Czech Technical University in Prague

Faculty of Nuclear Sciences and Physical Engineering

Diploma thesis

Martin Gajdoš

Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Investigation of the Properties of Generalized Perturbative Expansions in Quantum Chromodynamics

Martin Gajdoš

Department of Physics

Academic year:	2005/2006
Supervisor:	Prof. RNDr. Jiří Chýla, CSc.
Advisor:	Prof. RNDr. Jan Fischer, DrSc.

I declare that I have written this diploma thesis independently using the listed references.

Prague, May 12, 2006 ....

Signature

Prohlášení

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

Nemám závažný důvod proti užití tohoto školního díla ve smyslu §60 Zákona č. 121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

V Praze, dne 12. května 2006 ....

Podpis

Název práce: Výzkum vlastností zobecněných poruchových řad kvantové chromodynamiky

Autor: Martin Gajdoš

Katedra: Katedra fyziky, Fakulta jaderná a fyzikálně inženýrská, České vysoké učení technické v Praze

Obor: Matematické inženýrství

Vedoucí práce: Prof. RNDr. Jiří Chýla, CSc., Fyzikální ústav Akademie věd České republiky, chyla@fzu.cz

Konzultant: Prof. RNDr. Jan Fischer, DrSc., Fyzikální ústav Akademie věd České republiky, fischer@fzu.cz

Abstrakt: Fyzikální veličiny popisující tvrdé srážkové procesy ovlivněné silnou interakcí jsou v rámci poruchové QCD vyjádřeny poruchovými rozvoji v mocninách vazbového parametru α_s. Aplikace této metody je však komplikována divergencí těchto rozvojů, které mohou představovat v nejlepším případě pouze asymptotické rozvoje zkoumaných pozorovatelných. Vzhledem k těmto skutečnostem jsou nejprve představeny základní fakta o sumaci obecných mocninných řad a následně je podrobně rozebrána alternativní sumační metoda, navrhnuta Irinel Caprini a Janem Fischerem, využívající analytického pokračování v Borelově rovině pomocí konformního zobrazení oblasti holomorfnosti příslušné Borelovy transformace do jednotkového kruhu. Základní vlastnosti této metody jsou demonstrovány na modelových příkladech. Zároveň je tento přístup diskutován v případě fenomenologicky důležitých procesů rozpadu τ-leptonu a e^+e^- anihilace. Dále je studována závislost konečných aproximantů na renormalizační škále pro veličiny studované v rámci těchto dějů a konstruovaných na základě přístupu Caprini - Fischer. Získané výsledky jsou konfrontovány s výsledky, které plynou ze standardní poruchové teorie, kde jsou renormalizační škála a schéma voleny na základě přístupu PMS.

Klíčová slova: poruchová QCD, τ-lepton, e^+e^- anihilace, konformní zobrazení, analytické pokračování, Borelovská sumace

Title: Investigation of the Properties of Generalized Perturbative Expansions in Quantum Chromodynamics

Author: Martin Gajdoš

Department: Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague

Field of Study: Mathematical Engineering

Supervisor: Prof. RNDr. Jiří Chýla, CSc., Institute of Physics of the Academy of Sciences of the Czech Republic, chyla@fzu.cz

Advisor: Prof. RNDr. Jan Fischer, DrSc., Institute of Physics of the Academy of Sciences of the Czech Republic, fischer@fzu.cz

Abstract: Physical quantities describing hard processes affected by the strong interaction are, within the framework of perturbative QCD, expressed as power series in the strong coupling parameter α_s. The application of perturbative series is, however, complicated by their divergent nature. Therefore, perturbative series may represent, at best, only asymptotic expansions of studied observables. Following this pattern, basic facts about the summation of general power series are reviewed and novel, non-power, expansion of QCD quantities, invented by Irinel Caprini and Jan Fischer, is recalled. This approach makes use of an analytic continuation in the Borel plane by means of conformal mapping of the domain of holomorphy of a certain Borel transform considered. The basic features of this method are presented on some model examples. The Caprini - Fischer method is discussed with regard to phenomenologically interesting cases of the τ-lepton decay and e^+e^- annihilation. The renormalization scale dependence of finite order approximants constructed via this new expansion in these two cases is also investigated. These results are compared with the ones resulting from standard perturbation theory, where the renormalization scale and scheme are selected via PMS approach.

Keywords: perturbative QCD, τ-lepton, e^+e^- annihilation, conformal mapping, analytic continuation, Borel summation

Acknowledgements

I would like to thank my supervisor Prof. RNDr. Jiří Chýla, CSc. for his valuable advice, encouragement throughout the whole time and professional guidance. I am also deeply indebted to Prof. RNDr. Jan Fischer, DrSc. for his devoted help concerning the mathematical aspects of this work and innumerous stimulating discussions. I would like to express my gratitude to Dr. Irinel Caprini for many interesting remarks. My words of thanks belong also to Mgr. Miloš Rainiš for many creative argumentations.

Contents

List of Figures

Preface

Perturbative methods are one of the most commonly used frameworks in quantum theory calculations. This approximative technique provides the opportunity to search for solutions of certain problems by means of perturbative series in powers of the coupling constant[1].

Perturbative expansions in quantum theory, especially in the quantum theory of fields, are widely believed to be divergent. This observation was pointed out for the first time by F. J. Dyson in his pioneer work [1] in early fifties. Despite the fact that his conjecture has not yet been proven rigorously, we shall in the following assume the divergent nature of perturbative series, because perturbative expansions in quantum electrodynamics (QED) and quantum chromodynamics (QCD), the field theories which are of interest in the case of the phenomenologically important Standard model, are most likely to be of the divergent nature.

Dyson's arguments were later critically analyzed by many authors, e.g. [2, 3]. It was argued that Dyson's results in fact do not analyze the possible convergence or divergence of perturbative expansions but the possibility of recovering the exact results from these expansions. However, in QCD there are, unfortunately, no mathematically well-defined equations such as Schrödinger equation in quantum mechanics. Therefore, it is impossible to find an exact solution of any given QCD problem and then to compare it with the approximants obtained, e.g. from the perturbation theory. Nevertheless, the perturbative expansions are generally considered to be the asymptotic expansions of the exact results[2]. However, one has to bear in mind that an asymp-

[1] In the following sections, we shall use the word *couplant* instead of *coupling constant*.

[2] More extensive information concerning the clues indicating the divergent nature of perturbative expansions in QCD also with the mathematical comments on asymptotic series can be found in

totic series does not determine the expanded function uniquely, unless additional conditions are specified.

One of the attempts to give perturbative expansions in QCD a good meaning was invented by I. Caprini and J. Fischer [8–10] which is to be presented in Section 1.5. In this approach, the standard perturbative expansion in powers of the renormalized couplant a is replaced by a certain type of non-power series. The expansion functions are defined by the analytic continuation in the double cut Borel plane.

The essence of this thesis is to investigate the properties of these novel non-power expansions and to compare the results obtained by this method with the results given by the standard perturbative expansions in quantum chromodynamics.

The thesis is divided into three main parts. The first part entitled "The Mathematical Background" presents a brief overview on the mathematical aspects of general power expansions. The novel Caprini-Fischer method is presented and demonstrated on various model examples.

The second part of this work is carrying the name "Perturbative QCD". It summarizes the basic principles of QCD and reviews the renormalization scale and scheme dependence of finite order approximants of perturbative expansions of observables. Moreover, the application of the Caprini-Fischer method is demonstrated on the perturbative expansion of the Adler function.

The last part of the thesis "τ-lepton Decay and e^+e^- Annihilation in QCD" covers the most important results of the study. It presents the comparison of the perturbative expansions resulting from the standard perturbation theory with the Caprini-Fischer resummed perturbative expansion. This comparison is demonstrated on two phenomenologically interesting cases of τ-lepton decay and e^+e^- annihilation. The renormalization scale dependence of the approximants of ratio R_τ are shown together with the Q-dependence of the approximants of ratio $R_{e^+e^-}$. Moreover, in the case of $R_{e^+e^-}$, the Q-dependence of the Caprini-Fischer approximants compared with approximants resulting from standard perturbation theory computed in the PMS scheme is discussed.

Sections 1.4, 1.5 and in Chapter 2.

Chapter 1

The Mathematical Background

In order to understand the crucial part of this thesis, the treatment of perturbative expansions in quantum chromodynamics, let us briefly recall the mathematical formalism of general power expansions[1].

In the first place, it is necessary to define the meaning of the statement "summation of a series".

Definition 1.0.1 *Let $\{a_n\}$, $n = 0, 1, 2, \ldots$ be a set of numbers. To find* ***summation of the series***

$$\sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

means to determine the properties of the function

$$f_N(z) = \sum_{n=0}^{N} a_n z^n \tag{1.2}$$

around the point $z = 0$ for $N \to \infty$.

Definition 1.0.2 *Let $\mathcal{S}$ be a method of summation of the series (1.1). Then*

1. *$\mathcal{S}$ is said to be* ***regular*** *if*

$$\mathcal{S}\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n \tag{1.3}$$

whenever the righthand side of (1.3) converges,

[1]Definitions and theorems in this chapter are taken from [7].

2. *$\mathcal{S}$ is said to be* ***analytic*** *if*

$$\mathcal{S}\left(\sum_{n=0}^{\infty} a_n z^n\right) = f(z) \tag{1.4}$$

whenever $\mathcal{S}\left(\sum_{n=0}^{\infty} a_n z^n\right)$ exists, where $f(z)$ denotes the analytic continuation of $\sum_{n=0}^{\infty} a_n z^n$ out of its convergence disk.

1.1 Borel Summation

One of the intuitively clearest methods of summation of the series is the so called Borel summation. It arises from the idea of suppressing the coefficients a_n of the series (1.1) by dividing with $n!$. We shall start with the definition of the Borel transform of the series (1.1).

Definition 1.1.1 *Let (1.1) be an arbitrary power series assuming its convergence inside the circle K_R with radius R respecting the equation*

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{1.5}$$

Then the series

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \tag{1.6}$$

is called the ***Borel transform*** *of the series (1.1) and denoted by $B[f](z)$.*

Secondly, let us then define the Borel summability and the Borel sum of the series (1.1). Note that $\Gamma(n+1) = \int\limits_0^{\infty} \mathrm{e}^{-t} t^n \mathrm{d}t = n!$.

Definition 1.1.2 *Let (1.1) be an arbitrary power series satisfying the conditions stated in Definition 1.1.1. Then the series (1.1) is said to be* ***Borel summable*** *iff*

1. *$\exists \varepsilon > 0,\ \forall t \in \mathbb{C},\ |t| < \varepsilon$, such that $B[f](t)$ converges,*

2. *$B[f](t)$ has analytic continuation to an infinite strip bisected by the positive real semi-axis and nowhere shrinking to zero width,*

3. $\exists z \in \mathbb{C},\ z \neq 0$, *such that integral*

$$\tilde{f}(z) = \frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{t}{z}} B[f](t)\mathrm{d}t \tag{1.7}$$

converges.

*The function $\tilde{f}(z)$ is called **Borel sum** of the series (1.1).*

Before specifying the concluding theorem [5] of this section let us define the so-called Borel polygon.

Definition 1.1.3 *Let z_i be the position of the i-th singularity of $f(z)$, $i = 1, 2, \ldots, N$, where N is the number of singularities of $f(z)$ and R_i is the ray connecting z_i with the origin and continuing beyond z_i to infinity. Draw the perpendicular P_i to R_i at $z = z_i$ for each $i = 1, 2, \ldots, N$. The part of the complex plane that is closed up by the P_i and contains the origin is called the **Borel polygon**.*

Theorem 1.1.1 *Let $f(z)$ be the principal branch of an analytic function regular at $z = 0$,*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then the integral of this series

$$\int_0^\infty \mathrm{e}^{-t} \sum_{n=0}^{\infty} a_n \frac{(zt)^n}{n!} \mathrm{d}t = \int_0^\infty \mathrm{e}^{-t} B[f](zt)\mathrm{d}t, \tag{1.8}$$

exists in the Borel polygon and is there equal to the analytic function $f(z)$.

We can now demonstrate benefits and shortcomings of the Borel method on the following simple example. Take $a_n = 1,\ \forall n \in \mathbb{N}_0$ in (1.1) and set $f(z) = \frac{1}{1-z}$. This yields

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n. \tag{1.9}$$

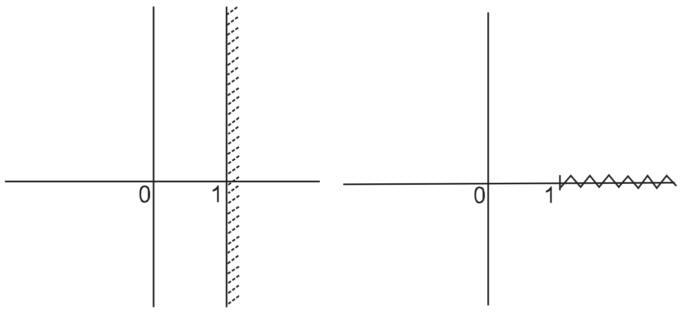
The sum on the righthand side of (1.9), which is the Taylor series of the function $f(z) = \frac{1}{1-z}$, converges in the open unit disk where we can carry out the following manipulations:

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^n \frac{1}{n!} \int_0^{\infty} \mathrm{e}^{-t} t^n \mathrm{d}t = \sum_{n=0}^{\infty} \int_0^{\infty} \mathrm{e}^{-t} \frac{(zt)^n}{n!} \mathrm{d}t =$$
$$= \int_0^{\infty} \mathrm{e}^{-t} \sum_{n=0}^{\infty} \frac{(zt)^n}{n!} \mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-(1-z)t} \mathrm{d}t. \tag{1.10}$$

The integral on the righthand side of (1.10) can be used to define the sum of the series $\sum_{n=0}^{\infty} z^n$ outside the unit disk. The sum is well-defined whenever the integral converges. This is true in the whole halfplane

$$\mathrm{Re}\, z < 1. \tag{1.11}$$

In the sense of Definition 1.0.2 the Borel method is regular and analytic because the integral on the righthand side of (1.10) is a holomorphic function in z.



(a) Borel method (b) μ-method introduced in the Section 1.2.2

Figure 1.1: Domains of summation of the series $\sum_{n=0}^{\infty} z^n$

It is clear that the integral defines the sum in a larger region than the standard Taylor expansion in z does. On the other hand, we can simultaneously see a weak point of this approach. The analytic continuation given by the Borel method is, due to Theorem 1.1.1, restricted to the Borel polygon, despite the fact that a large area of analyticity still spreads outside the polygon. The holomorphy region for the function $f(z) = \frac{1}{1-z}$ equals

$$\mathbb{C} \setminus \{1\}. \tag{1.12}$$

One question arises: "Is it possible to make analytic continuations beyond the Borel polygon?" There exist many opportunities how to give a positive answer to this question. One possibility is to make an extension of the Borel polygon into a larger region[2]. Another one is to make analytic continuations by means of conformal mappings. The former is introduced in the Section 1.2 and the latter in the Section 1.3.

1.2 Extension of the Borel Polygon

Let us consider the series (1.1) with nonzero radius of convergence[3]. The possibility of recovering a certain function from its power series outside its Borel polygon can be obtained by extending the Borel polygon to a larger domain completely containing the Borel polygon. In the following sections, we shall present two methods respecting this condition.

1.2.1 Generalization of the Borel Summation

Extensions of the Borel polygon presented in this section can be considered as the generalization of the Borel method reviewed in the Section 1.1. Just for illustration, it is possible to show the basic ideas of this method in the case of the series (1.9). The Borel method allows one to define the sum of (1.9) in the region (1.11) which has the border

$$\mathrm{Re}\, z = 1. \tag{1.13}$$

Expression (1.13) can be expressed equivalently as follows:

$$r = (\cos\varphi)^{-1}, \quad z = r\mathrm{e}^{i\varphi}, \quad |\varphi| < \frac{\pi}{2}. \tag{1.14}$$

[2] Figure 1.1 above shows the one possible extension of the Borel polygon of the function $f(z) = \sum_{n=0}^{\infty} z^n$ to a larger domain called the Mittag-Leffler (principal) star introduced by Definition 1.2.2. The picture was taken from the paper [12].

[3] Due to the fact that the character of this section is only informative we shall present the concluding ideas taken from the paper [12] very briefly.

An extension of the Borel polygon of the function $f(z) = \frac{1}{1-z}$ can be made by replacing the border (1.14) with

$$r = \left[\cos\left(\frac{\varphi}{\alpha}\right)\right]^{-\alpha}, \quad |\varphi| < \frac{\pi}{2}\alpha, \tag{1.15}$$

where $z = re^{i\varphi}$ and $\alpha > 0$ provided that Re $z < 1$ is extended into the region

$$\mathcal{B}_\alpha\left(\frac{1}{1-z}\right) = r^{\alpha^{-1}} \cos\left(\frac{\varphi}{\alpha}\right) < 1. \tag{1.16}$$

It is easy to check that for $\alpha = 1$ the region (1.14) is reproduced. For α small, region $\mathcal{B}_\alpha\left(\frac{1}{1-z}\right)$ spreads further to the righthand half-plane than the Borel polygon of the function $f(z) = \frac{1}{1-z}$.

Let us now define the region $\mathcal{B}_\alpha(f)$ for arbitrary function $f(z)$ regular at the origin [12].

Definition 1.2.1 *Let $f(z)$ be an arbitrary function regular at the origin and let $C(z_0)$ be a contour given by the relation*

$$r = r_0\left[\cos\frac{\theta-\theta_0}{\alpha}\right]^{-\alpha}, \quad |\theta-\theta_0| < \frac{\pi}{2}\alpha. \tag{1.17}$$

Let us draw the contour $C(z_s)$ for each singularity z_s of $f(z)$ and discarding from the complex plane $\mathbb{C}$ the domain $C^0(z_s)$ closed up by the contour, for which the sign $=$ in expression (1.17) is replaced by $>$. Then $\mathcal{B}_\alpha(f)$ denotes a starlike region containing the disk of convergence of the Taylor series of $f(z)$ centered at the origin.

The concluding theorem of this section generalizing the Borel method has the following form:

Theorem 1.2.1 *Let $f(z)$ be the principal branch of an analytic function, regular at the origin,*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then the integral

$$I(z) = \int_0^\infty t^{\beta-1} \mathrm{e}^{-t} \sum_{n=0}^{\infty} a_n \frac{(t^\alpha z)^n}{\Gamma(\alpha n + \beta)} \mathrm{d}t, \quad 0 < \alpha \leq 2, \ \beta > 0, \tag{1.18}$$

converges iff $z \in \mathcal{B}_\alpha(f)$. *Then*

$$f(z) = I(z). \tag{1.19}$$

The convergence is absolute and uniform on any bounded subset of $\mathcal{B}_\alpha(f)$ *with nonzero distance from the boundary of* $\mathcal{B}_\alpha(f)$. *For the first derivative* $f'(z)$ *of* $f(z)$ *the following representation*

$$f'(z) = \int_0^\infty t^{\beta-1} \mathrm{e}^{-t} \sum_{n=0}^{\infty} n a_n z^{n-1} \frac{(t^\alpha)^n}{\Gamma(\alpha n + \beta)} \mathrm{d}t \tag{1.20}$$

holds.

Theorem 1.2.1 shows the price payed for the extension of the Borel polygon. As we have already mentioned, the extension is done for $\alpha < 1$ and the suppression of the coefficients a_n in (1.1) is obtained by dividing a_n by $\Gamma(\alpha n + \beta)$ instead of $n!$. Therefore, the resulting suppression is weaker than the n!-like suppression since

$$n! > \Gamma(\alpha n + \beta), \quad 0 < \alpha < 1, \ \beta > 0. \tag{1.21}$$

Thus, for α small this method is applicable to a smaller class of sequences of coefficients $\{a_n\}_{n=0}^{\infty}$. Note that for $\alpha = \beta = 1$ one obtains the Borel method and therefore $\mathcal{B}_1(f)$ is equal to the Borel polygon of a function $f(z)$.

Remark 1.2.1 *The domain* $\mathcal{B}_\alpha(f)$ *is invariant under differentiation, i.e.* $\mathcal{B}_\alpha(f') = \mathcal{B}_\alpha(f)$. *Inserting first derivative* $f'(z)$ *of* $f(z)$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

in Theorem 1.2.1 one obtains another integral representation of $f'(z)$, *different from (1.20)*

$$f'(z) = \int_0^\infty t^{\beta-1} \mathrm{e}^{-t} \sum_{n=1}^{\infty} n a_n \frac{(t^\alpha z)^{n-1}}{\Gamma[\alpha(n-1) + \beta]} \mathrm{d}t. \tag{1.22}$$

1.2.2 Moment Constant Summability Method

In the previous section, we have constructed the summation method that extends the Borel polygon. This method along with the standard Borel summation is one member of the so-called moment constant summability methods [5]. Nevertheless, there exists a moment constant summability method generalizing the previous method of the Borel polygon extension providing an analytic continuation of a function regular at the origin onto its Mittag-Leffler (principal) star.

We shall start with the definition of the Mittag-Leffler (principal) star [7].

Definition 1.2.2 *Let z_i, R_i and N have the same meaning as in Definition 1.1.3. Cut the complex z-plane along that part of each ray R_i that lies behind the singularity z_i, and remove these cuts from the complex z-plane. The part of the z-plane remaining after these cuts are removed is called the* ***Mittag-Leffler (principal) star*** *of the function $f(z)$. We shall denote it by* $\mathrm{MLS}(f)$[4].

It is useful to notice that for $f(z) = \frac{1}{1-z}$, the Mittag-Leffler star $\mathrm{MLS}\left(\frac{1}{1-z}\right)$ equals $\mathbb{C} \setminus \langle 1, +\infty)$.

Theorem 1.2.1 provides, in contrast to Theorem 1.1.1, a considerable improvement. As we have already mentioned in Section 1.2.1, the region $\mathcal{B}_\alpha(f)$ for $\alpha < 1$, where we can carry out the summation (1.18), is larger than the Borel polygon $\mathcal{B}_1(f)$ of the function $f(z)$. It is clear that for $\alpha > 0$ arbitrarily small, the domain $\mathcal{B}_\alpha(f)$ approaches arbitrarily close the Mittag-Leffler star $\mathrm{MLS}(f)$. However, it is forbidden to choose $\alpha = 0$. Therefore, the function $I(z)$ defined by (1.18) can never represent the sum of $f(z)$ in the whole $\mathrm{MLS}(f)$. Nevertheless, the moment constant summability method representing the expanded function $f(z)$ in the whole $\mathrm{MLS}(f)$, which existence was mentioned in the beginning of this section, exists and is constructed using the following theorem [12].

Theorem 1.2.2 *Let $f(z)$ be the principal branch of an analytic function regular at origin*

[4]Figure 1.2 bellow used to illustrate Definitions 1.1.3, 1.2.2 and Theorems 1.2.1, 1.2.2 is taken from [7].

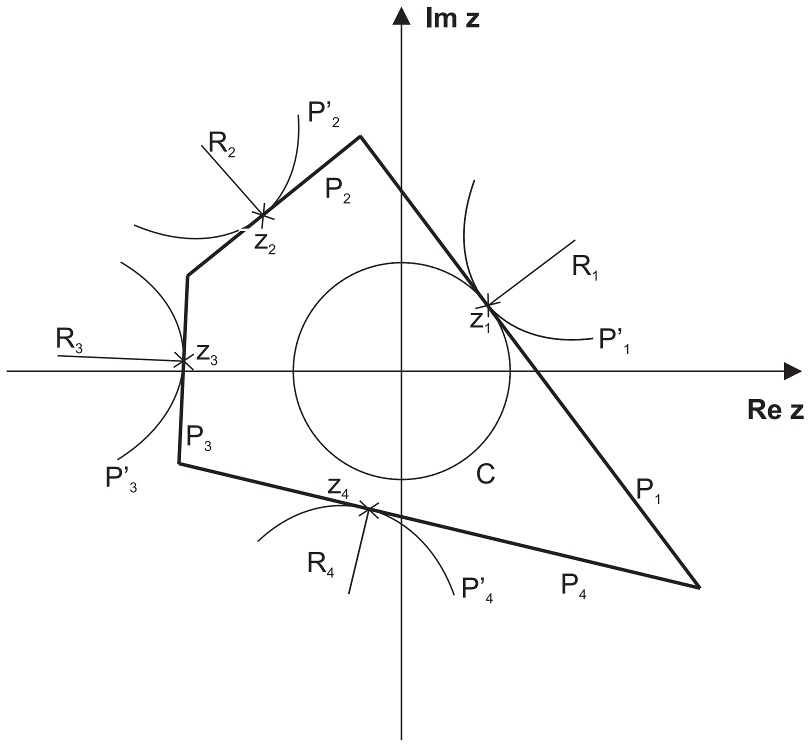


Figure 1.2: Explanation of symbols: C ... convergence circle of the series (1.1),
$P_1,\ P_2,\ P_3,\ P_4$ ... the Borel polygon from Definition 1.1.3,
$P_1',\ P_2',\ P_3',\ P_4'$ ... extension of the Borel polygon from Theorem 1.2.1,
$R_1,\ R_2,\ R_3,\ R_4$ ... boundary of $\mathrm{MLS}(f)$ from Definition 1.2.2 and Theorem 1.2.2.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then the integral $J(z)$

$$J(z) = \int\limits_0^{\infty} \exp(-\exp t) \sum_{n=0}^{\infty} a_n \frac{(zt)^n}{\mu(n)} \mathrm{d}t, \tag{1.23}$$

converges iff $z \in \mathrm{MLS}(f)$, where $\{\mu(n)\}_{n=0}^{\infty}$ is the Stieltjes moment sequence generated by the measure $\mathrm{d}\chi = \exp(-\exp t)\mathrm{d}t$ and defined as

$$\mu(n) = \int\limits_0^{\infty} \exp(-\exp t) t^n \mathrm{d}t. \tag{1.24}$$

Then

$$f(z) = J(z). \tag{1.25}$$

The convergence is absolute and uniform in any bounded subset of MLS(f) *with nonzero distance from the boundary of* MLS(f). *For the first derivative $f'(z)$ of $f(z)$ the following representation*

$$f'(z) = \int_0^\infty \exp(-\exp t) \sum_{n=1}^{\infty} n a_n z^{n-1} \frac{t^n}{\mu(n)} \mathrm{d}t \tag{1.26}$$

holds.

It can be shown that the moment constant summability methods defined in [5] are automatically regular in the sense of Definition 1.0.2. However not analytic in general. Nevertheless, it was proven [12] that the moment constant method based on the above Theorem 1.2.2 provides an analytic continuation of (1.1).

As in the case of Theorem 1.2.1, the price payed for this powerful result yielding the analytic continuation of the power series (1.1) in the whole MLS(f) is a relatively weaker suppression of the coefficients a_n by the moments $\mu(n)$. This phenomenon leads from the fact that moments $\mu(n)$ grow asymptotically slower for $n \to \infty$ than the gamma function $\Gamma(\alpha n + \beta)$ for any $\alpha > 0$ [12].

Remark 1.2.2 *The domain* MLS(f) *is invariant under differentiation, i.e.* MLS$(f') =$ MLS(f). *Inserting first derivative $f'(z)$ of $f(z)$*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

in Theorem 1.2.2 one obtains another integral representation of $f'(z)$, different from (1.26)

$$f'(z) = \int_0^\infty \exp(-\exp t) \sum_{n=1}^{\infty} n a_n \frac{(tz)^{n-1}}{\mu(n-1)} \mathrm{d}t. \tag{1.27}$$

1.3 The Use of Conformal Mapping

Take the series (1.1) convergent on the disk K_R in the complex z-plane. The radius R is equal to the distance of the nearest singularity of a function $f(z)$ from the origin [5].

A conformal mapping $w(z)$ of the z-plane can be used to improve the convergence of the series (1.1) in the sense that,

1. the power series
$$f(z) = f(z(w)) = \sum_{n=0}^{\infty} c_n w^n \tag{1.28}$$
is convergent in a region C_w of the complex z-plane that is conformally mapped onto the disk K_w in the w-plane by $w(z)$, is larger than K_R and fully contains it, that means $K_R \subset C_w$, and

2. the series (1.28) has a faster convergence rate than (1.1) at every point $z \in K_R$. The proof can be found in a paper by Ciulli and Fischer [16], including the result that if $C_{\tilde{w}}$ is the domain of analyticity of the expanded function and $\tilde{w}(z)$ is the conformal mapping that maps $C_{\tilde{w}}$ onto the unit disk in the $\tilde{w}$-plane, then the series
$$f(z) = f(z(\tilde{w})) = \sum_{n=0}^{\infty} \tilde{c}_n \tilde{w}^n \tag{1.29}$$
is convergent at every point of $C_{\tilde{w}}$ and yields the fastest convergence rate at every point. This conformal mapping $\tilde{w}(z)$ and the corresponding power series (1.29) are called optimal.

For reasons which will become clear later we shall restrict ourselves to such functions $w(z)$ that

$$w(z)|_{z=0} = 0 \ \wedge \ w(z)|_{z=z_1} = 1, \tag{1.30}$$

where z_1 is the location of the singularity nearest to the origin.

[5] Note that R can be calculated using the following formula $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Let us constrain for a moment on the situation when all the singularities of a function (1.1) are located on the real positive semi-axis, $0 < z_1 \leq z \leq +\infty$. Such conformal mapping respecting the requirements mentioned above is to be of the form

$$w(z) = \frac{1 - \sqrt{1 - \dfrac{z}{z_1}}}{1 + \sqrt{1 - \dfrac{z}{z_1}}}. \tag{1.31}$$

$w(z)$ transforms the whole complex z-plane cut along $\langle z_1, +\infty\rangle$ onto the unit disk centered at the origin. The mapping (1.31) also satisfies the conditions (1.30) and the points of the cut are mapped onto the boundary of the unit disk.

To obtain analytic continuation for (1.1) outside the convergence domain K_R we have to expand $f(z)$ in powers of $w(z)$:

$$f(z) = f(z(w)) = \sum_{n=0}^{\infty} c_n w^n. \tag{1.32}$$

This can be done by means of the following steps. Take the series (1.1) truncated at the order N

$$f_N(z) = \sum_{n=0}^{N} a_n z^n \tag{1.33}$$

and expand powers z^n in powers of $w(z)$. By truncating this expansion at the order N we get

$$z_N^n = \sum_{j=n}^{N} b_{nj} w^j. \tag{1.34}$$

The coefficients b_{nj} can be calculated by expressing z in terms of w as

$$z(w) = \frac{4 z_1 w}{(w+1)^2}. \tag{1.35}$$

Then, inserting the equation (1.34) into (1.33) and changing the order of summations yields

$$
\begin{aligned}
f_N(z) &= \sum_{n=0}^{N} a_n z^n = a_0 + \sum_{n=1}^{N} a_n z^n = a_0 + \sum_{n=1}^{N} a_n \sum_{j=n}^{N} b_{nj} w^j \\
&= a_0 + \sum_{n=1}^{N} \sum_{j=n}^{N} a_n b_{nj} w^j = a_0 + \sum_{j=1}^{N} \sum_{n=1}^{j} a_n b_{nj} w^j = a_0 + \sum_{j=1}^{N} c_j w^j \\
&= \sum_{j=0}^{N} c_j w^j, \qquad (1.36)
\end{aligned}
$$

where

$$c_0 = a_0, \quad c_j = \sum_{n=1}^{j} a_n b_{nj}, \quad j \in \mathbb{N}. \qquad (1.37)$$

Finally, considering the limit of the equation (1.36) for $N \to +\infty$ one obtains the sought expansion of $f(z)$ in powers of the conformal mapping $w(z)$

$$\lim_{N\to+\infty} \sum_{j=0}^{N} c_j w^j = \sum_{j=0}^{\infty} c_j w^j = f(z). \qquad (1.38)$$

This expansion converges inside the unit disk of the w-plane. Thus it converges in the whole cut z-plane. As pointed out in [7], the fastest asymptotic rate of convergence is achieved when a conformal mapping w is optimal in the above mentioned sense.

1.3.1 A Simple Example

To illustrate the method explained in the previous subsection, let us consider the function (1.9). In this situation $z_1 = 1$ and therefore (1.35) becomes

$$z(w) = \frac{4w}{(w+1)^2}. \qquad (1.39)$$

Equation (1.34) yields

$$z_N^n = \sum_{j=n}^{N} \frac{1}{j!} \frac{\mathrm{d}^j}{\mathrm{d}w^j} \left[\frac{4w}{(w+1)^2} \right]^n \Bigg|_{w=0} w^j, \qquad (1.40)$$

and (1.36), (1.37), (1.38) imply

$$f(z) = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{n=1}^{j} \frac{\mathrm{d}^j}{\mathrm{d}w^j} \left[\frac{4w}{(w+1)^2} \right]^n \Bigg|_{w=0} w^j. \qquad (1.41)$$

If we take a closer look at the last equation we find the calculation of the coefficients c_j defined according to (1.37) as

$$c_j = \frac{1}{j!}\sum_{n=1}^{j} \frac{\mathrm{d}^j}{\mathrm{d}w^j}\left[\frac{4w}{(w+1)^2}\right]^n\Bigg|_{w=0} \tag{1.42}$$

rather cumbersome. It is due to the j-th derivation of a composite function entangled in (1.42). We shall carry out these calculations in several steps.

Let us first write z^n by means of $w(z)$ in the form

$$z^n = 4^n w^{n-1}\frac{w}{(w+1)^{2n}}. \tag{1.43}$$

Then using the Leibniz chain-rule for the j-th derivation of the product of two functions one obtains

$$\begin{aligned}
\frac{\mathrm{d}^j z^n}{\mathrm{d}w^j} &= \frac{\mathrm{d}^j}{\mathrm{d}w^j}\left[4^n w^{n-1}\frac{w}{(w+1)^{2n}}\right] = 4^n\frac{\mathrm{d}^j}{\mathrm{d}w^j}\left[w^{n-1}\frac{w}{(w+1)^{2n}}\right]\\
&= 4^n\sum_{l=0}^{j}\frac{j!}{(j-l)!l!}\left[\frac{\mathrm{d}^l w^{n-1}}{\mathrm{d}w^l}\right]\frac{\mathrm{d}^{j-l}}{\mathrm{d}w^{j-l}}\left[\frac{w}{(w+1)^{2n}}\right]\\
&= 4^n\sum_{l=0}^{j}\frac{j!}{(j-l)!l!}\left[\frac{(n-1)!}{(n-l-1)!}w^{n-l-1}\right]\left[(-1)^{j-l}\frac{(2n+j-l-2)!}{(2n-2)!}\frac{w-\frac{j-l}{2n-1}}{(w+1)^{2n+j-l}}\right]\\
&= \sum_{l=0}^{j}(-1)^{j-l}4^n\frac{j!}{(j-l)!l!}\frac{(n-1)!}{(n-l-1)!}\frac{(2n+j-l-2)!}{(2n-2)!}\frac{w^{n-l}-w^{n-l-1}\frac{j-l}{2n-1}}{(w+1)^{2n+j-l}}.
\end{aligned} \tag{1.44}$$

Formulae

$$\frac{\mathrm{d}^l w^{n-1}}{\mathrm{d}w^l} = \frac{(n-1)!}{(n-l-1)!}w^{n-l-1} \tag{1.45}$$

$$\frac{\mathrm{d}^{j-l}}{\mathrm{d}w^{j-l}}\left[\frac{w}{(w+1)^{2n}}\right] = (-1)^{j-l}\frac{(2n+j-l-2)!}{(2n-2)!}\frac{w-\frac{j-l}{2n-1}}{(w+1)^{2n+j-l}} \tag{1.46}$$

can easily be proven by induction. Now inserting (1.44) into (1.41) yields

$$f(z) = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{n=1}^{j} \sum_{l=0}^{j} (-1)^{j-l} 4^n \frac{j!}{(j-l)!l!} \frac{(n-1)!}{(n-l-1)!} \frac{(2n+j-l-2)!}{(2n-2)!} \times \left. \frac{w^{n-l} - w^{n-l-1} \frac{j-l}{2n-1}}{(w+1)^{2n+j-l}} \right|_{w=0} w^j. \tag{1.47}$$

Moreover, the sum

$$\sum_{l=0}^{j} (-1)^{j-l} 4^n \frac{j!}{(j-l)!l!} \frac{(n-1)!}{(n-l-1)!} \frac{(2n+j-l-2)!}{(2n-2)!} \left. \frac{w^{n-l} - w^{n-l-1} \frac{j-l}{2n-1}}{(w+1)^{2n+j-l}} \right|_{w=0} \tag{1.48}$$

can be easily simplified. In fact, after we set $w = 0$, every term will vanish except for the term belonging to the index $l = n - 1$, and (1.48) is reduced to the single term

$$(-1)^{j-n} 4^n \frac{j!}{(2n-1)!} \frac{(n+j-1)!}{(j-n)!}. \tag{1.49}$$

As a consequence, the coefficient c_j defined by (1.42) is of the form

$$c_j = \frac{1}{j!} \sum_{n=1}^{j} (-1)^{j-n} 4^n \frac{j!}{(2n-1)!} \frac{(n+j-1)!}{(j-n)!} = 4j, \tag{1.50}$$

and finally we obtain the expansion of $f(z) = \frac{1}{1-z}$ in powers of its optimal conformal mapping w. Equation (1.41) yields the following expression:

$$\frac{1}{1-z} = 1 + \sum_{j=1}^{\infty} 4j w^j(z). \tag{1.51}$$

The final step in this exercise is to check the convergence of the new expansion (1.51). For this purpose we shall use the extreme case of the d'Alembert quotient criterion.

For all z in the whole cut z-plane (1.12) there exists a real $A < 1$ such that the following expression for $w(z)$ holds

$$|w(z)| \leq A. \tag{1.52}$$

Then using this inequality one derives

$$\left|\frac{4(j+1)w^{j+1}}{4jw^j}\right| = \left|\frac{j+1}{j}w\right| \le \frac{j+1}{j}A \xrightarrow{j\to+\infty} A < 1. \tag{1.53}$$

Therefore (1.51) for all z in the whole cut z-plane as a numerical series and by the definition it is convergent as a functional series.

1.4 Asymptotic Series

The divergence of perturbative expansions in quantum field theory was already considered in the Preface. Extensive studies of this phenomenon were performed leading to the conclusion that the standard QCD perturbative expansions in powers of the renormalized couplant are divergent, Borel non-summable and having zero radius of convergence [17, 18]. The factorial growth of perturbative coefficients at large orders is also present and, moreover, the coefficients do not alternate signs [19–25].

The fact that the perturbative series' radius of convergence equals zero means that the searched function $f(z)$ has a singularity at the origin $z = 0$. As we have already mentioned, such a power series cannot be understood in the sense of a Taylor expansion, but it can be interpreted as an asymptotic series. Let us then very briefly recall the definition of an asymptotic series.

Definition 1.4.1 *Let S be a set of points with the origin $z = 0$ as an accumulation point, and $f(z)$ a function defined on S. We shall say that $f(z)$ has on S the asymptotic series $\sum_{n=0}^{\infty} a_n z^n$ for $z \to 0$ and denote it by*

$$f(z) \approx \sum_{n=0}^{\infty} a_n z^n, \quad z \to 0, \quad z \in S \tag{1.54}$$

if

$$f(z) - \sum_{n=0}^{N} a_n z^n = o(z^N), \quad z \to 0, \quad z \in S. \tag{1.55}$$

Perturbation theory yields, at least in principle, all the expansion coefficients of the expanded Green function which is, in fact, unknown. As we have pointed out, a certain power series is asymptotic to an infinite number of different functions. This can be easily seen from the fact that the function

$$g(z,a,c) = c\mathrm{e}^{-\frac{a}{z}}, \quad c \in \mathbb{C}, a > 0 \tag{1.56}$$

has the asymptotic series of the form

$$\sum_{n=0}^{\infty} a_n z^n, \quad \forall n \in \mathbb{N}_0,\ a_n = 0 \tag{1.57}$$

for $z \to 0$, in any angle $|\arg z| < \frac{\pi}{2}$. Therefore, any function of the form $f(z) + g(z,a,c)$ has the asymptotic series equal to the asymptotic series (1.54) of the function $f(z)$ itself.

These ambiguities can be treated with additional conditions which have to be imposed on the asymptotic series. These conditions are needed to make the determination of the expanded function unique, or at least, to reduce the ambiguity [6]. Let us recall one example of such conditions which guarantee the expanded function to be equal to the Borel sum of its asymptotic power series. This subject was thoroughly studied by Watson leading to the statement of the following theorem [26] [7]:

Theorem 1.4.1 *Let $f(z)$ be analytic in a sector $|\arg z| < \frac{1}{2}\pi + \varepsilon,\ |z| < \eta$, for some $\varepsilon > 0$ and $\eta > 0$. Let $f(z)$ have there the asymptotic expansion*

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + R_N(z) \tag{1.58}$$

with

$$|R_N(z)| \leq A\sigma^N N! |z|^N \tag{1.59}$$

uniformly in N and z in the sector. Then

1. *the Borel transform $B(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$ converges on the disk $|t| < \frac{1}{\sigma}$,*

2. *$B(t)$ has an analytic continuation to the sector $|\arg t| < \varepsilon$,*

[6] The uniquely determined function is understood as the exact solution of a studied QCD problem, which we in fact do not know. For further explanations, please see Preface and Chapter 2.

[7] For a more complete overview of additional conditions needed for the expanded function to be uniquely recovered from its asymptotic power series please see [7, 52].

3. the integral

$$g(z) = \frac{1}{z} \int_0^\infty \mathrm{e}^{-\frac{t}{z}} B(t) \mathrm{d}t \tag{1.60}$$

converges absolutely in the open disk $\mathrm{Re} z^{-1} > \eta^{-1}$ *and there equals* $f(z)$[8].

Unfortunately, the conditions required by this theorem are too strong to be fulfilled by renormalizable theories such as QED or QCD [17]. Also, in the case of the improved theorem given by Nevanlinna [27,28] and Sokal [29], to the best our knowledge, neither QED nor QCD are able to offer conditions which would be sufficient for a unique determination of the expanded function $f(z)$ from the asymptotic series (1.54).

Some improvement can be achieved through the analytic continuation of the Borel transform of perturbative series by means of the optimal conformal mapping introduced by I. Caprini and J. Fischer [8–10]. Since this thesis is devoted to the application of this method on certain QCD processes, we shall present its theoretical background more extensively in the following section.

1.5 Optimal Expansion of the Borel Integral

In Section 1.3 we presented one of the many possibilities of performing an analytic continuation of an arbitrary power series of the form of (1.1) by means of the optimal conformal mapping of the domain of analyticity of the function $f(z)$ onto the unit disk. We also recalled the result proved by S. Ciulli and J. Fischer [16] that the asymptotically fastest convergence rate is achieved by expanding in powers of the optimal conformal mapping.

In perturbative QCD, only little information is available about the singularities of the Borel transform of Green functions. This information is obtained from the study of certain classes of Feynman diagrams and from many non-perturbative effects. As

[8]Note that the properties 1, 2 and 3 are, according to Definition 1.1.2, the three conditions of Borel summability.

was pointed out in [8], the Borel plane and the Borel transform seem to be very suitable objects as the starting point in our following discussions. One can expand the Borel transform in powers of its optimal conformal mapping variable and obtain a convenient non-power expansion of the Borel integral. Due to the the fact that the Borel transform of the QCD field correlators, e.g. the Adler function, has singularities placed on the positive real semi-axis[9], the condition 2 for the Borel summability in Definition 1.1.2 is not satisfied and a prescription specifying the integration path of the Borel integral is needed [8].

Let us give a simple example to illustrate the basic idea of introducing a non-power expansion replacing the expansion in powers of z. We make the following formal manipulations:

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n \frac{1}{n!} \int_0^{\infty} \mathrm{e}^{-t} t^n \mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-t} \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!} a_n}_{b_n} \overbrace{(zt)^n}^{u^n} \mathrm{d}t \\
&= \left| \begin{array}{l} \text{substitution } u = zt \text{ is valid only for } z \in \mathbb{R},\ z > 0, \\ \text{not for } z \in \mathbb{C} \text{ in general} \end{array} \right| \\
&= \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \sum_{n=0}^{\infty} b_n u^n \mathrm{d}u = \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \lim_{N\to\infty} \sum_{n=0}^{N} b_n u^n \mathrm{d}u \\
&= \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \lim_{N\to\infty} \sum_{n=0}^{N} b_n \sum_{k=0}^{N} c_k^{(n)} w^k(u) \mathrm{d}u \\
&= \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \lim_{N\to\infty} \sum_{k=0}^{N} \underbrace{\sum_{n=0}^{k} b_n c_k^{(n)}}_{c_k} w^k(u) \mathrm{d}u.
\end{aligned} \tag{1.61}$$

Let us consider, instead of this result, the expression

$$f(z) = \lim_{N\to\infty} \sum_{k=0}^{N} c_k \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} w^k(u) \mathrm{d}u, \tag{1.62}$$

which is formally obtained by interchanging the sum and the integration. This expansion can be rewritten as

[9] These singularities are called instantons and infrared renormalons.

$$f(z) = \sum_{k=0}^{\infty} c_k W_k(z), \tag{1.63}$$

where

$$W_k(z) = \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} w^k(u) \mathrm{d}u. \tag{1.64}$$

To give the sum (1.63) precise mathematical meaning, one has to consider the equation

$$W(z) = \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} B(u) \mathrm{d}u = \sum_{n=0}^{\infty} c_n W_n(z), \quad B(u) = \sum_{n=0}^{\infty} b_n u^n, \tag{1.65}$$

as the definition of the Borel integral provided that the integration path is properly defined and the series $\sum_{n=0}^{\infty} c_n W_n(z)$ is convergent. The generalized principal value prescription seems to be a reasonable, although not necessary, choice for the integration path. One can define $W_n^{PV}(z)$ as

$$W_n^{PV}(z) = \frac{1}{2z} \left[\int_{C_+} \mathrm{e}^{-\frac{u}{z}} w^n(u) \mathrm{d}u + \int_{C_-} \mathrm{e}^{-\frac{u}{z}} w^n(u) \mathrm{d}u \right], \quad n = 0, 1, 2, \ldots, \tag{1.66}$$

where C_+ and C_- are the lines parallel to the positive real axis, the former slightly above it and the latter slightly below it. Thus, for any $\epsilon > 0,\ \epsilon \ll 1$ one can rewrite the equation (1.66) in the form

$$W_n^{PV}(z) = \frac{1}{2z} \left[\int_0^{\infty} \mathrm{e}^{-\frac{u+i\epsilon}{z}} w^n(u+i\epsilon) \mathrm{d}u + \int_0^{\infty} \mathrm{e}^{-\frac{u-i\epsilon}{z}} w^n(u-i\epsilon) \mathrm{d}u \right], \quad n = 0, 1, 2, \ldots \tag{1.67}$$

Convergence conditions of the optimal expansion are investigated in detail in [9, 10]. Let us recall the results. For this purpose we introduce the notation

$$W_n^+(z) = \frac{1}{z} \int_{C_+} \mathrm{e}^{-F_n(u)} \mathrm{d}u, \tag{1.68}$$

$$W_n^-(z) = \frac{1}{z} \int_{C_-} \mathrm{e}^{-F_n(u)} \mathrm{d}u, \tag{1.69}$$

where

$$F_n(u) = \frac{u}{z} - n \ln w(u). \tag{1.70}$$

The relevant optimal conformal mapping is of the form[10]

$$w(u) = \frac{\sqrt{1+u} - \sqrt{1 - \frac{u}{2}}}{\sqrt{1+u} + \sqrt{1 - \frac{u}{2}}}. \tag{1.71}$$

Let us now write $z = |z|\mathrm{e}^{i\psi}$, $\psi = \arg z$. The asymptotic behavior of the integrals (1.68), (1.69) is then of the form

$$W_n^+(z) \approx n^{\frac{1}{4}} \zeta^n \mathrm{e}^{-2^{\frac{3}{4}}(1+i)\sqrt{\frac{n}{z}}}, \quad \psi > -\frac{\pi}{6}, \tag{1.72}$$

$$W_n^-(z) \approx n^{\frac{1}{4}} (\zeta^*)^n \mathrm{e}^{-2^{\frac{3}{4}}(1-i)\sqrt{\frac{n}{z}}}, \quad \psi < \frac{\pi}{6}, \tag{1.73}$$

where

$$\zeta = \frac{\sqrt{2}+i}{\sqrt{2}-i}, \quad \zeta^* = \frac{\sqrt{2}-i}{\sqrt{2}+i} \tag{1.74}$$

belong to the images of infinities in the upper and the lower u-plane given by the conformal mapping (1.71). Using the equations (1.72), (1.73) one obtains the asymptotic behavior for $W_n^{PV}(z)$ according to (1.67) of the form

$$W_n^{PV}(z) \approx n^{\frac{1}{4}} \zeta^n \mathrm{e}^{-2^{\frac{3}{4}}(1+i)\sqrt{\frac{n}{z}}} + n^{\frac{1}{4}} (\zeta^*)^n \mathrm{e}^{-2^{\frac{3}{4}}(1-i)\sqrt{\frac{n}{z}}}, \tag{1.75}$$

which is valid only for ψ satisfying the condition

$$|\psi| < \frac{\pi}{6}. \tag{1.76}$$

The conditions needed for the convergence of the expansion (1.63) can be obtained from the ratio[11]

[10] We shall discuss this particular choice later in Section 2.3.

[11] Note that the d'Alembert convergence criterion is used (d'Alembert ratio test).

$$\left|\frac{c_n W_n(z)}{c_{n-1} W_{n-1}(z)}\right| \tag{1.77}$$

considered for large n. This yields the following bounds for the coefficients:

1. $$\forall \varepsilon > 0,\ |c_n| < C\mathrm{e}^{\varepsilon n^{\frac{1}{2}}}, \tag{1.78}$$

2. $$\exists c > 0,\ |c_n| \approx \mathrm{e}^{c n^{\frac{1}{2}}}, \tag{1.79}$$

and the domains of convergence of (1.63) corresponding to these bounds are:

1. $$\mathrm{Re}\left[(1 \pm i) z^{-\frac{1}{2}}\right] > 0, \tag{1.80}$$

2. $$\mathrm{Re}\left[2^{\frac{3}{4}}(1 \pm i) z^{-\frac{1}{2}} + c\right] > 0. \tag{1.81}$$

We have to bear in mind that the condition (1.80) is equivalent to

$$|\psi| \le \frac{\pi}{2} - \delta, \quad \forall \delta > 0, \tag{1.82}$$

while the condition (1.76) is more restrictive. Thus, the former statement (1.78) about the convergence of the optimal expansion of the Borel integral is valid only in the domain (1.76) and the latter one (1.79) is valid only in the intersection

$$\psi \in \left\{\mathrm{Re}\left[2^{\frac{3}{4}}(1 \pm i) z^{-\frac{1}{2}} + c\right] > 0\right\} \cap \left(-\frac{\pi}{6}, \frac{\pi}{6}\right). \tag{1.83}$$

Finally, let us turn to the asymptotic expansions of $W_n(z)$ for small z. We shall recall this results only very briefly [10]. The asymptotic expansion for $W_n(z)$ is of the form

$$W_n(z) \approx \sum_{k=n}^{\infty} \xi_k^{(n)} k! z^k, \quad z \to 0+, \tag{1.84}$$

where

$$\xi_k^{(n)} = \frac{1}{k!} \frac{\mathrm{d}^k w^n(u)}{\mathrm{d}u^k}\bigg|_{u=0} \tag{1.85}$$

is the Taylor coefficient in the expansion of the n-th power of the optimal conformal mapping $w(u)$ in the powers of u

$$w^n(u) = \sum_{k=n}^{\infty} \xi_k^{(n)} u^k. \tag{1.86}$$

The analytic properties of $W_n(z)$ can be found in [10].

1.5.1 Simple Examples

Before proceeding to the main part of this thesis - the application of the Caprini-Fischer method in QCD - let us show some very illustrative examples of some common series and conformal mappings.

1.5.1.1 The Optimal Expansion of the Borel Integral by means of the Conformal Mapping with Single Cut

Let us consider the power series of the form

$$\sum_{n=0}^{\infty} n! z^n. \tag{1.87}$$

The Borel transform of (1.87) then equals

$$B(u) = \sum_{n=0}^{\infty} u^n \tag{1.88}$$

and this series converges surely inside the unit disk $|u| < 1$, where the sum is equal to the function

$$B(u) = \frac{1}{1-u}, \tag{1.89}$$

which is holomorphic in the whole complex u-plane except the pole at $u = 1$. The Borel integral of the series (1.87) has the form

$$\frac{1}{z}\int\limits_0^\infty \mathrm{e}^{-\frac{u}{z}} B(u)\mathrm{d}u = \frac{1}{z}\int\limits_0^\infty \mathrm{e}^{-\frac{u}{z}} \sum_{n=0}^\infty u^n \mathrm{d}u. \tag{1.90}$$

Since the series (1.88) is divergent for $|u| \geq 1$, the Borel integral (1.90) is ill-defined and thus the Borel method according to Theorem 1.1.1 is of no use. Nevertheless the integral can be given a precise meaning. This requires the following:

1. the Borel transfom (1.88) has to be analytically continued outside its convergence disk,

2. the integration path in the Borel integral (1.90) has to be properly defined.

The analytic continuation of the Borel transform is possible by means of the analytic function $\frac{1}{1-u}$. The analyticity domain of this function is (1.12).

The singularity $u = 1$ of the analytically continued Borel transform (1.89) causes the ill definition of the Borel integral (1.90). One can therefore define the integral by means of the generalized principal value prescription as[12]

$$\frac{1}{z}\mathrm{PV}\int\limits_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u}\mathrm{d}u = \frac{1}{2z}\left[\int\limits_0^\infty \mathrm{e}^{-\frac{u+i\epsilon}{z}} \frac{1}{1-(u+i\epsilon)}\mathrm{d}u + \int\limits_0^\infty \mathrm{e}^{-\frac{u-i\epsilon}{z}} \frac{1}{1-(u-i\epsilon)}\mathrm{d}u\right]. \tag{1.91}$$

Let us define then the function $I(z)$ by the equation

$$I(z) = \frac{1}{z}\mathrm{PV}\int\limits_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u}\mathrm{d}u. \tag{1.92}$$

Therefore, we have defined the sum of the series (1.87) as

$$\sum_{n=0}^\infty n!z^n = \frac{1}{z}\mathrm{PV}\int\limits_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u}\mathrm{d}u. \tag{1.93}$$

Another possibility of defining the sum of the series (1.87) is using the optimal expansion of the Borel integral. As mentioned in the previous Section 1.5, one has

[12] At this point, it is good to mention that the generalized PV prescription represents one of the many possibilities of giving precise mathematical meaning to the integrals of this type.

to expand the Borel transform (1.88) in powers of its optimal conformal mapping to obtain the analytic continuation outside its unit disk of convergence. For this purpose it is possible to remove the whole line $\langle 1, +\infty)$. Then the domain

$$D_{\frac{1}{1-u}} = \mathbb{C} \setminus \langle 1, +\infty) \tag{1.94}$$

can be mapped conformally into the unit disk. The optimal conformal mapping of $D_{\frac{1}{1-u}}$ has according to (1.31) the form

$$w_{\frac{1}{1-u}}(u) = \frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}. \tag{1.95}$$

To expand the Borel transform (1.88) in powers of the function (1.95) means to search for the solution of the Example 1.3.1. Then, according to (1.51) we have

$$B(u) = 1 + \sum_{j=1}^{\infty} 4j \left[w_{\frac{1}{1-u}}(u) \right]^j . \tag{1.96}$$

Therefore, the optimal expansion of the Borel integral (1.90) is due to (1.63) equal to

$$\frac{1}{z} \int\limits_0^\infty \mathrm{e}^{-\frac{u}{z}} B(u) \mathrm{d}u = 1 + \sum_{n=1}^{\infty} 4n W_n(z), \tag{1.97}$$

where

$$W_n(z) = \frac{1}{z} \mathrm{PV} \int\limits_0^\infty \mathrm{e}^{-\frac{u}{z}} \left[w_{\frac{1}{1-u}}(u) \right]^n \mathrm{d}u. \tag{1.98}$$

The sum of the series (1.87) is then defined by

$$\sum_{n=0}^{\infty} n! z^n = 1 + \sum_{n=1}^{\infty} 4n W_n(z). \tag{1.99}$$

The series on the righthand side of (1.99) is convergent at least in the region [54]

$$\mathrm{Re}\left[z^{-\frac{1}{3}} \right] > 0 \Rightarrow |\mathrm{Arg} z| < \frac{3}{2}\pi, \tag{1.100}$$

since the coefficients $c_n = 4n$ surely satisfy the condition

$$|c_n| < M\mathrm{e}^{\varepsilon k^{\frac{2}{3}}}, \quad \forall \varepsilon > 0. \tag{1.101}$$

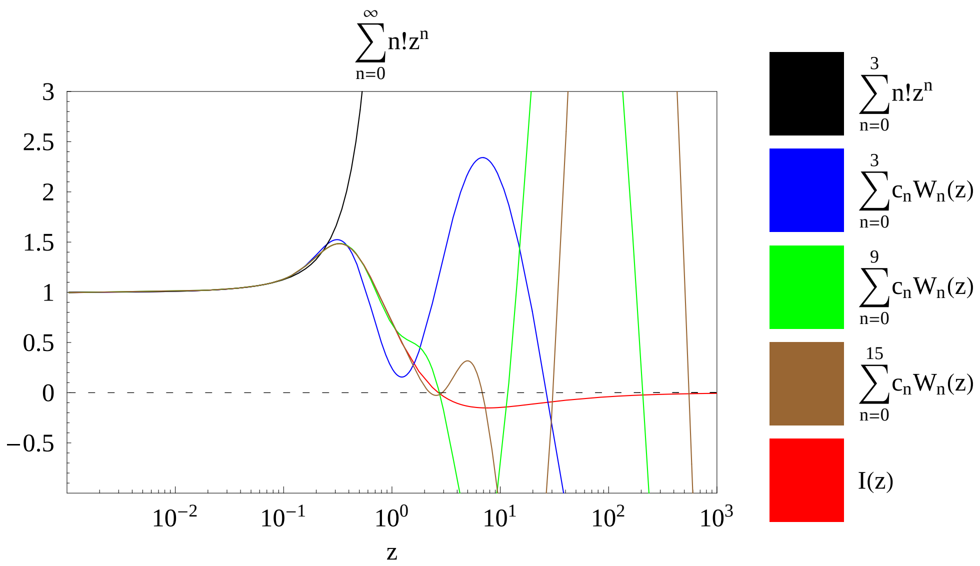


Figure 1.3: Summation of the power series $\sum_{n=0}^{\infty} n!z^n$.

In Figure 1.3 one can find a graphical comparison of two approaches of defining the sum of the series (1.87). Firstly by (1.93) and secondly by (1.99). The black curve represents standard approximant - partial sum truncated at the order $N = 3$, i.e. $\sum_{n=0}^{3} n!z^n$. The red curve represents the PV integral (1.93). Finally, the blue, green and brown curves represent the partial sums of (1.99) truncated at the order $N = 3, 9, 15$, i.e. $1 + \sum_{n=1}^{3} 4nW_n(z)$, $1 + \sum_{n=1}^{9} 4nW_n(z)$ and $1 + \sum_{n=1}^{15} 4nW_n(z)$, respectively.

As can be easily seen from the picture, the PV integral representation (1.93) of the sum (1.87) is, however, equal to the representation by the optimal expansion (1.99). Indeed, the higher the order N of the partial sum $1 + \sum_{n=1}^{N} 4nW_n(z)$, the better the approximation of the PV integral (1.93). It is good to recall at this point that it is not only the position of the singularities of the Borel transform, which is sufficient for defining the sum of a general power series, but also the character of these singularities is essential. The definition (1.93) is in fact the generalization of the approach (1.99), which also deals with information about the nature of the singularities present in the Borel plane. This can be done by factorization of these singularities of the Borel transform simply by rewriting the Borel transform (1.88). Since, in our case only one singularity is present, Borel transform (1.88) takes the following form[13]

$$B(u) = \frac{1}{1-u} \sum_{j=0}^{\infty} \tilde{b}_j u^j, \tag{1.102}$$

Then the sum (1.99) with respect to (1.97) has the form

$$1 + \sum_{n=1}^{\infty} 4nW_n(z) = \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} B(u) \mathrm{d}u = \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u} \sum_{n=0}^{\infty} \tilde{b}_n u^n \mathrm{d}u. \tag{1.103}$$

By expanding the righthand side in powers of optimal conformal mapping we obtain

$$1+\sum_{n=1}^{\infty} 4nW_n(z) = \tilde{b}_0 \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u} \mathrm{d}u + \sum_{j=1}^{\infty} c_j \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u} \left[w_{\frac{1}{1-u}}(u) \right]^j \mathrm{d}u, \tag{1.104}$$

where coefficients c_j are according to (1.50) of the form

$$c_j = \frac{1}{j!} \sum_{n=1}^{j} \tilde{b}_n (-1)^{j-n} 4^n \frac{j!}{(2n-1)!} \frac{(n+j-1)!}{(j-n)!}. \tag{1.105}$$

Since

$$\tilde{b}_0 = 1, \quad \tilde{b}_n = 0, \quad \forall n \in \mathbb{N}, \tag{1.106}$$

the coefficients c_j vanish, so

$$c_j = 0, \quad \forall j \in \mathbb{N}. \tag{1.107}$$

Therefore, the equation (1.104) transforms to

$$\sum_{n=0}^{\infty} n! z^n = 1 + \sum_{n=1}^{\infty} 4nW_n(z) = \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u} \mathrm{d}u. \tag{1.108}$$

[13]Every integral is considered to be defined using the PV prescription.

1.5.1.2 The Optimal Expansion of the Borel Integral by means of the Conformal Mapping with two Cuts

Let us now demonstrate the problem analogous to the one presented in the previous section. We shall consider the following power series

$$\sum_{n=0}^{\infty}(2n)!z^{2n}. \tag{1.109}$$

The Borel transform has the form

$$B(u)=\sum_{n=0}^{\infty}u^{2n}. \tag{1.110}$$

This can be easily verified when one rewrites the series (1.109) to take the form

$$\sum_{n=0}^{\infty}(2n)!z^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n+1}{2}n!z^n. \tag{1.111}$$

Then, it is very simple to check the validity of (1.110)

$$B(u)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(-1)^n+1}{2}n!u^n=\sum_{n=0}^{\infty}\frac{(-1)^n+1}{2}u^n=\sum_{n=0}^{\infty}u^{2n}. \tag{1.112}$$

The Borel transform (1.110) has two singularities: $-1, 1$. As in the case of the previous example, the convergence region is equal to the unit disk, where the Borel transform can be summed uniquely to

$$B(u)=\frac{1}{1-u^2}. \tag{1.113}$$

The analyticity region of the function (1.113) equals

$$\mathbb{C}\setminus\{-1,1\}. \tag{1.114}$$

Note, however, that neither (1.12), which appeared to be the case of the Borel transform with one sigularity (1.88), nor (1.114) can be conformally mapped onto a unit disk. We can avoid this difficulty by taking the function

$$w_{\frac{1}{1-u^2}}(u)=\frac{\sqrt{1+u}-\sqrt{1-u}}{\sqrt{1+u}+\sqrt{1-u}}, \tag{1.115}$$

which conformally maps the double-cut complex plane

$$D_{\frac{1}{1-u^2}} = \mathbb{C} \setminus \{(-\infty, -1\rangle \cup \langle 1, \infty)\} \tag{1.116}$$

onto the unit disk. By this, however, some information is lost, because (1.113) is holomorphic everywhere except $u = \pm 1$, while the mapping (1.115) is holomorphic everywhere except the rays $(-\infty, -1\rangle$ and $\langle 1, \infty)$[14].

According to the previous example, one can analytically continue the Borel transform (1.110) outside its region of convergence simply by the expression (1.113). Therefore the formal Borel integral suitable for the series (1.109) equals

$$\frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u^2} \mathrm{d}u. \tag{1.117}$$

Again, the integral has to be correctly defined since the singularity $u = 1$ lies on the integration path. We shall use once more the generalized PV prescription to do this, and define the function similar to $I(z)$ in (1.92) by

$$I(z) = \frac{1}{z}\mathrm{PV}\int_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u^2} \mathrm{d}u. \tag{1.118}$$

Finally, we have defined the sum of the series (1.118) as follows

$$\sum_{n=0}^{\infty} (2n)! z^{2n} = \frac{1}{z}\mathrm{PV}\int_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1-u^2} \mathrm{d}u. \tag{1.119}$$

Nevertheless, the analytic continuation of the Borel transform (1.118) can be obtained by means of the expansion in powers of the optimal conformal mapping (1.115). Moreover, as the essential information about the nature of the singularities is also known, it can be used properly. The maximum of the information was used in the definition (1.119) where the Borel transform (1.110) was reexpanded to take the form

$$B(u) = \frac{1}{1-u}\frac{1}{1+u}\sum_{n=0}^{\infty} \tilde{b}_n u^n, \tag{1.120}$$

[14]Similar remark has to be noted also with respect to the previous example. The information loss can be traced back to replacing the pole at $u = 1$ by the cut $\langle 1, \infty)$.

with singularities $u = \pm 1$ factorized out, where

$$\tilde{b}_0 = 1, \quad \tilde{b}_n = 0, \quad \forall n \in \mathbb{N} \tag{1.121}$$

similarly to (1.102). But another three possibilities of using the information about the singularities of the Borel transform arise[15]:

1. Nothing will be factorized out, only the position of the singularities will be used to construct the appropriate optimal conformal mapping. Then, the optimal expansion of the Borel integral

$$\frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}} \sum_{n=0}^{\infty} u^{2n} \mathrm{d}u \tag{1.122}$$

in powers of the mapping (1.115) will be performed,

2. The singularity $u = -1$ will be factorized out. The Borel transform will have the form

$$B(u) = \frac{1}{1+u} \sum_{n=0}^{\infty} u^n. \tag{1.123}$$

Then, the optimal expansion of the Borel integral

$$\frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1+u} \sum_{n=0}^{\infty} u^n \mathrm{d}u \tag{1.124}$$

in powers of the mapping (1.115) will be performed,

3. Finally, the singularity $u = 1$ will be factorized out. The Borel transform will have the form

$$B(u) = \frac{1}{1-u} \sum_{n=0}^{\infty} (-1)^n u^n. \tag{1.125}$$

Then, the optimal expansion of the Borel integral

[15] Every integral is considered to be defined by means of the generalized PV prescription.

$$\frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}}\frac{1}{1-u}\sum_{n=0}^\infty(-1)^n u^n \mathrm{d}u \tag{1.126}$$

in powers of the mapping (1.115) will be performed.

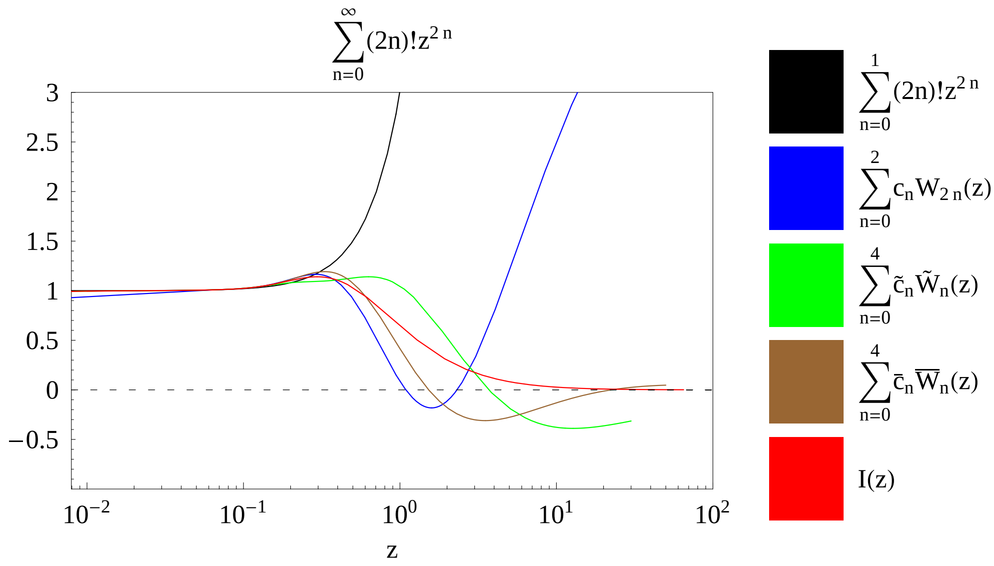


Figure 1.4: Summation of the power series $\sum_{n=0}^{\infty}(2n)!z^{2n}$, approximants to the order N=4

The appropriate expansion functions and optimal expansions have to be of the form[16]:

1.
$$\frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}}\sum_{n=0}^\infty u^{2n}\mathrm{d}u = 1+\sum_{n=1}^\infty 4nW_{2n}(z), \tag{1.127}$$

where

$$W_{2n}(z) = \frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}}\left[w_{\frac{1}{1-u^2}}(u)\right]^{2n}\mathrm{d}u, \tag{1.128}$$

2.
$$\frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}}\sum_{n=0}^\infty u^{2n}\mathrm{d}u = 1+\sum_{n=1}^\infty 2n\overline{W}_n(z), \tag{1.129}$$

[16]Every integral is considered to be defined by means of the generalized PV prescription.

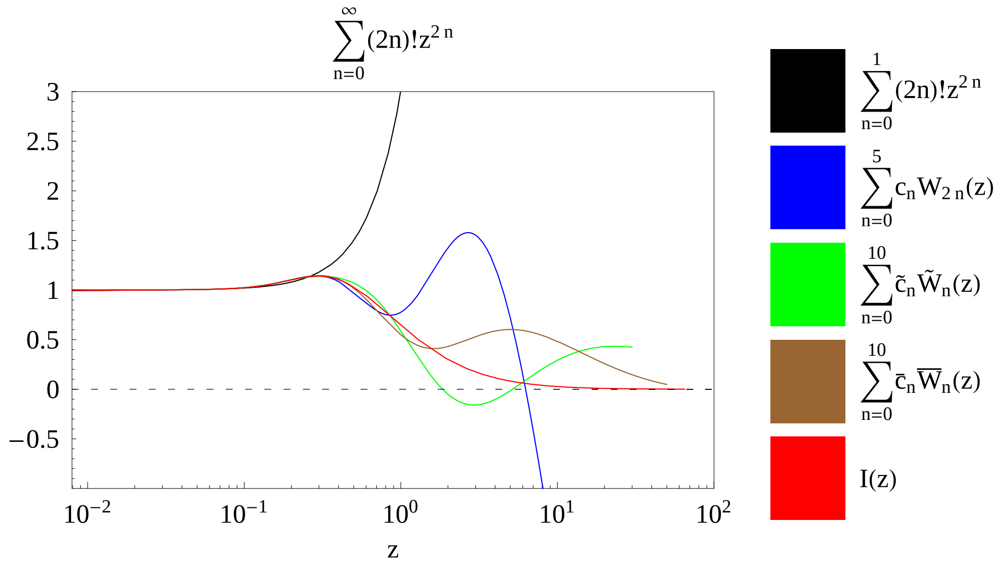


Figure 1.5: Summation of the power series $\sum_{n=0}^{\infty}(2n)!z^{2n}$, approximants to the order N=10

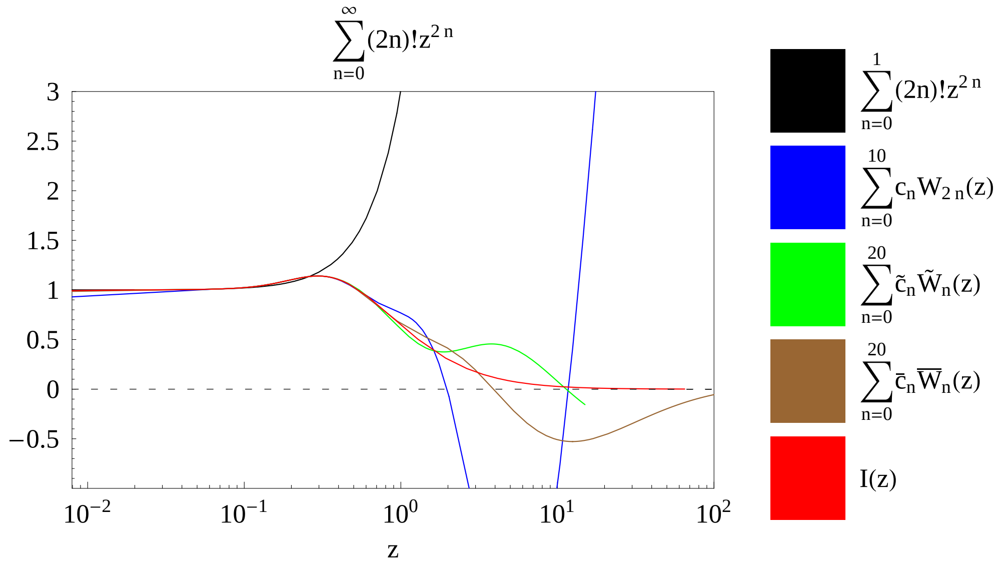


Figure 1.6: Summation of the power series $\sum_{n=0}^{\infty}(2n)!z^{2n}$, approximants to the order N=20

where

$$\overline{W}_n(z) = \frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}} \frac{1}{1+u}\left[w_{\frac{1}{1-u^2}}(u)\right]^n \mathrm{d}u, \tag{1.130}$$

3.
$$\frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}}\sum_{n=0}^\infty u^{2n}\mathrm{d}u = 1+\sum_{n=1}^\infty(-1)^n 2n\tilde{W}_n(z), \tag{1.131}$$

where

$$\tilde{W}_n(z) = \frac{1}{z}\int_0^\infty \mathrm{e}^{-\frac{u}{z}}\frac{1}{1-u}\left[w_{\frac{1}{1-u^2}}(u)\right]^n \mathrm{d}u. \tag{1.132}$$

Therefore, we have three definitions of the sum (1.109) different from (1.119):

$$\begin{aligned} &= 1+\sum_{n=1}^\infty 4nW_{2n}(z) && (1.133)\\ \sum_{n=0}^\infty (2n)!z^{2n} &= 1+\sum_{n=1}^\infty 2n\overline{W}_n(z) && (1.134)\\ &= 1+\sum_{n=1}^\infty(-1)^n 2n\tilde{W}_n(z). && (1.135)\end{aligned}$$

Figures 1.4, 1.5 and 1.6 compare the standard sum (1.109) truncated at the order $N=2$ (black line) with the sum (1.119) (red line) and with the approximants (1.133) (blue line), (1.134) (brown line) and (1.135) (green line) taken to the order $N = 5, 10, 20$. Similarly to the previous example, the approximants of the higher order provide better approximation of (1.119).

It is useful to point out that the information about the character of the singularities which lie in the old integration path, i.e. on the real semi-axis, is very important. The use of this information provides better numerical behaviour of the optimal expansion of the Borel integral. This can be seen again from the Figures 1.4, 1.5 and 1.6. The approximation, considered to a certain order N, of the red line by the green lines, which represents the approximants with the right-hand singularity factorized out, is much better than the approximation provided by the brown lines (only left singularity factorized out) or the blue lines (none of the singularities factorized out). However, the information about the position and the character of all singularities of the Borel transform in the whole Borel plane, not just the ones that lie on the real semi-axis, drastically reduces the order N of the certain approximant needed to approximate the Borel integral with a prescribed accuracy. Moreover, when the Borel transform is analytically continued outside the region of its convergence by means of the sum of its series inside the convergence disk, the optimal expansion of the certain Borel integral reduces only to the leading term (the red line approximants).

Chapter 2

Perturbative QCD

Quantum chromodynamics belongs to non-abelian gauge theories of quantum fields. It is the non-abelian character of gauge group SU(3) what makes the situation completely different from the case of abelian QED and leads to the following specific characteristics of QCD.

The first one is the so-called **colour confinement**. This phenomenon having no analogy in the classical physics is the consequence of the dynamical properties of the theory of strong interactions. These attributes yield the fact that only colourless combinations of quarks, antiquarks and gluons can be observed in Nature as hadrons with finite masses.

The second specific phenomenon of QCD is the so-called **asymptotic freedom**, which, roughly speaking, means that the renormalized couplant vanishes at short distances. This fundamental feature of QCD can be traced back to the selfinteraction of gluons yielding the behaviour of the renormalized couplant $a(\mu, \mathrm{RS})$ drastically different when compared to the case of the QED couplant. As a consequence, QCD is well-defined at short distances and the apparatus of perturbative theory can be used to compute cross sections of hard processes [1].

[1] These nontrivial aspects of QCD will be further discussed in following sections.

2.1 Basic Concepts in QCD

We shall comment briefly on the QCD Lagrangian to demonstrate the selfinteraction of gluons, the phenomenon yielding very specific characteristics of QCD mentioned above. The QCD Lagrangian is of the form

$$\mathcal{L}_{QCD} = -\frac{1}{4}\vec{G}_{\mu\nu}\vec{G}^{\mu\nu} + \bar{\mathbf{\Psi}}(i\not{\partial} - m_q)\mathbf{\Psi} + g\bar{\mathbf{\Psi}}\gamma_\mu\vec{T}\mathbf{\Psi}\vec{A}^\mu, \tag{2.1}$$

where the local field operator $\mathbf{\Psi}(x)$ describes the quark colour triplet of a particular flavour, i.e. $u,\ d,\ s,\ c,\ t,\ b$. Therefore, it can be represented by the colour 3-vector in the colour space

$$\mathbf{\Psi}(x) = \begin{pmatrix} \Psi^1(x) \\ \Psi^2(x) \\ \Psi^3(x) \end{pmatrix}. \tag{2.2}$$

Similarly, $\vec{A}_\mu$ describes an octet of coloured gluons and can be represented by the 8-vector[2]

$$\vec{A}_\mu(x) = \begin{pmatrix} A^1_\mu(x) \\ A^2_\mu(x) \\ \vdots \\ A^8_\mu(x) \end{pmatrix}. \tag{2.3}$$

Elements of the 8-vector $\vec{G}_{\mu\nu}$

$$\vec{G}_{\mu\nu}(x) = \begin{pmatrix} G^1_{\mu\nu}(x) \\ G^2_{\mu\nu}(x) \\ \vdots \\ G^8_{\mu\nu}(x) \end{pmatrix}. \tag{2.4}$$

[2] Also, there exists alternative 3×3 matrix representation of the gluon octet which is widely used and is defined using the generators of SU(3) as follows:

$$\mathbb{A}_\mu(x) = \sum_{a=1}^{8} A^a_\mu(x) T_a,$$

where $T_a,\ a = 1, 2, \ldots, 8$ are generators of SU(3).

represent tensors of field strength for each color field $A_\mu^a,\ a = 1, 2, \ldots, 8$. Every $G_{\mu\nu}^a,\ a = 1, 2, \ldots, 8$ is defined as follows

$$G_a^{\mu\nu} = F_a^{\mu\nu} + g f_a^{bc} A_b^\mu A_c^\nu = \frac{\partial A_a^\nu(x)}{\partial x_\mu} - \frac{\partial A_a^\mu(x)}{\partial x_\nu} + g f_a^{bc} A_b^\mu A_c^\nu. \tag{2.5}$$

The existence of the additional term $g f_a^{bc} A_b^\mu A_c^\nu$ on the right-hand side of (2.5) is a consequence of the non-abelian character of the gauge group SU(3).

Writing out $\vec{G}_{\mu\nu}\vec{G}^{\mu\nu}$ explicitly in terms of A_μ^a yields

$$\begin{aligned}\vec{G}_{\mu\nu}\vec{G}^{\mu\nu} &= \vec{F}_{\mu\nu}\vec{F}^{\mu\nu} \\ &+ g\left\{ f_a^{bc}\left(\frac{\partial A_\nu^a}{\partial x^\mu} - \frac{\partial A_\mu^a}{\partial x^\nu}\right) A_b^\mu A_c^\nu + f_{bc}^a \left(\frac{\partial A_a^\mu}{\partial x_\nu} - \frac{\partial A_a^\nu}{\partial x_\mu}\right) A_\mu^b A_\nu^c \right\} \\ &+ g^2 f_a^{bc} f_{de}^a A_b^\mu A_c^\nu A_\mu^d A_\nu^e. \end{aligned} \tag{2.6}$$

It can be seen easily from (2.6) that the QCD Lagrangian (2.1) differs from the QED one in the presence of two additional terms

$$g\left\{ f_a^{bc}\left(\frac{\partial A_\nu^a}{\partial x^\mu} - \frac{\partial A_\mu^a}{\partial x^\nu}\right) A_b^\mu A_c^\nu + f_{bc}^a \left(\frac{\partial A_a^\mu}{\partial x_\nu} - \frac{\partial A_a^\nu}{\partial x_\mu}\right) A_\mu^b A_\nu^c \right\} + g^2 f_a^{bc} f_{de}^a A_b^\mu A_c^\nu A_\mu^d A_\nu^e \tag{2.7}$$

describing the selfinteraction of three and four gluons. As we have already pointed out in the beginning of Chapter 2, the selfinteraction of gluons leads to the phenomenological consequences as the colour confinement and the asymptotic freedom.

2.2 Renormalization in QCD

In the following, let us assume all quarks to be massless[3]. Then, the Lagrangian (2.1) has only one free parameter g, the so-called **colour charge**. Applying the standard quantization procedure on (2.1), one obtains perturbative QCD, where any observable is represented by a series in powers g^2.

Straightforward calculations using the technique of Feynman diagrams in QCD however, similarly to QED, lead to the problem of divergencies of loop momentum

[3]We shall assume all quarks to be massless throughout this work.

Feynman integrals[4]. A method, invented by Feynman, Schwinger and Tomonaga, how to get rid of these divergencies is called **renormalization**. The essence of renormalization is **redefinition** of the original quantities, usually called as **bare** ones in the Lagrangian (2.1), i.e. colour charge, fermion masses and field operators $\mathbf{\Psi}(x)$, $\vec{A}_\mu(x)$. After applying this procedure, renormalized theory then works with the new, **renormalized** quantities instead of the bare ones. Technical realization of the renormalization procedure can be introduced in two steps, at first applying the so-called **regularization** procedure, which isolates ultraviolet divergences, followed by the introduction of **counterterms**, which cancel the original ultraviolet divergences. Many regularization methods have been invented recently, but presently the most widely used one is the so-called **dimensional regularization**.

Renormalization procedure removes physically meaningless infinite terms. However, because the cancellation of ultraviolet infinities of loop integrals does not determine the counterterms uniquely, we are left with a considerable freedom how to define renormalized quantities, which goes under the name **renormalization group** [5]. The requirement of self-consistency of the perturbation theory implies the invariance of the sum of perturbative series for physical observables with respect to this RG. Nevertheless, this condition guarantees an observable to be independent of a particular choice of renormalized quantities, only if the sum of the corresponding perturbative series is considered to all orders. Since in realistic calculations at most only first three terms of perturbative series are known, a selection of properly defined renormalized quantities is important and integral part of every perturbation theory application.

2.2.1 Renormalization Scale and Scheme Dependence of Perturbative Series

As we have already mentioned in QCD with massless quarks there is only one free parameter in the Lagrangian (2.1). Therefore, every observable depends only on renormalized colour charge g. Further we define the so-called **strong coupling**

[4]For instance, QED corrections to the photon and fermion propagator considered at one loop.

[5]We shall denote renormalization group as RG.

parameter as

$$\alpha_s = \frac{g^2}{4\pi}. \tag{2.8}$$

It it useful to notice that α_s is a function depending on a number of free parameters introduced in the process of renormalization. One of them, having the dimension of mass, is called **renormalization scale** and denoted as μ. The rest of them define the so-called **renormalization scheme** which will be denoted as RS. For brevity, the so-called **renormalized couplant**, or simply **couplant**, is defined having the following form[6]

$$a = \frac{\alpha_s}{\pi}. \tag{2.9}$$

The renormalization scale and scheme dependence of couplant a is defined by the **RG equation**[7]

$$\frac{\partial a(\mu, \mathrm{RS})}{\partial \ln \mu} \equiv \beta(a) = -ba^2(\mu, \mathrm{RS})\left[1 + ca(\mu, \mathrm{RS}) + c_2 a^2(\mu, \mathrm{RS}) + \cdots\right]. \tag{2.10}$$

The coefficients b and c are uniquely defined by the number of quark flavours n_f and colours N_c as

$$b = \frac{11N_c - 2n_f}{6}, \quad c = \frac{51N_c - 19n_f}{22N_c - 4n_f}, \tag{2.11}$$

while $c_i,\ i \geq 2$ are arbitrary finite numbers. It is very useful to make the parametrization of RS just by the set of free parameters $\{c_i\}$ [11]. Moreover, it is possible to derive from (2.10) the dependence of the couplant $a\left(\mu, \{c_i\}\right)$ on parameters $c_i,\ i \geq 2$ in the form

$$\frac{\partial a\left(\mu, \{c_i\}\right)}{\partial c_j} \equiv \beta_j = -\beta(a) \int\limits_0^a \frac{bx^{j+2}}{\beta(x)^2}\, \mathrm{d}x. \tag{2.12}$$

[6]The purpose of this definition is to eliminate the very frequent appearance of powers of number π.

[7]The right-hand side of the RG equation (2.10), i.e. $\beta(a)$, is called the $\boldsymbol{\beta}$**-function** of QCD.

The RG equation (2.10) has infinitely many solutions which differ by a certain boundary condition. We shall adopt the commonly used boundary condition

$$a(\mu = \Lambda) = \infty, \tag{2.13}$$

where Λ is a new free parameter distinguishing particular solutions of (2.10) for a specified $\beta(a)$ and having, similarly to μ, the dimension of mass[8]. Since the couplant a depends actually only on the ratio $\frac{\mu}{\Lambda}$, it is possible without the loss of generality to **fix** Λ and **vary** μ **only**.

We have pointed out so far that in QCD, contrary to quantum mechanics, even the expansion parameter, i.e. the couplant, is not well-defined. Precisely speaking, ambiguities in the definition of couplant occur as can be easily from the form of the $\beta(a)$ function on the right-hand side of (2.10). According to (2.10), $\beta(a)$ is defined by a formal power series as

$$\beta(a) = -ba^2\left(1 + ca + \sum_{i=2}^{\infty} c_i a^i\right). \tag{2.14}$$

Thus, it is very important to give the β-function unambiguous meaning making the right-hand side of (2.10) well-defined. This problem can be treated very easily by a certain choice of RS. Since any RS is fully defined by specifying the set $\{c_i\}$, it is easy to make the sum $\sum_{i=2}^{\infty} c_i a^i$ unambiguous, e.g. convergent in some area of the a-plane or simply equal to zero, making the couplant a well-defined. Moreover, it is possible to choose such scale μ and RS, that a perturbative series for a physical observable[9] is reduced to the leading term only [42]. Then, contrary to the previous case, the problems pointed out in Preface concern the Equation (2.10).

In general, respecting only the boundary condition (2.13), the solution of the RG equation (2.10) at LO reads[10]

[8]Note that the definition (2.13) cannot always be used. For instance, choosing $c_2 < 0$ and $c_i = 0,\ i \geq 3$ yields the limit $\lim_{\mu\to\Lambda} a(\mu, c_2)$ to be finite! Please, see Figure 2.1.

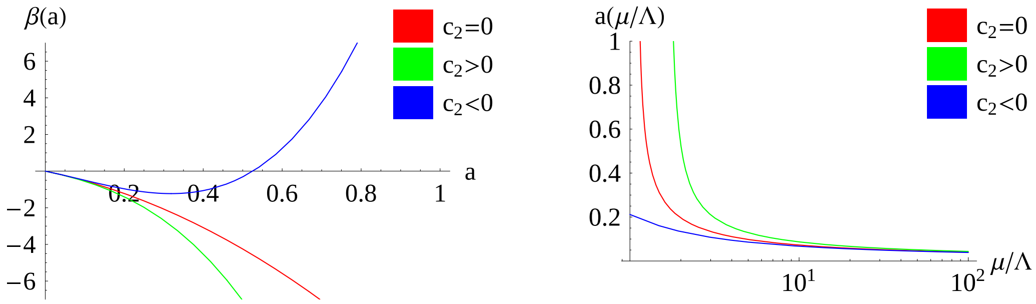
[9]A perturbative series for a physical observable may have the form (2.27), which we shall discuss more in detail.

[10]LO denotes the so-called leading term, i.e. the term having the smallest power of a on the right-hand side of (2.10).

$$a\left(\frac{\mu}{\Lambda}\right) = \frac{1}{b\ln\left(\frac{\mu}{\Lambda}\right)}. \tag{2.15}$$

Since in realistic QCD $N_c = 3$ and $n_f \leq 6$, the coefficient b is **positive**. Thus

$$a\left(\frac{\mu}{\Lambda}\right) \to 0, \quad \mu \to \infty. \tag{2.16}$$



(a) The behaviour of QCD β-function for three different choices of c_2.

(b) The behaviour of corresponding couplants a as a functions of the scale μ.

Figure 2.1: The behaviour of QCD β-function and corresponding couplants.

The expression (2.16) manifests the asymptotic freedom. Contrary to QED, this means that Λ is not the upper bound for the physically meaningful values of μ, but the **lower** one. Figure 2.1 sketches the behaviour of QCD β-function and corresponding couplants. It is important to notice here, that ambiguity in specifying the set $\{c_i,\ i \geq 2\}$ plays enormous role. In Figure 2.1, only first arbitrary parameter c_2 is taken into account and for three different choices of c_2, three different behaviours of couplant $a(\mu)$ at small μ occur. Since for $c_2 = 0$ and $c_2 > 0$, the couplant $a(\mu)$ blows up to infinity for $\mu \to \Lambda$, choosing $c_2 < 0$ leads to the so-called **couplant freezing**[11].

Considering (2.10) at NLO, the resulting equation is of the form[12]

$$\frac{\partial a(\mu)}{\partial \ln \mu} \equiv \beta(a) = -ba^2(\mu)\left[1 + ca(\mu)\right] \tag{2.17}$$

[11] Interesting discussion concerning the physical relevance of couplant freezing can be found in [4, 50].

[12] NLO denotes the so-called next-to-leading term.

The couplant a can be expressed analytically in closed-form in terms of the Lambert W function, which is defined implicitly as follows

$$W(z)\exp(W(z)) = z. \tag{2.18}$$

Then, using (2.18) yields

$$a(\mu) = -\frac{1}{c\left[1 + W_{-1}\left(-\frac{1}{\mathrm{e}}\left(\frac{\mu}{\Lambda}\right)^{-\frac{b}{c}}\right)\right]}, \tag{2.19}$$

where the subscript -1 on W denotes the branch of the Lambert W function required for asymptotic freedom. However, the couplant considered to NLO can be defined also implicitly as follows

$$b\ln\left(\frac{\mu}{\Lambda}\right) = \frac{1}{a} + c\ln\frac{ca}{1+ca}. \tag{2.20}$$

In general, considering (2.10) at an arbitrary order n, the implicit equation defining couplant a reads

$$b\ln\left(\frac{\mu}{\Lambda}\right) = \frac{1}{a} + c\ln\frac{ca}{1+ca} + \int\limits_0^a \mathrm{d}x\left[-\frac{1}{x^2 B_n(x)} + \frac{1}{x^2(1+cx)}\right], \tag{2.21}$$

where

$$B_n(x) = 1 + cx + c_2x^2 + \cdots + c_{n-1}x^{n-1}. \tag{2.22}$$

It is trivial to check, that for $n = 2$, the equation (2.21) takes the form (2.20). Moreover, setting $n = 3$, i.e. considering (2.10) at NNLO, one obtains[13]

$$b\ln\left(\frac{\mu}{\Lambda}\right) = \frac{1}{a} + c\ln\frac{ca}{1+ca} + \int\limits_0^a \mathrm{d}x\left[-\frac{1}{x^2(1+cx+c_2x^2)} + \frac{1}{x^2(1+cx)}\right]. \tag{2.23}$$

Performing the integration in (2.23) yields the implicit equation for couplant a considered at NNLO as

[13]NNLO denotes the so-called next-to-next-to-leading term.

$$b\ln\left(\frac{\mu}{\Lambda}\right)=\frac{1}{a}+c\ln\frac{ca}{\sqrt{1+ca+c_2a^2}}+f\left(a,c_2\right), \tag{2.24}$$

where

$$\begin{aligned}
f\left(a,c_2\right)&=\frac{2c_2-c^2}{d}\left[\arctan\frac{2c_2a+c}{d}-\arctan\frac{c}{d}\right],\\
d&=\sqrt{4c_2-c^2},\quad 4c_2>c^2,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
f\left(a,c_2\right)&=\frac{2c_2-c^2}{d}\left[\ln\left|\frac{2c_2a+c-d}{2c_2a+c+d}\right|-\ln\left|\frac{c-d}{c+d}\right|\right],\\
d&=\sqrt{c^2-4c_2},\quad 4c_2<c^2.
\end{aligned} \tag{2.26}$$

Further on, let us consider a perturbative series for an observable not specified in detail of the form

$$\mathcal{R}(Q)=a(\mu,c_i)\left[r_0+r_1(Q,\mu,c_i)a(\mu,c_i)+r_2(Q,\mu,c_i)a^2(\mu,c_i)+\cdots\right]. \tag{2.27}$$

The requirement that (2.27) has to be invariant with respect to RG means

$$\frac{\partial\mathcal{R}(Q)}{\partial\ln\mu}=0,\quad \frac{\partial\mathcal{R}(Q)}{\partial c_i}=0,\quad i\geq 2, \tag{2.28}$$

which applied to the finite sum

$$\begin{aligned}
\mathcal{R}_N&=a(\mu,c_i)\big[r_0+r_1(Q,\mu,c_i)a(\mu,c_i)+\\
&\quad+r_2(Q,\mu,c_i)a^2(\mu,c_i)+\cdots+r_{N-1}a^{N-1}(\mu,c_i)\big]=\\
&=\sum_{k=0}^{N-1}r_ka^{k+1}(\mu,c_i)
\end{aligned} \tag{2.29}$$

implies

$$\frac{\partial\mathcal{R}_N}{\partial\ln\mu}=\mathcal{O}\left(a^{N+1}\right),\quad \frac{\partial\mathcal{R}_N}{\partial c_i}=\mathcal{O}\left(a^{N+1}\right). \tag{2.30}$$

Thus, the N-th order partial sum of the series (2.27) is of the form

$$
\begin{aligned}
\mathcal{R}_N &= \sum_{k=0}^{N-1} r_k a^{k+1}(\mu, c_i) \\
&= \mathcal{F}(\mu, c_i, i \leq N-1; \rho, \rho_j, 2 \leq j < N-1),
\end{aligned} \tag{2.31}
$$

where the quantities ρ, $\rho_j,\ j \geq 2$ are the so-called RG **invariants** [11]. This means, that they are independent of μ and $c_i,\ i \geq 2$. Solving equations (2.30) yields the following expressions for coefficients $r_1(Q, \mu, c_i)$, $r_2(Q, \mu, c_i)$

$$
r_1(Q, \mu) = b \ln \frac{\mu}{\Lambda} - \rho\left(\frac{Q}{\Lambda}\right), \tag{2.32}
$$

$$
r_2(Q, \mu, c_2) = \rho_2 - c_2 + \left(r_1 + \frac{c}{2}\right)^2. \tag{2.33}
$$

The invariant ρ is defined in $\overline{\text{MS}}$ RS[14] as follows

$$
\rho\left(\frac{Q}{\Lambda_{\overline{\text{MS}}}}\right) = b \ln \frac{Q}{\Lambda_{\overline{\text{MS}}}} - r_1\left(\mu = Q, \overline{\text{MS}}\right). \tag{2.34}
$$

The invariant ρ_2, contrary to ρ, is only a pure number. When considering higher coefficients $r_k,\ k \geq 3$, higher invariants $\rho_k,\ k \geq 3$ have to be taken into account. These invariants, similarly to ρ_2, are also pure numbers.

Using (2.21) we can express (2.32), (2.33) and their higher order modifications it is possible to express only in terms of renormalized couplant a, RG invariants ρ_j and coefficients c_i. Explicitly, we find

$$
\mathcal{R}_{\text{NLO}}(a, \rho) = (1 + r_0)a + \left[c \ln \frac{ca}{1 + ca} - \rho\right] a^2 \tag{2.35}
$$

and

$$
\begin{aligned}
\mathcal{R}_{\text{NNLO}}(a, \rho, \rho_2, c_2) = (2 + r_0)a &+ \left[3\left(c \ln \frac{ca}{\sqrt{1 + ca + c_2 a^2}} + f(a, c_2) - \rho\right) + 2c\right] a^2 \\
&+ \left[\left(c \ln \frac{ca}{\sqrt{1 + ca + c_2 a^2}} + f(a, c_2) - \rho + \frac{c}{2}\right)^2 + \rho_2 - c_2\right] a^3.
\end{aligned} \tag{2.36}
$$

[14] This scheme will be described in the Section 2.2.2.1.

Equations (2.35), (2.36) imply that in QCD perturbative expansions in powers of renormalized couplant are in fact not power series, since the coefficients r_k of perturbative expansions are very complicated functions of a.

2.2.2 Commonly Used Choices of Renormalization Schemes

As we have pointed out, the meaningful phenomenological application of QCD requires renormalization of the theory. We have also mentioned that every renormalization process of a certain renormalizable field theory is preceded by regularization, which isolates the ultraviolet loop integral divergencies.

The commonly used regularization method in QCD preserving the gauge invariance, as pointed out in Section 2.2, is dimensional regularization. Moreover, this approach is generally the technically simplest one. The basic feature of dimensional regularization is that after regularizing our theory using dimensional regularization, powers of the term $\ln 4\pi - \gamma_E$ appear in the results. γ_E is the so-called Euler constant, which is an artefact of the power expansion of the Euler function Γ.

2.2.2.1 Renormalization Schemes MS and $\overline{\mathrm{MS}}$

After regularizing our theory, it is possible to proceed to the renormalization. As we already know, this can be made using the technique of counterterms. These are the additional terms introduced into the Lagrangian, which compensate divergencies of loop integrals. It is possible to introduce such terms in the following ways:

1. Counterterms introduced into the QCD Lagrangian compensate only divergent terms of perturbative expansion. All finite terms remain unchanged.

2. Counterterms introduced into the QCD Lagrangian compensate not only divergent terms of perturbative expansion, but also some terms which are finite.

The former RS is called the **Minimal Subtraction** scheme, denoted as MS. In the case the counterterms are chosen to cancel also the above mentioned terms $\ln 4\pi - \gamma_{\mathrm{E}}$, the latter is called the **Modified Minimal Subtraction** scheme, which is commonly denoted as $\overline{\mathrm{MS}}$.

2.2.2.2 Renormalization Scheme PMS - Principle of Minimal Sensitivity

Another choice of the renormalization scale and scheme is based on the **Principle of Minimal Sensitivity** and thus, it is denoted as PMS. This method was introduced by P. M. Stevenson [11]. The basic idea of PMS is to select the renormalization scale and scheme by imposing the conditions

$$\frac{\partial \mathcal{R}_N(Q,\mu,c_i)}{\partial \ln \mu} = 0, \quad \frac{\partial \mathcal{R}_N(Q,\mu,c_i)}{\partial c_i} = 0. \tag{2.37}$$

Therefore, we search for point having coordinates $[\mu, c_i]$ where the partial sum $\mathcal{R}_N$ is having, locally, the smallest variations with respect to changes of $\{\mu, \mathrm{RS}\} = \{\mu, c_i\}$. However, there is a possibility of existing of many such points fulfilling this requirement.

2.3 Adler Function of QCD

Let us now turn to the so-called **Adler function** of QCD[15] and recall its basic features, which will be used in Chapter 3.

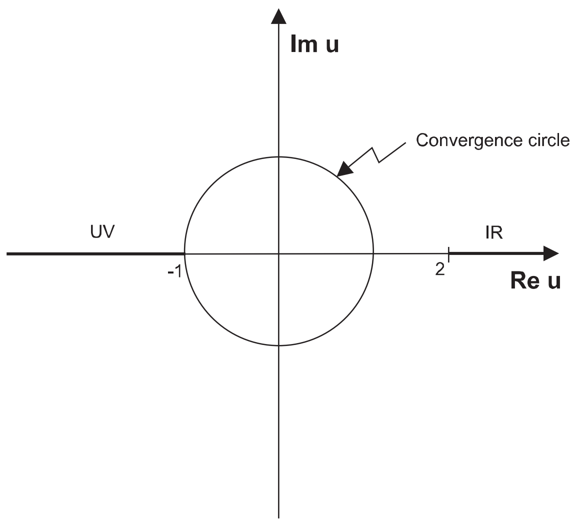


Figure 2.2: The Borel plane for the Adler function [10].

[15]For definition see [6].

Perturbative QCD yields the formal expansion of the Adler function in powers of renormalized strong coupling parameter $\alpha_s(\mu, \mathrm{RS})$ in the form

$$D(s) = 1 + \sum_{n=1}^{\infty} D_n\left(s, \mu, \mathrm{RS}\right) \left(\frac{\alpha_s\left(\mu, \mathrm{RS}\right)}{\pi}\right)^n, \tag{2.38}$$

where s is a kinematical variable equal to the external momentum squared. The coefficients $D_n(s, \mu, \mathrm{RS})$ are assumed to have the following specific large order behaviour

$$D_n\left(s, \mu, \mathrm{RS}\right) \sim \sum_k C_k\left(s, \mu, \mathrm{RS}\right) n! n^{\delta_k(s,\mu,\mathrm{RS})} \left(\frac{\pi\beta_0}{k}\right)^n, \tag{2.39}$$

where the sum runs over $k \in \mathbb{Z} \setminus \{0, 1\}$ [6]. The expression (2.38) can be rewritten using (2.9) as follows

$$D(s) = 1 + a \sum_{n=0}^{\infty} D_{n+1} a^n. \tag{2.40}$$

Then, the Borel transform of the function

$$D(s) - 1 = a \sum_{n=0}^{\infty} D_{n+1} a^n \tag{2.41}$$

is of the form

$$B[D](u) = \sum_{n=0}^{\infty} b_n u^n \tag{2.42}$$

and the coefficients b_n satisfy

$$b_n = \frac{D_{n+1}}{n!}. \tag{2.43}$$

Finally, the Adler function (2.40) can be formally obtained in the following integral representation

$$D(s) = 1 + \int_0^{\infty} \mathrm{e}^{-\frac{u}{a}} B[D](u) \mathrm{d}u. \tag{2.44}$$

The large order behaviour of the coefficients D_n (2.39) governs the existence of branch point singularities for the function $B[D](u)$ (2.42) in the Borel u-plane [6]. The corresponding dominant behaviour of $B[D](u)$ is of the form

$$B[D](u) \sim \sum_k C_k \Gamma(\delta_k + 1) \left(1 - \frac{u}{k}\right)^{-\delta_k - 1}. \tag{2.45}$$

Equation (2.45) implies that $B[D](u)$ becomes singular at the branch points located along the negative axis, the so-called **ultraviolet renormalons**, and along the positive axis, the so-called **infrared renormalons**. We mean the rays $u \leq -1$ and $u \geq 2$, respectively[16]. Precisely speaking, the renormalons are located at points $u = k,\ k \in \mathbb{Z} \setminus \{0, 1\}$.

The Borel transform (2.42) is renormalization scale and scheme dependent. However, it is generally assumed that position of its singularities in the Borel u-plane is renormalization scale and scheme independent. Moreover, as discussed in [6], it is independent of the external momenta s. The nature of the first two branch points, the UV renormalon $u = -1$ and the IR renormalon $u = 2$ was revealed in [39, 40]. It was shown that near the first UV renormalon $u = -1$ the function $B[D](u)$ behaves as [39]

$$B[D](u) \sim \frac{r_1}{(1+u)^{\gamma_1}}, \tag{2.46}$$

where r_1 is unknown and

$$\gamma_1 = 3 - 2\pi \frac{c}{b} + \lambda_1, \tag{2.47}$$

where λ_1 as well as b and c depend on the number of flavours considered. Similarly, near the first IR renormalon $u = 2$ $B[D](u)$ behaves as [40]

$$B[D](u) \sim \frac{r_2}{(2-u)^{\gamma_2}}, \tag{2.48}$$

where

$$\gamma_2 = 3 - 2\pi \frac{c}{b}. \tag{2.49}$$

Analogous to r_1, r_2 is also unknown.

[16]Please see Figure 2.2 [10].

2.3.1 Optimal Expansion of the Adler Function

Following the steps proposed in [8–10] and in Section 1.5 we shall expand the integral (2.44). The optimal expansion of the Adler function $B[D](u)$ (2.42) in powers of the optimal conformal mapping (1.71) provides an analytic continuation of the Borel transform outside its convergence disk [17]. Let us recall the definition of $w(u)$, which is of the form

$$w(u) = \frac{\sqrt{1+u} - \sqrt{1-\frac{u}{2}}}{\sqrt{1+u} + \sqrt{1-\frac{u}{2}}}$$

and its inverse

$$u = \frac{8w}{3w^2 - 2w + 3}. \tag{2.50}$$

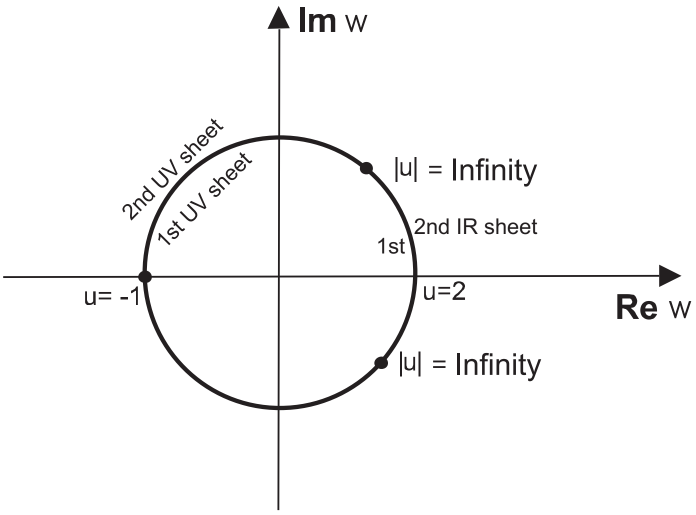


Figure 2.3: The complex w-plane for the Adler function [10].

The Borel u-plane with removed cuts $u \leq -1$ and $u \geq 2$ represents the domain of holomorphy $\mathcal{D}$ of (2.42)

$$\mathcal{D} = \mathbb{C} \setminus \{(-\infty, -1\rangle \cup \langle 2, +\infty)\}\,. \tag{2.51}$$

[17]Note that the radius of the convergence disk is such that the disk reaches the nearest singularity, i.e. $u = -1$. Thus, the radius equals one and $B[D](u)$ is convergent on the unit disk.

Since the Caprini-Fischer method requires the whole holomorphy domain (2.51) of (2.42) to be conformally mapped by the function (1.71) onto the unit disk, (1.71) possesses the same cuts as the Borel transform $B[D](u)$ (2.42). This yields the specific definition (1.71) of the optimal conformal mapping $w(u)$[18]. According to Section 1.5 and [8–10], the optimal expansion of the Borel integral (2.44) is done using (1.63) in the form

$$D(s) = 1 + \sum_{n=1}^{\infty} \widetilde{D}_n W_n(a), \tag{2.52}$$

where the expansion functions $W_n(a)$, $n \in \mathbb{N}_0$ are defined using (1.64), (1.66) and (1.67) as follows [38]

$$W_n(a) = \frac{1}{n!}\left(\frac{8}{3}\right)^n \frac{1}{a}\mathrm{PV}\int_0^{\infty} \mathrm{e}^{-\frac{u}{a}} w^n(u)\mathrm{d}u, \quad n \in \mathbb{N}_0. \tag{2.53}$$

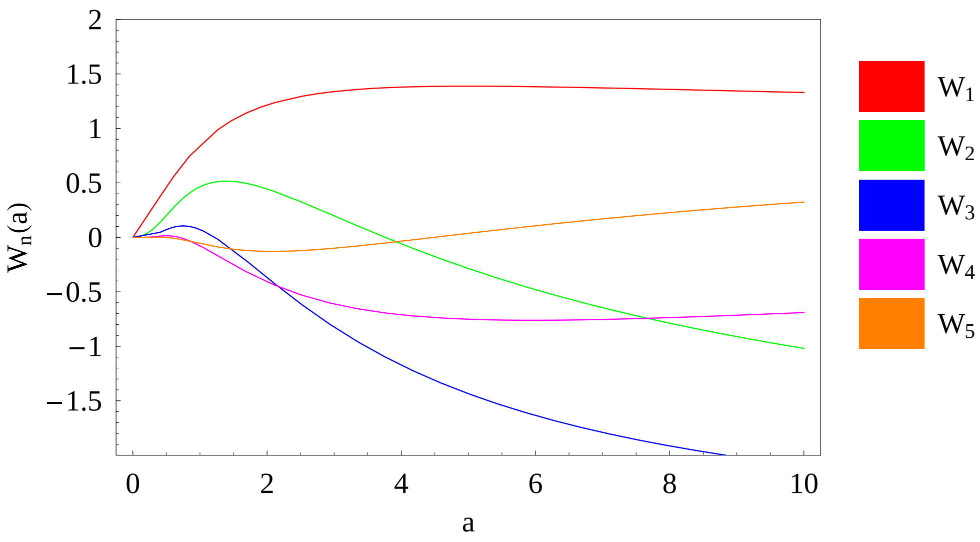


Figure 2.4: The shape of the expansion functions $W_n(a)$, $n = 1, 2, \ldots, 5$.

The shape of the functions $W_n(a)$ for $n = 1, 2, \ldots, 5$ is sketched in Figure 2.4. As was already pointed out, the functions (2.53) are singular at the point $a = 0$ and possess asymptotic expansions [10]

[18]Note that the mapping (1.71) is unique up to the rotations about the origin by an angle φ.

$$W_n(a) \approx \sum_{k=n}^{\infty} \zeta_{nk} a^k. \tag{2.54}$$

As can be easily seen, Equation (2.53) implies that $\zeta_{nn} = 1, \ n \in \mathbb{N}$. Therefore, the asymptotic expansions for the functions $W_n(a), \ n = 1, 2, 3$ read

$$W_1(a) \approx a - 0.5a^2 + 1.21875a^3 + \cdots , \tag{2.55}$$

$$W_2(a) \approx a^2 - 1.5a^3 + \cdots , \tag{2.56}$$

$$W_3(a) \approx a^3 + \cdots . \tag{2.57}$$

The expansion coefficients $\widetilde{D}_n$ in (2.52) can be easily computed from the original coefficients D_n in (2.38). For instance

$$\widetilde{D}_1 = D_1, \tag{2.58}$$

$$\widetilde{D}_2 = D_2 - D_1\zeta_{12}, \tag{2.59}$$

$$\begin{aligned}\widetilde{D}_3 &= D_3 - \widetilde{D}_2\zeta_{23} - D_1\zeta_{13} \\ &= D_3 - D_2\zeta_{23} + D_1\left(\zeta_{12}\zeta_{23} - \zeta_{13}\right).\end{aligned} \tag{2.60}$$

Up till now, only $D_n, \ n = 1, 2, 3$ were computed [44–49]. In the $\overline{\text{MS}}$ scheme with $n_f = 3$ and setting $\mu = \sqrt{s}$ their numerical values are

$$\begin{aligned}D_1 &= 1, \\ D_2 &= 1.63982, \\ D_3 &= 6.37101.\end{aligned} \tag{2.61}$$

Using Equations (2.55 - 2.60), (2.61) implies the following numerical values for $\widetilde{D}_n, \ n = 1, 2, 3$

$$\begin{aligned}\widetilde{D}_1 &= 1, \\ \widetilde{D}_2 &= 2.13982, \\ \widetilde{D}_3 &= 8.36199.\end{aligned} \tag{2.62}$$

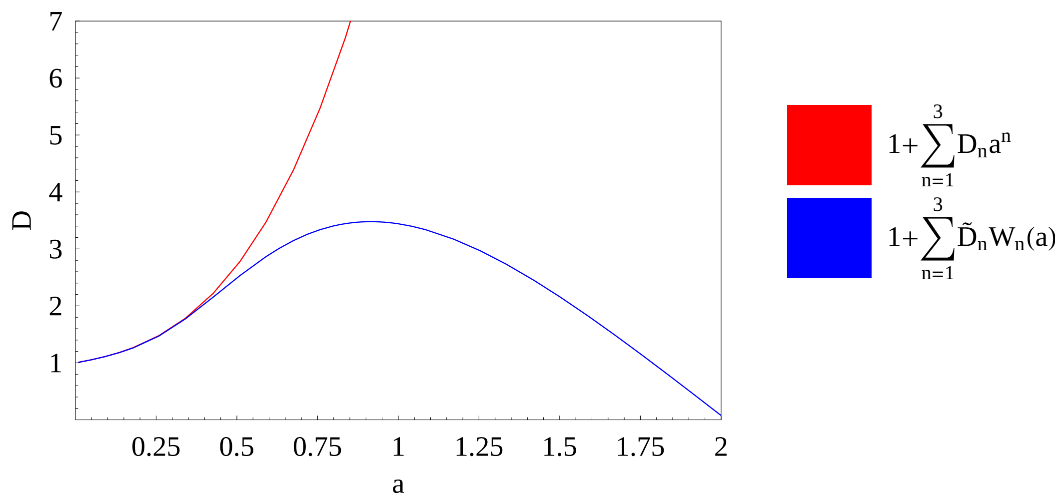


Figure 2.5: Comparison of the approximants of the Adler function $1 + \sum_{n=1}^{3} D_n a^n$ and $1 + \sum_{n=1}^{3} \widetilde{D}_n W_n(a)$.

The comparison of the expansions (2.38) and (2.52) considered to NNLO can be found in Figure 2.5.

However, there is another possibility of expanding $D(s)$ in alternative set of functions, which can be considered as a generalization of the set $W_n(a),\ n \in \mathbb{N}$. The motivation of this procedure is to use to most information known about the analytic properties of the Borel transform $B(u)$ in the Borel u-plane. Soper and Surguladze pointed out that the renormalons closest to the origin $u = 0$ govern the large order behaviour of perturbative expansion of the Adler function more significantly than the renormalons further to the right-hand or the left-hand side [43]. Moreover they suggest to factorize these singularities out introducing the so-called singularity softening. Since we know the character and the position of the first two renormalons, it is possible to follow these ideas and write the Borel transform in the form

$$B(u) = \frac{1}{(1+u)^{\gamma_1}(2-u)^{\gamma_2}} \sum_{n=0}^{\infty} \frac{\widehat{D}_{n+1}}{n!} u^n. \tag{2.63}$$

This yields, following similar steps as in the previous case, another set of expansion function $\widetilde{W}_n(a),\ n \in \mathbb{N}$ defined as follows

$$\widetilde{W}_n(a) = \frac{1}{n!}\left(\frac{8}{3}\right)^n \frac{1}{a}\mathrm{PV}\int_0^\infty \mathrm{e}^{-\frac{u}{a}}(1+u)^{-\gamma_1}\left(1-\frac{u}{2}\right)^{-\gamma_2} w^n(u)\mathrm{d}u, \quad n \in \mathbb{N}_0, \qquad (2.64)$$

which are displayed in Figure 2.6 for $n = 1, 2, \ldots, 5$.

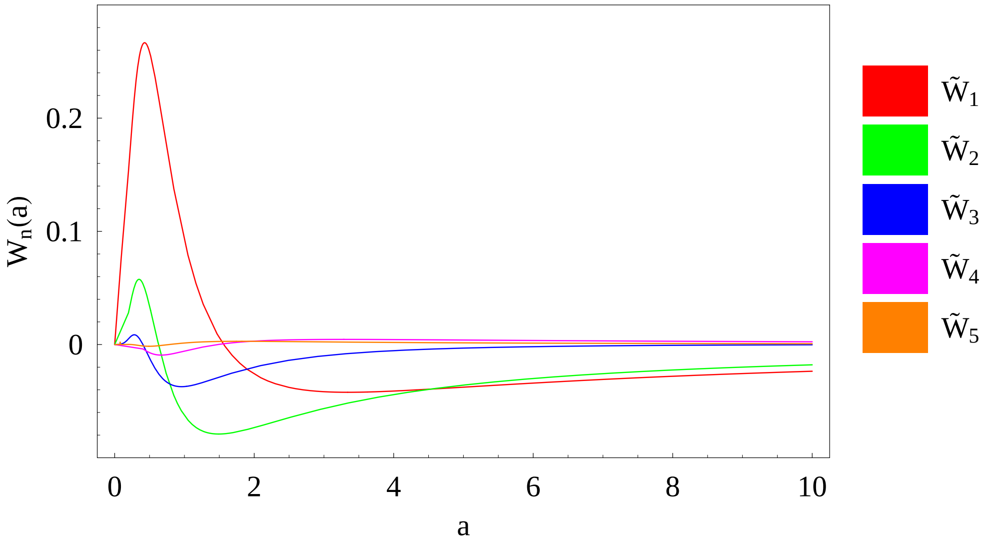


Figure 2.6: The shape of the expansion functions $\widetilde{W}_n(a)$, $n = 1, 2, \ldots, 5$.

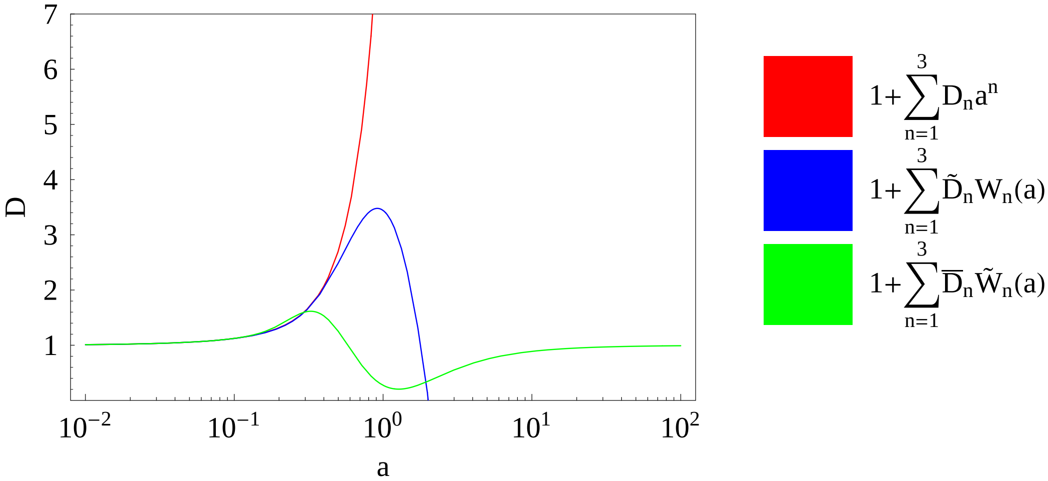


Figure 2.7: Comparison of the approximants of the Adler function $1 + \sum_{n=1}^{3} D_n a^n$, $1 + \sum_{n=1}^{3} \widetilde{D}_n W_n(a)$ and $1 + \sum_{n=1}^{3} \overline{D}_n \widetilde{W}_n(a)$.

The asymptotic expansions for the functions $\widetilde{W}_n(a),\ n = 1, 2, 3$ read

$$\widetilde{W}_1(a) \approx a - 3.098a^2 + 17.9315a^3 + \cdots , \tag{2.65}$$

$$\widetilde{W}_2(a) \approx a^2 - 5.397a^3 + \cdots , \tag{2.66}$$

$$\widetilde{W}_3(a) \approx a^3 + \cdots . \tag{2.67}$$

Therefore, we have constructed another expansion of the Adler function, which is of the form

$$D(s) = 1 + \sum_{n=1}^{\infty} \overline{D}_n \widetilde{W}_n(a), \tag{2.68}$$

where coefficients $\overline{D}_n,\ n \in \mathbb{N}$ can be calculated using the same procedure as have been used for coefficients $\widetilde{D}_n,\ n \in \mathbb{N}$. Comparison of three different approximants of the Adler function, i.e. (2.38), (2.52) and (2.68) can be found in Figure 2.7. The convergence conditions for the optimal expansion (2.68) were studied in [31] leading, however, to the same conclusion as in the case of expansion (2.52), which were discussed in Section 1.5 and studied in papers [9, 10]. The graphical comparison of the approximants in Figures 2.5 and 2.7 illustrates completely different behaviour of pertubative expansion (2.38) and the new optimized expansions (2.52), (2.68). It is useful to point out very specific behaviour of the approximant (2.68). Note that it varies quite slowly around the value $D(s) = 1$.

Chapter 3

τ-lepton Decay and e^+e^- Annihilation in QCD

The renormalization scale and scheme dependence of perturbative expansions considered to finite order was discussed more in detail in Subsection 2.2.1. As mentioned before, in this chapter we shall apply the Caprini-Fischer method of resummation proposed in Section 1.5 on perturbative expansions corresponding to the processes of τ-lepton decay and e^+e^- annihilation. The aim is to compare the renormaliztion scale dependence of the finite order approximants resulting from standard perturbation theory with the dependence of the approximants obtained by Caprini-Fischer resummation considered to the same order.

3.1 τ-lepton Decay

3.1.1 Basic Formulae

Let us consider the familiar ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau + \mathrm{e}^-\overline{\nu}_e)}, \tag{3.1}$$

which is fully computable within the framework of perturbative QCD as follows

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau + \mathrm{e}^-\overline{\nu}_e)} = 3(1+\delta_{\mathrm{EW}})(1+\mathcal{R}_\tau). \tag{3.2}$$

The term $\delta_{\rm EW}$ is the electroweak correction [32–34]

$$\delta_{\rm EW} = \left(\frac{5}{12} + 2\ln\frac{M_Z}{M_\tau}\right)\frac{\alpha(M_\tau)}{\pi}, \quad \alpha(M_\tau) = \frac{1}{133.29} \tag{3.3}$$

and its numerical value is approximately $\delta_{\rm EW} \simeq 0.019$. The QCD contribution $\mathcal{R}_\tau$ has the perturbative expansion of the form

$$\mathcal{R}_\tau\left(M_\tau\right) = a(\mu, c_k)\left[1 + r_1\left(M_\tau, \mu\right) a(\mu, c_k) + r_2\left(M_\tau, \mu, c_k\right) a^2(\mu, c_k) + \ldots\right]. \tag{3.4}$$

The rate R_τ is related to the corresponding Adler function as [35, 36]

$$R_\tau = \frac{3(1+\delta_{\rm EW})}{2\pi \mathrm{i}} \oint\limits_{|s|=M_\tau^2} \frac{\mathrm{d}s}{s}\left(1-\frac{s}{M_\tau^2}\right)^3\left(1+\frac{s}{M_\tau^2}\right)D(s). \tag{3.5}$$

Moreover, it was shown [37] that Equation (3.5) can be rewritten as follows

$$R_\tau = 3(1+\delta_{\rm EW})\left[1 + \int\limits_0^\infty \mathrm{e}^{-\frac{u}{a}} B[D](u) F_\tau\left(\frac{bu}{2}\right)\mathrm{d}u\right], \tag{3.6}$$

where

$$F_\tau(u) = \frac{-12\sin(\pi u)}{\pi u(u-1)(u-3)(u-4)}. \tag{3.7}$$

Therefore, the QCD contribution $\mathcal{R}_\tau$ can be expressed as follows

$$\mathcal{R}_\tau = \int\limits_0^\infty \mathrm{e}^{-\frac{u}{a}} B[D](u) F_\tau\left(\frac{bu}{2}\right)\mathrm{d}u. \tag{3.8}$$

The integration path in the integral (3.8) runs from 0 to ∞ and circumvents the singularities of the Borel transform $B(u)$, which create non-uniqueness of (3.8). To eliminate this non-uniqueness the universally adopted PV prescription is chosen.

3.1.2 Expansion Functions $W_n^{ij}(a)$

Similarly to the case of the optimal expansion of the Adler function considered in Subsection 2.3.1, we shall expand the integral representation (3.8) of $\mathcal{R}_\tau$. Inserting the optimal expansion of $B[D](u)$ into (3.8) one obtains the optimal expansion of (3.8) in the form

$$\mathcal{R}_\tau = \sum_{n=1}^{\infty} \bar{r}_n W_n(a), \tag{3.9}$$

where the expansion functions $W_n(a),\ n \in \mathbb{N}$ for the QCD contribution $\mathcal{R}_\tau$ are defined as follows [38]

$$W_n(a) = \frac{1}{n!}\left(\frac{8}{3}\right)^n \left(\frac{2}{b}\right)^n \frac{2}{ab}\mathrm{PV}\int_0^\infty \mathrm{e}^{-\frac{2u}{ab}} F_\tau(u) w^n(u)\mathrm{d}u. \tag{3.10}$$

However, similarly to the case of the Adler function, it is possible to define another sets of expansion functions using the so-called singularity softening [43]. For this purpose, we shall introduce new expansion functions $W_n^{ij}(a)$, where n is indexing the order of the function. Index i will denote how many pairs of renormalons are factorized out. Finally, j will index the shifting of the cuts of the optimal conformal mapping $w(u)$ to the left-hand and to the right-hand side by one, respectively. Moreover, we shall use index j also in the case of corresponding conformal mapping and denote it as $w_j(u)$.

In general, $W_n^{ij}(a)$ is defined as follows[1]

$$W_n^{ij}(a) = \frac{1}{a}\int_0^\infty \mathrm{e}^{-\frac{u}{a}} \prod_{k=0}^{i}\left(1+\frac{u}{k}\right)^{-\gamma_{2k-1}}\left(1-\frac{u}{k+1}\right)^{-\gamma_{2k}} w_j^n(u)\mathrm{d}u, \tag{3.11}$$

where $w_j(u)$ is of the form

$$w_j(u) = \frac{\sqrt{1+\frac{u}{j+1}} - \sqrt{1-\frac{u}{j+2}}}{\sqrt{1+\frac{u}{j+1}} + \sqrt{1-\frac{u}{j+2}}}. \tag{3.12}$$

Let us again emphasize that the integration is in the sense of PV. The formal term

$$\left(1+\frac{u}{0}\right)^{-\gamma_{-1}}\left(1-\frac{u}{1}\right)^{-\gamma_0},$$

[1]For brevity, it is useful to omit the normalization factors and the function F_τ. We shall write down the expansion functions relevant in the case of τ-lepton decay later on.

which enters the definition (3.11) corresponding to index $k = 0$ will be identified with unity

$$\left(1+\frac{u}{0}\right)^{-\gamma_{-1}}\left(1-\frac{u}{1}\right)^{-\gamma_0} \equiv 1. \tag{3.13}$$

Moreover, let us require that $i \geq j$. This condition avoids shifting of the cuts to forerun the factorization of renormalon pairs. Violation of this requirement yields that some renormalons will be omitted and thus, their contribution not properly counted in. The functions (3.11) are singular at the point $a = 0$ having there the asymptotic expansion of the form [10]

$$W_n^{ij}(a) \approx \sum_{k=n}^{\infty} \zeta_{nk}^{ij} a^k. \tag{3.14}$$

Further on, we shall work with the expansion functions of the type $W_n^{00}(a)$, $W_n^{10}(a)$ and $W_n^{11}(a)$. These functions are defined according to (3.11) as

$$W_n^{00}(a) = \frac{1}{a}\int_0^\infty e^{-\frac{u}{a}} w_0^n(u)\mathrm{d}u, \tag{3.15}$$

$$W_n^{10}(a) = \frac{1}{a}\int_0^\infty e^{-\frac{u}{a}}(1+u)^{-\gamma_1}\left(1-\frac{u}{2}\right)^{-\gamma_2} w_0^n(u)\mathrm{d}u, \tag{3.16}$$

where $w_0(u)$ is of the form

$$w_0(u) = \frac{\sqrt{1+u}-\sqrt{1-\frac{u}{2}}}{\sqrt{1+u}+\sqrt{1-\frac{u}{2}}} \tag{3.17}$$

and

$$W_n^{11}(a) = \frac{1}{a}\int_0^\infty e^{-\frac{u}{a}}(1+u)^{-\gamma_1}\left(1-\frac{u}{2}\right)^{-\gamma_2} w_1^n(u)\mathrm{d}u, \tag{3.18}$$

where $w_1(u)$ is of the form

$$w_1(u) = \frac{\sqrt{1+\frac{u}{2}}-\sqrt{1-\frac{u}{3}}}{\sqrt{1+\frac{u}{2}}+\sqrt{1-\frac{u}{3}}}. \tag{3.19}$$

The inverse mapping of (3.19) is of the form

$$u = \frac{24w_1}{5w_1^2 - 2w_1 + 5}. \tag{3.20}$$

The expansion functions (3.15), (3.16) and (3.18) can be easily modified for the case of τ-lepton decay. Taking into account Equation (3.8) yields the definition of corresponding expansion functions as follows

$$W_n^{00}(a) = \frac{1}{n!}\left(\frac{8}{3}\right)^n \left(\frac{2}{b}\right)^n \frac{2}{ab}\int_0^\infty \mathrm{e}^{-\frac{2u}{ab}} F_\tau(u) w_0^n(u)\mathrm{d}u, \tag{3.21}$$

$$W_n^{10}(a) = \frac{1}{n!}\left(\frac{8}{3}\right)^n \left(\frac{2}{b}\right)^n \frac{2}{ab}\int_0^\infty \mathrm{e}^{-\frac{2u}{ab}} (1+u)^{-\gamma_1}\left(1-\frac{u}{2}\right)^{-\gamma_2} F_\tau(u) w_0^n(u)\mathrm{d}u, \tag{3.22}$$

$$W_n^{11}(a) = \frac{1}{n!}\left(\frac{24}{5}\right)^n \left(\frac{2}{b}\right)^n \frac{2}{ab}\int_0^\infty \mathrm{e}^{-\frac{2u}{ab}} (1+u)^{-\gamma_1}\left(1-\frac{u}{2}\right)^{-\gamma_2} F_\tau(u) w_1^n(u)\mathrm{d}u. \tag{3.23}$$

It is easy to check that functions (3.21), (3.22) and (3.23) are normalized in such way that

$$\zeta_{nn}^{ij} = 1, \qquad n = 1,2,3, \qquad 0 \le j \le i \le 1. \tag{3.24}$$

Thus, the asymptotic expansions of (3.21), (3.22) and (3.23) for $n = 1,2,3$ are of the form

$$\begin{aligned} W_1^{00}(a) &\approx a + 6a^2 + 0.08a^3 + \cdots, \\ W_2^{00}(a) &\approx a^2 + 7.312a^3 + \cdots, \\ W_3^{00}(a) &\approx a^3 + \cdots, \end{aligned} \tag{3.25}$$

$$\begin{aligned} W_1^{10}(a) &\approx a + 0.154a^2 + 22.214a^3 + \cdots, \\ W_2^{10}(a) &\approx a^2 - 1.456a^3 + \cdots, \\ W_3^{10}(a) &\approx a^3 + \cdots \end{aligned} \tag{3.26}$$

and

$$
\begin{aligned}
W_1^{11}(a) &\approx a + 0.904a^2 + 19.013a^3 + \cdots \\
W_2^{11}(a) &\approx a^2 + 0.794a^3 + \cdots , \\
W_3^{11}(a) &\approx a^3 + \cdots ,
\end{aligned} \tag{3.27}
$$

respectively.

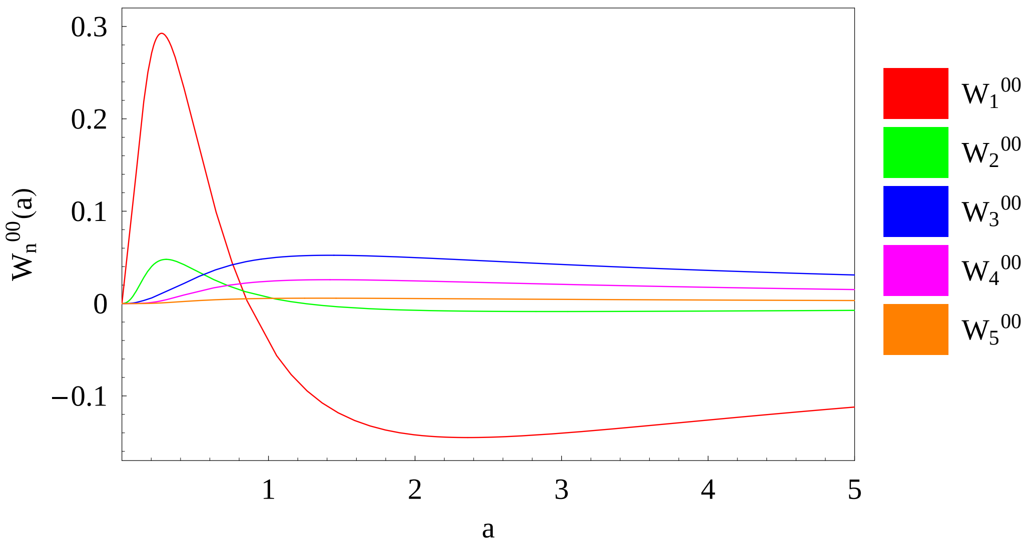


Figure 3.1: The shape of the expansion functions $W_n^{00}(a)$, $n = 1, 2, \ldots, 5$.

Figures 3.1, 3.2 and 3.3 shows the behaviour of expansion functions (3.21), (3.22) and (3.23), respectively, for $n = 1, 2, \ldots, 5$.

3.1.3 Renormalization Scale Dependence of $\mathcal{R}_\tau$ Approximants

We shall use the novel expansions

$$
\mathcal{R}_\tau = \sum_{n=1}^{\infty} r_{n-1}^{ij} W_n^{ij}(a) \tag{3.28}
$$

to investigate the renormalization scale dependence of finite order approximants of $\mathcal{R}_\tau$. In the case of the standard perturbation theory, the dependence was discussed in [37, 41]. The dependence in the case of expansion function (3.21) and (3.22) was

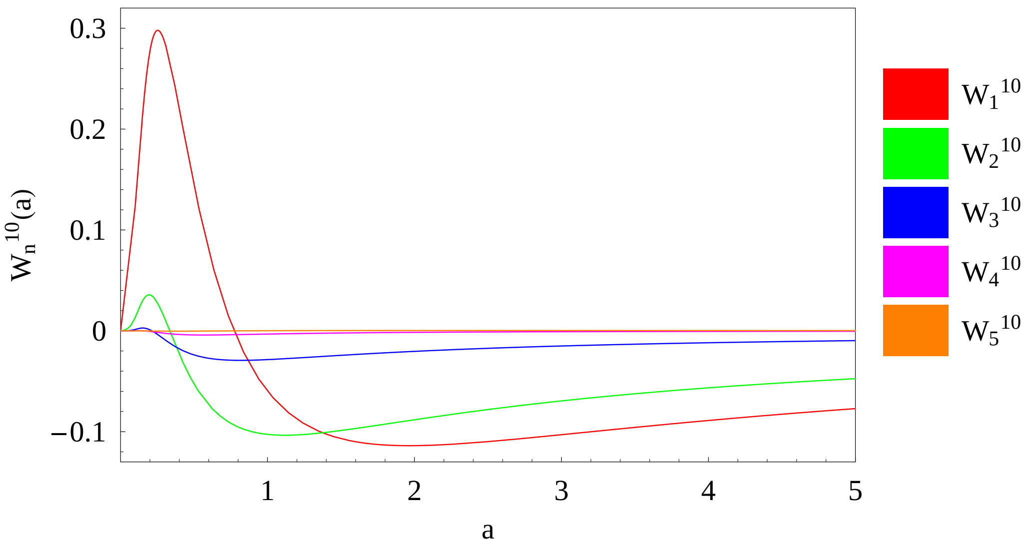


Figure 3.2: The shape of the expansion functions $W_n^{10}(a)$, $n = 1, 2, \ldots, 5$.

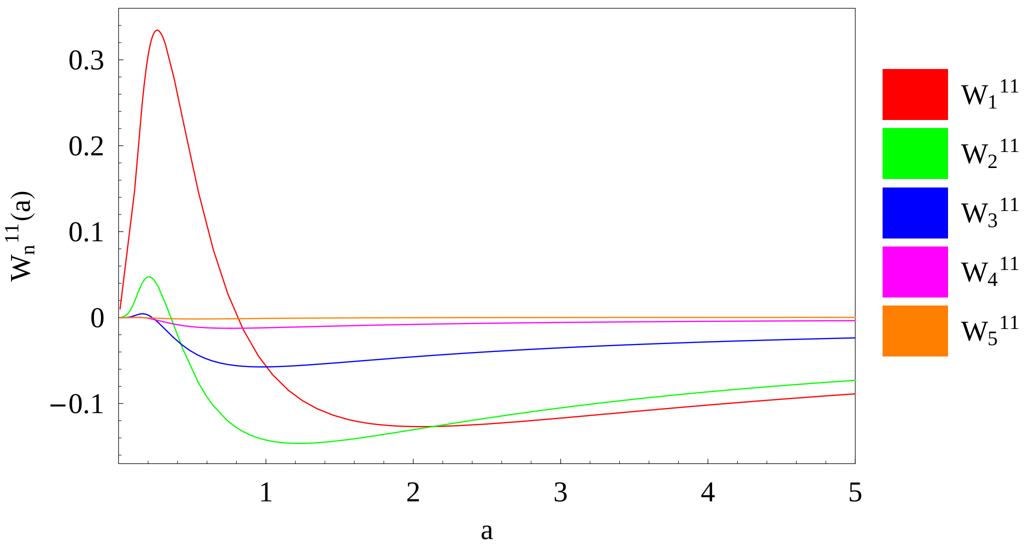


Figure 3.3: The shape of the expansion functions $W_n^{11}(a)$, $n = 1, 2, \ldots, 5$.

considered in [38]. Now, we shall compare the scale dependence of these approximants with the new ones resulting from the expansion in functions (3.23).

The coefficients r_n^{ij} can be easily computed, similarly to the coefficients $\widetilde{D}_n$ in the case of the Adler function using (2.58 - 2.60), from the coefficients r_n of the standard perturbative expansion (3.4) as follows

$$r_0^{ij} = r_0 = 1, \tag{3.29}$$

$$r_1^{ij} = r_1 - r_0\zeta_{12}^{ij} = r_1 - \zeta_{12}^{ij}, \tag{3.30}$$

$$\begin{aligned} r_2^{ij} &= r_2 - r_1^{ij}\zeta_{23}^{ij} - r_0\zeta_{13}^{ij} \\ &= r_2 - r_1\zeta_{23}^{ij} + r_0\left(\zeta_{12}^{ij}\zeta_{23}^{ij} - \zeta_{13}^{ij}\right) \\ &= r_2 - r_1\zeta_{23}^{ij} + \zeta_{12}^{ij}\zeta_{23}^{ij} - \zeta_{13}^{ij}. \end{aligned} \tag{3.31}$$

Let us construct the finite order order sums of (3.28) in the form

$$\mathcal{R}_\tau^{ij,(N)} = \sum_{n=1}^{N} r_{n-1}^{ij} W_n^{ij}(a). \tag{3.32}$$

It can be shown [38] that the approximants (3.32) obey the same property of formal internal consistency of the standard perturbation theory (2.30). This means that the derivatives of $\mathcal{R}_\tau^{ij,(N)}$ with respect to $\ln\mu$ are of the form

$$\frac{\partial\mathcal{R}_\tau^{ij,(N)}(\mu)}{\partial\ln\mu} = \sum_{k=N+1}^{\infty} s_k^{ij} W_k^{ij}(a), \tag{3.33}$$

where s_k^{ij} are some numbers. Therefore, Equation (3.33) can be considered as a generalization of (2.30).

Further on, we shall proceed to numerical tests of the renormalization scale dependence of approximants (3.32). The so-called **'t Hooft renormalization convention** will be used, which means that $c_i = 0,\ i \geq 2$. Therefore, the couplant a is defined by RG equation considered to NLO at every order of perturbative expansion. Moreover we set $N_c = 3$ and $n_f = 3$, which implies using (2.11)

$$b = \frac{9}{2}, \quad c = \frac{16}{9}.$$

The numerical value of Q is taken to be equal to M_τ, so

$$Q = M_\tau = 1.8\text{GeV}.$$

This choice yields [41]

$$r_1\left(\mu = Q, \overline{\text{MS}}\right) = 5.2.$$

In order to illustrate the scale dependence, the fundamental parameter Λ corresponding to $\overline{\text{MS}}$ and $n_f = 3$ is set equal to

$$\Lambda^{(3)}_{\overline{\text{MS}}} = 0.31 \text{ GeV}, \tag{3.34}$$

which is close to the current world average. This implies

$$\rho_1\left(\frac{Q}{\Lambda^{(3)}_{\overline{\text{MS}}}}\right) = b \ln \frac{Q}{\Lambda^{(3)}_{\overline{\text{MS}}}} - r_1\left(\mu = Q, \overline{\text{MS}}\right) = 2.72$$

and the RG invariant ρ_2 is equal to [41]

$$\rho_2 = -6.27.$$

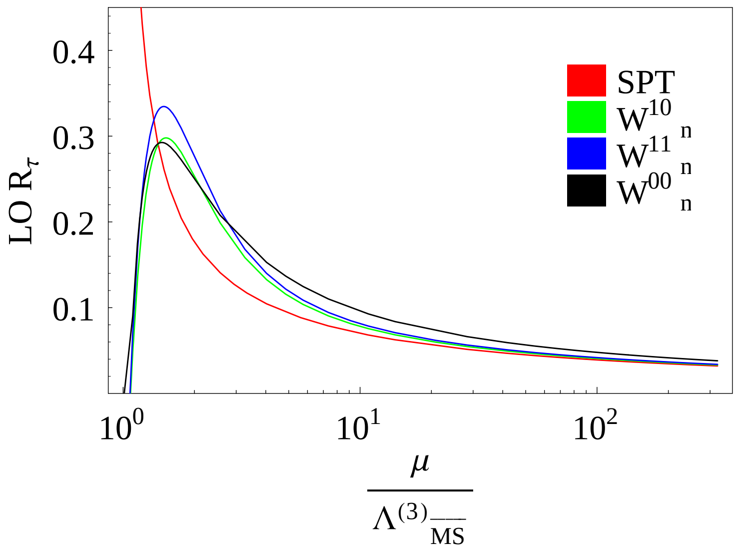


Figure 3.4: Renormalization scale dependence of $\mathcal{R}_\tau^{ij,(N)}$, $0 \leq j \leq i \leq 1$, considered to LO

The graphical comparison of the approximants $\mathcal{R}_\tau^{ij,(N)}$, $0 \leq j \leq i \leq 1$ with the conventional pertubative expansion of $\mathcal{R}_\tau$ considered to LO, NLO and NNLO can be found in Figures 3.4, 3.5 and 3.6, respectively. It is easy to see that the scale dependence of the $\mathcal{R}_\tau^{ij,(N)}$, $0 \leq j \leq i \leq 1$ approximants differs significantly from

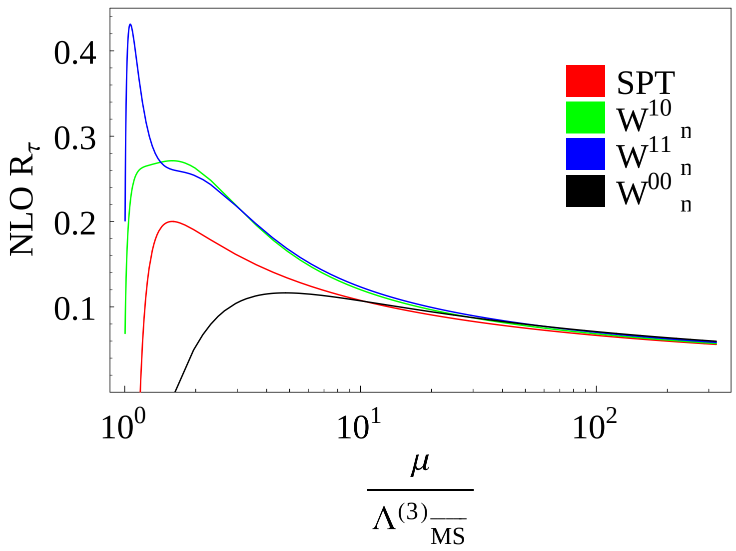


Figure 3.5: Renormalization scale dependence of $\mathcal{R}_\tau^{ij,(N)}$, $0 \leq j \leq i \leq 1$, considered to NLO

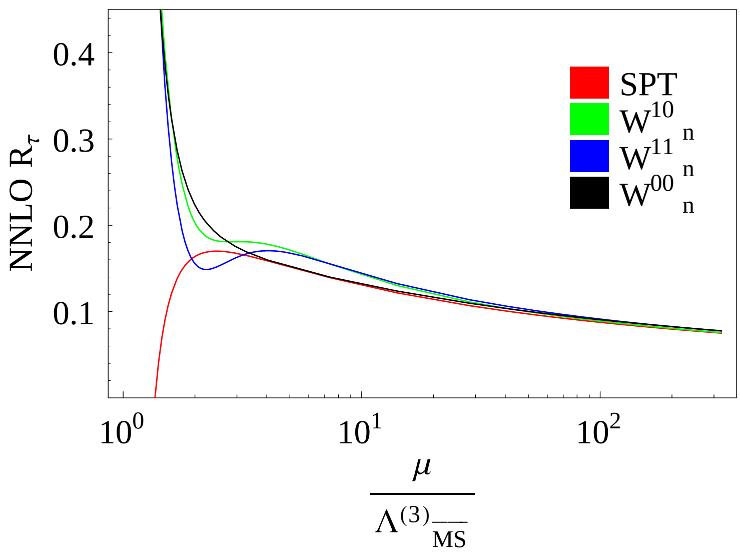


Figure 3.6: Renormalization scale dependence of $\mathcal{R}_\tau^{ij,(N)}$, $0 \leq j \leq i \leq 1$, considered to NNLO

the conventional ones denoted as SPT. Moreover, a remarkable difference between the approximants $\mathcal{R}_\tau^{00,(N)}$, $\mathcal{R}_\tau^{10,(N)}$and $\mathcal{R}_\tau^{11,(N)}$ can be observed. This implies that singularity softening plays an important role. This can be seen from the fact that the values of approximant $\mathcal{R}_\tau^{10,(3)}$ are very close to the PMS optimal point of the standard perturbative approximant considered to NNLO (note the distinctive plateau in the shape of $\mathcal{R}_\tau^{10,(3)}$). However, the values of $\mathcal{R}_\tau^{00,(3)}$ blow closely to this stationary point. The approximant $\mathcal{R}_\tau^{11,(3)}$ varies around this point quite rapidly. Therefore, it can be concluded that the shifting the cuts of optimal conformal mapping is not very useful despite the fact that a "knee" is remarkably present at NLO. Note the big numerical differences between approximants considered to NLO, which surprisingly vanish in NNLO.

There are arguments that it is safer to set $j = 0$ in the definition of functions $W_n^{ij}(a)0$. Consider the expanded function in the form

$$f(z) = \left[\prod_k \left(1 - \text{sign}(R_k)\frac{z}{|R_k|}\right)^{-\gamma_k}\right] h(z) + g(z), \tag{3.35}$$

where R_k are singularities[2], $R_k \in \mathbb{R}$, $\gamma_k > 0$ and $h(z)$, $g(z)$ holomorphic functions. Let us rewrite (3.35) in the following form

$$\begin{aligned} f(z) = &\left[\prod_k \left(1 - \text{sign}(R_k)\frac{z}{|R_k|}\right)^{-\gamma_k}\right] \\ &\times \left\{h(z) + \left[\prod_k \left(1 - \text{sign}(R_k)\frac{z}{|R_k|}\right)^{\gamma_k}\right] g(z)\right\}. \end{aligned} \tag{3.36}$$

This operation, however, generates new singularities by the term

$$\left[\prod_k \left(1 - \text{sign}(R_k)\frac{z}{|R_k|}\right)^{\gamma_k}\right] g(z). \tag{3.37}$$

Afterwards, expanding the function

$$h(z) + \left[\prod_k \left(1 - \text{sign}(R_k)\frac{z}{|R_k|}\right)^{\gamma_k}\right] g(z) \tag{3.38}$$

in powers of shifted conformal mapping, these new singularities are mapped inside the unit disk worsen the numerical behaviour of the optimal expansion. Therefore, using unshifted conformal mapping, new singularities are mapped again on the boundary of the unit disk. Thus, their influence on the numerical behaviour is neglected.

[2] In our case the UV and IR renormalons.

3.2 e^+e^- Annihilation

3.2.1 Basic Formulae

Another hard process, which can be considered within the framework of perturbative QCD is the e$^+$e$^-$ annihilation. Precisely, it is the familiar $R_{\mathrm{e^+e^-}}$-ratio

$$R_{\mathrm{e^+e^-}}(Q) = \frac{\sigma(Q, \mathrm{e^+e^-} \to \mathrm{hadrons})}{\sigma(Q, \mathrm{e^+e^-} \to \mu^+\mu^-)} = \left(3\sum_i e_i^2\right)\left(1 + \mathcal{R}_{\mathrm{e^+e^-}}(Q)\right), \tag{3.39}$$

where the term $3\sum_i e_i^2$ represents the QPM prediction and $\mathcal{R}_{\mathrm{e^+e^-}}(Q)$ is the QCD correction and Q is the center of mass energy.

The QCD contribution $\mathcal{R}_{\mathrm{e^+e^-}}(Q)$ obeys the following perturbative expansion

$$\mathcal{R}_{\mathrm{e^+e^-}}(Q) = a(\mu, c_i)\left[1 + r_1(Q, \mu, c_i)a(\mu, c_i) + r_2(Q, \mu, c_i)a(\mu, c_i)^2 + \cdots\right]. \tag{3.40}$$

However, Equation (3.39) can be rewritten as follows [6]

$$R_{\mathrm{e^+e^-}}(Q) = 12\pi\left(\sum_i e_i^2\right)\mathrm{Im}\Pi(Q^2 + \mathrm{i}\epsilon), \tag{3.41}$$

where Π is the electromagnetic correlator of two quark currents, which enters the definition of the Adler function. Therefore, it is possible to proceed similarly to the previous case of τ-lepton decay. Moreover, using (3.41) $\mathcal{R}_{\mathrm{e^+e^-}}(Q)$ can be formally represented in the integral form similar to (3.8) [6]. The relevant function $F_{\mathrm{e^+e^-}}(u)$ is as follows

$$F_{\mathrm{e^+e^-}}(u) = \frac{\sin \pi u}{\pi u}. \tag{3.42}$$

3.2.2 Expansion Functions $W_n^{ij}(a)$

Due to (3.42) the relevant expansion function $W_n^{ij}(a)$, $0 \le j \le i \le 1$ for $\mathcal{R}_{\mathrm{e^+e^-}}(Q)$ are defined as follows

$$W_n^{00}(a) = \frac{1}{n!}\left(\frac{8}{3}\right)^n\left(\frac{2}{b}\right)^n\frac{2}{ab}\int_0^\infty \mathrm{e}^{-\frac{2u}{ab}}F_{\mathrm{e^+e^-}}(u)w_0^n(u)\mathrm{d}u, \tag{3.43}$$

$$W_n^{10}(a) = \frac{1}{n!}\left(\frac{8}{3}\right)^n\left(\frac{2}{b}\right)^n\frac{2}{ab}\int_0^\infty \mathrm{e}^{-\frac{2u}{ab}}(1+u)^{-\gamma_1}\left(1-\frac{u}{2}\right)^{-\gamma_2}F_{\mathrm{e^+e^-}}(u)w_0^n(u)\mathrm{d}u, \tag{3.44}$$

$$W_n^{11}(a) = \frac{1}{n!}\left(\frac{24}{5}\right)^n\left(\frac{2}{b}\right)^n\frac{2}{ab}\int_0^\infty \mathrm{e}^{-\frac{2u}{ab}}(1+u)^{-\gamma_1}\left(1-\frac{u}{2}\right)^{-\gamma_2}F_{\mathrm{e^+e^-}}(u)w_1^n(u)\mathrm{d}u. \tag{3.45}$$

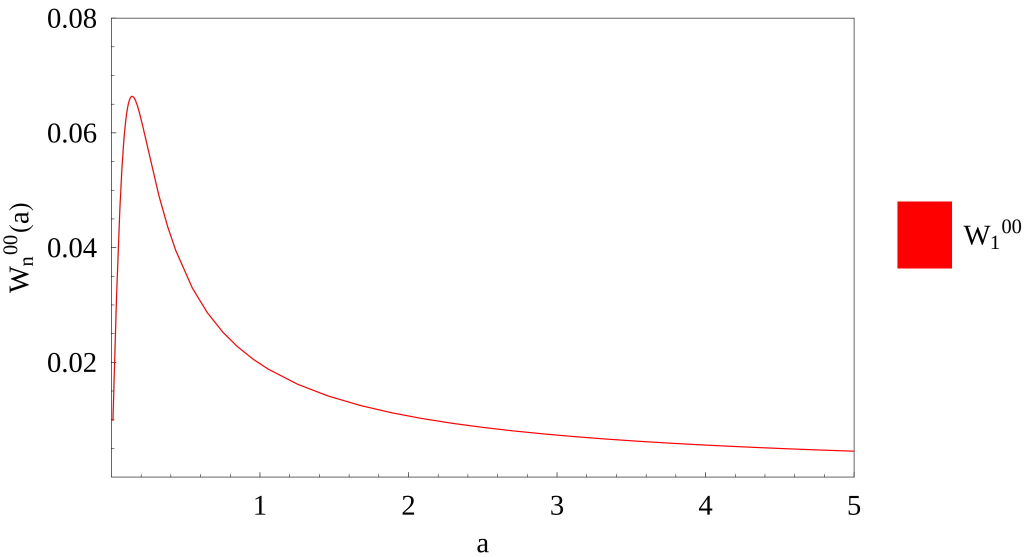


Figure 3.7: The shape of the expansion function $W_1^{00}(a)$.

Similarly to the previous case, it is easy to see that functions are normalized to fulfill the condition $\zeta_{nn}^{ij} = 1$. Thus, the asymptotic expansions of $W_n^{00}(a)$, $W_n^{10}(a)$ and $W_n^{11}(a)$ in the neighbourhood of $a = 0$ for $n = 1, 2, 3$ are of the form

$$W_1^{00}(a) \approx a - 1.125a^2 - 43.795a^3 + \cdots ,$$
$$W_2^{00}(a) \approx a^2 - 3.375a^3 + \cdots ,$$
$$W_3^{00}(a) \approx a^3 + \cdots , \tag{3.46}$$

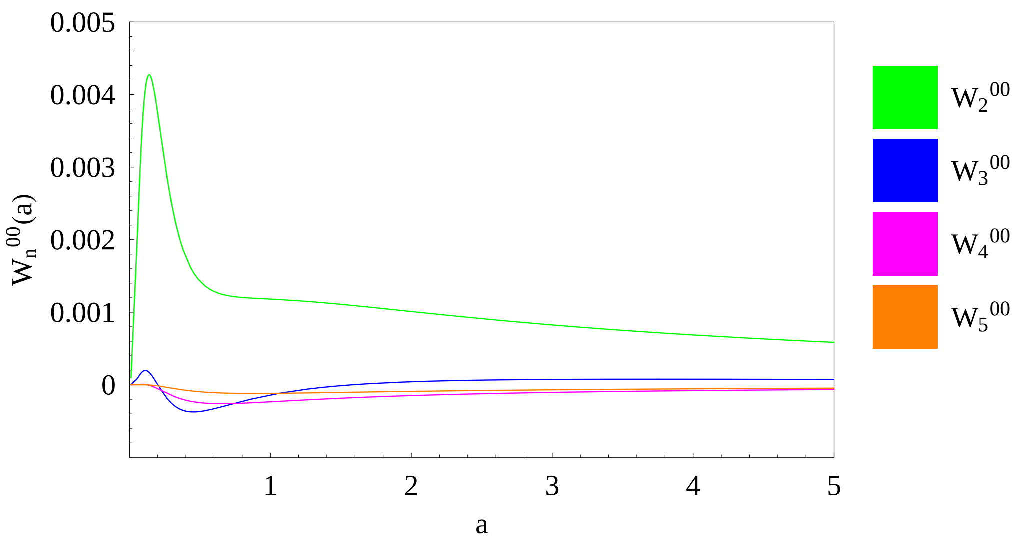


Figure 3.8: The shape of the expansion functions $W_n^{00}(a)$, $n = 2, \ldots, 5$.

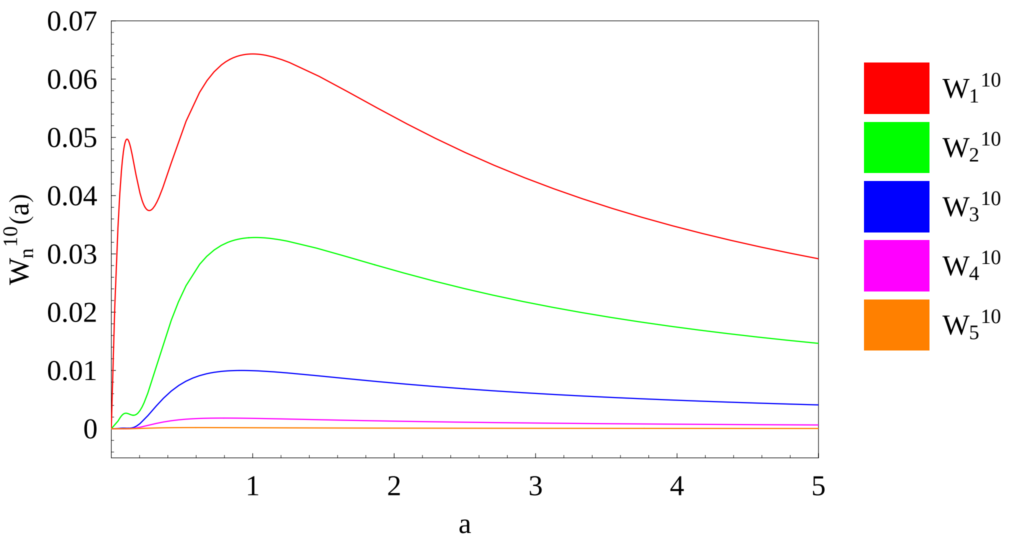


Figure 3.9: The shape of the expansion functions $W_n^{10}(a)$, $n = 1, 2, \ldots, 5$.

$$\begin{aligned} W_1^{10}(a) &\approx a - 6.97a^2 + 40.813a^3 + \cdots , \\ W_2^{10}(a) &\approx a^2 - 12.143a^3 + \cdots , \\ W_3^{10}(a) &\approx a^3 + \cdots \end{aligned} \tag{3.47}$$

and

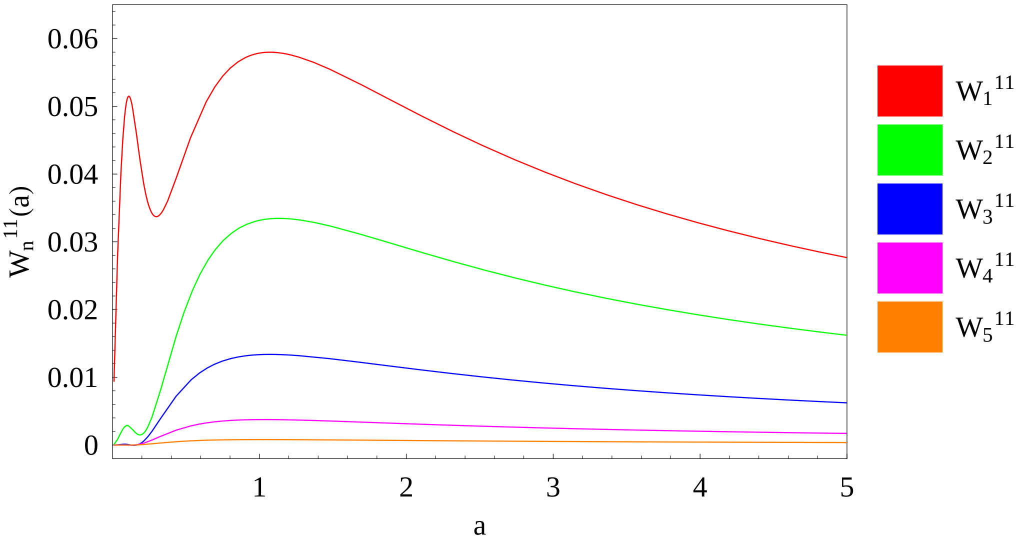


Figure 3.10: The shape of the expansion functions $W_n^{11}(a)$, $n = 1, 2, \ldots, 5$.

$$
\begin{aligned}
W_1^{11}(a) &\approx a - 6.22a^2 + 29.596a^3 + \cdots\,, \\
W_2^{11}(a) &\approx a^2 - 9.893a^3 + \cdots\,, \\
W_3^{11}(a) &\approx a^3 + \cdots\,,
\end{aligned}
\tag{3.48}
$$

respectively.

Figures 3.7, 3.8, 3.9 and 3.10 shows the behaviour of expansion functions (3.43), (3.44) and (3.45), respectively, for $n = 1, 2, \ldots, 5$.

3.2.3 Numerical Results in the Studies of $\mathcal{R}_{e^+e^-}$ Approximants

Using the expansion functions (3.43), (3.44) and (3.45) we shall expand $\mathcal{R}_{e^+e^-}(Q)$ in the following form

$$
\mathcal{R}_{e^+e^-} = \sum_{n=1}^{\infty} r_{n-1}^{ij} W_n^{ij}(a). \tag{3.49}
$$

Truncating the expansion (3.49) at finite order N yields the following optimized approximants

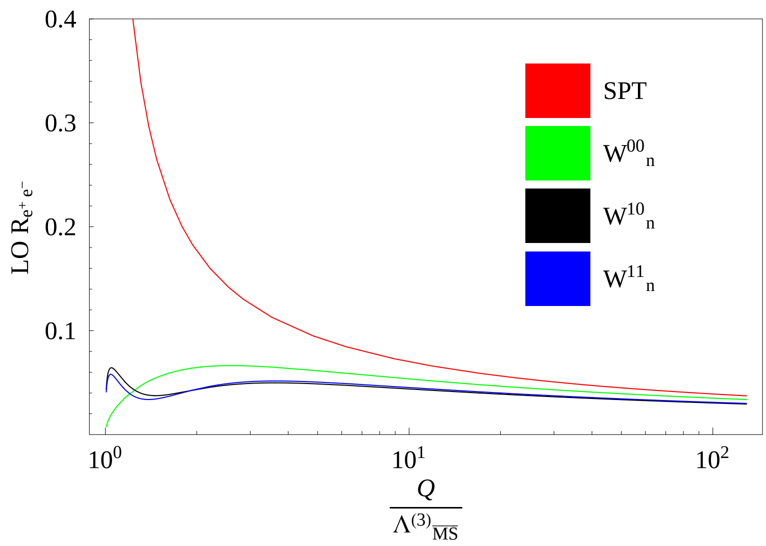


Figure 3.11: Energy dependence of $\mathcal{R}^{ij,(N)}_{e^+e^-}$, $0 \leq j \leq i \leq 1$, considered to LO

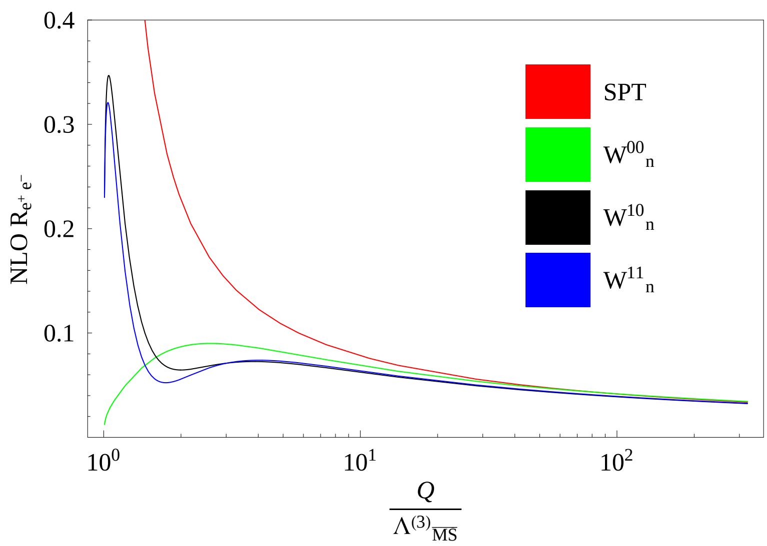


Figure 3.12: Energy dependence of $\mathcal{R}^{ij,(N)}_{e^+e^-}$, $0 \leq j \leq i \leq 1$, considered to NLO

$$\mathcal{R}^{ij,(N)}_{e^+e^-}(Q) = \sum_{n=1}^{N} r^{ij}_{n-1} W^{ij}_n(a). \tag{3.50}$$

The coefficients r^{ij}_n are computed identically as in the case of τ-lepton using (3.29 - 3.31).

In order to investigate the Q-dependence of (3.50) we set $\mu = Q$ in $a(\mu, c_k)$ as well as in the coefficients $r^{ij}_n(\mu, c_k, Q)$. We shall work in 't Hooft convention as well

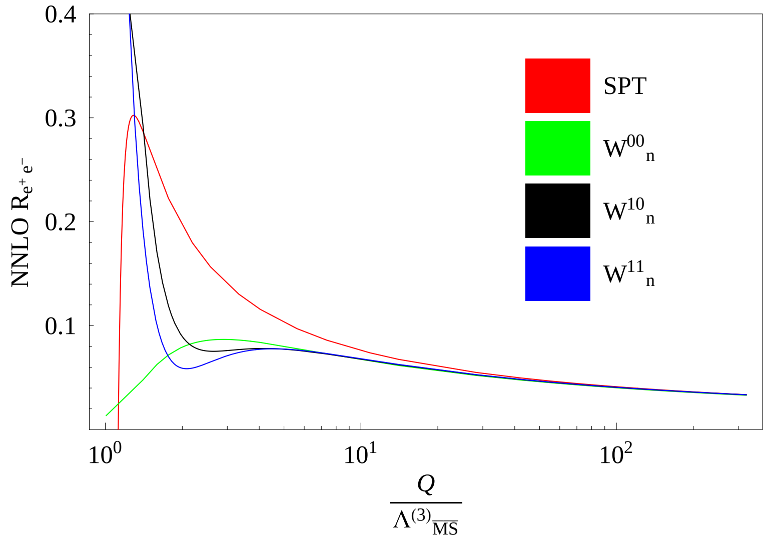


Figure 3.13: Energy dependence of $\mathcal{R}^{ij,(N)}_{e^+e^-}$, $0 \leq j \leq i \leq 1$, considered to NNLO

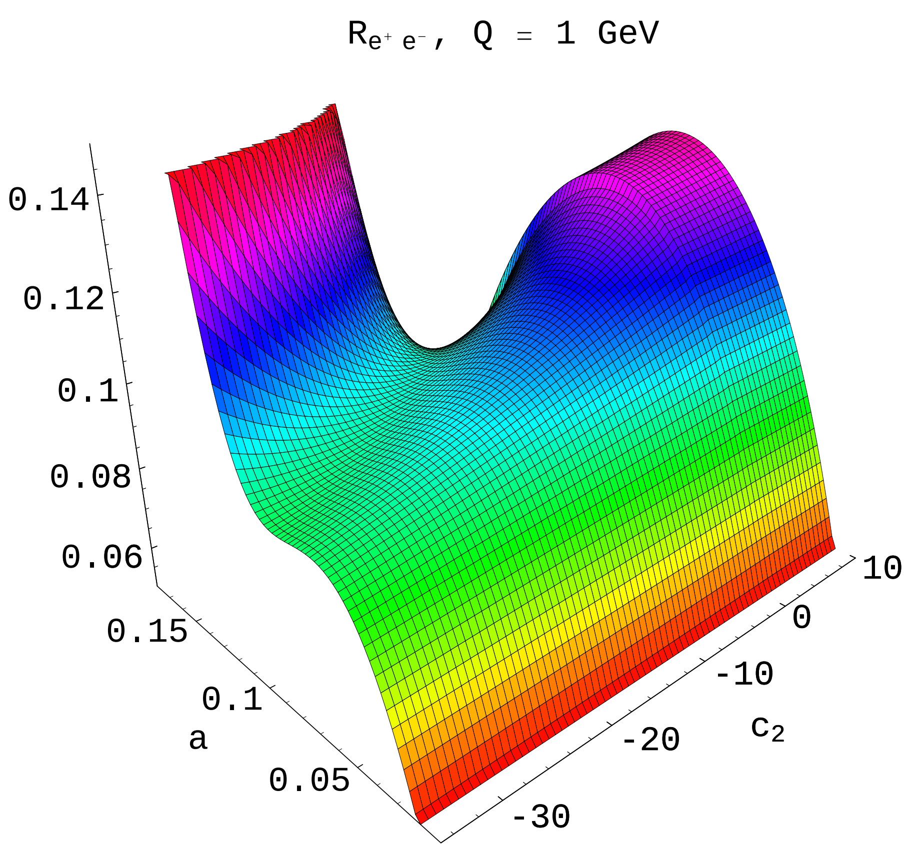


Figure 3.14: Dependence of the standard approximant $\mathcal{R}_{e^+e^-}(a, c_2)$ considered to NNLO as a function of a and c_2 for Q=1 GeV.

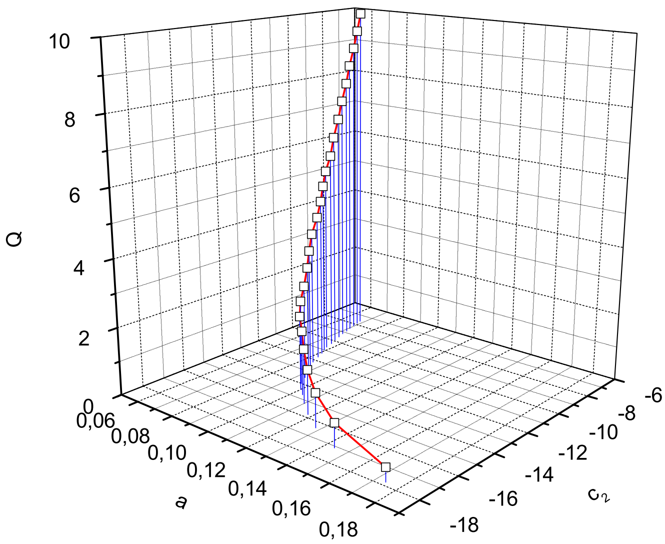


Figure 3.15: Position of the saddle point of the standard perturbation theory approximant $\mathcal{R}_{e^+e^-}(a, c_2)$ considered to NNLO.

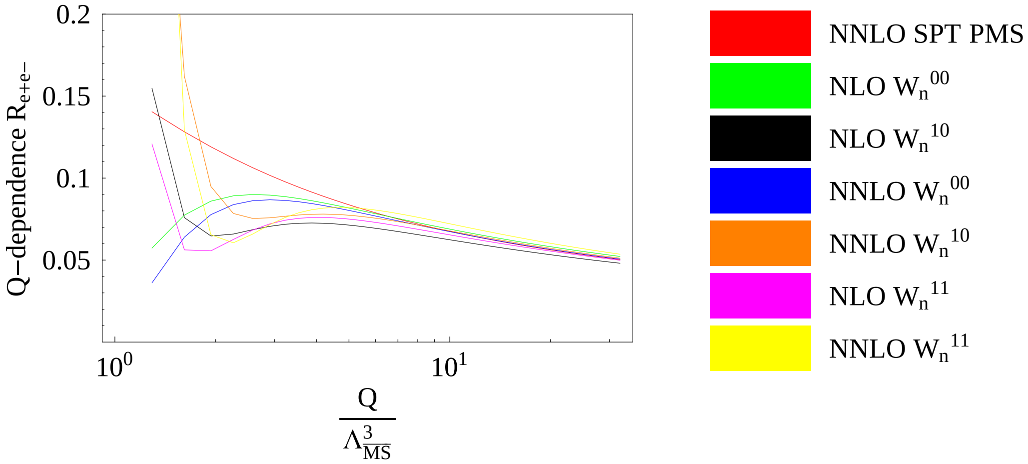


Figure 3.16: Energy dependence of $\mathcal{R}^{ij,(N)}_{e^+e^-}$, $0 \leq j \leq i \leq 1$, considered to NLO, NNLO compared with the standard perturbation theory NNLO approximant computed in PMS scheme with $c_2 \neq 0$.

as with $c_2 \neq 0$. Coefficient r_1 of the conventional expansion can be expressed in $\overline{\mathrm{MS}}$ as follows [41]

$$r_1(\overline{\mathrm{MS}}, \mu = Q) = 1.986 - 0.115 n_f \xrightarrow{n_f=3} r_1(\overline{\mathrm{MS}}, \mu = Q) = 1.941. \tag{3.51}$$

The invariant ρ_2 equals [41]

$$\rho_2 = -12.2. \tag{3.52}$$

Figures 3.11, 3.12 and 3.13 shows the behaviour of approximants $\mathcal{R}^{ij,(N)}_{e^+e^-}$, $0 \le j \le i \le 1$ with the conventional pertubative expansion of $\mathcal{R}_{e^+e^-}$ considered to LO, NLO and NNLO. The scale dependence of the $\mathcal{R}^{ij,(N)}_{e^+e^-}$, $0 \le j \le i \le 1$ approximants again, similatly to τ-lepton, differs significantly from the conventional approximants and so do the approximants $\mathcal{R}^{00,(N)}_{e^+e^-}$, $\mathcal{R}^{10,(N)}_{e^+e^-}$and $\mathcal{R}^{11,(N)}_{e^+e^-}$. The same comments concerning the singularity softening and cut shifting as in the previous case can be also adopted.

Taking into account $c_2 \ne 0$ yields another free parameter in the theory. Thus, the couplant has to be considered to NNLO and curves in Figures 3.11, 3.12 and 3.13 change to surfaces. These surfaces will be, for a certain value Q of the center of mass energy, functions of parameter c_2 and of the renormalized couplant a. According to PMS procedure, saddle points of these surfaces will be investigated. The renormalization scale μ and RS will be defined by the coordinates of these saddle points for a certain value of energy Q. The search for saddle points can be done in parameters $\{a, c_2\}$, since a corresponds to μ uniquely via the RG equation.

For illustration, Figure 3.14 shows distinctive saddle in the behaviour of the standard perturbation theory approximant considered to NNLO with nonzero c_2. The energy Q equals 1 GeV. Moreover, the value of the approximant in the saddle decreases with rising energy Q. This phenomenon is manifested in Figures 3.15 and 3.16.

Summary and Conclusions

Perturbative approach is widely used in QCD calculations of physical observables in studies of a large number of hard processes. Since at most first three terms of perturbative series are known, predictions are commonly trusted at high energies (UV region). However, the low-energy predictions (predictions for IR region) are not accepted in general, since they blow up.

In this work the renormalization scale dependence in the case of τ-lepton and e^+e^- annihilation was investigated. Finite order approximants of ratios $\mathcal{R}_\tau$ and $\mathcal{R}_\tau$ were constructed via the Fischer - Caprini method and many sets of expansion functions were tested.

The revised numerical tests of scale dependence in the case of τ-lepton decay shows the importance of the character of first renormalons to be incorporated in the definition of expansion functions. It can be concluded that the singularity softening plays a significant role. However, the cuts of the conformal mapping entering the definition of Caprini - Fischer expansion functions have to stay unshifted, since a strong numerical evidence was observed that this operation does not lead to better behaviour of resummed approximants at large distances.

Similar tests in the case of e^+e^- annihilation were performed, i.e. the Q-dependence of these approximants was investigated. These pioneer numerical results show again that the information about the character of renormalons is essential and has to be properly used in the definition of expansion functions. As in the previous case, the comparison of Q-dependence leads to the conclusion that the expansion functions, with the most of the information about character of renormalons used, have to be preferred. However, one has to bear in mind that cuts of the optimal conformal mapping have to stay unshifted again. Moreover, the Q-dependence of resummed approximants was compared with the approximants resulting from the standard perturbation theory computed in the PMS scheme leading to the same conclusions.

Despite the fact that the IR stability of finite order approximants of perturbative series in QCD was improved by the Caprini - Fischer resummation, we have to be aware of many other circumstances. One has to bear in mind that this method uses only the behaviour of perturbative coefficients at large orders and in realistic calculations we work only with a very small number of them. Moreover, the information known about the singularities of the Borel transform of the Adler function stems from the studies of certain classes of Feynman diagrams omitting infinitely many of them. We also do not know much about the behaviour of the Borel transform on other sheets of its Riemann surface behind the first renormalons, which appear to be very complicated branching points. Nevertheless, I consider this method a very promising step outside the standard perturbation theory bringing interesting new insights on application of perturbative QCD in the IR region.

Bibliography

[1] F. J. Dyson, Phys. Rev. 85 (1952) 4, 631

[2] P. M. Stevenson, Nucl. Phys. B 231 (1984), 65

[3] N. N. Khuri, Phys. Rev. D 16 (1977), 1754

[4] J. Chýla, Teoretická analýza nejednoznačností poruchové QCD

[5] G. Hardy, Divergent series, Oxford University Press, Oxford 1963

[6] M. Beneke, Phys. Rep. 317 (1999), 1

[7] J. Fischer, Fortschr. Phys. 42 (1994) 8, 665

[8] I. Caprini, J. Fischer, Phys. Rev. D 60 (1999), 054014 1

[9] I. Caprini, J. Fischer, Phys. Rev. D 62 (2000), 054007 1

[10] I. Caprini, J. Fischer, Eur. Phys. J. C 24 (2002), 127

[11] P. M. Stevenson, Phys. Rev. D 23 (1981) 12, 2916

[12] A. Moroz, Czech. J. Phys. 40 (1990), 705

[13] A. Moroz, Commun. Math. Phys. 133 (1990), 369

[14] A. Moroz, Czech. J. Phys. 42 (1992), 753

[15] A. Moroz, Quantum Field Theory as a Problem of Resummation, PhD thesis, Charles University Prague, 1991; hep-th/9206074

[16] S. Ciulli, J. Fischer, Nucl. Phys. 24 (1961), 465

[17] G. 't Hooft, in: The Whys of Subnuclear Physics, Proceedings of the 15th International School on Subnuclear Physics, Erice, Sicilly, 1977, edited by A. Zichihci (Plenum Pres, New York 1979), p. 943

[18] N. N. Khuri, Phys. Rev. D 23 (1981), 2285

[19] V. Zakharov, Nucl. Phys. B 385 (1992), 452

[20] A. H. Mueller, in: QCD - Twenty Years Later, Aachen 1992, edited by P. Zerwas, H. A. Kastrup (World Scientific, Singapore 1992)

[21] M. Beneke, V. I. Zakharov, Phys. Lett. B 312 (1993), 340

[22] G. Grunberg, Phys. Lett. B 304 (1993), 183

[23] M. Beneke, Phys. Lett. B 307 (1993), 154

[24] M. Beneke, Nucl. Phys. B 405 (1993), 424

[25] D. Broadhurst, Z. Phys. C 58 (1993), 339

[26] G. N. Watson, Phil. Trans. Roy. Soc. London, Ser, A 211 (1912), 279

[27] F. Nevanlinna, Zur Theorie der asymptotischen Potenzreihen, PhD thesis, Alexander University, Helsingfors, 1918; Ann. Acad. Sci. Fennicae, Ser. A 12, No. 3 (1918-1919)

[28] F. Nevanlinna, Jahrbuch Fort. Math. 46 (1916-1918), 1463

[29] A. D. Sokal, J. Math. Phys. 21 (1980), 261

[30] M. Rainiš, private communications

[31] M. Rainiš, Rozpad tau leptonu a vazbová konstanta silné interakce, diploma thesis, Charles University Prague

[32] E. Braaten, C. S. Li, Phys. Rev. D 42 (1990) 11, 3888

[33] E. Braaten, S. Narison, A. Pich, Nucl. Phys. B 373 (1992), 581

[34] W. J. Marciano, A. Stirlin, Phys. Rev. Lett. 61 (1988) 16, 1815

[35] E. Braaten, Phys. Rev. Lett. 60 (1988) 16, 1606

[36] E. Braaten, Phys. Rev. D 39 (1989) 5, 1458

[37] G. Altarelli, P. Nason, G. Ridolfi, Z. Phys. C 68 (1995), 257

[38] J. Fischer, J. Chýla, I. Caprini, Acta Phys. Slovaca 52 (2002) 6, 483

[39] M. Beneke, V. M. Braun, N. Kivel, Phys. Lett. B 404 (1997), 315

[40] A. Mueller, Nucl. Phys. B 250 (1985), 327

[41] J. Chýla, A. L. Kataev, S. A. Larin, Phys. Lett. B 267 (1991), 269

[42] G. Grunberg, Phys. Rev. D 29 (1984) 10, 2315

[43] D. E. Soper, L. R. Surguladze, Phys. Rev. D 54 (1996), 4566

[44] K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, Phys. Lett. B 85 (1979), 277

[45] M. Dine, J. Sapirstein, Phys. Rev. Lett. 43 (1979), 668

[46] W. Celmaster, R. Gonsalves, Phys. Rev. Lett. 44 (1980), 560

[47] S. G. Gorishny, A. L. Kataev, S. A. Larin, Phys. Lett. B 259 (1991), 144

[48] L. R. Surguladze, M. A. Samuel, Phys. Rev. Lett. 66 (1991), 560

[49] L. R. Surguladze, M. A. Samuel, Phys. Rev. Lett. 66 (1991), 2416(E)

[50] J. Chýla, Quarks, partons and Quantum Chromodynamics, www-hep.fzu.cz/chyla/lectures/text.pdf

[51] J. Chýla, Czech. J. Phys. 42 (1992) 3, 263

[52] J. Fischer, Int. J. Mod. Phys. A 12 (1997) 21, 3625

[53] J. Fischer, Čs. čas. fyz. 50 (2000), 300

[54] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, kap. 40,40A, Clarendon Press, Oxford 1989, 1993

[55] F. J. Yndurain, Quantum Chromodynamics, Springer-Verlag New York, Berlin, Heidelberg, Tokyo, 1983

[56] J. Zinn-Justin, Phys. Rep. C 70 (1981) 2, 109

[57] C. S. Lam, T. M. Yan, Phys. Rev. D 16 (1977) 3, 703

[58] M. Luo, W. J. Marciano, preprint BNL-47187 (1992)

[59] D. M. Howe, C. J. Maxwell, Phys. Rev. D 70 (2004), 014002