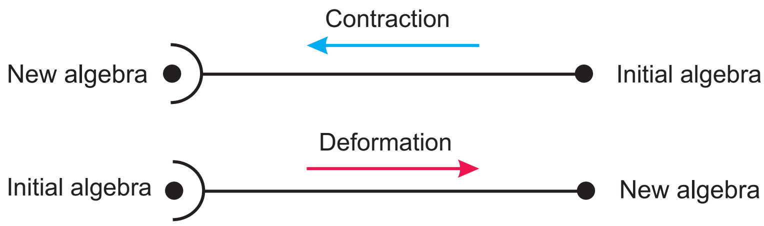


Figure 1.1: A contraction is a limiting process that gives rise to a new Lie algebra that is not isomorphic to the original one whereas throughout the whole process the output algebra and original algebra are isomorphic. This means that we do not get a new nonisomorphic algebra until the limit point is reached. Deformations are in this perspective inverse to contractions. Algebras are represented by dots. Lines represent isomorphic algebras.

we can deform the Galilei algebra into the Poincaré algebra. Moreover, Levy-Nahas [2] was able to show that the only Lie algebras which can give the Poincaré algebra by contraction are the semisimple Lie algebras of the de Sitter groups $SO(4,1)$ and $SO(3,2)$.

Deformations work with whole class of algebras which are not necessarily of finite dimension. Generalized Inönü-Wigner contractions work with finite-dimensional Lie algebras only, which might be the first hint that deformation-contraction relation is not trivial. Weimar-Woods showed in [4] that for every generalized Inönü-Wigner contraction there exists an inverse deformation of certain type and vice versa. In this work we will deal with deformations of infinite-dimensional associative algebras and Lie algebras and thus we cannot use generalized Inönü-Wigner contractions to construct inverse operations. There is however another approach to contract Lie algebras that even works with infinite dimensional algebras. It is called graded contraction and was introduced by de Montigny and Patera in [5] but in general connection between deformations and graded contractions has not been established yet.

1.3 Deformation Quantization

In the conventional formulation of quantum mechanics quantization means a radical change in the nature of observables. It implies that state representation and time evolution are treated in a completely different way in comparison with classical mechanics as well. Observables are represented by linear operators in Hilbert space whereas physical quantities are represented by eigenvalues of these operators and state representations are related to operator eigenfunctions. All this machinery is involved in order to introduce two principal features. Non-commutativity of the quantum mechanical observables and state representation in terms of probabilities which leads to situation where uncertainty is not an effect of approximative treatment (like in statistical physics) but a fundamental principle.

In deformation quantization, non-commutativity of observables is not realized by changing the nature of observables themselves but by introducing new algebra multiplication while the nature of observables remains the same. Uncertainty is implemented by describing physical states as distributions on phase space that are not sharply localized, in contrast to Dirac delta functions in classical approach (1.1).

Conventional classical pointwise multiplication of functions in a Poisson algebra of a Poisson manifold is modified in a continuous way using deformation. Thus deformation gives a one-parameter family of algebras where for an initial value of the parameter we get Poisson algebra that we have started with. As a result conversion from deformed structure back to the starting algebra is realized by straightforward limit in contrast to conventional formulation of quantum mechanics, where precise relationship between quantum and classical mechanics has remained obscure.

Chapter 2

Introduction to Deformation Theory

We introduce basic concepts concerning associative and Lie algebras cohomology. Then we follow the paper by Sternheimer and Dito [7] while our aim is to give more detailed discussion on the subject of algebra deformation.

2.1 Associative and Lie Algebras Cohomologies

Let $\mathcal{A}$ be an associative algebra over a field k. By the field k we mean the field of complex numbers $\mathbb{C}$ or that of the real numbers $\mathbb{R}$. Let $C^p(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}^p, \mathcal{A})$ be the space of p-multilinear maps from $\mathcal{A}$ to $\mathcal{A}$. The space $C^p(\mathcal{A})$ is called *the module of cochains* of degree p on $\mathcal{A}$ with values in $\mathcal{A}$. We define a coboundary operator

$$b : C^p(\mathcal{A}) \to C^{p+1}(\mathcal{A})$$

by

$$bf(u_1, \ldots, u_{p+1}) = u_1 f(u_2, \ldots, u_{p+1}) + \sum_{i=1}^{p} (-1)^i u(x_1, \ldots, u_i x_{i+1}, \ldots, u_{p+1}) +$$
$$+(-1)^{p+1} f(u_1, \ldots, u_p) u_{p+1}.$$

Appropriate cohomology complex ($bb = 0$) is given by

$$\mathcal{A} \xrightarrow{b} C(\mathcal{A}) \xrightarrow{b} C^2(\mathcal{A}) \xrightarrow{b} \cdots \xrightarrow{b} C^p(\mathcal{A}).$$

We say that p-cochain f is a *p-cocycle* if $bf = 0$. We denote by $Z^p(\mathcal{A}, \mathcal{A})$ the space of p-cocycles and by $B^p(\mathcal{A}, \mathcal{A})$ the space of those p-cocycles which are coboundaries (of a $(p-1)$-cochain). Then

$$H^p(\mathcal{A}, \mathcal{A}) := Z^p(\mathcal{A}, \mathcal{A})/B^p(\mathcal{A}, \mathcal{A}) \equiv \mathrm{Ker} b_p/\mathrm{Im} b_{p-1}$$

is the *p-th Hochschild cohomology group* of $\mathcal{A}$ with coeficients in $\mathcal{A}$. Considering dimensions one, two and three, for $u, v, w \in \mathcal{A}$ we get:

$$\begin{aligned} bu(v) &= vu - uv \\ bf(u,v) &= uf(v) - f(uv) + f(u)v \\ bg(u,v,w) &= ug(v,w) - g(uv,w) + g(u,vw) - g(u,v)w. \end{aligned} \tag{2.1}$$

Now let $\mathcal{A}$ be a Lie Algebra and ρ it's representation on the vector space V over a field k. Let $\Lambda^p(\mathcal{A}^*, V)$ denote the space of multilinear skewsymmetric p-forms:

$$\alpha : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{p} \to (V, \rho) \qquad \underbrace{(v, \ldots, w)}_{p} \to \alpha(v, \ldots, w) \in (V, \rho).$$

Now we introduce the coboundary operator

$$\delta : \Lambda^p(\mathcal{A}^*, V) \to \Lambda^{p+1}(\mathcal{A}^*, V) \qquad \alpha \to \delta\alpha$$

as follows:

$$\delta\alpha(u_1, \ldots, u_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} \rho(u_j) \alpha(u_1, \ldots, \widehat{u}_j, \ldots, u_{p+1}) +$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha(\{u_i, u_j\}, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, u_{p+1}) \quad u_j \in \mathcal{A}$$

In [3] Chevalley and Eilenberg showed that diagram

$$\Lambda^0(\mathcal{A}^*, V) \xrightarrow{\delta} \Lambda^1(\mathcal{A}^*, V) \xrightarrow{\delta} \Lambda^2(\mathcal{A}^*, V) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Lambda^n(\mathcal{A}^*, V)$$

describes a cohomology complex ($\delta\delta = 0$) with cohomology groups

$$H^p_C(\mathcal{A}, V) := Z^p_C(\mathcal{A}, V)/B^p_C(\mathcal{A}, V) \equiv \mathrm{Ker}\delta_p/\mathrm{Im}\delta_{p-1}$$

called *Lie algebra cohomology* or *Chevalley cohomology* of $\mathcal{A}$ with respect to representation (ρ, V).

Thus by definition of cohomology, a represenation of Lie algebra must be provided apart from Lie algebra itself. Let us consider adjoint representation $\rho(x) = \mathrm{ad}_x = \{x, .\}$ of $\mathcal{A}$. Then appropriate complex is:

$$\mathcal{A} \xrightarrow{\delta} \mathcal{L}(\mathcal{A}, \mathcal{A}) \xrightarrow{\delta} \Lambda^2(\mathcal{A}^*, \mathcal{A}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Lambda^n(\mathcal{A}^*, \mathcal{A}),$$

where operator δ takes the form:

$$\delta\alpha(u_1, \ldots, u_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1}\{u_j, \alpha(u_1, \ldots, \widehat{u}_j, \ldots, u_{p+1})\} +$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha(\{u_i, u_j\}, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, u_{p+1}) \quad u_j \in \mathcal{A}.$$

In the case of dimension one, two and three for $u, v, w \in \mathcal{A}$ one has:

$$\begin{aligned} \delta u(v) &= \{u, v\} \\ \delta\alpha(u, v) &= \{u, \alpha(v)\} - \{v, \alpha(u)\} - \alpha(\{u, v\}) \\ \delta\beta(u, v, w) &= \{u, \beta(v, w)\} - \{v, \beta(u, w)\} + \{w, \beta(u, v)\} \\ &- \beta(\{u, v\}, w) + \beta(\{u, w\}, v) - \beta(\{v, w\}, u) \\ &\equiv \sum_{\mathcal{P}(u,v,w)} \{u, \beta(v, w)\} + \beta(u, \{v, w\}). \end{aligned} \tag{2.2}$$

2.2 Deformations and Cohomology

Let $\mathcal{A}$ be an algebra. Only associative or Lie algebra are considered throughout this work. We will now introduce the definition of deformation and the way how cohomologies naturally arise in deformation theory.

Having a field k we will need to extend it to the ring $k[[\lambda]]$ of formal series in some parameter λ which gives the module $\tilde{\mathcal{A}} = \mathcal{A}[[\lambda]]$. On this module we can consider both structures of associative and Lie algebras.

Definition 2.2.1. A *deformation* of an algebra $\mathcal{A}$ is a $k[[\lambda]]$-algebra $\tilde{\mathcal{A}}$ such that $\tilde{\mathcal{A}}/\lambda\tilde{\mathcal{A}} \approx \mathcal{A}$. Two deformations $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ are called *equivalent* if they are isomorphic over $k[[\lambda]]$, and $\tilde{\mathcal{A}}$ is called *trivial* if it is isomorphic to the original algebra $\mathcal{A}$ considered by base field extension as a $k[[\lambda]]$-algebra.

Definition 2.2.1 tells us that there exists a new product $\star$ (resp. bracket $[.,.]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. $\{.,.\}$) this means that, for $u, v \in \mathcal{A}$ we have:

$$u \star v = uv + \sum_{r=1}^{\infty} \lambda^r C_r(u, v) \tag{2.3}$$

$$[u, v]_\star = \{u, v\} + \sum_{r=1}^{\infty} \lambda^r B_r(u, v) \tag{2.4}$$

where the C_r are the Hochschild 2-cochains and the B_r (skew-symmetric) Chevalley 2-cochains, such that for $u, v, w \in \mathcal{A}$ we have

$$(u \star v) \star w = u \star (v \star w) \tag{2.5}$$

and

$$\mathcal{P}[[u, v]_\star, w]_\star = 0, \tag{2.6}$$

where $\mathcal{P}$ denotes summation over cyclic permutations. We will look in detail what conditions (2.5) and (2.6) mean.

We treat the associative case first. Let $C_0(u,v) = uv$ then substituting (2.3) into (2.5) gives

$$\sum_{\substack{\mu+\nu=n \\ \mu,\nu\geq 0}} (C_\mu(C_\nu(u,v),w) - C_\nu(C_\mu(u,v),w)) = 0, \tag{2.7}$$

The condition (2.7) for $n=0$ is always satisfied as the algebra $\mathcal{A}$ is associative in this case. For $n=1$ one has:

$$uC_1(v,w) - C_1(uv,w) + C_1(u,vw) - C_1(u,v)w = 0,$$

and from (2.1) it is clear that:

$$uC_1(v,w) - C_1(uv,w) + C_1(u,vw) - C_1(u,v)w = 0 = bC_1(u,v,w). \tag{2.8}$$

Thus C_1 is a Hochschild 2-cocycle:

$$C_1 \in Z^2(\mathcal{A},\mathcal{A}).$$

C_1 is called the *infinitesimal deformation* of (2.3). Conversely an element of $Z^2(\mathcal{A},\mathcal{A})$ need not to be an infinitesimal deformation of a deformation of $\mathcal{A}$. If it is such, then we shall say that C_1 is *integrable* i.e. a 2-cochain is integrable if it is the first element C_1 of sequence $\{C_i\}$, which fulfills conditions (2.7).

By rewriting (2.7) in the form:

$$\sum_{\substack{\mu+\nu=n \\ \mu,\nu> 0}} (C_\mu(C_\nu(u,v),w) - C_\nu(C_\mu(u,v),w)) = bC_n(u,v,w), \tag{2.9}$$

and setting $n=2$ gives

$$C_1(C_1(u,v),w) - C_1(C_1(u,v),w)) = bC_2(u,v,w). \tag{2.10}$$

Whenever $C_1 \in Z^2(\mathcal{A},\mathcal{A})$, then function of three variables on the left is an element of $Z^3(\mathcal{A},\mathcal{A})$. From the preceding condition it follows that function on the lefthand side is in addition an element of $B^3(\mathcal{A},\mathcal{A})$. The cohomology class of this element is thus the first obstruction to the integration of C_1; if C_1 is integrable, then $H^3(\mathcal{A},\mathcal{A})$ has to be the zero class. Conversely if $H^3(\mathcal{A},\mathcal{A}) = 0$ then all obstructions vanish and every $C_1 \in Z^2(\mathcal{A},\mathcal{A})$ is integrable.

We proceed in a complete analogy in the Lie case. Let $B_0(u,v) = \{u,v\}$. Substituting (2.4) into (2.6) gives:

$$\sum_{\mathcal{P}(u,v,w)} \sum_{\substack{\mu+\nu=n \\ \mu,\nu\geq 0}} B_\mu(B_\nu(u,v),w) = 0, \tag{2.11}$$

We can rewrite (2.11):

$$\sum_{\mathcal{P}(u,v,w)} \sum_{\substack{\mu+\nu=n \\ \mu,\nu\geq 0}} B_\mu(B_\nu(u,v),w) + B_\nu(B_\mu(u,v),w) = 0. \tag{2.12}$$

The condition (2.12) for $n = 0$ is the Jacobi identity and therefore is always satisfied as the algebra $\mathcal{A}$ is a Lie algebra in this case. Setting $n = 1$ gives:

$$\sum_{\mathcal{P}(u,v,w)} B_1(\{u,v\},w) + \{B_1(u,v),w\} = 0,$$

and from (2.2) it follows that:

$$\sum_{\mathcal{P}(u,v,w)} B_1(\{a,b\},c) + \{F_1(a,b),c\} = 0 = \delta B_1(u,v,w). \tag{2.13}$$

Last equation tells that B_1 is a Chevalley 2-cocycle:

$$B_1 \in Z^2_C(\mathcal{A},\mathcal{A}).$$

By rewriting (2.11) in the form:

$$\sum_{\mathcal{P}(u,v,w)} \sum_{\substack{\mu+\nu=n \\ \mu,\nu>0}} B_\mu(B_\nu(u,v),w) = \delta B_n(u,v,w),$$

setting $n = 2$, one has

$$\sum_{\mathcal{P}(u,v,w)} B_1(B_1(u,v),w) = -\delta B_2(u,v,w). \tag{2.14}$$

And again, whenever is $C_1 \in Z^2_C(\mathcal{A},\mathcal{A})$, then function of three variables on the left is an element of $Z^3_C(\mathcal{A},\mathcal{A})$. From the preceding condition it follows that function on the lefthand side is in addition an element of $B^3_C(\mathcal{A},\mathcal{A})$. The cohomology class of this element is thus the first obstruction to the integration of B_1; if B_1 is integrable then $H^3_C(\mathcal{A},\mathcal{A})$ has to be the zero class. Conversely if $H^3_C(\mathcal{A},\mathcal{A}) = 0$, then all obstructions vanish and every $B_1 \in Z^2_C(\mathcal{A},\mathcal{A})$ is integrable.

Furthermore, assuming one has shown that (2.9) or (2.14) are satisfied to some order $n = t$, then it folows from preceding calculations that the left-hand sides for $n = t+1$ are then 3-cocycles, depending only on the cochains C_μ (resp. B_μ) of order $\mu \leq t$. If we want to extend the deformation up to order $n = t+1$ (i.e. to find the required 2-cochains C_{t+1} or B_{t+1}), this cocycle has to be a coboundary (the coboundary of the required cochain): The obstructions to extend a deformation from one step to the next lie in the 3-cohomology. In particular if one can manage to pass always through the null class in the 3-cohomology, a cocycle can be the infinitesimal deformation of a full-fledged deformation.

Equivalence in the definition 2.2.1 means that there is an isomorphism

$$T_\tau = I + \sum_{\mu=1}^{\infty} \tau^\mu T_\mu, \quad T_\mu \in \mathcal{L}(\mathcal{A},\mathcal{A})$$

so that in the associative case

$$T_\tau(u \star' v) = (T_\tau u \star T_\tau v),$$

denoting by $\star$ (resp. $\star'$) the deformed laws in $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{A}}'$). Similarly in the Lie case:

$$T_\tau[u,v]'_\star = [T_\tau u, T_\tau v]_\star,$$

where $[.,.]_\star$ (resp. $[.,.]'_\star$) are the deformed laws in $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{A}}'$). In this sense a deformation of an associative algebra (resp. Lie algebra) is trivial if the deformed multiplication has the form

$$T_\tau(u \star v) = T_\tau(u)T_\tau(v) \quad \text{(resp. } T_\tau[u,v]_\star = \{T_\tau(u), T_\tau(v)\}\text{)} \tag{2.15}$$

Consider a trivial deformation of an associative algebra to the first order ie. $T_\tau = I + \tau T_1$. Substituting into (2.15) gives:

$$\begin{aligned} u \star v &= uv + \tau(uT_1(v) - T_1(uv) + T_1(u)v + \cdots) \\ &= uv + \tau bT_1(u,v). \end{aligned}$$

It follows that in the case of trivial deformation $C_1 \in B^2(\mathcal{A},\mathcal{A})$. Conversely an element of $B^2(\mathcal{A},\mathcal{A})$ is integrable and hence defines a deformation.

Let $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ be equivalent deformations and again $T_\tau = I + \tau T_1$. To the first order in τ one has:

$$u \star' v - u \star v = \tau(u \star T_1(v) - T_1(u \star v) + T_1(u) \star v).$$

This equation gives conditions for components C_i resp. C'_i corresponding to deformations $\tilde{\mathcal{A}}$ resp. $\tilde{\mathcal{A}}'$. First components thus satisfy:

$$C'_1(u,v) - C_1(u,v) = uT_1(v) - T_1(uv) + T_1(u)v = bT_1(u,v).$$

The difference between components C_1, C'_1 of two equivalent deformations is hence given by a coboundary from $B^2(\mathcal{A},\mathcal{A})$.

The Lie case is analogous to the associative one. Considering $T_\tau = I + \tau T_1$ and substituting into (2.15) gives:

$$\begin{aligned} [u,v]_\star &= \{u,v\} + \tau(\{u,T_1(v)\} - T_1(\{u,v\}) - \{T_1(u),v\} + \cdots) \\ &= \{u,v\} + \tau\delta T_1(u,v), \end{aligned}$$

ie. $B_1 \in B^2_C(\mathcal{A},\mathcal{A})$.

Let $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ be equivalent deformations and again $T_\tau = I + \tau T_1$. To the first order in τ one has:

$$[u,v]'_\star - [u,v]_\star = \tau([u,T_1(v)]_\star - T_1([u,v]_\star) - [T_1(u),v]_\star).$$

This equation gives conditions for components B_i resp. B'_i corresponding to deformations $\tilde{\mathcal{A}}$ resp. $\tilde{\mathcal{A}}'$. First components thus satisfy:

$$B'_1(u,v) - B_1(u,v) = \{u,T_1(v)\} - T_1(\{u,v\}) - \{T_1(u),v\} = \delta T_1(u,v).$$

The difference between components B_1, B'_1 of two equivalent deformations is hence given by a coboundary from $B^2_C(\mathcal{A},\mathcal{A})$.

More generally, we can show that if two deformations are equivalent up to some order t, the condition to extend the equivalence one step further is that a 2-cocycle (defined using the T_k $k \leq t$) is the coboundary of the required T_{t+1} and therefore the obstructions to equivalence lie in the 2-cohomology. In particular, if that space is null, all deformations are trivial.

We have seen that a deformation by definition preserves associativity of an algebra as well as Lie algebra will again be a Lie algebra. Some other basic algebraic properties remain unchanged under deformation, for example a unital algebra will continue to be unital. However, a deformation of a commutative algebra need not to remain commutative.

Chapter 3

Deformation Quantization Overview

Three papers [7], [12] and [6] are followed in this chapter with some notes taken from [17]. We establish the star-product on a Poisson manifold as a special case of algebra deformation. Then basic ideas of deformation quantization are sketched. The last two sections deal with relationship between deformation quantization and conventional quantum mechanics. Part concerning Heisenberg uncertainty relations analogue follows the discussion in [13].

3.1 Star Product

The construction of deformations in a step-by-step manner by solving (2.9) requires knowledge of Hochschild cohomology of $\mathcal{A}$ and it is evident that this is not a simple task to deal with. In one particular case, however, all integration obstructions vanish.

Definition 3.1.1. A *derivation* D of $\mathcal{A}$ is a linear mapping of $\mathcal{A}$ into itself such that $D(ab) = (Da)b + a(Db)$.

It follows that elements of $Z^1(\mathcal{A}, \mathcal{A})$ are derivations. Let $D', D'' \in Z^1(\mathcal{A}, \mathcal{A})$ and let $C_n(u, v) = \frac{1}{n!} D'^n u D''^n v$. Assuming in addition that D' and D'' are commuting derivations then one has

$$bC_n(u,v,w) = \frac{1}{n!}\left[uD'^n vD''^n w - D'^n(uv)D''^n w + D'^n uD''^n vw - (D'^n uD''^n v)w\right] =$$

$$= \frac{1}{n!}\left[uD'^n vD''^n w - (D'^n u)vD''^n w - uD'^n vD''^n w + \right.$$

$$\left. + D'^n u(D''^n v)w + (D'^n u)vD''^n w - D'^n u(D''^n v)w\right] = 0,$$

ie. $C_n \in Z^2(\mathcal{A},\mathcal{A})$. Similarly it is easy to show that

$$\sum_{\substack{\mu+\nu=n\\ \mu,\nu>0}} \left[\frac{1}{\mu!}D'^\mu\left(\frac{1}{\nu!}D'^\nu uD''^\nu v\right)D''^\mu w - \frac{1}{\nu!}D'^\nu\left(\frac{1}{\mu!}D'^\mu uD''^\mu v\right)D''^\nu w\right] =$$

$$= \sum_{\substack{\mu+\nu=n\\ \mu,\nu>0}} \frac{1}{\mu!}\frac{1}{\nu!}\left[D'^\mu D'^\nu uD''^\nu vD''^\mu w + D'^\nu uD'^\mu D''^\nu vD''^\mu w - \right.$$

$$\left. - D'^\mu D'^\nu uD''^\mu vD''^\nu w - D'^\nu uD'^\mu D''^\mu vD''^\nu w\right] = 0.$$

This means that our choice of C_n satisfies (2.9) and deformed multiplication

$$u \star v = uv + \lambda D'uD''v + \frac{\lambda^2}{2!}D'^2uD''^2v + \frac{\lambda^3}{3!}D'^3uD''^3v + \cdots$$

yields a deformation as it preserves associativity. Preceding form of multiplication can be written as:

$$u \star v = ue^{\lambda\overleftarrow{D'}\overrightarrow{D''}}v. \tag{3.1}$$

More generally, if $D'_1,\ldots,D'_r$, $D''_1,\ldots,D''_r$ are all mutually commuting derivations then

$$u \star v = ue^{\lambda\sum_{i=1}^r \overleftarrow{D'_i}\overrightarrow{D''_i}}v$$

is again an associative multiplication.

Assuming composition of two derivations $D'D''$ it can be shown in a similar way that

$$T = T_\tau = e^{\tau\lambda D'D''}$$

is an equivalence between deformations. Resulting deformation is then given by

$$T_\tau^{-1}(T_\tau(u) \star T_\tau(v)) = u \star_\tau v = u e^{\lambda \overleftarrow{D'} \overrightarrow{D''} - \tau\lambda(\overleftarrow{D'} \overrightarrow{D''} - \overleftarrow{D''} \overrightarrow{D'})} v \tag{3.2}$$

Now that we have constructed a deformation which is exactly the relevant one as we shall see shortly we may return to a Poisson manifold M. In a Poisson algebra of such a Poisson manifold there are two structures as we already know. Associative (commutative) pointwise multiplication of functions and Poisson structure which is a Lie algebra of functions. We want to define a special type of deformation that would anchor the new product to the given structure of the Poisson manifold. In other words having a new product $\star$ which is a deformation of associative algebra of functions we require the commutator

$$[f, g]_\star = f \star g - g \star f \quad f, g \in C^\infty(M) \tag{3.3}$$

to be a deformation of initial Poisson structure. Appropriate definition is as follows

Definition 3.1.2. Let M be a Poisson manifold with a Poisson structure $\{.,.\}$. A *star-product* on M is a deformation of the associative algebra of functions $C^\infty(M)$ of the form

$$\star = \sum_{n=0}^{\infty} \lambda^n C_n,$$

where the C_n are bidifferential operators such that for $u, v \in C^\infty(M)$

(i) $C_0(u, v) = uv$

(ii) $C_1(u, v) - C_1(v, u) = \{u, v\}$

Here we take the parameter λ of the deformation to be $\lambda = i\hbar$. Property (i) means that

$$\lim_{\hbar \to 0} f \star g = fg$$

whereas considering (3.3), property (ii) gives

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}[f,g]_\star = \{f,g\}.$$

This is the correct form of correspondence principle. In classical limit deformed Poisson structure reduces to the Poisson bracket.

An extensive effort has been put on the question of existence of star product for a priori we do not know if a star product exists for a given Poisson manifold. First the existence of star product for symplectic manifolds whose third cohomology group is trivial was proved. Then for larger and larger class of symplectic manifolds existence was showed until Fedosov [8] proved existence of star-product for every symplectic manifold i.e. for every regular Poisson manifold. Further generalization was needed as we work with Poisson manifolds in general. Finally solution came from Kontsevich [9] who proved that classes of star-products correspond to classes of deformations of the Poisson manifold. This means that every Poisson manifold admits a deformation quantization.

From now on let $M = \mathbb{R}^{2n}$. Then ordinary differentiation in the algebra of infinitely differentiable functions on $\mathbb{R}$ is a derivation. Setting $D' = \partial_q$, $D'' = \partial_p$ and $\lambda = i\hbar$ then (3.1) becomes

$$f \star_S g = f e^{i\hbar \overleftarrow{\partial}_q \overrightarrow{\partial}_p} g \tag{3.4}$$

and is called the *standard star-product*. It follows that $q \star p = qp + i\hbar$ and $p \star q = pq$, so

$$[p,q]_\star = q\star p - p\star q = i\hbar. \tag{3.5}$$

For $\tau = 1$ we get from (3.2) *antistandard star-product*

$$f \star_N g = f e^{i\hbar \overleftarrow{\partial}_p \overrightarrow{\partial}_q} g \tag{3.6}$$

and finally setting $\tau = \frac{1}{2}$ we get the *Groenewold-Moyal star-product*

$$f \star_M g = f e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)} g. \tag{3.7}$$

Indeed all the forms (3.4), (3.6) and (3.7) are equivalent in the sense of cohomology. In order to stress the fact that such an equivalence does not imply any kind of physical equivalence, from now on we shall call this equivalence the *c-equivalence*.

$T_\lambda(p) = p$ as well as $T_\lambda(q) = q$ imply that commutation relations (3.5) will not change while passing between two c-equivalent star-products. In fact commutator of two phase space variables is fixed by the property (ii) in the star-product definition.

3.2 Deformation Quantization on $\mathbb{R}^{2n}$

Noncomutativity is now well settled up by star-product as we can see from (3.5). So it seems that it remains to incorporate the non-locality feature of quantum mechanics. If we rewrite the Groenewold-Moyal product

$$\begin{aligned} f \star_M g &= f e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)} g \\ &= \sum_{m,n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^{m+n} \frac{(-1)^m}{m!n!} (\partial_p^m \partial_q^n f)(\partial_p^n \partial_q^m g) \end{aligned} \tag{3.8}$$

then we observe that star product of the functions f and g at the point x involves higher derivatives at this point. But for a smooth function, knowledge of all higher derivatives at a given point is equivalent to knowledge of the function on entire space. Non-locality is thus already established by star-product. This might be argued even better using integral representation of star product which is used widely since Baker's work [10]. It can be derived by utilizing Fourier convolution theorem and by writing the Fourier transforms of the functions f and g explicitly that

$$\begin{aligned}(f \star_M g)(q,p) &= \frac{1}{4\pi^2}\int d\tau d\sigma d\xi d\eta dq_1 dp_1 dq_2 dp_2 e^{i\sigma q} e^{i\tau p} \\ &\times \; e^{\frac{i\hbar}{2}(\eta(\sigma-\xi)-\xi(\tau-\eta))} e^{-i\xi q_1 - i\eta p_1} f(q_1,p_1) e^{-i(\sigma-\xi)q_2 - i(\tau-\eta)p_2} g(q_2,p_2) \\ &= \frac{1}{4\pi^2}\int d\tau d\sigma d\xi d\eta dq_1 dp_1 dq_2 dp_2 f(q_1,p_1) g(q_2,p_2) \\ &\times \; \left(i\sigma(q + \frac{\hbar}{2}\eta - q_2) + i\tau(p - \frac{\hbar}{2}\xi - p_2) - i\xi q_1 - i\eta p_1 + i\xi q_2 + i\eta p_2 \right) \\ &= \frac{1}{4\pi^2}\int d\xi d\eta dq_1 dp_1 dq_2 dp_2 f(q_1,p_1) g(q_2,p_2) \\ &\times \; \delta(-q - \frac{\hbar}{2}\eta + q_2)\delta(p - \frac{\hbar}{2}\xi - p_2) exp[-i\xi q_1 - i\eta p_1 + i\xi q_2 + i\eta p_2].\end{aligned}$$

Considering $\delta(-q - \frac{\hbar}{2}\eta + q_2) = \left(\frac{2}{\hbar}\right)\delta(\eta + \frac{2}{\hbar}q - \frac{2}{\hbar}q_2)$ and similarly for the second delta function we can now perform the ξ and η integrations which finally lead to

$$\begin{aligned}(f \star_M g)(q,p) &= \frac{1}{\hbar^2\pi^2}\int dq_1 dp_1 dq_2 dp_2 f(q_1,p_1) g(q_2,p_2) \\ &\times \; exp\left(\frac{2}{i\hbar}(p(q_1 - q_2) + q(p_2 - p_1) + (q_2 p_1 - q_1 p_2)) \right). \end{aligned} \tag{3.9}$$

From this equation it is clear that knowledge of the functions f and g on the whole space is necessary to determine the value of the star product at the point (q,p). We shall now sketch an idea introduced in [11] that allows to establish spectrality and non-local state representation in deformation quantization.

Consider a Hamiltonian function H then the key is the *star exponential*

$$\mathrm{Exp}(Ht) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)(H)^{\star n}, \tag{3.10}$$

where $(H)^{\star n} = \underbrace{H \star H \star \cdots \star H}_{n\times}$. We have always considered star-products as formal series and looked for convergence only in specific examples, generally in the sense of distributions. The same applies to star exponentials, as long as each coefficient in the formal series is well defined. Star exponential is the solution of

$$i\hbar\frac{d}{dt}\mathrm{Exp}(Ht) = H \star \mathrm{Exp}(Ht), \tag{3.11}$$

which is the equation that corresponds to Schrödinger equation. This expresses that the star exponential of Hamiltonian function is the generator of the time-evolution of the system. We can now write the Fourier-Stieltjes transform $d\mu$ (in the distribution sense) as

$$\mathrm{Exp}(Ht) = \int e^{-\frac{iEt}{\hbar}} d\mu(E),$$

which defines the spectrum of $(\frac{H}{\hbar})$ as the support of the measure μ. More precisely, suppose for simplicity that the Hamiltonian is such that the series in (3.10) defines a periodic distribution in t, then the measure μ is atomic and the Fourier-Stieltjes transform reduces to the expression called *Fourier-Dirichlet expansion*

$$\mathrm{Exp}(Ht) = \sum_E \pi_E e^{-\frac{iEt}{\hbar}}. \tag{3.12}$$

Functions π_E are distributions on the phase space that are normalized

$$\frac{1}{2\pi\hbar}\int \pi_E(q,p)dqdp = 1 \tag{3.13}$$

and idempotent in the sense that

$$(\pi_E \star \pi_{E'})(q,p) = \delta_{E,E'}\pi_E(q,p). \tag{3.14}$$

Distributions π_E called *projectors* thus represent the orthonormal eigenstates and E are the eigenvalues of H

$$(H \star \pi_E)(q,p) = E\pi_E(q,p). \tag{3.15}$$

This equation corresponds to the time-independent Schrödinger equation and is sometimes called $\star$-*genvalue equation.* As discussed in [11] we should consider

$$\pi_E^* = \pi_E$$

in order to get solutions of physical interest from the equation (3.15).

The spectral decomposition of the Hamiltonian function is given by

$$H(q,p) = \sum_E E\pi_E(q,p), \tag{3.16}$$

where the summation sign may indicate an integration if the spectrum is continuous.

Let H be a general observable. Then it follows from (3.10) that in general for different star-products different eigenstates and eigenvalues will be obtained although all the star-products are c-equivalent in the case $M = \mathbb{R}^{2n}$. Ambiguity in choosing a specific c-equivalent product corresponds to similar problem with operator ordering in ordinary quantum mechanical approach. The c-equivalent star products determine different quantization schemes that lead to slightly different spectra for the observables. For $M = \mathbb{R}^{2n}$ the Groenewold-Moyal product is preferred due to symmetry and coholomogical uniqueness. In general, however, it turns out that choice of specific quantization scheme can only be motivated by further physical requirements. A recent attempt to generalize these requirements and to introduce a selection principle was developed by Gerstenhaber in [12].

Quantum mechanical version of (1.1) is obtained by integrating (3.15)

$$E = \frac{1}{2\pi\hbar}\int (H \star \pi_E)(q,p)dqdp = \frac{1}{2\pi\hbar}\int H(q,p)\pi_E(q,p)dqdp.$$

3.3 Connection with Conventional Quantum Mechanics Formulation

In conventional quantization the process of assigning appropriate quantum mechanical observables to the classical ones is in general a difficult task as it cannot be done in a systematic way. The only guiding principle is the corresponding principle: we want the quantum mechanical relations to reduce somehow to the classical relations in an appropriate limit. However, several attempts to find a unique assignment between quantum and classical quantities were made. This effort could not be successful due to operator-ordering ambiguity in conventional approach to quantum mechanics. This problem comes from the fact that we a priori don't know what to assign to for example qp^2. Should we consider $\hat{Q}^2P$ or $\hat{Q}\hat{P}\hat{Q}$? So if there would exist an assignment process then it cannot be unique because different orderings of operators $\hat{Q}$ and $\hat{P}$ all reduce to the original phase space function.

But a general scheme for associating phase space functions and Hilbert space operators called *Weyl map* and denoted here by ϑ is of a great importance as it shows equivalence between conventional quantization and deformation quantization. Let us consider the so-called *Weyl ordering*

$$\vartheta(q^2p) = \frac{1}{3}\left(\hat{Q}^2\hat{P} + \hat{Q}\hat{P}\hat{Q} + \hat{P}\hat{Q}^2\right).$$

It can be shown that this generalizes to

$$\vartheta(e^{aq+bp}) = e^{a\hat{Q}+b\hat{P}}.$$

By using this expression in usual Fourier formula we get

$$\vartheta(f) = \int \tilde{f}(\xi,\eta)e^{-i\left(\xi\hat{Q}+\eta\hat{P}\right)}d\xi d\eta, \tag{3.17}$$

where $\tilde{f}$ is the Fourier transform of f and the integral is taken in the weak operator topology.

The inverse $\mathcal{W}$ of the Weyl map ϑ is known as the *Weyl transform*. It is a process of finding the phase space function that corresponds to a given operator $\hat{f}$, for the special case of Weyl ordering given by

$$\mathcal{W}(\hat{f}) = f(q,p) = \hbar\int \left\langle q + \frac{\hbar}{2}\xi\right| \hat{f} \left|q - \frac{\hbar}{2}\xi\right\rangle e^{-i\xi p}d\xi. \tag{3.18}$$

Whenever considering conventional quantization we will restrict ourselves to the case of pure states only. Then the density matrix $\hat{\rho}$ has the form

$$\hat{\rho} = \left|\psi\right\rangle\left\langle\psi\right|.$$

Substituting this expression into the Weyl transform (3.18) gives

$$\mathcal{W}(\hat{\rho}) = \hbar\int \psi^*\left(q - \frac{\hbar}{2}\xi\right)e^{-ip\xi}\psi\left(q + \frac{\hbar}{2}\xi\right)d\xi.$$

After normalization we get the so called *Wigner function*

$$\rho = \frac{\mathcal{W}(\hat{\rho})}{2\pi\hbar}. \tag{3.19}$$

Assuming we are working with Groenewold-Moyal product, from now on we will denote a Wigner function calculated from a known Schrödinger wave function by $\rho^{(W)}$ with W for Weyl ordering.

We will now show that

$$\vartheta(f)\vartheta(g) = \vartheta(f \star_M g). \tag{3.20}$$

Using equation (3.17) one has

$$\begin{aligned}
\vartheta(f)\vartheta(g) &= \int d\xi_1 d\eta_1 d\xi_2 d\eta_2 \tilde{f}(\xi_1, \eta_1)\tilde{g}(\xi_2, \eta_2) \\
&\times \exp\left[-i\left(\xi_1\hat{Q} + \eta_1\hat{P}\right)\right] \exp\left[-i\left(\xi_2\hat{Q} + \eta_2\hat{P}\right)\right] \\
&= \int d\xi_1 d\eta_1 d\xi_2 d\eta_2 \tilde{f}(\xi_1, \eta_1)\tilde{g}(\xi_2, \eta_2) \\
&\times \left[-i\left((\xi_1 + \xi_2)\hat{Q} + (\eta_1 + \eta_2)\hat{P}\right)\right] \exp\left[\frac{-i\hbar}{2}(\xi_1\eta_2 - \nu_1\xi_2)\right],
\end{aligned} \tag{3.21}$$

where the truncated Campbell-Baker-Hausdorff formula

$$e^{\hat{A}}e^{\hat{B}} = e^{\left(\hat{A}+\hat{B}\right)}e^{\frac{1}{2}\left[\hat{A},\hat{B}\right]}$$

was used. We expand the last exponential in (3.21), make the substitution of variables $\xi = \xi_1 + \xi_2$, $\eta = \eta_1 + \eta_2$, and obtain

$$\begin{aligned}
\vartheta(f)\vartheta(g) &= \int d\xi d\eta e^{-i\left(\xi\hat{Q}+\eta\hat{P}\right)} \int d\xi_1 d\eta_1 \sum_{m,n=0}^{\infty} \left[\frac{(-1)^m}{m!n!}\left(\frac{i\hbar}{2}\right)^{m+n}\right. \\
&\times \left. \xi_1^m \eta_1^n \tilde{f}(\xi_1, \eta_1)(\xi - \xi_1)^n(\eta - \eta_1)^m \tilde{g}(\xi - \xi_1, \eta - \eta_1)\right]
\end{aligned}$$

The integral with respect to ξ_1 and η_1 is by Fourier convolution theorem just the Fourier transform of the expression for the Groenwold-Moyal product in equation (3.8). Hence

$$\vartheta(f)\vartheta(g) = \int d\xi d\eta (\widetilde{f \star_M g}) e^{-i\left(\xi\hat{Q}+\eta\hat{P}\right)} = \vartheta(f \star_M g).$$

We can now see more precisely from preceding derivations that there is a connection between freedom in choosing operator ordering and the problem of picking out a star-product from a class of c-equivalent star-products. Although we have shown the equation (3.20) for the Groenewold-Moyal product and Weyl ordering it holds for any other possible ordering and was proved by Groenewold. Therefore we can write

$$\theta(f)\theta(g) = \theta(f \star g),$$

where θ is a linear assignment between operator algebras and star-product algebras of the same kind as Weyl map. This crucial formula tells as that the quantum mechanical algebra of observables is a representation of the star product algebra. Because in the algebraic approach to quantum theory all the information concerning the quantum system may be extracted from the algebra of observables, specifying the star-product completely determines the quantum system.

3.4 Wigner Functions and Uncertainty Principle

Considering Groenewold-Moyal product then every distribution from section (3.2) has to be a Wigner function. Phase-space distributions are, however, normalized in the sense that (3.13) while for Wigner functions it obviously follows

$$\int \rho(q,p)dqdp = 1.$$

When working with distributions normalized according to equation (3.2) then idempotency is expressed in the elegant form by equation (3.14). For Wigner functions representing pure states we similarly get

$$(\rho_a \star_M \rho_b)(q,p) = \frac{1}{2\pi\hbar}\delta_{a,b}\rho_a(q,p). \tag{3.22}$$

Definition (3.19) leads to the following equations for probability densities

$$\langle q \rangle = |\psi(q)|^2 = \int \rho(q,p)dp \quad \langle p \rangle = |\psi(p)|^2 = \int \rho(q,p)dq.$$

Using (3.9) it follows that

$$\int (f \star_M g)(q,p)dqdp = \int (g \star_M f)(q,p)dqdp = \int f(q,p)g(q,p)dqdp. \tag{3.23}$$

Then operator's expectation value is given by

$$\left\langle \hat{F} \right\rangle = \int (\rho \star_M f)(q,p)dqdp = \int \rho(q,p)f(q,p)dqdp,$$

where $f = \mathcal{W}(\hat{F})$.

It is straightforward to show that for Wigner functions which go negative for an arbitrary function g, $\langle | \, g \, |^2 \rangle$ need not to be non-negative. This is in fundamental contradiction with classical probability distribution theory. It turns out, however, that $\langle | \, g \, |^2 \rangle$ should be replaced by $\langle g^* \star_M g \rangle$. Using (3.22) and (3.23) we can write

$$\begin{aligned} \langle g^* \star_M g \rangle &= \int dqdp (g^\star \star_M g)(q,p)\rho(q,p) \\ &= \frac{1}{2\pi\hbar} \int dqdp (g^\star \star_M g)(q,p)(\rho \star_M \rho)(q,p) \\ &= \frac{1}{2\pi\hbar} \int dqdp \left[(\rho \star_M g^*) \star_M (g \star_M \rho) \right](q,p) \\ &= \frac{1}{2\pi\hbar} \int dqdp \left| g \star_M \rho \right|^2 (q,p) \end{aligned}$$

This means that

$$\langle g^* \star_M g \rangle \geq 0. \tag{3.24}$$

Now let us consider

$$g = a + bq + cp,$$

where a, b, c are arbitrary complex coefficients. Substituting into (3.24) gives semi-definite positive quadratic form

$$\begin{aligned} 0 \;\; &\leq \;\; a^* a + b^* b \langle q \star_M q \rangle + c^* c \langle p \star_M p \rangle + (a^* b + b^* a) \langle q \rangle \\ &+ \;\; (a^* c + c^* a) \langle p \rangle + c^* b \langle p \star_M q \rangle + b^* c \langle q \star_M p \rangle . \end{aligned}$$

The eigenvalues of the corresponding matrix are then non-negative and thus the same holds for its determinant. Groenewold-Moyal product gives

$$q \star_M q = q^2 \quad p \star_M p = p^2$$

$$p \star_M q = pq - \frac{i\hbar}{2} \quad q \star_M p = pq + \frac{i\hbar}{2}.$$

Considering

$$(\Delta q)^2 \equiv \left\langle (q - \langle q \rangle)^2 \right\rangle , \quad (\Delta p)^2 \equiv \left\langle (p - \langle p \rangle)^2 \right\rangle ,$$

then positivity condition on the 3×3 matrix determinant leads to

$$(\Delta q)^2 (\Delta p)^2 \geq \frac{\hbar 2}{4} + \big(\langle (q - \langle q \rangle)(p - \langle p \rangle) \rangle \big)^2$$

and hence

$$\Delta q \Delta p \geq \frac{\hbar}{2}.$$

Chapter 4

Basic Deformation Quantization Applications

Typically, deformation quantization is demonstrated by solving harmonic oscillator. Here we present calculation techniques as they appear in [6] considering the time-dependent case. Quantization of a free-particle moving in one dimension is omitted in most papers. This might be caused by the fact that we have to deal with continuous energy spectrum, which requires treatment in terms of distributions. Results from [15] are provided here for time-independent case only.

4.1 Harmonic Oscillator

The simple one-dimensional harmonic oscillator is characterized by the classical Hamiltonian function

$$H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2.$$

In terms of *holomorphic variables*

$$a = \sqrt{\frac{m\omega}{2}} \left(q + i \frac{p}{m\omega} \right), \quad \bar{a} = \sqrt{\frac{m\omega}{2}} \left(q - i \frac{p}{m\omega} \right)$$

the Hamiltonian function becomes

$$H = \omega a\bar{a}.$$

Our aim is to calculate the time-evolution function. We first choose a quantization scheme characterized by the standard star-product

$$f \star_N g = f e^{\hbar \overleftarrow{\partial_a} \overrightarrow{\partial_{\bar{a}}}} g.$$

We then have

$$\bar{a} \star_N a = a\bar{a}, \quad a \star_N \bar{a} = a\bar{a} + \hbar$$

so that

$$[a, \bar{a}]_{\star_N} = \hbar,$$

which is equivalent to $[q, p]_{\star_N} = i\hbar$. Equation (3.11) for this case is

$$i\hbar \frac{d}{dt} \mathrm{Exp}_N(Ht) = (H + \hbar\omega\bar{a}\partial_{\bar{a}}) \mathrm{Exp}_N(Ht). \tag{4.1}$$

This is a simple first order partial differential equation. Let $s = a\bar{a}$ then (4.1) becomes

$$i\hbar \frac{d}{dt} F(s, t) = (\omega s + \hbar\omega s \partial_s) F(s, t)$$

with the solution (that must have the value 1 at $t = 0$)

$$F(s, t) = e^{\frac{-s}{\hbar}} \exp\left(e^{-i\omega t} \frac{s}{\hbar}\right),$$

so we have

$$\mathrm{Exp}_N(Ht) = e^{\frac{-a\bar{a}}{\hbar}} \exp\left(e^{-i\omega t} \frac{a\bar{a}}{\hbar}\right). \tag{4.2}$$

By expanding the last exponential in equation (4.2), we obtain the Fourier-Dirichlet expansion

$$\mathrm{Exp}_N(Ht) = e^{\frac{-a\bar{a}}{\hbar}} \sum_{n=0}^{\infty} \frac{1}{\hbar^n n!} \bar{a}^n a^n e^{-in\omega t}. \tag{4.3}$$

If we compare coefficients in equations (3.12) and (4.3), we find

$$\begin{aligned}
\pi_0^{(N)} &= e^{-\frac{a\bar{a}}{\hbar}} \\
\pi_n^{(N)} &= \frac{1}{\hbar^n n!}\pi_0^N \bar{a}^n a^n = \frac{1}{\hbar^n n!}\bar{a}^n \star_N \pi_0^{(N)} \star_N a^n \\
E_n &= n\hbar\omega.
\end{aligned} \tag{4.4}$$

Note that the spectrum obtained in equation (4.4) does not include the zero point energy. The projector onto the ground state $\pi_0^{(N)}$ satisfies

$$a \star_N \pi_0^{(N)} = 0.$$

The spectral decomposition of the Hamiltonian function is according to equation (3.16)

$$H = \sum_{n=0}^{\infty} n\hbar\omega \left(\frac{1}{\hbar^n n!} e^{-\frac{a\bar{a}}{\hbar}} \bar{a}^n a^n \right) = \omega a\bar{a}.$$

The factor $\hbar$ in the spectrum comes from the deformation parameter in the star-product and Hamiltonian of course remains a classic quantity.

We now consider the Groenewold-Moyal quantization scheme. Expression (3.7) rewritten in terms of holomorphic coordinates has the form

$$f \star_M g = f e^{\frac{\hbar}{2}(\overleftarrow{\partial_a}\overrightarrow{\partial_{\bar{a}}} - \overleftarrow{\partial_{\bar{a}}}\overrightarrow{\partial_a})} g.$$

Here we have

$$\bar{a} \star_M a = a\bar{a} + \frac{\hbar}{2}, \quad a \star_M \bar{a} = a\bar{a} - \frac{\hbar}{2}$$

and again

$$[a, \bar{a}]_{\star_M} = \hbar.$$

This is an expected result according to the note at the end of section 3.1. The standard star-product is c-equivalent to the Groenewold-Moyal star product with the transition operator

$$T = e^{\frac{\hbar}{2}\overrightarrow{\partial_a}\overrightarrow{\partial_{\bar{a}}}}.$$

We can use this operator to transform the standard product version of the $\star$-genvalue equation (3.15) into the corresponding Groenewold-Moyal product according to (3.2). The result is

$$H \star_M \pi_n^{(M)} = \omega \left(\bar{a} \star_M a + \frac{\hbar}{2} \right) \star_M \pi_n^{(M)} = \hbar\omega \left(n + \frac{1}{2} \right) \pi_n^{(M)},$$

with

$$\pi_0^{(M)} = T\pi_0^{(N)} = 2e^{-\frac{2a\bar{a}}{\hbar}} \tag{4.5}$$

$$\pi_n^{(M)} = T\pi_n^{(N)} = \frac{1}{\hbar^n n!} \bar{a}^n \star_M \pi_0^{(M)} \star_M a^n. \tag{4.6}$$

The projector onto the ground state $\pi_0^{(M)}$ satisifies

$$a \star_M \pi_0^{(M)} = 0.$$

The spectrum is now

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega, \tag{4.7}$$

which is the text book result. We conclude that for this problem the Groenewold-Moyal quantization scheme is the correct one.

We can now follow two different paths in order to get explicit formula for π_n^M. First possibility is direct computation of (4.6) whereas the same result should be achieved by solving equation (3.11) considering Groenewold-Moyal product. Let us calculate $\pi_n^{(M)} = T\pi_n^{(N)}$.

$$\begin{aligned} T\pi_n^{(N)} &= \exp\left(-\frac{\hbar}{2} \partial_a \partial_{\bar{a}} \right) \frac{1}{\hbar^n n!} \bar{a}^n a^n e^{-\frac{a\bar{a}}{\hbar}} \\ &= \frac{1}{\hbar^n n!} \bar{a}^n a^n \exp\left(-\frac{\hbar}{2} \left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a + \overrightarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} \right) \right) e^{-\frac{a\bar{a}}{\hbar}} \end{aligned}$$

When using the Campbell-Baker-Hausdorff formula one has to include appropriate exponential with commutator. In this case, however, commutator vanishes. Hence, we can factor out the last term in the exponent and apply equation (4.5) to get

$$
\begin{aligned}
& \frac{2}{\hbar^n n!}\bar{a}^n a^n \exp\left(-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a\right)\right) e^{-2\frac{a\bar{a}}{\hbar}} \\
= & \frac{2}{\hbar^n n!}\bar{a}^n a^n \exp\left(-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} +\right)\right) \exp\left(-\frac{\hbar}{2}\left(\overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a\right)\right) e^{-2\frac{a\bar{a}}{\hbar}} \\
= & \frac{2}{\hbar^n n!}\bar{a}^n a^n \exp\left(-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} +\right)\right) \exp\left(\overleftarrow{\partial}_{\bar{a}} \bar{a}\right) e^{-2\frac{a\bar{a}}{\hbar}}
\end{aligned}
$$

Commutator $\left[-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}\right), \overleftarrow{\partial}_{\bar{a}} \bar{a}\right]$ vanishes so we can exchange the order of the two exponentials in the last equation and then carry out the operations indicated by the first exponential

$$
\begin{aligned}
& \frac{2}{\hbar^n n!}\bar{a}^n a^n \left(\overleftarrow{\partial}_{\bar{a}} \bar{a}\right) \exp\left(-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}\right)\right) e^{-2\frac{a\bar{a}}{\hbar}} \\
= & \frac{2}{\hbar^n n!}(\bar{a}+\bar{a})^n a^n \exp\left(-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}\right)\right) e^{-2\frac{a\bar{a}}{\hbar}} \\
= & \frac{2}{\hbar^n n!}2^n \bar{a}^n a^n \exp\left(-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overleftarrow{\partial}_{\bar{a}}\right)\right) \exp\left(\overleftarrow{\partial}_a a\right) e^{-2\frac{a\bar{a}}{\hbar}}.
\end{aligned}
$$

Here we have used the Taylor formula in the form

$$
f(x+a) = e^{a\partial_x} f(x). \tag{4.8}
$$

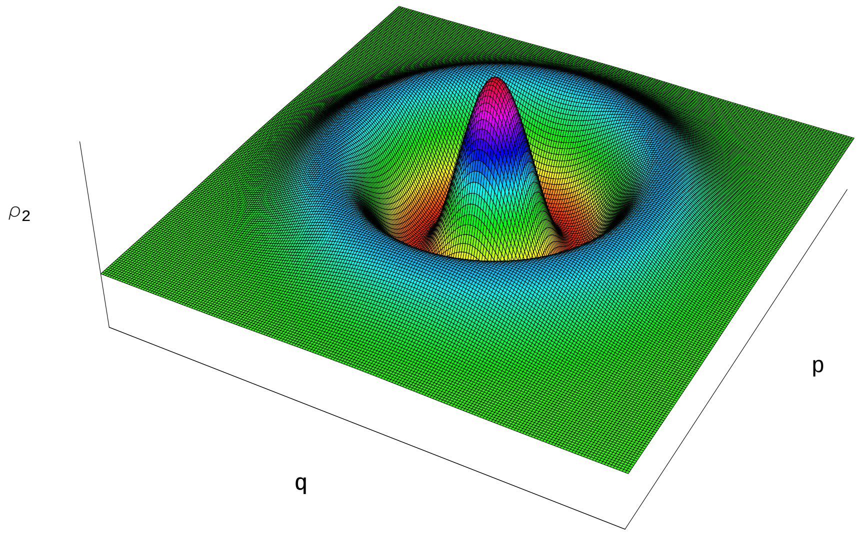
The first exponential can now be expanded

$$
\begin{aligned}
& \frac{2}{\hbar^n n!}2^2 \left(\sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{\hbar}{2}\right)\left(\partial_a^k a^n\right)\left(\partial_{\bar{a}}^k \bar{a}^n\right)\right) \exp\left(\overleftarrow{\partial}_a a\right) e^{-2\frac{a\bar{a}}{\hbar}} \\
= & \; 2\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\left(\frac{2}{\hbar}\right)^{n-k} \frac{n!}{(n-k)!(n-k)!} a^{n-k}\bar{a}^{n-k}\right) \exp\left(\overleftarrow{\partial}_a a\right) e^{-2\frac{a\bar{a}}{\hbar}} \\
= & \; (-1)^n 2 L_n\left(\frac{2a\bar{a}}{\hbar}\right) \exp\left(\overleftarrow{\partial}_a a\right) e^{-2\frac{a\bar{a}}{\hbar}} \\
= & \; (-1)^n 2 L_n\left(\frac{4a\bar{a}}{\hbar}\right) e^{-2\frac{a\bar{a}}{\hbar}}
\end{aligned} \tag{4.9}
$$

where we have used the definition of the Laguerre polynomials

$$
L_n(x) = \sum_{m=0}^{n} (-1)^m \frac{n!}{(n-m)!m!m!} x^m.
$$

Figure 4.1: WF for the harmonic oscillator, 2nd excited state.



Expression (4.9) is usually written in the form

$$\pi_n^{(M)} = 2(-1)^n e^{-\frac{2H}{\hbar\omega}} L_n \left(\frac{4H}{\hbar\omega}\right). \tag{4.10}$$

Corresponding Wigner function charts for 2st resp. 3rd excited state are given in figures 4.1 resp. 4.2.

As mentioned already $\pi_n^{(M)}$ can be again obtained by solving (3.11). Applying the Groenewold-Moyal product gives

$$i\hbar\frac{d}{dt}\mathrm{Exp}_N(Ht) = \left(H + \frac{\hbar}{2}\omega(\bar{a}\partial_{\bar{a}} - a\partial_a) - \frac{\hbar^2}{4}\omega\partial^2_{\bar{a}a}\right)\mathrm{Exp}_N(Ht).$$

Following the case of standard product we will again use substitution $s = \bar{a}a$

$$i\hbar\frac{d}{dt}F(s,t) = \omega\left(s - \frac{\hbar^2}{4}\partial_s - \frac{\hbar^2}{4}s\partial_s^2\right)F(s,t).$$

The solution is

$$F(s,t) = \frac{1}{\cos\frac{\omega t}{2}}\exp\left[\left(\frac{2s}{i\hbar}\right)\tan\frac{\omega t}{2}\right]$$

ie.

$$\mathrm{Exp}_N(Ht) = \frac{1}{\cos\frac{\omega t}{2}} \exp\left[\left(\frac{2H}{i\hbar\omega}\right)\tan\frac{\omega t}{2}\right].$$

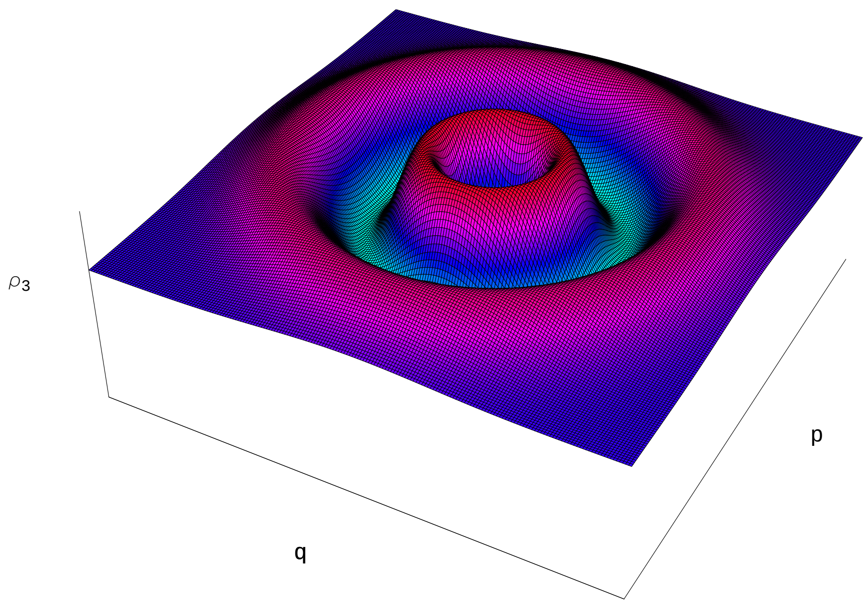
We can rewrite this solution in the following way

$$\begin{aligned} & \frac{2e^{-\frac{i\omega t}{2}}}{1+e^{-i\omega t}} \exp\left[\left(\frac{4H}{i\hbar\omega}\right)\frac{e^{-i\omega t2}-1}{2\left(1+e^{-i\omega t2}\right)}\right] \\ = \; & 2e^{-\frac{i\omega t}{2}} e^{-\frac{2H}{\hbar\omega}} \frac{1}{1+e^{-i\omega t}} \exp\left[\left(\frac{4H}{\hbar\omega}\right)\frac{e^{-i\omega t}}{1+e^{-i\omega t}}\right]. \end{aligned}$$

The last expression can be brought in the form of the Fourier-Dirichlet expansion by using the generating function for the Laguerre polynomials

$$\frac{1}{1+z}\exp\left[\frac{xz}{1+z}\right] = \sum_{n=0}^{\infty} z^n(-1)^n L_n(x).$$

Figure 4.2: WF for the harmonic oscillator, 3rd excited state.



By substituting $s = e^{-i\omega t}$ and $x = \frac{4H}{\hbar\omega}$ we get

$$\mathrm{Exp}_N(Ht) = \sum_{n=0}^{\infty} 2(-1)^n e^{-\frac{2H}{\hbar\omega}} L_n\left(\frac{4H}{\hbar\omega}\right) e^{-\frac{i\omega\hbar\left(n+\frac{1}{2}\right)}{\hbar}}.$$

Comparing with (3.12) we can see that projectors have exactly the form (4.10) and the spectrum is the same as in the equation (4.7).

4.2 Free Particle

We shall take units such that $\hbar = 2m = 1$. Let us present solution of $\star$-genvalue equation (3.15) for the Hamiltonian function

$$H = p^2.$$

Carrying out the Groenewold-Moyal product gives

$$\left(p - \frac{i}{2}\partial_q\right)^2 \rho(q,p) = E\rho(q,p). \tag{4.11}$$

Here we have used the shift formula

$$(f \star_M g)(q,p) = f\left(q + \frac{i\hbar}{2}\overrightarrow{\partial}_p, p - \frac{i\hbar}{2}\overrightarrow{\partial}_q\right) g(q,p).$$

This expression is used very often in deformation quantization and can be obtained from the definition (3.8), by repeated applications of the Taylor formula in the form given in expression (4.8). It follows that the imaginary part of this equation is

$$p\partial_q\rho(q,p) = 0, \tag{4.12}$$

while

$$\left(p^2 - \frac{1}{4}\partial_q^2\right)\rho(q,p) = E\rho(q,p) \tag{4.13}$$

is the real part.

It can be seen from equation (4.12) that we can conclude when $p \neq 0$ and $\partial_q\rho(q,p) = 0$. The case $p = 0$ requires further treatment. Substituting the ansatz

$$\rho(q,p) = f(p) + \delta(p)g(q)$$

into equation (4.13) leads to

$$\left(p^2 - E\right) f(p) - \delta(p)\left(E + \frac{1}{4}\partial_q^2\right) g(q) = 0. \tag{4.14}$$

Imposing $\rho = \rho^*$ and considering $p = 0$, then one has

$$g(q) = be^{2i\sqrt{E}q} + b^* e^{-2i\sqrt{E}q}. \tag{4.15}$$

Equation (4.14) then reduces to

$$(p^2 - E)f(p) = 0$$

with solution

$$f(p) = a_+\delta(p - \sqrt{(E)}) + a_-\delta(p + \sqrt{E}), \tag{4.16}$$

where $a_\pm$ ar arbitrary real constants.

The terms of (4.16) correspond to plane waves of momentum $\sqrt{E}$, as this can be verified by solving

$$p \star_M \rho = \sqrt{E}\rho.$$

The expression

$$\rho = a_+\delta(p - \sqrt{E}) + a_-\delta(p + \sqrt{E})$$

is the Wigner function of a mixed state of two momentum eigenstates. The terms from the right side of (4.15) are necessary for coherent superpositions of two momentum eigenstates, and they represent interference between them.

The general result is

$$\rho(q, p) = \delta(p)\left(be^{2i\sqrt{E}q} + b^* e^{-2i\sqrt{E}q}\right) + a_+\delta(p - \sqrt{E}) + a_-\delta(p + \sqrt{E})$$

In order to get a pure-state Wigner function we shall request

$$(\rho \star_M \rho)(q, p) \propto \delta(0)\rho(q, p). \tag{4.17}$$

This is the requirement valid for Wigner functions corresponding to non-normalizable pure states. Groenewold-Moyal product gives

$$\begin{aligned}(\rho \star_M \rho)(q,p) &= \delta(0)\Big((a_+^2 + |b|^2)\delta(p-\sqrt{E}) + (a_-^2 + |b|^2)\delta(p+\sqrt{E}) \\ &+ (a_+ + a_-)\delta(p)\left[be^{2i\sqrt{E}q} + b^* e^{-2i\sqrt{E}q}\right]\Big).\end{aligned}$$

In (4.17), this result gives rise to condition

$$|b|^2 = a_+ a_- \quad \Rightarrow b = \sqrt{a_+ a_-}e^{i\phi}, \quad \phi \in \mathbb{R}.$$

Finally we can write the general pure-state solution to the free-particle $\star$-genvalue equation (4.11) in the form

$$\rho(q,p) = a_+\delta(p-\sqrt{E}) + a_-\delta(p+\sqrt{E}) + 2\sqrt{a_+a_-}\delta(p)\cos(2\sqrt{E}q+\phi) \qquad (4.18)$$

Considering the pure-state wave function

$$\psi = \alpha_+ e^{i\sqrt{E}q} + \alpha_- e^{-i\sqrt{E}q} \qquad (4.19)$$

and substituting into (3.19) yields

$$\rho^{(W)}(q,p) = |\alpha_+|^2\delta(p-\sqrt{E}) + |\alpha_-|^2\delta(p+\sqrt{E}) + \delta(p)\left(\alpha_+^*\alpha_- e^{-2i\sqrt{E}q} + \alpha_+\alpha_-^* e^{2i\sqrt{E}q}\right).$$

By comparing this expression with (4.18) we obtain following relations

$$\alpha_\pm = \sqrt{a_\pm}e^{i\phi_\pm}, \quad \phi - \phi_+ + \phi_- = 0.$$

This is an expected result since only relative phase $\phi_+ - \phi_-$ of (4.19) is relevant to the Wigner function.

Chapter 5

Infinite Walls and Wells

Infinite wall and infinite square well are represented by potentials that are solved easily in conventional quantum mechanical approach. It turns out, however, that treatment of these potentials in deformation quantization is far from being straightforward. In this chapter we present computation methods introduced in [15] for the time-independent case.

5.1 Introduction

Treating systems with simple potentials like infinite wall or well in deformation quantization is not s simple task as it is in conventional quantum mechanics. Let us consider equation (3.15) more carefully in the case of infinite wall for example, i.e. the potential is given by

$$V(q) = \begin{cases} 0, & q < 0 \\ \infty, & q > 0 \end{cases}$$

Following the Schrödinger approach one would restrict to $q < 0$, and impose the boundary condition $\pi(0, p) = 0$. It is the $\star$-genvalue equation for a free particle in the case of $q < 0$ with its real (4.13) and imaginary (4.12) parts that should be solved considering this boundary condition. However, this procedure doesn't give any sensible results. The expected result is given by Weyl transform of the following Schrödinger wave function

$$\psi(q) = \mathcal{H}(-q)\left(e^{i\sqrt{E}q} - e^{-i\sqrt{E}q}\right),$$

where $\mathcal{H}(q)$ is the Heaviside step function. So using (3.19) one has

$$\rho^{(W)} = \mathcal{H}(-q)\bar{\rho}(q,p),$$

with

$$\begin{aligned}\bar{\rho}(q,p) = \frac{2\sin\left(2q(p+\sqrt{E})\right)}{p+\sqrt{E}} + \frac{2\sin\left(2q(p-\sqrt{E})\right)}{p-\sqrt{E}} + \\ +2\cos(2q\sqrt{E})\frac{2\sin(2qp)}{p}.\end{aligned} \tag{5.1}$$

But this $\bar{\rho}(q,p)$ does not satisfy the imaginary part of free particle $\star$-genvalue equation given by (4.12).

Recently several attempts were made to solve this problem. In [14] it was shown that (5.1) is the solution of the modified $\star$-genvalue equation

$$\left(\left(p^2 + \delta'_q\right) \star_M \rho\right)(q,p) = E\rho(q,p),$$

where the generalized distribution $\delta_-(q)$ is defined by

$$\int dq\delta_-(q)\tilde{t}(q) = \lim_{\varepsilon\to 0^+}\int dq\delta(q)\tilde{t}(q-\varepsilon).$$

Kryukov and Walton [15] were able to introduce a computing procedure which makes it possible to derive the differential equation that is solved by (5.1) from the unchanged $\star$-genvalue equation. This approach seems to treat the problem in a more systematic way and therefore we will present their results here. We want to stress the fact that Kryukov and Walton treated the time-independent case only and application of their method for time-dependent cases may not be straightforward.

In order to be able to work with simple potentials like infinite wall or well, in a systematic and consistent way further study will be needed.

In the whole chapter units are taken so that $\hbar = 2m = 1$.

5.2 Infinite Wall

We shall consider the Liouville Hamiltonian

$$H_\alpha = p^2 + e^{2\alpha q}. \tag{5.2}$$

Taking the $\alpha \to \infty$ limit of $V(q) = e^{2\alpha q}$ gives an infinite wall with

$$V(q) = \begin{cases} 0, & q < 0 \\ \infty, & q > 0 \end{cases}$$

Pure deformation has already been carried out in [16] and performing the preceding limit yields correct Wigner function. Our effort here is, however, to find a differential equation from the $\star$-genvalue equation that would lead to the physical Wigner function.

Using the shift formula for the Groenewold-Moyal product gives

$$(H_\alpha \star_M \rho)(q,p) = \left[\left(p - \frac{i}{2}\partial_q\right)^2 + e^{2\alpha\left(q+\frac{i}{2}\partial_p\right)}\right]\rho(q,p) = E\rho(q,p).$$

The imaginary part is

$$\left(-p\partial_q + e^{2\alpha q}\sin(\alpha\partial_p)\right)\rho(q,p) = 0$$

while the real part read

$$\left(p^2 - E - \frac{1}{4}\partial_x^2 + e^{2\alpha q}\cos(\alpha\partial_p)\right)\rho(q,p) = 0.$$

Formally, these equations can be rewritten as

$$e^{-2\alpha q}\partial_q\rho(q,p) = -\frac{i}{2p}\left(\rho(q,p+i\alpha) - \rho(q,p-i\alpha)\right) \tag{5.3}$$

and

$$e^{-2\alpha q}\left(p^2 - E - \frac{1}{4}\partial_q^2\right)\rho(q,p) + \frac{1}{2}\left(\rho(q,p+i\alpha) + \rho(q,p-i\alpha)\right) = 0. \tag{5.4}$$

Using equation (5.3) to find $\partial_q^2 \rho(q,p)$, and substituting the result into the equation (5.4) leads to an expression without derivatives

$$
\begin{aligned}
0 &= (p^2 - E)\rho(q,p) + \frac{1}{p}\left(\frac{e^{2\alpha q}}{4}\right)^2 \\
&\times \left(\frac{\rho(q,p+2i\alpha) - \rho(q,p)}{p+i\alpha} + \frac{\rho(q,-2i\alpha) - \rho(q,p)}{p-i\alpha}\right) \\
&- \frac{ie^{2\alpha q}}{4p}\left(\rho(q,p+i\alpha) - \rho(q,p-i\alpha)\right) \\
&+ \frac{e^{2\alpha q}}{2}\left(\rho(q,p+i\alpha) + \rho(q,p-i\alpha)\right). \qquad (5.5)
\end{aligned}
$$

We can consider this result as a difference equation in the momentum variable whereas only imaginary shifts of the momentum arguments are involved. This means that besides $\rho(q,p)$, equation (5.5) involves the four quantities

$$\rho(q,p \pm i\alpha), \quad \rho(q,p \pm 2i\alpha). \qquad (5.6)$$

The limit $\alpha \to \infty$ of (5.5) fishy. We can trade, however, the four quantities of (5.6) for the derivatives

$$\partial_q^m \rho(q,p), \quad m = 1,2,3,4. \qquad (5.7)$$

The resulting differential equation will possess a well-defined limit $\alpha \to \infty$ as we will see.

Four equations relating the "variables" of (5.6) to those of (5.7) are required. Two of them are already provided by equations (5.3) and (5.4). The other two can be obtained by taking derivatives of equation (5.4). The resulting expressions have the form

$$
\begin{aligned}
0 &= \partial_q^3 \rho - 4(p^2 - E)\partial_q \rho(q,p) \\
&+ 4\alpha e^{2\alpha q}\left(\rho(q,p+i\alpha) + \rho(q,p-i\alpha)\right) \\
&- 2e^{2\alpha q}\left(\partial_q \rho(q,p+i\alpha) + \partial_q \rho(q,p-i\alpha)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \partial_q^4 \rho(q,p) - 4(p^2 - E)\partial_q^2 \rho(q,p) \\
&- 8\alpha^2 e^{2\alpha q}\left(\rho(q,p+i\alpha) + \rho(q,p-i\alpha)\right) \\
&- 4\alpha i e^{4\alpha q}\left(\frac{\rho(q,p+2i\alpha) - \rho(q,p)}{p+i\alpha} + \frac{\rho(q,-2i\alpha) - \rho(q,p)}{p-i\alpha}\right) \\
&+ 2e^{2\alpha q}\Big[\left(4(p+i\alpha)^2 - E\right)\rho(q,p+i\alpha) \\
&+ 2e^{2\alpha q}\left(\rho(q,p+2i\alpha) + \rho(q,p)\right) \\
&+ \left(4(p-i\alpha)^2 - E\right)\rho(q,p-i\alpha) \\
&+ 2e^{2\alpha q}\left(\rho(q,p-2i\alpha) + \rho(q,p)\right)\Big].
\end{aligned}
$$

Kryukov-Walton were able to find a simple differential equation

$$
\begin{aligned}
0 &= \frac{1}{16}\partial_q^4 \rho(q,p) + \frac{1}{2}(p^2 + E)\partial_q^2 \rho(q,p) \\
&+ (p^4 - 2Ep^2 + E^2)\rho(q,p) - e^{4\alpha q}\rho(q,p).
\end{aligned}
$$

The limit $\alpha \to \infty$ leads to the new equation

$$
\left(\frac{1}{16}\partial_q^4 + \frac{1}{2}(p^2 + E)\partial_q^2 + (p^4 - 2Ep^2 + E^2)\right)\rho(q,p) = 0, \tag{5.8}
$$

which is valid for $q < 0$. The Wigner function (5.1) satisfies equation (5.8) so we conclude that we have found what we were looking for.

5.3 Infinite Square Well

Once we have solved the infinite wall case, infinite square well should be straightforward as it can be constructed form two infinite walls. More precisely we will consider the $\alpha \to \infty$ limit of the $\star$-genvalue equation following from the sinh-Gordon Hamiltonian

$$
H_\alpha = p^2 + e^{-2\alpha(x+1)} + e^{2\alpha(x-1)}.
$$

The limit gives rise to potential

$$V(q) = \begin{cases} \infty & q < -1 \\ 0, & q \in]-1, 1[\\ \infty, & q > 1 \end{cases}$$

Again we start with $\star$-genvalue equation and use the shift formula

$$\begin{aligned} (H_\alpha \star_M) \rho(q,p) &= \left[\left(p - \frac{i}{2}\partial_q \right)^2 + 2e^{-2\alpha} \cosh\left(2\alpha \left(x + \frac{i}{2}\partial_p \right) \right) \right] \rho(q,p) \\ &= E\rho(q,p). \end{aligned}$$

The imaginary part read

$$\partial_q \rho(q,p) = -\frac{ie^{-2\alpha}}{p} \left(\rho(q, p+i\alpha) - \rho(q, p-i\alpha) \right) \sinh(2\alpha q)$$

whereas the real part is

$$\begin{aligned} 0 &= \left(p^2 - E - \frac{1}{4}\partial_q^2 \right) \rho(q,p) \\ &+ e^{-2\alpha} \left(\rho(q, p+i\alpha) + \rho(q, p-i\alpha) \right) \cosh(2\alpha q). \end{aligned}$$

By combining the two preceding equations we get

$$\begin{aligned} 0 &= (p^2 - E)\rho(q,p) \\ &+ \frac{e^{-4\alpha}}{4p} \cosh^2(2\alpha q) \left(\frac{\rho(q,p+2i\alpha) - \rho(q,p)}{p+i\alpha} + \frac{\rho(q,p) - \rho(q,p-2i\alpha)}{p-i\alpha} \right) \\ &- \frac{i\alpha e^{-2\alpha}}{2p} \cosh(2\alpha q) \left(\rho(q,p+i\alpha) - \rho(q,p-i\alpha) \right) \\ &- e^{-2\alpha} \cosh(2\alpha q) \left(\rho(q,p+i\alpha) + \rho(q,p-i\alpha) \right) \end{aligned}$$

This equations involves no derivatives, but only $\rho(q,p)$ and the quantities

$$\rho(q, p \pm i\alpha), \quad \rho(q, p \pm 2i\alpha).$$

We again wish to replace these quantities by derivatives

$$\partial_q^m \rho(q,p), \quad m = 1, 2, 3, 4.$$

For this purpose we need to compute additional two derivatives similarly to the infinite wall case. Resulting expressions are as follows

$$
\begin{aligned}
\partial_q^3 \rho(q,p) &= 4(p^2-E)\partial_q \rho(q,p) \\
&+ 2\alpha e^{-2\alpha}\sinh(2\alpha q)\left(\rho(q,p+i\alpha)+\rho(q,p-i\alpha)\right) \\
&- 4ie^{-4\alpha}\cosh(2\alpha q)\sinh(2\alpha q)\times \\
&\times \left(\frac{\rho(q,p+2i\alpha)-\rho(q,p)}{p+i\alpha}+\frac{\rho(q,p)-\rho(q,p-2i\alpha)}{p-i\alpha}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_q^4 \rho(q,p) &= 4(p^2-E)\partial_q^2 \rho(q,p) \\
&+ 16\alpha^2 e^{-2\alpha}\cosh(2\alpha q)\left(\rho(q,p+i\alpha)+\rho(q,p-i\alpha)\right) \\
&+ 16i\alpha e^{-4\alpha}\sinh^2(2\alpha q)\times \\
&\times \left(\frac{\rho(q,p+2i\alpha)-\rho(q,p)}{p+i\alpha}+\frac{\rho(q,p)-\rho(q,p-2i\alpha)}{p-i\alpha}\right) \\
&+ e^{-2\alpha}\cosh(2\alpha q)\Big[\left(4(p+i\alpha)^2-E\right)\rho(q,p+i\alpha) \\
&- e^{-2\alpha}\cosh(2\alpha q)\left(\rho(q,p+2i\alpha)+\rho(q,p)\right) \\
&+ \left(4(p-i\alpha)^2-E\right)\rho(q,p-i\alpha) \\
&- e^{-2\alpha}\cosh(2\alpha q)\left(\rho(q,p)+\rho(q,p-2i\alpha)\right)\Big].
\end{aligned}
$$

From these expressions we get a quite complicated new equation. This equation is well defined with respect to the limiting process $\alpha \to \infty$ for $x \in]-1,1[$. The limit gives

$$
\left(\frac{1}{16}\partial_q^4 + \frac{1}{2}(p^2+E)\partial_q^2 + (p^4 - 2Ep^2 + E^2)\right)\rho = 0. \tag{5.9}
$$

We conclude that we have acquired the same equation as in the one-wall case.

Let us consider Schrödinger wave function for the infinite square well

$$
\psi(q) = \mathcal{H}(-q+1)\mathcal{H}(q+1)\cos(\sqrt{E_n}q), \quad E_n = \frac{n^2\pi^2}{4}
$$

Applying the Weyl transform and using (3.19) gives

$$
\rho^{(W)}(q,p) = \mathcal{H}(-x+1)\mathcal{H}(x+1)\bar{\rho}(q,p),
$$

where

$$\bar{\rho}(q,p) = \frac{\sin\left((2p+n\pi)(1-|q|)\right)}{2p+n\pi} + \frac{\sin\left((2p-n\pi)(1-|q|)\right)}{2p-n\pi} + \frac{\cos(n\pi q)\sin\left(2p(1-|x|)\right)}{q},$$

or equivalently

$$\bar{\rho}(q,p) = \frac{\sin\left(2(p+\sqrt{E_n})(1-|q|)\right)}{2(p+\sqrt{E_n})} + \frac{\sin\left(2(p-\sqrt{E_n})(1-|q|)\right)}{2(p-\sqrt{E_n})} + \frac{\cos\left(2\sqrt{E}q\right)\sin\left(2p(1-|q|)\right)}{p}.$$

This Wigner function is the solution of (5.9) valid for $V = 0$.

Another Schrödinger wave function for the infinite square well

$$\psi(q) = \mathcal{H}(-q+1)\mathcal{H}(q+1)\sin(\sqrt{E_n}(q-1)), \quad E_n = \frac{n^2\pi^2}{4}$$

that by means of the Weyl transform (3.19) leads to the Wigner function

$$\rho^{(W)}(q,p) = \mathcal{H}(-x+1)\mathcal{H}(x+1)\bar{\rho}(q,p),$$

where

$$\begin{aligned}\bar{\rho}(q,p) \;&=\; \frac{n}{-8p^3+2n^2p\pi^2}\Big[n\pi\cos\left(n\pi(|q|-1)\right)\sin\left(2p(|q|-1)\right) \\ &-\; 2p\cos\left(2p(|q|-1)\right)\sin\left(n\pi(|q|-1)\right)\Big]\end{aligned}$$

solves (5.9) for $V = 0$. Figures 5.1, 5.2, 5.3 and 5.4 correspond to Wigner function charts for 1st, 2nd, 3rd and 4th excited state respectively.

Figure 5.1: WF for the infinite square well, 1st excited state.

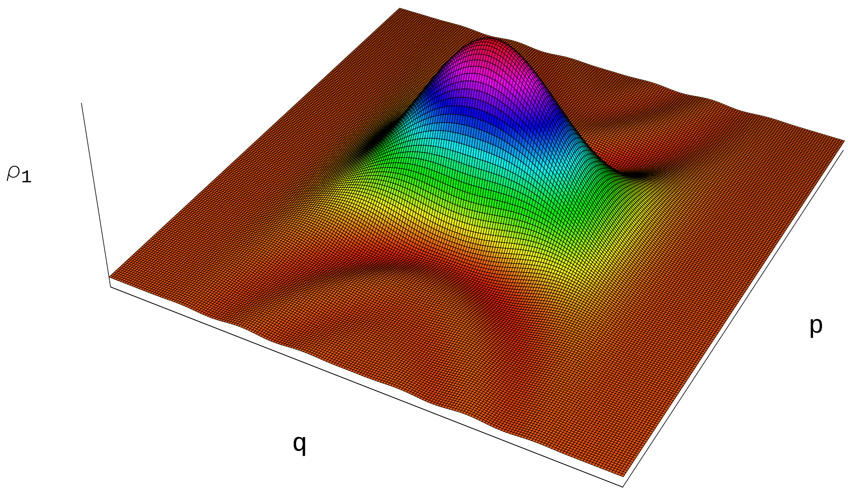


Figure 5.2: WF for the infinite square well, 2nd excited state.

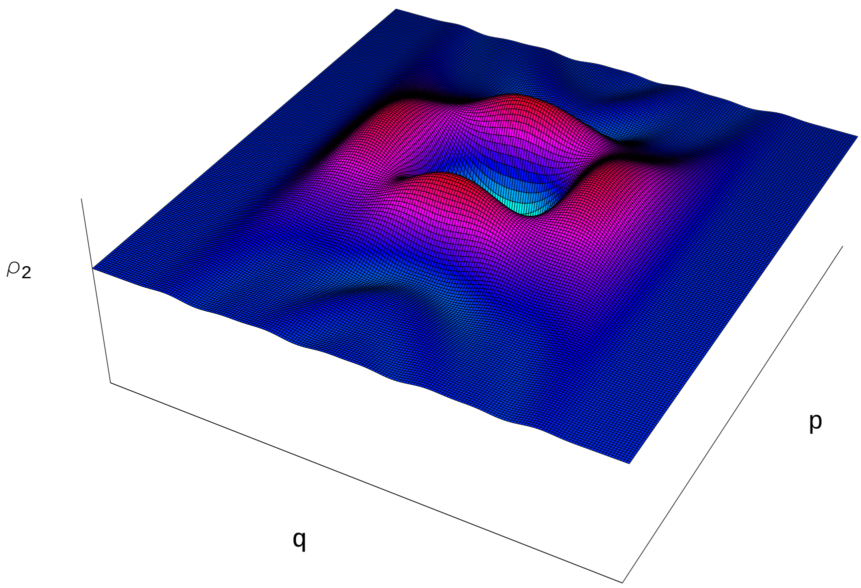


Figure 5.3: WF for the infinite square well, 3rd excited state.

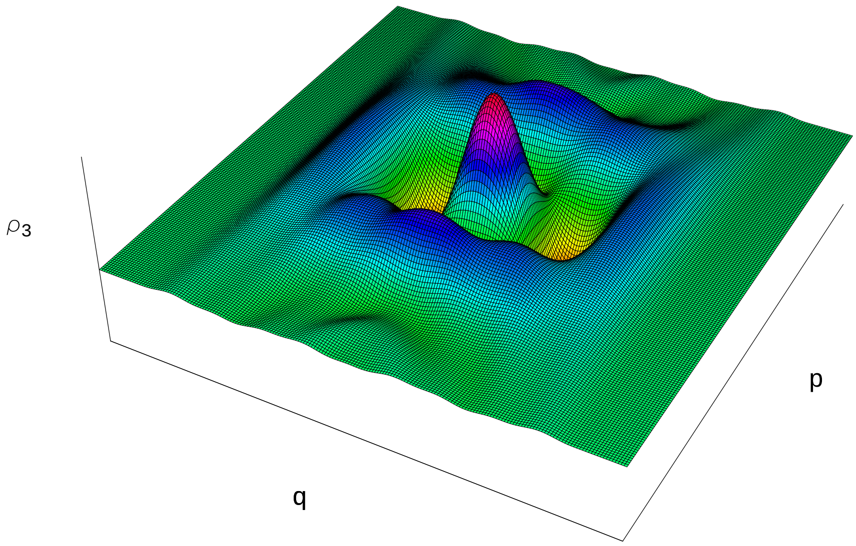
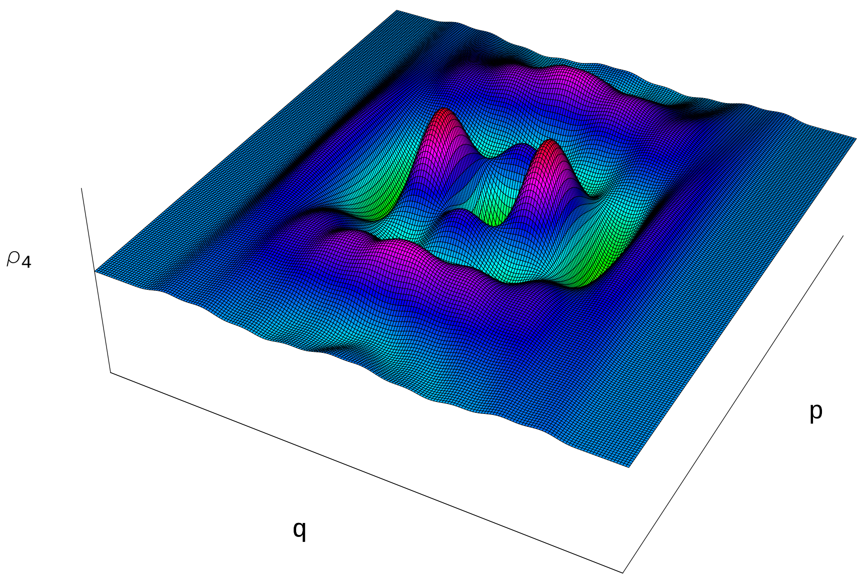


Figure 5.4: WF for the infinite square well, 4th excited state.



Conclusion

Deformations of associative algebras and Lie algebras were introduced. The aim was, in the case of a Poisson algebra of phase-space functions, to construct the $\star$-product as a special type of deformation. From mathematical point of view equivalence classes of $\star$-product were established whereas two equivalent $\star$-products were called c-equivalent, where "c" stands for cohomology. This is supposed to emphasize the fact that c-equivalent $\star$-products need not to be physically equivalent.

Considering phase space $\mathbb{R}^{2n}$, then all $\star$-products are c-equivalent but lead to slightly different spectra. Equivalence of deformation quantization to the conventional formulation of quantum mechanics is given by Weyl correspondence and in this case is expressed by the fact that distinct $\star$-products are related to different operator orderings in conventional quantum mechanical approach.

Basic deformation quantization principles were then given without further detail. We would like to stress out that objects considered in this phase-space theory require careful mathematical analysis. But likewise in any other physical theory, the existence of a quantity is often determined by calculating it, which is the case of examples provided here.

It turns out that deformation quantization and ordinary quantum mechanical formulation are not only equivalent but moreover the quantum mechanical algebra of observables is a representation of the $\star$-product algebra. It follows that specifying the $\star$-product completely determines the quantum system because in the algebraic approach to quantum theory (C^*-algebra of bounded operators) all the information concerning the quantum system may be extracted from the algebra of observables.

Complete time-dependent solution for harmonic oscillator is given using first principles and is followed by time-independent solutions of one-dimensional free particle, infinite wall and infinite square well. In particular in the case of two last named examples we have seen that solution in terms of deformation quantization is far from being straightforward which is in contrast with simple treatment in Schrödinger formalism. In many other cases the situation might be opposite to this one which underlines the fact that conventional quantum mechanical approach and deformation quantization rather complement each other.

In the case of infinite wall and infinite square well a recent method introduced by Krykov-Walton is presented and a new solution of their differential equation is proposed.

This work as well as the study of extended literature lead us to the conclusion that deformation quantization formulation of quantum mechanics is far from complete. While the algebra of observables is established by the $\star$-product, quantum state space as the space of normed linear functionals on the algebra, remains to be defined in a mathematically rigorous way.

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