QUANTIZATION AND COHERENT STATES

PETR LUFT

Contents

1. **Introduction**

The notion of coherent states belongs to the most important tools in many applications of quantum physics. They found many applications in quantum optics, quantum field theory, condensed matter physics, atomic physic etc. There exists a big number of various definitions and approaches to the coherent states dependent on author and application.

Our main reference is [9], where the canonical coherent states are described, and in analogy with this paper we will formulate our approach to construction of coherent states. Our approach is based on the notion of Mackey's irreducible systems of imprimitivity. Using the irreducible system of imprimitivity we shall construct an irreducible set of Weyl operators, which will act on a fiducial vacuum state.

In section 2 we recall the formalism of Mackey's systems of imprimitivity as described in [11], [3] and [13]. We start with the configuration manifold, which will be a homogeneous $\mathbf{G}$-space of a group $\mathbf{G}$. After recalling the construction of a projection-valued measure and projective representation, we define quantum position and momentum observables. This will be the starting point for construction of the set of Weyl operators. Special subsection is devoted to imprimitivity systems on finite configuration spaces.

In section 4 we describe construction of imprimitivity systems and systems of coherent states on finite configuration manifold $\mathbf{Z}_n$, with the symmetry group $\mathbf{D}_n$. We apply the method of paper [14], where quantization on $\mathbf{Z}_n$ with Abelian symmetry group $\mathbf{Z}_n$ was introduced. Here we shall extend the Abelian cyclic symmetry group to the non-commuting dihedral group $\mathbf{D}_n$.

Section 5 is devoted to the construction of coherent states on the circle. This problem was already solved in [4], [8] and [1]. Each of these papers has different approach to the construction of coherent states. We follow our approach and construct coherent states based on the notion of imprimitivity system. In analogy with [9], we construct the coherent states of Perelomov type on the circle $\mathbf{S}^1$.

2. Mackey's quantization and systems of imprimitivity

In this chapter we want to introduce briefly Mackey's quantization method, and its formalism. The idea of Mackey's quantization is based on construction of systems of imprimitivity for a locally compact and separable group $\mathbf{G}$. This method is appropriate for homogeneous configuration manifolds, with the symmetry group $\mathbf{G}$.

The formalism of Mackey's quantization is comprehensively described for example in [11] or in [3]. In these works the most general algebraic formulation of Mackey's quantization and its assumptions is given. We do not find necessary to define here all algebraic structures, however they are elegant and necessary for the formulation of symmetry in quantum mechanics. We can do with the main result of this theory, which will be used as a basis for our further work. We will also restrict this theory on pure states, even though the theory is formulated also for density operators — self-adjoint positive bounded operators with trace equal to one.

2.1. **Configuration manifold and symmetry group.** Let us now consider a connected smooth configuration manifold $\mathbf{M}$. We will suppose that $\mathbf{M}$ is a homogeneous $\mathbf{G}$-space of some finite-dimensional Lie group $\mathbf{G}$. This means that $\mathbf{G}$ is a symmetry group of configuration manifold $\mathbf{M}$, and there exists a transitive action $\rhd$ of symmetry group $\mathbf{G}$ on manifold $\mathbf{M}$:

$$(\mathbf{G}\times\mathbf{M})\to\mathbf{M}: \quad g_1\cdot(g_2\rhd m)=(g_1\cdot g_2)\rhd m, \quad g_1,g_2\in\mathbf{G},\ m\in\mathbf{M},$$
$$e\rhd m=m, \quad \forall m\in\mathbf{M}. \tag{2.1}$$

Transitivity of this action means that any two elements of configuration manifold can be connected by the group action:

$$\forall m_1,m_2\in\mathbf{M}\ \exists g\in\mathbf{G}:\ m_1=g\rhd m_2. \tag{2.2}$$

We will suppose that $\mathbf{G}$ is locally compact and separable to fulfil the assumption of Mackey's theorem.

Now we define a subgroup of stability $\mathbf{H}_m$ of element $m\in\mathbf{M}$ as a subgroup of symmetry group $\mathbf{G}$, which does not shift the element m:

$$\mathbf{H}_m:=\{g\in\mathbf{G}|g\rhd m=m\}. \tag{2.3}$$

Thanks to the transitivity of the action $\rhd$ we can easily show that all symmetry groups $\mathbf{H}_m$ for all $m\in\mathbf{M}$ are isomorphic. We may also consider only one symmetry subgroup $\mathbf{H}:=\mathbf{H}_{m_0}$ for some arbitrary $m_0\in\mathbf{M}$. Subgroup of stability $\mathbf{H}$ is also called isotropy subgroup. It is known that $\mathbf{H}$ is a closed Lie subgroup of $\mathbf{G}$.

The quotient space $\mathbf{G}/\mathbf{H}$ of left cosets $g\mathbf{H}$ may be endowed with factor topology. Hence, we have a differentiable structure on quotient space $\mathbf{G}/\mathbf{H}$, and thanks to the mapping

$$\phi:\mathbf{G}/\mathbf{H}\to\mathbf{M}: g\mathbf{H}\to g\rhd m_0 \tag{2.4}$$

we can identify quotient space $\mathbf{G}/\mathbf{H}$ with configuration manifold $\mathbf{M}$:

$$\mathbf{G}/\mathbf{H}\cong\mathbf{M}. \tag{2.5}$$

We shall show that all transitive systems of imprimitivity on $\mathbf{M}$ will be completely classified using the notion of projective representation of subgroup of stability $\mathbf{H}$.

2.2. **Projection-valued measure, projective representation and system of imprimitivity.** Let us denote by $\mathcal{H}$ the Hilbert space conected with our quantum system. On this Hilbert space we will now define Mackey's system of imprimitivity for configuration manifold $\mathbf{M}$ and its symmetry group $\mathbf{G}$. Mackey's way of quantization has two starting points: projection-valued measure $\mathbf{E}$ and projective representation $\mathbf{V}$ of the symmetry group $\mathbf{G}$.

As we will see, the position operator will be defined using a system of orthogonal projectors — the projection-valued measure. In quantum mechanics the result of position measurement is point (or a subset) in configuration space $\mathbf{M}$. Therefore we construct the projection-valued measure as a mapping from the system of Borel subsets $\mathcal{B}(\mathbf{M})$ in $\mathbf{M}$ to the set of orthogonal projectors in Hilbert space $\mathcal{H}$. The projection-valued measure $\mathbf{E}$ is defined as follows:

$$\mathbf{E}: \ \mathcal{B}(\mathbf{M}) \to \mathcal{B}(\mathcal{H}), \tag{2.6}$$

where the following conditions hold:

$$\mathbf{E}(\mathbf{M}) = \widehat{\mathbf{I}}, \tag{2.7}$$

$$\mathbf{E}(S_1 \cap S_2) = \mathbf{E}(S_1)\mathbf{E}(S_2), \tag{2.8}$$

$$\mathbf{E}(S_1 \cup S_2) = \mathbf{E}(S_1) + \mathbf{E}(S_2) - \mathbf{E}(S_1 \cap S_2), \ S_1, S_2 \in \mathcal{B}(\mathbf{M}), \tag{2.9}$$

$$\mathbf{E}(\bigcup_{k=1}^{\infty} S_k) = \sum_{k=1}^{\infty} \mathbf{E}(S_k). \tag{2.10}$$

In (2.10) we assume that subsets $S_i \in \mathcal{B}(\mathbf{M})$ are mutually disjoint. Projectors $\mathbf{E}(S)$ should be also self-adjoint:

$$\mathbf{E}(S) = \mathbf{E}^{\star}(S). \tag{2.11}$$

Here $\mathcal{B}(\mathbf{M})$ is the Borel structure on configuration manifold $\mathbf{M}$, and $\mathcal{B}(\mathcal{H})$ is the set of all bounded operators in Hilbert space $\mathcal{H}$.

The operators of projection-valued measure form, according to [11], an orthocomplemented lattice of projectors on Hilbert space $\mathcal{H}$.

As we can find in [3], the conditions on symmetry transformations of the quantum mechanical description, together with the well known Wigner's theorem, lead to the fact that each element of symmetry group $g \in \mathbf{G}$ is associated with some unitary operator in Hilbert space $\mathcal{H}$. That means that we obtain a mapping $\mathbf{V}$:

$$\mathbf{V} : \mathbf{G} \to \mathcal{U}(\mathcal{H}), \tag{2.12}$$

where $\mathcal{U}(\mathcal{H})$ is the group of all unitary operators in $\mathcal{H}$ with strong topology. Moreover, it is possible to show that mapping $\mathbf{V}$ is a projective representation of symmetry group $\mathbf{G}$ on $\mathcal{H}$. If we define a projective group of Hilbert space $\mathcal{H}$ as a quotient group

$$\mathcal{P}(\mathcal{H}) := \mathcal{U}(\mathcal{H})/\mathcal{Z}, \tag{2.13}$$

where $\mathcal{Z}$ is a center of group $\mathcal{U}(\mathcal{H})$

$$\mathcal{Z} := \{\lambda \cdot \mathbb{I} \in \mathcal{U}(\mathcal{H}) | \lambda \in \mathbf{U}(1)\}, \tag{2.14}$$

and π is the natural projection homomorphism from $\mathcal{U}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H})$, then the conditions on symmetry transformations of the quantum mechanical description lead to finding that the mapping

$$\pi \circ \mathbf{V} : \mathbf{G} \to \mathcal{P}(H) \tag{2.15}$$

is a homomorphism. Therefore $\mathbf{V}$ is a projective representation of $\mathbf{G}$ in $\mathcal{H}$.

However, the mapping $\mathbf{V}$ is not determined uniquely. This is the reason why we define equivalence of two projective representations $\mathbf{V}_1$ and $\mathbf{V}_2$. Representations $\mathbf{V}_1$ and $\mathbf{V}_2$ are equivalent, if there exists a measurable mapping $z : \mathbf{G} \to \mathbf{U}(1)$ such that

$$\mathbf{V}_2(g) = z(g)\mathbf{V}_1(g), \quad \forall g \in \mathbf{G}. \tag{2.16}$$

The equivalence can also be formulated for multipliers of projective representation. Multiplier of a projective representation $\mathbf{V}$ is a measurable mapping

$$m : \mathbf{G} \times \mathbf{G} \to \mathbf{U}(1), \tag{2.17}$$

such that

$$\begin{gathered} m(g_1g_2, g_3)m(g_1, g_2) = m(g_1, g_2g_3)m(g_2, g_3), \\ m(g, e) = m(e, g) = 1, \end{gathered} \tag{2.18}$$

and

$$\mathbf{V}(g_1)\mathbf{V}(g_2) = m(g_1, g_2)\mathbf{V}(g_1g_2). \tag{2.19}$$

We can now formulate the condition of equivalence (2.16) for multipliers. Multipliers $m_1(g)$ and $m_2(g)$ of projective representations $\mathbf{V_1}$ and $\mathbf{V_2}$ are equivalent, if there exists a measurable mapping $z : \mathbf{G} \to \mathbf{U}(1)$ such that

$$m_2(g_1, g_2) = z^{-1}(g_1g_2)z(g_1)z(g_2)m_1(g_1, g_2). \tag{2.20}$$

The set of all multipliers of group $\mathbf{G}$ forms an Abelian group, the corresponding quotient group (with respect to equivalence (2.20)) is called multiplier group for group $\mathbf{G}$. The multiplier group then plays an important role in construction of inequivalent systems of imprimitivity, as we shall see later.

The last result of the conditions on symmetry transformations of the quantum mechanical description is so called imprimitivity condition

$$\mathbf{E}(g \triangleright S) = \mathbf{V}(g)\mathbf{E}(S)\mathbf{V}(g)^{-1}, \quad g \in \mathbf{G}, S \in \mathcal{B}(\mathcal{M}), \tag{2.21}$$

which connects the projection-valued measure $\mathbf{E}$ with projective representation $\mathbf{V}$.

We are now prepared to give the definition of a system of imprimitivity.

Definition: *A pair* $(\mathbf{V}, \mathbf{E})$ *of projective representation* $\mathbf{V}$ *and projection-valued measure* $\mathbf{E}$ *on a configuration manifold* $\mathbf{M}$ *with symmetry group* $\mathbf{G}$ *is called projective system of imprimitivity, if condition (2.21) is fulfilled.*

Two projective systems of imprimitivity $(\mathbf{V}, \mathbf{E})$ and $(\mathbf{V}', \mathbf{E}')$ are called equivalent, if $\mathbf{V}$ is equivalent to $\mathbf{V}'$, and $\mathbf{E} = \mathbf{E}'$.

The last notion that we define is a notion of irreducibility of projective system of imprimitivity. Here we cannot do with the irreducibility of projective representation $\mathbf{V}$ only. We define a commuting ring $\mathcal{C}(\mathbf{V}, \mathbf{E})$ as a set of all bounded operators in

$\mathcal{H}$, which commute at once with all operators of projective representation $\mathbf{V}(g)$ and operators of projection-valued measure $\mathbf{E}(S)$ for all $g \in \mathbf{G}$ and $S \in \mathcal{B}(\mathbf{M})$

$$\mathcal{C}(\mathbf{V},\mathbf{E}) := \{A \in \mathcal{B}(\mathcal{H}) | [A, \mathbf{E}(S)] = [A, \mathbf{V}(g)] = \widehat{0},\ \forall g \in \mathbf{G}, \forall S \in \mathcal{B}(\mathbf{M})\}. \tag{2.22}$$

Projective system of imprimitivity $(\mathbf{V},\mathbf{E})$ is called irreducible if the commuting ring $\mathcal{C}(\mathbf{V},\mathbf{E})$ consists only of multiples of unit operator $\widehat{\mathbf{I}}$.

2.3. **Canonical construction of transitive systems of imprimitivity.** In previous sections we have defined such notions like projective representation of symmetry group $\mathbf{G}$ and projection-valued measure on Borel structure of configuration manifold $\mathbf{M}$. If these two objects fulfil the condition of imprimitivity (2.21), then they form a pair called projective system of imprimitivity. Now we will show, how we can construct the projective system of imprimitivity, if we have configuration manifold $\mathbf{M}$ and its symmetry group $\mathbf{G}$ acting transitively on $\mathbf{M}$. This procedure was introduced by G.W.Mackey in [6], and is called canonical construction of transitive systems of imprimitivity. In this procedure we construct not only the system of imprimitivity, but even the Hilbert space of our quantum mechanical system.

The idea of canonical construction of transitive systems of imprimitivity is based on construction of a projective representation induced from the stability subgroup $\mathbf{H}$. Here we see that for classification of all projective systems of imprimitivity important role is played by the set of all projective unitary representations of stability subgroup $\mathbf{H}$, and also the multiplier group for symmetry group $\mathbf{G}$.

Let the symmetry group $\mathbf{G}$ be a locally compact separable group, $\mathbf{H}$ be a closed subgroup of stability of the transitive $\mathbf{G}$-action on manifold $\mathbf{M}$. On the quotient space $\mathbf{G}/\mathbf{H} (\cong \mathbf{M})$ we can define a quasi-invariant measure μ, that means that measures μ and $\mu \circ g$ are mutually absolutely continuous for any element g of symmetry group $\mathbf{G}$. According to [11], all quasi-invariant σ-finite measures on quotient space $\mathbf{G}/\mathbf{H}$ are mutually absolutely continuous.

Thus we choose a quasi-invariant measure μ, then we choose a projective unitary representation $\mathbf{L}$ of stability subgroup $\mathbf{H}$ with multiplier m (restricted on $\mathbf{H} \times \mathbf{H}$). Projective representation $\mathbf{L}$ acts on some separable Hilbert space $\mathcal{H}^{\mathbf{L}}$. First of all, we construct Hilbert space $\mathcal{H}$, on which the induced representation will act. Hilbert space $\mathcal{H}$ is a space of vector valued functions ψ on symmetry group $\mathbf{G}$ with values in Hilbert space $\mathcal{H}^{\mathbf{L}}$ satisfying conditions

$$\psi : \mathbf{G} \to \mathcal{H}^{\mathbf{L}}, \tag{2.23}$$

the mapping

$$g \mapsto \langle \psi(g), f \rangle \tag{2.24}$$

is a Borel function for all vectors $f \in \mathcal{H}^{\mathbf{L}}$ and $g \in \mathbf{G}$. Next condition is

$$\psi(ah) = m(g,h)\mathbf{L}^{-1}(h)\psi(g),\ \forall h \in \mathbf{H}, \tag{2.25}$$

$$\|\psi\| < \infty. \tag{2.26}$$

Here the norm $\|\cdot\|$ is induced by the inner product on $\mathcal{H}$, which is defined using the inner product $<\cdot,\cdot>$ in Hilbert space $\mathcal{H}^{\mathbf{L}}$, and by the quasi-invariant measure on $\mathbf{G}/\mathbf{H}$

$$(\psi,\varphi) := \int_{\mathbf{G}/\mathbf{H}} <\psi(g),\varphi(g)> d\mu(g). \tag{2.27}$$

It is possible to show, that integral (2.27) is well-defined, because the integrated function $<\psi(g),\varphi(g)>$ is constant on all left cosets $a\mathbf{H}\in\mathbf{G}/\mathbf{H}$.

Now, when we have defined the Hilbert space $\mathcal{H}$, we construct the projection-valued measure using characteristic function of subsets $S\in\mathbf{G}/\mathbf{H}$. The projection-valued measure $\mathbf{E}^{\mathbf{L}}:\mathcal{B}(\mathbf{G}/\mathbf{H})\to\mathcal{B}(\mathcal{H}^{\mathbf{L}})$ is defined by

$$[\mathbf{E}^{L}(S)\psi](g) := \chi_S(g)\psi(g),\ S\in\mathcal{B}(\mathbf{G}/\mathbf{H}),\ g\in\mathbf{G}. \tag{2.28}$$

The characteristic function $\chi_S(g)$ is

$$\chi_S(a) := \begin{cases} 0 & \text{if } g\mathbf{H}\notin S\\ 1 & \text{if } g\mathbf{H}\in S. \end{cases} \tag{2.29}$$

The projective representation $\mathbf{V}^{\mathbf{L}}$ of symmetry group $\mathbf{G}$ is defined by

$$[\mathbf{V}^{L}(g)\psi](a) := \sqrt{\frac{d\mu}{d\mu\circ g}}(g^{-1}a\mathbf{H})\cdot m(a^{-1},g)\cdot\psi(g^{-1}a),\ a,g\in\mathbf{G}. \tag{2.30}$$

Here $d\mu/d\mu\circ g$ is the Radon -Nikodym derivative. Unitarity of projective representation $\mathbf{V}^{\mathbf{L}}$ follows from the unitarity of projective representation $\mathbf{L}$.

The pair $(\mathbf{V}^{\mathbf{L}},\mathbf{E}^{\mathbf{L}})$ fulfills the condition of imprimitivity (2.21), hence it forms a projective system of imprimitivity. The system of imprimitivity $(\mathbf{V}^{\mathbf{L}},\mathbf{E}^{\mathbf{L}})$ is independent of the choice of quasi-invariant measure μ, it depends only on the projective representation $\mathbf{L}$. Finally, we can state the famous Imprimitivity Theorem, first proposed by G. W. Mackey. We can find it for example in [3]. The Imprimitivity Theorem states that any projective transitive system of imprimitivity can be obtained by canonical construction of transitive systems of imprimitivity described above, i.e. the classification of all inequivalent system of imprimitivity is done by classification of all inequivalent projective unitary representations of stability subgroup $\mathbf{H}$.

The Imprimitivity Theorem : *Let $\mathbf{G}$ be a locally compact and separable group, $\mathbf{H}$ its closed subgroup and m a multiplier of $\mathbf{G}$. Let a pair $(\mathbf{V},\mathbf{E})$ be a projective system of imprimitivity for $\mathbf{G}$ based on $\mathbf{G}/\mathbf{H}$ with multiplier m. Then there exists a projective representation $\mathbf{L}$ with multiplier m of subgroup $\mathbf{H}$, such that system of imprimitivity $(\mathbf{V},\mathbf{E})$ is equivalent to the canonically constructed system of imprimitivity $(\mathbf{V}^{\mathbf{L}},\mathbf{E}^{\mathbf{L}})$. For any two projective representations $\mathbf{L}$, $\mathbf{L}'$ of the subgroup $\mathbf{H}$ the corresponding canonical systems of imprimitivity $(\mathbf{V}^{\mathbf{L}},\mathbf{E}^{\mathbf{L}})$, $(\mathbf{V}^{\mathbf{L}'},\mathbf{E}^{\mathbf{L}'})$ are equivalent if and only if $\mathbf{L}$ and $\mathbf{L}'$ are equivalent. The commuting rings $\mathcal{C}(\mathbf{V}^{\mathbf{L}},\mathbf{E}^{\mathbf{L}})$ and $\mathcal{C}(\mathbf{L})$ are isomorphic.*

The last statement that $\mathcal{C}(\mathbf{V}^{\mathbf{L}},\mathbf{E}^{\mathbf{L}})$ and $\mathcal{C}(\mathbf{L})$ are isomorphic means that the system of imprimitivity $(\mathbf{V}^{\mathbf{L}},\mathbf{E}^{\mathbf{L}})$ is irreducible if and only if the projective representation $\mathbf{L}$ of stability subgroup $\mathbf{H}$ is irreducible.

2.4. **Definition of position and momentum operators.** Having the irreducible system of imprimitivity $(\mathbf{V},\mathbf{E})$ and associated Hilbert space $\mathcal{H}$, we define quantum position and momentum observables [11]. To perform that, we take classical position and momentum observables. Classical observable is in general a real measurable function on cotangent bundle $(\mathbf{T}^*\mathbf{M},\pi,\mathbf{M},\mathbb{R}^n)$, where n is a dimension of configuration manifold $\mathbf{M}$.

Classical position observable $\tilde{f}$ is such function on cotangent bundle, which is constant on fiber $\pi^{-1}(m)$ for all $m\in\mathbf{M}$. That means that position observable $\tilde{f}$ is fully determined by a real function f on configuration manifold

$$f:\mathbf{M}\to\mathbb{R}:m\mapsto\tilde{f}(\pi^{-1}(m)). \tag{2.31}$$

The starting point to define a position operator in $\mathcal{H}$ is the projection-valued measure $\mathbf{E}$. We define spectral function

$$\mathbf{E}^f_\lambda:=\mathbf{E}(f^{-1}(\lambda)),\ \lambda\in\mathbb{R}. \tag{2.32}$$

Here $\mathbf{E}$ is the projection-valued measure of our system of imprimitivity $(\mathbf{V},\mathbf{E})$. The self-adjoint position operator, associated to classical position observable, is given by

$$\widehat{Q}^f\psi:=\int_{\mathbb{R}}\lambda d\mathbf{E}^f_\lambda\psi,\ \psi\in\mathcal{H}. \tag{2.33}$$

The domain of position operator $\widehat{Q}^f$ is formed by all functions with finite norm in $\mathcal{H}$

$$\|\widehat{Q}^f\psi\|=\int_{\mathbb{R}}\lambda^2 d(\psi,\mathbf{E}^f_\lambda\psi)<\infty. \tag{2.34}$$

To define momentum operator, we will use the projective unitary irreducible representation $\mathbf{V}$ of the system of imprimitivity $(\mathbf{V},\mathbf{E})$. We assume that the symmetry group $\mathbf{G}$ is a Lie group. We choose an element X from the Lie algebra $\underline{\mathbf{G}}$ associated to the Lie group $\mathbf{G}$. The element $X\in\underline{\mathbf{G}}$ generates one-parameter subgroup $\gamma_X(t)$, $t\in\mathbb{R}$ of symmetry group $\mathbf{G}$

$$\gamma_X(t)=exp(tX). \tag{2.35}$$

This one-parameter subgroup acts on Hilbert space $\mathcal{H}$ through the projective representation $\mathbf{V}$ of the system of imprimitivity $(\mathbf{V},\mathbf{E})$. Since we know that the projective representation $\mathbf{V}$ is unitary, we can use Stone's theorem, which guarantees the existence of self-adjoint operator $\widehat{P}(X)$ defined by

$$\mathbf{V}(\gamma_X(t))=exp(-it\widehat{P}(X)). \tag{2.36}$$

Thanks to the condition of imprimitivity (2.21) we can derive commutation relations for position operator associated with classical observable f, and momentum operator associated with element of the Lie algebra $X\in\underline{\mathbf{G}}$. We just denote $\dot{q}_X$ the vector field on configuration manifold $\mathbf{M}$,which is generated by $X\in\underline{\mathbf{G}}$ through the action of group $\mathbf{G}$ on $\mathbf{M}$. Then the commutation relation is

$$[\widehat{P}(X),\widehat{Q}^f]=-i\widehat{Q}^{\dot{q}_X f}. \tag{2.37}$$

2.5. **Systems of imprimitivity for finite groups.** In this section we will briefly examine the case when the configuration manifold $\mathbf{M}$ and its symmetry group $\mathbf{G}$ are finite [13]. We denote the configuration manifold $\mathbf{M}$

$$\mathbf{M} := \{m_1, m_2, ..., m_n\}, \quad n = |\mathbf{M}| \tag{2.38}$$

Let $\mathbf{G}$ be a finite group acting transitively on $\mathbf{M}$, $\mathbf{H}$ the subgroup of stability. Let $\mathbf{L}$ be a unitary irreducible representation of subgroup $\mathbf{H}$ on Hilbert space $\mathcal{H}^{\mathbf{L}}$. According to [7], we can consider only unitary representations of $\mathbf{H}$, because each representation of finite group on Hilbert space is equivalent to some unitary representation. The irreducibility of representation $\mathbf{L}$ can be assumed thanks to Maschke's theorem: Every unitary reducible representation of a finite group on some Hilbert space is completely reducible [7].

If the stability subgroup $\mathbf{H}$ is Abelian, then all its irreducible representations $\mathbf{L}$ are one-dimensional, and the quantum mechanical Hilbert space $\mathcal{H}^{\mathbf{L}}$ is n-dimensional complex space

$$\mathcal{H}^{\mathbf{L}} := \mathbb{C}^n. \tag{2.39}$$

The definition of projection-valued measure leads to the system of diagonal matrices

$$\mathbf{E}(m_i) := diag(0, 0, ..., 1, ..., 0), \ i = 1, 2, ..., n. \tag{2.40}$$

The only non-vanishing element is at i-th position.

The Imprimitivity Theorem holds in the following form [13]:

Theorem : *A unitary representation $\mathbf{V}$ of finite group $\mathbf{G}$ in Hilbert space $\mathcal{H}$ possesses the imprimitivity system $(\mathbf{V}, \mathbf{E})$ based on $\mathbf{G}/\mathbf{H}$ if and only if $\mathbf{V}$ is equivalent to an induced representation $Ind_{\mathbf{H}}^{\mathbf{G}}(\mathbf{L})$ for some unitary representation $\mathbf{L}$ of subgroup $\mathbf{G}$.*

Hence, the construction of unitary representation $\mathbf{V}$ can be performed directly by the construction of induced representation. This procedure is described for example in [7].

Let $\mathbf{G}$ be a finite group of order r, $\mathbf{H}$ its subgroup of order s. Suppose, that $\mathbf{L}$ is a representation of subgroup $\mathbf{H}$. We can write the group $\mathbf{G}$ by means of left cosets

$$\mathbf{G} = \{\bigcup_{j=1}^{r/s} t_j \cdot \mathbf{H} | t_j \in \mathbf{G}, t_1 = e\}. \tag{2.41}$$

Group elements t_j are arbitrarily chosen representatives of left cosets. If the dimension of representation $\mathbf{L}$ is $l := dim(\mathbf{L})$, then the induced representation $\mathbf{V}$ of group $\mathbf{G}$ is defined by

$$(\mathbf{V}(g))_{ij} := \begin{cases} \mathbf{L}(h) & \text{if} \quad t_i^{-1} \cdot g \cdot t_j = h \text{ for some } h \in \mathbf{H}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.42}$$

Here $(\mathbf{V}(g))_{ij}$ are $l \times l$ matrices which serve as building blocks of

$$\mathbf{V}(g) = Ind_{\mathbf{H}}^{\mathbf{G}}(\mathbf{L}). \tag{2.43}$$

2.6. **Hilbert space as a space of sections.** In previous section, we have constructed the Hilbert space $\mathcal{H}$ as a space of vector functions. We can also consider this Hilbert space as a space of section in associated fiber bundle [13]. This point of view may be in some cases more useful, as for example in the case of systems of imprimitivity on $\mathbf{S}^2$.

We start with the principal fiber bundle $(\mathbf{G}, \pi, \mathbf{M}; \mathbf{H})$, where $\mathbf{M}(\cong \mathbf{G}/\mathbf{H})$ is a homogeneous $\mathbf{G}$-space, $\mathbf{H}$ is the stability subgroup of $\mathbf{G}$ and π is the natural projection

$$\pi : \mathbf{G} \to \mathbf{H} : g \mapsto g\mathbf{H};\ g \in \mathbf{G}. \tag{2.44}$$

In order to construct the associated fiber bundle to principal bundle $(\mathbf{G}, \pi, \mathbf{M}; \mathbf{H})$, we choose representation $\mathbf{L}$ of stability subgroup $\mathbf{H}$. Base space of the associated fiber bundle is configuration manifold $\mathbf{M}$, fiber of the associated bundle is the career Hilbert space, say $\mathcal{H}^{\mathbf{L}}$, of representation $\mathbf{L}$. The total space $E^{\mathbf{L}}$ of associated fiber bundle consists of equivalence classes in $\mathbf{G} \times \mathcal{H}^{\mathbf{L}}$. We define a right action of stability subgroup $\mathbf{H}$ on $\mathbf{G} \times \mathcal{H}^{\mathbf{L}}$

$$(\mathbf{G} \times \mathcal{H}^{\mathbf{L}}) \times \mathbf{H} \to (\mathbf{G} \times \mathcal{H}^{\mathbf{L}}) : ((g, v), h) \mapsto (gh, \mathbf{L}^{-1}(h)v),\ h \in \mathbf{H},\ g \in \mathbf{G},\ v \in \mathcal{H}^{\mathbf{L}}. \tag{2.45}$$

Using this action, we define equivalence relation $\sim_{\mathbf{L}}$ on $\mathbf{G} \times \mathcal{H}^{\mathbf{L}}$ by

$$(g, v) \sim_{\mathbf{L}} (gh, \mathbf{L}^{-1}(h)v), \text{or equivalently } (gh, v) \sim_{\mathbf{L}} (g, \mathbf{L}(h)v). \tag{2.46}$$

This equivalence relation corresponds to condition (2.25). The total $E^{\mathbf{L}}$ space is then defined as quotient space

$$E^{\mathbf{L}} := \mathbf{G} \times \mathcal{H}^{\mathbf{L}} / \sim_{\mathbf{L}}. \tag{2.47}$$

The Hilbert space $\mathcal{H}$ is a set of all Borel sections in the associated fiber bundle $(E^{\mathbf{L}}, \tilde{\pi}, \mathbf{M}; \mathcal{H}^{\mathbf{L}})$, where the projection $\tilde{\pi}$ is given by

$$\tilde{\pi} : E^{\mathbf{L}} \to \mathbf{M} : (g, v) \mapsto \pi(g) = g \cdot \mathbf{H}. \tag{2.48}$$

Mapping π is the projection from the principal bundle $(\mathbf{G}, \pi, \mathbf{M}; \mathbf{H})$.

3. **Coherent states**

The notion of "coherent states" is used for a wide class of quantum mechanical states, and it appears in a huge number of quantum theories. Thanks to this fact, there exists a big number of various approaches to coherent states and their definition. This is also the reason why there does not exist a unique definition of the notion of coherent states. In this section, we shall first summarize several mathematical properties which are, according to [5], common to all definitions of coherent states. Then we give the definition of Perelomov's group related coherent states, which will be our starting point to our approach to coherent states. We shall also briefly mention canonical coherent states on the real line, which is the fundamental notion in the coherent state theory, and also a good starting point for our work.

3.1. **General definition of coherent states.**

In every case the family of coherent states consists of vectors of some separable Hilbert space $\mathcal{H}$, labeled by some parameter $x \in \mathcal{X}$. We denote the coherent states by $|x\rangle \in \mathcal{H}$. The nature of the family of coherent states depends on the approach to coherent states and its application, same as the nature of the label space $\mathcal{X}$. As we shall see, the label space $\mathcal{X}$ will have various algebraic and topological properties, but in general we assume that $\mathcal{X}$ is endowed with a notion of topology; hence $\mathcal{X}$ is a topological space.

There are in essence two properties that are common for all coherent states. The first property is the condition of continuity:

The set of vectors $|x\rangle$ is a strongly continuous function of the label x.

In other words we assume the property

$$\lim_{n\to\infty} x_n = x \text{ in } \mathcal{X} \quad \Rightarrow \| \, |x_n\rangle - |x\rangle \| \to 0 \text{ in } \mathcal{H}. \tag{3.1}$$

Usually the set of states $\{|x\rangle \ \ | \, x \in \mathcal{X}\}$ forms a continuous connected manifold in the Hilbert space $\mathcal{H}$. The coherent states are then very rarely mutually orthogonal. This condition rules out several families of vectors in $\mathcal{H}$, like for example any orthogonal basis of $\mathcal{H}$.

The second property common for all families of coherent states is the resolution of unity:

There exists a positive measure $d\mu(x)$ on $\mathcal{X}$ such that the unit operator $\widehat{I}$ fulfills

$$\widehat{I} = \int_{\mathcal{X}} |x\rangle\langle x| d\mu(x). \tag{3.2}$$

As we know, (3.2) appears like a resolution of unity for self-adjoint operators. However, here the one-dimensional projection operators $|x\rangle\langle x|$ are not in general mutually orthogonal, and the existence of the resolution of unity (3.2) must be verified for each family of coherent states. The resolution of unity then often leads to restrictions on the choice of the label space $\mathcal{X}$ as we shall see in this work.

The direct consequence of the resolution of unity (3.2) is the completeness of the set of coherent states:

The closed linear span of the family of coherent states $\{|x\rangle \ \ | \, x \in \mathcal{X}\}$ is the entire Hilbert space $\mathcal{H}$.

In other words, the family of coherent states is a total set of vectors in Hilbert space $\mathcal{H}$. This condition also means that any vector in Hilbert space $\mathcal{H}$ may be represented as a linear superposition of coherent states. If we realize that the label space may be uncountable, and the Hilbert space $\mathcal{H}$ is separable, as we assumed, then we can raise the question if there exists some (countable) subspace $\mathcal{X}'$ of the (uncountable) label space $\mathcal{X}$ such that the resolution of unity would hold also for $\mathcal{X}'$

$$\widehat{I} = \int_{\mathcal{X}'} |x\rangle\langle x| d\mu'(x) \tag{3.3}$$

This problem was solved for example by Perelomov for canonical coherent states in [10].

The notion of resolution of unity enables us to introduce the functional coherent states representation of the Hilbert space $\mathcal{H}$. Due to (3.2) we have

$$|\psi\rangle = \int_{\mathcal{X}} |x\rangle\langle x|\psi\rangle d\mu(x), \quad |\psi\rangle \in \mathcal{H}. \tag{3.4}$$

The functional representation then consists of complex-valued functions $\psi(x) := \langle x|\psi\rangle$ on the label space $\mathcal{X}$, which are square integrable

$$\langle\psi|\psi\rangle = \int_{\mathcal{X}} |\langle x|\psi\rangle|^2 d\mu(x) = \int_{\mathcal{X}} |\psi(x)|^2 d\mu(x) < \infty. \tag{3.5}$$

Due to (3.2) each admissible function of the coherent states representation satisfies the integral equation

$$\langle x|\psi\rangle = \int_{\mathcal{X}} \langle x|x'\rangle\langle x'|\psi\rangle d\mu(x') \Leftrightarrow \psi(x) = \int_{\mathcal{X}} \mathcal{K}(x,x')\psi(x') d\mu(x'). \tag{3.6}$$

This proposition is called the reproducing kernel property, because the function $\mathcal{K}(x,x')$ on $\mathcal{X} \times \mathcal{X}$ is the (reproducing) kernel of an integral operator, which is equal to the unit operator in the coherent states representation. The reproducing kernel $\mathcal{K}(x,x')$ is given by

$$\mathcal{K}(x,x') := \langle x|x'\rangle. \tag{3.7}$$

Our approach to the construction of coherent states will use the Perelomov definition of coherent states [9], also called group related coherent states, where we assume that label space $\mathcal{X}$ has a group structure:

Let $\mathcal{H}$ be a separable Hilbert space, $\mathbf{G}$ be a group, $T(g)$ be an arbitrary representation of group $\mathbf{G}$ on the Hilbert space $\mathcal{H}$, and $|0\rangle$ be an arbitrary normalized vector in $\mathcal{H}$. Then the set of states $|g\rangle$ defined by

$$|g\rangle := T(g)|0\rangle, \ g \in \mathcal{G} \tag{3.8}$$

is called the system of coherent states related to representation T. The state $|0\rangle$ is then called the vacuum state.

In this definition it is usually assumed that $\mathbf{G}$ is a Lie group, and the notions of irreducibility and unitarity of representation T may be assumed as well.

3.2. **Canonical coherent states on the real line.** In this section we shall briefly describe the canonical coherent states on the real line, which were first considered by Schrödinger in 1926. In quantum mechanics the canonical coherent states describe nonspreading wave packets for the harmonic oscillator. We shall describe them in the sense of previous definition, and we shall show several important properties of these states, which will be one of the aims in our following work. A more detailed description of canonical coherent states can be found for example in [9].

The Hilbert space of quantum mechanics on the real line $\mathbb{R}$ is the well-known $\mathbf{L}^2(\mathbb{R})$, as group $\mathbf{G}$ in the definition (3.8) we consider the complex additive group $\mathbb{C}$. We define a set of unitary operators $\mathbf{D}(\alpha)$ in the Hilbert space $\mathbf{L}^2(\mathbb{R})$

$$\mathbf{D}(\alpha) := exp(\alpha \mathbf{a}^+ - \bar{\alpha}\mathbf{a}),\ \alpha \in \mathbb{C}, \tag{3.9}$$

where $\mathbf{a}^+$ and $\mathbf{a}$ are the creation and annihilation operators defined by position operator $\widehat{\mathbf{Q}}$ and momentum operator $\widehat{\mathbf{P}}$

$$\mathbf{a}^+ := \frac{\widehat{\mathbf{Q}} - i\widehat{\mathbf{P}}}{\sqrt{2}}, \quad \mathbf{a} := \frac{\widehat{\mathbf{Q}} + i\widehat{\mathbf{P}}}{\sqrt{2}}. \tag{3.10}$$

Using the notion of position and momentum operators we can write (3.9) equivalently as

$$\mathbf{D}(\alpha) = exp(i\sqrt{2}Im(\alpha)\widehat{\mathbf{Q}} + i\sqrt{2}Re(\alpha)\widehat{\mathbf{P}}). \tag{3.11}$$

We want to show that the system of operators $\mathbf{D}(\alpha)$ forms a projective representation of the additive group $\mathbb{C}$. To do that we shall use the Baker-Campbell-Hausdorff formula ([9]):

Theorem : *Let operators $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ fulfil the comutation relations*

$$[[\widehat{\mathbf{A}}, \widehat{\mathbf{B}}], \widehat{\mathbf{A}}] = [[\widehat{\mathbf{A}}, \widehat{\mathbf{B}}], \widehat{\mathbf{B}}] = 0, \tag{3.12}$$

then the formula

$$exp(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}) = exp(-\frac{1}{2}[\widehat{\mathbf{A}}, \widehat{\mathbf{B}}])exp(\widehat{\mathbf{A}})exp(\widehat{\mathbf{B}}). \tag{3.13}$$

holds.

Because of (3.13) we obtain the multiplication law for operators $\mathbf{D}(\alpha)$

$$\mathbf{D}(\alpha)\mathbf{D}(\beta) = exp(iIm(\alpha\bar{\beta}))\mathbf{D}(\alpha + \beta),\ \alpha, \beta \in \mathbb{C}, \tag{3.14}$$

and

$$\mathbf{D}(\alpha)\mathbf{D}(\beta) = exp(2iIm(\alpha\bar{\beta}))\mathbf{D}(\beta)\mathbf{D}(\alpha). \tag{3.15}$$

Actually, with the notation

$$p := \sqrt{2}Im(\alpha), \qquad q := \sqrt{2}Re(\alpha), \tag{3.16}$$

equality (3.15) takes the form of the Weyl commutation relation

$$exp(iq\widehat{\mathbf{P}})exp(ip\widehat{\mathbf{Q}}) = exp(ipq)exp(ip\widehat{\mathbf{Q}})exp(iq\widehat{\mathbf{Q}}). \tag{3.17}$$

Thus we see that the set of operators $\mathbf{D}(\alpha)$ forms a unitary projective representation of additive group $\mathbb{C}$ on the Hilbert space $\mathbf{L}^2(\mathbb{R})$. This representation is, according to [9], also irreducible. The fact that the representation $\mathbf{D}(\alpha)$ is projective is essential.

The vacuum state $|0\rangle \in \mathbf{L}^2(\mathbb{R})$ is in this case the ground state of the harmonic oscillator

$$|0\rangle = \frac{1}{\sqrt[4]{\pi}} exp(-\frac{x^2}{2}), \tag{3.18}$$

which is annihilated by the annihilation operator

$$\mathbf{a}|0\rangle = 0. \tag{3.19}$$

The canonical coherent states on $\mathbf{L}^2(\mathbb{R})$ are then defined by the action of the representation $\mathbf{D}(\alpha)$

$$|\alpha\rangle := \mathbf{D}(\alpha)|0\rangle, \ \alpha \in \mathbb{C}. \tag{3.20}$$

The explicit form of canonical coherent states in $\mathbf{L}^2(\mathbb{R})$ is then

$$|\alpha\rangle = \exp(-iRe(\alpha)Im(\alpha)) \exp(i\sqrt{2}Im(\alpha)x) \exp(-(x - \sqrt{2}Re(\alpha))^2). \tag{3.21}$$

Finally, we shall briefly summarize several important properties of canonical coherent states on the real line.

It can be proven that the canonical coherent states are eigenvectors of the annihilation operator $\mathbf{a}$

$$\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{3.22}$$

Also the resolution of unity holds in the form

$$\widehat{\mathbf{I}} = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha |\alpha\rangle\langle\alpha|. \tag{3.23}$$

The next interesting property of canonical coherent states is the fact that their inner product (overlap) never vanishes:

$$\langle\beta|\alpha\rangle = exp(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \bar{\beta}\alpha), \tag{3.24}$$

and

$$|\langle\beta|\alpha\rangle| = exp(-\frac{1}{2}|\alpha - \beta|^2) \neq 0 \ \forall\alpha, \beta \in \mathbb{C}. \tag{3.25}$$

It can be shown that states $|\alpha\rangle$ minimize the Heisenberg uncertainty relations

$$\triangle_{|\alpha\rangle}\widehat{\mathbf{Q}}\triangle_{|\alpha\rangle}\widehat{\mathbf{P}} = \frac{1}{2} \quad \forall\alpha \in \mathbb{C}. \tag{3.26}$$

3.3. **Our approach to the construction of coherent states.** Our approach to the construction of coherent states is based on the notion of irreducible imprimitivity system and also on the formalism of canonical coherent states described above.

We assume the configuration manifold to be a homogeneous $\mathbf{G}$-space of some symmetry group $\mathbf{G}$. Then we shall construct the imprimitivity system in a canonical way and we shall define the quantum position and momentum operators. In the analogy with the canonical coherent states, our aim will be to construct the Weyl operators in the form (3.17). To perform that, we shall employ the projective representation $\mathbf{V}$ of symmetry group $\mathbf{G}$ from the imprimitivity system, and we will define the set of unitary Weyl operators

$$\widehat{\mathbf{W}}(\alpha, g) := e^{i\alpha\widehat{\mathbf{Q}}}\mathbf{V}(g),\ g \in \mathbf{G},\ \alpha \in \mathcal{L}, \tag{3.27}$$

where $\mathcal{L}$ is some label space specified later. If $\mathbf{G}$ is a Lie group, then the relation between representation $\mathbf{V}$ and momentum operator $\widehat{\mathbf{P}}$ is given by the Stone theorem (2.36).

The label space $\mathcal{L}$ will be some subset of real numbers and usually will have a group structure. This set will be specified individually for each case of coherent states and will be defined by the requirement of validity of the resolution of unity. The coherent states will be labeled by the label set $\mathcal{L} \times \mathbf{G}$, which is closely related to the symmetry group $\mathbf{G}$ of the configuration space. The label set $\mathcal{L} \times \mathbf{G}$ will usually be a group.

If the position and momentum operators fulfills the Heisenberg commutation relation

$$[\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}] = i\mathbf{I}, \tag{3.28}$$

then using the Baker-Campbell-Hausdorff formula (3.13) we can show that the family of operators $\widehat{\mathbf{W}}(\alpha, g)$ satisfies the Weyl commutation relation similar to (3.17). Hence the set of operators $\widehat{\mathbf{W}}(\alpha, g)$ will form a projective representation of $\mathcal{L} \times \mathbf{G}$. Then the family of coherent states will be group related.

Strictly said, the system of operators $\widehat{\mathbf{W}}(\alpha, g)$ will form only projective representation of $\mathcal{L} \times \mathbf{G}$. Therefore we have to consider a central extension of group $\mathcal{L} \times \mathbf{G}$. However, this extension will not bring any new state to the resulting set of coherent states. On the other hand, we will see for the case of non-commuting finite dihedral groups that there will not be satisfied the condition (3.17), and the system of operators $\widehat{\mathbf{W}}(\alpha, g)$ will not form a projective representation of $\mathcal{L} \times \mathbf{G}$.

The choice of the vacuum state $|0\rangle$ will be done in analogy with (3.19). If we rewrite (3.19) in exponential form, then using the notion of position and momentum operators we demand

$$e^{\widehat{\mathbf{Q}}+i\widehat{\mathbf{P}}}|0\rangle = |0\rangle. \tag{3.29}$$

In general we will aim to construct our vacuum state by solving this equation. The family of coherent will then be constructed by action of the set of the Weyl operators $\widehat{\mathbf{W}}(\alpha, g)$ on the vacuum state $|0\rangle$.

When the family of coherent states is constructed, we shall investigate several properties satisfied for the canonical coherent states. We shall be interested in the resolution unity and in the Heisenberg uncertainty relation. Also the inner product of two coherent states will be investigated, because we shall try to find out whether the coherent states are mutually orthogonal or not.

Our approach to the construction of coherent states gives us one important advantage. If we consider an irreducible system of imprimitivity, then the notion of irreducibility is transferred to the set of the Weyl operators $\widehat{\mathbf{W}}(\alpha, g)$, i.e. the set of all bounded operators commuting with $\widehat{\mathbf{W}}(\alpha, g)$ for all $\alpha \in \mathcal{L}$ $g \in \mathbf{G}$ is formed only by the multiples of unit operator in the Hilbert space.

4. **Coherent states on dihedral groups**

In this section we will first introduce quantization on a periodic chain. As the configuration space we will consider the Abelian cyclic group $\mathbf{Z_n}$. In paper [14] quantization on $\mathbf{Z_n}$ was introduced, where the symmetry group was identical with the configuration space. Here, we are going to extend the symmetry group to the non-Abelian dihedral group $\mathbf{D_n}$. That means that we shall consider also mirror symmetries.

In the second part of this section construction of coherent states over $\mathbf{Z_n} \times \mathbf{D_n}$ will be introduced. This construction will be an extension of the construction of coherent states introduced in [14].

4.1. **Structure of dihedral groups.**

Before we perform the quantization, it is essential to describe the structure of dihedral groups $\mathbf{D_n}$, where $\mathbf{n} \in \mathbb{N}$. $\mathbf{D_n}$ is a finite group, it is constructed as a semidirect product of two cyclic groups:

$$\mathbf{D_n} = \mathbf{Z_n} \triangleright \mathbf{Z_2}. \tag{4.1}$$

It is straightforward that the number of elements of $\mathbf{D_n}$ is equal to $2n$. The elements of the group $\mathbf{Z_2}$ are denoted as $+1$ (unit element in $\mathbf{Z_2}$) and -1, elements of $\mathbf{Z_n}$ will be denoted as $r_i,\ i = 0, 1, ...n-1$.

$$\mathbf{Z_2} = \{+1, -1\}; \quad \mathbf{Z_n} = \{e = r_0, r_1, ..., r_{n-1}\}. \tag{4.2}$$

Group operation in $\mathbf{Z_2}$ is addition modulo 2, in $\mathbf{Z_n}$ modulo n.

Group $\mathbf{D_n}$ is thus a set of pairs of elements (a, x), where $a \in \mathbf{Z_n}$ and $x \in \mathbf{Z_2}$. The multiplication law of the semidirect product of two groups is determined by a fixed homomorphism, let be denoted f, from group $\mathbf{Z_2}$ to group of all automorphisms of the group $\mathbf{Z_n}$:

$$f : \mathbf{Z_2} \to Aut(\mathbf{Z_n}). \tag{4.3}$$

The group multiplication law has then the following form:

$$(r_i, x) \cdot (r_j, y) = (r_i \cdot f(x)(r_j), x \cdot y),\ x, y \in \mathbf{Z_2},\ r_i, r_j \in \mathbf{Z_n}. \tag{4.4}$$

Let us now specify the mapping f, so we can define the multiplication law in $\mathbf{D_n}$:

$$f : (+1) \mapsto Id, \quad f : (-1) \mapsto Inv, \tag{4.5}$$

where Id is an identical mapping on $\mathbf{Z_n}$, Inv is an automorphism on $\mathbf{Z_n}$, which maps element of $\mathbf{Z_n}$ into its inverse element:

$$Inv : a \mapsto a^{-1}, \qquad a \in \mathbf{Z_n}. \tag{4.6}$$

The multiplication law (4.4) has thus the following explicit form:

$$(r_i, +1) \cdot (r_j, x) = (r_i \cdot r_j, x) = (r_{(i+j)_{mod(n)}}, x), \tag{4.7}$$

$$(r_i, -1) \cdot (r_j, x) = (r_i \cdot (r_j)^{-1}, x) = (r_{(i-j)_{mod(n)}}, x). \tag{4.8}$$

Let us now look at the geometrical interpretation of elements of $\mathbf{D_n}$. Performing that, it will be clear also how does the transitive action of $\mathbf{D_n}$ on $\mathbf{Z_n}$ looks like.

We can divide the group $\mathbf{D_n}$ in two subsets. First subset $\{(r_i, +1),\ i = 0, 1, ..., n-1\}$ is in fact a subgroup of $\mathbf{D_n}$ isomorphic to the group $\mathbf{Z_n}$. Therefore we can consider an element $(r_i, +1)$ as a clockwise rotation of an n-sided regular polygon through an angle of $2\pi/n$. The second subset of $\mathbf{D_n}$ is formed by elements $(r_i, -1)$, $i = 0, 1, ..., n-1$. Geometrical interpretation of these elements is a little more difficult. The group element $(r_i, -1)$ can be interpreted as a mirror symmetry, where the mirror is at angle π/n in the n-sided polygon. If n is odd, then each axis of some mirror symmetry passes through one vertex of the n-sided polygon, if n is even, then only one half of all mirror symmetries has an axis passing through two opposite vertices, the remaining axes are axes of two opposite sides of the polygon.

It is straightforward that the group $\mathbf{D_n}$ contains n rotation symmetries and n mirror symmetries. That's why we can set up a new notation of these group elements. For rotation symmetries we will use the symbol $\mathbf{R}_i$, $i = 0, 1, ..., n-1$; mirror symmetries will be denoted $\mathbf{M}_i$, $i = 0, 1, ..., n-1$. Now we have

$$\mathbf{R}_i := (r_i, +1),\ i = 0, 1, ..., n-1, \tag{4.9}$$

$$\mathbf{M}_i := (r_i, -1),\ i = 0, 1, ..., n-1. \tag{4.10}$$

This denotation seems to be more distinct thanks to its direct geometrical significance. The multiplication law has now the following form:

$$\mathbf{R_i} \cdot \mathbf{R_j} = \mathbf{R_{(i+j)_{mod(n)}}},\ i, j = 0, 1, ..., n-1, \tag{4.11}$$

$$\mathbf{R_i} \cdot \mathbf{M_j} = \mathbf{M_{(i+j)_{mod(n)}}},\ i, j = 0, 1, ..., n-1, \tag{4.12}$$

$$\mathbf{M_i} \cdot \mathbf{R_j} = \mathbf{M_{(i-j)_{mod(n)}}},\ i, j = 0, 1, ..., n-1, \tag{4.13}$$

$$\mathbf{M_i} \cdot \mathbf{M_j} = \mathbf{R_{(i-j)_{mod(n)}}},\ i, j = 0, 1, ..., n-1. \tag{4.14}$$

Equations (4.11) - (4.14) are direct consequence of multiplication law (4.7), (4.8), and also definition relations (4.9) and (4.10).

4.2. **Quantization on $\mathbf{Z_n}$ with $\mathbf{D_n}$ as a symmetry group.** The action of symmetry group $\mathbf{D_n}$ on configuration space $\mathbf{Z_n}$ will be clear, if we identify the $\mathbf{Z_n}$ phase space with set of all vertices of regular n-sided polygon. The $\mathbf{D_n}$ acts on $\mathbf{Z_n}$ as a group of rotations and mirror symmetries, as discussed in previous section. Transitivity of this action is straightforward.

Thanks to transitivity of the group action, we can find the stability subgroup $\mathbf{H_n}$ in $\mathbf{D_n}$:

$$\mathbf{H_n} = \mathbf{Z_2} \quad \forall n \in \mathbb{N}. \tag{4.15}$$

Hence we can write

$$\mathbf{Z_n} \cong \mathbf{D_n}/\mathbf{Z_2}. \tag{4.16}$$

Note that the stability subgroup is independent of the order of symmetry group $\mathbf{D_n}$. Number of all inequivalent quantum mechanics on $\mathbf{Z_n}$ is determined by the number of all inequivalent unitary irreducible representation of the group of stability $\mathbf{Z_2}$. $\mathbf{Z_2}$ is an Abelian group, therefore it has only one-dimensional irreducible representations, and the number of all irreducible representation is then equal to the order of $\mathbf{Z_2}$. $\mathbf{Z_2}$

has two elements, so we have two inequivalent irreducible representations of $\mathbf{Z_2}$ and two inequivalent systems of imprimitivity on $\mathbf{Z_n}$ with the symmetry group $\mathbf{D_n}$. The first representation is trivial:

$$\mathbf{T_1} : \mathbf{Z_2} \to \mathbb{R} : \{+1, -1\} \mapsto \{+1, +1\}, \tag{4.17}$$

the second one is the so called alternating representation, and it is in fact the defining representation of $\mathbf{Z_2}$:

$$\mathbf{T_2} : \mathbf{Z_2} \to \mathbb{R} : \{+1, -1\} \mapsto \{+1, -1\}. \tag{4.18}$$

In both cases the Hilbert space of quantum mechanics is the space of complex functions on the configuration space $\mathbf{Z_n}$. This Hilbert space is in fact isomorphic to n-dimensional complex linear space:

$$\mathcal{H}_\mathbf{n} \cong \mathbb{C}^n. \tag{4.19}$$

System of imprimitivity is a pair $(\mathbf{E}, \mathbf{V})$, where $\mathbf{V}$ is a unitary representation of the symmetry group $\mathbf{D_n}$. This representation is constructed as a representation induced from the subgroup of stability $\mathbf{Z_2}$. In our case, it is induced from irreducible representations of $\mathbf{T_1}$ and $\mathbf{T_2}$. $\mathbf{E}$ is a projection-valued measure on configuration space $\mathbf{Z_n}$. First, we will construct the projection-valued measure. This measure is common for both representations $\mathbf{T_1}$ and $\mathbf{T_2}$, and moreover it is in fact the same measure as constructed in [13] in section 5.7.

If we follow the denotation (4.2) of group elements of $\mathbf{Z_n}$, then the generators of projection-valued measure have the following form:

$$\mathbf{E}(r_i) = \begin{array}{cc} & \begin{array}{ccccc} & & i+1 & & \end{array} \\ i+1 & \begin{pmatrix} & & & & \\ & & \cdot & & \\ \cdot & \cdot & 1 & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix} \end{array}, \quad i = 0, 1, ..., n-1. \tag{4.20}$$

$\mathbf{E}(r_i)$ is an $n \times n$ diagonal matrix with only one nonvanishing element in $(i+1)$-th row and $(i+1)$-th column. This set of operators on $\mathbb{C}^n$ generates a set of orthogonal projectors on this space — the projection-valued measure. Measure of an empty set in $\mathbf{Z_n}$ is the vanishing operator on $\mathbb{C}^n$, measure of the whole configuration space is the unit operator. Each of the operators of the projection-valued measure has diagonal form.

Construction of unitary representations of the symmetry group $\mathbf{D_n}$ on Hilbert space $\mathbb{C}_n$ is a little more difficult than construction of projection-valued measure. According to Mackey's imprimitivity theorem, the representation $\mathbf{V}$ is constructed as representation induced from some irreducible representation of the stability subgroup. In our case, we have stability subgroup $\mathbf{Z_2}$ and its two irreducible representations $\mathbf{T_1}$ and $\mathbf{T_2}$, so we shall construct two induced representations on $\mathbb{C}^n$, say $\mathbf{V_1}$ and $\mathbf{V_2}$:

$$\mathbf{V_1} = Ind_{\mathbf{Z_2}}^{\mathbf{D_n}}(\mathbf{T_1}), \qquad \mathbf{V_2} = Ind_{\mathbf{Z_2}}^{\mathbf{D_n}}(\mathbf{T_2}). \tag{4.21}$$

Construction of representations induced from groups of finite order is described for example in [7]. First, we have to decompose the symmetry group $\mathbf{D_n}$ into left cosets:

$$\mathbf{D_n} = \{\bigcup_{m=1}^{n} t_m \cdot \mathbf{Z_2} | t_m \in \mathbf{D_n},\ t_1 = e.\} \tag{4.22}$$

In our case we have

$$\mathbf{Z_2} = \{\mathbf{R}_0,\, \mathbf{M}_0\}; \qquad \mathbf{Z_2} \subset \mathbf{D_n}, \tag{4.23}$$

and as representatives t_m of the cosets we choose

$$t_m = \mathbf{R_{m-1}},\; m = 0, 1, ..., n-1. \tag{4.24}$$

Now we have

$$\mathbf{D_n} = \{\{\mathbf{R}_0, \mathbf{M}_0\} \cup \{\mathbf{R}_1, \mathbf{M}_1\} \cup \,...\, \cup \{\mathbf{R}_{n-1}, \mathbf{M}_{n-1}\}\}. \tag{4.25}$$

Matrices of induced representation are constructed in the following way: dimension of both representations $\mathbf{V_1}$ and $\mathbf{V_2}$ is n:

$$dim(\mathbf{V_i}) = \frac{|\mathbf{D_n}|}{|\mathbf{Z_2}|} \cdot dim(\mathbf{T_i}) = \frac{2n}{2} \cdot 1 = n, \quad i = 1, 2, \tag{4.26}$$

and elements of these matrices have the following form:

$$(V_i(g))_{ij} = \begin{cases} \mathbf{T_i(h)} & \text{if } t_i^{-1} \cdot g \cdot t_j = h \text{ for some } h \in \mathbf{Z_2}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

In our case $t_i = \mathbf{R}_{i-1}$, so the matrix element $(\mathbf{V}_i(g)_{ij})$ does not vanish if and only if the following equality holds:

$$\mathbf{R}_{(1-i)_{modn}} \cdot g \cdot \mathbf{R}_{(j-1)_{mod(n)}} \in \{\mathbf{R}_0,\, \mathbf{M}_0\}. \tag{4.28}$$

Let us now construct the induced representation $\mathbf{V}_1$, first for rotations $\mathbf{R}_k$. Condition (4.28) then has the form

$$\mathbf{R}_{(1-i)_{modn}} \cdot \mathbf{R}_k \cdot \mathbf{R}_{(j-1)_{modn}} = \mathbf{R}_{(-i+j+k)_{modn}} \in \{\mathbf{R}_0,\, \mathbf{M}_0\} \Leftrightarrow (j+k)_{mod(n)} = i. \tag{4.29}$$

The matrix element (4.27) of rotation $\mathbf{R_k}$ is then

$$(\mathbf{V_1}(\mathbf{R}_k))_{ij} = \delta_{i,(j+k)_{mod(n)}}, \tag{4.30}$$

and the entire matrix is

$$\mathbf{V_1}(\mathbf{R}_k) = \begin{array}{c} \\ \\ \\ \\ \\ \\ {\scriptstyle k+1} \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccccc} &&&&&{\scriptstyle k+1}&&&& \end{array} \\ \begin{pmatrix} &&&&& 1 &&&& \\ &&&&&& 1 &&& \\ &&&&&&& \cdot && \\ &&&&&&&& \cdot & \\ &&&&&&&&& 1 \\ 1 &&&&&&&&& \\ & 1 &&&&&&&& \\ && \cdot &&&&&&& \\ &&& \cdot &&&&&& \\ &&&& 1 &&&&& \end{pmatrix} \end{array} \tag{4.31}$$

It is evident that $\mathbf{V}_1(\mathbf{R}_0) = \widehat{\mathbf{I}}$, representation of $\mathbf{R}_k$ is then just the unit matrix with shifted diagonal.

Consider now the representation of mirror symmetries $\mathbf{M}_k$. Due to (4.11) - (4.14), the condition (4.28) acquires now this form:

(4.32)
$$\mathbf{R}_{(1-i)_{mod(n)}} \cdot \mathbf{M}_k \cdot \mathbf{R}_{(j-1)_{modn}} = \mathbf{M}_{(2-i-j+k)_{mod(n)}} \in \{\mathbf{R}_0,\, \mathbf{M}_0\} \Leftrightarrow i = k - j + 2.$$

The matrix element (4.27) of mirror symmetry $\mathbf{M}_k$ is

$$(\mathbf{V_1}(\mathbf{M_k}))_{ij} = \delta_{i,-j+k+2}. \tag{4.33}$$

The matrix $\mathbf{V_1}(\mathbf{M}_k)$ has this explicit form:

$$\mathbf{V_1}(\mathbf{M_k}) = \left(\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & \cdot & & & & & \\ {\scriptstyle k+1} & & 1 & & & & & & \\ & \cdot & & & & & & & \\ 1 & & & & & & & & \\ & & & & & & & & 1 \\ & & & & & & & \cdot & \\ & & & & & & \cdot & & \\ & & & & & 1 & & & \end{array} \right) \tag{4.34}$$

Representation of mirror symmetries has also antidiagonal form, the antidiagonal intersects the diagonal in the (k+1)-th row and the (k+1)-th column. The second representation $\mathbf{V_2}$ can be obtained similarly, but it is straightforward that representation $\mathbf{V_2}$ is connected with representation $\mathbf{V_1}$ in the following way:

$$\mathbf{V_2}(\mathbf{R_k}) = \mathbf{V_1}(\mathbf{R_k}),\ \mathbf{V_2}(\mathbf{M_k}) = -\mathbf{V_1}(\mathbf{M_k}). \tag{4.35}$$

It is easy to show that both these representations $\mathbf{V_1}$ and $\mathbf{V_2}$ are unitary.

We have shown that there exist two unitary systems of imprimitivity $(\mathbf{E}, \mathbf{V}_1)$ and $(\mathbf{E}, \mathbf{V_2})$ on phase space $\mathbf{Z_n}$ with symmetry group $\mathbf{D_n}$. These two systems are not equivalent. As carrier Hilbert space we have in both cases the complex n-dimensional space $\mathbb{C}^n$.

4.3. **Quantum observables.** The definition of position operator is based on knowledge of projection-valued measure and, of course, on classical position observable. Classical position observable is in general a Borel mapping from configuration space, in our case from $\mathbf{Z_n}$, to the set of real numbers. We shall choose the following position observable:

$$f : \mathbf{Z_n} \to \mathbb{R} : r_i \mapsto i, \quad i = 0, 1, ..., n-1. \tag{4.36}$$

Quantization of position observable is then realized as follows. Position operator $\widehat{\mathbf{Q}}^f$ associated to classical position observable f has the integral form:

$$\widehat{\mathbf{Q}}^f := \int_{\mathbb{R}} \lambda d\mathbf{E}(\mathbf{f^{-1}}(\lambda)). \tag{4.37}$$

In our case the configuration space is discrete, therefore we have

$$\widehat{\mathbf{Q}} := \sum_{k=0}^{n-1} k \cdot \mathbf{E}(r_k). \tag{4.38}$$

Here we obtain, in analogy with [14], this explicit form of position operator:

$$\widehat{\mathbf{Q}} = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & n-1 \end{pmatrix} \tag{4.39}$$

This construction of position operator is independent of representations $\mathbf{V_1}$,$\mathbf{V_2}$ constructed in previous section, therefore the position operator $\widehat{\mathbf{Q}}$ is common for both quantum mechanics.

The construction of momentum operator raises several problems. First, existence of the momentum operator is guaranteed by Stone's theorem, but under the assumed existence of one-parameter subgroup of symmetry group. We have a discrete symmetry group $\mathbf{D_n}$. However, the momentum operator can be evaluated by direct computation, but as we shall see, this operator is not uniquely determined.

Stone's theorem says, that if we have a one-parameter subgroup $\gamma(t)$ of a symmetry group, then there exists a self-adjoint operator $\widehat{\mathbf{P}}$, such that the following condition holds:

$$\mathbf{V}(\gamma(t)) = exp(-it\widehat{\mathbf{P}}),\ t \in \mathbb{R}. \tag{4.40}$$

In the following, we shall look for self-adjoint operators $\widehat{\mathbf{P}}_g$ on $\mathbb{C}^n$ such that

$$\mathbf{V_i}(g) = exp(-i\widehat{\mathbf{P}}_g), \quad i = 1, 2,\ g \in \mathbf{D_n}. \tag{4.41}$$

From now on we shall consider only the representation $\mathbf{V_1}$. Results for representation $\mathbf{V_2}$ and for the second system of imprimitivity will be similar and will be discussed later.

We may try to compute the operators $\widehat{\mathbf{P}}_g$ by inverting the *exp* function in (4.41). We get

$$\widehat{\mathbf{P}}_g = i \cdot ln(\mathbf{V_1}(g)). \tag{4.42}$$

The problem is that the exponential function is not invertible, and therefore the operator $\widehat{\mathbf{P}}_g$ will not be determined uniquely.

To compute matrix functions we shall use the so-called Lagrange-Sylvester formula:

Theorem Let $\mathbb{A}$ be an $n \times n$ matrix with spectrum $\sigma(\mathbb{A}) = \{\lambda_1, \lambda_2, ..., \lambda_s\}$, $s \leq n$. Let q_j be the order of eigenvalue λ_j, $j = 1, 2, ..., s$.Let $\Omega \subset \mathbb{C}$ be an open subset of the complex plane such that $\sigma(\mathbb{A}) \subset \mathbb{C}$. Then the formula

$$f(\mathbb{A}) = \sum_{j=1}^{s} \sum_{k=0}^{q_j - 1} \frac{f^{(k)}(\lambda_j)}{k!} (\mathbb{A} - \lambda_j \mathbb{I})^k \mathbb{P}_j \tag{4.43}$$

holds for every function f holomorphic on Ω. Here $\mathbb{P}_j$ is the orthogonal projector onto the subspace of $\mathbb{C}^n$ which is generated by the set of all eigenvectors with eigenvalue λ_j:

$$\mathbb{P}_j := \prod_{l=1, l \neq j}^{s} \frac{\lambda_l \mathbb{I} - \mathbb{A}}{\lambda_l - \lambda_j}. \tag{4.44}$$

Now we will apply formula (4.43) to equation (4.42) and will try to evaluate operators $\widehat{\mathbf{P}}_g$ in this way. But first of all, we have to calculate the spectrum of operators $\mathbf{V_1}(\mathbf{R_i})$ and $\mathbf{V_1}(\mathbf{M}_i)$, $i = 0, 1, ..., n-1$.

Let us first look at eigenvalues of operator $\mathbf{V_1}(\mathbf{R_1})$. In other words we solve the equation

$$det(\lambda\mathbb{I} - \mathbf{V_1}(\mathbf{R_1})) = 0. \tag{4.45}$$

After the computation we obtain that (4.45) is equivalent to solving

$$\lambda^n - 1 = 0. \tag{4.46}$$

Hence we have

$$\sigma(\mathbf{V_1}(\mathbf{R_1})) = \{\lambda_j = e^{\frac{2\pi ij}{n}} | j = 0, 1, ..., n-1\}. \tag{4.47}$$

The spectra of operators $\mathbf{V_1}(\mathbf{R_i})$ can be obtained from the spectrum of operator $\mathbf{V_1}(\mathbf{R_1})$:

$$\sigma(\mathbf{V_1}(\mathbf{R_k})) = \sigma(\mathbf{V_1}((\mathbf{R_1})^k)) = (\sigma(\mathbf{V_1}(\mathbf{R_1})))^k. \tag{4.48}$$

Similarly we obtain spectra of operators $\mathbf{V_1}(\mathbf{M_i})$. We shall now solve

$$det(\lambda\mathbb{I} - \mathbf{V_1}(\mathbf{M_i})) = 0. \tag{4.49}$$

It is now necessary to consider separately two cases. First, if n is odd, then (4.49) becomes

$$(1-\lambda)(\lambda^2 - 1)^{\frac{n-1}{2}} = 0. \tag{4.50}$$

In this case the spectrum has two elements:

$$\sigma(\mathbf{V_1}(\mathbf{M_k})) = \{-1, +1\}, \tag{4.51}$$

where the order of eigenvalue $+1$ is $\frac{n+1}{2}$, the order of eigenvalue -1 is equal to $\frac{n-1}{2}$.

If n is even, then the characteristic polynomial of operator $\mathbf{V_1}(\mathbf{R_k})$ depends, in addition to dimension n, also on parameter k. At this point, we also must distinguish if k is odd or even. In the analogy with the geometrical interpretation of group elements of $\mathbf{D_n}$, we have to distinguish if the axis of mirror symmetry $\mathbf{M}_k$ passes through vertices of the n-sided regular polygon (k is even), or if it is an axis of two opposite sides of the regular polygon (k is odd).

So if n is even then (4.49) has following form:

$$0 = \begin{cases} (1-\lambda)^{\frac{n}{2}+1}(1+\lambda)^{\frac{n}{2}-1} & \text{if } k \text{ is even,} \\ (1-\lambda)^{\frac{n}{2}}(1+\lambda)^{\frac{n}{2}} & \text{if } k \text{ is odd.} \end{cases} \tag{4.52}$$

The spectra for both cases are the same as if n is odd, but the orders of eigenvalues are different. If k is even, the order of eigenvalue $+1$ is $\frac{n}{2}+1$, the order of eigenvalue -1 is $\frac{n}{2}-1$. If k is odd, then the order of both eigenvalues is $\frac{n}{2}$. To be able to use the Lagrange-Sylvester formula (4.43), we have to determine the projectors $\mathbb{P}_k$ for each representation element $\mathbf{V_1}(\mathbf{g})$, $g \in \mathbf{D_n}$. This will be performed during particular computation of operators $\widehat{\mathbf{P}}_g$.

Let us first start with evaluating the operators $\widehat{\mathbf{P}}_{\mathbf{R}_k}$. Here is the situation quite simple, because elements $\mathbf{R}_k$ form an Abelian subgroup $\mathbf{Z_n}$ of the group $\mathbf{D_n}$. That is the reason why the following equality holds:

$$exp(-i\widehat{\mathbf{P}}_{\mathbf{R}_k}) = \mathbf{V_1}(\mathbf{R_k}) = (\mathbf{V_1}(\mathbf{R_1}))^k = exp(-ik\widehat{\mathbf{P}}). \tag{4.53}$$

Here we have defined

$$\widehat{\mathbf{P}} := \widehat{\mathbf{P}}_{\mathbf{R}_1}. \tag{4.54}$$

This is an advantage, because we are able to replace a system of n operators by one self-adjoint operator $\widehat{\mathbf{P}}$. Spectrum of operator $\mathbf{V_1}(\mathbf{R}_1)$ is determined in (4.48), it remains only to find projectors $\mathbb{P}_j$. Naturally, we may use formula (4.44), but it would be quite difficult. It is more effective to find the system of eigenvectors $|j\rangle$ of operator $\mathbf{V_1}(\mathbf{R}_1)$. Because we have n different eigenvalues $\lambda_j = e^{\frac{2\pi ij}{n}}$, order of each eigenvalue is equal to 1, and projectors $\mathbb{P}_j$ can be written in this simple form:

$$\mathbb{P}_j = |k\rangle\langle k|. \tag{4.55}$$

It is easy to see that the explicit form of eigenvectors $|k\rangle$ is

$$|k\rangle = \frac{1}{\sqrt{n}} \begin{pmatrix} \lambda_k^{n-1} \\ \lambda_k^{n-2} \\ \cdot \\ \cdot \\ \lambda_k \\ 1 \end{pmatrix}, \tag{4.56}$$

where λ_k is eigenvalue pertinent to eigenvector $|k\rangle$. These eigenvectors are normalized:

$$\langle k|k\rangle = 1. \tag{4.57}$$

Using (4.55), it is straightforward that matrix elements of projector $\mathbb{P}_k$ can be written as

$$(\mathbb{P}_k)_{lm} = \frac{1}{n}\lambda_k^{n-l}\overline{\lambda_k^{n-m}} = \frac{1}{n}e^{\frac{2\pi ik(m-l)}{n}}. \tag{4.58}$$

Now, we have all prepared for direct evaluation of operator $\widehat{\mathbf{P}}$. According to (4.43), we may write

$$\begin{aligned} (\widehat{\mathbf{P}})_{lm} &= i \cdot \sum_{j=0}^{n-1} ln(e^{\frac{2\pi ij}{n}})\frac{1}{n}e^{\frac{2\pi ij(m-l)}{n}} \\ (\widehat{\mathbf{P}})_{lm} &= -\frac{2\pi}{n^2}\sum_{j=0}^{n-1} je^{\frac{2\pi ij(m-l)}{n}} \\ (\widehat{\mathbf{P}})_{lm} &= \frac{2\pi}{n}\frac{1}{1-e^{\frac{2\pi i(m-l)}{n}}}. \end{aligned} \tag{4.59}$$

But this result is valid only for the case $m \neq n$. For the other case the computation is more simple:

$$
\begin{aligned}
(4.60) \qquad (\widehat{\mathbf{P}})_{lm} &= -\frac{2\pi}{n^2}\sum_{j=0}^{n-1} j \\
(\widehat{\mathbf{P}})_{lm} &= -\frac{2\pi}{n}\frac{(n-1)}{2}.
\end{aligned}
$$

Thus we conclude with

$$
(4.61) \qquad (\widehat{\mathbf{P}})_{lm} = \begin{cases} \frac{2\pi}{n}\frac{1}{1-e^{\frac{2\pi i(m-l)}{n}}} & m \neq l, \\ -\pi\frac{n-1}{n} & m = l. \end{cases}
$$

The same result was obtained in [14]. There the momentum operator $\widehat{\mathbf{P}}$ was obtained by Fourier transform of position operator $\widehat{\mathbf{Q}}$. This way of computation is a little more elegant, but in our case we shall compute also the explicit form of operators $\widehat{\mathbf{P}}_{\mathbf{M}_k}$ of mirror symmetries.

In the case of mirror symmetry, we have to use the Lagrange-Sylvester formula (4.43). Since equation (4.42) gives us

$$
(4.62) \qquad \widehat{\mathbf{P}}_{\mathbf{M}_k} = i \cdot ln(\mathbf{V_1}(\mathbf{M}_k)),
$$

we can put down, due to (4.43) and (4.51), the following equality:

(4.63)

$$
\widehat{\mathbf{P}}_{\mathbf{M}_k} = i \cdot \sum_{j=0}^{q_{(+)}-1} \frac{ln^{(j)}(+1)}{j!}(\mathbf{V_1}(\mathbf{M_k}) - \mathbb{I})^k \widehat{\mathbb{P}}_{+1} + i \cdot \sum_{j=0}^{q_{(-)}-1} \frac{ln^{(j)}(-1)}{j!}(\mathbf{V_1}(\mathbf{M_k}) + \mathbb{I})^k \widehat{\mathbb{P}}_{-1}.
$$

Strictly said, here the assumption of the Lagrange-Sylvester formula (4.43) is not satisfied, because function ln is not holomorphic on the negative part of the real line $(-\infty, 0\rangle$, and -1 is in the spectrum of $\mathbf{V_1}(\mathbf{M_k})$. However, we shall see that the computation of operator $\widehat{\mathbf{P}}_{\mathbf{M}_k}$ is possible to perform in a formal way. Then we have to verify validity of (4.41) using again (4.43). Here function *exp* is holomorphic, thus the problem will not arise.

Numbers $q_{(+)}$ and $q_{(-)}$ are multiplicities of eigenvalues $+1$ and -1, as was discussed before. We will show that the explicit value of these two numbers is not important for following computation. It is necessary now to find projectors $\widehat{\mathbb{P}}_{+1}$ and $\widehat{\mathbb{P}}_{-1}$. These operators are orthogonal projectors on subspaces generated by eigenvectors of eigenvalues $+1$ and -1. Because of the low number of eigenvalues, it will be essential to use directly definition of these operators (4.44):

$$
\begin{aligned}
(4.64) \qquad \widehat{\mathbb{P}}_{+1} &= \frac{(\mathbf{V_1}(\mathbf{M_k}) + \mathbb{I})}{2}, \\
\widehat{\mathbb{P}}_{-1} &= -\frac{(\mathbf{V_1}(\mathbf{M_k}) - \mathbb{I})}{2}.
\end{aligned}
$$

If we put this into equality (4.63), we obtain

$$\widehat{\mathbf{P}}_{\mathbf{M}_k} = i \cdot \sum_{j=0}^{q_{(+)}-1} \frac{ln^{(j)}(+1)}{j!}(\mathbf{V_1}(\mathbf{M_k}) - \mathbb{I})^k \frac{(\mathbf{V_1}(\mathbf{M_k}) + \mathbb{I})}{2} - \tag{4.65}$$
$$-i \cdot \sum_{j=0}^{q_{(-)}-1} \frac{ln^{(j)}(-1)}{j!}(\mathbf{V_1}(\mathbf{M_k}) + \mathbb{I})^k \frac{(\mathbf{V_1}(\mathbf{M_k}) - \mathbb{I})}{2}.$$

This equation will be more simple, if we realize that thanks to the group multiplication law the elements $\mathbf{V_1}(\mathbf{M_k})$ are nilpotent elements. Now we can write

$$(\mathbf{V_1}(\mathbf{M_k}) - \mathbb{I})(\mathbf{V_1}(\mathbf{M_k}) + \mathbb{I}) = (\mathbf{V_1}(\mathbf{M_k}))^2 - \mathbb{I} = \widehat{0}. \tag{4.66}$$

This means that all elements in sum (4.65) vanish, excluding the case $j = 0$! Now we have

$$\begin{aligned} \widehat{\mathbf{P}}_{\mathbf{M}_k} &= i \cdot (\frac{ln(+1)}{2}(\mathbf{V_1}(\mathbf{M_k}) + \mathbb{I}) - \frac{ln(-1)}{2}(\mathbf{V_1}(\mathbf{M_k}) - \mathbb{I})) \\ \widehat{\mathbf{P}}_{\mathbf{M}_k} &= \frac{\pi}{2}(\mathbf{V_1}(\mathbf{M_k}) - \mathbb{I}). \end{aligned} \tag{4.67}$$

So far we considered only representation $\mathbf{V_1}$. We may also ask how the generators of mirror symmetries $\widehat{\mathbf{P}}_{\mathbf{M}_k}$ and generator of rotation $\widehat{\mathbf{P}}$, obtained in (4.61) and (4.67), look like, if we consider the second quantum mechanics, i.e. if we consider representation $\mathbf{V_2}$. The answer will be clear, if we use relations (4.35). Then we see that generator of rotation $\widehat{\mathbf{P}}$ has the same form as in (4.61). Derivation of generators of mirror symmetries for the representation $\mathbf{V}_2$ seems to be a little more complicated. But if we compute spectra of operators $\mathbf{V_2}(\mathbf{M_k})$, we find $\sigma(\mathbf{V_2}(\mathbf{M_k})) = \{-1, +1\}$. This is the same result as in (4.51) for operators $\mathbf{V_1}(\mathbf{M_k})$. Only multiplicities of eigenvalues $+1$ ad -1 are different, but this fact is not relevant for for further computation. Finally, we find the same result as in (4.67):

$$\widehat{\widehat{\mathbf{P}}}_{\mathbf{M_k}} = \frac{\pi}{2}(\mathbf{V_2}(\mathbf{M_k}) - \mathbb{I}). \tag{4.68}$$

We denoted the generator of mirror symmetry with tilde to distinguish representations $\mathbf{V_1}$ and $\mathbf{V_2}$.

In this section we have found an explicit form of quantum observables, i.e. we have found operators $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{P}}$, and $\widehat{\mathbf{P}}_{\mathbf{Z_k}}$. Now, we shall look for commutation relations of these operators. Let us look first at commutator $[\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}]$. Using (4.39), we obtain

$$(\widehat{\mathbf{Q}})_{jk} = (j-1)\delta_{j,k}. \tag{4.69}$$

Relations (4.69) and (4.61) gives us after simple computation

$$([\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}])_{jk} = (j-k)(\widehat{\mathbf{P}})_{jk}. \tag{4.70}$$

Now we see that the Heisenberg commutation relation does not hold. Similar result we obtain for commutator $[\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}_{\mathbf{Z_k}}]$. Using (4.69), (4.30), and (4.67), we obtain

$$([\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}_{\mathbf{M_k}}])_{jl} = \frac{\pi}{2}([\widehat{\mathbf{Q}}, \mathbf{V_1}(\mathbf{M}_k)])_{jl}. \tag{4.71}$$

Note, that operators $\widehat{\mathbf{P}}$ and $\widehat{\mathbf{P}}_{\mathbf{M}_k}$ are not uniquely determined. This is caused by properties of exponential mapping used in definition of these operators (4.41), exponential mapping is not one to one. Here may arise a question of relevance of these operators. As we shall see in the following section, more important for us will be thu unitary operators of representations $\mathbf{V_1}$ and $\mathbf{V_2}$.

4.4. **Construction of the coherent states over $\mathbf{Z}_n \times \mathbf{D}_n$.** In the previous section we have constructed quantum mechanics on discrete group $\mathbf{Z_n}$ with symmetry group $\mathbf{D}_n$. We obtained two inequivalent quantum mechanics represented by two inequivalent systems of imprimitivity $(\mathbf{E}, \mathbf{V_1})$ and $(\mathbf{E}, \mathbf{V_2})$. For both cases we have constructed systems of quantum observables: $\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}, \widehat{\mathbf{P}}_{\mathbf{Z}_k}$ for the first system of imprimitivity, $\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}, \widehat{\widehat{\mathbf{P}}}_{\mathbf{Z}_k}$ for the other system of imprimitivity. Now, we utilize these results in construction of coherent states. Because of the fact that we have two inequivalent quantum mechanics, we shall obtain two systems of coherent states. Let us first discuss the first case for representation $\mathbf{V_1}$.

To construct coherent states of Perelomov type over $\mathbf{Z}_n \times \mathbf{D}_n$, we shall first construct a system of unitary operators $\widehat{\mathbf{W}}(a, g)$, where $(a, g) \in \mathbf{Z}_n \times \mathbf{D}_n$:

$$\widehat{\mathbf{W}}(a,g) := exp(\frac{2\pi ia}{n}\widehat{\mathbf{Q}})exp(-i\widehat{\mathbf{P}}_g) = e^{ia\widehat{\mathbf{Q}}}\mathbf{V_1}(g); \quad a \in \mathbf{Z}_n,\ g \in \mathbf{D}_n. \tag{4.72}$$

Elements of unitary representation $\mathbf{V_1}(g)$ were discussed in (4.34) and (4.31). Matrix elements of operator $exp(\frac{2/piia}{n}\widehat{\mathbf{Q}})$ can be easily derived because of the diagonal form of position operator $\widehat{\mathbf{Q}}$:

$$(e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}})_{jk} = \delta_{j,k}e^{\frac{2\pi ia}{n}(j-1)}, \quad exp(\frac{2\pi ia}{n}\widehat{\mathbf{Q}}) = \begin{pmatrix} 1 & & & & \\ & e^{2\pi\frac{ia}{n}} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{\frac{2\pi ia(n-1)}{n}} \end{pmatrix}. \tag{4.73}$$

Here we have a unitary representation of our group $\mathbf{Z}_n \times \mathbf{D}_n$, which acts irreducibly in the Hilbert space $\mathbb{C}^n$ [13]. Now, it would be essential to derive one important property of operators $\widehat{\mathbf{W}}(a, g)$:

$$e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}}e^{im\widehat{\mathbf{P}}} = e^{\frac{2\pi iam}{n}}e^{im\widehat{\mathbf{P}}}e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}}. \tag{4.74}$$

Using (4.30) and (4.73), we shall prove that for each matrix element:

$$\begin{aligned}(e^{\frac{2/piia}{n}\widehat{\mathbf{Q}}}e^{im\widehat{\mathbf{P}}})_{jk} &= \sum_{l=1}^{n}(e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}})_{jl}(\mathbf{V_1}(\mathbf{R_{-m}}))_{lk} = \\ = \sum_{l=1}^{n} e^{\frac{2\pi ia}{n}(j-1)}\delta_{j,l}\delta_{l,(k-m)(mod\,n)} &= e^{\frac{2\pi ia}{n}(j-1)}\delta_{j,(k-m)(mod\,n)}.\end{aligned} \tag{4.75}$$

On the other hand, we have

$$(e^{im\widehat{\mathbf{P}}}e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}})_{jk} = \sum_{l=1}^{n}(\mathbf{V_1}(\mathbf{R_{-m}}))_{jl}(e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}})_{lk} = \tag{4.76}$$

$$= \sum_{l=1}^{n}\delta_{j,(l-m)(mod\,n)}e^{\frac{2\pi ia}{n}(l-1)}\delta_{l,k} = \delta_{j,(k-m)(mod\,n)}e^{\frac{2\pi ia}{n}(k-1)} =$$

$$= e^{\frac{2\pi i}{n}am}e^{\frac{2\pi ia}{n}(j-1)}\delta_{j,(k-m)(mod\,n)}$$

Comparing (4.75) and (4.76), we obtain (4.74).

Note, that to derive (4.74), we cannot use the Baker-Campbell-Hausdorff formula (3.13), because the commutation relation condition is not fulfilled due to (4.70).

Unfortunately, if we want to construct a similar relation for operator $\widehat{\mathbf{P}}_{\mathbf{M_k}}$, we do not get the group property of operators $\widehat{\mathbf{W}}(a,g)$. If we perform just the same computation as for operator $\widehat{\mathbf{P}}$, we get the following equality:

$$(e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}}e^{i\widehat{\mathbf{P}}_{\mathbf{M_m}}})_{jk} = e^{\frac{2\pi ia}{n}(2m-2k)}(e^{i\widehat{\mathbf{P}}_{\mathbf{M_m}}}e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}})_{jk}. \tag{4.77}$$

Due to k-dependence of thr multiplier, we cannot formulate any equality similar to (4.74).

To construct the system of coherent states in $\mathbb{C}^n$, we have to determine the fiducial 'vacuum' vector $|0\rangle$. Having this vector, we try to generalize Perelomov's definition of the system of coherent states in the following way:

$$|a,g\rangle := \widehat{\mathbf{W}}(a,g)|0\rangle; \quad a \in \mathbf{Z}_n,\ g \in \mathbf{D}_n. \tag{4.78}$$

$$(\Rightarrow |0,e\rangle = |0\rangle)$$

In analogy with continuous case, where the coherent states are eigenvectors of the annihilation operator and the vacuum vector belongs to eigenvalue 0, we can demand our fiducial vacuum vector to fulfill the following condition:

$$e^{\frac{2\pi}{n}\widehat{\mathbf{Q}}}e^{i\widehat{\mathbf{P}}}|0\rangle = |0\rangle. \tag{4.79}$$

However, this equation cannot be fulfilled, because 1 is not an eigenvalue of operator $e^{\frac{2\pi}{n}\widehat{\mathbf{Q}}}e^{i\widehat{\mathbf{P}}}$. In our approach, we shall make a little change in equation (4.79). In analogy with [14], we shall admit such vacuum vector, where condition

$$e^{\frac{2\pi}{n}\widehat{\mathbf{Q}}}e^{i\widehat{\mathbf{P}}}|0\rangle = \lambda|0\rangle \tag{4.80}$$

is fulfilled for some complex λ. Hence, we have a problem to find the set of eigenvalues and eigenvectors of operator $e^{\frac{2\pi}{n}\widehat{\mathbf{Q}}}e^{i\widehat{\mathbf{P}}}$. Doing that, we simply find

$$\sigma(e^{\frac{2/pi}{n}\widehat{\mathbf{Q}}}e^{i\widehat{\mathbf{P}}}) = \{\lambda_k = e^{\frac{2\pi(n-1)}{2}}e^{\frac{2\pi ik}{n}}|k=0,1,..,n-1\}. \tag{4.81}$$

Also we obtain not only one, but a whole set of vacuum states satisfying (4.80):

$$e^{\frac{2\pi}{n}\widehat{\mathbf{Q}}}e^{i\widehat{\mathbf{P}}}|0\rangle^{(k)} = \lambda_k|0\rangle^{(k)}. \tag{4.82}$$

Looking for vacuum vectors $|0\rangle^{(k)}$, we should solve

$$\begin{pmatrix} -\lambda_k & & & & & 1 \\ e^{\frac{2\pi}{n}} & -\lambda_k & & & & \\ & e^{\frac{4\pi}{n}} & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & -\lambda_k & \\ & & & & e^{\frac{2\pi(n-1)}{n}} & -\lambda_k \end{pmatrix} |0\rangle^{(k)} = \vec{0}. \tag{4.83}$$

We define $k_j^{(k)}$ the j-th element of vector $|0\rangle^{(k)}$:

$$|0\rangle^{(k)} = \begin{pmatrix} g_1^{(k)} \\ g_2^{(k)} \\ \cdot \\ \cdot \\ g_n^{(k)} \end{pmatrix}. \tag{4.84}$$

Now, we can (4.83) rewrite as a system of equations

$$\begin{aligned} e^{\frac{2\pi j}{n}} g_j^{(k)} - e^{\frac{2\pi(n-1)}{n}} e^{\frac{2\pi ik}{n}} g_{j+1}^{(k)} &= 0, \quad j = 1, 2, .., n-1 \\ g_n^{(k)} - e^{\frac{2\pi(n-1)}{n}} e^{\frac{2\pi ik}{n}} g_1^{(k)} &= 0. \end{aligned} \tag{4.85}$$

If we assume

$$g_1^{(k)} = 1. \tag{4.86}$$

we may solve (4.85):

$$g_j^{(k)} = e^{\frac{\pi(j-n+1)(j-1)}{n} - (j-1)\frac{2\pi ik}{n}}, \qquad j, k = 1, 2, ..., n. \tag{4.87}$$

If we compute norms of these vacuum vectors, we obtain a system of admissible vacuum vectors $|0\rangle^{(k)}$:

$$|0\rangle^{(k)} = \mathcal{A}_n \begin{pmatrix} 1 \\ e^{\frac{\pi(1-n)}{n}} e^{\frac{2\pi ik}{n}} \\ \cdot \\ \cdot \\ e^{\frac{\pi(n-1)}{n}} e^{\frac{2\pi ik}{n}} \end{pmatrix}, \tag{4.88}$$

where $\mathcal{A}_n$ is a normalization constant:

$$\mathcal{A}_n = \frac{1}{\sqrt{\sum_{j=1}^{n} e^{\frac{2\pi}{n}(j-1)(j-n+1)}}}. \tag{4.89}$$

Now all is prepared to construct n families of coherent states, which are labeled by parameter k. We will use (4.78) first for $g = \mathbf{R_l}$:

$$\begin{aligned} (|a, \mathbf{R_m}\rangle^{(k)})_j &= (\widehat{\mathbf{W}}(a, \mathbf{R_m})|0\rangle^{(k)})_j = \\ = (e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}}\widehat{\mathbf{V_1}}(\mathbf{R_m})|0\rangle^{(k)})_j &= e^{\frac{2\pi iaj}{n}} g^{(k)}_{(j+m)_{mod(n)}}. \end{aligned} \tag{4.90}$$

For $g = \mathbf{M_l}$ we obtain

$$(4.91) \qquad (|a, \mathbf{M_m}\rangle^{(k)})_j = (\widehat{\mathbf{W}}(a, \mathbf{M_m})|0\rangle^{(k)})_j = = (e^{\frac{2\pi ia}{n}\widehat{\mathbf{Q}}}\widehat{\mathbf{V_1}}(\mathbf{M_m})|0\rangle^{(k)})_j = e^{\frac{2\pi iaj}{n}} g^{(k)}_{(m-j+2)_{mod(n)}}.$$

We may bring to notice that condition (4.80) is in fact independent of the order of operators $e^{\frac{2\pi}{n}\widehat{\mathbf{Q}}}$ and $e^{i\widehat{\mathbf{P}}}$. This is caused by formula (4.74), where we choose $a = -i$ and $m = 1$. Thanks this to if these operators would be in (4.80) in a reverse order, the vacuum states $|0\rangle^{(k)}$ would not change.

If we replace in (4.80) operator $\widehat{\mathbf{P}}$ by operator $\widehat{\mathbf{P}}_{\mathbf{M}_k}$ for some $k = 0, 1, ..., n-1$, then we shall obtain new set of vacuum vectors. But here we have not any formula similar to (4.74), as shown above, so the result would be dependent on the order of operators $e^{\frac{2\pi}{n}\widehat{\mathbf{Q}}}$ and $e^{i\widehat{\mathbf{P}}}$.

Now we ask how the system of coherent states looks like for the second quantum mechanics with representation $\mathbf{V_2}$. Thanks to (4.35) we find that the computation is straightforward. We can see that the set of coherent states for representation $\mathbf{V_1}$ will be only multiplied by -1 to obtain coherent states for representation $\mathbf{V_2}$.

4.5. **Properties of coherent states.** One of the most important properties of coherent states is a resolution of unity:

$$(4.92) \qquad \sum_{(a,g)\in\mathbf{Z_n}\times\mathbf{D_n}} |a, g\rangle^{(k)}\langle a, g|^{(k)} = c_k\widehat{\mathbb{I}},$$

where c_k is some nonvanishing complex number. Let us now check this property for our coherent states. Thanks to (4.90) and (4.91) we get

$$(4.93) \qquad \sum_{(a,g)\in\mathbf{Z_n}\times\mathbf{D_n}} |a, g\rangle^{(k)}\langle a, g|^{(k)} =$$

$$= \sum_{a\in\mathbf{Z_n},m=0,..,n-1} |a, \mathbf{R_m}\rangle^{(k)}\langle a, \mathbf{R_m}|^{(k)} + \sum_{a\in\mathbf{Z_n},m=0,..,n-1} |a, \mathbf{M_m}\rangle^{(k)}\langle a, \mathbf{M_m}|^{(k)}.$$

Matrix element of the first sum on the right side of (5.19) is

$$(4.94) \qquad (\sum_{a,m} |a, \mathbf{R_m}\rangle^{(k)}\langle a, \mathbf{R_m}|^{(k)})_{jl} = \sum_{a,m}(|a, \mathbf{R_m})\rangle^{(k)})_j(\langle a, \mathbf{R_m}|^{(k)})_l =$$

$$= \sum_{a,m} e^{\frac{2\pi ia}{n}(j-l)} g^{(k)}_{(j+m)(mod\, n)}\overline{g^{(k)}_{(l+m)(mod\, n)}} = n\delta_{j,l}\langle 0|0\rangle = \frac{n\delta_{j,l}}{\mathcal{A}_n^2}.$$

The same result is obtained for the second sum on the right hand side of (5.19):

$$(\sum_{a,m} |a, \mathbf{M_m}\rangle^{(k)}\langle a, \mathbf{M_m}|^{(k)})_{jl} = \sum_{a,m} e^{\frac{2\pi ia}{n}(j-l)} g^{(k)}_{(m-j+2)(mod\, n)}\overline{g^{(k)}_{(m-l+2)(mod\, n)}} =$$

$$(4.95) \qquad = n\delta_{j,l}\sum_{m} g^{(k)}_{(m-j+2)(mod\, n)}\overline{g^{(k)}_{(m-l+2)(mod\, n)}} = \frac{n\delta_{j,l}}{\mathcal{A}_n^2}.$$

Now we conclude that the resolution of unity is fulfilled:

$$\sum_{(a,g)\in\mathbf{Z_n}\times\mathbf{D_n}} |a,g\rangle^{(k)}\langle a,g|^{(k)} = \frac{2n}{\mathcal{A}_n^2}\widehat{\mathbb{I}}. \tag{4.96}$$

This result holds for both representations $\mathbf{V_1}$ and $\mathbf{V_2}$.

For the overlap (inner product) of two coherent states we have the formula

$$\begin{aligned}
\langle a,\mathbf{R_p}|b,\mathbf{R_q}\rangle^{(k)} &= \sum_{j=1}^{n} e^{\frac{2\pi ij}{n}(b-a)} g^{(k)}_{(j+p)(mod\,n)}\overline{g^{(k)}_{(j+q)(mod\,n)}}, \\
\langle a,\mathbf{M_p}|b,\mathbf{M_q}\rangle^{(k)} &= \sum_{j=1}^{n} e^{\frac{2\pi ij}{n}(b-a)} g^{(k)}_{(p-j+2)(mod\,n)}\overline{g^{(k)}_{(q-j+2)(mod\,n)}}, \\
\langle a,\mathbf{R_p}|b,\mathbf{M_q}\rangle^{(k)} &= \sum_{j=1}^{n} e^{\frac{2\pi ij}{n}(b-a)} g^{(k)}_{(j+p)(mod\,n)}\overline{g^{(k)}_{(q-j+2)(mod\,n)}}.
\end{aligned} \tag{4.97}$$

This result is also common for both representation $\mathbf{V_1}$ and $\mathbf{V_2}$.

4.6. **Conclusion.** In this section we have constructed systems of imprimitivity on the finite configuration space $\mathbf{Z}_n$ — homogeneous space of the dihedral group $\mathbf{D}_n$. We have shown that there exist two inequivalent irreducible systems of imprimitivity $\mathbf{V_1}$ and $\mathbf{V}_2$. Using these systems of imprimitivity, we have constructed families of coherent states. Unfortunately, due to (4.77) we have lost the group property of the set of operators $\widehat{\mathbf{W}}(a,m)$, i.e. these operators do not form a projective representation of the group $\mathbf{Z}_n \times \mathbf{D}_n$.

In spite of this fact, for each system of imprimitivity n families of coherent states were obtained. For all families of coherent states we have shown that the resolution of unity holds as required. We have also evaluated the inner product of two coherent states in the form of the sum (4.97).

5. Coherent states over $\mathbf{U(1)} \times \mathbb{Z}$

In this section we shall construct coherent states on the circle $\mathbf{S^1}$ according to our approach. The problem of coherent states on the circle was solved also in the paper [8] by M. Del Olmo and J. A. González, based on Perelomov's canonical coherent states. They used the Weil-Berezin-Zak transform $\mathcal{T}$

$$\mathcal{T} : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbf{S}^1 \times \mathbf{S}^{1*}) \tag{5.1}$$

defined by

$$(\mathcal{T}\psi)(q,k) := \sum_{n=-\infty}^{\infty} e^{iank}\psi(q-na), \tag{5.2}$$

$$q \in \mathbf{S}^1 = \langle 0, a), \ k \in \mathbf{S}^{1*} = \langle 0, \frac{2\pi}{a}), \ \psi \in \mathbf{L}^2(\mathbb{R}).$$

Applying the Weil-Berezin-Zak transform on canonical coherent states on the real line they obtained a set of coherent states on the circle labeled by the cylinder $\mathbb{R} \times \mathbf{S^1}$. The resolution of unity and the uncertainty relations were then investigated.

A different approach to construction of coherent states on the circle was introduced by C. J. Isham and J. R. Klauder in [4]. They considered coherent states on the circle labeled by the Eucledean group $E(2)$, which is the semi-direct product of groups $\mathbb{R}^2$ and $SO(2)$. However, they observed that there does not exist an irreducible representation of $E(2)$, such that the resolution of unity holds. Therefore they considered only reducible representations. Their method was then extended to the most general case for the n-dimensional sphere.

In our approach, we construct the family of coherent states using the notion of the imprimitivity system on the circle $\mathbf{S^1}$. Our set of coherent states will be of the Perelomov type, they will be labeled by the group $\mathbb{Z} \times \mathbf{U}(1)$. We shall show that these coherent states fulfil the resolution of unity, and we shall investigate also several properties of them.

5.1. System of imprimitivity on $\mathbf{S^1}$.

In this section we shall construct system of imprimitivity, as described in [11]. Let us also consider the circle $\mathbf{S^1}$ as the configuration manifold. The symmetry group of $\mathbf{S^1}$ is the group $\mathbf{U(1)}$ of all unitary operators on the complex plane. In this case, the symmetry group $\mathbf{U(1)}$ is topologically homeomorphic to the configuration manifold $\mathbf{S^1}$. If we consider the natural transitive action of $\mathbf{U(1)}$ on $\mathbf{S^1}$, then it is straightforward that the subgroup of stability is trivial ($\mathbf{H} = \{e\}$). Classification of all systems of imprimitivity is then simple, because there exists only one irreducible representation of trivial group. This representation is one-dimensional:

$$\rho : \{e\} \mapsto \mathbb{C} : e \to 1. \tag{5.3}$$

The multiplier group for $\mathbf{U(1)}$ is also trivial [11], also projective representation cannot bring any new inequivalent system of imprimitivity. Hence, if we consider the symmetry group $\mathbf{U(1)}$, so we obtain only one quantum mechanics on $\mathbf{S^1}$ via the Mackey's quantization. However, if we change the symmetry group, the number of inequivalent systems of imprimitivity may change. According to [11] and [3], the Hilbert space of quantum mechanics on $\mathbf{S^1}$ is

$$\mathcal{H} = L^2(\mathbf{S^1}, d\varphi), \tag{5.4}$$

the induced unitary representation of $\mathbf{U(1)}$ on the Hilbert space $L^2(\mathbf{S^1}, d\varphi)$ has the following form:

$$[\mathbf{V}(\alpha)\psi](\beta) = \psi(\beta - \alpha),\ \psi \in L^2(\mathbf{S}^1, d\varphi),\ \alpha, \beta \in U(1). \tag{5.5}$$

Operators of representation $\mathbf{V}(\alpha)$ shift the argument of functions in $L^2(\mathbf{S}^1, d\varphi)$. Position operator we define in the natural form

$$(\widehat{\mathbf{Q}}\psi)(\varphi) = \varphi\psi(\varphi), \tag{5.6}$$

momentum operator is

$$\widehat{P} = -i\frac{d}{d\varphi}. \tag{5.7}$$

One should keep in mind that the well known commutation relation for position and momentum operator formally holds

$$[\widehat{Q}, \widehat{P}] = i\mathbb{I}, \tag{5.8}$$

but they do not have a common dense domain in $\mathcal{H}$.

5.2. **Construction of coherent states over $\mathbf{U(1)} \times \mathbb{Z}$.** To define coherent states, we have to first construct a system of unitary operators, labeled by elements of group $\mathbf{U(1)}\times\mathbb{Z}$, and then it is also necessary to find fiducial 'vacuum' vector $|0\rangle$. The system of unitary operators will be obtained using representation $\mathbf{V}$ defined in (5.5):

$$\widehat{W}(m, \alpha) := e^{im\widehat{Q}}e^{-i\alpha\widehat{P}} = e^{im\widehat{Q}}\mathbf{V}(\alpha),\ \alpha \in U(1),\ m \in \mathbb{Z}. \tag{5.9}$$

For operator $\widehat{W}(m, \alpha)$ a property analogous to (4.74) holds:

$$e^{im\widehat{\mathbf{Q}}}e^{-i\alpha\widehat{\mathbf{P}}} = e^{im\alpha}e^{-i\alpha\widehat{\mathbf{P}}}e^{im\widehat{\mathbf{Q}}},\ \alpha \in \mathbf{U(1)}, m \in \mathbb{Z}. \tag{5.10}$$

Namely, action of operator $e^{im\widehat{Q}}$ is simply

$$e^{im\widehat{Q}}\psi(\varphi) = e^{im\varphi}\psi(\varphi). \tag{5.11}$$

Formula (5.10) can then be derived from (5.5):

$$\begin{aligned} e^{im\widehat{\mathbf{Q}}}e^{-i\alpha\widehat{\mathbf{P}}}\psi(\varphi) &= e^{im\widehat{\mathbf{Q}}}[\mathbf{V}(\alpha)\psi](\varphi) = \\ = e^{im\widehat{\mathbf{Q}}}\psi(\varphi - \alpha) &= e^{im\varphi}\psi(\varphi - \alpha), \\ \psi \in L^2(\mathbf{S}^1, d\varphi). & \end{aligned} \tag{5.12}$$

On the other hand

$$\begin{aligned} e^{-i\alpha\widehat{\mathbf{P}}}e^{im\widehat{\mathbf{Q}}}\psi(\varphi) &= \mathbf{V}(\alpha)e^{im\widehat{\mathbf{Q}}}\psi(\varphi) = \\ = [\mathbf{V}(\alpha)](e^{im\varphi}\psi(\varphi)) &= e^{im(\varphi-\alpha)}\psi(\varphi - \alpha), \\ \psi \in L^2(\mathbf{S}^1, d\varphi), & \end{aligned} \tag{5.13}$$

and formula (5.10) follows.

Here, the system of operators $\widehat{W}(m,\alpha)$ does not create a representation of group $\mathbf{U(1)}\times\mathbb{Z}$, but thanks to (5.10) we can see that we obtained a projective representation of $\mathbf{U(1)}\times\mathbb{Z}$.

The vacuum vector $|0\rangle$ will be chosen in analogy with canonical coherent states on $L^2(\mathbb{R})$. We demand the vacuum state to be eigenvector of anihilation operator with eigenvalue 1. So, if we write this condition in exponential form, we have

$$e^{\widehat{Q}+i\widehat{P}}|0\rangle = |0\rangle. \tag{5.14}$$

From the Baker-Campbell-Hausdorff formula (3.13) and (5.10) we see that operator $e^{\widehat{Q}+i\widehat{P}}$, because of (5.14), can be separated in product of two operators $e^{\widehat{Q}}$ and $e^{i\widehat{P}}$ in arbitrary order. Such change will not influence the final vacuum state $|0\rangle$.

It is easy to see that this condition leads to the Gauss exponential function

$$|0\rangle = \mathcal{A}e^{-\frac{\varphi^2}{2}},\ \varphi \in < -\pi, \pi > . \tag{5.15}$$

It is clear, that this vacuum state is an element of our Hilbert space: $|0\rangle \in L^2(\mathbf{S}^1, d\varphi)$. For the normalizing constant $\mathcal{A}$ we have

$$\mathcal{A} = \frac{1}{\sqrt{\int_{-\pi}^{\pi} exp(-\varphi^2)d\varphi}} \doteq 0.751128. \tag{5.16}$$

The system of coherent states on $L^2(\mathbf{S}^1, d\varphi)$ will be obtained by the action of the system of operators $\widehat{W}(m,\alpha)$ on the vacuum state $|0\rangle$:

$$|m,\alpha\rangle := \widehat{W}(m,\alpha)|0,0\rangle. \tag{5.17}$$

Using (5.5) and (5.11) we find the functional form of our coherent states :

$$|m,\alpha\rangle = \mathcal{A}e^{im\varphi}e^{-\frac{(\varphi-\alpha)^2}{2}},\ \varphi \in < -\pi, \pi > . \tag{5.18}$$

5.3. **Properties of coherent states on $L^2(\mathbf{S}^1, d\varphi)$.** In this section we shall check several properties of coherent states which were derived for canonical coherent states on $L^2(\mathbb{R})$. First of all, we shall look at the resolution of unity, i.e. we shall try to prove the following equality for our coherent states:

$$\sum_{k\in\mathbb{Z}} \int_{\mathbf{S}^1} |k,\alpha\rangle\langle k,\alpha| d\alpha = c\widehat{I}, \tag{5.19}$$

where c is an arbitrary nonvanishing constant.

Let us choose an arbitrary vector η from our Hilbert space $\eta \in L^2(\mathbf{S}^1, d\varphi)$. Then the inner product of vector $|\eta\rangle$ with some coherent state $|k,\alpha\rangle$ has the following integral form:

$$\langle k,\alpha|\eta\rangle = \mathcal{A}\int_{\mathbf{S}^1} e^{-ik\varphi}e^{-\frac{(\varphi-\alpha)^2}{2}}\eta(\varphi)d\varphi. \tag{5.20}$$

If we denote the operator on left side of (5.19) by symbol $\widehat{A}$

$$\widehat{A} := \sum_{k\in\mathbb{Z}} \int_{\mathbf{S}^1} |k,\alpha\rangle\langle k,\alpha| d\alpha, \tag{5.21}$$

then we have

$$\widehat{A}\eta(\omega) = \mathcal{A}^2 \sum_{k\in\mathbb{Z}} \int_{\mathbf{S}^1} e^{ik\omega} e^{-\frac{(\omega-\alpha)^2}{2}} [\int_{\mathbf{S}^1} e^{-ik\varphi} e^{-\frac{(\varphi-\alpha)^2}{2}} \eta(\varphi) d\varphi] d\alpha. \tag{5.22}$$

The orthonormal basis of the Hilbert space $L^2(\mathbf{S}^1, d\varphi)$ is formed by functions

$$\frac{1}{\sqrt{2\pi}} exp(ik\varphi),\ \varphi \in \mathbf{S}^1,\ k \in \mathbb{Z}, \tag{5.23}$$

then the equality (5.20) is in fact the k-th element of a Fourier decomposition function $exp(-\frac{(\varphi-\alpha)^2}{2})\eta(\varphi)$ (up to a factor $\sqrt{2\pi}$) in the orthonormal basis (5.23). Denoting this element $a_k(\omega)$:

$$a_k(\omega) := \langle k, \alpha | \eta \rangle, \tag{5.24}$$

we have

$$\widehat{A}\eta(\omega) = \mathcal{A}^2 \int_{\mathbf{S}^1} exp(-\frac{(\omega-\alpha)^2}{2}) \sum_{k\in\mathbb{Z}} \sqrt{2\pi} exp(ik\omega) a_k d\alpha. \tag{5.25}$$

If we evaluate the sum in (5.25), then we obtain, thanks to the theory of Fourier transform, again the function $exp(-\frac{(\varphi-\alpha)^2}{2})\eta(\varphi)$. So we get

$$\widehat{A}\eta(\omega) = (2\pi)\mathcal{A}^2 \int_{\mathbf{S}^1} exp(-(\omega-\alpha)^2)\eta(\omega) d\alpha. \tag{5.26}$$

The integral (5.26) leads in fact to the norm of coherent state $|m, \alpha\rangle$. Thanks to unitarity of operators $\widehat{W}(m, \alpha)$ we know that this norm is equal to norm the of vacuum state $|0\rangle$. Hence we conclude

$$\widehat{A}\eta(\omega) = 2\pi\eta(\omega). \tag{5.27}$$

For the constant c in (5.19) we have

$$c = 2\pi, \tag{5.28}$$

and finally

$$\sum_{k\in\mathbb{Z}} \int_{\mathbf{S}^1} |k, \alpha\rangle\langle k, \alpha| d\alpha = 2\pi\widehat{I}. \tag{5.29}$$

The resolution of unity for our set of coherent states is fulfilled.

Let us now evaluate the inner product (overlap) of two different coherent states on $L^2(\mathbf{S}^1, d\varphi)$. Here, it is necessary to realize, how the operator $exp(-i\alpha\widehat{P})(=V(\alpha))$ acts on the Hilbert space $L^2(\mathbf{S}^1, d\varphi)$ if we identify one-dimensional sphere $\mathbf{S^1}$, our configuration space, with the closed interval $\langle -\pi, \pi\rangle$. Then the action of operator $exp(-i\alpha\widehat{P})$ on some function $\psi(\varphi) \in L^2(\mathbf{S}^1, d\varphi)$ has (for positive α) the form :

$$e^{-i\alpha\widehat{P}}\psi(\varphi) = \begin{cases} \psi(\varphi-\alpha) & \varphi \in < -\pi+\alpha, \pi >, \\ \psi(\varphi-\alpha+2\pi) & \varphi \in < -\pi, -\pi+\alpha > \end{cases} \tag{5.30}$$

$$\alpha \in \langle 0, \pi\rangle.$$

For negative α we have

$$e^{-i\alpha\widehat{P}}\psi(\varphi) \;=\; \begin{cases}\psi(\varphi-\alpha) & \varphi \in <-\pi, \pi+\alpha>,\\ \psi(\varphi-\alpha-2\pi) & \varphi \in <\pi+\alpha, \pi>\end{cases} \tag{5.31}$$

$$\alpha \in \langle -\pi, 0\rangle.$$

In other words, we have to consider addition modulo 2π in the argument of function ψ. This is the reason why we cannot calculate the inner product according to

$$\langle m,\alpha|n,\beta\rangle \neq \mathcal{A}^2 \int_{-\pi}^{\pi} e^{-i\varphi(n-m)} e^{-\frac{(\varphi-\alpha)^2}{2}} e^{-\frac{(\varphi-\beta)^2}{2}} d\varphi. \tag{5.32}$$

From now on we shall restrict ourselves only to the case, when α and β are positive numbers:

$$\alpha \in \langle 0,\pi\rangle, \quad \beta \in \langle 0,\pi\rangle. \tag{5.33}$$

Without loss of generality we may also suppose

$$\beta \geq \alpha. \tag{5.34}$$

Regarding (5.30) and (5.31), we have to split the inner product of two coherent states into two terms

$$\langle m,\alpha|n,\beta\rangle = \mathcal{A}I_1(\alpha,\beta,n-m) + \mathcal{A}I_2(\alpha,\beta,n-m), \tag{5.35}$$

where

(5.36)

$$I_1(\alpha,\beta,n-m) := \int_{\alpha-\pi}^{\beta-\pi} exp(i\varphi(n-m)) exp(-\frac{(\varphi-\alpha)^2}{2}) exp(-\frac{(\varphi-\beta+2\pi)^2}{2}) d\varphi,$$

and for the other integral we have

$$I_2(\alpha,\beta,n-m) := \int_{\beta-\pi}^{\pi+\alpha} exp(i\varphi(n-m)) exp(-\frac{(\varphi-\alpha)^2}{2} exp(-\frac{(\varphi-\beta)^2}{2})) d\varphi. \tag{5.37}$$

In the following, we shall be evaluating integrals $I_1(\alpha,\beta,n-m)$ and $I_2(\alpha,\beta,n-m)$. To do that, we shall need to remind the definition of the error function, as a holomorphic function of a complex variable:

$$erf(z) := \frac{2}{\sqrt{\pi}} \int_{\Gamma(z)} exp(-\eta^2) d\eta. \tag{5.38}$$

Here $\Gamma(z)$ is an arbitrary continuous path of a finite length which connects number $0 \in \mathbb{C}$ with a complex number $z \in \mathbb{C}$. Because the Gauss function $exp(-\eta^2)$ is analytic, the definition of error function (5.38) is independent of the choice of path $\Gamma(z)$.

To evaluate the integral $I_1(\alpha,\beta,n-m)$ we apply substitution

$$\omega = \varphi + \pi - \frac{\alpha+\beta}{2}. \tag{5.39}$$

Then $I_1(\alpha,\beta,n-m)$ gets the form

$$I_1(\alpha,\beta,n-m) = exp(-(\frac{\beta-\alpha}{2})^2-\pi)exp(i(\frac{\alpha+\beta}{2}-\pi)(m-n)) \cdot$$
$$\cdot \int_{\frac{\alpha-\beta}{2}}^{\frac{\beta-\alpha}{2}} exp(i\omega(n-m))exp(-\omega^2)d\omega \tag{5.40}$$

To evaluate the integral $I_2(\alpha,\beta,n-m)$ we use a similar substitution as for integral $I_1(\alpha,\beta,n-m)$:

$$\omega = \varphi - \frac{\alpha+\beta}{2}. \tag{5.41}$$

Performing that, we obtain:

$$I_2(\alpha,\beta,n-m) = exp(-(\frac{\beta-\alpha}{2})^2)exp(i(\frac{\alpha+\beta}{2})(m-n)) \cdot$$
$$\cdot \int_{-\pi-\frac{\alpha-\beta}{2}}^{\pi-\frac{\beta-\alpha}{2}} exp(i\omega(n-m))exp(-\omega^2)d\omega. \tag{5.42}$$

Finally we obtain for integral $I_1(\alpha,\beta,n-m)$

$$I_1(\alpha,\beta,n-m) = (-\frac{\sqrt{\pi}}{2})e^{-(\frac{\beta-\alpha}{2})^2-\pi}e^{i(\frac{\alpha+\beta}{2}-\pi)(m-n)}e^{-\frac{(n-m)^2}{4}} \cdot$$
$$\cdot[erf(\frac{\alpha-\beta}{2}+\frac{i(n-m)}{2})+erf(\frac{\alpha-\beta}{2}-\frac{i(n-m)}{2})], \tag{5.43}$$

and for integral $I_2(\alpha,\beta,n-m)$ we have

$$I_2(\alpha,\beta,n-m) = (-\frac{\sqrt{\pi}}{2})e^{-(\frac{\beta-\alpha}{2})^2}e^{i(\frac{\alpha+\beta}{2})(m-n))}e^{-\frac{(n-m)^2}{4}} \cdot$$
$$\cdot[erf(\frac{\alpha-\beta}{2}-\pi+\frac{i(n-m)}{2})+erf(\frac{\alpha-\beta}{2}-\pi-\frac{i(n-m)}{2})]. \tag{5.44}$$

Unfortunately, we do not see any way how to simplify the integrals $I_1(\alpha,\beta,n-m)$ and $I_2(\alpha,\beta,n-m)$ to show in an analytical way, whether the coherent states are mutually non-orthogonal or not. However, we may plot the absolute value of the inner product for several cases to have some visualization of it. At the end of this work we attach several graphs of the absolute value of overlap (Fig. 1-4) as a function of parameter α and β. Parameter $n-m$ is fixed for each graph. Looking at these graphs we see that for our chosen cases the overlap never vanishes.

Next important property of canonical coherent states on $L^2(\mathbb{R})$ is that for the coherent states equality in the Heisenberg uncertainty relations is reached. We are going to check this situation for our family of coherent states on $L^2(\mathbf{S}^1, d\varphi)$. So let us find operators $\Delta_{|m,\alpha\rangle}\widehat{Q}$ and $\Delta_{|m,\alpha\rangle}\widehat{P}$ to determine the product

$$\Delta_{|m,\alpha\rangle}\widehat{Q}\cdot\Delta_{|m,\alpha\rangle}\widehat{P}. \tag{5.45}$$

Here

$$\Delta_{|m,\alpha\rangle}\widehat{A} := \sqrt{\langle\widehat{A}^2\rangle_{|m,\alpha\rangle}-\langle\widehat{A}\rangle^2_{|m,\alpha\rangle}} = \sqrt{\langle m,\alpha|\widehat{A}^2|m,\alpha\rangle-\langle m,\alpha|\widehat{A}|m,\alpha\rangle^2}. \tag{5.46}$$

For a moment we suppose that $\alpha \geq 0$.

Now, we first compute the dispersion for the position operator in the state $|m,\alpha\rangle$. To do that we have to evaluate the mean values of operators $\widehat{Q}$ and $\widehat{Q^2}$ in the state $|m,\alpha\rangle$. During the computation, we have to remember the problem with the action of operator $exp(-i\alpha\widehat{P})$ (see (5.30), (5.31)). For the mean value of position operator in the state $|m,\alpha\rangle$ we have

$$(5.47)\qquad \langle m,\alpha|\widehat{Q}|m,\alpha\rangle = \mathcal{A}^2\int_{-\pi}^{-\pi+\alpha}\varphi e^{-(\varphi-\alpha+2\pi)^2}d\varphi + \mathcal{A}^2\int_{-\pi+\alpha}^{\pi}\varphi e^{-(\varphi-\alpha)^2}d\varphi.$$

After few minutes of computation we obtain

$$(5.48)\qquad \langle m,\alpha|\widehat{Q}|m,\alpha\rangle = \alpha - \mathcal{A}^2\sqrt{\pi^3}(erf(\pi) - erf(\pi-\alpha)).$$

To simplify further our computation we define function $q_1^{(+)}(\alpha)$:

$$(5.49)\qquad q_1^{(+)}(\alpha) := -\mathcal{A}^2\sqrt{\pi^3}(erf(\pi) - erf(\pi-\alpha)),$$

also

$$(5.50)\qquad \langle m,\alpha|\widehat{Q}|m,\alpha\rangle = \alpha + q_1^{(+)}(\alpha).$$

We use index '+' in definition of function $q_1^{(+)}$ to remember that this result holds only for positive values of parameter α.

The mean value of square of position operator in state $|m,\alpha\rangle$ is

$$(5.51)\qquad \langle m,\alpha|\widehat{Q}^2|m,\alpha\rangle = \mathcal{A}^2\int_{-\pi}^{-\pi+\alpha}\varphi^2 e^{-(\varphi-\alpha+2\pi)^2}d\varphi + \mathcal{A}^2\int_{-\pi+\alpha}^{\pi}\varphi^2 e^{-(\varphi-\alpha)^2}d\varphi.$$

The computation gives us

$$(5.52)\qquad \langle m,\alpha|\widehat{Q}^2|m,\alpha\rangle = \alpha^2 + \tfrac{1}{2} + $$
$$+\mathcal{A}^2[\pi(e^{-\pi^2} - 2e^{-(\pi-\alpha)^2}) + 2\sqrt{\pi^3}(\pi-\alpha)(erf(\pi) - erf(\pi-\alpha))].$$

After substitution of

$$(5.53)\qquad q_2^{(+)}(\alpha) := \mathcal{A}^2[\pi(e^{-\pi^2} - 2e^{-(\pi-\alpha)^2}) + 2\sqrt{\pi^3}(\pi-\alpha)(erf(\pi) - erf(\pi-\alpha))],$$

we have

$$(5.54)\qquad \langle m,\alpha|\widehat{Q}^2|m,\alpha\rangle = \alpha^2 + \frac{1}{2} + q_2^{(+)}(\alpha).$$

Finally, the dispersion of the position operator is

$$(5.55)\qquad \Delta_{|m,\alpha\rangle}\widehat{Q} = \sqrt{\langle m,\alpha|\widehat{Q}^2|m,\alpha\rangle - \langle m,\alpha|\widehat{Q}|m,\alpha\rangle^2} =$$
$$= \sqrt{\alpha^2 + \frac{1}{2} + q_2^{(+)}(\alpha) - (\alpha + q_1^{(+)}(\alpha))^2} = (\frac{1}{2} + q_2^{(+)}(\alpha) - 2\alpha q_1^{(+)}(\alpha) - q_1^{(+)}(\alpha)^2).$$

This result is independent of parameter m, it depends only on α. If we look at the dispersion of position operator in the case of canonical coherent states on $L^2(\mathbb{R})$, where $\Delta_{|\alpha\rangle}\widehat{Q} = \sqrt{\frac{1}{2}}$, we see that our result is very similar. If we let $q_1 = q_2 = 0$, we would obtain the same result. However, functions q_1 and q_2 do not vanish.

Now we look at the dispersion for momentum operator $\widehat{P}$. Here it is necessary to point out one problem with domains of position and momentum operators. In our case the position operator is even bounded, that means that the domain of position operator is the whole space $L^2(\mathbf{S}^1, d\varphi)$. (Note that in quantum mechanics on real line the position operator is unbounded in $L^2(\mathbb{R})$.) However, the momentum operator is unbounded, and moreover, we have to add one condition to definition of its domain which does not appear in the case of $L^2(\mathbb{R})$. The domain of the momentum operator is:

(5.56)
$$Dom(\widehat{P}) := \{\psi | \psi \in L^2(\langle -\pi, \pi \rangle),\ \psi \textit{ is absolutely continuous},\ \psi(-\pi) = \psi(\pi)\}.$$

How does this fact affect our coherent states ? If we examine the Heisenberg uncertainty relation, we cannot use the Heisenberg uncertainty theorem, which guarantees the well known inequality, because our states do not fulfil assumptions of this theorem. According to [2] we have to assume that our coherent states are in the domain of operators $\widehat{Q}$, $\widehat{Q^2}$, $\widehat{P}$, $\widehat{P}^2$, $\widehat{Q}\widehat{P}$ and $\widehat{P}\widehat{Q}$. And our coherent states, as we can see, are not in general in the domain of operators $\widehat{P}\widehat{Q}$ and $\widehat{P}^2$:

$$|m, \alpha\rangle \notin Dom(\widehat{P}\widehat{Q}), Dom(\widehat{P}^2). \tag{5.57}$$

This is caused by the condition

$$\psi(-\pi) = \psi(\pi) \tag{5.58}$$

in the definition of domain of momentum operator (5.56). Due to this fact, we can not guarantee the Heisenberg inequality; moreover we have troubles with evaluation of dispersion of momentum operator $\widehat{P}$. However, we may consider $\widehat{P}^2$ as a formal operator of second derivative, which is not self-adjoint, and compute the dispersion of $\widehat{P}$ in following way.

For the mean value of momentum operator in state $|m, \alpha\rangle$ we have

$$\begin{aligned} \langle m, \alpha | \widehat{P} | m, \alpha \rangle = \mathcal{A}^2 \int_{-\pi}^{-\pi+\alpha} (m + i(\varphi - \alpha + 2\pi)) e^{-(\varphi-\alpha+2\pi)^2} d\varphi + \\ + \mathcal{A}^2 \int_{-\pi+\alpha}^{\pi} (m + i(\varphi - \alpha)) e^{-(\varphi-\alpha)^2} d\varphi. \end{aligned} \tag{5.59}$$

After a computation we obtain

$$\langle m, \alpha | \widehat{P} | m, \alpha \rangle = m. \tag{5.60}$$

Note, that this result is the same as for canonical coherent states on $L^2(\mathbb{R})$.

The mean value of square of momentum operator in state $|m, \alpha\rangle$ is determined by computation of the integral:

$$\begin{aligned} \langle m, \alpha | \widehat{P}^2 | m, \alpha \rangle = \mathcal{A}^2 \int_{-\pi}^{-\pi+\alpha} [1 + (m + i(\varphi - \alpha + 2\pi))^2] e^{-(\varphi-\alpha+2\pi)^2} d\varphi + \\ + \mathcal{A}^2 \int_{-\pi+\alpha}^{\pi} [1 + (m + i(\varphi - \alpha))^2] e^{-(\varphi-\alpha)^2} d\varphi. \end{aligned} \tag{5.61}$$

We obtain

$$\langle m, \alpha | \widehat{P}^2 | m, \alpha \rangle = m^2 + \frac{1}{2} + \mathcal{A}^2 \pi exp(-\pi^2). \tag{5.62}$$

With substitution

$$p_2^{(+)} := \mathcal{A}^2 \pi exp(-\pi^2) \tag{5.63}$$

we have

$$\langle m,\alpha|\widehat{P}^2|m,\alpha\rangle = m^2 + \frac{1}{2} + p_2^{(+)}. \tag{5.64}$$

Finally, the dispersion of the momentum operator in state $|m,\alpha\rangle$ is

$$\begin{aligned}\Delta_{|m,\alpha\rangle}\widehat{P} &= \sqrt{\langle m,\alpha|\widehat{P}^2|m,\alpha\rangle - \langle m,\alpha|\widehat{P}|m,\alpha\rangle^2} = \\ &= \sqrt{m^2 + \frac{1}{2} + p_2^{(+)} - m^2} = \sqrt{\frac{1}{2} + p_2^{(+)}}.\end{aligned} \tag{5.65}$$

This result is — like for the dispersion for position operator — independent of parameter m.

Finally, the Heisenberg uncertainty relations have the following form:

$$\Delta_{|m,\alpha\rangle}\widehat{Q}\cdot\Delta_{|m,\alpha\rangle}\widehat{P} = \sqrt{(\frac{1}{2} + p_2^{(+)})}\cdot\sqrt{(\frac{1}{2} + q_2^{(+)}(\alpha) - 2\alpha q_1^{(+)}(\alpha) - q_1^{(+)}(\alpha)^2)}. \tag{5.66}$$

Now we look at the case, when parameter α is negative: $\alpha \le 0$. According to (5.31), the integrals for mean values of position and momentum operators are different to equalities (5.47, 5.51) etc. For the mean value of position operator we then have

$$\langle m,\alpha|\widehat{Q}|m,\alpha\rangle = \mathcal{A}^2\int_{\pi+\alpha}^{\pi}\varphi e^{-(\varphi-\alpha-2\pi)^2}d\varphi + \mathcal{A}^2\int_{-\pi}^{\pi+\alpha}\varphi e^{-(\varphi-\alpha)^2}d\varphi. \tag{5.67}$$

If we define

$$q_1^{(-)}(\alpha) := \mathcal{A}^2\sqrt{\pi^3}(erf(\pi) - erf(\pi+\alpha)), \tag{5.68}$$

we have

$$\langle m,\alpha|\widehat{Q}|m,\alpha\rangle = \alpha + q_1^{(-)}(\alpha). \tag{5.69}$$

For the mean value of square of position operator we get

$$\langle m,\alpha|\widehat{Q}^2|m,\alpha\rangle = \mathcal{A}^2\int_{\pi+\alpha}^{\pi}\varphi^2 e^{-(\varphi-\alpha-2\pi)^2}d\varphi + \mathcal{A}^2\int_{-\pi}^{\pi+\alpha}\varphi^2 e^{-(\varphi-\alpha)^2}d\varphi, \tag{5.70}$$

$$q_2^{(-)}(\alpha) := \mathcal{A}^2[\pi(e^{-\pi^2} - 2e^{-(\pi-\alpha)^2}) + 2\sqrt{\pi^3}(\pi+\alpha)(erf(\pi) - erf(\pi+\alpha))], \tag{5.71}$$

$$\langle m,\alpha|\widehat{Q}^2|m,\alpha\rangle = \alpha^2 + \frac{1}{2} + q_2^{(-)}(\alpha). \tag{5.72}$$

For the mean value of momentum operator we have

$$\begin{aligned}\langle m,\alpha|\widehat{P}|m,\alpha\rangle &= \mathcal{A}^2\textstyle\int_{\pi+\alpha}^{\pi}(m + i(\varphi-\alpha-2\pi))e^{-(\varphi-\alpha-2\pi)^2}d\varphi + \\ &+\mathcal{A}^2\textstyle\int_{-\pi}^{\pi+\alpha}(m + i(\varphi-\alpha))e^{-(\varphi-\alpha)^2}d\varphi,\end{aligned} \tag{5.73}$$

$$\langle m,\alpha|\widehat{P}|m,\alpha\rangle = m. \tag{5.74}$$

This result is equal to the result for positive value of parameter α. Finally, for the mean value of square of momentum operator we get

$$\begin{aligned}\langle m,\alpha|\widehat{P}^2|m,\alpha\rangle = \mathcal{A}^2\int_{\pi+\alpha}^{\pi}&[1 + (m + i(\varphi-\alpha-2\pi))^2]e^{-(\varphi-\alpha-2\pi)^2}d\varphi + \\ &+\mathcal{A}^2\int_{-\pi}^{\pi+\alpha}[1 + (m + i(\varphi-\alpha))^2]e^{-(\varphi-\alpha)^2}d\varphi,\end{aligned} \tag{5.75}$$

$$p_2^{(-)} := \mathcal{A}^2 \pi exp(-\pi^2) = p_2^{(+)}, \tag{5.76}$$

$$\langle m, \alpha | \widehat{P}^2 | m, \alpha \rangle = m^2 + \frac{1}{2} + p_2^{(-)}. \tag{5.77}$$

So we may conclude with

$$\Delta_{|m,\alpha\rangle} \widehat{Q} \cdot \Delta_{|m,\alpha\rangle} \widehat{P} = \sqrt{(\frac{1}{2} + p_2^{(-)})} \cdot \sqrt{(\frac{1}{2} + q_2^{(-)}(\alpha) - 2\alpha q_1^{(-)}(\alpha) - q_1^{(-)}(\alpha)^2)}. \tag{5.78}$$

At the end of this work we find two graphs (Figs. 5,6) of the function $|\Delta_{|m,\alpha\rangle} \widehat{Q} \cdot \Delta_{|m,\alpha\rangle} \widehat{P}|$ as a function of parameter α. The first graph is for a negative value of parameter α, the second one for a positive value of parameter α.

If we look at the graphs of uncertainty relations for our coherent states we see that function $|\Delta_{|m,\alpha\rangle} \widehat{Q} \cdot \Delta_{|m,\alpha\rangle} \widehat{P}|$ achieves its minimum for $\alpha = 0$. Let us now look at the explicit value for $\alpha = 0$. For the functions q_1, q_2, and p_2 we have

$$q_1^{\pm}(0) = 0, \ q_2^{\pm}(0) = -\mathcal{A}^2 \pi e^{-\pi^2}, \ p_2^{\pm} = \mathcal{A}^2 \pi e^{-\pi^2}. \tag{5.79}$$

Using (5.66) and (5.78) we than get

$$\begin{aligned} \Delta_{|m,0\rangle} \widehat{Q} \cdot \Delta_{|m,0\rangle} \widehat{P} \quad &= \sqrt{(\tfrac{1}{2} + p_2^{(\pm)})} \cdot \sqrt{(\tfrac{1}{2} + q_2^{(\pm)}(0))} = \\ &= \sqrt{(\tfrac{1}{2} + \mathcal{A}^2 \pi e^{-\pi^2})} \cdot \sqrt{(\tfrac{1}{2} - \mathcal{A}^2 \pi e^{-\pi^2})} = \\ &= \sqrt{\tfrac{1}{4} - \mathcal{A}^2 \pi^2 e^{-2\pi^2}} = \sqrt{\tfrac{1}{4} - \tfrac{e^{-2\pi^2}}{erf^4(\pi)}} < \tfrac{1}{2}. \end{aligned} \tag{5.80}$$

Here we find the explicit value for $\alpha = 0$ $|\Delta_{|m,0\rangle} \widehat{Q} \cdot \Delta_{|m,0\rangle} \widehat{P}| \doteq 0.4999999973$.

Now we see, that there exist coherent states in our family, which do not fulfill the Heisenberg uncertainty inequality. This is caused by the problem with domains of operators $\widehat{P}\widehat{Q}$ and $\widehat{P}^2$.

5.4. **Extension of the symmetry group $\mathbf{U(1)}$ to the covering group $\mathbb{R}$.** So far, we have considered group $\mathbf{U(1)}$ as a symmetry group of our configuration manifold $\mathbf{S^1}$. However, group $\mathbf{U(1)}$ is not simply connected. This fact could in general bring several troubles. According to [11], if the symmetry group is connected, simply connected and semisimple, then the multiplier group is trivial. In our case, symmetry group $\mathbf{U(1)}$ is neither simply connected, nor semisimple. Nevertheless, multiplier group of $\mathbf{U(1)}$ is, according to [11], trivial. In the following we shall replace group $\mathbf{U(1)}$ by its universal covering group $(\mathbb{R}, +)$, which is simply connected. Action σ of symmetry group $\mathbb{R}$ on $\mathbf{S^1}$ is natural:

$$\sigma : \mathbb{R} \times \mathbf{S^1} \to \mathbf{S^1} : (x, e^{i\varphi}) \mapsto e^{i(x+\varphi)}, \ e^{i\varphi} \in \mathbf{S^1}, x \in \mathbb{R}. \tag{5.81}$$

Action σ of symmetry group $\mathbb{R}$ is evidently transitive, subgroup of stability $\mathbf{H}$ of action σ is discrete group $\mathbf{H} = \mathbb{Z}$,

$$\mathbf{S^1} \cong \mathbb{R}/\mathbb{Z}. \tag{5.82}$$

The set of all inequivalent irreducible representations of subgroup of stability $\mathbb{Z}$ is labeled by parameter $\phi \in \mathbf{S^1}$; all of these representations are onedimensional:

$$L^{\phi} : \mathbb{Z} \to \mathbb{C} : k \mapsto exp(ike\phi). \tag{5.83}$$

interpretation of constant e will be done later.

Hence we have an uncountable set of inequivalent quantum mechanics labeled by parameter $\phi \in \mathbf{S^1}$. Hilbert space $\mathcal{H}^{\phi}$, corresponding to parameter ϕ, contains Borel complex function on $\mathbb{R}$ with finite norm, which fulfil condition of quasiperiodicity:

$$\mathcal{H}^{\phi} := \{\psi : \mathbb{R} \to \mathbb{C} | \psi(x + 2k\pi) = exp(-ike\phi)\psi(x), \|\psi\| < \infty\}. \tag{5.84}$$

Inner product of two functions in $\mathcal{H}^{\phi}$ has the following form:

$$< \psi, \chi >:= \int_a^{a+2\pi} \psi(x)\overline{\chi}(x)dx,\ a \in \mathbb{R},\ \psi, \chi \in \mathcal{H}^{\phi}. \tag{5.85}$$

Induced unitary representation of group $\mathbb{R}$ on $\mathcal{H}^{\phi}$ has the form:

$$\mathbf{V}^{\phi}(t) = exp(-it\widehat{P}^{\phi}),\ t \in \mathbb{R}. \tag{5.86}$$

The self-adjoint operator $\widehat{P}^{\phi}$ is the derivative:

$$\widehat{P}^{\phi} := -i\frac{d}{dx}. \tag{5.87}$$

The position operator on $\mathcal{H}^{\phi}$ is a little more complicated. Position operator here is not multiplication by independent variable, but multiplication by a saw-shaped function on $\mathbb{R}$:

$$(\widehat{Q}\psi)(x) := (x)_{mod(2/pi)}\psi(x), \quad \psi \in \mathcal{H}^{\phi}. \tag{5.88}$$

Thank to this, the function $\widehat{Q}\psi$ remains quasiperiodic and fulfills the condition in definition of Hilbertspace $\mathcal{H}^{\phi}$ in (5.84). Note that here the momentum operators $\widehat{P}^{\phi}$ have the same form for all $\phi \in \mathbf{S^1}$.

It is possible to identify Hilbert spaces $\mathcal{H}^{\phi}$ with Hilbert space $L^2(\mathbf{S}^1, d\varphi)$ using unitary mapping $\mathcal{U}^{\phi}$:

$$\mathcal{U}^{\varphi} : \mathcal{H}^{\phi} \to L^2(\mathbf{S}^1, d\varphi) : \psi(x) \mapsto exp(\frac{i\phi e\varphi}{2\pi})\psi(\varphi). \tag{5.89}$$

The transformed operators $\widehat{P}^{\phi}$ have on Hilbert space $L^2(\mathbf{S}^1, d\varphi)$ the form:

$$\mathcal{U}^{\phi}\widehat{P}^{\phi}(\mathcal{U}^{\phi})^{-1} = -i\frac{d}{d\varphi} - \frac{e\phi}{2\pi}. \tag{5.90}$$

The position operator on $\mathcal{H}^{\phi}$ acts by multiplying by independent variable, as in (5.6). Here we can see that the Hilbert space $L^2(\mathbf{S}^1, d\varphi)$ is the same for all inequivalent quantum mechanics, but momentum operators are different.

Hence using the universal covering group as a symmetry group we obtain a bigger family of inequivalent quantum mechanics. Parameter ϕ can be interpreted as a flux of magnetic field through the circle, which is the trajectory of our quantum particle with electric charge e. See also [11].

Let us now define a family of coherent states for quantum mechanics labeled by the flux ϕ. We shall proceed identically to the previous chapter and generalize our results for all parameters ϕ.

It is easy to see that the equality 5.10 holds in the same form:

$$e^{im\widehat{Q}}e^{-i\alpha\widehat{P^{\phi}}} = e^{im\alpha}e^{-i\alpha\widehat{P}^{\phi}}e^{im\widehat{Q}}, \;\alpha \in \mathbf{U(1)}, m \in \mathbb{Z}, \;\phi \in \mathbf{S^1}. \tag{5.91}$$

We may also look for the vacuum state. Solving

$$exp(\widehat{Q} + i\widehat{P}^{\phi})|0,\phi\rangle = |0,\phi\rangle \tag{5.92}$$

we find the vacuum state

$$|0,\phi\rangle := \mathcal{A}_{\phi}exp(-\frac{(\varphi - \frac{ie\phi}{2\pi})^2}{2}). \tag{5.93}$$

The normalization constant $\mathcal{A}_{\phi}$ can be determined by direct computation $\mathcal{A}_{\phi} = \mathcal{A}exp(-\frac{e^2\phi^2}{2\pi^2})$, where constant $\mathcal{A}$ was given in (5.16). Coherent states are defined by the action of operators $\widehat{W}^{\phi}(m,\alpha)$:

$$\widehat{W}^{\phi}(m,\alpha) := exp(im\widehat{Q})exp(-i\alpha\widehat{P}^{\phi}). \tag{5.94}$$

Then the coherent states have the following explicit form:

$$\begin{aligned} |m,\alpha,\phi\rangle &= \widehat{W}^{\phi}(m,\alpha)|m,\alpha,\phi\rangle = \\ &= exp(im\varphi)exp(-\frac{(\varphi - \alpha - \frac{ie\phi}{2\pi})^2}{2}), \;\varphi \in < -\pi,\pi > . \end{aligned} \tag{5.95}$$

We have extended the symmetry group $\mathbf{U}(1)$ to symmetry group group $\mathbb{R}$. That means that the parameter α in definition of operator (5.94) may be an arbitrary real number. However, system of operators $\widehat{W}^{\phi}(m,\alpha)$ would be in this case periodic in parameter α with period 2π. That is the reason why we may restrict back to interval $\langle -\pi,\pi\rangle$, because the rest of the real line will not bring any new state to our family of coherent states. As we shall see later, this restriction is even necessary to avoid several problems with divergence of integrals.

Now we may start with examining properties of coherent states for quantum mechanics labeled by parameter ϕ. We start with resolution of unity. To do that, we define operator $\widehat{A}^{\phi}$ as follows:

$$\widehat{A}^{\phi} = \sum_{k\in\mathbb{Z}}\int_{\mathbf{S}^1}|k,\alpha,\phi\rangle\langle k,\alpha,\phi|d\alpha. \tag{5.96}$$

Now we choose an arbitrary function $\eta \in L^2(\mathbf{S}^1, d\varphi)$, and let act operator $\widehat{A}^{\phi}$ onto this state:

$$\widehat{A}\eta(\omega) = \sum_{k\in\mathbb{Z}}\int_{\mathbf{S}^1}|k,\alpha,\phi\rangle\langle k,\alpha,\phi|\eta\rangle d\alpha = \tag{5.97}$$

$$= \sum_{k\in\mathbb{Z}}\int_{\mathbf{S}^1}e^{ik\omega}e^{-\frac{(\omega-\alpha-\frac{ie\phi}{2\pi})^2}{2}}\cdot[\int_{\mathbf{S}^1}e^{-ik\varphi}e^{-\frac{(\varphi-\alpha+\frac{ie\phi}{2\pi})^2}{2}}\eta(\varphi)d\varphi]d\alpha.$$

If we perform the same computation as in (5.22), we finally obtain

$$\widehat{A}^{\phi}\eta(\omega) = (2\pi)exp(\frac{e^2\phi^2}{4\pi^2})\int_{\mathbf{S}^1}exp(-(\omega-\alpha)^2)\eta(\omega)d\alpha = \frac{2\pi}{\mathcal{A}_{\phi}^2}\eta(\omega). \tag{5.98}$$

Hence the resolution of unity is

$$\sum_{k\in\mathbb{Z}}\int_{\mathbf{S}^1}|k,\alpha\rangle\langle k,\alpha|d\alpha=\frac{2\pi}{\mathcal{A}_\phi^2}\widehat{I}. \tag{5.99}$$

This result agrees with (5.29).

Next we examine the inner product of two coherent states. We will keep our restrictions on parameters α and β (5.33), (5.34), and then we divide the inner product in two integrals. This procedure is similar to (5.35):

$$\langle m,\alpha,\phi|n,\beta,\phi\rangle=\mathcal{A}_\phi I_1(\alpha,\beta,n-m,\phi)+\mathcal{A}_\phi I_2(\alpha,\beta,m-n,\phi), \tag{5.100}$$

where we have

$$I_1(\alpha,\beta,n-m,\phi):=\int_{\alpha-\pi}^{\beta-\pi}e^{i\varphi(n-m)}e^{-\frac{(\varphi-\alpha+\frac{ie\phi}{2\pi})^2}{2}}e^{-\frac{(\varphi-\beta+2\pi-\frac{ie\phi}{2\pi})^2}{2}}d\varphi, \tag{5.101}$$

and

$$I_2(\alpha,\beta,n-m,\phi):=\int_{\beta-\pi}^{\pi+\alpha}e^{i\varphi(n-m)}e^{-\frac{(\varphi-\alpha+\frac{ie\phi}{2\pi})^2}{2}}e^{-\frac{(\varphi-\beta-\frac{ie\phi}{2\pi})^2}{2}}d\varphi. \tag{5.102}$$

Computation of $I_1(\alpha,\beta,n-m,\phi)$ and $I_2(\alpha,\beta,n-m,\phi)$ gives us

$$\begin{aligned} I_1(\alpha,\beta,n-m,\phi):=\, & e^{\frac{e^2\phi^2}{4\pi^2}}e^{-\frac{ie\phi}{2\pi}(\beta-\alpha+2\pi)}(-\frac{\sqrt{\pi}}{2})e^{-(\frac{\beta-\alpha}{2})^2-\pi}e^{i(\frac{\alpha+\beta}{2}-\pi)(m-n)}e^{-\frac{(n-m)^2}{4}}\cdot \\ & \cdot[erf(\frac{\alpha-\beta}{2}+\frac{i(n-m)}{2})+erf(\frac{\alpha-\beta}{2}-\frac{i(n-m)}{2})], \end{aligned} \tag{5.103}$$

for the second integral we have

$$\begin{aligned} I_2(\alpha,\beta,n-m,\phi):=\, & e^{\frac{e^2\phi^2}{4\pi^2}}e^{-\frac{ie\phi}{2\pi}(\beta-\alpha)}(-\frac{\sqrt{\pi}}{2})e^{-(\frac{\beta-\alpha}{2})^2}e^{i(\frac{\alpha+\beta}{2})(m-n))}e^{-\frac{(n-m)^2}{4}}\cdot \\ & \cdot[erf(\frac{\alpha-\beta}{2}-\pi+\frac{i(n-m)}{2})+erf(\frac{\alpha-\beta}{2}-\pi-\frac{i(n-m)}{2})]. \end{aligned} \tag{5.104}$$

If we compare this result with the result obtained in (5.43) and (5.44), we may write

$$I_1(\alpha,\beta,n-m,\phi):=e^{\frac{e^2\phi^2}{4\pi^2}}e^{-\frac{ie\phi}{2\pi}(\beta-\alpha+2\pi)}I_1(\alpha,\beta,n-m), \tag{5.105}$$

and

$$I_2(\alpha,\beta,n-m,\phi):=e^{\frac{e^2\phi^2}{4\pi^2}}e^{-\frac{ie\phi}{2\pi}(\beta-\alpha)}I_2(\alpha,\beta,n-m). \tag{5.106}$$

The inner product for two coherent states is finally

$$\langle m,\alpha,\phi|n,\beta,\phi\rangle=e^{\frac{e^2\phi^2}{4\pi^2}}e^{-\frac{ie\phi}{2\pi}(\beta-\alpha+2\pi)}I_1(\alpha,\beta,n-m)+e^{\frac{e^2\phi^2}{4\pi^2}}e^{-\frac{ie\phi}{2\pi}(\beta-\alpha)}I_2(\alpha,\beta,m-n). \tag{5.107}$$

At the end of this section we would like to say few words about the Heisenberg uncertainty relations and how they change with the parameter ϕ. If we perform the same computation for the mean values for momentum operator and square of position operator, we find that the only change will be in the coefficient $e^{\frac{e^2\phi^2}{4\pi^2}}$ in integrals (5.48) and (5.52). This factor will only change the normalization constant $\mathcal{A}$ to normalization constant $\mathcal{A}_\phi$. The same change holds for the mean value of momentum operator $\widehat{P}^\phi$

and its square. If we assume that parameter α is positive, we may generalize functions $q_1^{(+)}$,$q_2^{(+)}$ and $p_2^{(+)}$ in (5.50), (5.53) and (5.63):

$$q_1^{(+)\phi}(\alpha) := \mathcal{A}_\phi^2\sqrt{\pi^3}(erf(\pi) - erf(\pi - \alpha)), \tag{5.108}$$

$$q_2^{(+)\phi}(\alpha) := \mathcal{A}_\phi^2[\pi(e^{-\pi^2} - 2e^{-(\pi-\alpha)^2}) + 2\sqrt{\pi^3}(\pi - \alpha)(erf(\pi) - erf(\pi - \alpha))],$$

$$p_2^{(+)\phi} := \mathcal{A}_\phi^2 \pi exp(-\pi^2).$$

The uncertainty relations have then the following form:

$$\Delta_{|m,\alpha,\phi\rangle}\widehat{Q} \cdot \Delta_{|m,\alpha,\phi\rangle}\widehat{P}^\phi = \sqrt{(\frac{1}{2} + p_2^{(+)\phi})} \cdot \sqrt{(\frac{1}{2} + q_2^{(+)\phi}(\alpha) - 2\alpha q_1^{(+)\phi}(\alpha) - q_1^{(+)\phi}(\alpha)^2)}. \tag{5.109}$$

Computation for negative values of parameter α is straightforward.

Here the same problem arises with domain of operator $\widehat{P}^\phi$ in $L^2(\mathbf{S}^1, d\varphi)$ as in the previous section. For the domain of momentum operator we have

$$Dom(\widehat{\mathbf{P}^\phi}) := \{\psi|\psi, \psi' \in L^2(\langle -\pi, \pi\rangle),\, \psi(-\pi) = e^{-ie\phi}\psi(\pi)\}. \tag{5.110}$$

(Strictly said we mean operator $\mathcal{U}^\phi\widehat{P}^\phi(\mathcal{U}^\phi)^{-1}$.) Then the discussion about possible relevance of Heisenberg uncertainty relation is similar to previous section.

5.5. **Conclusion.** Using the notion of imprimitivity systems on the circle $\mathbf{S^1}$ we have constructed systems of coherent states. First we considered the symmetry group $\mathbf{U}(1)$. We saw, that there exists just one system of imprimitivity, up to equivalence. Using this imprimitivity system, we constructed the system of coherent states in the sense of our approach to construction of coherent states, which is discussed above. For our family of coherent states we have shown that the resolution of unity holds. We have also evaluated the inner product of two arbitrary coherent states. This inner product was expressed using the analytic error function $erf(z)$, and we have shown the absolute value of the inner product on the graphs for several cases of parameters. The error function also appears in the computation of the Heisenberg uncertainty relations. It shows that the Heisenberg uncertainty relation is independent of the discrete parameter m. Thanks to this we were able to plot the relations graphically as a function of the parameter α.

If the covering group $\mathbb{R}$ was considered as a symmetry group of our configuration manifold $\mathbf{S^1}$, we obtained an uncountable set of mutually inequivalent systems of imprimitivity labeled by parameter ϕ. Parameter ϕ has the meaning of the magnetic flux through the circle. After construction of coherent states, we looked at their properties. During the computation of the resolution of unity, of the inner product of two coherent states and of the Heisenberg uncertainty relation we obtained similar results to the results for symmetry group $\mathbf{U}(1)$. The only change appears in the normalization constant $\mathcal{A}_\phi$. If the parameter ϕ vanishes, then the results fully correspond with the results for symmetry group $\mathbf{U}(1)$, as expected.

6. **Conclusion**

This work was devoted to the problem of construction of coherent states. We began with the notion of Mackey's irreducible imprimitivity system. We used this formalism to construct an irreducible set of Weyl operators, which were then the starting point to construction of coherent states. We were interested in two cases.

First we generalized quantum mechanics on finite Abelian groups $\mathbf{Z}_n$ [14]. We extended the Abelian symmetry group $\mathbf{Z}_n$ to non-commuting dihedral group $\mathbf{D}_n$. Thanks to this extension we obtained two irreducible quantum mechanics and also a bigger set of coherent states.

The second example is devoted to coherent states on the circle. We used the imprimitivity systems constructed in [11], where is shown that there exists an uncountable family of mutually inequivalent imprimitivity systems on the circle. We constructed families of coherent states for each such imprimitivity system, and then studied their properties, the overlap of two coherent states and the Heisenberg uncertainty relation.

Figure 1. *Inner product of coherent states on the circle for $m - n = 0$.*

Figure 2. *Inner product of coherent states on the circle for $m - n = 1$.*

Figure 3. *Inner product of coherent states on the circle for $m - n = 4$.*

Figure 4. *Inner product of coherent states on the circle for $m - n = 7$.*

7. **Appendix**

Figure 5. *Heisenberg uncertainty relation for negative values of α.*

Figure 6. *Heisenberg uncertainty relation for positive values of α.*

Acknowledgements

The author gratefully acknowledges stimulating and helpful conversations on the subject matter of this work with Prof. Ing. Jiří Tolar, DrSc. and Doc. Ing. Goce Chadzitaskos, CSc.

References

1. S. De Biévre, Coherent states over symplectic homogenous spaces, *J. Math. Phys.* **30** (1989), 1401-1407
2. J. Blank, P. Exner, M. Havlíček, *Lineární Operátory v Kvantové Fyzice*, Karolinum, Prague, 1993
3. H.D.Doebner,P. Šťovíček, J. Tolar, Quantization of kinematics on configuration manifolds, *Reviews in Mathematical Physics*, **13**(7) (2001), 799-845
4. C. J. Isham, J. R. Klauder, Coherent states for n-dimensional Eucledean groups $E(n)$ and their application, *J. Math. Phys.* **32**(3), (1991), 607-620
5. J. R. Klauder, Bo-Sture Skagerstam: *Coherent States*, World Scientific, Singapore, 1985
6. G. W. Mackey, *Induced Representations and Quantum Mechanics*, W. A. Benjamin, New York, 1968
7. J. Niederle, Group theory, part I, Lecture notes, Miaramare-Trieste, 1969
8. M. A. Del Olmo, José González: Coherent states on the circle and quantization, 207-221
9. A. M. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag, Berlin, 1986
10. A. M. Perelomov, On the completeness of a system of coherent states, *Teor. Mat. Fiz.* **6**(2) (1971), 213-224
11. P. Šťovíček, J. Tolar, Topologie gonfigurační variety a kvantová mechanika, Acta Polytechnica-Práce ČVUT v Praze (6(IV,1)), (1984),37-75
12. P. Šťovíček, J. Tolar, Quantum mechanics in a discrete space-time, *Rep. Math. Phys.* **20** (1984), 157-170
13. J. Tolar: Quantization methods, Preliminary lecture notes, Clausthal, 1978
14. J. Tolar, G. Chadzitaskos, Quantization on $\mathbf{Z}_M$ and coherent states over $\mathbf{Z}_M \times \mathbf{Z}_M$, *J. Phys. A: Math. Gen.* **30** (1997), 2509-2517