Czech Technical University in Prague
Faculty of Nuclear Sciences and
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Diploma thesis

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Stability of time-dependent quantum systems

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Podpis

Název práce: **Stabilita časově závislých kvantových systémů**

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Vedoucí práce: Prof. Ing. Pavel Šťovíček, DrSc., Katedra matematiky, FJFI, ČVUT
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Abstrakt: Studujeme stabilitu časově zavislých kvantových systémů. Zejména se zabýváme periodicky časově zavislými systémy a vztahem mezi spektrálními vlastnostmi Floquetova operátoru a vázanými a volnými stavy. Uvádíme geometrickou a topologickou definici vázaných respektive volných stavů a shrnujeme základní výsledky této teorie. Dále analyzujeme jednodimenzionální harmonický oscilátor narušený skoro periodickou vnější silou. Zabýváme se nabitou častice v rovině a pod vlivem homogenního magnetického pole a periodicky časově závislého Aharonova-Bohmova toku. Příslušný Hamiltonián zkoumáme jak na úrovni kvantové tak klasické.

Klíčová slova: Matematická fyzika, kvantová stabilita, Floquetů operator, skoro periodické funkce, Aharonovův-Bohmův efekt.

Title: **Stability of time-dependent Quantum Systems**

Author: Tomáš Kalvoda

Abstract: We study stability of time-dependent quantum systems. Especially we are interested in periodically time-dependent systems and relation between spectral properties of the Floquet operator and the bound and free states. Two ways how to deal with bound and free states are considered. In particular we present geometrical and topological definitions of bound and free states in quantum theory. Next we analyse one dimensional harmonic oscillator perturbed by almost periodic external force. Further we investigate charged particle in the plane under influence of homogenneous magnetic field and time-periodic Aharonov-Bohm flux, in both classical and quantum framework.

Keywords: Mathematical physics, quantum stability, Floquet operator, almost periodic functions, Aharonov-Bohm effect.

Contents

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List of Symbols

Symbol	Meaning
M'	accumulation points of the set M
$\mathbb{C}$	complex numbers
δ_{kn}	Kronecker's symbol
$\operatorname{dom} A$	domain of operator A
$\mathscr{H}$	separable Hilbert space
$\mathscr{H}_{\pm}^{p}$	states with precompact trajectories
$\mathscr{H}_{\pm}^{f}$	free states
$\mathbb{I}$	identity operator
$\Im$	imaginary part of complex number
$\imath$	imaginary unit
$\langle \cdot, \cdot \rangle$	inner product in $\mathscr{H}$
$[a, b]$	closed interval
$\overline{K}$	closure of operator or set
$\overline{a}$	complex conjugate of a
(a, b)	open interval
$\mathscr{M}_{\pm}^{f}$	geometrically free states
$\mathscr{M}_{\pm}^{bd}$	geometrically bounded states
$\hat{n}$	$\{1, 2, \ldots, n\}$
$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	natural numbers with zero
$\Re$	real part of complex number
$\mathbb{R}$	real numbers
$\restriction$	restriction
∂_t	partial derivative with respect to t
$\mathbb{R}_+$	positive real numbers
$\mathscr{S}$	Schwartz space of rapidly decreasing functions defined on $\mathbb{R}$
$\subset$	inclusion
$\sum$	summation
$\cup$, $\bigcup$	union
$\mathbb{Z}$	integral numbers

1.
Introduction

In this work we study time-dependent quantum systems, especially we are interested in characterisation of bound states and propagating (or free, scattering) states. Mathematically, quantum system is described by Hilbert space $\mathscr{H}$ and family of self-adjoint operators $H(t)$, $t \in \mathbb{R}$ acting in $\mathscr{H}$. Each state of the system corresponds to some vector ψ in $\mathscr{H}$. Time evolution of the vector ψ is described by mapping $t \mapsto \psi(t) \in \mathscr{H}$ with initial condition $\psi(s) = \psi$ for some time $s \in \mathbb{R}$. This evolution is governed by Schrödinger equation

$$\imath \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = H(t)\psi(t).$$

This initial value problem can be reformulated using the notion of unitary propagator. Family of unitary operators $U(t,s)$ obeying

$$U(t,t) = \mathbb{I} \tag{1.1}$$

$$U(t,s) = U(t,r)U(r,s), \ \forall t,s,r \in \mathbb{R}, \tag{1.2}$$

and

$$\imath\partial_t U(t,s) = H(t)U(t,s), \tag{1.3}$$

where the meaning of the last equation is to be specified, is called unitary propagator. In the particular situation we obtain $H(t)$ by physical reasoning and we face the question of existence of unitary propagator and validity of (1.3). This problem can be solved under some additional assumptions interposed on $H(t)$. As a example we cite here Theorem X.70 from [RS75].

Theorem 1.1: *Let $H(t)$ be a self-adjoint operator-valued function of $t \in \mathbb{R}$ such that*

(i) the domain $\mathscr{D} = \operatorname{dom} H(t)$ is independent of t,

(ii) the function

$$(t,s) \mapsto (t-s)^{-1}\left[(\imath + H(t))(\imath + H(s))^{-1} - \mathbb{I}\right]$$

extends to a jointly strongly continuous bounded operator-valued function on $\mathbb{R}^2$.

Then there exists a unique propagator U satisfying (1.1), (1.2) such that $U(t,s)\mathscr{D} \subset \mathscr{D}$ and

$$\imath\partial_t U(t,s)\psi = H(t)U(t,s)\psi, \ \forall \psi \in \mathscr{D}.$$

In the last chapter of this document we present example when assumptions of this theorem are not satisfied.

In this work we are especially intrested in T-periodic Hamiltonians, i.e. $H(t) = H(t+T)$. If in this case the last theorem is applicable, then the uniqueness of propagator implies, that

$$U(t+T, s+T) = U(t,s), \ \forall t,s \in \mathbb{R}.$$

Such a unitary propagator is then called T-periodic. Operator $U(s+T,s)$ is called the monodromy operator, or Floquet operator. Obviously it is unitary equivalent to $U(T,0)$ for any $s \in \mathbb{R}$. Its importance is obvious from relation

$$U(t+nT,s) = U(t,s)\,[U(s+T,s)]^n\,,\ n \in \mathbb{Z},$$

i.e. it is sufficient to know $U(t,s)$ for one period $[s,s+T]$. In particular the spectral properties of the Floquet operator are important. Let us now make small detour.

It is useful to point out main aspects of time-independent case. The system is now described by self-adjoint operator H with domain $\operatorname{dom} H \subset \mathscr{H}$. Time evolution of vector $\psi \in \mathscr{H}$ is given by Schrödinger equation, which takes form

$$\imath \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = H\psi(t),\ \psi(0) = \psi. \tag{1.4}$$

This can be solved with aid of Stone's theorem (cf. [RS72]). The unitary propagator in this case is $U(t,s) = \exp(-\imath H(t-s))$, or shortly $U(t) = \exp(-\imath Ht)$. Stone's theorem now states that $\psi(t) = U(t)\psi$ is solution of (1.4).

Bounded states and propagating states are related to spectral properties of Hamiltonian H. More precisely, we define pure point respectively continuous part of Hilbert space $\mathscr{H}$ with respect to H by

$$\begin{aligned}\mathscr{H}^{pp}(H) &= \overline{\operatorname{span}\{\text{eigenvectors of } H\}},\\ \mathscr{H}^{cont}(H) &= \mathscr{H}^{pp}(H)^{\perp}.\end{aligned}$$

These sets are directly related to the spectral properties of H. Equivalent definition of pure point part and continuous part of Hilbert space is as follows. Denote P_λ the partition of unity corresponding to H, i.e.

$$H = \int\limits_{\mathbb{R}} \lambda \mathrm{d}P_\lambda.$$

Furthermore set $\mu_\psi(\lambda) = \langle \psi, P_\lambda \psi \rangle$. We say that $\psi \in \mathscr{H}^{pp}(H)$ if μ_ψ is pure point, $\psi \in \mathscr{H}^{ac}(H)$ if μ_ψ is absolutely continuous with respect to Lebesgue measure and $\psi \in \mathscr{H}^{sc}(H)$ if μ_ψ is singular continuous with respect to Lebesgue measure. Continuous subspace is $\mathscr{H}^{cont}(H) = \mathscr{H}^{ac}(H) \oplus \mathscr{H}^{sc}(H)$. It holds, that

$$\begin{aligned}\sigma_{pp}(H) &= \sigma(H \restriction \mathscr{H}^{pp}(H)),\\ \sigma_{cont}(H) &= \sigma(H \restriction \mathscr{H}^{cont}(H)),\\ \sigma_{ac}(H) &= \sigma(H \restriction \mathscr{H}^{ac}(H)),\\ \sigma_{sc}(H) &= \sigma(H \restriction \mathscr{H}^{sc}(H)).\end{aligned}$$

There are many ways how to describe bound or free states. Mainly we are interested in certain properties of trajectories $U(t,0)\psi$. Two of these approaches are summarised in Chapter 2. Briefly, the main result is (when we are considering time-independent Hamiltonians), that the bound states are from $\mathscr{H}^{pp}(H)$ and the free states are from $\mathscr{H}^{cont}(H)$. Classical result is due to Ruelle, Amrein, Georgescu and Enss (cf. [RS75]).

Theorem 1.2 (RAGE Theorem): *Let H be self-adjoint operator on Hilbert space $\mathscr{H}$ and C bounded operator which is relatively compact to H, i.e. the operator*

$$C(H+\imath)^{-1}$$

is compact. Then it holds that

(i) For all ψ in $\mathscr{H}^{cont}(H)$

$$\lim_{\tau\to\infty} \frac{1}{2\tau} \int\limits_{-\tau}^{\tau} \|Ce^{-\imath tH}\psi\|^2 \ \mathrm{d}t = 0.$$

(ii) For all ψ in $\operatorname{dom} H$

$$\frac{1}{2\tau} \int\limits_{-\tau}^{\tau} \|Ce^{-\imath tH} P^{cont}\psi\|^2 \ \mathrm{d}t \le \varepsilon(\tau)\|(H+\imath)\psi\|^2,$$

where P^{cont} is orthogonal projector onto $\mathscr{H}^{cont}(H)$ and $\varepsilon(\tau) \to 0$ as $\tau \to \infty$.

In the Chapter 2 it will be shown that in the time-dependent case the role of H is transferred to the Floquet operator $U(T,0)$. Therefore we will be interested in decomposition

$$\mathscr{H} = \mathscr{H}^{pp}(U(T,0)) \oplus \mathscr{H}^{cont}(U(T,0)) \tag{1.5}$$

In particular, it will be shown that states from $\mathscr{H}^{pp}(U(T,0))$ have precompact trajectories and that states from $\mathscr{H}^{cont}(U(T,0))$ leave any compact subset of $\mathscr{H}$. I.e.[1]

$$\mathscr{H}^{pp}(U(T,0)) = \mathscr{H}^{p}_{\pm}(U),$$
$$\mathscr{H}^{cont}(U(T,0)) = \mathscr{H}^{f}_{\pm}(U).$$

We will study geometrically bounded and free states $\mathscr{M}^{f}_{\pm}$, $\mathscr{M}^{bd}_{\pm}$. There the distinction is made using the probability of measurement of position.

[1] For exact definition see Chapter 2.

2.
Bound states and propagating states in quantum theory

In this Chapter we will review geometrical and topological approach to bound states and scattering states. We mainly follow [EV83] and [dO95].

2.1 Topological approach

We will start with definition of states with precompact trajectories $\mathscr{H}_\pm^p$, and its complement $\mathscr{H}_\pm^f$.

Definition 2.1: *Vector $\psi \in \mathscr{H}$ belongs to the set $\mathscr{H}_\pm^p$, if and only if the set $\{U(t,0)\psi | t \gtrless 0\}$ is precompact.*[1]

Definition 2.2: *We say that vector $\psi \in \mathscr{H}$ belongs to the set $\mathscr{H}_\pm^f$, if and only if*

$$\lim_{\tau\to\pm\infty} \frac{1}{\tau} \int_0^\tau \|KU(t,0)\psi\|^2 \, \mathrm{d}t = 0 \tag{2.1}$$

holds for any compact operator K.

The trajectory $\{\psi(t)|t \gtrless 0\}$ of the state $\psi \in \mathscr{H}_\pm^p$ is precompact, thus approximately finite-dimensional. On the other hand state from $\mathscr{H}_\pm^f$ will leave any finite-dimensional set during its time evolution. It follows that the set $\mathscr{H}_\pm^p$, respectively $\mathscr{H}_\pm^f$ can be thought of as a set of bounded and propagating states. If we want to emphasise that we work with propagator U we will write $\mathscr{H}_\pm^f(U)$ respectively $\mathscr{H}_\pm^p(U)$. These two sets are closed linear subspaces of $\mathscr{H}$, mutually orthogonal. This is contents of following Lemma and Theorem.

Lemma 2.3: *Let the set $M \subset \mathscr{H}$ be precompact. Then for each positive ε there exists finite-dimensional orthogonal projector C, such that $(1-C)M \subset B_\varepsilon$, where B_ε is a ball of radius ε and centre in origin.*

Proof. For given $\varepsilon > 0$ there is finite $\frac{\varepsilon}{2}$-net corresponding to the set M (we will denote it by $\{x_i\}_{i=1}^n$). Let C be orthogonal projector onto linear span of M. Choose $y \in M$ arbitrarily. We can find $i \in \hat{n}$ obeying $\|x_i - y\| < \frac{\varepsilon}{2}$. Thus

$$\|Cy - y\| \le \underbrace{\|Cy - x_i\|}_{=\|C(y-x_i)\|\le\varepsilon/2} + \|x_i - y\| \le \frac{\varepsilon}{2} \le \varepsilon.$$

In other words $(1-C)y \in B_\varepsilon$ for any $y \in M$. □

[1] *The sign + corresponds to > and − to <. Similar convention will be used throughout this work.*

Theorem 2.4: *(i) For any $\psi \in \mathscr{H}^f_\pm(U)$ and arbitrary compact operator K it holds*

$$\lim_{\tau \to \pm\infty} \frac{1}{\tau} \int_0^\tau \|K \; U(t,0)\psi\| \; \mathrm{d}t = 0.$$

(ii) $\mathscr{H}^f_\pm(U)$ and $\mathscr{H}^p_\pm(U)$ are closed orthogonal subspaces of $\mathscr{H}$.

Proof. (i) Choose $\psi \in \mathscr{H}^f_\pm(U)$, compact operator K and $\tau > 0$. Schwartz inequality implies that

$$\frac{1}{\tau} \int_0^\tau \|K \; U(t,0)\psi\| \; \mathrm{d}t \le \left(\frac{1}{\tau} \int_0^\tau \|KU(t,0)\psi\|^2 \; \mathrm{d}t \right)^{1/2}.$$

This proofs case (i).

(ii) Both sets are obviously linear subspaces. We will now show closeness of $\mathscr{H}^f_\pm(U)$. Let $\phi \in \overline{\mathscr{H}^f_\pm(U)}$, $\varepsilon > 0$ and K be compact operator. Then we can find $\psi \in \mathscr{H}^f_\pm(U)$ fullfilling $\|\psi - \phi\| < \varepsilon$. Thus

$$\begin{aligned}
\frac{1}{\tau} \int_0^\tau \|KU(t,0)\phi\|^2 \; \mathrm{d}t &\le \frac{1}{\tau} \int_0^\tau \left(\|KU(t,0)(\phi-\psi)\| + \|KU(t,0)\psi\|\right)^2 \; \mathrm{d}t \\
&\le \frac{1}{\tau} \int_0^\tau \left(\|K\| \cdot \|\phi-\psi\| + \|KU(t,0)\psi\|\right)^2 \; \mathrm{d}t \\
&\le \|K\|^2 \|\phi-\psi\|^2 + \frac{1}{\tau} \int_0^\tau \|KU(t,0)\psi\|^2 \; \mathrm{d}t + 2\|K\|^2 \|\psi\| \cdot \|\phi-\psi\|.
\end{aligned}$$

This expression can be made arbitrarily small by choosing τ sufficiently large and ε small. Therefore $\phi \in \mathscr{H}^f_\pm(U)$. The closeness of $\mathscr{H}^f_\pm(U)$ is proved.

The closeness of $\mathscr{H}^p_\pm(U)$ can be treated by similar manner. Let $\phi \in \overline{\mathscr{H}^p_\pm(U)}$ and $\varepsilon > 0$. Then there is $\psi \in \mathscr{H}^p_\pm(U)$ such as $\|\psi - \phi\| < \varepsilon$. Therefore

$$\{U(t,0)\phi | t \gtrless 0\} \subset \{U(t,0)(\phi-\psi) | t \gtrless 0\} + \{U(t,0)\psi | t \gtrless 0\}. \tag{2.2}$$

The first set on the right hand side of (2.2) is arbitrarily small and the second is precompact. Thus the set on the left hand side of (2.2) is also precompact. Indeed. If $N = \{x_i\}_{i=1}^n$ denotes finite ε-net of the set $\{U(t,0)\psi | t \gtrless 0\}$, then we can find $i \in \hat{n}$ such that

$$\|U(t,0)\psi - x_i\| < \varepsilon$$

for any $t \gtrless 0$. Therefore

$$\|U(t,0)\phi - x_i\| \le \|\phi - \psi\| + \|U(t,0)\psi - x_i\| < 2\varepsilon.$$

We see that the set N is finite 2ε-net corresponding to the trajectory $\{U(t,0)\phi | t \gtrless 0\}$. This is therefore precompact and $\mathscr{H}^p_\pm(U)$ is closed subspace.

Let now $\varphi \in \mathscr{H}_\pm^f(U)$ and $\eta \in \mathscr{H}_\pm^p(U)$. To check the orthogonality we rewrite the inner product as

$$|\langle \varphi, \eta \rangle| = \frac{1}{\tau} \int_0^\tau |\langle U(t,0)\varphi, U(t,0)\eta \rangle| \, \mathrm{d}t.$$

For each $\varepsilon > 0$ there is a finite-dimensional projector P_ϵ fullfilling $(1 - P_\epsilon)U(t,0)\eta \in B_\varepsilon$ for any $t \gtrless 0$ (cf. Lemma 2.3). According to (i) in Theorem 2.4 we can write, for sufficiently large τ,

$$\begin{aligned} |\langle \varphi, \eta \rangle| &\leq \frac{1}{\tau} \int_0^\tau |\langle P_\epsilon U(t,0)\varphi, U(t,0)\eta \rangle| \, \mathrm{d}t + \frac{1}{\tau} \int_0^\tau |\langle U(t,0)\varphi, (1 - P_\epsilon)U(t,0)\eta \rangle| \, \mathrm{d}t \\ &\leq \|\eta\| \frac{1}{\tau} \int_0^\tau \|P_\epsilon U(t,0)\varphi\| \, \mathrm{d}t + \|\varphi\| \frac{1}{\tau} \int_0^\tau \|(1 - P_\epsilon)U(t,0)\eta\| \, \mathrm{d}t \\ &\leq \|\eta\|\varepsilon + \|\varphi\|\varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary the orthogonality was proved.

□

In this paragraph we will show that in case of T-periodic propagator the Hilbert space decays to orthogonal sum $\mathscr{H}_\pm^f \oplus \mathscr{H}_\pm^f$ and that this decay corresponds to (1.5). We will need following Lemma.

Lemma 2.5: *Let $U(t,s)$ be T-periodic propagator and P^{cont} orthogonal projector onto subspace $\mathscr{H}^{cont}(U(T,0))$. Then for any compact operator C it holds that*

$$\lim_{|\tau| \to \infty} \left\| \frac{1}{\tau} \int_0^\tau U^*(t,0) C U(t,0) P^{cont} \, \mathrm{d}t \right\| = 0. \tag{2.3}$$

Proof. Choose $\tau \in \mathbb{R}$, $\tau = \sigma + nT$ and $\sigma \in [0, T)$. Then

$$\left\| \frac{1}{\tau} \int_0^\tau U^*(t,0) C U(t,0) P^{cont} \, \mathrm{d}t \right\| \leq \frac{\sigma}{\tau} \|C\| + \left\| \frac{1}{nT} \int_0^{nT} U^*(t,0) C U(t,0) P^{cont} \, \mathrm{d}t \right\| \tag{2.4}$$

The first term in the last equation tends to zero as $\tau \to \infty$, because $\sigma < T$. The expression in the norm on the right hand side can be recast in

$$\begin{aligned} &\frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{T} \int_{jT}^{(j+1)T} U^*(t,0) C U(t,0) P^{cont} \, \mathrm{d}t = \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{T} \int_0^T U^*(t + jT, 0) C U(t + jT, 0) \, \mathrm{d}t. \end{aligned} \tag{2.5}$$

Because U is T-periodic it holds true that

$$\begin{aligned} U(t + jT, 0) &= U(t,0)[U(T,0)]^j, \\ U(t + jT, 0)^* &= [U(T,0)^*]^j U(t,0)^*. \end{aligned}$$

So (2.5) is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1} U^*(T,0)^j \underbrace{\left(\frac{1}{T}\int_0^T U^*(t,0)CU(t,0)\ \mathrm{d}t\right)}_{C'} U(T,0)^j P^{cont}.$$

Operator C' is again compact[2] and therefore it can be arbitrarily accurately approximated by finitedimensional operator. More precisely, there exists operators C_0' and D such as C_0' is finitedimensional and $C' = C_0' + D$. $\|D\|$ can be chosen as small as we please independently on τ. Therefore the second term in (2.4) can be estimated by

$$\left\| \frac{1}{n}\sum_{j=0}^{n-1} U^*(T,0)^j C_0' U(T,0)^j P^{cont} \right\| + \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} \|D\|}_{<\frac{1}{\tau}}.$$

Without loss of generality it is sufficient to consider only one onedimensional operator C''. We can find $\psi, \phi \in \mathscr{H}$, $\|\psi\| = 1$ such that

$$C'' = \langle \phi, \cdot \rangle \psi \text{ and } C''^* = \langle \psi, \cdot \rangle \phi.$$

Next, mind that if A is bounded operator then it holds true

$$\|AA^*\| = \sup_{\|\varphi\|=1} \langle \varphi, AA^*\varphi \rangle = \sup_{\|\varphi\|=1} \|A^*\varphi\|^2 = \|A^*\|^2 = \|A\|^2.$$

We will apply this equality to $A = \frac{1}{n}\sum_{j=0}^{n-1} U^*(T,0)^j C'' U(T,0)^j P^{cont}$. First observe, that for any $\eta \in \mathscr{H}$ it holds that

$$A^*\eta = \frac{1}{n}\sum_{j=0}^{n-1} P^{cont} U^*(T,0)^j \langle \psi, U(T,0)^j \eta \rangle \phi,$$

and

$$AA^*\eta = \frac{1}{n^2}\sum_{j,k=0}^{n-1} \langle \psi, U(T,0)^k \eta \rangle \langle \phi, U(T,0)^j P^{cont} U^*(T,0)^k \phi \rangle U^*(T,0)^j \psi,$$

[2] Indeed, let $A : [a,b] \to \mathscr{B}(\mathscr{H})$ be norm continuous, then the integral

$$B = \int_a^b A(t)\ \mathrm{d}t$$

exists in Riemann sense. The operator B is given by relation

$$\langle \varphi, B\psi \rangle = \int_a^b \langle \varphi, A(t)\psi \rangle\ \mathrm{d}t,\ \forall \varphi, \psi \in \mathscr{H}.$$

If moreover $A(t)$ is compact for any t then B is compact too.

because $U(T,0)$ and P^{cont} commute. The norm of the last expression can be estimated by

$$\|AA^*\eta\| \leq \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\langle P^{cont}\phi, U(T,0)^j U^*(T,0)^k P^{cont}\phi\rangle\right| \|\eta\|.$$

And for the second term in equation (2.4) we finally have

$$\|A\|^2 = \|AA^*\| \leq \frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\langle P^{cont}\phi, U(T,0)^{j-k} P^{cont}\phi\rangle\right| \leq \tag{2.6}$$

$$\leq \left\{\frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\langle P^{cont}\phi, U(T,0)^{j-k}, P^{cont}\phi\rangle\right|^2\right\}^{1/2}. \tag{2.7}$$

Obviously it is sufficient to consider only $\phi \in \mathscr{H}^{cont}(U(T,0))$. We will use spectral theorem for unitary operators. The monodromy operator is unitary and therefore there is partition of unity (on unit circle in the complex plane) $E(\lambda)$ such as

$$U(T,0) = \int_0^{2\pi} e^{\imath\lambda}\mathrm{d}E(\lambda).$$

We will denote $\nu_\phi(\lambda) := \langle \phi, E(\lambda)\phi\rangle$. Than it holds true that

$$\langle \phi, f(U(T,0))\phi\rangle = \int_0^{2\pi} f(e^{\imath\lambda})\mathrm{d}\nu_\phi(\lambda)$$

for measurable f.

Let us turn our attention to the last term in (2.6), using above note we can write

$$\frac{1}{n^2}\sum_{j,k=0}^{n-1}\left|\langle \phi, U(T,0)^{j-k}\phi\rangle\right|^2 = \int_0^{2\pi}\int_0^{2\pi}\underbrace{\left|\frac{1}{n}\sum_{j=0}^{n-1} e^{\imath j(\lambda-\mu)}\right|^2}_{f_n(\lambda-\mu)}\mathrm{d}\nu_\phi(\lambda)\mathrm{d}\nu_\phi(\mu). \tag{2.8}$$

The sum of geometric series is just

$$\sum_{j=0}^{n-1} e^{\imath j(\lambda-\mu)} = \frac{1-e^{\imath n(\lambda-\mu)}}{1-e^{\imath(\lambda-\mu)}}.$$

Therefore

$$f_n(x) = \left|\frac{1}{n}\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right|^2.$$

Next observe, that for any natural n and real x it holds that

$$\left|\frac{\sin nx}{n\sin x}\right| \leq 1.$$

This means that $f_n(\lambda-\mu)\le 1$, moreover

$$f_n(\lambda-\mu)\le \frac{1}{n^2\sin^2 x},$$

where $\lambda,\mu\in[0,2\pi]$ and $x=\frac{\lambda-\mu}{2}\in[-\pi,\pi]$. Choose $\delta\in(0,\pi)$. Then for x which obeys inequality $|\frac{x}{2}-\pi k|>\frac{1}{2}\delta$, $k=1,0,1$ is $\sin^2 x\ge C_\delta>0$. In other words

$$\frac{1}{n^2\sin^2 x}\le\frac{\text{const}(\delta)}{n^2},\quad\text{for } x\in\left[-\pi+\frac{\delta}{2},-\frac{\delta}{2}\right]\cup\left[\frac{\delta}{2},\pi-\frac{\delta}{2}\right].$$

Thus for such λ,μ we have following inequality

$$|f_n(\lambda-\mu)|\le\frac{\text{const}(\delta)}{n^2}.$$

And for (2.8) we have

$$\int_0^{2\pi}\int_0^{2\pi} f_n(\lambda-\mu)\mathrm{d}\nu_\phi(\lambda)\mathrm{d}\nu_\phi(\mu)=\sum_{k=-1}^{1}\Bigg\{\iint\limits_{|\lambda-\mu-2k\pi|<\delta} f_n(\lambda-\mu)\mathrm{d}\nu_\phi(\lambda)\mathrm{d}\nu_\phi(\mu)+ \tag{2.9}$$

$$+\underbrace{\iint\limits_{|\lambda-\mu-2k\pi|>\delta} f_n(\lambda-\mu)\mathrm{d}\nu_\phi(\lambda)\mathrm{d}\nu_\phi(\mu)}_{\le\frac{\text{const}(\delta)}{n^2}}\Bigg\}. \tag{2.10}$$

We can naturally assume that $E(\lambda)=0$ for $\lambda<0$ and $E(\lambda)=I$ for $\lambda>2\pi$. The first term in curly bracket is smaller then or equal to

$$\iint\limits_{|\lambda-\mu-2k\pi|<\delta}\mathrm{d}\nu_\phi(\lambda)\mathrm{d}\nu_\phi(\mu)=\int\limits_{\mathbb{R}}\mathrm{d}\nu_\phi(\lambda)\left(\int\limits_{|\lambda-\mu-2k\pi|<\delta}\mathrm{d}\nu_\phi(\mu)\right)=\int\limits_{\mathbb{R}}\mathrm{d}\nu_\phi(\lambda)\Big[\nu_\phi(\mu)\Big]_{\lambda-2k\pi-\delta}^{\lambda-2k\pi+\delta}.$$

The second term in (2.9) tends to zero as $n\to\infty$. Recall that $\delta>0$ can by chosen arbitrarily small. Because $\phi\in\mathscr{H}^{cont}(U(T,0))$ the function $\lambda\mapsto\langle\phi,E(\lambda)\phi\rangle$ is continuous and therefore

$$\lim_{\delta\to 0}\left[\nu_\phi(\mu)\right]_{\lambda-2k\pi-\delta}^{\lambda-2k\pi+\delta}=0.$$

The Lemma is proved. □

Corollary 2.6: *Let C be compact operator and $\psi\in\mathscr{H}^{cont}(U(T,0))$, then*

$$\frac{1}{\tau}\int_0^\tau\|CU(t,0)\psi\|\ \mathrm{d}t\le f(\tau)\|\psi\|, \tag{2.11}$$

where $f(\tau)\to 0$ as $|\tau|\to\infty$.

Proof. Using Schwartz inequality[3] it holds true that for $\psi \in \mathscr{H}^{cont}(U(T,0))$

$$\left(\frac{1}{\tau}\int_0^\tau \|CU(t,0)\psi\| \,\mathrm{d}t\right)^2 \leq \frac{1}{\tau}\int_0^\tau \|CU(t,0)\psi\|^2 \,\mathrm{d}t =$$
$$= \left\langle \psi, \frac{1}{\tau}\int_0^\tau U^*(t,0)C^*CU(t,0)P^{cont} \,\mathrm{d}t\psi \right\rangle \leq$$
$$\leq \left\| \frac{1}{\tau}\int_0^\tau U^*(t,0)C^*CU(t,0)P^{cont} \,\mathrm{d}t \right\| \|\psi\|^2 \equiv f^2(\tau)\|\psi\|^2.$$

The operator C^*C is compact, so from the preceding Lemma it follows that $\lim_{|\tau|\to\infty} f(\tau) = 0$. □

Theorem 2.7: *Let U be T-periodic unitary propagator, then*

$$\mathscr{H}^p_{\pm} = \mathscr{H}^{pp}(U(T,0)). \tag{2.14}$$

Proof. We will show both inclusions.

- $\mathscr{H}^{pp}(U(T,0)) \subset \mathscr{H}^p_{\pm}$: Let ψ be eigenvector of $U(T,0)$. I.e. there is $\alpha \in \mathbb{R}$ such as $U(T,0)\psi = e^{-\imath\alpha}\psi$. Therefore the set
$$\left\{U(t,0)\psi \middle| t \in \mathbb{R}\right\} = \left\{e^{-\imath n(t)\alpha}U(\sigma,0)\psi \middle| t = \sigma + nT \in \mathbb{R},\ \sigma \in [0,T)\right\}$$
is subset of $M = S_1 \cdot \{U(t,0)\psi | t \in [0,T]\}$. The mapping $t \mapsto U(t,0)\psi : [0,T] \to \mathscr{H}$ is continuous and the interval $[0,T]$ is comapact, consequently the set M is compact. This also means that any finite linear combination of eigenvectors of Floquet operator lies in $\mathscr{H}^p_{\pm}$, because any finite sum of compact sets is compact. And because $\mathscr{H}^{pp}(U(T,0))$ is closed subspace, the inclusion holds.

[3] It is true that

$$\int_a^b \|A(t)\psi\|^2 \,\mathrm{d}t = \int_a^b \langle \psi, A(t)^*A(t)\psi\rangle \,\mathrm{d}t = \tag{2.12}$$
$$= \left\langle \psi, \int_a^b A(t)^*A(t) \,\mathrm{d}t\psi \right\rangle \leq \left\| \int_a^b A(t)^*A(t) \,\mathrm{d}t \right\| \|\psi\|^2. \tag{2.13}$$

If $A = \int_a^b A(t)\,\mathrm{d}t$ then for $B \in \mathscr{B}(\mathscr{H})$ we have

$$BA = \int_a^b BA(t) \,\mathrm{d}t,\ AB = \int_a^b A(t)B \,\mathrm{d}t,$$

because for $\psi, \phi \in \mathscr{H}$ we have

$$\langle \phi, \int_a^b BA(t) \,\mathrm{d}t\psi\rangle = \int_a^b \langle B^*\phi, A(t)\psi\rangle = \langle \phi, BA\psi\rangle.$$

The second equality can be shown similarly

- $\mathscr{H}^p_\pm \subset \mathscr{H}^{pp}(U(T,0))$: Let $\phi \in \mathscr{H}^p_+$, $\|\phi\| = 1$ and $\varepsilon > 0$. Then the set $M = \{U(t,0)\phi | t > 0\}$ is precompact and by virtue of Lemma 2.3 there is a finite-dimensional orthogonal projector C such as $(C-1)M \subset B_\varepsilon$. Thus

$$\sup_{t>0} \|(C-1)U(t,0)\phi\| < \varepsilon.$$

For $\psi \in \mathscr{H}^{cont}(U(T,0))$, $\|\psi\| = 1$ there is $\tau > 0$ such as[4]

$$\frac{1}{\tau}\int_0^\tau \|CU(t,0)\psi\| \, \mathrm{d}t < \varepsilon.$$

It follows that there is some $t_\varepsilon < \tau$ fullfilling

$$\int_0^\tau \|CU(t,0)\psi\| \, \mathrm{d}t = \tau \|CU(t_\varepsilon,0)\psi\|,$$

and therefore $\|CU(t_\varepsilon,0)\psi\| < \varepsilon$. Finally

$$\begin{aligned}|\langle\phi,\psi\rangle| &= |\langle U(t_\varepsilon,0)\phi, U(t_\varepsilon,0)\psi\rangle| = \\ &= |\langle(1-C)U(t_\varepsilon,0)\phi, U(t_\varepsilon,0)\psi\rangle + \langle U(t_\varepsilon,0)\phi, CU(t_\varepsilon,0)\psi\rangle| \le \\ &\le \|(1-C)U(t_\varepsilon,0)\phi\| + \|CU(t_\varepsilon,0)\psi\| \le 2\varepsilon.\end{aligned}$$

ε can be chosen as small as we please, thus

$$\mathscr{H}^p_\pm \perp \mathscr{H}^{cont}(U(T,0)).$$

By the definition we have $\mathscr{H} = \mathscr{H}^{pp}(U(T,0)) \oplus \mathscr{H}^{cont}(U(T,0))$, thus necessarily $\mathscr{H}^p_\pm \subset \mathscr{H}^{pp}(U(T,0))$.

□

Theorem 2.8: *Let U be T-periodic propagator on Hilbert space $\mathscr{H}$. Then*

$$\mathscr{H}^f_\pm(U) = \mathscr{H}^{cont}(U(T,0)), \tag{2.15}$$

in particular $\mathscr{H} = \mathscr{H}^p_\pm(U) \oplus \mathscr{H}^f_\pm(U)$.

Proof. By Theorem 2.4 and Theorem 2.7 we showed, that $\mathscr{H}^f_\pm(U) \subset \mathscr{H}^{cont}(U(T,0))$. The opposite inclusion is to be proved. Choose $\psi \in \mathscr{H}^{cont}(U(T,0))$ and K compact operator. Then it holds

$$\frac{1}{\tau}\int_0^\tau \|KU(t,0)\psi\|^2 \, \mathrm{d}t = \frac{1}{\tau}\int_0^\tau \langle U(t,0)\psi, K^*KU(t,0)\psi\rangle \, \mathrm{d}t \le \|\psi\| \frac{1}{\tau}\int_0^\tau \|K^*KU(t,0)\psi\| \, \mathrm{d}t \to 0$$

as $\tau \to \infty$, because K^*K is compact and we used Corollary 2.6.. □

[4] cf. Corollary 2.6, C is finite-dimensional, and therefore compact.

2.2 Geometrical approach

We will now define geometrically bounded and propagating states, following [EV83]. Let $\{P_r\}_{r\geq 0}$ by family of bounded linear operators defined on separable Hilbert space $\mathscr{H}$ and obeying following conditions

$$P_r = P_r^*, \quad \|P_r\| \leq 1 \quad \text{a} \quad \operatorname*{s-lim}_{r\to\infty} P_r = \mathbb{I}. \tag{2.16}$$

In most cases we will have $\mathscr{H} = \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ and for $\{P_r\}$ we will choose family of orthogonal projectors[5] $\{F(|Q| < R)\}$.

Definition 2.9: *We say that $\psi \in \mathscr{M}_{\pm}^{bd}(P)$ if and only if*

$$\lim_{r\to\infty} \sup_{t\gtrless 0} \|(1 - P_r)U(t,0)\psi\| = 0. \tag{2.17}$$

Set $\mathscr{M}_{\pm}^{bd}(P)$ is called the set of geometrically bounded states.

The definition obviously depends on our choice of family P_r. The motivation of previous definition in the case when $\mathscr{H}$ is L^2 space and $P_R = F(|Q| < R)$ is clear. The probability to find element of $\mathscr{M}_{\pm}^{bd}(P)$ in some instant t located outside ball of radius R tends to zero when the radius R is large. To simplify our notation we will write only $\mathscr{M}_{\pm}^{bd}$ if L^2 space and $F(|Q| < R)$ case is under consideration. Same note will hold true for set of geometrically propagating states defined below. Let's now start with examination of basic properties of the set $\mathscr{M}_{\pm}^{bd}(P)$. Notice, that in the proof of the following theorem there is no need of T-periodicity of propagator $U(t,s)$

Theorem 2.10: *Set $\mathscr{M}_{\pm}^{bd}(P)$ is closed linear subspace of Hilbert space $\mathscr{H}$ and it holds that $\mathscr{H}_{\pm}^{p} \subset \mathscr{M}_{\pm}^{bd}(P)$.*

Proof. Obviously $\mathscr{M}_{\pm}^{bd}(P)$ are linear subspaces. Closeness is nontrivial. Let $\psi \in \overline{\mathscr{M}_{\pm}^{bd}(P)}$ then for each $\varepsilon > 0$ there is $\phi \in \mathscr{M}_{\pm}^{bd}(P)$ such that $\|\psi - \phi\| < \varepsilon$. It holds

$$\|(1-P_r)U(t,0)\psi\| \leq \underbrace{\|(1-P_r)U(t,0)(\psi - \phi)\|}_{\leq 2\varepsilon} + \|(1-P_r)U(t,0)\phi\|.$$

Because $\varepsilon > 0$ can be chosen arbitrarily small and $\phi \in \mathscr{M}_{\pm}^{bd}(P)$ it implies that also $\psi \in \mathscr{M}_{\pm}^{bd}(P)$.

Let now $\psi \in \mathscr{H}_{\pm}^{p}$ than for $\varepsilon > 0$ there is finite ε-net $N = \{x_i\}_{i=1}^n$ of the set $\{U(t,0)\psi | t \gtrless 0\}$. Therefore for given $t \in \mathbb{R}$ there is i such that $\|x_i - U(t,0)\psi\| < \varepsilon$. On the other hand for each i exists $r_0(i)$ such that $\forall r > r_0(i)$ we have $\|(1-P_r)x_i\| < \varepsilon$, because of our assumption that $\operatorname{s-lim}_{r\to\infty} P_r = 1$. Thus for any given $t \in \mathbb{R}$ it holds

$$\|(1-P_r)U(t,0)\psi\| \leq \|(1-P_r)x_i\| + \|(1-P_r)(U(t,0)\psi - x_i)\| \leq \varepsilon + 2\varepsilon.$$

Summarising we see that for any $\varepsilon > 0$ and $\forall r > \max_i r_0(i)$ it is true that $\sup_{t\gtrless 0} \|(1 - P_r)U(t,0)\psi\| < 3\varepsilon$. This is equivalent to $\psi \in \mathscr{M}_{\pm}^{bd}(P)$. □

The opposite inclusion can be proved if we put additional condition on propagator U.

[5] $F(|Q| < R)$ is multiplication operator on $\mathrm{L}^2(\mathbb{R}^n, \mathrm{d}^n x)$ by open ball of radius R.

Definition 2.11: *We will say that the family $\{P_r\}$ is relatively compact with respect to U at $\pm\infty$ (or shortly $\pm U$-compact) if the set*

$$\{P_r U(t,0)\psi | t \lesseqgtr 0\} \tag{2.18}$$

is precompact in $\mathscr{H}$ for any r and all $\psi \in \mathscr{H}$.

Theorem 2.12: *Let U propagator on $\mathscr{H}$ and let family $\{P_r\}$ be $\pm U$-compact. Then*

$$\mathscr{M}_{\pm}^{bd}(P) = \mathscr{H}_{\pm}^{p}. \tag{2.19}$$

Proof. It suffices to prove inclusion $\mathscr{M}_{\pm}^{bd}(P) \subset \mathscr{H}_{\pm}^{p}$. Put $\psi \in \mathscr{M}_{\pm}^{bd}(P)$, then it holds

$$\{U(t,0)\psi | t \gtrless 0\} \subset \{P_r U(t,0)\psi | t \gtrless 0\} + \{(1-P_r)U(t,0)\psi | t \gtrless 0\}.$$

Due to our assumptions the first set on right hand side of the last equation is precompact. The second one is arbitrarily small, because r can be chosen arbitrarily large. Thus the left hand side is precompact (cf. proof of Theorem 2.4 (ii)). □

We will now turn to propagating states.

Definition 2.13: *We say that $\psi \in \mathscr{M}_{\pm}^{f}(P)$ if and only if*

$$\lim_{\tau\to\pm\infty} \frac{1}{\tau} \int_0^{\tau} \|P_r U(t,0)\psi\| \, \mathrm{d}t = 0, \ \forall r \geq 0. \tag{2.20}$$

Set $\mathscr{M}_{\pm}^{f}(P)$ is called the set of geometrically unbounded states.

In case when $P_R = F(|Q| < R)$ this means that if $\psi \in \mathscr{M}_{\pm}^{f}$ than the mean time value of probability to find the system in open ball of any radius R is equal to zero. In other words such a state will leave any geometrically bounded region of configuration space. In the rest of this section we will describe the connection betwen $\mathscr{M}_{\pm}^{f}(P)$ and $\mathscr{M}_{\pm}^{bd}(P)$.

Theorem 2.14: *Set $\mathscr{M}_{\pm}^{f}(P)$ is closed linear subspace, orthogonal to $\mathscr{M}_{\pm}^{bd}(P)$, i.e.*

$$\mathscr{M}_{\pm}^{f}(P) \perp \mathscr{M}_{\pm}^{bd}(P) \tag{2.21}$$

Proof. The linearity and closeness can be easily checked. Let now $\psi \in \mathscr{M}_{\pm}^{f}(P)$, $\phi \in \mathscr{M}_{\pm}^{bd}(P)$, $\|\psi\| = \|\phi\| = 1$ and $\varepsilon > 0$. Because

$$\lim_{\tau\to\pm\infty} \frac{1}{\tau} \int_0^{\tau} \|P_r U(t,0)\psi\| \, \mathrm{d}t = 0$$

there exists t_ε such that $\|P_r U(t_\varepsilon,0)\psi\| < \varepsilon$. It also holds $\sup_{t\gtrless 0} \|(1-P_r)U(t,0)\phi\| < \varepsilon$ for some $r > 0$. Thus

$$|\langle\psi,\phi\rangle| \leq \|P_r U(t_\varepsilon,0)\psi\| + \sup_{t\gtrless 0} \|(1-P_r)U(t,0)\phi\| < 2\varepsilon.$$

Therefore (2.21) is true. □

We are now ready to prove the main result of this section.

Theorem 2.15 (Abstract RAGE theorem): *Let U be T-periodical propagator on separable Hilbert space $\mathscr{H}$. Let the family $\{P_r\}$ be $\pm U$-compact. Then it holds that*

$$\mathscr{M}^{f}_{\pm}(P) = \mathscr{H}^{cont}(U(T,0)), \tag{2.22}$$
$$\mathscr{M}^{bd}_{\pm}(P) = \mathscr{H}^{pp}(U(T,0)). \tag{2.23}$$

In particular

$$\mathscr{M}^{f}_{\pm}(P) \oplus \mathscr{M}^{bd}_{\pm} = \mathscr{H}. \tag{2.24}$$

Proof. Let $\psi \in \mathscr{H}^{cont}(U(T,0))$, $\varepsilon > 0$ and $r > 0$. Due to our assumptions the set

$$\{P_r U(t,0)\psi | t \gtrless 0\}$$

is precompact in $\mathscr{H}$ and thus by the Lemma 2.3 there is a finitedimensional projector P_ϵ such as

$$\sup_{t \gtrless 0} \|(1-P_\epsilon)P_r U(t,0)\psi\| < \varepsilon.$$

Therefore

$$\frac{1}{\tau}\int_0^\tau \|P_r U(t,0)\psi\| \,\mathrm{d}t \le \frac{1}{\tau}\int_0^\tau \underbrace{\|(1-P_\epsilon)P_r U(t,0)\psi\|}_{\le\varepsilon} \,\mathrm{d}t + \frac{1}{\tau}\int_0^\tau \|P_\epsilon P_r U(t,0)\psi\| \,\mathrm{d}t. \tag{2.25}$$

Operator $P_\epsilon P_r$ is compact and according to Corollary 2.6 the second term in the last equation is smaller then ε for τ sufficiently large. Thus $\psi \in \mathscr{M}^{f}_{\pm}(P)$ and the inclusion $\mathscr{H}^{cont}(U(T,0)) \subset \mathscr{M}^{f}_{\pm}(P)$ holds.

On the other hand, by virtue of Theorem 2.7 and Theorem 2.12 we have

$$\mathscr{H}^{pp}(U(T,0)) \overset{2.7}{=} \mathscr{H}^{p}_{\pm} \overset{2.12}{=} \mathscr{M}^{bd}_{\pm}(P) \perp \mathscr{M}^{f}_{\pm}(P)$$

so $\mathscr{M}^{f}_{\pm}(P) \subset \mathscr{H}^{cont}(U(T,0))$. We have proved (2.22). □

Remark 2.16 (Time-independent case): Let us consider time-independent Hamiltonian H and corresponding propagator $U(t,s) = e^{-\imath H(t-s)}$. It holds that

$$\mathscr{H}^{pp}(H) = \mathscr{H}^{pp}(e^{-\imath HT}),$$
$$\mathscr{H}^{cont}(H) = \mathscr{H}^{cont}(e^{-\imath HT})$$

and therefore the Theorem 2.7 gives $\mathscr{H}^{pp}(H) = \mathscr{H}^{p}_{\pm}$.

Notice that the condition of relative compactness in time-independent case

$$P_r(H+\imath)^{-1} \text{ is compact } \forall r > 0,$$

where $\{P_r\}$ satisfies (2.16), implies $\pm U$-compactness (Definition 2.11). Indeed, take $\psi \in \operatorname{dom} H$, then

$$P_r U(t,0)\psi = P_r(H+\imath)^{-1} \underbrace{e^{-\imath Ht}(H+\imath)\psi}_{\text{bounded for } t\gtrless 0}.$$

Therefore relative compactness implies decay of Hilbert space to subspaces of geometrically bounded and free states.

3.
Almost periodically perturbed harmonic oscillator

In the Section 3 of this Chapter we work with almost periodic function, thus it is useful to start with review of their basic properties.

3.1 Basic definitions and theorems

As the name suggests, almost periodic functions are generalisation of periodic functions. Main definition is based on the notion of relatively dense sets.

Definition 3.1: *Set $M \subset \mathbb{R}$ is called relatively dense in $\mathbb{R}$, if and only if there exists $L > 0$ (so called inclusion length) such as each interval of length L contains at least one element of M.*

Definition 3.2: *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. We say that f is almost periodic, if and only if for each $\varepsilon > 0$ the set*

$$M(\varepsilon, f) := \{\tau \in \mathbb{R} | \sup_{x \in \mathbb{R}} |f_\tau(x) - f(x)| < \varepsilon\}, \tag{3.1}$$

where $f_\tau(x) = f(x+\tau)$, is relatively dense in $\mathbb{R}$. Elements $M(\varepsilon, f)$ are called ε-almost periods of almost periodic function f. The set of all almost periodic functions is denoted by AP.

We now state some basic results concerning almost periodic functions. Proofs and other details can be found in [DS62, DS66, Bes54]. By definition we have $AP \subset C(\mathbb{R})$.

Theorem 3.3: *Almost periodic function is bounded.*

Proof. Let f be almost periodic function. It suffices to choose $\varepsilon = 1$ and denote the maximum of function $|f(x)|$ on interval $(0, L)$, where L is the inclusion length of the set $M(1, f(x))$, by M. Let x be some real number, then we can find $\tau \in M(1, f)$ such as $x + \tau \in (0, L)$, so

$$|f(x)| \le |f(x+\tau)| + |f(x+\tau) - f(x)| \le M + 1.$$

□

Theorem 3.4: *Almost periodic function is uniformly continuous.*

Proof. Let f be almost periodic function, $\varepsilon > 0$ and L_ε inclusion length of $M(\varepsilon, f)$. Choose $0 < \delta < 1$ such that $f(x_1) - f(x_2) < \varepsilon$ whenever $x_1, x_2 \in (0, L_\varepsilon + 1)$ and $|x_1 - x_2| < \delta$. Take x', x'' fullfilling $|x' - x''| < \delta$. Than there is $\tau \in M(\varepsilon, f)$ such as $x' + \tau, x'' + \tau \in (0, L_\varepsilon + 1)$. Thus

$$|f(x' + \tau) - f(x'' + \tau)| < \varepsilon$$

and

$$|f(x+\tau)-f(x)|<\varepsilon,\ \forall x\in\mathbb{R}.$$

Finally

$$|f(x')-f(x'')|\le|f(x')-f(x'+\tau)|+|f(x'+\tau)-f(x''+\tau)|+|f(x''+\tau)-f(x'')|<3\varepsilon.$$

□

For further study of the structure of set AP is crucial following Bochner Theorem.

Theorem 3.5 (Bochner)**:** *Continuous function $f\in C(\mathbb{R})$ is almost periodic if and only if the set $\{f_\lambda\}_\lambda$ is precompact in $C(\mathbb{R})$.*

Corollary 3.6: *The set of all almost periodic functions AP forms vector space. When equipped with supremum norm*

$$\|f\|_C=\sup_{x\in\mathbb{R}}|f(x)| \tag{3.2}$$

it forms Banach space.

Is the derivative of almost periodic function almost periodic? The following theorem gives answer to this question.

Theorem 3.7: *Let f be almost periodic function. If it's derivative exists and is uniformly continuous, then it is almost periodic.*

And finally we present another way how to characterise almost periodic functions.

Theorem 3.8 (Bohr)**:** *Continuous function defined on $\mathbb{R}$ is almost periodic if and only if it can by arbitrarily precisely uniformly approximated by trigonometric polynomials (i.e. by finite linear combinations of $e^{\imath\lambda x}$, $\lambda\in\mathbb{R}$). In other words*

$$AP=\overline{\operatorname{span}\{e^{\imath\lambda x}|\lambda\in\mathbb{R}\}}, \tag{3.3}$$

where the closure is with respect to the supremum norm $\|\cdot\|_C$.

3.2 Fourier series of almost periodic functions

The space of almost periodic functions AP can be equipped with inner product

$$\langle f,g\rangle_{AP}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\overline{f(x)}g(x)\ \mathrm{d}x. \tag{3.4}$$

It can be easily shown, using Bohr's theorem, that this limit exists for any almost periodic functions f and g and satisfies all properties of inner product.

Lemma 3.9: *The set $\{e_a(x):=e^{iax}|a\in\mathbb{R}\}$ is orthonormal total subset of AP.*

Proof. Orthonormality is obvious. Let $f\in AP$ such as $\langle e_\lambda,f\rangle_{AP}=0$ for all $\lambda\in\mathbb{R}$. For any $\varepsilon>0$ there exists trigonometric polynomial $g(x)=\sum_{k=1}^n c_k e^{i\lambda_k x}$ obeying $\|f-g\|_C<\varepsilon$. Thus

$$|\langle e_\lambda,g\rangle_{AP}|=|\langle e_\lambda,g-f\rangle_{AP}|\le\|e_\lambda\|_{AP}\|g-f\|_{AP}\le\|g-f\|_C<\varepsilon.$$

ε is arbitrarily small and therefore $\langle e_\lambda,g\rangle_{AP}=0$ for any $\lambda\in\mathbb{R}$. This means that $c_k=0$ for $k=1,\dots,n$ and therefore $g=0$. So for any $\varepsilon>0$ we have $\|f\|_C<\varepsilon$, i.e. $f=0$. □

Lemma 3.10: *Let f be almost periodic and $\lambda_1, \dots, \lambda_N$ N-tuple of arbitrary real numbers. We will denote Fourier coefficients of almost periodic function f by symbol*

$$a(\lambda) := \langle e_\lambda, f \rangle_{AP}.$$

Further let $b_1, \dots, b_N$ be arbitrary complex numbers. Then it holds that

$$\|f - \sum_{n=1}^{N} b_n e_{\lambda_n}\|_{AP}^2 = \|f\|_{AP} - \sum_{n=1}^{N} |a(\lambda_n)|^2 + \sum_{n=1}^{N} |b_n - a(\lambda_n)|^2.$$

Proof. This can be checked by direct computation. See page 16 in [Bes54]. □

It follows that trigonometric polynomial $\sum b_n e_{\lambda_n}$ is the best approximation to almost periodic function f (with respect to norm $\| \cdot \|_{AB}$ induced by inner product $\langle \cdot, \cdot \rangle_{AB}$) if $b_n = a(\lambda_n)$. In this case for any N-tuple $\lambda_1, \dots, \lambda_N$ is

$$\sum_{n=1}^{N} |a(\lambda_n)|^2 \leq \|f\|_{AP}^2.$$

Therefore there exists at most countable set $\{\lambda_n\}$ such as $a(\lambda_n) \neq 0$. We will denote these λ_n by Λ_n and we will write $a(\Lambda_n) = A_n$. Λ_n are called Fourier exponents and A_n Fourier coefficients corresponding to almost periodic function f. We have shown that the Bessel inequality holds, i.e.

$$\sum_n |A_n|^2 \leq \|f\|_{AP}^2.$$

The formal series $\sum_n A_n e_{\Lambda_n}$ is called Fourier series of almost periodic function f. Next theorem gives conditions under which the Fourier series is uniformly convergent. Notice also that in the case when f is periodic these definitions coincide with usual one.

Theorem 3.11: *If the almost periodic function f can be written as uniformly convergent trigonometric series $f = \sum_{n=1}^{\infty} a_n e_{\lambda_n}$ than it coincides with the Fourier series corresponding to f.*

Proof. For any $\lambda \in \mathbb{R}$ it holds

$$\langle e_\lambda, f \rangle_{AP} = \sum_{n=1}^{\infty} \|e_{\lambda_n - \lambda}\|_{AP}^2 = \sum_{n=1}^{\infty} a_n \delta_{\lambda, \lambda_n}.$$

So $\langle e_\lambda, f \rangle_{AP} = 0$ for $\lambda \neq \lambda_n$ and $\langle e_{\lambda_n}, f \rangle_{AP} = a_n$. □

Theorem 3.12 (Parseval equality for almost periodic function)**:** *Let f be almost periodic and $\{A_k\}$ and $\{\Lambda_k\}$ its Fourier coefficients and exponents respectively. Then the Parseval equality holds*

$$\sum_{k=1}^{\infty} |A_k|^2 = \|f\|.$$

Proof. Cf. [Bes54] Chapter I, section 4. □

We now state one approximation theorem.

Theorem 3.13 (Bochner-Fejér): *Let f be almost periodic function with Fourier coefficients and exponents A_k and Λ_k respectively. Then for any $\varepsilon > 0$ there exists Bochner-Fejér trigonometric polynomial*

$$\sigma = \sum_{k=1}^{n} d_k A_k e_{\Lambda_k},$$

where $0 \leq d_n \leq 1$, which obeys $\|f - \sigma\|_C < \varepsilon$.

Proof. Cf. [Bes54] Chapter I., Section 9. □

3.3 Harmonic oscillator with almost periodic perturbation

We consider one-dimensional harmonic oscillator with almost periodic perturbation. The system is described by time-dependent Hamiltonian, formally given by

$$H(t) = \frac{1}{2}P^2 + \frac{1}{2}\omega Q^2 + f(t)Q, \tag{3.5}$$

on Hilbert space $\mathscr{H} = \mathrm{L}^2(\mathbb{R},\ \mathrm{d}x)$. Q a P are position and momentum operators, acting like

$$(P\psi)(x) = -\imath\frac{\mathrm{d}\psi}{\mathrm{d}x}(x),$$
$$(Q\psi)(x) = x\psi(x).$$

We suppose that f is almost periodic. Unitary propagator exists and can be computed exactly[1]

$$U(t,s) = \exp\left(-\imath\varphi_1(t,s)Q\right)\exp\left(\imath\varphi_2(t,s)\frac{P}{\omega}\right)\exp\left(-\imath H_\omega(t-s) + \imath\psi(t,s)\right), \tag{3.6}$$

where

$$H_\omega = \frac{1}{2}P^2 + \frac{1}{2}\omega Q^2, \tag{3.7}$$

$$\varphi_1(t,s) = \int_s^t f(\tau)\cos\omega(\tau - t)\mathrm{d}\tau, \tag{3.8}$$

$$\varphi_2(t,s) = -\int_s^t f(\tau)\sin\omega(\tau - t)\mathrm{d}\tau, \tag{3.9}$$

$$\psi(t,s) = -\frac{1}{2}\int_s^t \left(\varphi_1(\tau,s)^2 - \varphi_2(\tau,s)^2\right)\mathrm{d}\tau. \tag{3.10}$$

Proposition 3.14: *$\varphi_1(t,0)$ a $\varphi_2(t,0)$ are bounded if and only if $\mathscr{H}_\pm^{\mathrm{p}} = \mathscr{H}$.*

[1] This holds for any continuous and bounded f, cf. [EV83].

Proof. Notice that boundedness of $\varphi_1(t,0)$ is equivalent to boundedness of $\varphi_2(t,0)$. Indeed, it suffices to check following equality

$$\varphi_2(t,0) = -\int_0^t f(\tau)\ \underbrace{\sin\omega(\tau-t)}_{-\cos(\omega(\tau-t)+\frac{\pi}{2})}\ \mathrm{d}\tau = +\varphi_1\left(t-\frac{\pi}{2\omega},0\right)+$$
$$+\int_{t-\frac{\pi}{2\omega}}^{t} f(\tau)\cos\left(\omega\left(\tau-t+\frac{\pi}{2\omega}\right)\right)\mathrm{d}\tau.$$

Because f is almost periodic it is also bounded, say $|f|<K$. Then

$$|\varphi_2(t,0)| \le \left|\varphi_1\left(t-\frac{\pi}{2\omega},0\right)\right| + K\ \frac{\pi}{2\omega}.$$

Similar inequality for φ_1 can be derived analogously.

With the aid of relation

$$e^{tA}Be^{-tA} = e^{t\mathrm{ad}_A}B \equiv \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathrm{ad}_A^k B,\ \ \mathrm{ad}_A B \equiv [A,B] \tag{3.11}$$

we derive

$$Q(t) \equiv U(0,t)QU(t,0) = Q\cos\omega t + \frac{P}{\omega}\sin\omega t - \frac{1}{\omega}\varphi_2(t,0),$$
$$P(t) \equiv U(0,t)PU(t,0) = -\omega Q\sin\omega t + P\cos\omega t - \varphi_1(t,0).$$

For $\psi\in\mathscr{S}$ it holds

$$\|P^nU(t,0)\psi\|^2 = \langle\psi, P(t)^{2n}\psi\rangle, \tag{3.12}$$
$$\|Q^nU(t,0)\psi\|^2 = \langle\psi, Q(t)^{2n}\psi\rangle,\ n\in\mathbb{N}_0. \tag{3.13}$$

Thus $\varphi_1(t,0)$ is bounded if and only if $\|P^nU(t,0)\psi\|^2$ is bounded and $\varphi_2(t,0)$ is bounded if and only if $\|Q^nU(t,0)\psi\|^2$ is bounded.

For any t and $\psi\in\mathscr{S}$ it holds

$$\begin{aligned}\|U(t,0)\psi - F(|Q|<R)F(|P|<R)U(t,0)\psi\| &\le \|(I - F(|Q|<R))U(t,0)\psi\| + \\ &+ \|F(|Q|<R)(I-F(|P|<R))U(t,0)\psi\| \le \\ &\le \|F(|Q|\ge R)U(t,0)\psi\| + \|F(|P|\ge R)U(t,0)\psi\|.\end{aligned} \tag{3.14}$$

The rest of the proof is based on notion of time-bounded energy[2]. Choose $f(\lambda) := \lambda^2$. We have showed that $\|Q^2U(t,0)\psi\|$ is bounded in t for any ψ from Schwartz space, i.e.

$$M := \sup_{t\gtreqless 0}\|f(Q)U(t,0)\psi\| < \infty,\ \forall\psi\in\mathscr{S}.$$

[2]

Definition 3.15: *We say that a propagator U has time-bounded energy H_1 at $\pm\infty$, if there is a total set S such as for all $\psi\in S$*

$$\lim_{\lambda\to\infty}\sup_{t\gtreqless 0}\|F(|H_1|>\lambda)U(t,0)\psi\| = 0,$$

where H_1 is some self-adjoint operator.

For proof of following Lemma 3.16 consult [EV83], Lemma 3.3.

Using Lemma 3.16 we deduce that U has time-bounded energy Q at $\pm\infty$, thus

$$\lim_{R\to\infty} \sup_{t\gtrless 0} \|F(|Q| \geq R)U(t,0)\psi\| = 0.$$

Analogously for P we obtain

$$\lim_{R\to\infty} \sup_{t\gtrless 0} \|F(|P| \geq R)U(t,0)\psi\| = 0.$$

According to (3.14) we see that for any $\varepsilon > 0$ and sufficiently large $R > 0$ it holds

$$\|U(t,0)\psi - F(|Q| < R)F(|P| < R)U(t,0)\psi\| \leq \varepsilon.$$

$F(|Q| < R)F(|P| < R)$ is compact operator. Thus for any $R > 0$ the set $\{U(t,0)\psi | t \gtrless 0\}$ is arbitrarily precisely approximated by compact set and is therefore compact (for any ψ). $\square$

Lemma 3.17: *Let f, g be bounded real functions of real variable and suppose that f is continuous T-periodic and that in interval $[0,T]$ it has at most finite number of simple roots*[3]. *Then*

$$\lim_{s\to\infty} \frac{1}{s} \int_0^s e^{-(|\varphi(t)|-R)^2}\ \mathrm{d}t = 0,\ \forall R > 0,$$

where $\varphi(t) = tf(t) + g(t)$.

Proof. For any $s \in \mathbb{R}$ such that $nT < s < (n+1)T$ holds

$$\frac{1}{s}\int_0^s h(t)\ \mathrm{d}t \leq \frac{1}{nT}\int_0^{(n+1)T} h(t)\ \mathrm{d}t = \frac{n+1}{n}\frac{1}{(n+1)T}\int_0^{(n+1)T} h(t)\ \mathrm{d}t.$$

It is therefore sufficient to check the convergence of the right hand side of the last equation. Thus

$$\frac{1}{nT}\int_0^{nT} e^{-(|\varphi(t)|-R)^2}\ \mathrm{d}t = \frac{1}{nT}\sum_{j=0}^{n-1}\int_0^T e^{-(|\varphi(t+jT)|-R)^2}\ \mathrm{d}t \leq \frac{C}{nT}\sum_{j=0}^{n-1}\int_0^T e^{-(jT|f(t)|-(G+R))^2}\ \mathrm{d}t,$$

where G is constant such that $|tf(t) + g(t)| < G$ for any $t \in [0,T]$. It is sufficient to study

$$\frac{1}{n}\int_0^n \mathrm{d}y \int_0^T e^{-(y|f(t)|-A)^2}\ \mathrm{d}t,$$

Lemma 3.16: *U has a time-bounded energy H_1 if and only if*

$$M := \sup_{t\gtrless 0} \|f(H_1)U(t,0)\psi\| < \infty$$

for all ψ in some total set S and real nonnegative function f, possibly depending on ψ, such that $f(\lambda) \to \infty$ as $\lambda \to \infty$.

[3] *i.e.* $0 = f(x) \neq f'(x)$

where $A > 0$ and f is again bounded on $[0,T]$. Let us denote $\{x_i\}_{i=1}^N \subset [0,T]$ simple roots of f, i.e. $f(x_i) = 0 \neq f'(x_i)$. Let $\varepsilon > 0$, then there is $K(\varepsilon) > 0$ which satisfies $|f(t)| > K(\varepsilon)$ for any $t \in M := [0,T] \setminus \bigcup_{i=1}^N (x_i - \varepsilon, x_i + \varepsilon)$. Thus

$$\frac{1}{n}\int\limits_0^n \mathrm{d}y \int\limits_0^T e^{-(y|f(t)|-A)^2}\ \mathrm{d}t = \frac{1}{n}\int\limits_0^n \mathrm{d}y \int\limits_M e^{-(y|f(t)|-A)^2}\ \mathrm{d}t + \frac{1}{n}\int\limits_0^n \mathrm{d}y \int\limits_{[0,T]\setminus M} e^{-(y|f(t)|-A)^2}\ \mathrm{d}t \leq$$

$$\leq \frac{1}{n}\int\limits_M \mathrm{d}t \int\limits_0^{n|f(t)|} e^{-(y-A)^2}\ \underbrace{\frac{1}{|f(t)|}}_{\leq 1/K(\varepsilon)}\ \mathrm{d}y + \frac{1}{n}\int\limits_0^n \mathrm{d}y \int\limits_{[0,T]\setminus M} \mathrm{d}t \leq \frac{T\sqrt{\pi}}{nK(\varepsilon)} + 2\varepsilon N.$$

We have proved that for any $\varepsilon > 0$ there exists $N(\varepsilon) = \frac{T\sqrt{\pi}}{\varepsilon K(\varepsilon)}$ such as for all $n > N(\varepsilon)$ it is true that

$$\frac{1}{n}\int\limits_0^n \mathrm{d}y \int\limits_0^T e^{-(y|f(t)|-A)^2}\ \mathrm{d}t < (1+2N)\varepsilon.$$

Lemma is therefore proved. □

Remark 3.18: Notice that claims of Theorem 2.10 holds for any unitary propagator U. We are especially interested in inclusion $\mathscr{H}_\pm^p \subset \mathscr{M}_\pm^{bd}$. Thus the system described by Hamiltonian (3.5) and propagator (3.6) has only geometrically bounded states if the function

$$\varphi_2(t,0) = -\int\limits_0^t f(x)\sin\omega(x-t)\ \mathrm{d}x$$

is bounded.

On the other hand, take

$$\mathscr{T} := \left\{\psi_a \in \mathscr{S} \middle| \psi_a(x) := \exp\left(-\frac{\omega}{2}(x-a)^2\right),\ a \in \mathbb{R}\right\}.$$

It is well known that for $\psi_a \in \mathscr{T}$ it holds

$$|\exp(-\imath H_\omega t)\psi_a(x)|^2 = \exp(-\omega(x - a\cos\omega t)^2).$$

So

$$h(t) := \|F(|Q|<R)U(t,0)\psi_a\|^2 = \int\limits_{B_R} \exp\left[-\omega\left(x + \frac{\varphi_2(t,0)}{\omega} - a\cos\omega t\right)^2\right]\ \mathrm{d}x \leq$$

$$\leq \mathrm{const}\exp\left[-\omega\left(|\frac{\varphi_2(t,0)}{\omega} - a\cos\omega t| - R\right)^2\right].$$

Thus, by Lemma 3.17 the considered system has only geometrically unbounded states in case when the function $\varphi_2(t,0)$ can be written in form $tF(t) + G(t)$, where F is periodic, bounded and in interval of period length has at most finite number if simple roots and G is bounded.

Theorem 3.19: *Suppose that almost periodic function f has Fourier coefficients $0 \neq A_k \in \mathbb{C}$ and exponents Λ_k satisfying*

$$\sum_{k=1}^{\infty} \frac{1}{\Lambda_k^2} < \infty. \tag{3.15}$$

Than it holds

(i) If $\omega \neq \Lambda_k$, $\forall k \in \mathbb{N}$, then our system has only geometrically bounded states and each trajectory is precompact.

(ii) Suppose in addition that f' is uniformly continuous on $\mathbb{R}$. Then the Fourier series is uniformly convergent.If $\omega = \Lambda_k$ for some $k \in \mathbb{N}$, then the system has only geometrically unbounded states.

Proof. (i): Let $t \in \mathbb{R}_+$. If $\omega \neq \Lambda_k$, $\forall k \in \mathbb{N}$ then there exists Bochner-Fejér polynomial

$$\sigma(x) = \sum_{k=1}^{N} d_k A_k e^{i\Lambda_k x},$$

where $0 \leq d_k \leq 1$, satisfying $\|f - \sigma\|_C < 1/t$. So

$$\varphi_2(t,0) = -\int_0^t (f(x) - \sigma(x)) \sin \omega(x - t) \, \mathrm{d}x - \int_0^t \sigma(x) \sin \omega(x - t) \, \mathrm{d}x,$$

and using Schwartz inequality one obtains

$$|\varphi_2(t,0)| \leq t\|f - \sigma\|_C + \sum_{k=1}^{N} d_k |A_k| \frac{2\omega + |\Lambda_k|}{|\Lambda_k^2 - \omega^2|} \leq$$

$$\leq 1 + \sqrt{\sum_{k=1}^{\infty} |A_k|^2} \sqrt{\sum_{k=1}^{\infty} \frac{4\omega^2 + 4\omega|\Lambda_k| + |\Lambda_k|^2}{\Lambda_k^2 - \omega^2}^2}$$

The first sum in the last expression is equal to $\|f\|_{AP} < \infty$ by Parseval equality. The second one can be estimated by

$$\sum_{k=1}^{\infty} \frac{1}{\Lambda_k^4} \frac{4\omega^2 + 4\omega|\Lambda_k| + |\Lambda_k|^2}{(1 - (\omega/\Lambda_k)^2)^2} \leq \text{const} \sum_{k=1}^{\infty} \frac{1}{\Lambda_k^2} < \infty.$$

We conclude that $\varphi_2(t,0)$ is bounded. The assertions of item (i) follow from Remark 3.18.

(ii): By Theorem 3.7 f' is almost periodic. We will show that under our assumptions the Fourier series of function f is uniformly convergent. Integrating *per partes* one obtains

$$\int_0^T f(x) e^{-i\Lambda_k x} \, \mathrm{d}x = \left[f(x) \frac{e^{-\imath\Lambda_k x}}{-\imath\Lambda_k} \right]_0^T + \int_0^T f'(x) \frac{e^{-\imath\Lambda_k x}}{\imath\lambda_k} \, \mathrm{d}x,$$

thus[4]

$$A_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x) e^{-\imath\Lambda_k x} \, \mathrm{d}x = \frac{1}{\imath\Lambda_k} \lim_{T\to\infty} \frac{1}{T} \int_0^T f'(x) e^{-\imath\Lambda_k x} \, \mathrm{d}x = \frac{A'_k}{\imath\Lambda_k}$$

Parseval equality now implies

$$\sum_{k=1}^{\infty} |A_k| \le \sqrt{\sum_{k=1}^{\infty} \frac{1}{\Lambda_k^2} \cdot \sum_{k=1}^{\infty} |A'_k|^2} < \infty$$

Thus the Fourier series $\sum_k A_k e^{\imath\Lambda_k x}$ converges uniformly to f. We can therefore write

$$\varphi_2(t,0) = -\frac{\imath A_k}{\omega}(\omega t e^{\imath\omega t} - \sin\omega t) - \sum_{n=1, n\neq k}^{\infty} A_n \int_0^t e^{\imath\Lambda_n x} \sin\omega(x-t) \, \mathrm{d}x.$$

By the same argument as in item (i) it can be shown that the last term is bounded in t. We conclude[5] that $\varphi_2(t,0)$ has exactly the form required by Remark 3.18 and therefore the system has only geometrically unbounded states.

□

Theorem 3.20: *Let f be almost periodic function with real positive Fourier coefficients, $A_k > 0$, and exponents $\Lambda_k \in \mathbb{R}$. Then the Fourier series is uniformly convergent and it holds*

(i) If[6] $\pm\omega \notin (\{\Lambda_k\}_{k=1}^{\infty})'$ and $\omega \notin \{\Lambda_k\}_{k=1}^{\infty}$, then the system has only geometrically bounded states and each trajectory is precompact.

(ii) If $\omega = \Lambda_k$ for some $k \in \mathbb{N}$ a $\pm\omega \notin (\{\Lambda_k\}_{k=1}^{\infty})'$ the the system has only geometrically unbounded states.

Proof. If the Fourier coefficients of almost periodic function f are positive, then it can by shown using Bochner-Fejér approximation that $\sum_k A_k < \infty$. And therefore the respective Fourier series is uniformly convergent to f. For proof see [Bes54], Chapter I., Section 10.

(i): By direct computation

$$\varphi_2(t,0) = \sum_{k=1}^{\infty} \frac{A_k}{\omega^2 - \Lambda_k^2}(\omega e^{\imath\Lambda_k t} - \omega\cos\omega t - \imath\Lambda_k \sin\omega t).$$

Thus

$$|\varphi_2(t,0)| \le \sum_{k=1}^{\infty} A_k \frac{2\omega + |\Lambda_k|}{|\Lambda_k^2 - \omega^2|}.$$

Using our assumptions there is some constant $K > 0$ such that $\frac{2\omega+|\Lambda_k|}{|\Lambda_k^2-\omega^2|} < K$, so

$$|\varphi_2(t,0)| \le K \sum_{k=1}^{\infty} A_k < \infty.$$

[4] A'_k are Fourier coefficients corresponding to f'.

[5] Notice that $\varphi_2(t,0) = \Re\varphi_2(t,0)$.

[6] *M' denotes the set of all accumulation points of the set M.*

(ii): Let us suppose that $\omega = \Lambda_j$ for some j and that $\pm\omega$ is not an accumulation point of the set of all Fourier exponents corresponding to function f. If this is the case then

$$\varphi_2(t,0) = -A_j \frac{\imath}{\omega}(\omega t e^{\imath\omega t} - \sin \omega t) - \sum_{k=1, k\neq j}^{\infty} \int_0^t e^{\imath \Lambda_k x} \sin \omega(x-t) \, \mathrm{d}x.$$

Again it can be shown, as in item (i), that the second term is bounded in t. And by the Remark 3.18 we see, that the system has only geometrically unbounded states.

□

4. Periodically time-dependent Aharonov-Bohm effect

4.1 Classical framework

In this chapter we will investigate motion of a charged classical particle in the plane under influence of a homogeneous magnetic field and a periodically time-dependent Aharonov-Bohm flux. Especially we will show that there is interesting resonant phenomenon depending on the strength of the field and the frequency of flux. Our treatment is not completely rigorous. Reader will be properly warned in the text.

4.2 Transformation to action-angle variables

Let's consider charged massive classical particle in constant magnetic field and Aharonov-Bohm flux. The configuration space is simply $\mathbb{R}^2 - \{0\}$. Vector potential is a sum of two parts. For charged particle in homogeneous magnetic field $\vec{A}_M$ and Aharonov-Bohm flux $\vec{A}_{AB}$ we have

$$\begin{aligned} \vec{A}_M \equiv (A_{M1}, A_{M2}) &= \frac{-b}{2}(-x_2, x_1), \\ \vec{A}_{AB} &= \frac{\Phi(t)}{2\pi|\vec{x}|^2}(-x_2, x_1), \\ \vec{A} &= \vec{A}_M + \vec{A}_{AB}, \end{aligned}$$

where, without loss of generality, we assume $b > 0$. It is very useful to use polar coordinates

$$\begin{aligned} x_1 &= r\cos\theta, \\ x_2 &= r\sin\theta. \end{aligned}$$

Lagrangian for particle in electromagnetic field given by vector potential $\vec{A}$ reads[1]

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}m\dot{\vec{x}}^2 + e\vec{A}\cdot\dot{\vec{x}}.$$

For now on we set $m = e = 1$. And in polar coordinates

$$L(\theta, r, \dot{\theta}, \dot{r}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \left(\frac{\Phi(t)}{2\pi} - \frac{b}{2}\right)\dot{\theta}.$$

[1] Dots will always denote time derivatives.

Corresponding Hamiltonian is obtained through Legendre transformation and reads

$$H(r,\theta,p_r,p_\theta,t)=\frac{1}{2}\left(p_r^2+\left(\frac{p_\theta-\frac{\Phi(t)}{2\pi}}{r}+\frac{br}{2}\right)^2\right).$$

Coordinate θ is cyclic, thus p_θ is integral of motion

$$\dot{p}_\theta=-\frac{\partial H}{\partial\theta}=0.$$

Our system is therefore reduced to one-dimensional problem. We will denote $p=p_r$ and set

$$V(r)=\frac{1}{2}\left(\frac{a}{r}+\frac{br}{2}\right)^2,$$
$$a(t)=p_\theta-\frac{\Phi(t)}{2\pi}.$$

The Hamiltonian equations for the one-dimensional Hamiltonian are

$$\dot{r}=p,\quad \dot{p}=\frac{a(t)^2}{r^3}-\frac{b^2}{4}r.$$

Equivalently

$$\ddot{r}+\frac{b^2}{4}r=\frac{a(t)^2}{r^3}. \tag{4.1}$$

The minimum of V for $r>0$ is

$$V_{min}=\min_{r>0}V(r)=\begin{cases}V\left(\sqrt{\frac{2a}{b}}\right)=ab & a>0,\\ V\left(\sqrt{\frac{2|a|}{b}}\right)=0 & a<0.\end{cases}$$

Now we will construct action-angle coordinates in case when $a(t)=a$ is constant, i.e. our Hamiltonian is independent of time[2]. For a fixed energy level $E>V_{min}$ the motion is constrained to the interval $[r_+,r_-]$. These constraints are obtained as a solution of equation $V(r)=E$. Thus we have

$$E-V(r)=\frac{b^2}{8r^2}(r_+^2-r^2)(r^2-r_-^2),$$

where

$$r_\pm^2=\frac{2}{b^2}\left(2E-ab\pm\sqrt{(2E-ab)^2-a^2b^2}\right). \tag{4.2}$$

It is useful to explicitly write out some combinations of r_+ and $r_\pm$

$$\begin{aligned} r_+^2+r_-^2&=\tfrac{4}{b^2}(2E-ab), & r_+^2r_-^2&=\tfrac{4a^2}{b^2},\\ r_+r_-&=\tfrac{2|a|}{b} & (r_+-r_-)^2&=\tfrac{8E}{b^2}-\tfrac{8a}{b}\vartheta(a),\end{aligned} \tag{4.3}$$

[2]This is the case when there is now Aharonov-Bohm flux. Then every trajectory is circle.

where $\vartheta(x)$ is Heavyside step function. The action is defined by integral[3]

$$I(E) = \frac{1}{\pi}\int\limits_{r_-}^{r_+} \sqrt{2(E - V(r))}\,\mathrm{d}\rho = \frac{b}{2\pi}\int\limits_{r_-}^{r_+} \frac{1}{r}\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)}\,\mathrm{d}\rho = \tag{4.4}$$

$$= \frac{b}{4\pi}\int\limits_{r_-^2}^{r_+^2} \frac{1}{x}\sqrt{(r_+^2 - x)(x - r_-^2)}\,\mathrm{d}x = \frac{b}{8}(r_+ - r_-)^2 = \frac{1}{b}(E - \vartheta(a)ab) = \frac{1}{b}(E - V_{min}). \tag{4.5}$$

Integral involved with the above computation is explicitly evaluated in the following remark.

Remark 4.1: For $0 < r_- < r < r_+$ it holds

$$\int\limits_{r_-^2}^{r^2} \frac{1}{x}\sqrt{(r_+^2 - x)(x - r_-^2)}\,\mathrm{d}x = \frac{\pi}{4}(r_+ - r_-)^2 + \sqrt{(r_+^2 - r^2)(r^2 - r_-^2)} -$$

$$\frac{1}{2}(r_-^2 + r_+^2)\arctan\left(\frac{\frac{1}{2}(r_-^2 + r_+^2) - r^2}{\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)}}\right) +$$

$$+ r_- r_+ \arctan\left(\frac{2r_-^2 r_+^2 - (r_+^2 + r_-^2)r^2}{2r_- r_+\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)}}\right).$$

In particular

$$\int\limits_{r_-^2}^{r_+^2} \frac{1}{x}\sqrt{(r_+^2 - x)(x - r_-^2)}\,\mathrm{d}x = \frac{\pi}{2}(r_+^2 - r_-^2)$$

Generating function of the transformation reads

$$S(r, I) = \int\limits_{r_-}^{r} \sqrt{2(E - V(\rho))}\mathrm{d}\rho = \frac{b}{2}\int\limits_{r_-}^{r} \frac{1}{\rho}\sqrt{(r_+^2 - \rho^2)(\rho^2 - r_-^2)}\mathrm{d}\rho =$$

$$= \frac{b}{4}\int\limits_{r_-^2}^{r^2} \frac{1}{x}\sqrt{(r_+^2 - x)(x - r_-^2)}\,\mathrm{d}x.$$

[3]For more details consult [Arn89].

Again using Remark 4.1 this integral can be evaluated explicitly

$$S(r,I) = \frac{b}{4}\Bigg[\frac{\pi}{4}(r_+ - r_-)^2 + \sqrt{(r_+^2 - r^2)(r^2 - r_-^2)} - \tag{4.6}$$

$$- \frac{1}{2}(r_+^2 + r_-^2)\arctan\left(\frac{\frac{1}{2}(r_+^2 + r_-^2) - r^2}{\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)}}\right) +$$

$$+ r_- r_+ \arctan\left(\frac{2r_-^2 r_+^2 - (r_+^2 + r_-^2)r^2}{2r_- r_+\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)}}\right)\Bigg].$$

From equation (4.4) we can express E by means of I and recast formulae (4.2) and (4.3) into

$$r_\pm = \frac{2}{\sqrt{b}}\sqrt{I + \frac{|a|}{2} \pm \sqrt{I(I + |a|)}}, \tag{4.7}$$

$$r_+^2 + r_-^2 = \frac{8}{b}\left(I + \frac{|a|}{2}\right), \tag{4.8}$$

$$(r_+ - r_-)^2 = \frac{8I}{a}, \tag{4.9}$$

$$r_+ r_- = \frac{2|a|}{b}. \tag{4.10}$$

We will drop first term in (4.6), because it coresponds only to shift of angle variable φ by the value $\frac{\pi}{2}$. Further simplification is achieved by using the identity

$$\arctan x - \arctan y = \arctan\left(\frac{x - y}{1 + xy}\right).$$

So finally we have

$$S(r,I) = \frac{1}{4}\sqrt{8bIr^2 - (br^2 - 2|a|)^2} - I\arctan\left(\frac{4I - br^2 + 2|a|}{\sqrt{8bIr^2 - (br^2 - 2|a|)^2}}\right) - \tag{4.11}$$

$$- \frac{|a|}{2}\arctan\left(\frac{(br^2 + 2|a|)\sqrt{8bIr^2 - (br^2 - 2|a|)^2}}{b^2r^4 - 4bIr^2 + 4|a|^2}\right).$$

The induced transformation of variables $(r,p) = \Psi(\varphi, I)$ is defined as follows: $\Psi = F \circ G^{-1}$, where the transformations $(r,p) = F(u,v)$ and $(\varphi, I) = G(u,v)$ are given respectively by the relations

$$r = u,\ p = \frac{\partial S(u,v)}{\partial u} \text{ and } \varphi = \frac{\partial S(u,v)}{\partial v},\ I = v.$$

By direct computation we get

$$r = \frac{2}{\sqrt{b}}\sqrt{I + \frac{|a|}{2} + \sqrt{I(I + |a|)}\sin\varphi}, \tag{4.12}$$

$$p = \frac{\sqrt{bI(I + |a|)}\cos\varphi}{\sqrt{I + \frac{|a|}{2} + \sqrt{I(I + |a|)}\sin\varphi}}, \tag{4.13}$$

and conversely,

$$\varphi = -\arctan\left(\frac{1}{bpr}\left(p^2 + \frac{a^2}{r^2} - \frac{b^2r^2}{4}\right)\right), \tag{4.14}$$

$$I = \frac{1}{b}(H - V_{min}) = \frac{1}{2b}\left(p^2 + \left(\frac{|a|}{r} - \frac{br}{2}\right)^2\right). \tag{4.15}$$

Let us switch to the time-dependent case with a Hamiltonian $H(r,p,t)$. Seeking the action-angle variables for the frozen Hamiltonian at each moment of time one in fact constructs a time-dependent transformation of variables. Hence the generating function of the transformation, $S(u,v,t)$, is time-dependent as well. One arrives again at a Hamiltonian system with a Hamiltonian $K(\varphi,I,t)$ and it holds

$$K(\varphi,I,t) = H(\Psi(\varphi,I,t),t) + \left.\frac{\partial S(u,I,t)}{\partial t}\right|_{u=\Psi_r(\varphi,I,t)},$$

where Ψ_r denotes component of Ψ belonging to r. Our Hamiltonian depends on time t only through function $a(t)$

$$H(r,p,t) = \frac{1}{2}\left(p^2 + \left(\frac{a(t)}{r} + \frac{br}{2}\right)^2\right).$$

New Hamiltonian now reads

$$K(\varphi,I,t) = bI + \begin{cases} a(t)b - \arctan\left(\frac{\sqrt{I}\cos\varphi}{\sqrt{I+a(t)}+\sqrt{I}\sin\varphi}\right)\dot{a}(t), & a(t) > 0, \\ \arctan\left(\frac{\sqrt{I}\cos\varphi}{\sqrt{I-a(t)}+\sqrt{I}\sin\varphi}\right)\dot{a}(t), & a(t) < 0. \end{cases} \tag{4.16}$$

And equations of motion are given by

$$\dot{\varphi}(t) = \frac{\partial K(\varphi,I,t)}{\partial I}, \quad \dot{I}(t) = -\frac{\partial K(\varphi,I,t)}{\partial \varphi}.$$

This leads to

$$\dot{\varphi} = b - \frac{a\dot{a}}{2}\frac{\cos\varphi}{\sqrt{I(I+|a|)}}\frac{1}{2I+|a|+2\sqrt{I(I+|a|)}\sin\varphi}, \tag{4.17}$$

$$\dot{I} = -\frac{\operatorname{sgn} a}{2}\left(\dot{a} - \frac{|a|\dot{a}}{2I+|a|+2\sqrt{I(I+|a|)}\sin\varphi}\right). \tag{4.18}$$

From (4.12) and (4.7) we see, that

$$r^2 = \frac{1}{2}(r_+^2 + r_-^2) + \frac{1}{2}(r_+^2 - r_-^2)\sin\varphi.$$

Thus if φ grows then r^2 oscillates between r_-^2 and r_+^2. Moreover if $a(t)$ is bounded and $I \to \infty$ as $t \to \infty$ then obviously[4] $r_+ \to \infty$ and

$$r_-^2 = \frac{4}{b^2}\frac{a(t)^2}{r_+^2} \to 0,$$

as $t \to \infty$. This means that, in this very case, during the time evolution the particle will be located arbitrarily close to and arbitrarily far from origin. In the next part of this chapter we will try to find out if this phenomenon can occur, i.e. if the action variable grows to infinity.

[4] c.f. (4.7)

4.3 Simplification of equations of motion

As a first step we substitute

$$F = 2I + |a|, \tag{4.19}$$
$$\phi = \varphi - bt. \tag{4.20}$$

Equations (4.17) convert to

$$\dot{F} = \frac{\dot{a}a}{F + \sqrt{F^2 - a^2}\sin(bt+\phi)}, \tag{4.21}$$
$$\dot{\phi} = -\frac{\cos(bt+\phi)}{\sqrt{F^2-a^2}} \frac{\dot{a}a}{F + \sqrt{F^2 - a^2}\sin(bt+\phi)}. \tag{4.22}$$

Until now the flux function Φ was arbitrary. We consider time-periodic case

$$\Phi(t) = 2\pi\varepsilon \sin \Omega t, \tag{4.23}$$

where $\varepsilon > 0$ and $\Omega > 0$. Thus $a(t) = p_\theta - \varepsilon \sin \Omega t$. Let's assume that ϵ is "small". If this is the case we will consider only first term in Taylor expansion of right sides of equations (4.21) and (4.22). In this approximation equations of motion are

$$\dot{F} = -p_\theta \varepsilon \Omega \frac{\cos \Omega t}{F + \sqrt{F^2 - p_\theta^2}\sin(bt+\phi)} + o(\varepsilon^2), \tag{4.24}$$
$$\dot{\phi} = p_\theta \varepsilon \Omega \frac{\cos \Omega t}{\sqrt{F^2 - p_\theta^2}} \frac{\cos(\phi + bt)}{F + \sqrt{F^2 - p_\theta^2}\sin(bt+\phi)} + o(\varepsilon^2). \tag{4.25}$$

In this paragraph approximative solution of equations (4.24) and (4.25) will be found. We start with the key idea formulated in following proposition.

Proposition 4.2: *Let $\psi_A(x)$ be real function of real variable given by*

$$\psi_A(x) = \frac{1}{A + \sqrt{A^2 - C^2}\sin(x)},$$

where $A > |C|$ and $C \in \mathbb{R}$ are constants. Then

$$\psi_A \longrightarrow 2\pi \delta_{\frac{3\pi}{2}}$$

whenever $A \to +\infty$ in the space of generalised functions $\mathscr{D}'((0, 2\pi))$. The symbol $\delta_{\frac{3\pi}{2}}$ denotes Dirac delta function shifted to $\frac{3\pi}{2}$.

Proof. The function ψ_A is integrable on $(0, 2\pi)$ for every $A > |C|$ and therefore can be thought of as a regular generalised function. It's primitive function is given by

$$f_A(x) = \int \psi_A(x)\ \mathrm{d}x = 2\arctan\left(\sqrt{A^2 - C^2} + A \tan\frac{x}{2}\right).$$

Let $\varphi \in \mathscr{D}((0,2\pi))$ be arbitrary test function, then we are interested in

$$(\psi_A, \varphi) \equiv \int\limits_0^{2\pi} \psi_A(x)\varphi(x)\ \mathrm{d}x.$$

Substitution $z = f_A(x)$ leads to

$$(\psi_A, \varphi) = \int\limits_{-\pi}^{\pi} \varphi(f_A^{-1}(z))\mathrm{d}z,$$

where (mind the discontinuity of f_A at π, for illustration[5] there is Figure 4.1)

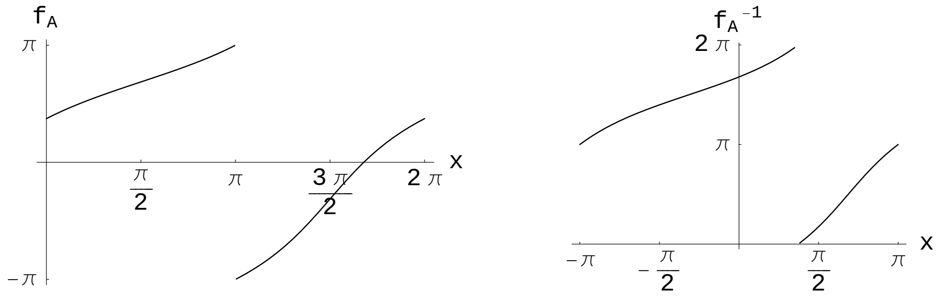

Figure 4.1: f_A and it's inversion f_A^{-1}

$$f_A^{-1}(z) = \begin{cases} 2\arctan\left(\frac{1}{A}\tan\frac{z}{2} - \sqrt{1-\frac{C^2}{A^2}}\right), & z \in \left(2\arctan\sqrt{A^2-C^2}, \pi\right), \\ 2\arctan\left(\frac{1}{A}\tan\frac{z}{2} - \sqrt{1-\frac{C^2}{A^2}}\right) + 2\pi, & z \in \left(-\pi, 2\arctan\sqrt{A^2-C^2}\right). \end{cases}$$

By the Lebesgue theorem (φ is smooth with compact support and therefore bounded) we get

$$\lim_{A\to+\infty}(\psi_A,\varphi) = \int\limits_{-\pi}^{\pi} \varphi\left(-\frac{\pi}{2}+2\pi\right)\mathrm{d}z = 2\pi\varphi\left(\frac{3}{2}\pi\right) = \left(2\pi\delta_{\frac{3}{2}\pi}, \varphi\right).$$

$\square$

Inspired by the previous proposition we will perform a bit heuristic step. Let's consider ordinary differential equation of the form

$$\dot{g}(t) = \varphi(t)\psi_{g(t)}(t).$$

Then in region where $g(t) \gg |C|$ the solution will behave like

$$g(t) - g(0) = \int\limits_0^t \dot{g}(\tau)\mathrm{d}\tau \sim 2\pi \sum_{k=0}^{n(t)} \varphi\left(\frac{3}{2}\pi + 2k\pi\right),$$

[5] All plots in this work were created using the *Mathematica*.

where $n(t)$ is integral part of $\frac{t+\pi/2}{2\pi}$.

We will apply this intuition to our system (4.24), (4.25). Using above argument we see that the second equation (4.25) is reduced[6] to $\phi(t) - \phi(0) \sim 0$. Thus it is sufficient to study equation

$$\dot{F} = \frac{-\varepsilon p_\theta \Omega \cos \Omega t}{F + \sqrt{F^2 - p_\theta^2} \sin(bt + \phi_0)}, \tag{4.26}$$

where $\phi_0 \in [0, 2\pi)$ is constant. More convenient is to use rescaled time $\tau(t) = bt + \phi_0$, and denoting $G(\tau) = F(t(\tau))$ we obtain

$$\dot{G} = \frac{-\varepsilon p_\theta \Omega}{b} \frac{\cos \frac{\Omega}{b}(\tau - \phi_0)}{G + \sqrt{G^2 - p_theta^2} \sin \tau}.$$

Thus

$$\begin{aligned} G(\tau) - G(0) &= -\frac{\varepsilon p_\theta \Omega}{b} \int\limits_0^\tau \frac{\cos \frac{\Omega}{b}(x - \phi_0)}{G + \sqrt{G^2 - p_theta^2} \sin(x)} \, \mathrm{d}x \sim \\ &\sim -\frac{\varepsilon p_\theta \Omega}{b} \sum_{k=0}^{n-1} 2\pi \cos \frac{\Omega}{b} \left(\frac{3}{2}\pi + 2\pi k - \phi_0 \right), \end{aligned}$$

where $n = n(\tau)$ is integral part of $\frac{\tau + \frac{\pi}{2}}{2\pi}$. Using formulae

$$\sum_{k=0}^{n} \sin kx = \frac{\sin \frac{1}{2}nx \sin \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x}, \tag{4.27}$$

$$\sum_{k=0}^{n} \cos kx = \frac{\cos \frac{1}{2}nx \sin \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x}, \tag{4.28}$$

which are just imaginary and real part of sum of geometric series $\sum_{k=0}^{n} e^{ikx}$, we obtain

$$G(\tau) - G(0) = \begin{cases} -\frac{2\pi\varepsilon p_\theta \Omega}{b} \frac{\sin \frac{\Omega\pi}{b} n}{\sin \frac{\Omega\pi}{b}} \cos \frac{\Omega}{b}(\frac{3}{2}\pi + \pi(n-1) - \phi_0), & \frac{\Omega}{b} \notin \mathbb{N}, \\ -\frac{2\pi\varepsilon p_\theta \Omega}{b} n \cos \frac{\Omega}{b} \left(\frac{3}{2}\pi - \phi_0\right), & \frac{\Omega}{b} \in \mathbb{N}. \end{cases}$$

The approximative solution of the original equation (4.26) is

$$F(t) = F(0) + \begin{cases} -\frac{2\pi\varepsilon p_\theta \Omega}{b} \frac{\sin \frac{\Omega\pi}{b} n}{\sin \frac{\Omega\pi}{b}} \cos \frac{\Omega}{b}(\frac{3}{2}\pi + \pi(n-1) - \phi_0), & \frac{\Omega}{b} \notin \mathbb{N}, \\ -\frac{2\pi\varepsilon p_\theta \Omega}{b} n \cos \frac{\Omega}{b} \left(\frac{3}{2}\pi - \phi_0\right), & \frac{\Omega}{b} \in \mathbb{N}, \end{cases} \tag{4.29}$$

where $n = n(t) = \left[\frac{bt + \phi_0 + \frac{\pi}{2}}{2\pi}\right]$. This is valid for $F(t) \gg |p_\theta|$.

We are led to the following conclusion.

Conclusion 4.3: *In the region where $F \gg p_\theta$ equation (4.26) has approximative solution given by (4.29). Qualitative behaviour depends on constants b, Ω and ϕ_0:*

[6] Mind the term $\cos(bt + \phi)$.

(i) If $\frac{\Omega}{b} \notin \mathbb{N}$ *or* $\frac{\Omega}{b} \in \mathbb{N}$ *and* $\cos\frac{\Omega}{b}\left(\frac{3}{2}\pi - \phi_0\right) = 0$ *then the solution is bounded.*

(ii) If $\frac{\Omega}{b} \in \mathbb{N}$ *and* $\cos\frac{\Omega}{b}\left(\frac{3}{2}\pi - \phi_0\right) < 0$ *then the solution is increasing.*

(iii) If $\frac{\Omega}{b} \in \mathbb{N}$ *and* $\cos\frac{\Omega}{b}\left(\frac{3}{2}\pi - \phi_0\right) > 0$ *then the solution is decreasing.*

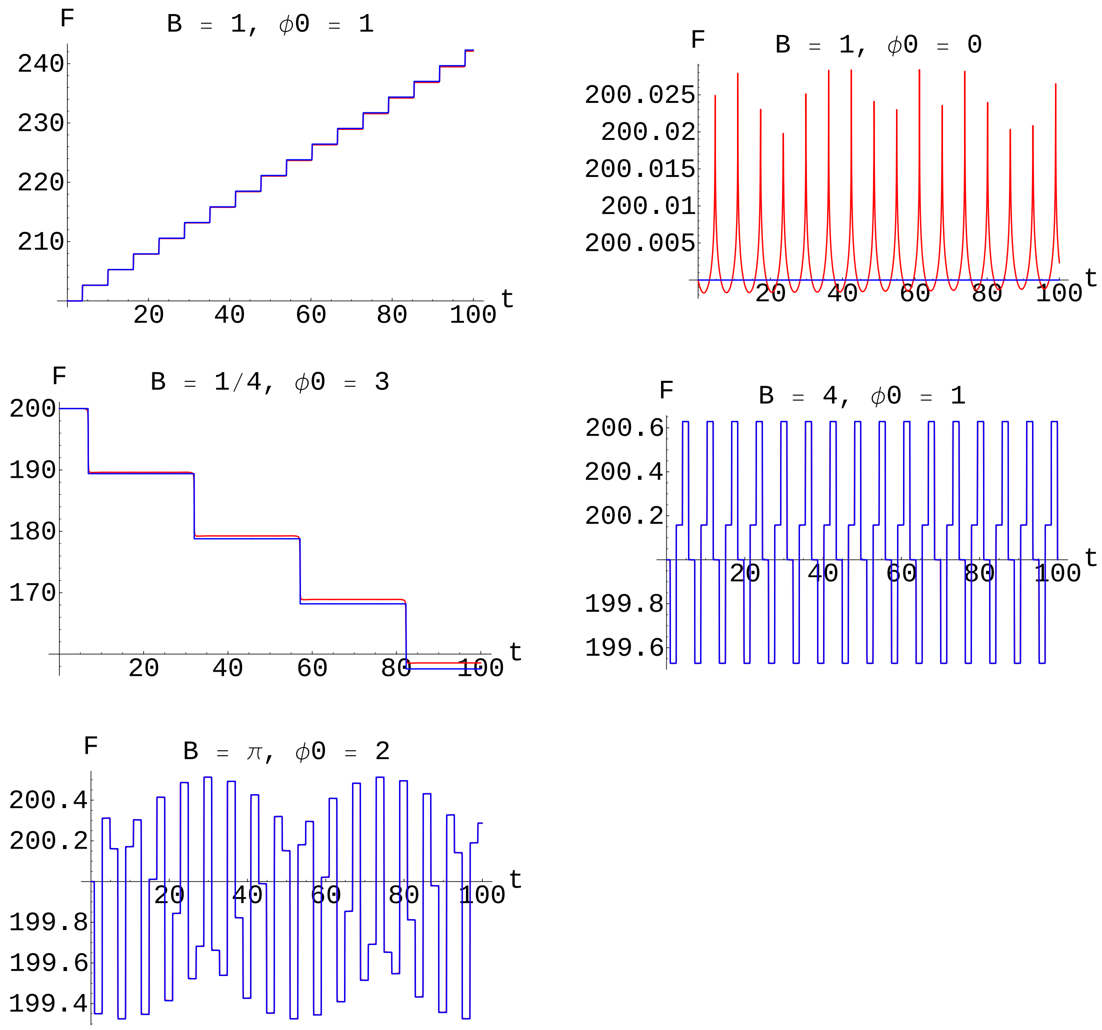

Figure 4.2: Red - numerical solution of (4.26) with initial condition $F(0) = 200$, Blue - approximative solution (4.29)

Because of a bit vague formulation of condition $F \gg |p_\theta|$ we present few plots. In Figure 4.2 we choose $\varepsilon = \frac{1}{2}$, $F(0) = 200$. Particular values of b, Ω and ϕ_0 are depicted above each plot. As a demonstration of behaviour of solution for smaller F there are two more plots in Figure 4.3. Initial condition is set to 10.

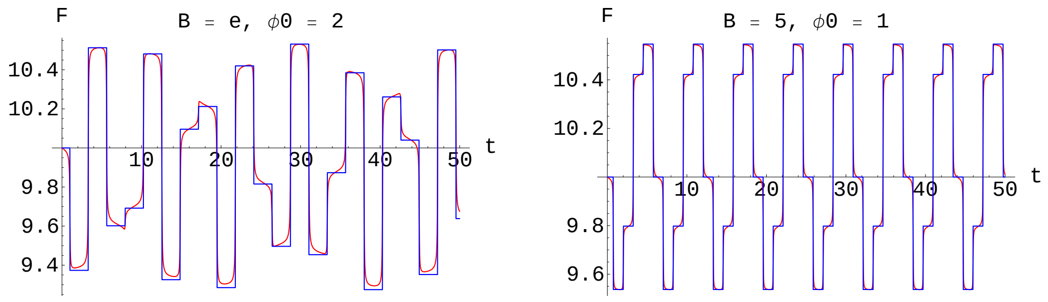

Figure 4.3: Red - numerical solution of (4.26) with initial condition $F(0) = 10$, Blue - approximative solution (4.29)

4.4 Solution of the original system

Our treatment cannot be used for situations when F (or I) is approaching $|p_\theta|$. Numerical analysis shows, that in the resonant case ($\Omega/b \in \mathbb{N}$) the system will end in the increasing mode. In the Figure 4.4 there are numerical solutions of (4.24) and (4.25). Our choice of initial conditions and other constants is depicted for each row. First row corresponds to case (iii) in Conclusion 4.3 (i.e. there is no fall on origin) and the second to case (i). Of course when $\Omega/b \in \mathbb{N}$. Again we choose $\varepsilon = \frac{1}{2}$.

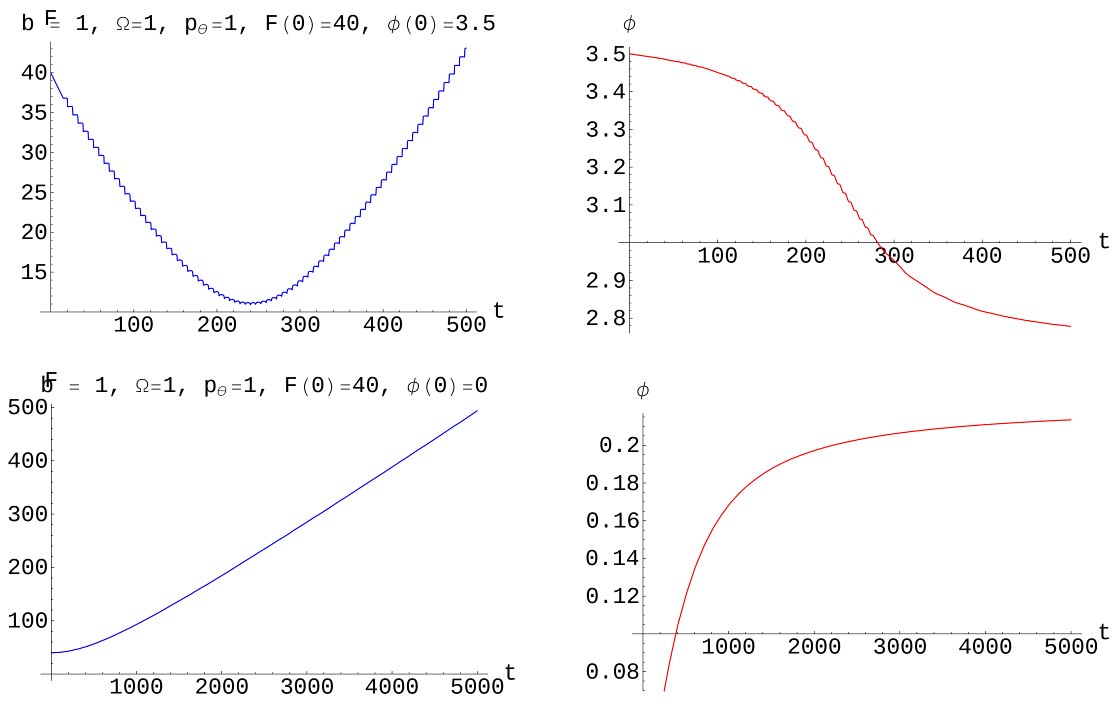

Figure 4.4: In the resonant case the action will grow to infinity.

Radial motion of particle influenced by homogeneous magnetic field and time-periodic Aharonov-Bohm flux is described by equation (4.1)

$$\ddot{r} + \frac{b^2}{4} r = \frac{a(t)^2}{r^3}$$

and by specifying initial values $r(0) = r_0$ and $\dot{r}(0) = \dot{r}_0$. We have constructed approximative solution

$$r(t) = \frac{2}{\sqrt{b}} \left(I(t) + \frac{|a(t)|}{2} + \sqrt{I(t)(I(t) + |a(t)|)} \sin\varphi(t) \right)^{1/2}, \tag{4.30}$$

where $\varphi(t) = bt + \phi_0$,

$$I(t) = I_0 + \frac{1}{2}(|p_\theta| - |a(t)|) + \begin{cases} -\frac{\pi\varepsilon p_\theta\Omega}{b} \frac{\sin\frac{\Omega\pi}{b} n}{\sin\frac{\Omega\pi}{b}} \cos\frac{\Omega}{b}(\frac{3}{2}\pi + \pi(n-1) - \phi_0), & \frac{\Omega}{b} \notin \mathbb{N}, \\ -\frac{\pi\varepsilon p_\theta\Omega}{b} n \cos\frac{\Omega}{b}\left(\frac{3}{2}\pi - \phi_0\right), & \frac{\Omega}{b} \in \mathbb{N}, \end{cases}$$

and $n = n(t) = \left[\frac{bt+\phi_0+\frac{\pi}{2}}{2\pi}\right]$. Initial conditions I_0, ϕ_0 and r_0, $\dot{r}_0$ are related by

$$\phi_0 = -\arctan\left(\frac{1}{b\dot{r}_0 r_0}\left(\dot{r}_0^2 + \frac{p_\theta^2}{r_0^2} - \frac{b^2 r_0^2}{4}\right)\right),$$

$$I_0 = \frac{1}{2b}\left(\dot{r}_0^2 + \left(\frac{|p_\theta|}{r_0} - \frac{br_0}{2}\right)^2\right).$$

We assumed that $a(t) = p_\theta - \epsilon\sin\Omega t$. In Figure 4.5 we compared numerical solution of (4.1) and approximative solution (4.30). In the right column the difference of these two functions is plotted. In our study of classical particle influenced by homogeneous magnetic field of strength b and periodic time-dependent Aharonov-Bohm flux with frequency Ω we have come to hypothesis that if Ω/b is natural number then the particle will get arbitrarily close to and arbitrarily far from the origin. In other cases the trajectory will be bounded.

4.5 Quantum framework

We now turn back to the quantum case.

4.6 Propagators weakly associated to a family of Hamiltonians

In [AHŠ05] the notion of propagator weakly associated to Hamiltonian was proposed. To every unitary propagator $U(t,s)$ on Hilbert space $\mathscr{H}$ one can relate a unique self-adjoint[7] operator K in $\mathscr{K} = \mathrm{L}^2(\mathbb{R}, \mathscr{H},\ \mathrm{d}t)$ which is the generator of the one-parameter group of unitary operators $\exp(-\imath\sigma K)$, $\sigma \in \mathbb{R}$, defined by

$$\left(e^{-\imath\sigma K} f\right)(t) = U(t, t-\sigma) f(t-\sigma).$$

K is called the quasienergy operator. Furthermore it holds that the relation between propagators and quasienergy operators is one-to-one.[8]

[7] cf. [How74]

[8] See [How74] or [AHŠ05].

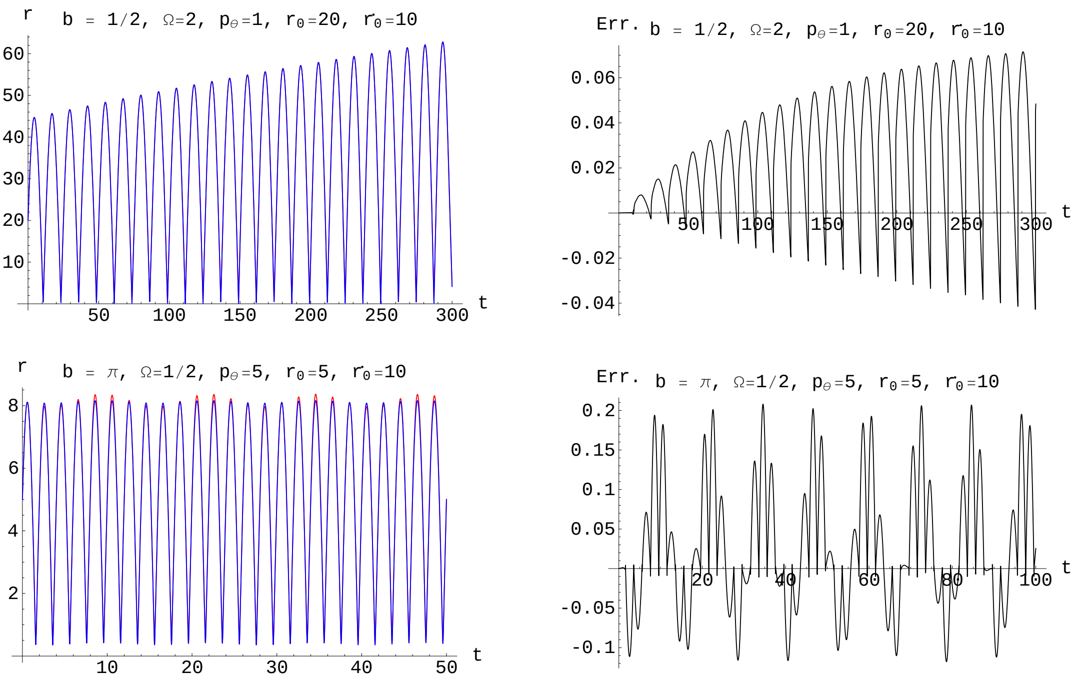

Figure 4.5: Blue - numerical solution of (4.1) with indicated initial condition, Red - approximative solution (4.30)

Definition 4.4: *We shall say that a propagator $U(t,s)$ is weakly associated to $H(t)$ if*

$$K = \overline{-\imath\partial_t + \mathfrak{H}}, \tag{4.31}$$

where[9]

$$\mathfrak{H} = \int_{\mathbb{R}}^{\oplus} H(t)\,\mathrm{d}t. \tag{4.32}$$

The equality (4.31) is equivalent to the following two conditions:

(i) $-\imath\partial_t + \mathfrak{H} \subset K$,

(ii) $-\imath\partial_t + \mathfrak{H}$ is essentially self-adjoint.

It is important to note that the definition still guarantees the uniqueness. If $U(t,s)$ and $\tilde{U}(t,s)$ are weakly associated to $H(t)$ then by (4.31) it holds that $K = \tilde{K}$. But due to the one-to-one correspondence between the propagators and quasienergy operators we have $U(t,s) = \tilde{U}(t,s)$. In the analysis of current model we will need two Lemmas from [AHŠ05].

Lemma 4.5: *Let $A(t)$ be a family of bounded self-adjoint operators in $\mathscr{H}$ which is locally bounded. Let $C(t,s)$ be the propagator associated to $A(t)$ via the Dyson formula. Let $\mathscr{D} \subset \mathscr{H}$*

[9] *$f \in \mathscr{K}$ is in domain of $\mathfrak{H}$ if and only if $f(t) \in \operatorname{dom} H(t)$ for almost all $t \in \mathbb{R}$ and $t \mapsto \|H(t)f(t)\|$ is square integrable. Then $(\mathfrak{H}f)(t) = H(t)f(t)$.*

be a dense linear subspace and let $T(t)$ be a strongly continuous family of unitary operators in $\mathscr{H}$ obeying the following conditions

(i) $\forall t \in \mathbb{R}$, $T(t)\mathscr{D} = \mathscr{D}$,

(ii) $\forall \psi \in \mathscr{D}$, $T(t)\psi$ *is continuously differentiable,*

(iii) $\forall t \in \mathbb{R}$, $X(t) = \imath\dot{T}(t)T(t)^{-1}$, *with* $\operatorname{dom} X(t) = \mathscr{D}$, *is a self-adjoint operator.*

Then tho propagator $T(t)C(t,s)T(s)^{-1}$ *is weakly associated to the family*

$$X(t) + T(t)A(t)T(t)^{-1}.$$

Lemma 4.6: *Suppose that* $V(t)$, $t \in \mathbb{R}$, *is a family of unitary operators which is continuously differentiable in the strong sense. Let* $\tilde{H}(t)$, $t \in \mathbb{R}$, *be a family of self-adjoint operators such that* $\operatorname{dom} \tilde{H}(t) = \mathscr{D}$ *for all* $t \in \mathbb{R}$. *Set*

$$H(t) = V(t)\tilde{H}(t)V(t)^{-1} + \imath\dot{V}(t)V(t)^{-1}.$$

If the propagator $\tilde{U}(t,s)$ *is weakly associated to* $\tilde{H}(t)$ *then the propagator*

$$U(t,s) = V(t)\tilde{U}(t,s)V(s)^{-1}$$

is weakly associated to $H(t)$.

We now show that a unitary operator weakly associated to T-periodic Hamiltonian $H(t)$ is also T-periodic.

Proposition 4.7: *Let* $U(t,s)$ *be unitary propagator weakly associated to* T*-periodic Hamiltonian* $H(t) = H(t+T)$. *Then the propagator is* T*-periodic, in particular*

$$U(t,s) = U(t+T, s+T),\ \forall s,t \in \mathbb{R}.$$

Proof. The proof is based on the uniqueness of weakly associated propagators. We will show that the propagator

$$\tilde{U}(t,s) := U(t+T, s+T)$$

is also weakly associated to $H(t)$.

Denote by $\tilde{K}$ the quasienergy operator generated by $\tilde{U}(t,s)$ and the time translation operator $(\mathscr{T}_a f)(t) = f(t+a)$. The evolution group generated by $\tilde{K}$ acts on $\mathscr{K}$ by

$$\begin{aligned}\left(e^{-\imath\sigma\tilde{K}} f\right)(t) &= \tilde{U}(t, t-\sigma)f(t-\sigma) = U(t+T, t+T-\sigma)(\mathscr{T}_T^* f)(t+T-\sigma) = \\ &= \left(e^{-\imath\sigma K}\mathscr{T}_T^* f\right)(t+T) = \left(\mathscr{T}_T e^{-\imath\sigma K}\mathscr{T}_T^* f\right)(t).\end{aligned}$$

Thus

$$e^{-\imath\sigma\tilde{K}} = \mathscr{T}_T e^{-\imath\sigma K}\mathscr{T}_T^*. \tag{4.33}$$

Stone's theorem states, that $f \in \operatorname{dom} K$ if and only if there exists

$$\imath\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{1}{\sigma}\left(e^{-\imath\sigma K} f\right)(t)\Big|_{\sigma=0} \tag{4.34}$$

and if this is the case then $(Kf)(t)$ is equal to (4.34). From (4.33) it follows that $f \in \operatorname{dom} \tilde{K}$ if and only if there exists

$$\imath \frac{\mathrm{d}}{\mathrm{d}\sigma} \frac{1}{\sigma} \left(e^{-\imath\sigma\tilde{K}} f \right)(t) \Big|_{\sigma=0} = \imath \frac{\mathrm{d}}{\mathrm{d}\sigma} \frac{1}{\sigma} \left(e^{-\imath\sigma K} \mathscr{T}_T^* f \right)(t+T) \Big|_{\sigma=0}.$$

So, f belongs to domain of $\tilde{K}$ if and only if $\mathscr{T}_T^* f$ is in domain of K and then it holds

$$\tilde{K} f = \mathscr{T}_T K \mathscr{T}_T^* f. \tag{4.35}$$

Our hypothesis means that $K = \overline{-\imath\partial_t + \mathfrak{H}}$, where

$$\mathfrak{H} = \int_{\mathbb{R}}^{\oplus} H(t)\ \mathrm{d}t.$$

Notice that we have

$$\mathscr{T}_T^* \partial_t \mathscr{T}_T = \partial_t$$

and the T-periodicity of $H(t)$ implies

$$\mathfrak{H} = \mathscr{T}_T^* \mathfrak{H} \mathscr{T}_T.$$

To simplify our equations we will denote $\mathfrak{K} := -\imath\partial_t + \mathfrak{H}$. Therefore $f \in \operatorname{dom} \mathfrak{K}$ if and only if $\mathscr{T}_T^* f \in \operatorname{dom} \mathfrak{K}$. If one takes $f \in \operatorname{dom} \mathfrak{K}$ then also $g = \mathscr{T}_T^* f \in \operatorname{dom} \mathfrak{K}$ and using (4.35) and the fact that K is extension of $\mathfrak{K}$ we obtain

$$\mathscr{T}_T^* \tilde{K} \mathscr{T}_T g = Kg = \mathfrak{K} g = \mathscr{T}_T^* \mathfrak{K} \mathscr{T}_T g.$$

This means

$$\tilde{K} f = -\imath\partial_t f + \mathfrak{H} f$$

for any $f \in \operatorname{dom} \mathfrak{K}$. Because $\mathfrak{K}$ is essentially self-adjoint it holds that $K = \tilde{K}$, which was to be proved. $\square$

4.7 Landau Hamiltonian with a periodically time-dependent Aharonov-Bohm flux

We consider charged particle in the homogeneous magnetic field of strength $B > 0$ under the influence of periodically time-dependent Aharonov-Bohm flux $\Phi(t)$. Thus analogously to Chapter 4.1 the model is described, in polar coordinates, by Hamiltonian acting in $\mathrm{L}^2(\mathbb{R}_+ \times [0, 2\pi), r\mathrm{d}r\mathrm{d}\theta)$ given by differential expression

$$\frac{\hbar^2}{2m} \left(-\frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2} \left(-\imath\partial_\theta - \frac{e}{\hbar c}\frac{Br^2}{2} - \frac{e}{\hbar c}\Phi \right)^2 \right).$$

We fix angular momentum sector defined by $-\imath\partial_\theta e^{\imath m\theta} = m e^{\imath m\theta}$, $m \in \mathbb{Z}$, and set all physical parameters $e, \hbar, c, 2m$ equal to one. Let the period of Φ be T. Therefore we are interested in the family of operators

$$H_\zeta(t) = H(\zeta(t)) = -\frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2} \left(\zeta(t) + \frac{Br^2}{2} \right)^2,$$

where $\zeta(t) = -m + \Phi(t)$, acting in $\mathrm{L}^2(\mathbb{R}_+, r\ \mathrm{d}\rho)$. The Hamiltonian $H(t)$ is unambiguously determined by specifying a complete set of eigenfunctions with corresponding eigenvalues

$$\begin{aligned} H(t)\varphi_n(t) &= \lambda_n(t)\varphi_n(t),\ n \in \mathbb{N}_0,\ \text{where} \\ \lambda_n(t) &= B(t + t + 2n + 1), \\ \varphi_n(t; r) &= c_n(t) r^{|t|} L_n^{(|t|)}\left(\frac{Br^2}{2}\right)\exp(-\frac{Br^2}{4}), \end{aligned}$$

and

$$c_n(t) = \left(\frac{B}{2}\right)^{(|t|+1)/2}\left(\frac{2n!}{\Gamma(n + |t| + 1)}\right)^{1/2}$$

are normalisation constants and $L_n^{(\alpha)}$ are generalised Laguerre polynomials. The dynamics of the model should be defined by

$$\imath\partial_t U(t, s)\psi = H_\zeta(t) U(t, s)\psi,\ U(s, s)\psi = \psi, \tag{4.36}$$

where U is unitary and ψ is an initial condition from the domain of $H_\zeta(s)$. However problems arises from the fact that domain of $H_\zeta(t)$ is not constant in t. See [AHŠ05] for discussion.

Consider unitary operator $V(t)$ which takes all eigenfunctions at time 0 to eigenfunctions at time $\zeta(t)$, i.e

$$V(t)\varphi_n(\zeta(0)) = \varphi_n(\zeta(t)),\ \forall n \in \mathbb{N}_0.$$

Denote the formal solution of (4.36) by $\psi(t)$, $\psi(0) = \psi$ and take $\tilde{\psi}(t) = V(t)\psi(t)$. Then it follows that $\tilde{\psi}(t)$ should satisfy

$$\imath\partial_t\tilde{\psi}(t) = \left(\underbrace{V(t)H_\zeta(t)V(t)^*}_{H_d(t)} + \underbrace{\imath V(t)^*(\partial_t V(t))}_{Q(t)}\right)\tilde{\psi}(t). \tag{4.37}$$

For simplicity we will write φ_n instead of $\varphi_n(\zeta(0))$. The matrix entries of $H_d(t)$ and $Q(t)$ in the basis φ_n are, respectively

$$\begin{aligned} \langle\varphi_n, H_d(t)\varphi_k\rangle &= B(\zeta(t) + |\zeta(t)|2n + 1)\delta_{nk}, \\ \langle\varphi_n, Q(t)\varphi_k\rangle &= \begin{cases} 0, & n = m, \\ \imath\zeta'(t)\langle\varphi_n(\zeta(t)), \dot{\varphi}_k(\zeta(t))\rangle, & n \neq m. \end{cases} \end{aligned} \tag{4.38}$$

From Lemma 4 and Lemma 6 in [AHŠ05] it follows that for $n \neq k$

$$\langle\varphi_n(\zeta(t)), \dot{\varphi}_k(\zeta(t))\rangle = \frac{\operatorname{sgn}(\zeta(t))}{2(k - n)}\min\left\{\frac{\gamma_n(\zeta(t))}{\gamma_k(\zeta(t))}, \frac{\gamma_k(\zeta(t))}{\gamma_n(\zeta(t))}\right\}, \tag{4.39}$$

where

$$\gamma_n(t) = \left(\frac{\Gamma(n + |t| + 1)}{n!}\right)^{1/2}$$

and that $Q(t)$ is bounded operator with

$$\|Q(t)\| \leq \zeta'(t)\left(\frac{\pi}{2} + 12|\zeta(t)| + \frac{1}{2}|\zeta(t)|(1 + |\zeta(t)|)^{(3+|\zeta(t)|)/2}\right)$$

Let P_n be orthogonal projector onto $\mathbb{C}\varphi_n$. Furthermore set

$$W(t) = \sum_{n=0}^{\infty} \lambda_n(\zeta(t))P_n,$$
$$\Omega(t) = \int_0^t W(u)\mathrm{d}u.$$

From (4.37) it follows that we seek propagator related to the Hamiltonian $H_d(t) + Q(t)$ and because $Q(t)$ is bounded we can pass to interaction picture of time evolution and use Dyson formula. More precisely let $C(t, t_0)$ be unitary propagator related to the Hamiltonian $-Q_\zeta(t) = -\exp(\imath\Omega(t))Q(t)\exp(-\imath\Omega(t))$ via Dyson formula

$$C(t, t_0) = \mathbb{I} + \sum_{n=1}^{\infty} \imath^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n Q_\zeta(t_1)Q_\zeta(t_2)\cdots Q(t_n). \tag{4.40}$$

Then

$$U(t, t_0) = V(t)\exp(-\imath\Omega(t))C(t, t_0)\exp(\imath\Omega(t_0))V(t_0)^*, \tag{4.41}$$

should be related to $H_\zeta(t)$.

Proposition 4.8: *The propagator $U(t, t_0)$ in (4.40) is weakly associated to $H_\zeta(t)$.*

Proof. Set $A(t) = -Q_\zeta(t)$, $\mathscr{D} = \operatorname{dom} H(\zeta(0))$, $T(t) = \exp(-\imath\Omega(t))$ and

$$X(t) = \imath\left(\partial_t e^{-\imath\Omega(t)}\right)e^{\imath\Omega(t)} = W(t).$$

By applying Lemma 4.5 we conclude that the propagator $\exp(-\imath\Omega(t))C(t, t_0)\exp(\imath\Omega(t_0))$ is weakly associated to

$$W(t) - e^{-\imath\Omega(t)}Q_\zeta(t)e^{\imath\Omega(t)} = W(t) - Q(t).$$

Now put $\tilde{H}(t) = W(t) - Q(t)$ and $\tilde{U}(t, t_0) = \exp(-\imath\Omega(t))C(t, t_0)\exp(\imath\Omega(t_0))$. It follows from Lemma 4.6, that $U(t, t_0) = V(t)\tilde{U}(t, t_0)V(t_0)^*$ is weakly associated to

$$V(t)W(t)V(t)^* - V(t)Q(t)V(t)^* + \imath(\partial_t V(t))V(t)^* = H_\zeta(t),$$

because $Q(t) = \imath V(t)^*(\partial_t V(t))$. □

We now choose $\Phi(t) = \varepsilon \sin\omega t$, ω, $\varepsilon > 0$, i.e. $\zeta(t) = -m + \varepsilon\sin\omega t$. The first term in expansion (4.40) is related to case when $\varepsilon = 0$. We now consider first approximation to the first order of ε. Thus we have T-periodic propagator, $T = \frac{2\pi}{\omega}$,

$$U_1(t, s) = V(t)\exp\left(-\imath(\Omega(t) - \Omega(s))\right)V(s)^* +$$
$$+ \imath\omega\varepsilon V(t)\exp(-\imath\Omega(t)) \int_s^t \cos\omega t_1 \exp(\imath\Omega(t_1))\tilde{Q}(m)\exp(-\imath\Omega(t_1))\mathrm{d}t_1 \exp(\imath\Omega(s))V(s)^*,$$

where $\tilde{Q}(m)$ is bounded operator given by (4.38) where we drop $\zeta'(t)$ and put $\zeta(t) = -m$. For any $n \in \mathbb{N}_0$ we now have

$$U(t,0)\varphi_n = e^{-\imath\Omega_n(t)}\varphi_n(\zeta(t)) + \\ + \imath\omega\varepsilon \sum_{k=0}^{\infty} e^{-\imath\Omega_k(t)} \underbrace{\left[\int_0^t \cos\omega t_1 e^{-\imath(\Omega_n(t_1)-\Omega_k(t_1))}\mathrm{d}t_1\right]}_{\xi_{nk}(t)} \langle\varphi_k, \tilde{Q}(m)\varphi_n\rangle\varphi_k(\zeta(t)),$$

where $\Omega_n(t) = \int_0^t \lambda_n(\zeta(u))\mathrm{d}u$. Notice that $\Omega_n(t) - \Omega_k(t) = 2B(n-k)t$. If $\omega \neq 2Bj$ for all $j \in \mathbb{N}$ then the integral in the last expression is equal to

$$\xi_{nk}(t) = \frac{1}{\omega^2 - 4B^2(n-k)^2}\left(-2B\imath(n-k) + e^{\imath(n-k)2Bt}(2B\imath(n-k)\cos\omega t + \omega\sin\omega t)\right).$$

So

$$\|F(r>R)U_1(t,0)\varphi_n\| \leq \|F(r>R)\varphi_n(\zeta(t))\| + \\ + \varepsilon\omega \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \Xi_{nk} \cdot |\langle\varphi_k, \tilde{Q}(m)\varphi_n\rangle| \cdot \|F(r>R)\varphi_k(\zeta(t))\|,$$

where

$$\Xi_{nk} = \frac{1}{|\omega^2 - 4B^2(n-k)^2|}(4B|n-k| + \omega)$$

and $F(r>R)$ is multiplication operator by characteristic function of interval (R,∞). Notice also that for any $n \in N_0$

$$\sum_{\substack{k=0 \\ k\neq n}}^{\infty} \Xi_{nk}^2 < \infty$$

and that for all $t \in \mathbb{R}$ and $k \in \mathbb{N}_0$ it holds that $\lim_{R\to\infty} \|F(r>R)\varphi_k(\zeta(t))\|$. Thus

$$\lim_{R\to\infty} \sup_{t\gtrless 0} \|F(r>R)U_1(t,0)\varphi_n\| = 0.$$

We conclude that in the first approximation the system has only geometrically bound states if $\omega \neq 2Bj$ for all $j \in \mathbb{N}$.

Conclusion

In this work we studied time-dependent quantum systems. The Chapter 2 reviewed basic results concerning T-periodic quantum systems. We presented topological and geometrical approach to bound and free states in quantum theory and showed their relation to spectral properties of Floquet operator $U(T, 0)$.

In the Chapter 3 we studied one-dimensional harmonic oscillator perturbed by external almost periodic force. We showed that this system has either only geometrically bound states or geometrically free states depending on the frequency of oscillator and Fourier coefficients of almost periodic function (see Theorems 3.19 and 3.20).

Then we considered charged particle in the plane under influence of homogeneous magnetic field of strength b and T-periodic Aharonov-Bohm flux. In the first part of Chapter 4 we studied corresponding Hamiltonian in the framework of classical Hamiltonian mechanics. Motivated by transformation to action-angle variables we constructed time-dependent canonical transformation which simplified equations of motion. We considered flux function given by $\Phi(t) = 2\pi\epsilon \sin \Omega t$. Using Proposition 4.2 and numerical simulation we concluded that the trajectory is either bounded or it passes arbitrarily close to origin and infinity, depending on the ratio of b and Ω. In the second part of Chapter 4 we turned to quantum case. We first reviewed notion of unitary propagator weakly associated to family of self-adjoint operators $H(t)$. In Proposition 4.7 we proved that unitary propagator weakly associated to T-periodic Hamiltonian is also T-periodic. Again we choose the flux function $\Phi(t) = \varepsilon \sin \omega t$. It was then shown that in the first approximation the system has only geometrically bound states if $\omega \neq 2Bj$, $j \in \mathbb{N}$.

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