CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

Department of Physics

Operator Algebraic Approach to Quantum Theory

Research project

Supervisor: Prof. RNDr. Jan Hamhalter, CSc.,
Czech Technical University, Faculty of Electrical Engineering,
Department of Mathematics

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Author: Jaroslav Kučera

Abstract: Hidden variables theory is studied in W^*-algebras. Possible generalization of this theory to Jordan algebras is investigated. We show that type I_2 JBW algebras always admit $0-1$ state. On the other hand, our main result states that there is no $0-1$ state on JBW algebra without associative and Type I_2 direct summand. We also generalize the *Kochen-Specker* Theorem to JBW algebras. A counterexample for the Function Principle in algebra of matrices $M_2(\mathbb{C})$ is demonstrated.

Key words: Operator algebras, Jordan algebras, C^*-algebras, W^*-algebras, JBW algebras, spin factor, hidden variables, quantum measure, quantum logic

CONTENTS

1. INTRODUCTION

1.1 Hilbert lattice as quantum logic

As we know classical probability theory allows us to assign to any element of a given set of events Ω, the *sample space*, a probability for its realization which we want to express on the algebra $\mathcal{D}$ given by all possible *logical combinations* of events. We represent logical combinations with the *set-theoretic* operations of intersection (which represents *and*), union (which represents *or*), and complement (which represents *not*). Thus $\mathcal{D}$ is obtained by Ω closing it under these operations. *Kolmogorov's axioms*[1] allow us to extend a probability measure p from Ω to the entire algebra $\mathcal{D}$.

Quantum probability theory is deduced in the following way: given a separable Hilbert space $\mathcal{H}$, every one-dimensional subspace (every normalized vector) corresponds to a simple event, so that $\mathcal{H}$ can be considered the sample space. We generate an algebra of events $\mathcal{L}$ [2] by closing the sample space under the operations of span, intersection and orthogonal complement. These operations correspond to the lattice-theoretic operations of *join* (denoted by $\vee$), meet (denoted by $\wedge$) and orthocomplement (denoted by $^{\perp}$). Vector can be represented as the (one-dimensional) subspace that is spans which in turn can be represented by the projection operator which projects on it. The probability measure p satisfies Kolmogorov's axiom 3 only when the subspaces representing the events are orthogonal. Hence we see that even though in classical probability theory the feature such as *orthogonal* (*compatible*) *statements* does not exist, it is not possible to neglect this in the quantum probability theory.

1.2 The Birkhoff-von Neumann concept of quantum logic

In 1936 Birkhoff and von Neumann postulated that the quantum logic, i.e. the algebraic structure that should replace the logic (Boolean algebra) of a classical system has the same structure as an *abstract projective geometry.* By an abstract projective geometry Birkhoff and von Neumann meant an *orthocomplemented*, *modular* lattice. They were aware of the fact that the lattice of projections on an infinite dimensional Hilbert space is not modular, thus it was not the Hilbert lattice to represent the logic of quantum system. They were not only searching for a non-commutative (i.e. quantum) logic but also for a non-commutative generalisation of classical probability theory. To be able to obtain normalised measure, the projection structure should be modular. Since with non-modularity of $\mathcal{P}(\mathcal{H})$, with $\mathcal{H}$

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[1] *Kolmogorov's axioms*:
(1) $p(\emptyset) = 0$,
(2) $p(\neg a) = 1 - p(a)$,
(3) $p(a \cup a') = p(a) + p(a') - p(a \cap a')$.

[2] The algebra of quantum-mechanical events is denoted by $\mathcal{L}$ because it forms a *lattice*.

infinite dimensional, there is no *dimension function*[3], it was considered by von Neumann as an obstacle to accepting the usual Hilbert space quantum mechanics as a non-commutative probability theory.

Back in 1927, the paper of Hilbert, Nordheim and von Neumann attempts an axiomatic description of quantum mechanics starting with the *amplitude*[4] $\phi(x, y; F_1, F_2)$ of the *density for relative probability.* The quantity w defined by

$$\phi(x, y; F_1, F_2)\overline{\phi}(x, y; F_1, F_2) = w(x, y; F_1, F_2) \tag{1.1}$$

is assumed to give the *probability density* for the probability that for a fixed value y of the quantity F_2 the value of the quantity F_1 lies between a and b, i.e. this probability is given by

$$\int_a^b w(x, y; F_1, F_2)dx\,. \tag{1.2}$$

This probability is called relative because w is not normalised. In the above mentioned paper the amplitudes are identified with kernels of integral operators, and since the assumption is made that every operator is an integral operator, the Dirac function must be allowed as a kernel. In particular, the relative probability density for the probability that a quantity has simultaneously the value x and y turns out to be given by $\delta(x - y)$. The interpretation of this probability density given in the paper is that the probability relation between a quantity to itself distinguishes the value of the quantity infinitely sharply. Although this seems to be reasonable, the authors were fully aware that the Dirac function as they used it was not mathematically legitimate. It turned out, however, that a new, conceptual difficulty arises even if the "problematic" Dirac function is eliminated from the formalism: infinite probabilities appear in the theory.

In this paper crucial role is played by "statistical Ansatz" stating that the relative probability that the values of the pairwise commuting quantities S_i lie in the intervals I_i if the values of the pairwise commuting quantities R_j lie in the intervals J_j are given by

$$Tr\big(E_1(I_1)E_2(I_2)\ldots E_n(I_n)F_1(J_1)F_2(J_2)\ldots F_m(J_m)\big) \tag{1.3}$$

$E_i(I_i), F_j(J_j)$ being the spectral projections (belonging to the respective intervals) of the corresponding operators S_i and R_j. The reason why von Neumann calls the probability in (1.3) *relative* is discussed in [12].

As long as the probability in (1.3) is finite, no distinction between relative and absolute probability is significant since in this case it can be normalised. However, von Neumann realizes that the relative probability in (1.3) can be infinite. This happens if any of I_i or J_j contains parts of the continuous spectrum of R_i or S_j. Von Neumann justified the usage of infinite probability in one of his papers published in 1927 and showed that each *elementary*

[3] Dimension function is a map $d : \mathcal{L} \mapsto [0, 1]$ having the following properties:
(i) $d(\mathbb{I}) = 1$; $d(A) = 0$ if and only if $A = 0$, $A \in \mathcal{L}$;
(ii) if $A \wedge B = 0$ then $d(A + B) = d(A) + d(B)$;
(iii) if A, B are *perspective* (i.e. if they have a *common complement*), then and only then $d(A) = d(B)$.

[4] In quantum mechanics, a *probability amplitude* is a complex-valued function that describes an uncertain or unknown quantity. For example, each particle has a probability amplitude describing its position. This amplitude is then called *wave function.* This is a complex-valued function of the position coordinates.

unordered ensemble can be described by a *statistical operator*, a positive, non-zero operator U today known as *density matrix* describing *ensembles* of systems prepared in different quantum states. In particular, the density matrix is defined as

$$\rho = \sum_i \lambda_i P_i \,, \tag{1.4}$$

where P_i are one dimensional projections determined by a single state vector ν_i. The expectation value of an observable Q in the statistical ensemble described by ρ is given by

$$< Q >_\rho = Tr(\rho Q) = \sum_j < \nu_j, \rho Q \nu_j > \ . \tag{1.5}$$

This interpretation has serious conceptual difficulties: the main problem is that the statistical operator I is not normalised, i.e. its trace is infinite, as it is the trace of any infinite dimensional projection. This concept was fortunately successfully completed in 1936 by classifying the factors and by discovering the type II_1 factor: there exists an a priori probability on this lattice which is given uniquely by the trace. Von Neumann really became unfaithful to the Hilbert space formalism which was clear from his famous letter to Birkhoff, where he writes:

> I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space anymore. After all Hilbert space (as far as quantum mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of conserving the validity of all normal rules ... Now we begin to believe that it is not the *vectors* which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical *states*, but they do it redundantly, up to a complex factor only, 2) and besides, the states are merely a derived notion, the primitive (phenomenogically given) notion being the equalities which correspond to the *linear closed subspaces*. But if we wish to generalize the lattice of all linear closed subspaces to a Euclidean space to infinitely many dimensions, then one does not obtain Hilbert space, but that configuration which Murray and I called case II_1. (The lattice of all linear closed subspaces of Hilbert space is our 'case I_∞'.)[5]

It is the uniquely determined trace τ on a type II_1 factor which von Neumann interpreted as the proper a priori probability in quantum mechanics. A trace is just the *unique* (positive, linear, normalized) functional invariant with respect to all *unitary transformation*. Since the physical symmetries of the system are generally expressed as representations of the symmetry group on the algebra of observables by unitaries, the existence of unique trace means physically that the probability is determined uniquely as the only (positive, linear) assignment of values in $[0, 1]$ to the events that is invariant with respect to any conceivable symmetry.

Another reason why von Neumann and Birkhoff found lattices very significant (in von Neumann's words [6]) is:

[5] Reader can refer to [12] for more detailed discussion over this topic as this is not the prime concern of this work. We only wanted to sketch out briefly various approaches to quantum logic and its history.

[6] This excerpt quoted from [12]

> Essentially if a state of a system is given by one vector, the transition probability in another state is the inner product of the two which is the square of cosine of the angle between them. In other words, probability corresponds precisely to introducing the angles geometrically. Furthermore, there is only one way to introduce it. The more so because in the quantum mechanical machinery the negation of a statement, so the negation of a statement which is represented by a linear set of vectors, corresponds to the orthogonal complement of this linear space. And therefore, as soon as you have introduced into the projective geometry the ordinary machinery of logics, you must have introduced the concept of orthogonality. ... In order to have probability all you need is a concept of all angles, I mean angles other than 90°. Now it is perfectly quite true that in geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, leave all angles intact, in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logic are fixed under some isomorphic transformation, all of probability theory is already fixed.

NOTE: In this work we use the Birkhoff and von Neumann approach. Simply put, the basic role is played by the convex set of positive norm-one functionals called *states*; quantum propositions[7] are identified with projections in Hilbert spaces. The basic axioms of C^*-algebraic quantum mechanics can be summarized into these points:

(i) The set of all observables of a quantum system $\mathcal{S}$ is the self-adjoint part of a C^*-algebra $\mathcal{A}$.

(ii) The set of all states of a quantum system $\mathcal{S}$ is the state space, $\mathcal{S}(\mathcal{A})$, of the C^*-algebra $\mathcal{A}$.

(iii) The value $\rho(a)$, where $\rho \in \mathcal{S}(\mathcal{A})$ and $a \in \mathcal{A}_{s.a.}$ is the expectation value of an observable a on the condition that a system $\mathcal{S}$ is prepared in the state ρ.

(iv) Evolution of a system $\mathcal{S}$ is given by a specified class of morphisms of the C^*-algebra $\mathcal{A}$ (unitary maps, automorphisms, completely positive maps).

(v) Given independent quantum systems $\mathcal{S}_1$ and $\mathcal{S}_2$ represented by C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, the smallest composite system containing $\mathcal{S}_1$ and $\mathcal{S}_2$ is given by the minimal tensor product[8] $\mathcal{S}_1 \otimes_{min} \mathcal{S}_2$.

1.3 THE PROBLEM OF HIDDEN VARIABLES

An attack towards the orthodox view stating quantum mechanics to be a complete theory came in 1935 from Einstein, Podolsky and Rosen (so-called *EPR paradox*) in their

[7] an observable with two possible values 0,1. In C^*-algebraic quantum mechanics the system of observables is given by the self-adjoint part of a C^*-algebra.

[8] See definition of the tensor product in the section *Operator algebras: C^*-algebras* or [7].

famous paper "*Can quantum-mechanical description of physical reality be considered complete?*". This motivated the search for alternative interpretations of quantum theory based on the assumption of *hidden variables*. This problem is not purely physical as one might have thought but has commonly been associated with the question of validity of certain very general philosophical (metaphysical) principles, such as the principle of determinism or the philosophical standpoint known as realism. This problem has not been settled once for all but has always been rephrased within every new framework of quantum theory that has been created. Simply put, some theoreticians suggested the existence of *hidden space* on which the states would possess zero or at least an arbitrarily small pointwise dispersion, which is convenient to be viewed as a *"measure of statistical character of quantum states"*.

A turning point which was subject to controversy was so-called *von Neumann's impossibility proof*, which is the first definite result in this field falsifying determinism. Von Neumann investigates the statistical character of quantum mechanics and the problem of quantum mechanical probabilities in connection with the probabilities occurring in classical statistical mechanics. The significance of keeping in mind this context of von Neumann's proof comes from the fact that the concept of state in classical statistical mechanics already raises the problem of determinism in that it contradicts the state concept in classical mechanics: the classical statistical mechanical state is (in today's terminology) a *probability measure*. The reason why this probability measure should indeed be viewed as a physical state is that it can be identified with thermodynamical states - as far as one can derive thermodynamic relations with its help - on the other hand, this state concept is incompatible with the state concept of classical mechanics because, unlike in the case of classical mechanical state (which is identified with a single point in the phase space), the physical quantities possess a non-zero dispersion in a statistical mechanical state.

Physicists insisted on accepting classical statistical mechanical states as physical states. It was due to accepting the following reasoning: A classical statistical mechanical system consists of a large number of particles interacting and moving according to the laws of classical mechanics and, therefore, the point representing the whole system in the phase space also moves deterministically. To describe the motion of the phase point would require both the exact knowledge of the initial states of all the particles and the ability to solve large number of differential equations. But one is unable to solve so many equations of motions, since the initial conditions are not known either. For this reason it was *logical* to describe the system by probabilities, although the real physical system is at every moment in some well defined state as understood in classical mechanics.

These well defined real physical states in the sense of classical mechanics, i.e. the single phase point can be identified with the *Dirac measures* concentrated at these points with the Dirac measures being nothing but *dispersion-free states* in classical statistical mechanics. The classical mechanical states in this interpretation can be considered as our inability to give the precise (*pure*) state the system is in.

The historic significance in von Neumann's impossibility proof was that pure quantum states are not dispersion-free, thus one has to meet the challenge of interpreting probability in quantum mechanics. This conclusion was critisized by theoreticians who tried to refutate von Neumann's impossibility proof by constructing hidden variable theories. There also were attempts aiming at producing stronger impossibility proofs by weakening von Neumann's assumptions that had been questioned. This resulted in two different axiomatic approaches to quantum mechanics after 1932: the one based on lattice theory and the other one formulated in terms of the operator algebra theory. It is the theory of *von Neumann algebras* that connects these two approaches. Later, von Neumann's definition of hidden variables and impossibility proof was generalized in the operator algebraic approach by Misra.

It should be mentioned that three main theorems against hidden variables hypothesis (so-called *no-go theorems*) are *von Neumann's theorem*, *the Bell inequalities* and *Kochen-Specker theorem*[9].

1.4 JORDAN-ALGEBRAIC FORMULATION OF QUANTUM MECHANICS

One of the essential "ingredients" of quantum mechanics are observables. In C^*-algebraic approach they are taken to be the self-adjoint part of a C^*-algebra $\mathcal{A}$. The C^*-algebra contains a lot of non-self-adjoint elements and can therefore hardly be considered the most essential structure for the observables. Since the C^*-product of two self-adjoint elements in general is not self-adjoint, the observables do not even form a C^*-subalgebra but only a real subspace. But they do form an algebra under the product

$$a \circ b = \frac{1}{2}(ab + ba) \tag{1.6}$$

which is commutative but not fully associative anymore. Instead, one has

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2) \tag{1.7}$$

which is equivalent to the power-associativity of $\mathcal{A}$. The classification of finite-dimensional, formally real Jordan algebras[10] gives two classes of special algebras: self-adjoint matrix algebras over the reals, the complexes or the *quaternions* and *spin factors*. In addition, there is one *exceptional* (i.e. non-special Jordan algebra, the self-adjoint 3×3-matrices over the octonions.)

The Jordan-algebraic approach to quantum mechanics was initiated by Jordan, von Neumann and Wigner who studied finite dimensional *formally real* Jordan algebras. The restriction to finite dimensions was removed by von Neumann. Jordan operator algebras (linear spaces of self-adjoint operators on a Hilbert space closed under the Jordan product) first were studied by Segal, Topping and Størmer. The self-adjoint part of a C^*-algebra or a von Neumann algebra is a special case of a JB algebra or a JBW algebra, respectively.

[9] see Theorem 164

[10] See definitions in this work.

2. OPERATOR ALGEBRAS

2.1 Preliminaries in topology

Starting point in the theory of topology is definition of *neighborhood*. Let V be a normed vector space. Then for arbitrary $x \in V$ and $\varepsilon > 0$ the set $U_\varepsilon(x) := \{y \in V \,|\, ||x - y|| < \varepsilon\}$ is called *ε-neighborhood* of x. Let $F \subseteq V$, then for each $\varepsilon > 0$ the set $U_\varepsilon(F) := \{y \in V \,|\, ||y - x|| < \varepsilon\,,\ \text{for all } x \in F\}$ is called *ε-neighborhood* of the set $F \subseteq V$.

Definition 1 *A topological space, also called an abstract topological space, is the set X together with a collection of open subsets τ that satisfies the four conditions: (i) $\emptyset \in \tau$ (ii) $X \in \tau$ (iii) $\bigcap_{\alpha=1}^{n} a_\alpha \in \tau$ for finite $n \in \mathbb{N}$ and $(a_\alpha)_{\alpha=1,\cdots,n} \subseteq \tau$ (iv) for an arbitrary system $(a_\alpha)_\alpha \subseteq \tau$ it holds that $\bigcup_\alpha a_\alpha \in \tau$.*

Definition 2 *Topological space (X, τ) is said to fulfil (separation) axiom: (i) T_1, if for each pair of points $x, y \in X$, $x \neq y$ there exists neighborhood $U(x)$ such that $y \notin U(x)$ (ii) T_2, if for each pair of points $x, y \in X$, $x \neq y$ there exist disjoint neighborhoods $U(x)$ and $U(y)$ (iii) T_3, if for each $x \in X$ and closed set F such that $x \notin F$ there exist disjoint neighborhoods $U(x)$ and $U(F)$ (iv) T_4, if for each pair of disjoint closed sets F, G there exist disjoint neighborhoods $U(F)$ and $U(G)$. Space, in which axioms T_1 and T_j , $j = 1, \ldots, 4$ hold, is called T_j-space.*

Definition 3 *T_2-space is called a Hausdorff space.*

Definition 4 *A topological space (X, τ) is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected. A totally disconnected space is a space in which all subsets with more than one element are disconnected. A topological space is said to be compact if each of its open covers has a finite subcover.*

A subset K of vector space K is called *convex* if $\alpha x + (1-\alpha)y \in K$ whenever $x, y \in K$ and $\alpha \in [0, 1]$. A *face* F in K is a convex subset of K such that if $\alpha x + (1 - \alpha)y \in F$ for $x, y \in K$ and $\alpha \in [0, 1]$, then $x, y \in F$. An element $x \in K$ is called an *extreme point* of the set K if the set $F = \{x\}$ is a face of K, i.e. if $x = \alpha y + (1-\alpha)z$, $y, z \in K$, $\alpha \in [0, 1]$, implies $y = z = x$.

Definition 5 *Let X be a Hausdorff locally convex topological vector space. A subset A of X is called* a cone *in X if $x \in A$ and $\lambda \in \mathbb{R}$, $\lambda > 0$, implies $\lambda x \in A$, and $(-A) \bigcap A = \{0\}$, where $-A = \{-x : x \in A\}$. If A is a convex subset we talk about* convex cone.

2.2 C^*-algebras

Definition 6 *A vector space $\mathcal{A}$ over a field $\mathcal{T}$ with a binary composition $(a, b) \mapsto a \cdot b \in \mathcal{A}$ is called algebra if it satisfies*

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a,$$
$$\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b),$$

for all $a, b, c \in \mathcal{A}$, $\alpha \in \mathcal{T}$. If $\mathcal{T} = \mathbb{C}$ or $\mathcal{T} = \mathbb{R}$ we call $\mathcal{A}$ a complex or real algebra, respectively. $\mathcal{A}$ is called unital if there is unit, $e \in \mathcal{A}$, such that $a \cdot e = e \cdot a = a$, for all $a \in \mathcal{A}$. $\mathcal{A}$ is called associative if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in \mathcal{A}$. $\mathcal{A}$ is called commutative (abelian) if $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{A}$.

Let Ω be a set of operations defined on algebra $\mathcal{A}$. Such algebra is usually denoted by ordered doublet $(\mathcal{A}, \Omega)$. Each set $\mathcal{D} \subseteq \mathcal{A}$ such that $\mathcal{D}$ is algebra (with respect to the same operations) is called subalgebra of algebra $\mathcal{A}$. Generally, unit in $\mathcal{A}$ does not have to be an element of $\mathcal{D}$.

Since operator algebras are our prime concern, the elements of algebras will always be operators (matrices or projections in special cases, see def.).

Definition 7 *An associative complex algebra $\big(\mathcal{A}, +, \cdot, *\big)$ is called $*$-algebra (involutory, involutive) if, for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, the operation $*$ (involution) obeys the following rules: (i) $a^{**} = a$, (ii) $(a + b)^* = a^* + b^*$, (iii) $(\lambda a)^* = \overline{\lambda} a^*$, (iv) $(ab)^* = b^* a^*$.*

Note 8 *An element $a^* \in \mathcal{A}$ is called the adjoint of $a \in \mathcal{A}$. Let $\mathcal{A}$ be $*$-algebra, $\mathcal{S} \subset \mathcal{A}$. Denote by $\mathcal{S}$ adjoint part of $\mathcal{A}$, i.e. $\mathcal{S}^* := \{a^* \,|\, a \in \mathcal{S}\}$. We say that $\mathcal{S}$ is self-adjoint if, and only if, $\mathcal{S}^* = \mathcal{S}$. An element a such that $a = a^*$ is called hermitian (self-adjoint).*

Definition 9 *Let $\mathcal{A}$ be a complex $*$-algebra with the norm $||\cdot||$. $\mathcal{A}$ is said to be an involutive Banach algebra if the following conditions are satisfied for all $a, b \in \mathcal{A}$: (i) $||ab|| \leq ||a|| \cdot ||b||$ and (ii) $||a^*|| = ||a||$. A C^*-algebra is an involutive Banach algebra satisfying (iii) $||a^* a|| = ||a||^2$ for all $a \in \mathcal{A}$. A C^*-subalgebra B of a C^*-algebra $\mathcal{A}$ is defined as the normed closed $*$-subalgebra of $\mathcal{A}$.*

For an arbitrary set $\mathcal{S} \subset \mathcal{A}$ denote by $\mathcal{A}(\mathcal{S})$ an algebra generated by $\mathcal{S}$, which is defined as the smallest subalgebra (in terms of inclusion) of $\mathcal{A}$ containing $\mathcal{S}$. In particular, C^*-algebra generated by the elements $a_1, a_2, \ldots, a_n$ will be denoted by $C^*(a_1, \ldots, a_n)$.

Note 10 *Since the condition (i) with (iii) implies (ii) in the previous definition, it suffices that an involutive Banach algebra satisfies (i) and (iii) only.*[1]

Despite the fact that generally C^*-algebras do not have to be unital, we will (without loss of generality) assume only unital C^*-algebras in this work unless stated otherwise. This is possible due to the fact that if $\tilde{\mathcal{A}}$ is an algebra obtained by appending the unit to nonunital C^*-algebra $\mathcal{A}$, then the norm $||\cdot||_{\mathcal{A}}$ in $\mathcal{A}$ extends uniquely to a norm $||\cdot||_{\tilde{\mathcal{A}}}$ in $\tilde{\mathcal{A}}$, which makes $\tilde{\mathcal{A}}$ a C^*-algebra.

An example of finite-dimensional C^*-algebra is the complex algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$ endowed with the standard arithmetic operations, matrix norm and the involution sending a matrix $a \in M_n(\mathbb{C})$ to its adjoint matrix a^*. Finite-dimensional C^*-algebras are direct sums of algebras of this type.

Another very important example is the $*$-algebra of all bounded operators on a Hilbert space $\mathcal{H}$ denoted by $\mathcal{B}(\mathcal{H})$. Such algebra with the operator norm

$$||a|| := sup\{||a\xi|| \,|\, \xi \in \mathcal{H},\ ||\xi|| = 1\}$$

[1] Indeed, $||a||^2 = ||a^* a|| \leq ||a|| \cdot ||a^*|| \Rightarrow ||a|| \leq ||a^*||$ and $||a^*||^2 = ||a^{**} a^*|| \leq ||a^{**}|| \cdot ||a^*|| \Rightarrow ||a^*|| \leq ||a^{**}|| = ||a||$. Thus we have $||a|| \leq ||a^*|| \leq ||a||$ which implies $||a^*|| = ||a||$.

is a C^*-algebra.

Let $\mathcal{A} = M_n(\mathbb{C})$, $\mathcal{B} = M_m(\mathbb{C})$ be matrix algebras. Then their *tensor product* denoted by $\mathcal{A} \otimes \mathcal{B}$ is the algebra $M_{mn}(\mathbb{C})$. For $a = (a_{ij}) \in \mathcal{A}$ and $b = (b_{kl}) \in \mathcal{B}$, the tensor product $a \otimes b$ is the $nm \times nm$ matrix

$$a \otimes b = \begin{pmatrix} b_{11}a & b_{12}a & \dots & b_{1m}a \\ b_{21}a & b_{22}a & \dots & b_{2m}a \\ \dots & & & \\ b_{m1}a & b_{m2}a & \dots & b_{mm}a \end{pmatrix}.$$

Let $\mathcal{A}$ be a C^*-algebra. The tensor product $\mathcal{A} \otimes \mathcal{B}(\mathcal{H}_n)$, where $\dim\mathcal{H}_n = n$, can be identified with the algebra $M_n(\mathcal{A})$ of all $n \times n$ matrices with entries in $\mathcal{A}$ and usual matrix operations. A more general than this is the tensor product $\mathcal{A}\otimes\mathcal{B}(\mathcal{H})$ which can be identified with the algebra of infinite matrices over $\mathcal{A}$.

Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$. A commutant of $\mathcal{S}$ is the set $\mathcal{S}' := \{B \in \mathcal{B}(\mathcal{H}) : BC = CB, C \in \mathcal{S}\}$. Bicommutant of $\mathcal{S}$ is the set $\mathcal{S}'' := \big(\mathcal{S}'\big)'$.

Definition 11 *The center, $\mathcal{Z}(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is the set $\mathcal{Z}(\mathcal{A}) := \{a \in \mathcal{A} \,|\, a \cdot b = b \cdot a, \forall\, b \in \mathcal{A}\}$.*

Definition 12 *A projection in a C^*-algebra $\mathcal{A}$ is a self-adjoint idempotent, i.e. an element $p \in \mathcal{A}$ such that $p = p^2 = p^*$. The set of all projections in a C^*-algebra $\mathcal{A}$ will be denoted by the symbol $P(\mathcal{A})$.*

Indivisible part of this work is the function calculus in the theory of C^*-algebras, which is a consequence of the Gelfand Theorem and turns out to be very practical tool that helps us to view C^*-algebras as the algebras of continuous functions on locally compact Hausdorff spaces (see def.).

Let X be a locally compact Hausdorff space. Let us say that a continuous function f on X vanishes at infinity, if for each $\epsilon > 0$ the set $\{x \in X \,|\, |f(x)| \geq \epsilon\}$ is compact. Let us denote by $C_0^{\mathbb{C}}(X)$ (resp. $C_0(X)$) the $*$-algebra of all continuous complex (resp. real) functions on X vanishing at infinity, where the $*$-operation assigns to each function its complex conjugate. Let us endow $C_0(X)$ with the norm $||f|| := sup_{x\in X}|f(x)|$. Then $C_0(X)$ becomes an abelian C^*-algebra. In particular, if X is a compact space, then the algebra $C_0(X)$ coincides with the algebra $C(X)$ of all continuous complex functions on X.

Definition 13 *Let $\mathcal{A}$ be abelian C^*-algebra. A character ω on $\mathcal{A}$ is a nonzero linear map $\omega : \mathcal{A} \mapsto \mathbb{C}$ for which $\omega(a \cdot b) = \omega(a) \cdot \omega(b)$ and $\omega(a^*) = \overline{\omega(a)}$.*

Let $\mathcal{A}$ be a C^*-algebra, $x \in \mathcal{A}$. The spectrum of x (in symbols $\sigma(x)$) is the set

$$\sigma(x) := \{\lambda \in \mathbb{C} \,|\, (x - \lambda \cdot \mathbb{I}) \text{ is not invertible in } \mathcal{A}\},$$

where $\mathbb{I}$ denotes the unit in $\mathcal{A}$. The spectrum $\Omega(\mathcal{A})$ of an algebra $\mathcal{A}$ is the set of all characters on $\mathcal{A}$ endowed with the topology of pointwise convergence on elements of $\mathcal{A}$. It can be proved that $\Omega(\mathcal{A})$ is a locally compact Hausdorff space (see def.) and that $\Omega(\mathcal{A})$ is compact if and only if $\mathcal{A}$ is unital. The Gelfand transform is the map $\tau : \mathcal{A} \mapsto C_0\big(\Omega(\mathcal{A})\big)$ defined for all $a \in \mathcal{A}$ by the formula

$$\tau(a)(\omega) = \omega(a),$$

for all $\omega \in \Omega(\mathcal{A})$.

Theorem 14 (GELFAND) *For each abelian C^*-algebra $\mathcal{A}$ the Gelfand transform is a $*$-preserving isometric isomorphism of $\mathcal{A}$ onto $C_0\big(\Omega(\mathcal{A})\big)$.*

As a corollary of the Gelfand Theorem it is possible to identify the algebra $C^*(x)$ with the algebra $C_0\big(\sigma(x) \setminus \{0\}\big)$, where $x \in \mathcal{A}$ is a normal element, i.e. $x^*x = xx^*$. It is not constitutive to go into details here. Reader can refer to [9]. For a normal element $x \in \mathcal{A}$ we denote by $f(x)$ the element in $C^*(x)$ that corresponds to the function $f \in C_0\big(\sigma(x) \setminus \{0\}\big)$. Then the assignment $f \to f(x)$ is the function calculus. If x is a normal operator on a Hilbert space $\mathcal{H}$ and $x\xi = \lambda\xi$ for a unit vector $\xi \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then $f(x)\xi = f(\lambda)\xi$ for any $f \in C_0\big(\sigma(x) \setminus \{0\}\big)$.

Theorem 15 (COMPLEX SPECTRAL THEOREM) *Let $\mathcal{A}$ be a unital C^* algebra and x a self-adjoint element in $\mathcal{A}$. Let $C^*(x)$ denote the C^* subalgebra of $\mathcal{A}$ generated by x and the unit $\mathbb{I}$. Then there is a canonical isometric isomorphism of $C^*(x)$ onto $C^{\mathbb{C}}(\sigma(x))$.*

Definition 16 *A closed subset $\mathcal{I}$ of $\mathcal{A}$ is called the closed right (left) ideal if $\mathcal{I} \cdot \mathcal{A} \subset \mathcal{I}$ $\big(\mathcal{A} \cdot \mathcal{I} \subset \mathcal{I}\big)$. A closed subset in $\mathcal{A}$ is called closed (two-sided) ideal if it is simultaneously left and right closed ideal. Zero subalgebra $\{0\} \subset \mathcal{A}$ is referred to as a trivial ideal.*

Definition 17 *The C^*-algebra having no nontrivial ideals is called simple.*

Definition 18 *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. A linear map $\pi : \mathcal{A} \mapsto \mathcal{B}$ is called $*$-homomorphism if it satisfies $\pi(a \cdot b) = \pi(a) \cdot \pi(b)$ and $\pi(a^*) = \pi(a)^*$, for all $a, b \in \mathcal{A}$.*

Note 19 *Since for any $*$-homomorphism it holds that $||\pi(x)|| \leq ||x||$ for all $x \in \mathcal{A}$, it is continuous.*

Definition 20 *$*$-homomorphism of C^*-algebra $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ is called $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. An injective $*$-homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ is called $*$-isomorphism. If the kernel of a $*$-representation satisfies $Ker\,\pi = \{a \in \mathcal{A} \,|\, \pi(a) = 0\} = \{0\}$ then π is called faithful.*

2.2.1 STATES AND REPRESENTATIONS

In this work a crucial role is played by investigation of properties of positive functionals on C^*-algebras which is due to the fact that (under certain conditions) a probability measure on such algebra can be represented by a positive functional.

Definition 21 *An element x of a C^*-algebra $\mathcal{A}$ is called* positive *(or* non-negative*) if $x = a^*a$ for some $a \in \mathcal{A}$.*

Definition 22 *A linear form $f : \mathcal{A} \mapsto \mathbb{C}$ on a C^*-algebra $\mathcal{A}$ is called a positive functional if $f(a) \geq 0$ whenever $a \geq 0$.*

Note 23 *Any positive functional on C^*-algebra is bounded.*

Definition 24 *A real non-negative functional $f : V \mapsto \mathbb{R}$ is called* seminorm *(or* pseudonorm*) if it holds that*

$$f(x+y) \leq f(x) + f(y), \quad f(\alpha x) = |\alpha| f(x)\,,$$

for all $x, y \in V$, $\alpha \in \mathbb{C}$.

Definition 25 *A positive functional ρ on a C^*-algebra is called a state if $||\rho|| = 1$.*

The convex set of all states on $\mathcal{A}$ will be denoted by $S(\mathcal{A})$ and called the *state space* of $\mathcal{A}$. The extreme points of $S(\mathcal{A})$ are called *pure states*. A subset S of the state space $S(\mathcal{A})$ is called *order determining* if, and only if, $\rho(a) \geq 0$ for all $\rho \in S$ implies that $a \geq 0$.

Definition 26 *Let $\mathcal{A}$ be a C^*-algebra. A vector $\psi \in \mathcal{H}$ is called cyclic for the representation π of $\mathcal{A}$ on Hilbert space $\mathcal{H}$ if the set $\pi(\mathcal{A})\psi = \{\pi(a)\psi \,|\, a \in \mathcal{A}\}$ is dense in $\mathcal{H}$.*

Definition 27 A partially ordered vector space *is a real vector space A with a proper convex cone A^+. We write $a \geq b$, or $b \leq a$ for $a, b \in A$, if $a - b \in A^+$. An element $e \in A^+$ is called an* order unit *for A if for all $a \in A$ there is $\lambda > 0$ such that $-\lambda e \leq \lambda e$. We say A is* Archimedean *if $na \leq e$ for all $n \in \mathbb{N}$ implies $a \leq 0$. In this case A has a norm given by*

$$||a|| = inf\{\lambda > 0 : \, -\lambda e \leq a \leq \lambda e\} \, .$$

This norm is called the order norm. *A is said to be an* order unit space *if A has an order unit and A is Archimedean. If A is furthermore a Banach space with respect to the order norm then A is called* a complete order unit space.

Proposition 28 *Let ρ be a positive functional on unital C^*-algebra $\mathcal{A}$ and let $a, b \in \mathcal{A}$ then*

$$|f(a^*b)|^2 \leq f(a^*a)f(b^*b). \quad (\mathrm{Cauchy-Schwarz\,inequality}) \tag{2.1}$$

The following Theorem deserves a special attention as it constructs very useful tool for obtaining the value of a positive functional by an inner product. It is known as the **Gelfand-Naimark-Segal** (G.N.S.) construction.

Theorem 29 (Gelfand-Naimark-Segal) *For any positive functional f on a unital C^*-algebra $\mathcal{A}$ there is a Hilbert space $\mathcal{H}$, $*$-morphism $\pi : \, \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and a cyclic vector, $\psi_0 \in \mathcal{H}$, such that*

$$f(x) = \big(\pi(x)\psi_0, \psi_0\big) \, . \tag{2.2}$$

Moreover, the triplet $\big(\pi, \mathcal{H}, \psi_0\big)$ is unique up to a unitary transformation between the corresponding Hilbert spaces.

Proof. The Hilbert space shall be constructed starting with C^*-algebra $\mathcal{A}$. Recall that inner product is linear in the first slot and conjugate linear in the second. Let f be a positive functional on $\mathcal{A}$. Then the map $\varphi : \, \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ such that $\varphi(a, b) := f(b^*a)$ defines a positive conjugate symmetric sesquilinear form. Indeed, for all $\alpha \in \mathbb{C}$ and $a, b, c \in \mathcal{A}$ we have that (i-a) $\varphi\big(\alpha a + b, c\big) = f\big(c^*(\alpha a + b)\big) = f\big(\alpha c^* a + c^* b\big) = \alpha f\big(c^* a\big) + f\big(c^* b\big) = \alpha\varphi\big(a, c\big) + \varphi\big(c, b\big)$, (i-b) $\varphi\big(a, \alpha b + c\big) = f\big((\alpha b + c)^* a\big) = f\big(\overline{\alpha} b^* a + c^* a\big) = \overline{\alpha} f\big(b^* a\big) + f\big(c^* a\big) = \overline{\alpha}\varphi\big(a, b\big) + \varphi\big(a, c\big)$, (ii) $\varphi\big(a, b\big) = f\big(b^* a\big) = f\big((a^* b)^*\big) = \overline{f\big(a^* b\big)} = \overline{\varphi\big(b, a\big)}$, and finally (iii) $\varphi(a, a) = f(a^* a) \geq 0$[2].

However $\varphi(a, a) = 0$ does not generally imply that $a = 0$. So far, except for the strict positivity, φ has the properties of an inner product. Such limitation can be superseded by factoring $\mathcal{A}$ [3]. Define a set $\mathcal{I}_f := \{a \in \mathcal{A} | f(a^* a) = 0\}$. Now we will show that $\mathcal{I}_f$ is the left ideal in $\mathcal{A}$ [4] ($\mathcal{A} \cdot \mathcal{I}_f \subset \mathcal{I}_f$), i.e. we need to show that $f\big((ab)^*(ab)\big) = 0$ for all $(a, b) \in \mathcal{A} \times \mathcal{I}_f$, which is just a matter of straightforward computation using the Cauchy-Schwarz inequality:

[2] By definition of positive functional $f \in \mathcal{A}^*$.

[3] Our aim is to construct Hilbert space which is a complete space endowed with the scalar product.

[4] By definition of ideal I_f is automatically a subspace in $\mathcal{A}$.

$$0 \leq |f\big((ab)^*(ab)\big)|^2 = |f\big(b^*a^*ab\big)|^2 = |f\big(b^*(a^*ab)\big)|^2 \leq$$
$$\leq f\big((a^*ab)^*(a^*ab)\big)f\big(bb^*\big) = 0,$$

implying that $f\big((ab)^*(ab)\big) = 0$.

Define an inner product on quotient $\mathcal{A}|_{\mathcal{I}_f} = \mathcal{A} + \mathcal{I}_f$ by

$$\big(\tilde{a}, \tilde{b}\big) := f\big(b^*a\big)\,,$$

where a, b are representatives of equivalence classes $\tilde{a}, \tilde{b} \in \mathcal{A}|_{\mathcal{I}_f}$, respectively. This definition must be correct independently of the representatives $a \in \tilde{a}$, $b \in \tilde{b}$. For this, it is enough to show that $f(b^*a) = 0$ whenever at least one of elements a, b is an element of $\mathcal{I}_f$. This condition is fulfilled since the righthand side of inequality

$$0 \leq |f(b^*a)|^2 \leq f(bb^*)f(a^*a)$$

is (by definition of $\mathcal{I}_f$) zero whenever either a or b is an element of $\mathcal{I}_f$. Because a and b were chosen arbitrarily, we are done with this. The Hilbert space $\mathcal{H}$ will be obtained by completion of $\mathcal{A}|_{\mathcal{I}_f}$.

The next point will be construction of the representation π. For all $a \in \mathcal{A}$ define map $\pi_0 : \mathcal{A}|_{\mathcal{I}_f} \mapsto \mathcal{A}|_{\mathcal{I}_f}$ by $\pi_0(a)\widetilde{b} = \widetilde{a \cdot b}$, where b is the element representing the class $\widetilde{b} \in \mathcal{A}|_{\mathcal{I}_f}$. The definition is correct, since $\mathcal{I}_f$ is the left closed ideal in $\mathcal{A}$. We have that

(i) $\pi_0(a)(\lambda\widetilde{b} + \widetilde{c}) = a \cdot (\lambda b + c)^\sim = \lambda\widetilde{a \cdot b} + \widetilde{a \cdot c} = \lambda\pi_0(a)\widetilde{b} + \pi_0(a)\widetilde{c}$

(ii) $||\pi_0(a)\widetilde{b}||^2 = ||\widetilde{ab}||^2 = (\widetilde{ab}, \widetilde{ab}) = f((ab)^*(ab)) =$
$= f(b^*a^*ab) \leq f(b^*eb)||a^*a||_{\mathcal{A}} = f(b^*b)||a^*a||_{\mathcal{A}} = ||b||_{\mathcal{A}|_{\mathcal{I}_f}}||a||_{\mathcal{A}} < \infty,$

for arbitrary $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{A}_{\mathcal{I}|_f}$, $\lambda \in \mathbb{C}$, where e is the unit in $\mathcal{A}|_{\mathcal{I}_f}$. Hence, π_0 is linear bounded and $||\pi_0(a)|| \leq ||a||_{\mathcal{A}}$. Since by construction $\mathcal{H} = \overline{\mathcal{A}|_{\mathcal{I}_f}}$[5], π_0 extends uniquely to $\pi \in \mathcal{B}(\mathcal{H})$ such that $||\pi_0|| = ||\pi||$. It is necessary to verify that π is a representation, i.e. $*$-homomorphism, which (because of the existence of a continuous extension of π_0) is enough to be done on $\mathcal{A}|_{\mathcal{I}_f}$ only. Linearity of π follows from linearity of π_0 (which has already been shown) and the existence of its continuous extension (by definition of π). It is sufficient to show that for arbitrary $a, b \in \mathcal{A}$, $\tilde{c}, \tilde{d} \in \mathcal{A}/\mathcal{I}_f$, the formulas (i) $\pi(a \cdot b)\tilde{c} = \pi(a)\pi(b)\tilde{c}$ and (ii) $\pi(a^*)\tilde{d} = \pi(a)^*\tilde{d}$ hold, which is true:

(i) $\pi(a \cdot b)\tilde{c} = (a \cdot b \cdot c)^\sim = (a \cdot b)^\sim\tilde{c} = \tilde{a} \cdot \tilde{b} \cdot \tilde{c} = \pi(a)\pi(b)\,\tilde{c}$

(ii) $(\tilde{c}, \pi(a^*)\tilde{d}) = (\tilde{c}, (a^*d)^\sim) = f(c^*a^*d) = (\widetilde{ac}, \tilde{d}) = (\pi(a)\tilde{c}, \tilde{d})$.

We will prove the existence of cyclic vector ψ_0 of the representation π. Let e be the unit in $\mathcal{A}|_{\mathcal{I}_f}$ and $\tilde{e}$ the corresponding equivalence class. Since $\pi(a)\widetilde{e} = \widetilde{ae} = \widetilde{e}$ for all $a \in \mathcal{A}|_{\mathcal{I}_f}$, we have that $\pi(\mathcal{A}|_{\mathcal{I}_f})\widetilde{e} = \mathcal{A}|_{\mathcal{I}_f}$. From the construction of $\mathcal{H}$, definition of cyclic vector and $\mathcal{H} = \overline{\mathcal{A}|_{\mathcal{I}_f}} = \overline{\pi(\mathcal{A})\widetilde{e}}$ we have that $\widetilde{e}$ is the cyclic vector of the representation π. We will denote the cyclic vector by the symbol ψ_0. For all $a \in \mathcal{A}|_{\mathcal{I}_f}$ we have that

$$(\psi_0, \pi(a)\psi_0) = (\widetilde{e}, \pi(a)\widetilde{e}) = (\tilde{e}, \tilde{a}) = f(e^*a) = f(ea) = f(a)\,.$$

To summarize the procedure, we have shown that for each positive functional $f \in \mathcal{A}^*$ there is at least one ordered triplet $\{\mathcal{H}, \pi, \Psi_0\}$ with the properties stated in the Theorem.

[5] $\mathcal{H}$ is also a Banach space.

Such ordered triplet will be called a G.N.S.-triplet and π a G.N.S.-representation.

To complete the proof it remains to show that if $\{\mathcal{H}', \pi', \psi_0'\}$ is another triplet with the properties stated in the Theorem, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ such that $\psi_0' = U\psi_0$ and $U\pi(a) = \pi'(a)U$ for all $a \in \mathcal{A}$. Suppose $\{\mathcal{H}', \pi', \psi_0'\} \neq \{\mathcal{H}, \pi, \psi_0\}$ is another G.N.S.-triplet. Put $\mathcal{H}_0 = \pi(\mathcal{A}|_{\mathcal{I}_f})\psi_0 = \mathcal{A}|_{\mathcal{I}_f}$, $\mathcal{H}_0' = \pi'(\mathcal{A}|_{\mathcal{I}_f})$ and define bijection $U_0 : \mathcal{H}_0 \mapsto \mathcal{H}'_0$ by $U_0\pi(a)\psi_0 = \pi'(a)\psi_0'$, for all $a \in \mathcal{A}|_{\mathcal{I}_f}$. We will show that U_0 is an isometry.

(i) U_0 is norm-preserving:

$$\begin{gathered}||U_0\pi(a)\psi_0||^2_{\mathcal{H}'} = ||\pi'(a)\psi_0'||^2_{\mathcal{H}'} = (\pi'(a)\psi_0', \pi'(a)\psi_0') = \\ = (\psi_0', (\pi'(a))^*\pi'(a)\psi_0') = (\psi_0', \pi'(a^*)\pi'(a)\psi_0') = (\psi_0', \pi'(a^*a)\psi_0') = \\ = f(a^*a) = (\psi_0, \pi(a^*a)\psi_0) = (\psi_0, \pi(a^*)\pi(a)\psi_0) = \\ = (\psi_0, \pi(a)^*\pi(a)\psi_0) = (\pi(a)\psi_0, \pi(a)\psi_0) = ||\pi(a)\psi_0||^2.\end{gathered}$$

(ii) U_0 is linear:

$$\begin{gathered}U_0[(\lambda\pi(a) + \pi(b))\psi_0] = U_0(\pi(\lambda a + b)\psi_0) = \pi'(\lambda a + b)\psi_0' = \\ = \lambda\pi'(a)\psi_0' + \pi'(b)\psi_0' = \lambda U_0\pi(a)\psi_0 + U_0\pi(b)\psi_0.\end{gathered}$$

Both U_0 being an isometry with $\overline{\mathcal{H}_0} = \overline{\mathcal{A}|_{\mathcal{I}_f}} = \mathcal{H}$ and $\overline{\mathcal{H}_0'} = \mathcal{H}'$ imply the existence of unique continuous extension of $U_0 \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_0')$ to $U \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$. Setting $a = e$ yields $U_0\pi(e)\psi_0 = \pi'(e)\psi_0'$. Because $\pi(e) = \mathbb{I}$, then $U\psi_0 = \psi_0'$. We get that $U\pi(a)\psi_0 = \pi'(a)\psi_0' = \pi'(a)U\psi_0$, which after application of the operator $\pi'(b)$ yields

$$\begin{gathered}\pi'(b)U\pi(a)\psi_0 = \pi'(b)\pi'(a)U\psi_0 = \pi'(ba)U\psi_0 = \\ = \pi'(ba)\psi_0' = U\pi(ba)\psi_0 = U\pi(b)\pi(a)\psi_0 \ ,\end{gathered}$$

ψ_0 is a cyclic vector in $\mathcal{H}$, that is $\overline{\pi(\mathcal{A})\psi_0} = \mathcal{H}$. In other words, the subspace $\pi(\mathcal{A})\psi_0$ is dense in $\mathcal{H}$. From here and

$$\pi'(b)U(\pi(a)\psi_0) = U\pi(b)(\pi(a)\psi_0)$$

we have equality of the operators $\pi'(b)U = U\pi(b)$ for all $b \in \mathcal{A}$, which completes the proof. $\square$

Definition 30 *Let π be a $*$-representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. A closed subspace $F \subset \mathcal{H}$ is called invariant for a representation π if $\pi(F) \subset F$.*

Note 31 *The previous definition is equivalent to saying that the orthogonal projection p of $\mathcal{H}$ onto F is in the commutant $\pi(\mathcal{A})'$. In this case π can be decomposed into the direct sum*

$$\pi = p\pi \oplus (1 - p)\pi \, . \tag{2.3}$$

The representation $p\pi$ is called a subrepresentation of π.

Definition 32 *Two $*$-representations π_1 and π_2 of a C^*-algebra $\mathcal{A}$ are called equivalent if there is a $*$-isomorphism $\tau : \pi_1(\mathcal{A})'' \mapsto \pi_2(\mathcal{A})''$ between bicommutants such that $\pi_2 = \tau \circ \pi_1$. The representations π_1 and π_2 are called unitarily equivalent if the automorphism τ above is implemented by a unitary map. The representation π_2 is called subequivalent to a representation π_2 if it is equivalent to some subrepresentation of π_2. A $*$-representation is called irreducible if it has no nontrivial invariant subspace, i.e. if it cannot be written as* *nontrivial direct sum of* $*$*-representations. Two representations* π_1 *and* π_2 *of* $\mathcal{A}$ *are called disjoint if no subrepresentation of* π_1 *is equivalent to any subrepresentation of* π_2.

Note 33 *Two irreducible representations are either unitarily equivalent or disjoint.*

Proposition 34 (SEGAL) *Let* $\mathcal{A}$ *be a unital* C^**-algebra. A state* $\rho \in S(\mathcal{A})$ *is pure if, and only if, its G.N.S. representation is irreducible.*

2.2.2 LATTICE THEORY AND PROJECTION STRUCTURE

Definition 35 *We say that relation* R *is ordering on the set* A *if, and only if,* R *is reflexive, weakly antisymmetric and transitive on* A*. We say that elements* $a, b \in A$ *are comparable with respect to* R *if* $< x, y > \in R$ or $< y, x > \in R$*. If* $< x, y > \in R$*, then we write* $x \leq_r y$.

Note 36 *We shall write* $\leq$ *instead of* $\leq_r$*. The set* A *endowed with partial ordering* $\leq$ *shall be denoted by doublet* $(A, \leq)$ *and called partially ordered set (poset in the abbreviation). If* $a \leq b$ *then we say "*a *is smaller than* b*", or "*b *is greater than* a*". If* $a \leq b$ *but* $a \neq b$ *then we write* $a < b$ *and say "*a *is strictly smaller than* b*".*

Definition 37 *Let* $\leq$ *be an ordering on the set* A *and let* $X \subseteq A$*. We say that* $a \in A$ *is an upper (a lower) bound of* X *if* $b \leq a$ $(a \leq b)$ *for all* $b \in A$*. The element* a *is the least upper bound (the greatest lower bound) of* A *if* $a \leq a'$ $(a' \leq a)$ *for any* a' *which is an upper (a lower) bound of* A.

Note 38 *Any subset of poset has at most one least upper (resp. greatest lower) bound.*

Definition 39 *The partially ordered set* $(\mathcal{L}, \leq)$ *is called a lattice if for any two elements* $a, b \in \mathcal{L}$ *there exists the least upper bound denoted by* $a \vee b$ *and the greatest lower bound denoted by* $a \wedge b$ *of the set* $\{a, b\}$*. The lattice is said to have zero and unit elements if there are elements* $0, 1 \in a$ *such that* $0 \leq a$ *and* $a \leq 1$ *for every* $a \in \mathcal{L}$.

Note 40 *We shall always assume every lattice to have a zero and unit elements. Such lattices are called bounded.*

Proposition 41 *In a lattice* $\mathcal{L}$ *the following equalities hold*

(i) *idempotency:* $a \wedge a = a$, $a \vee a = a$,

(ii) *commutativity:* $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,

(iii) *associativity:* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$,

(iv) *absorption:* $a \wedge (a \vee c) = a$, $a \vee (a \wedge b) = a$,

furthermore, (i) - (iv) determine the lattice completely.

Definition 42 *The lattice* $\mathcal{L}$ *is called complete if any subset of* $\mathcal{L}$ *has both the greatest lower and the least upper bound. The lattice* $\mathcal{L}$ *is called* σ*-lattice if any countable subset of* $\mathcal{L}$ *has the greatest lower and the least upper bound.*

Definition 43 *The element* $a \in \mathcal{L}$ *is an atom in* $\mathcal{L}$ *if* $b \leq a$ *implies* $b = a$ *or* $b = 0$*. The lattice* $\mathcal{L}$ *is called atomic if for any* $b \in \mathcal{L}$ *there exists an atom* a *such that* $a \leq b$*. The* *lattice is called completely atomistic if any element is equal to the least upper bound of all the atoms it majorizes, i.e. if for any* $0 \neq a \in \mathcal{L}$ *it holds that*

$$b = \bigvee_i a_i\,, \quad a_i \leq b \tag{2.4}$$

where a_i *is atom.*

Definition 44 *The lattice* $\mathcal{L}$ *is called*

(i) distributive if for any $a, b, c \in \mathcal{L}$ *it holds that*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\,. \tag{2.5}$$

(ii) modular if the following condition holds

$$\text{if } a \leq b \text{ then } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\,. \tag{2.6}$$

If $a \leq b$ *then* $a \vee b = b$ *and so the modularity equality is equivalent to*

$$\text{if } a \leq b \text{ then } a \vee (b \wedge c) = b \wedge (a \vee c)\,. \tag{2.7}$$

Definition 45 *Let* $\mathcal{L}$ *be a lattice. The map*

$$a \mapsto a^{\perp} \tag{2.8}$$

is called orthocomplementation and $a^{\perp}$ *is called the orthocomplement of* a *if the following properties are satisfied:*

(i) $(a^{\perp})^{\perp} = a$,

(ii) If $a \leq b$ *then* $b^{\perp} \leq a^{\perp}$,

(iii) $a \wedge a^{\perp} = 0$,

(iv) $a \vee a^{\perp} = 1$.

If an orthocomplementation is defined on a lattice $\mathcal{L}$*, then the lattice* $\mathcal{L}$ *is called an orthocomplemented lattice. If* a *and* b *are elements in an orthocomplemented lattice, then they are called orthogonal if* $a \leq b^{\perp}$.

Definition 46 *Boolean algebra is an orthocomplemented, distributive lattice. A Boolean algebra which is also a* σ*-lattice is called Boolean* σ*-algebra.*

Within the lattice theory, [12] defines *state* as the map $\rho : \mathcal{L} \mapsto [0, 1]$ additive on orthogonal elements of an orthocomplemented lattice $\mathcal{L}$ such that $\rho(\mathbb{I}) = 1$.

Definition 47 *The state* ρ *on a lattice* $\mathcal{L}$ *is called a Jauch-Piron state if the condition*

$$\rho(a) = \rho(b) = 0$$

implies

$$\rho(a \vee b) = 0\,.$$

The lattice $\mathcal{L}$ *is called Jauch-Piron lattice if every state on* $\mathcal{L}$ *is a Jauch-Piron state.*

The set $P(\mathcal{A})$ of all projections in a C^*-algebra $\mathcal{A}$ is a poset. It holds that $e \leq f$ in $P(\mathcal{A})$ if, and only if, $e\,f = f\,e = e$. If $\mathcal{A}$ is unital, then the structure $P(\mathcal{A})$ is an orthomodular poset with the complement $p^{\perp} = \mathbb{I} - p$. Further we shall need to define relation of equivalence in $P(\mathcal{A})$, which can be done in the sense of Murray-von Neumann.

Definition 48 *An element $v \in \mathcal{A}$ is called a partial isometry if*

$$v^* v = p \;, \tag{2.9}$$

where $p \in P(\mathcal{A})$.

Definition 49 *Two projections $p, q \in P(\mathcal{A})$ are called (Murray-von Neumann) equivalent (in symbols $p \sim q$) if*

$$p = v^* v \;\text{ and }\; q = v v^* \tag{2.10}$$

for some $v \in \mathcal{A}$.

Proposition 50 *The relation $\sim$ is an equivalence on the set $P(\mathcal{A})$.*

Definition 51 *The projection p is said to be subequivalent to a projection q (in symbols $p \lesssim q$) if there exists projection u such that $p \sim u$ and $u \leq q$. The projections p and q in a unital C^*-algebra $\mathcal{A}$ are said to be unitarily equivalent (in symbols $p \sim_u q$) if there is a unitary map $u \in \mathcal{A}$ such that $p = u^* q u$.*

Note 52 *Orthogonal equivalent projections are always unitarily equivalent.*

Definition 53 *A projection p in a C^*-algebra $\mathcal{A}$ is said to be infinite if there is a projection $q \in \mathcal{A}$ such that $p \sim q < p$. If p is not infinite, then p is said to be finite. A C^*-algebra $\mathcal{A}$ is finite if, and only if, all projections in $\mathcal{A}$ are finite. A unital C^*-algebra is said to be finite (resp. infinite) if its unit is a finite (resp. infinite) projection. A projection in $\mathcal{B}(\mathcal{H})$ is finite if, and only if, it is finite-dimensional. Hence, $\mathcal{B}(\mathcal{H})$ is finite if, and only if, $\dim \mathcal{H} < \infty$.*

Definition 54 *A unital C^*-algebra is said to have real rank zero if the invertible elements are dense in the set of all self-adjoint elements.*

2.3 von Neumann algebras

The standard notation as in the previous part is preserved. In this paragraph $\mathcal{H}$ will be an infinite-dimensional separable Hilbert space, i.e. a Hilbert space with countable orthonormal basis. As we will see, it is possible to define von Neumann algebras either topologically or algebraically (see von Neumann bicommutant Theorem). Before doing this we shall need to introduce some new definitions.

2.3.1 Compact operators and traces

Definition 55 *The operator $x \in \mathcal{B}(\mathcal{H})$ is called compact if it maps the unit ball of $\mathcal{H}$ onto a pre-compact set. The set of all compact operators on $\mathcal{H}$ will be denoted by $K(\mathcal{H})$.*

Theorem 56 (Riesz-Schauder) *Let $x \in \mathcal{B}(\mathcal{H})$ be a compact operator. Then (i) each nonzero point of $\sigma(x)$ is an eigenvalue of x, (ii) each nonzero eigenvalue of a has a finite* *multiplicity* [6], *(iii)* $\sigma(x)$ *has at most one limiting point* $\lambda = 0$, *(iv) the set of eigenvalues is countable:*

$$\sigma(x) = \{\lambda_j : j = 1, \ldots, N\}, \quad N \leq \infty\,,$$

and it is always possible to achieve that $|\lambda_j| \geq |\lambda_{j+1}|$, $j = 1, 2, \ldots$; *if* $N = \infty$, *then* $\lim_{j\to\infty} \lambda_j = 0$.

Theorem 57 (HILBERT-SCHMIDT) *For each normal compact operator* $x \in \mathcal{B}(\mathcal{H})$ *there is an orthonormal basis consisting of the eigenvectors of* x.

Definition 58 *Let* $x \in \mathcal{B}(\mathcal{H})$ *and*

$$||x||_1 = \sum_{e\in E} (|x|e, e)\ , \tag{2.11}$$

where E *is an orthonormal basis of* $\mathcal{H}$ *and* $|x| = \sqrt{x^*x}$. *The operator* $x \in \mathcal{B}(\mathcal{H})$ *is said to be the trace class operator if* $||x||_1 < \infty$. *The set of all trace class operators on* $\mathcal{H}$ *shall be denoted by* $L^1(\mathcal{H})$.

Note 59 *The positive number* $||x||_1$ *is the same for all orthonormal bases of* E *and is called the Hilbert-Schmidt norm of the operator* x.

$\big(L^1(\mathcal{H}), ||\cdot||_1\big)$ is an involutive $*$-algebra with respect to the usual adjoint operation such that $L^1(\mathcal{H}) \subset K(\mathcal{H})$. $L^1(\mathcal{H})$ is a self-adjoint ideal in $\mathcal{B}(\mathcal{H})$. A self-adjoint operator x is of trace class if, and only if, $x = \sum_n \lambda_n p_n$, where (p_n) is a sequence of pairwise orthogonal one-dimensional projections and $\sum_n |\lambda_n| < \infty$. For each $x \in L^1(\mathcal{H})$ the sum $\sum_{e\in E} (xe, e)$ converges and the number

$$tr\, x = \sum_{e\in E} (xe, e) \tag{2.12}$$

is called the trace of x. Moreover if $x \in L^1(\mathcal{H})$, then $tr(xy) = tr(yx)$ for all $y \in \mathcal{B}(\mathcal{H})$.

Definition 60 *The weak operator topology on* $\mathcal{B}(\mathcal{H})$ *is given by the system of seminorms* $a \in \mathcal{B}(\mathcal{H}) \mapsto |(ax, y)|$, *where* $x, y \in \mathcal{H}$. *The strong operator topology is given by the system of seminorms* $a \in \mathcal{B}(\mathcal{H}) \mapsto ||ax||$, *where* $x \in \mathcal{H}$. *The* σ*-weak (ultraweak) topology is given by the system of seminorms induced by the trace class operators, that is* $a \in \mathcal{B}(\mathcal{H}) \mapsto |tr(ta)|$, *where* $t \in L^1(\mathcal{H})$.

Definition 61 C^**-algebra that can be faithfully represented as a strongly operator closed* $*$*-subalgebra of* $\mathcal{B}(\mathcal{H})$ *is called the von Neumann algebra (*W^**-algebra).*

On a $*$-subalgebra M of $\mathcal{B}(\mathcal{H})$ the weak operator, strong operator and ultraweak closedness are equivalent.

Theorem 62 (VON NEUMANN BICOMMUTANT) *A unital* $*$*-subalgebra* M *of* $\mathcal{B}(\mathcal{H})$ *is a von Neumann algebra if, and only if,* $M'' = M$, *where* M'' *is the bicommutant of* M.

The center $\mathcal{Z}(M) = M \bigcap M'$ of von Neumann algebra M is an abelian von Neumann subalgebra of M.

Definition 63 *A von Neumann algebra* M *is called the factor if* $\mathcal{Z}(M)$ *consists of scalar multiples of the unit of* M *only. For each projection* $p \in P(M)$ *we define the central cover,* $c(e)$, *of* e *as the smallest central projection majorizing* e.

[6] The *geometric multiplicity* of an eigenvalue λ of a is the dimension of the subspace of vectors x for which $ax = \lambda x$.

If z is a central projection and $p \in P(M)$, then $c(zp) = zc(p)$. Moreover, $c(e) = sup\{u^*eu \,|\, u \in U(M)\}$.

2.3.2 Normal states and homomorphisms

Definition 64 *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A bounded functional φ on M is called normal if $\varphi(u_\alpha) \to \varphi(u)$ whenever $u_\alpha \nearrow u$ in M_{sa}.*

Theorem 65 *Let M be a von Neumann algebra. The following conditions are equivalent: (i) φ is a normal functional (ii) φ is weakly operator continuous on the unit ball of M (iii) φ is strongly operator continuous on the unit ball of M (iv) φ is a completely additive measure on the projection lattice $P(M)$, i.e.*

$$\varphi\Big(\sum_\alpha p_\alpha\Big) = \sum_\alpha \varphi(p_\alpha)\,, \tag{2.13}$$

for any system of pairwise orthogonal projections, $\{p_\alpha\}$, in M (v) φ is continuous in the ultraweak topology.

It follows from the previous proposition that any normal functional on M is given by a trace class operator $t \in L^1(\mathcal{H})$ such that

$$\varphi(x) = tr(tx)\,, \tag{2.14}$$

for each $x \in M$.

Definition 66 *Let φ be a state on von Neumann algebra M. The projection p is called the support of φ if $\varphi(\mathbb{I} - p) = 0$ and $\varphi(q) > 0$ for any subprojection q of p.*

Note 67 *Every normal state has a support.*

Note 68 *The set of all normal states on M will be denoted by $S_n(M)$. It is a convex compact set that is weak*-dense in the state space $S(M)$.*

Definition 69 *Let $\pi : M \mapsto N$ be a $*$-homomorphism between von Neuman algebras M and N. The map π is called normal if $u_\alpha \nearrow u$ in M_{sa} implies that $\pi(u_\alpha) \nearrow \pi(u)$ in N.*

If π is a normal $*$-homomorphism, then the algebra $\pi(M)$ is a von Neumann subalgebra of N. If π is a faithful $*$-homomorphism of M onto N, then π is a normal $*$-isomorphism. A state on a von Neumann algebra is normal if, and only if, its G.N.S. representation is normal.

2.3.3 Projection lattice

The projection lattice $P(M)$ of von Neumann algebra M is a complete orthomodular lattice endowed with the equivalence relation $\sim$.

Definition 70 *A projection $e \in M$ is called σ-finite (relative to M) if each orthogonal family of nonzero subprojections of e is countable. M is said to be a σ-finite algebra if the unit of M is a σ-finite projection.*

Note 71 *Every algebra acting on the separable Hilbert space is σ-finite, the converse is not true.*

Proposition 72 (Paralellogram rule, Kaplansky formula) *Let e, f be arbitrary projections in the von Neumann algebra M. Then*

$$(e \vee f) - f \sim e - (e \wedge f)\,. \tag{2.15}$$

Let $0 \neq I \subset \mathbb{N}$, $(e_\alpha)_{\alpha\in I}$ and $(f_\alpha)_{\alpha\in I}$ be two families of pairwise orthogonal projections such that $e_\alpha \sim f_\alpha$,$\forall\alpha \in I$. Then

$$\sum_{\alpha\in I} e_\alpha \sim \sum_{\alpha\in I} f_\alpha\ . \tag{2.16}$$

Theorem 73 (Comparability theorem) *For any pair $e, f \in P(M)$, there exists a central projection $z \in M$ such that*

$$ze \lesssim zf \quad and \quad (\mathbb{I} - z)e \gtrsim (\mathbb{I} - z)f\ . \tag{2.17}$$

Definition 74 *A projection $e \in M$ is called abelian if the hereditary subalgebra eMe is abelian.*

2.3.4 Classification theory

We shall classify von Neumann algebras according to decomposition into direct sum with its direct summands being of certain types. Those express something like the "degree of noncommutativity" of such von Neumann algebra.

Definition 75 *A von Neumann algebra is said to be Type I algebra if there is an abelian projection $e \in M$ such that $c(e) = \mathbb{I}$.*

Definition 76 *Let $n \in \mathbb{N}$ be cardinal number. If the unit in a von Neumann algebra M can be written as a sum of n equivalent abelian projections, then M is said to be of Type I_n. Each algebra of Type I_n is called homogenous Type I algebra.*

Note 77 *Type I_1 algebras are just abelian von Neumann algebras.*

For every von Neumann algebra M of type I and each cardinal less than card M there is a unique central projection z_α such that $z_\alpha M$ is zero or Type I_α and such that $\sum_\alpha z_\alpha = \mathbb{I}$. All Type I_α algebras are $*$-isomorphic copies of $\mathcal{A}\overline{\bigotimes}\mathcal{B}(\mathcal{H}_\alpha)$, where $\mathcal{A}$ is an abelian von Neumann algebra and $\mathcal{H}_\alpha$ is a Hilbert space such that dim $H_\alpha = \alpha$. Type I algebra is finite if, and only if, it is a direct sum of countably many Type I_n, $n < \infty$ algebras. Let $e \in P(M)$ where M is a Type I algebra. Then there are orthogonal projections e_1, e_2, e_3 such that $e = e_1 + e_2 + e_3$, $e_1 \sim e_2$ and e_3 is abelian.

Definition 78 *A totally disconnected compact Hausdorff topological space is called a Stone space (stonean space). The space is called hyperstonean if it is stonean and if for any nonzero positive real function $f \in C(X)$ there is a positive normal functional ρ on $C(X)$ with $\rho(f) \neq 0$.*

The finite Type I homogenous algebras can be indentified with the algebras $M_n(C(X))$, where X is a hyperstonean space. This algebra can also be represented as algebra $C(X, M_n(\mathbb{C}))$ of all continuous functions on a hyperstonean space X with values in matrix algebra $M_n(\mathbb{C})$.

The algebras of Type I contain many abelian hereditary subalgebras. Next, von Neumann algebras that have no nonzero abelian hereditary subalgebra shall be investigated further.

Definition 79 *A von Neumann algebra is said to be of Type II if it has no nonzero abelian projection but has a finite projection e such that $c(e) = \mathbb{I}$. The finite algebras of Type II are called Type II_1, the infinite ones are called Type II_∞. A von Neumann algebra is called semifinite if it is a Type I or Type II or a direct sum of algebras of these types.*

Definition 80 *Let M be a von Neumann algebra with the center $\mathcal{Z}(M)$. The center-valued trace $T : M \mapsto \mathcal{Z}(M)$ is a linear map such that: (i) $T(xy) = T(yx)$ for all $x, y \in M$ (ii) T is identity on $\mathcal{Z}(M)$ (iii) $T(x) > 0$ whenever $x \in M$ is positive operator.*

Theorem 81 *Any finite von Neumann algebra admits exactly one center-valued trace, T. Moreover, T has the following properties: (i) $T(zx) = zT(x)$ for all $z \in \mathcal{Z}(M)$ and $x \in M$ (ii) $T(e) \leq T(f)$ for projections e, f if, and only if, $e \lesssim f$ (iii) $||T|| \leq 1$ (iv) T is ultraweakly continuous.*

The following proposition classifies von Neumann algebras in terms of the traces.

Proposition 82 *Let $T : M \mapsto \mathcal{Z}(M)$ be a center-valued trace on M. Then the following holds:*

(i) If M is of Type I_n, then $T(P(M))$ consists of all elements of the form

$$\frac{1}{n} z_1 + \frac{2}{n} z_2 + \cdots + \frac{n-1}{n} z_{n-1} + z_n \ , \tag{2.18}$$

where $z_1, \ldots, z_n$ are pairwise orthogonal central projections.

(ii) If M is of Type II_1, then $T(P(M))$ consists of all positive elements in the unit ball of $\mathcal{Z}(M)$.

A state ρ is called *tracial* (or a *trace*) if $\rho(x^*x) = \rho(xx^*)$. Significant property of von Neumann algebras is that if ρ is a state on $\mathcal{Z}(M)$, then $\rho \circ T$ is a unique tracial state on M extending ρ.

Let M be a factor. Type I_n factors are matrix algebras $M_n(\mathbb{C})$. Here, the center-valued trace is normalized matrix trace, i.e. $T(e) \in \{\frac{k}{n} \,|\, 0 \leq k \leq n\}$ for all projections $e \in M$. This is not the case of Type II_1 factors. The restriction of faithful tracial state on Type II_1 factor to projection lattice attains all values in the interval $[0, 1]$. Thus Type II_1 factors may be thought of as "continuous dimension" or "continuous matrix" algebras.

Definition 83 *A von Neumann algebra is said to be of Type III (or purely infinite) if it contains no nonzero finite projection.*

Proposition 84 *Every von Neumann algebra is uniquely decomposable into direct sum of algebras of Type I, II_1, II_∞ and III. In particular any von Neumann factor is one of the Type I, II_1, II_∞ and III.*

Proposition 85 *If von Neumann algebra has zero Type I finite part, then for every projection $e \in M$ there exist orthogonal equivalent projections $e_1, \ldots, e_n$ such that $e = \sum_{i=1}^n e_i$. In particular, for any projection e in an arbitrary von Neumann algebra there are projections e_1, e_2 and e_3 such that $e = e_1 + e_2 + e_3$, $e_1 \sim e_2$ and e_3 is abelian.*

Proof can be found in [9].

2.3.5 Measures and quasi-functionals

Definition 86 *Let $P(M)$ be a projection lattice of a von Neumann algebra M, X a Banach space, and let $\mu : P(M) \mapsto X$ be a map satisfying*

$$\mu(e+f) = \mu(e) + \mu(f) \tag{2.19}$$

whenever $ef = 0$. Then μ is said to be a (finitely additive) X-valued measure on $P(M)$. If, moreover, sup$\{||\mu(e)|| \mid e \in P(M)\} < \infty$, we say that μ is bounded. Further, if X is the Banach space of all complex or real numbers, we call μ the complex or real measure on $P(M)$, respectively.

Theorem 87 (Gleason Theorem) *Let M be a von Neumann algebra with no direct summand of Type I_2. Then each bounded complex measure on $P(M)$ extends to a bounded linear functional on M.*

Proof of the Gleason Theorem is in [9].

Theorem 88 *Let M be a von Neumann algebra with no direct summand of type I_2 and let X be a Banach space. Then each bounded X-valued measure μ on $P(M)$ extends uniquely to a bounded linear operator T from M to X.*

Proof. μ is bounded if, and only if, there is $K \geq 0$ such that $||\mu(p)|| \leq K$ for all $p \in P(M)$. We want to show that μ extends to a bounded linear map on $V(M) = \{P(M)\}_{lin}$. Suppose that $x \in V(M)$ has the form $x = \sum_{j=1}^{n} \lambda_j p_j$. We have to show that the definition of μ is correct, i.e. not depending on the way of writing x as a linear combination of the projections.

Let $\varphi \in X^*$ and $\mu : P(M) \mapsto X$ bounded X-valued measure. Since each functional on a Banach space is bounded, $\varphi \circ \mu$ is a bounded measure on $P(M)$. By the previous Theorem $\varphi \circ \mu$ extends to a linear functional $\widehat{\varphi} \in M^*$. By [9] we have that $||\widehat{\varphi}|| \leq 4 \sup\{|\widehat{\varphi}(p)| \mid p \in P(M)\}$. Applying this we obtain

$$|\varphi \circ \mu(x)| \leq ||\varphi \circ \mu|| \cdot ||x|| \leq 4 \cdot ||x|| \sup\{|(\varphi \circ \mu)(p) \mid p \in P(M)\} \leq$$
$$^{7} \leq 4||x|| \cdot \sup\{||\varphi|| \cdot |\mu(p)| \mid p \in P(M)\} = 4||x|| \cdot ||\varphi|| \cdot \sup\{|\mu(p)| \mid p \in P(M)\} = 4||x|| \cdot ||\varphi|| \cdot K.$$

By the Hahn-Banach Theorem there is $\psi \in X^*$, $||\psi|| = 1$ such that

$$||\sum_{i}^{n} \lambda_i \mu(p_i)|| = |\psi(\sum_{i=1}^{n} \lambda_i \mu(p_i))| \leq 4||\psi|| \cdot ||x|| \cdot K = 4K||x||\,.$$

This inequality immediately implies that if $\sum_{i=1}^{k} \alpha_i q_i$ is another expression of x as a linear combination of projections $q_1, \ldots, q_k$, then

$$0 \leq ||\mu(\sum_{i=1}^{n} \lambda_i p_i) - \mu(\sum_{i=1}^{k} \alpha_i q_i)|| = ||\sum_{i=1}^{n} \lambda_i \mu(p_i) - \sum_{i=1}^{k} \alpha_i \mu(q_i)|| \leq$$

$$\leq 4K||x - x|| = 0\,,$$

[7] $|(\varphi \circ \mu)(p) \leq |\varphi| \cdot |\mu(p)|$

which implies that $\sum_{i=1}^{n} \lambda_i \mu(p_i) = \sum_{i=1}^{k} \alpha_i \mu(q_i)$.

Define a map $T : V(M) \mapsto X$ by

$$T\Big(\sum_{i=1}^{n} \lambda_i p_i\Big) = \sum_{i=1}^{n} \lambda_i \mu(p_i)\,.$$

Since

$$||Tx|| = ||\sum_{i=1}^{n} \lambda_i \mu(p_i)|| \leq 4K \cdot ||x||$$

and by definition $||T|| = sup\{||Tx|| : ||x|| = 1\} = 4K$, we have that $||T|| \leq 4K$ and hence T is bounded.

By definition, $V(M)$ is dense in M. To complete the proof it is enough to apply continuous extension theorem, which says that T extends uniquely to a bounded operator from M into X, $\mathcal{B}(M, X)$. □

Definition 89 *Let M be a von Neumann algebra. A map $\rho : M \mapsto \mathbb{C}$ is called a quasi-linear functional (also quasi-functional) if it satisfies:*

(i) ρ is a linear functional on any abelian von Neumann subalgebra $\mathcal{A}$ of M,

(ii) $\rho(x + iy) = \rho(x) + i\rho(y)$ for all self-adjoint elements $x, y \in M$,

(iii) ρ is bounded on the unit sphere of M.

Moreover, we say that ρ is self-adjoint if $\rho(x) \in \mathbb{R}$ whenever x is a self-adjoint element of M.

The following proposition is a part of the proof of the Gleason Theorem.

Proposition 90 *Let M be a von Neumann algebra and $\mu : P(M) \mapsto \mathbb{C}$ a bounded measure. Then μ extends uniquely to a quasi-linear functional $\widehat{\mu}$ on M. If μ is real, then $\widehat{\mu}$ is self-adjoint.*

Proof is in [9].

Now we will show an interesting application of the Gleason Theorem to multimeasures. This can be viewed as another generalization of the Gleason Theorem. Before we get to the application, a few definitions are needed to be pointed out.

Definition 91 *Let $A_1, \ldots, A_n$ be C^*-algebras with $P(A_1), \ldots, P(A_n)$ being their respective projection structures. Let X be a Banach space. The map $m : P(A_1) \times \cdots \times P(A_n) \mapsto X$ is called an X-*valued multimeasure *(in short a* multimeasure*) if m is separately finitely additive, meaning that, for each $j = 1, \ldots, n$,*

$$\begin{aligned} &m(p_1, \ldots, p_{j-1}, q_1 + q_2, p_{j+1}, \ldots, p_n) \\ &= m(p_1, \ldots, p_{j-1}, q_1, p_{j+1}, \ldots, p_n) + m(p_1, \ldots, p_{j-1}, q_2, p_{j+1}, \ldots, p_n)\,, \end{aligned} \tag{2.20}$$

where $p_i \in P(A_i)$, q_1, q_2 are orthogonal projections in $P(A_j)$. If $X = \mathbb{C}$, we call m the complex multimeasure. Completely additive and σ-additive multimeasures are defined in the standard way as measures separately completely additive and σ-additive, respectively.

In case when $n = 2$ we call the multimeasure the bimeasure. Moreover, if $A = A_1 = A_2$ we say that m is a bimeasure on A. A bimeasure m on A is said to be hermitian if

$$m(p,q) = \overline{m(q,p)} \tag{2.21}$$

for all projections $p, q \in P(A)$.

Definition 92 *Let $X_1, \ldots, X_n$ be Banach spaces. Let $F : X_1 \times \ldots \times X_n \mapsto X$ be an n-linear map. F is said to be bounded if there is a constant $K \in R_0^+$ such that, for each $x_i \in X_i$,*

$$||F(x_1, \ldots, x_n)|| \leq K||x_1|| \cdots ||x_n|| \,. \tag{2.22}$$

A smallest constant K for which this inequality holds is said to be the norm of F (in symbols $||F||$).

Theorem 93 *Let $M_1, \ldots, M_n$ be von Neumann algebras, each having no direct summand of Type I_2. Suppose that $m : P(M_1) \times \ldots \times P(M_n) \mapsto X$ is a bounded X-valued multimeasure. There is a unique bounded n-linear map $F : M_1 \times \ldots \times M_n \mapsto X$ which extends m.*

Proof is in [9].

2.4 Jordan operator algebras

Consider an arbitrary (in general non-associative) algebra $\mathcal{A}$. For each $a, b \in \mathcal{A}$ define a map $\circ : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ by

$$a \circ b = \frac{1}{2}(ab + ba) \,. \tag{2.23}$$

The map $\circ$ is

(i) linear: $(a+\lambda b) \circ c = \frac{1}{2}((a+\lambda b)c + c(a+\lambda b)) = \frac{1}{2}(ac+ca+\lambda(bc+cb)) = a \circ c + \lambda b \circ c$,

(ii) commutative: $a \circ b = \frac{1}{2}(ab+ba) = \frac{1}{2}(ba+ab) = b \circ a$.

Linearity and commutativity both imply bilinearity. Also: (iii) $a \circ (b+c) = a \circ b + a \circ c$, (iv) $(b+c) \circ a = b \circ a + c \circ a$, (v) $\lambda(a \circ b) = (\lambda a) \circ b = a \circ (\lambda b)$. Hence $\circ$ defines a bilinear, commutative product on $\mathcal{A}$. Thus $\mathcal{A}^J$, which by definition is the vector space $\mathcal{A}$ with the product $\circ$, is a commutative algebra. If $\mathcal{A}$ is associative, we call $\circ$ the *special Jordan product* in $\mathcal{A}$. In general, associativity of $\mathcal{A}$ does not imply associativity of $\mathcal{A}^J$ as the following example shows. Let $\mathcal{A} = M_2(\mathbb{R})$ and let

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $(ab)c = a(bc) = 0$, but $0 = (a \circ b) \circ c \neq a \circ (b \circ c) = \frac{1}{4}c$.

However, the product $\circ$, if $\mathcal{A}$ is associative, satisfies the following weak form of associativity:

$$a \circ (b \circ a^2) = (a \circ b) \circ a^2 \,. \tag{2.24}$$

Definition 94 *Let $\mathcal{A}$ be an algebra with the product written $(a,b) \mapsto a \circ b$. $\mathcal{A}$ is called a Jordan algebra if the following two identities are satisfied for all $a, b \in \mathcal{A}$:*

$$a \circ b = b \circ a \text{ (commutativity)} \tag{2.25}$$

$$a \circ (b \circ a^2) = (a \circ b) \circ a^2 \text{ (Jordan axiom)} \,. \tag{2.26}$$

Jordan algebras are *power associative*. The powers are defined inductively by $a^0 = 1$ and $a^{n+1} = a \circ a^n$ $(n \geq 1)$. Put $a^{\circ k} := \underbrace{a \circ \cdots \circ a}_{k-times}$, $k \geq 2$. Note that $a^{\circ 2} = \frac{1}{2}(a^2 + a^2) = a^2$. By induction we get that $a^{\circ k} = a^k$. Because of this we shall write a^k instead of $a^{\circ k}$. For further notation it is convenient to introduce the multiplication operator $T_a : \mathcal{A} \mapsto \mathcal{A}$ for $a \in \mathcal{A}$ by

$$T_a b = a \circ b \,. \tag{2.27}$$

The power associativity can be put in the form of Lemma.

Lemma 95 *Let $\mathcal{A}$ be a Jordan algebra and $a \in \mathcal{A}$. Then, for $m, n \in \mathbb{N}$,*
(i) $a^{m+n} = a^m \circ a^n$,
(ii) $T_{a^m} T_{a^n} = T_{a^n} T_{a^m}$.

Definition 96 *A Jordan subalgebra is called* reversible *if it is closed under the Jordan multiple product for all $n \in \mathbb{N}$, and* irreversible *otherwise.*

Definition 97 *Let $\mathcal{A}$ be an associative algebra. By a Jordan subalgebra of $\mathcal{A}$ we mean a subalgebra of $\mathcal{A}^J$, i.e. a linear subspace of $\mathcal{A}$ which is closed under the Jordan product $\circ$. Any algebra isomorphic to a Jordan subalgebra of an associative algebra will be called a* special Jordan algebra.

From here and definition of Jordan algebra it immediately follows that any special Jordan algebra is a Jordan algebra. The converse is not true. The Jordan algebras which are not special will be called *exceptional*. Example of exceptional Jordan algebra is $\mathbb{H}_3(\mathbb{O})$, where $\mathbb{O}$ is the field of *octonions*[8].

Definition 98 *Let $\mathcal{A}$ be an associative algebra. The Jordan triple product is defined by*

$$\{abc\} = \frac{1}{2}(abc + cba) \,. \tag{2.28}$$

The Jordan triple product can be expressed in terms of the Jordan product

$$\{abc\} = (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b \,. \tag{2.29}$$

The Jordan multiple product is defined by

$$\{a_1, \ldots, a_n\} = \frac{1}{2}\{a_1 \cdots a_n + a_n \cdots a_1\} \,, \tag{2.30}$$

and cannot be expressed in terms of the Jordan product if $n \geq 4$. There exist Jordan subalgebras of associative algebras which are not closed under the multilinear product.

The following *Macdonalds's* Theorem has applications in extending validity of identities to more general structures. We only mention its rephrased version.

Theorem 99 (REPHRASED MACDONALD'S THEOREM) *Any polynomial identity in three variables, with degree at most 1 in the third variable, and which holds in all special Jordan algebras, holds in all Jordan algebras.*

Theorem 100 (SHIRSHOV-COHN) *Any Jordan algebra generated by two elements (and $\mathbb{I}$, if unital) is special.*

[8] see appendix

For examples illustrating the use of the Macdonald's and Shirshov-Cohn's theorem, see [10], section 2.4.16., page 35.

Consider an associative algebra $\mathcal{A}$. Recall that by definition $\mathcal{A}^J$ is commutative but not associative. This suggests two conditions: for any $c \in \mathcal{A}^J$, $a \circ (c \circ b) = (a \circ c) \circ b$; or the subalgebra $\mathcal{A}^J$ generated by given a, b is associative. Rewriting the first condition explicitly yields $abc - bac - cab + cba = 0$. Put $[a, b] := ab - ba$, then the equation above can be rewritten as $\big[[a, b], c\big] = 0$. It is evident that this is implied by the relation $[a, b] = 0$. As we see now, the associativity of $\mathcal{A}^J$ is closely related to commutation relation. Before putting this in the form of Lemma, we need to introduce commutation in a Jordan algebra.

Definition 101 *Two elements a, b in a Jordan algebra $\mathcal{A}$ are said to operator commute if the operators T_a, T_b commute, i.e. if $(a \circ c) \circ b = a \circ (c \circ b)$ for all $c \in \mathcal{A}$. By the center of $\mathcal{A}$ we mean the set of all elements of $\mathcal{A}$ which operator commute with every other element of $\mathcal{A}$.*

Lemma 102 *The center of a Jordan algebra is an associative subalgebra.*

In Jordan algebra $\mathcal{A}$ an element $p \in \mathcal{A}$ is called *idempotent* if $p^2 = p \circ p = p$; $p, q \in \mathcal{A}$ are said to be *orthogonal* if $p \circ q = q \circ p = 0$. Let $\mathcal{A}$ be a unital, associative algebra containing orthogonal idempotents $p_1, \dots, p_n$ with sum $\mathbb{I}$. Then we have a decomposition $\mathcal{A} = \bigoplus_{i,j} \mathcal{A}_{ij}$, where $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$. We ask if this is also possible for Jordan algebras. Note that $p_i \mathcal{A} p_j$ does not make sense in the Jordan algebra, but $\{p_i \mathcal{A} p_j\}$ does.

It is convenient to introduce the operator

$$U_{a,c}(b) = \{abc\} . \tag{2.31}$$

Let p be an idempotent in $\mathcal{A}$. Let $p^\perp = \mathbb{I} - p$. Then

$$p \circ a = \frac{1}{2}(a + \{pap\} - \{p^\perp a p^\perp\}) , \tag{2.32}$$

which can be rewritten as $T_p a = \frac{1}{2}(i_d a + U_p a - U_{p^\perp} a)$, or

$$T_p = \frac{1}{2}(i_d + U_p - U_{p^\perp}) , \tag{2.33}$$

where i_d is identity. U_p and $U_{p^\perp}$ are orthogonal idempotent maps. We can rewrite the previous formula as

$$T_p = U_p + \frac{1}{2}(i_d - U_p - U_{p^\perp}) + 0 \cdot U_{p^\perp} . \tag{2.34}$$

U_p, $i_d - U_p - U_{p^\perp}$ and $U_{p^\perp}$ are mutually orthogonal idempotent maps with sum $\mathbb{I}$. Thus T_p has eigenvalues in the set $\{0, \frac{1}{2}, 1\}$ and we have so called the Peirce decomposition of $\mathcal{A}$ with respect to p:

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_{\frac{1}{2}} \oplus \mathcal{A}_0 , \tag{2.35}$$

where $\mathcal{A}_j$ is the eigenspace of T_p corresponding to the eigenvalue $j \in \{0, \frac{1}{2}, 1\}$.

In Jordan algebras, two idempotents p, q will be called *orthogonal* if $p \circ q = 0$. The following theorem generalizes the Peirce decomposition to the case of several orthogonal idempotents. It is called the Peirce decomposition with respect to the set $\{p_j\}_{j=1,\dots,n} \subseteq \mathcal{A}$.

Theorem 103 *Let $\mathcal{A}$ be a unital Jordan algebra. Suppose $p_1, \ldots, p_n$ are pairwise orthogonal idempotents in $\mathcal{A}$ with sum $\mathbb{I}$. Let $\mathcal{A}_{ij} = \{p_i \mathcal{A} p_j\}$. Then $\mathcal{A}_{ij} = \mathcal{A}_{ji}$, and we have Peirce decomposition with respect to $p_1, \ldots, p_n$*

$$\mathcal{A} = \bigoplus_{1 \le i \le j \le n} \mathcal{A}_{ij} \,. \tag{2.36}$$

Furthermore, the following multiplication properties hold:

$$\mathcal{A}_{ij} \circ \mathcal{A}_{kl} = 0 \;\text{ if } \{i, j\} \cap \{k, l\} = \varnothing \,, \tag{2.37}$$

$$\mathcal{A}_{ij} \circ \mathcal{A}_{kj} \subseteq \mathcal{A}_{ik} \;\text{if } i, j, k \text{ are all distinct} \,, \tag{2.38}$$

$$\mathcal{A}_{ij} \circ \mathcal{A}_{ij} \subseteq \mathcal{A}_{ii} + \mathcal{A}_{jj} \,, \tag{2.39}$$

$$\mathcal{A}_{ii} \circ \mathcal{A}_{ij} \subseteq \mathcal{A}_{ij} \,. \tag{2.40}$$

$M_n(\mathbb{R})$, $M_n(\mathbb{C})$ shall denote the algebra of all matrices over $\mathbb{R}$ or $\mathbb{C}$, respectively. For convenience we shall use the symbol M_n in the case when specification of field is not necessary. Product in M_n is nothing but usual matrix product $(a_{ij})(b_{jk}) = \sum_j a_{ij} b_{jk}$. The Hermitian part of M_n is denoted by $\mathbb{H}_n$. Defining product in $\mathbb{H}_n$ by $a \circ b = \frac{1}{2}(ab + ba)$ we have Jordan algebra structure (the field of real or complex numbers is associative algebra, however this condition is not necessary). $\mathbb{H}_n$ shall be called a Jordan matrix algebra.

The matrix units in M_n are the elements e_{ij}, where $1 \le i \le n$ and $1 \le j \le n$. e_{ij} is a matrix in M_n whose (i, j) entry is 1, the others being zero. Obviously, $e_{ij}^* = e_{ji}$, $\sum_{i=1}^n e_{ii} = \mathbb{I} \in M_n$ and $e_{ij} e_{kl} = 0$ if $j \ne k$, $e_{ij} e_{jk} = e_{ik}$. Arbitrary matrix $(a_{ij}) \in M_n$ can be written as a linear combination of matrix units, i.e. $(a_{ij}) = \sum_{i,j} a_{ij} e_{ij}$.

Definition 104 *Let p, q be orthogonal idempotents in a Jordan algebra $\mathcal{A}$. They are said to be strongly connected if there exists $v \in \{p \mathcal{A} q\}$ such that $v^2 = p + q$.*

The following theorem gives condition under which a unital Jordan algebra is a Hermitian Jordan matrix algebra, i.e. $\mathcal{A} \cong \mathbb{H}_n$.

Theorem 105 (THE COORDINATIZATION THEOREM) *Let $\mathcal{A}$ be a unital Jordan algebra. Suppose that $\mathcal{A}$ contains $n \ge 3$ pairwise orthogonal strongly connected idempotents with sum $\mathbb{I}$. Then $\mathcal{A}$ is isomorphic to $\mathbb{H}_n(R)$ for some $*$-algebra R.*

Definition 106 *A Jordan algebra $\mathcal{A}$ over the field $\mathbb{R}$ is called* formally real *if $a_i \in \mathcal{A}$ for all $i = 1, \ldots n$ and*

$$\sum_{i=1}^{n} a_i^2 = 0 \textit{ implies } a_1 = \ldots = a_n = 0 \,. \tag{2.41}$$

Proposition 107 *Suppose $R = \mathbb{R}$, $\mathbb{C}$, $\mathbb{O}$ or $\mathbb{Q}$[9]. Let $n \ge 2$. If $R = \mathbb{O}$, assume $n \le 3$. Then $\mathbb{H}_n(R)$ is formally real Jordan algebra.*

Definition 108 *Let $\mathcal{A}$ be arbitrary ring or algebra. A nonzero idempotent $p \in \mathcal{A}$ such that for any nonzero idempotent $q \in \mathcal{A}$ with $p \cdot q = q \cdot p = q$ implies $q = p$ is called minimal.*

One of the major contributions to the theory of Jordan algebras is the following theorem made by Jordan, von Neumann and Wigner when they classified all simple finite-dimensional formally real Jordan algebras.

[9] Here $\mathbb{Q}$ denotes the field of *quaternions* (see appendix).

Theorem 109 *Every finite-dimensional, formally real, unital Jordan algebra $\mathcal{A}$ is a direct sum of simple algebras. If $\mathcal{A}$ is simple then it contains $n \geq 1$ pairwise orthogonal and strongly connected minimal idempotents with sum $\mathbb{I}$. If $n = 1$, $\mathcal{A} \simeq \mathbb{R}$. If $n \geq 3$, $\mathcal{A}$ is isomorphic to one of $\mathbb{H}_n(\mathbb{R})$, $\mathbb{H}_n(\mathbb{C})$, $\mathbb{H}_n(\mathbb{Q})$ or, if $n = 3$, $\mathbb{H}_n(\mathbb{O})$.*

Formally real Jordan algebra with $n = 2$ produces so-called spin factor which has specific properties. Spin factors are discussed in the section 2.4.4

In the next section we introduce JB algebras, which can be viewed as a proper Banach algebra version of the formally real Jordan algebras. Then we continue by JW and JBW algebras.

Definition 110 (JC ALGEBRA) *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})_{sa}$ denote the special Jordan algebra of all self-adjoint operators in $\mathcal{B}(\mathcal{H})$ equipped with the operator norm. By a JC algebra we shall mean any norm-closed Jordan subalgebra of $\mathcal{B}(\mathcal{H})_{sa}$. By a JC algebra we shall often call any normed Jordan algebra isometrically isomorphic to a JC algebra.*

Consider finite-dimensional algebras $\mathbb{H}_n(R)$, where $R = \mathbb{R}$ or $\mathbb{C}$. Since $\mathbb{H}_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)_{sa}$, where $\mathbb{C}^n$ is the n-dimensional Hilbert space, $\mathbb{H}_n(\mathbb{C})$ is by definition a JC algebra. $\mathbb{H}_n(\mathbb{R})$ is a norm-closed Jordan subalgebra of $\mathbb{H}_n(\mathbb{C})$ since the JC property of $\mathbb{H}_n(\mathbb{R})$ is hereditary from $\mathbb{H}_n(\mathbb{C})$. Ultimately, we have that in finite dimensions, $\mathbb{H}_n(\mathbb{R})$ and $\mathbb{H}_n(\mathbb{C})$ are all JC algebras.

Definition 111 (JORDAN BANACH ALGEBRA) *A Jordan Banach algebra is a real Jordan algebra $\mathcal{A}$ equipped with a complete norm satisfying*

$$||a \circ b|| \leq ||a|| \cdot ||b||, \quad a\,, b \in \mathcal{A}\,. \tag{2.42}$$

Definition 112 (JB ALGEBRA) *A JB algebra is a Jordan Banach algebra $\mathcal{A}$ in which the norm satisfies the following two additional conditions for $a, b \in \mathcal{A}$:*

$$||a^2|| = ||a||^2\,, \tag{2.43}$$

$$||a^2|| \leq ||a^2 + b^2||\,. \tag{2.44}$$

As usual, if $\mathcal{A}$ is unital, we shall denote the identity by $\mathbb{I}$. Because $\mathbb{I}^2 = \mathbb{I}$, by the previous definition we have that $||\mathbb{I}|| = ||\mathbb{I}^2|| = ||\mathbb{I}||^2$ which (together with nonnegativity of the norm $||\cdot||$) implies that $||\mathbb{I}|| = 1$. Each JC algebra is a JB algebra. All finite-dimensional (unital) formally real Jordan algebras are JB algebras, which is false in infinite dimensions.

2.4.1 SPECTRAL THEORY

In this section we shall give a brief review of the spectral theory of the JB algebras. Two spectral theorems will be included, one for general associative JB algebras and one for singly generated ones.

Let X be a locally compact Hausdorff space. By $C_0^{\mathbb{C}}(x)$ (resp. $C_0(X)$) we shall denote the set of continuous complex (resp. real) functions on X vanishing at ∞. If X is compact, we shall write $C(X)$ instead of $C_0(X)$. $C_0^{\mathbb{C}}(X)$ along with pointwise multiplication, $*$ operation $f^*(\omega) = \overline{f(\omega)}$, and norm $||f|| = sup_{\omega\in X}|f(\omega)|$ is an abelian C^*-algebra with self-adjoint part $C_0(X)$. Furthermore, if X is compact and $f \in C_0^{\mathbb{C}}(X)$ then spectrum $\sigma(f) = f(X)$.

$C_0(X)$ is an associative JC algebra under pointwise multiplication and supremum norm. The following theorem says that the converse is true, too.

Theorem 113 (THE SPECTRAL THEOREM I.) *Let $\mathcal{A}$ be an associative JB algebra. Then there is a locally compact Hausdorff space X such that $\mathcal{A}$ is isometrically isomorphic to $C_0(X)$. Furthermore, $\mathcal{A}$ is unital if and only if X is compact.*

Proof. To make the proof more transparent let us summarize the idea of the proof. We will endow $\tilde{\mathcal{A}}$ with certain operations, involution and norm to become an abelian C^*-algebra. Both, the Gelfand transform theorem and characterization of abelian C^*-algebras then complete the proof.

Let $\tilde{\mathcal{A}}$ denote the complexification of $\mathcal{A}$, i.e. $\tilde{\mathcal{A}} := \{a + ib : \ a, b \in \mathcal{A}\}$. Endowing $\tilde{\mathcal{A}}$ with the product $(a+ib)(c+id) = (a \circ c - b \circ d) + i(a \circ d + b \circ c)$ and involution $(a+ib)^* = a - ib$, we obtain an involutive abelian complex algebra (note that Jordan product is always commutative). Define

$$||a + ib|| := ||a^2 + b^2||^{1/2}, \quad a, b \in \mathcal{A}. \tag{2.45}$$

We shall show that $||\ \ ||$ is a norm on $\tilde{\mathcal{A}}$ making $\tilde{\mathcal{A}}$ into an abelian C^* algebra containing $\mathcal{A}$ as its self-adjoint part.

Let $z = a + ib \in \tilde{\mathcal{A}}$. Obviously $||z|| = 0$ if and only if $z = 0$. Let $\lambda \in \mathbb{C}$, then $||\lambda z|| = |\lambda| ||z||$. Indeed, let $\lambda = \eta + i\xi$, where $\eta, \xi \in \mathbb{R}$, then

$$||\lambda z|| = ||(\eta a - \xi b) + i(\xi a + \eta b)|| = ||(\eta a - \xi b)^2 + (\xi a + \eta b)^2||^{1/2} =$$
$$= ||\eta^2 a^2 - \xi^2 b + \xi^2 a^2 + \eta^2 b^2||^{1/2} = ||(\eta^2 + \xi^2)(a^2 + b^2)||^{1/2} =$$
$$= (\eta^2 + \xi^2)^{1/2} ||a^2 + b^2||^{1/2} = |\lambda| ||z||.$$

We also have that $||z||^2 = ||a^2 + b^2|| = ||(a - ib)(a + ib)|| = ||z^* z||$. Moreover if $w \in \tilde{\mathcal{A}}$ then by the C^* property of $||\ \ ||$,

$$||zw||^2 = ||w^* z^* z w|| = ||(z^* z) \circ (w^* w)|| \leq ||z^* z|| \, ||w^* w|| = ||z||^2 ||w||^2,$$

which shows $||zw|| \leq ||z|| \, ||w||$ for all $z, w \in \tilde{\mathcal{A}}$. It remains to show the triangle inequality. Let $z = a + ib$, $w = c + id$ belong to $\tilde{\mathcal{A}}$. Then by definition of the product in $\tilde{\mathcal{A}}$ we have that

$$z^* w + w^* z = 2a \circ c + 2b \circ d \in \mathcal{A}\,.$$

Furthermore, by the defining properties of JB algebras and associativity of $\mathcal{A}$ we find

$$||a \circ c + b \circ d||^2 \leq ||(a \circ c + b \circ d)^2 + (a \circ d - b \circ c)^2|| =$$
$$= ||(a^2 + b^2) \circ (c^2 + d^2)|| \leq ||a^2 + b^2|| \, ||c^2 + d^2|| = ||z||^2 ||w||^2.$$

Now we will evaluate $||z + w||^2$. Note that $z, w \in \mathcal{B}(\mathcal{H})$. Norm of operator $a \in \mathcal{B}(\mathcal{H})$ is defined as $||a|| := sup\{||ax|| \, : \, x \in \mathcal{H}, \ ||x|| = 1\}$. The norm $||\ ||$ is induced by an inner product on $\mathcal{H}$ in the way $||(z+w)x||^2 =< (z+w)x, (z+w)x >$, which being estimated yields

$$< (z + w)x, (z + w)x >= ||zx||^2 + ||wx||^2 + < zx, wx > + < wx, zx >=$$
$$= ||zx||^2 + ||wx||^2 + < zx, wx > + \overline{< zx, wx >} \leq$$

$$\leq ||zx||^2 + ||wx||^2 + |<zx, wx>| + \underbrace{|\overline{<zx, wx>}|}_{=|<zx,wx>|} =$$

$$= ||zx||^2 + ||wx||^2 + 2|<zx, wx>|\,.$$

Employing the Schwartz inequality on the last term, $|<zx, wx>| \leq ||zx|| ||wx||$, we can write

$$||(z+w)x||^2 \leq \big(||zx|| + ||wx||\big)^2. \tag{2.46}$$

Since (2.46) holds true in $\mathcal{H}$ it also does for all $x \in \mathcal{H}$ such that $||x|| = 1$. Put

$$S_1 := \{||(z+w)x||^2 \;:\; x \in \mathcal{H},\, ||x|| = 1\},$$

$$S_2 := \{\big(||zx|| + ||wx||\big)^2 \;:\; x \in \mathcal{H},\, ||x|| = 1\}.$$

It is evident that $S_1 \leq S_2$ for all $x \in \mathcal{H}$ such that $||x|| = 1$. Because $z, w \in \mathcal{B}(\mathcal{H})$ are bounded, we have that $sup\, S_2 < \infty$. Hence $sup\, S_1$ exists and with the help of the fact that suprema preserve inequalities we get that

$$sup\, S_1 \leq sup\, S_2\,,$$

which is nothing but

$$||z+w||^2 \leq \big(||z|| + ||w||\big)^2\,,$$

proving the triangle inequality.

$\mathcal{A}$ is norm-complete, i.e. every Cauchy sequence $(a_\alpha)_\alpha$ has limit in $\mathcal{A}$ with respect to metric induced by norm, i.e. $||a_\alpha|| \overset{n\to\infty}{\to} ||a||$. Now let $(z_n)_n$ be a Cauchy sequence in $\tilde{\mathcal{A}}$. Put $z_n = a_n + ib_n$ for all $n \in \mathbb{N}$, where $(a_n)_n, (b_n)_n$ are Cauchy sequences in $\mathcal{A}$ with limits $a, b \in \mathcal{A}$, respectively. Employing the norm defined as above, we obtain

$$||z_n|| = ||a_n^2 + ib_n^2||^{1/2} \overset{n\to\infty}{\to} ||a^2 + ib^2||^{1/2} = ||z|| \in \tilde{\mathcal{A}}\,,$$

hence $\tilde{\mathcal{A}}$ is complete. Thus we have shown that $\tilde{\mathcal{A}}$ is an abelian C^* algebra. By the Gelfand transform theorem we have that $\tilde{\mathcal{A}}$ is isometrically isomorphic to $C_0\big(\Omega(\tilde{\mathcal{A}})\big)$, where $\Omega(\tilde{\mathcal{A}})$ is the spectrum of algebra $\tilde{\mathcal{A}}$. Put $X := \Omega(\tilde{\mathcal{A}})$. We already know that X is a locally compact Hausdorff space and is compact if and only if $\tilde{\mathcal{A}}$ is unital. $\square$

If $\mathcal{A}$ is a unital Jordan algebra and $a \in \mathcal{A}$, we denote by $C(a)$ the smallest norm-closed Jordan subalgebra of $\mathcal{A}$ containing a and $\mathbb{I}$. Evidently $C(a)$ is associative. The spectrum $\sigma(a)$ of a is defined in a usual way, i.e. to be the set of $\lambda \in \mathbb{R}$ such that $a - \lambda\mathbb{I}$ does not have an inverse in $C(a)$.

Theorem 114 (THE SPECTRAL THEOREM II.) *Let $\mathcal{A}$ be a unital Jordan Banach algebra. Let $a \in \mathcal{A}$ and suppose $C(a)$ is a JB algebra in an equivalent norm $|||\;|||$. Then $C(a)$ is isometrically isomorphic to $C(\sigma(a))$ with respect to $|||\;|||$.*

Proof. By the previous spectral theorem there is a compact Haudsorff space X such that $C(a)$ with the norm $|||\;|||$ is mapped isometrically and isomorphically onto $C(X)$. Thus $\sigma(a)$ as defined above equals the spectrum of a as an element in $C(X)$. Thus by the complex spectral theorem $C(a)$ is isometrically isomorphic to $C(\sigma(a))$. $\square$

Proposition 115 *Let $\mathcal{A}$ be a unital Jordan Banach algebra such that $C(a)$ is a JB algebra in the given norm for each $a \in \mathcal{A}$. Then $\mathcal{A}$ is a JB algebra.*

Proposition 116 *Suppose $\mathcal{A}$ is a complete order unit space which is a Jordan algebra for which the distinguished order unit acts as an identity element, and suppose*

$$-\mathbb{I} \leq a \leq \mathbb{I} \Rightarrow 0 \leq a^2 \leq \mathbb{I}, \tag{2.47}$$

for all $a \in \mathcal{A}$. Then $\mathcal{A}$ is a JB algebra in the order norm.

The set of all states (positive, norm-one linear functionals) in $\mathcal{A}$ will be denoted by $S(\mathcal{A})$. The set $S(\mathcal{A})$ is convex for any JB algebra $\mathcal{A}$. For any JB algebra, whether unital or not, we call any extreme point of $S(\mathcal{A})$ a pure state.

Lemma 117 *Let $\mathcal{A}$ be a JB algebra and ρ a state on $\mathcal{A}$. Then if $a, b \in \mathcal{A}$, we have that*
(i) $\rho(a \circ b)^2 \leq \rho(a^2)\rho(b^2)$,
(ii) $\rho(a)^2 \leq \rho(a^2)$.
In particular, the map $a \mapsto \rho(a^2)^{\frac{1}{2}}$ is a seminorm on $\mathcal{A}$.

Proposition 118 *If $\mathcal{A}$ is a unital JB algebra, then $\mathcal{A}$ is a complete order unit space with the ordering induced by $\mathcal{A}^+$ and order unit the identity $\mathbb{I}$. The order norm is the given one, and $a \in \mathcal{A}$ satisfies $-\mathbb{I} \leq a \leq \mathbb{I} \Rightarrow 0 \leq a^2 \leq \mathbb{I}$.*

Hence we see that the implication in the proposition 116 can be reversed in unital JB algebras. For non-unital JB algebras this is not that clear. One might suggest that adjoining an identity to a non-unital JB algebra would lead to the same result. However, the problem is that while this is easy for C^*-algebras, it is not so simple to prove that we get a JB algebra when we adjoin a unit to a non-unital JB algebra.

The next proposition gives a GNS representation of Jordan matrix algebras analogously to the case of C^*-algebras.

Proposition 119 *Let $A = \mathbb{H}_n(R)$, $n \geq 2$, be a Jordan matrix algebra which is also a JB algebra. Assume R is associative. Then there exists a $*$-representation of $M_n(R)$ on a complex Hilbert space carrying $\mathcal{A}$ isometrically onto a reversible JC algebra.*

2.4.2 JBW ALGEBRAS

The Jordan analogue to (abstract) von Neumann algebras are JBW algebras. Von Neumann algebras are by definition ultraweakly closed C^*-algebras on a Hilbert space. The corresponding Jordan algebras will be called JW algebras. JW algebras is a smaller class than that of JBW algebras (see definition below).

Definition 120 *Let M be a JB algebra. M is said to be monotone complete if each bounded increasing set (a_α) in M has a least upper bound $a \in M$. A bounded linear functional ρ on M is called normal if $\rho(a_\alpha) \to \rho(a)$ for each net (a_α) as above. A set of functionals is called separating if for any nonzero $a \in M$ there is a functional ρ in the set satisfying $\rho(a) \neq 0$.*

Definition 121 (JBW ALGEBRA) *Let M be a JB algebra. M is said to be a JBW algebra if M is monotone complete and has a separating set of normal positive functionals.*

Definition 122 *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the von Neumann algebra of all bounded operators on $\mathcal{H}$ equipped with the ultraweak topology. A Jordan subalgebra M of $\mathcal{B}(\mathcal{H})_{sa}$ is called a JW algebra if M is ultraweakly closed.*

JW algebra may also mean a JB algebra which is isomorphic to a JW algebra like M above. Since ultraweakly continuous linear functionals on $\mathcal{B}(\mathcal{H})$ are normal and since the least upper bound of a bounded increasing net in $\mathcal{B}(\mathcal{H})_{sa}$ is the ultraweak limit of the net, JW algebra is a JBW algebra.

In the rest of the chapter, K will denote the set of all normal states. Note that state is a norm-one positive functional, even if the JB algebra is non-unital. We denote by V the real vector subspace of M^* spanned by K.

As we see, topology is an integral part of the theory of ultraweakly closed algebras. Let M be a JBW algebra. The *weak topology* is defined in the usual way (see von Neumann algebras section). The *strong topology* on M is the locally convex topology defined by the seminorms $a \mapsto \rho(a^2)^{\frac{1}{2}}$, $\rho \in K$. If $\rho \in K$, then $\rho(a^2)^{\frac{1}{2}} \leq ||a^2||^{\frac{1}{2}} = (||a||^2)^{\frac{1}{2}} = ||a||$, hence norm convergence implies strong convergence. Since $\rho(a)^2 \leq \rho(a^2)$, $\rho(a^2)^{\frac{1}{2}} \to 0$ implies $\rho(a) \to 0$. Hence strong convergence implies weak convergence. Furthermore, if (a_α) is a bounded increasing net in M with least upper bound $a \in M$, written $a_\alpha \nearrow a$, then (a_α) converges strongly to a. Indeed, if $\rho \in K$, then $a - a_\alpha \geq 0$, so that

$$\rho\big((a-a_\alpha)^2\big) \leq ||a-a_\alpha||\rho(a-a_\alpha) \leq ||a||\rho(a-a_\alpha) \to 0\,.$$

In the second equality we assumed $a_\alpha \geq 0$ without loss of generality.

JBW algebras have one non-trivial property, namely that a JBW algebra is unital. For proof and discussion, see [10].

Let M be a JBW algebra and $a \in M$. We denote by $W(a)$ the weak closure of $C(a)$. If $b, c \in W(a)$, then there are nets (b_α) and (c_β) in $C(a)$ which converge weakly to b and c respectively. By [10] the operators T_a and U_a are weakly continuous for all $a \in M$, where M is a JBW algebra. This implies that $b_\alpha \circ c_\beta \to b \circ c_\beta$ weakly with α, so evidently $b \circ c_\beta \in W(a)$. Again, $b \circ c_\beta \to b \circ c$ weakly, so $b \circ c \in W(a)$. Thus $W(a)$ is a subalgebra of M. Norm topology is finer than the weak topology, which implies that $W(a)$ is norm closed, hence is a JB algebra. If (b_α) is an increasing net in $W(a)$ with least upper bound b, then $b_\alpha \to b$ strongly, hence weakly. Thus $b \in W(a)$, and $W(a)$ is monotone complete. It is thus a JBW algebra, since the states in K restrict to normal states on $W(a)$. We may conclude, that $W(a)$ is a JBW algebra generated by an element a and $\mathbb{I}$ in a JBW algebra.

Lemma 123 *Let M be a JBW algebra and $a \in M$. Then $W(a)$ is an associative JBW subalgebra of M isometrically isomorphic to a monotone complete $C(X)$, where X is a compact Hausdorff space.*

An idempotent in a JB algebra will be called a projection. Just as in the theory of von Neumann algebras they play important role in the classification theory. We wish to modify the existing theory for von Neumann algebras to JBW algebras.

Lemma 124 *Let $\mathcal{A}$ be a JB algebra and p, q projections in $\mathcal{A}$. Then the following conditions are equivalent:*

(i) p and q are orthogonal.
(ii) $p + q$ is a projection.
(iii) $p + q \leq \mathbb{I}$.
(iv) $U_p q = 0$.
(v) $U_p U_q = 0$.

Proposition 125 *Let M be a JBW algebra, $a \in M$, and $\epsilon > 0$. Then there exist pairwise orthogonal projections $p_1, \dots, p_n$ in $W(a)$ and real numbers $\lambda_1, \dots, \lambda_n$ such that $||a - \sum_{i=1}^{n} \lambda_i p_i|| < \epsilon$.*

Lemma 126 *Let M be a JBW algebra and $a \in M$. If p is a projection in M, the following conditions are equivalent:*

(i) p operator commutes with a.

(ii) p operator commute with all elements in $W(a)$.

(iii) p operator commutes with all projections in $W(a)$.

In classification theory the existence of the least upper and greatest lower bound of family of projections is investigated. It shows that these bounds do exist in a JBW algebra.

The center of a Jordan algebra $\mathcal{A}$ is the set of elements in $\mathcal{A}$ which operator commute with all elements in $\mathcal{A}$. Since multiplication is separately weakly continuous in a JBW algebra M, then center $\mathcal{Z}$ of M is an associative JBW subalgebra of M. Symmetry in M is an operator s such that $s^2 = \mathbb{I}$.

Lemma 127 *Let M be a JBW algebra with center $\mathcal{Z}$. If $a \in M$ then $a \in \mathcal{Z}$ if and only if $U_s a = a$ for all symmetries $s \in M$.*

Definition 128 *Let M be a JBW algebra with center $\mathcal{Z}$. If $p \in M$ is a projection, then its central support $c(p)$ is the smallest projection in Z majorizing p.*

Lemma 129 *Let $\mathcal{M}$ be a von Neumann algebra and M a weakly closed Jordan subalgebra of $\mathcal{M}_{sa}$, and assume that M generates $\mathcal{M}$. Then the center of M is contained in the center of $\mathcal{M}$.*

2.4.3 CLASSIFICATION THEORY

In the present section we shall generalize the projection lattice theory of von Neumann algebras to JBW algebras.

Let M be a JBW algebra. We denote by $P(M)$ the lattice of projections in M. $\mathcal{Z}$ will denote the center of M, and $P_{\mathcal{Z}} := P \cap \mathcal{Z}$ the set of central projections in M. As before, we write $p^{\perp} = \mathbb{I} - p$, if $p \in P(M)$. $P(M)$ is then an orthocomplemented lattice, since $p \leq q$ implies $q \wedge p^{\perp} = q - p$ and so $q = p \vee (q \wedge p^{\perp})$.

If $s \in M$ is a symmetry, then U_s is an automorphism of M and these automorphisms generate a group $IntM$ called the group of inner automorphisms of M.

Two projections $p, q \in M$ are called *equivalent* if there is $\alpha \in IntM$ such that $q = \alpha(p)$. We then write $p \sim q$. If α can be written as $\alpha = U_{s_1} U_{s_2}, \dots, U_{s_n}$, we write $p \sim_n q$. If $n = 1$, we say that p and q are *exchanged by a symmetry*. Note that in contrast with equivalence of projections in a von Neumann algebra, $p \sim q$ implies $p^{\perp} \sim q^{\perp}$ in JBW algebras. Moreover we have that $c(p) = \bigvee\{q \in P : q \sim p\}$.

Let M be a JBW algebra and $p \in M$ a projection. Denote by $M_p := U_p(M)$ a JBW subalgebra of M. A projection $p \in M$ is called *Abelian* if M_p is associative; p is *modular* if the projection lattice $[0, p]$ of M_p is modular. If $\mathbb{I}$ is modular, M itself is called modular.

Define central projections e_I and e_{III} in M by

$e_I := \bigvee\{q \in P : \ p \text{ is Abelian}\}$,

$e_{III}^{\perp} := \{p \in P : \ p \text{ is modular}\}$.

Definition 130 *Let M be a JBW algebra and let $e_{II} = \mathbb{I} - e_I - e_{III}$ with e_I, e_{II} and e_{III} defined as above. M is said to be of type I (resp. II, III) if $e_I = \mathbb{I}$ (resp. $e_{II} = \mathbb{I}$, $e_{III} = \mathbb{I}$).*

Theorem 131 *Let M be a JBW algebra. Then M can be split uniquely into a direct sum of parts of Type I, II and III, the different parts being characterized as follows:*

(i) M is of Type I if and only if there is an abelian projection $p \in M$ with $c(p) = \mathbb{I}$.

(ii) M is of Type II if and only if there is a modular projection $p \in M$ with $c(p) = \mathbb{I}$, and M contains no nonzero abelian projection.

(iii) M is of Type III if and only if it contains no nonzero modular projection.

Just like in the case of von Neumann algebras it is possible to define a finer decomposition. Any JBW algebra has the largest central modular projection. If this is 0, M is called purely nonmodular; if it is $\mathbb{I}$, M is called modular. We say M is of Type II_1 if it is modular and of Type II, and it is of Type II_∞ if it is purely nonmodular and of Type II. If M is of Type III, it is purely nonmodular.

Lemma 132 *Let M be a JBW algebra with projection lattice P. If $p, q \in P$, then there is a symmetry $s \in M$ such that $U_s\{pqp\} = \{qpq\}$.*

Proposition 133 *Let M be a JBW algebra with projection lattice P. Let $p, q \in P$ and suppose $p \perp q$ and $p \sim_2 q$. Then $p \sim_1 q$.*

The character of the following propositions is purely technical; they shall be used later only as technical tool in proofs of other theorems.

Proposition 134 *Let M be a JBW algebra with projection lattice P. Suppose $p, q \in P$ are nonzero and equivalent. Then there are nonzero $p_1, q_1 \in P$ such that $p_1 \leq p$, $q_1 \leq q$ and $p \sim_1 q_1$.*

Proposition 135 *Let M be a JBW algebra with projection lattice P, and let J be an index set. Let $p, q, p_\alpha, q_\alpha \in P$, $\alpha \in J$, satisfy $p \perp q$, $p = \sum_{\alpha \in J} p_\alpha$, $q = \sum_{\alpha \in J} q_\alpha$ and $p_\alpha \sim_1 q_\alpha$ for all $\alpha \in J$. Then $p \sim_1 q$.*

Proposition 136 *Let M be a JBW algebra with projection lattice P. Let $p_i, q_i \in P$, $i = 1, 2$, satisfy $p_1 + p_2 = p \in P$, $q_1 + q_2 = q \in P$, $p_1 \perp q_2$, $p_2 \perp p_1$ and $p_i \sim_1 q_i$. Then $p \sim_1 q$.*

Lemma 137 (The halving lemma) *Let M be a JBW algebra with projection lattice P. Suppose M has no direct summand of Type I. Then there is $p \in P$ with $p \sim_1 p^\perp$.*

Proposition 138 *Let M be a JBW algebra with projection lattice P. Suppose M has no direct summand of Type I. Then there are $p_i \in P$, $i = 1, 2, 3, 4$, such that $p_1 + p_2 + p_3 + p_4 = \mathbb{I}$, and $p_i \sim_1 p_j$ for all i, j.*

Now we shall classify and discuss JBW algebras of type I, which may be quite different from type I von Neumann algebras (for example spin factors).

Definition 139 *Let M be a JBW algebra and n cardinal number. We say M is of Type I_n if there is a family $(p_\alpha)_{\alpha \in J}$ of abelian projections such that $c(p_\alpha) = \mathbb{I}$, $\sum_{\alpha \in J} p_\alpha = \mathbb{I}$ and card $J = n$. We also say M is of Type I_∞ if M is a direct sum of JBW algebras of Type I_n with n infinite.*

Note 140 *If M is a JBW algebra of type I_n and e is a nonzero central projection in M then Me is of type I_n.*

Theorem 141 *Each JBW algebra of Type I has a unique decomposition*

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_\infty\,, \tag{2.48}$$

where each M_n is either 0 or is a JBW algebra of Type I_n.

Definition 142 *A JBW algebra is called a JBW factor if its center consists of real multiples of the identity only.*

Proposition 143 *Let M be a JBW factor of Type I. Then there exists $n \in \mathbb{N} \cup \{\infty\}$ such that M is of Type I_n. Let M be a JBW factor of Type I_n, $3 \leq n < \infty$. Then M is isomorphic to one of $\mathbb{H}_n(\mathbb{R})$, $\mathbb{H}_n(\mathbb{C})$, $\mathbb{H}_n(\mathbb{Q})$ or $\mathbb{H}_n(\mathbb{O})$ in the case $n = 3$.*

2.4.4 SPIN FACTORS

Let $\mathcal{H}$ be a real Hilbert space, and suppose $a : \mathcal{H} \mapsto \mathcal{A}$ is a linear map of $\mathcal{H}$ into a C^*-algebra $\mathcal{A}$ satisfying the cannonical anticommutation relations:

$$2a(f) \circ a(g) = a(f)a(g) + a(g)a(f) = 0\,, \tag{2.49}$$

$$2a(f) \circ a(g)^* = a(f)a(g)^* + a(g)^*a(f) = (f,g)\mathbb{I}\,, \tag{2.50}$$

for all $f, g \in \mathcal{H} = L^2(\mathbb{R})$. Setting $b(f) = a(f) + a(f)^*$ we obtain

$$b(f) \circ b(g) = a(f) \circ a(g)^* + a(f)^* \circ a(g) + \big(a(f) \circ a(g) + a(f)^* \circ a(g)^*\big)\,.$$

The first term in the bracket is by (2.49) zero. The second term in the bracket satisfies

$$a(f)^* \circ a(g)^* = \frac{1}{2}\big(a(f)^*a(g)^* + a(g)^*a(f)^*\big) =$$

$$= \frac{1}{2}\Big(\big(a(g)a(f)\big)^* + \big(a(f)a(g)\big)^*\Big) = \Big(\frac{a(g)a(f) + a(f)a(g)}{2}\Big)^* =$$

$$= \Big(\frac{a(f)a(g) + a(g)a(f)}{2}\Big)^* = \big(a(f) \circ a(g)\big)^* = 0^* = 0\,,$$

thus

$$b(f) \circ b(g) = a(f) \circ a(g)^* + a(f)^* \circ a(g)\,.$$

Adding (2.50) to its conjugation

$$a(g) \circ a(f)^* = a(f)^* \circ a(g) = \frac{1}{2}\overline{(f,g)}\mathbb{I} = \frac{1}{2}(f,g)\mathbb{I}\,,$$

yields

$$b(f) \circ b(g) = (f,g)\mathbb{I} \quad f, g \in \mathcal{H}, \tag{2.51}$$

in particular $||b(f)|| = ||f||$, and b is an isometry of $\mathcal{H}$ into $\mathcal{A}_{sa}$, such that $b(\mathcal{H}) + \mathbb{R}\mathbb{I}$ is a JC algebra A. If $(f_n)_{n\in J}$ is an orthonormal basis for $\mathcal{H}$ then the set $\{\mathbb{I}, b(f_n)\}_{n\in J}$ is a set of symmetries with the properties $b(f_n) \circ b(f_m) = \delta_{mn}\mathbb{I}$, whose closed linear span is A. The JC algebra A and the set $\{b(f_n)\}_{n\in J}$ shall be called a spin factor and a spin system, respectively.

Now let B be a real unital Jordan algebra. A spin system in B is a collection $\mathcal{P}$ of at least two symmetries different from $\pm\mathbb{I}$ such that $s \circ t = 0$ whenever $s \neq t$ in $\mathcal{P}$. Let H_0 denote the linear span of $\mathcal{P}$ in B. Then any two elements $a, b \in \mathcal{H}_0$ can be written as $a = \sum_{i=1}^{n} \alpha_i s_i$, $b = \sum_{i=1}^{n} \beta_i s_i$, where $s_1, \ldots, s_n$ are distinct symmetries in $\mathcal{P}$. We have that

$$a \circ b = \sum_{i,j=1}^{n} \alpha_i \beta_j (s_i \circ s_j) = \sum_{i=1}^{n} \alpha_i \beta_i (s_i \circ s_i) = (\sum_{i=1}^{n} \alpha_i \beta_i)\mathbb{I}\,,$$

thus $\mathcal{H}_0$ is a pre-Hilbert space with inner product defined by

$$< a, b > \mathbb{I} = a \circ b\,. \tag{2.52}$$

We are however interested in the algebra obtained from the subalgebra $\mathcal{H}_0 + \mathbb{R}\mathbb{I}$ of B by completing $\mathcal{H}_0$.

Proposition 144 *Let $\mathcal{H}$ be a real Hilbert space of dimension at least* 2*. Let $A = \mathcal{H} \oplus \mathbb{R}\mathbb{I}$ have the norm $||a + \lambda\mathbb{I}|| = ||a|| + |\lambda|$, $a \in \mathcal{H}$, $\lambda \in \mathbb{R}$. Define a product in A by*

$$(a + \lambda\mathbb{I}) \circ (b + \mu\mathbb{I}) = (\mu a + \lambda b) + (< a, b > + \lambda\mu)\mathbb{I}\,, \tag{2.53}$$

where $a, b \in \mathcal{H}$, $\lambda, \mu \in \mathbb{R}$. Then A is a JB algebra.

Definition 145 *A unital JB algebra generated as a JB algebra by a spin system will be called a spin factor.*

Proposition 146 *For each cardinal number $n \geq 2$ there is, up to isomorphism, a unique spin factor generated by the spin system of cardinality n.*

Theorem 147 *Any finite-dimensional, formally real, unital Jordan algebra which is also simple and contains two minimal projections with sum $\mathbb{I}$ is a finite-dimensional spin factor.*

Algebras $\mathbb{H}_2(\mathbb{R})$ and $\mathbb{H}_2(\mathbb{C})$ are finite-dimensional spin factors. Orthonormal basis for the inner product space $\mathbb{H}(\mathbb{R})$, $\mathbb{H}(\mathbb{C})$ consists of the first two (resp. three) matrices of the following list:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.54}$$

Note that these matrices are the Pauli spin matrices in $M_2(\mathbb{C})$. This is why we denote our spin systems by $\mathcal{P}$. Also note that these matrices satisfy the definition of a spin system above.

Definition 148 *We call a symmetry in a JB algebra nontrivial if it is not $\pm\mathbb{I}$.*

Let $A = \mathcal{H} \oplus \mathbb{R}\mathbb{I}$ be a spin factor. If $a \in \mathcal{H}$, $\lambda \in \mathbb{R}$, then $\sigma(a + \lambda\mathbb{I}) = \{\lambda - ||a||, \lambda + ||a||\}$. By this it is evident that nontrivial symmetries in a spin factor $A = \mathcal{H} \oplus \mathbb{R}$ are the unit vectors in $\mathcal{H}$.

Theorem 149 *Let M be a JBW algebra. Then M is a JBW factor of Type I_2 if and only if M is a spin factor.*

Theorem 150 *Let A be a JB algebra of real dimension at least* 3*. Then A is a spin factor if, and only if, A is a JW factor of Type I_2.*

So far we have that each JBW factor of type I_2 is a spin factor. Now we shall review general JBW algebras of type I_2 and global characterizations of the previous results.

If M is a JBW algebra of Type I_2 with center $\mathcal{Z}$, we may assume that $\mathcal{Z} = C(X)$, where X is a compact Hausdorff space. Any element $a \in M$ can be considered as a continuous function on X with values in JBW factors of Type I_2, i.e. for each $t \in X$ we have $a(t) = x(t)p(t) + y(t)q(t)$, where $x(t), y(t) \in \mathbb{R}$, and $p(t)$ and $q(t)$ are abelian projections in a I_2 factor with sum $\mathbb{I}$.

Lemma 151 *Let M be a JBW algebra of Type I_2 with center $\mathcal{Z}$. If $a \in M$ then there exist $x, y \in \mathcal{Z}$ and a projection $p \in M$ with $c(p) = c(p^\perp) = \mathbb{I}$ such that $a = xp + yp^\perp$. In particular, if s is the symmetry $s = 2p - \mathbb{I}$ then $a = z + ws$ with $z, w \in \mathcal{Z}$.*

If $s = 2p - \mathbb{I}$ is a symmetry then $s + \mathbb{I} = 2p$ and $s - \mathbb{I} = 2p^\perp$. We write $c(s \pm \mathbb{I})$ for the central supports of p, $p^\perp$ respectively. Thus the symmetry from the previous lemma has the property that $c(s \pm \mathbb{I}) = \mathbb{I}$.

Lemma 152 *Let M be a JBW algebra of type I_2 with center $\mathcal{Z}$. Let*

$$N = \{ws : w \in \mathcal{Z}, s\,\text{a symmetry in M with}\, c(s \pm \mathbb{I}) = \mathbb{I}\}\,.$$

If s, t are symmetries in N then $s \circ t \in \mathcal{Z}$.

Proposition 153 *Let M be a JBW algebra of type I_2 with center $\mathcal{Z}$. Let N be as in the previous lemma. Then N is a vector subspace of M, and $M = \mathcal{Z} \bigoplus N$.*

Definition 154 *Let M be a JBW algebra of type I_2. A spin system $(s_\alpha)_{\alpha\in J}$ in M is called locally maximal if for every nonzero central projection $e \in M$ the family $(es_\alpha)_{\alpha\in J}$ is a maximal spin system in eM.*

Lemma 155 *Let M be a JBW algebra of type I_2 with center $\mathcal{Z}$. Suppose $(s_1, \ldots, s_k)$, $k \in \mathbb{N}$, is a locally maximal spin system in M. Then every operator $a \in M$ can be written uniquely in the form*

$$a = z_0 + \sum_{i=1}^{k} z_i s_i\,, \quad z_0, \ldots, z_k \in \mathcal{Z}\,. \tag{2.55}$$

Definition 156 *Let M be a JBW algebra of type I_2. Let $k \in \{2, 3, \ldots\} \cup \{\infty\}$. We say M is of type $I_{2,k}$ if there is a locally maximal spin system $(s_\alpha)_{\alpha\in J}$ in M with card $J = k$ if $k < \infty$, and card J is infinite if $k = \infty$.*

Proposition 157 *Let M be a JBW algebra of type $I_{2,k}$, $k < \infty$. Let $\mathcal{Z}$ denote the center of M, and X be a compact Hausdorff space such that $\mathcal{Z} \simeq C(X)$. Then $M \simeq C(X, V_k)$, where V_k is a spin factor of dimension $k + 1$, $k \geq 2$ contained in $M_{2^n}(\mathbb{C})$ if $k \in \{2n - 1, 2n\}$.*

Theorem 158 *Any JBW algebra of type I_2 is a direct sum of JBW algebras of type $I_{2,k}$.*

We have not covered JBW algebras of type I_n, $3 \leq n < \infty$ yet. The following theorem proposes decomposition of such algebras in terms of direct summands.

Theorem 159 *Let M be a JBW algebra of type I_n, $3 \leq n < \infty$. Then M is a direct sum*

$$M = M_1 \oplus M_2 \oplus M_3 \oplus M_4\,, \tag{2.56}$$

where $M_4 = 0$ if $n \neq 3$, such that all factor representations of M_i are onto JBW factors isomorphic to $\mathbb{H}_n(\mathbb{R})$, $\mathbb{H}_n(\mathbb{C})$, $\mathbb{H}_n(\mathbb{Q})$ and $\mathbb{H}_n(\mathbb{O})$.

3. HIDDEN VARIABLES THEORY

3.1 Hidden variables in von Neumann algebras

Definition 160 *Let L be an orthomodular lattice. By a dispersion-free state on L we mean a finitely additive probability measure on L with values in the set $\{0, 1\}$.*

Theorem 161 (Hamhalter, (1993)) *The projection lattice $P(M)$ of a von Neumann algebra M which has neither a non-zero abelian nor a type I_2 direct summand admits no dispersion-free state.*

The following proposition reduces the investigation of dispersion-free states to simple matrix algebras instead of applying the Gleason Theorem for von Neumann algebras of infinite dimension.

Theorem 162 *Let M be a von Neumann algebra with no non-zero abelian direct summand and no type I_2 direct summand. The following statements hold:*

(i) Any subalgebra of M which is $*$-isomorphic to $M_2(\mathbb{C})$ is contained in a subalgebra $C \oplus D$ of M satisfying the following properties: C is either zero or it is $*$-isomorphic to $M_4(\mathbb{C})$; D is either zero or it is a copy of $M_2(\mathbb{C})$ contained in another subalgebra of M which is $*$-isomorphic to $M_3(\mathbb{C})$.

(ii) M contains a unital subalgebra $*$-isomorphic to one of the following matrix algebras: $M_2(\mathbb{C})$, $M_3(\mathbb{C})$, $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

Statement (i) is summarized and statement (ii) proved in [9].

As a result of the previous theorem it is enough to prove the non-existence of dispersion-free states on algebras $M_3(\mathbb{C})$ and $M_4(\mathbb{C})$.

Definition 163 *A Hidden space of a given quantum system is a set, Ω, with a σ-field, $\mathcal{A}$ of subsets of Ω with the following properties: for each quantum observable A and for each quantum state ρ there is an $\mathcal{A}$-measurable function $f_A : \Omega \mapsto \mathbb{R}$ and a probability measure μ_ρ on $\mathcal{A}$, such that the following conditions are fulfilled:*

(i) For each Borel set $\mathcal{B} \subset \mathbb{R}$ the probability that the value of an observable A is in $\mathcal{B}$ equals $\mu_\rho(f_A^{-1}(\mathcal{B}))$, provided that the system is in the state ρ.

*(ii) (*Function Principle*) If A and B are observables such that $B = g(A)$, where g is a real Borel function, then $f_B = g \circ f_A$.*

Condition (ii) means preserving transformation rules for observables.

Theorem 164 (Kochen-Specker) *Let $\mathcal{B}(\mathcal{H})$ be an algebra of bounded operators on a separable Hilbert space $\mathcal{H}$ of dimension at least 3. There is no σ-field $(\Omega, \mathcal{A})$ and a map*

$a \mapsto f_a$ assigning to each self-adjoint element $a \in M$ an $\mathcal{A}$-measurable real function f_a on Ω such that $f_{g(a)} = g \circ f_a$ for any real continuous function g on $\mathbb{R}$.

In 2004, Döring proved that hidden space does not exist for any von Neumann algebra without a type I_2 and a non-zero abelian direct summand that acts on a separable Hilbert space. This result can be extended to all von Neumann algebras without abelian and a type I_2 direct summand. The following theorem cited from [7] tells us that only the validity of the Function Principle is enough for excluding a hidden space without being necessary to specify the set of states.

Theorem 165 *Let M be a von Neumann algebra without a Type I_2 direct summand and with no non-zero abelian direct summand. There is no σ-field $(\Omega, \mathcal{A})$ and a map $a \mapsto f_a$ assigning to each self-adjoint element $a \in M$ an $\mathcal{A}$-measurable real function f_a on Ω such that $f_{g(a)} = g \circ f_a$ for any real continuous function g on $\mathbb{R}$.*

Proof. We shall prove that existence of such σ-field $(\Omega, \mathcal{A})$ implies existence of a dispersion-free state on $P(M)$ which shall be in contradiction with Theorem 164.

Let $(\Omega, \mathcal{A})$ be the σ-field with the properties stated above. Fix $\omega \in \Omega$. Let us consider the map $s : M_{sa} \mapsto \mathbb{R}$, $s(a) = f_a(\omega)$. If $p \in P(M)$, then $p^2 = p$. Put $m(x) = x^2$, we have that

$$f_p(\omega) = f_{p^2}(\omega) = f_{m(p)}(\omega) = m\big(f_p(\omega)\big) = f_p(\omega)^2\,,$$

hence $s(p) \in \{0, 1\}$.
Let us now take orthogonal non-zero projections $p, q \in M$ and put $x = p + \frac{1}{2}q$. Let g, h be continuous real functions on $\mathbb{R}$ such that $g(1) = h(\frac{1}{2}) = 1$ and $g(\frac{1}{2}) = h(1) = 0$. Since

$$g(x) = g\big(p + \frac{1}{2}q\big) = g(1)p + g\big(\frac{1}{2}\big)q\,,$$

(the same holds analogously for h) we have that $g(x) = p$ and $h(x) = q$. Setting $u = g + h$ and using the Function Principle we obtain

$$\begin{aligned} s(p+q) &= s\big(g(x) + h(x)\big) = s\big((g+h)(x)\big) = s\big(u(x)\big) = f_{u(x)}(a) = u\big(f_x(\omega)\big) = \\ &= g\big(f_x(\omega)\big) + h\big(f_x(\omega)\big) = f_{g(x)}(\omega) + f_{h(x)}(\omega) = s\big(g(x)\big) + s\big(h(x)\big) = s(p) + s(q). \end{aligned}$$

Hence s is a finitely-additive dispersion-free measure on $P(M)$. It remains to show that $s(\mathbb{I}) = 1$. It is a consequence of the fact that $s(\mathbb{I}) = f_{\mathbb{I}}(\omega) = 1$ for all $\omega \in \Omega$, where $\mathbb{I} = b(\mathbb{I})$ and b is a constant unit function on $\mathbb{R}$. We have showed that s induces a dispersion-free state on $P(M)$, which is in contradiction with Theorem 161. □

The problem of hidden variables is solved for von Neumann algebras without Type I_2 or non-zero abelian direct summand. It turns out that hidden variables do not exist in these type of algebras. In the case of C^*-algebras we have the following theorem.

Theorem 166 (HAMHALTER, (2004)) *Let $\mathcal{A}$ be a simple infinite unital C^*-algebra. Then $\mathcal{A}$ does not admit any dispersion-free quasi-state.*

How does this theory apply to direct measurement of quantum observables? Measurement of quantum systems is always accompanied by an error. Hence it is very strict to demand a dispersion-free state. It is more natural to ask whether there is a hidden space on which the quantum states would have smaller, or even better, arbitrarily small dispersion.

The latter is called the problem of *approximate hidden variables* and was introduced by G. W. Mackey in 1968.

Definition 167 *Let ρ be a state on the projection structure $P(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$. The overall dispersion, $\sigma(\rho)$ of ρ is defined by*

$$\sigma(\rho) = \sup\{\rho(p) - [\rho(p)]^2 \mid p \in P(\mathcal{A})\}\,. \tag{3.1}$$

Note 168 *By the previous definition $\sigma(p) \in [0, \frac{1}{4}]$ for all $p \in P(\mathcal{A})$ and $\sigma(p) = 0$ iff $\rho(p) \in \{0, 1\}$, i.e. iff ρ is dispersion-free state.*

Theorem 169 (HAMHALTER, (2004)) *Let $\mathcal{A}$ be a unital real rank-zero algebra having no representation onto an abelian C^*-algebra. Then*

$$\sigma(\rho) \geq \frac{2}{9}\,, \tag{3.2}$$

for any state ρ on $\mathcal{A}$.

3.2 HIDDEN VARIABLES IN JBW ALGEBRAS

In this section we seek generalization of the results of hidden variables theory to JBW algebras.

Theorem 170 *Let M be a JBW algebra without associative and Type I_2 direct summand. Then there is no finitely-additive dispersion-free state on $P(M)$.*

Proof. Let M be a JBW algebra with the properties stated above. Suppose that there exists finitely additive dispersion-free measure on $P(M)$. Let us decompose M as follows

$$M = z_1 M \oplus z_2 M\,,$$

where $z_1 M$ is of type I modular part and $z_2 M$ has no type I modular part. Let ρ be a non-zero $0-1$ state on M.

1) Suppose $\rho(z_2) = 1$. By Proposition 138 there are pairwise orthogonal projections $p_1, \ldots, p_4 \in P(M)$ such that $z_2 = p_1 + \cdots + p_4$ and $p_i \sim_1 p_j$ for all $i, j \in \{1, \ldots, 4\}$. Hence these projections are contained in a subalgebra of isomorphic copy of algebra of matrices $M_4(\mathbb{R})$, which we denote by $\{p_1, \ldots, p_4\} \subseteq M_4(\mathbb{R})$. Since the existence of non-zero $0-1$ state on algebra of real symmetric matrices 4×4 is excluded, we have that $\rho(a) = 0$ for all $a \in M_4(\mathbb{R})$. This property is hereditary to $\{p_1, \ldots, p_4\} \subseteq M_4(\mathbb{R})$, thus ρ must be identically equal zero, i.e. $\rho(z_2) = 0$ which is a contradiction.

2) Now assume that M is of type I. By Theorem 141 M can be uniquely decomposed in the way

$$M = \bigoplus_{n=3}^{\infty} M_n\ ,$$

where M_n is either zero or type I_n, $3 \leq n < \infty$. Now let M_n be a direct summand in the sum above. M_n is either zero or type I_n. If the latter is true then there are n orthogonal abelian projections $q_{j,n} \sim_1 q_{k,n}$ for all $j, k \in \{1, \ldots, n\}$ such that for the unit $\mathbb{I}_n \in M_n$ we have that

$$\sum_{j=1}^{n} q_{j,n} = \mathbb{I}_n \,, \quad 3 \leq n < \infty \,.$$

Put $k := [\frac{n}{3}]$ being the whole part of $\frac{n}{3}$; $n = 3k + r$, where $r \in \{0, 1, 2\}$ satisfies $n \equiv r (mod\, 3)$. Now put

$$f_n := \sum_{j=1}^{k} q_{j,n} \,, \quad g_n := \sum_{j=k+1}^{2k} q_{j,n} \,,$$

$$h_n := \sum_{j=2k+1}^{3k} q_{j,n} \,, \quad u_n := \sum_{j=3k+1}^{n} q_{j,n} \,.$$

On employing Proposition 135 we obtain $f_n \sim_1 g_n \sim_1 h_n$ and $u_n \lesssim_1 h_n$. We can write

$$\mathbb{I}_n = f_n + g_n + h_n + u_n \,.$$

Now put

$$f := \sum_{n=3}^{\infty} f_n \,, \quad g := \sum_{n=3}^{\infty} g_n \,,$$

$$h := \sum_{n=3}^{\infty} h_n \,, \quad u := \sum_{n=3}^{\infty} u_n \,.$$

By Proposition 135 $h \sim_1 g \sim_1 f$ and $u \lesssim_1 h$, such that $h + g + f + u = \mathbb{I}$. Now consider the following two possibilities:

(i) $\rho(u) = 0$, $\{h, g, f\} \subseteq M_3(\mathbb{R})$. If $\rho(h + g + f) \neq 0$, then ρ is non-zero on $M_3(\mathbb{R})$, which is a contradiction.

(ii) $\rho(u) = 1$ and $\{h, g, f, u\} \subseteq M_4(\mathbb{R})$, then u is contained in $M_4(\mathbb{R})$ thus $\rho(u) = 0$ which is a contradiction again.

We have showed that if ρ is a $0 - 1$ state on M, then it is identically zero, proving the statement of the Theorem. $\square$

Theorem 171 *Type I_2 JBW algebras always admit* $0 - 1$ *state.*

Proof. Let us denote by U_k a spin factor $U_k = \mathcal{H}_k \oplus \mathbb{R}\mathbb{I}$, where $\mathcal{H}_k$ is the Hilbert space of dimension $k \in \mathbb{N} \cup \{\infty\}$ and $\mathbb{I}$ the identity operator. We shall start the proof by investigating the explicit form of orthogonal projections in U_k.

Let $p \in U_k$ be a projection. Since $U_k = \mathcal{H}_k \oplus \mathbb{R}\mathbb{I}$, there exists a unique pair $(\xi, \lambda\mathbb{I}) \in \mathcal{H}_k \times \mathbb{R}\mathbb{I}$ such that $p = \xi + \lambda\mathbb{I}$. Projection is idempotent, i.e.

$$\begin{aligned} \xi + \lambda\mathbb{I} = p = p^2 = p \circ p = (\xi + \lambda\mathbb{I}) \circ (\xi + \lambda\mathbb{I}) = \\ = 2\lambda\xi + (< \xi, \xi > + \lambda^2)\mathbb{I} \,. \end{aligned}$$

Solving the equation above we get

$$\xi(2\lambda - \mathbb{I}) + (< \xi, \xi > + \lambda^2 - \lambda)\mathbb{I} = 0 \Rightarrow \lambda = \frac{1}{2}, \ ||\xi|| = \frac{1}{2} \,.$$

The vector $\xi \in \mathcal{H}_k$ such that $||\xi|| = \frac{1}{2}$ can be written in the form $\xi = \frac{1}{2}x$, where $x \in \mathcal{H}_k$ such that $||x|| = 1$. Hence if p and q are two minimal orthogonal projections in U_k with sum $\mathbb{I}$ then

$$p = \frac{1}{2}x + \frac{1}{2}\mathbb{I}\,,$$

$$q = -\frac{1}{2}x + \frac{1}{2}\mathbb{I}\,.$$

Define $\rho : U_k \mapsto \mathbb{R}$ by

$$\rho(\alpha x + \beta\mathbb{I}) = \alpha + \beta\,,$$

where x is the unit vector in $\mathcal{H}_k$ and $\alpha, \beta \in \mathbb{R}$. We will show that ρ is a finitely-additive probability measure on U_k with values in the set $\{0, 1\}$:

(i) $\rho(\mathbb{I}) = \rho(0 + 1 \cdot \mathbb{I}) = 0 + 1 = 1$,

(ii) $\rho(0) = \rho(0 + 0 \cdot \mathbb{I}) = 0 + 0 = 0$,

(iii) Let $n \in \mathbb{N}$ and $\{a_j\}_{j\in\{1,\dots,n\}} \subset U_k$ be set of pairwise orthogonal projections. It is evident that only two of them are non-zero, for instance $a_1 := p$, $a_2 :=$ q and $a_k = 0$ for $k \in \{3, \dots, n\}$. If p, q are non-zero orthogonal projections in U_k then $p + q = \mathbb{I}$. Hence $\rho\big(\sum_{j=1}^{n} a_j\big) = \rho(p + q) = \rho(\mathbb{I}) = 1 = 1 + 0 + \dots + 0 = \rho(p) + \rho(q) + \rho(0) + \dots + \rho(0) = \sum_{j=1}^{n} \rho(a_j)$.

(iv) Positivity of ρ follows from the fact that ρ takes values only in the set $\{0, 1\}$.

To complete the proof we shall need to extend the results to whole Type I_2 algebra with the help of the fact that Type I_2 algebra is isomorphic to direct sum of $C(X, U_k)$. Fix $x \in X$ and define $f, g \in C(X, U_k)$ to be minimal orthogonal projections with sum $\mathbb{I}$, i.e. $f(x) \circ g(x) = (f \circ g)(x) = 0$ and $(f + g)(x) = f(x) + g(x) = \mathbb{I}$. We have that

$$1 = \rho(\mathbb{I}) = \rho((f + g)(x)) = \rho(f(x) + g(x)) = \rho(f(x)) + \rho(g(x))\,,$$

$$\rho(0) = \rho((f \circ g)(x)) = \rho(f(x) \circ g(x)) = 0\,.$$

Thus ρ is a dispersion-free state on $C(X, U_k)$. The proof is complete. □

Corollary 172 *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be JBW algebra without associative and Type I_2 direct summand. There is no σ-field $(\Omega, \mathcal{D})$ and a map $a \mapsto f_a$ assigning to each self-adjoint element $a \in \mathcal{A}$ a $\mathcal{D}$-measurable real function f_a on Ω such that $f_{g(a)} = g \circ f_a$ for real quadratic polynomial g on $\mathbb{R}$.*

Proof. The proof is analogous to the proof in Theorem 165, i.e. assuming such field exists we find dispersion-free state on $\mathcal{A}$ which is a contradiction. □

Theorem 173 *$M_2(\mathbb{C})$ admits the Function principle.*

Proof. We have already proved that algebra $M_2(\mathbb{C})$ admits $0-1$ state (recall that it is a finitely-additive probability measure with values in the set $\{0,1\}$, hence it is bounded). By Proposition 90 such states extend uniquely to a $0-1$ quasi-functional on $M_2(\mathbb{C})$. Denote by $M_{s.a.}$ a self-adjoint part of $M_2(\mathbb{C})$ and by S the separating set of $0-1$ quasi-functionals on $M_2(\mathbb{C})$ defined on orthogonal pairs. For each $A \in M_{s.a.}$ define map $F_A(\mu) : S \mapsto \mathbb{R}$ by $F_A(\mu) = \mu(A)$. Since $A \in M_{s.a.}$, by virtue of spectral decomposition we have that $A = \lambda_1 p_1 + \lambda_2 p_2$, where p_1, p_2 are orthogonal idempotents with sum $\mathbb{I}$. Let $f \in C(\mathbb{R})$. Matrix function then has the form $f(A) = f(\lambda_1)p_1 + f(\lambda_2)p_2$. Hence

$$F_{f(A)}(\mu) = \mu(f(A)) = \mu(f(\lambda_1)p_1 + f(\lambda_2)p_2) = {}^1 f(\lambda_1)\mu(p_1) + f(\lambda_2)\mu(p_2)\,,$$

$$f \circ F_A(\mu) = f(\mu(A)) = f(\mu(\lambda_1 p_1 + \lambda_2 p_2)) = f(\lambda_1\mu(p_1) + \lambda_2\mu(p_2))\,.$$

Since $\mu \in S$ and p_1, p_2 have the properties stated above, it is either

$$\mu(p_1) = 1 \text{ and } \mu(p_2) = 0$$

or vice versa. Without loss of generality we shall assume that the first is true. Then evidently

$$F_{f(A)}(\mu) = f(\lambda_1) = f \circ F_A(\mu)\,,$$

completing the proof. □

Note 174 *Theorem 171 can be viewed as a result of Theorem 173. Indeed, the Function principle implies the existence of non-zero* $0-1$ *state.*

[1] Recall that quasi-functional is additive with respect to commuting elements. Indeed, p_1, p_2 being orthogonal projections with sum $\mathbb{I}$ commute with each other,

$$p \circ q = q \circ p = pq = qp = 0\,.$$

4. CONCLUSION

In this work we have studied and seeked possible generalization of Kochen-Specker theorem to Jordan algebras. The reason for doing this is simple: JBW algebras are more natural for conceptual fundamentals of quantum mechanics (as briefly summarized in the section 1.4). Going deep into highly non-trivial structural theory of Jordan algebras we have proved that JBW algebras without associative and Type I_2 direct summand do not admit $0-1$ state. Generalization of Kochen-Specker Theorem to JBW algebras has been proved (see Corollary 172, p. 43). Thus, interpreting quantum mechanics in JBW algebras in terms of hidden variables is not possible in dimensions equal or greater than 3.

The Function Principle (FP) has very remarkable statute in the theory of hidden variables because it is very closely related to existence of $0-1$ states; FP implies the existence of $0-1$ state. FP is not admitted in JBW algebras without associative and Type I_2 direct summand. By contrast to this the algebra of complex matrices $M_2(\mathbb{C})$, spin factor and Type I_2 JBW algebra admit the FP.

APPENDIX

ALGEBRAS OF QUATERNIONS AND OCTONIONS

I. INTRODUCTION

In this paragraph we briefly summarize the algebras of complex numbers, quaternions and octonions. Essential for all these algebras is the existence of the real unit element e_0 and a different number of adjoined hyper-complex units e_n. For the case of complex numbers $n = 1$, for quaternions $n = 3$ and for octonions $n = 7$. The square of the unit element e_0 is always positive and the squares of the *hyper-complex units* e_n can be positive or negative

$$e_n^2 = \pm e_0 \,. \tag{.1}$$

Mainly the negative sign in (.1) is used, in that case norm of the algebra is positively defined. Taking positive sign in (.1) leads to so-called split algebras with the equal number of terms with the positive and negative signs in the definition of their norms.

II. QUATERNIONS

William Hamilton's discovery of quaternions in 1843 was the first time in history when the concept of two-dimensional numbers was successfully generalized.

General element of the quaternion algebra can be written using only the two basis elements i and j in the form

$$q = a + bi + (c + di)j \,,$$

where $a, b, c, d \in \mathbb{R}$. The third basis element of the quaternion (ij) is possible to be obtained by composition of the first two.

The quaternion reverse to q, the conjugated quaternion q^* can be constructed using the properties of the basis units under the conjugation (reflection)

$$i^* = -i, \quad j^* = -j, \quad (ij)^* = -(ij) \,.$$

When the basis elements i and j are imaginary, i.e. $i^2 = j^2 = -1$ we have Hamilton's quaternion with the positively defined norm

$$N = qq^* = q^*q = a^2 + b^2 + c^2 + d^2 \,.$$

For the positive squares $i^2 = j^2 = 1$ we have the algebra of *split-quaternions*. The unit elements i, j have the properties of real unit vectors. The norm of a split-quaternion

$$N = qq^* = q^*q = a^2 - b^2 - c^2 + d^2 \,,$$

has (2+2)-signature and in general is not positively defined.

The quaternion algebra is associative and therefore can be represented by matrices. We get the simplest non-trivial representation of the split-quaternion basis units if we choose the real Pauli matrices accompanied by the unit matrix.

The independent unit basis elements of split quaternions i and j have the following matrix representation

$$i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The third unit basis element (ij) is formed by multiplication of i and j and has the representation

$$(ij) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The squares of Pauli matrices give the unit matrix

$$(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with different signs, i.e.

$$i^2 = j^2 = (1) \,, \quad (ij)^2 = -(1) \,.$$

Conjugation of the unit elements i and j means changing of signs of matrices i, j and (ij), thus

$$ii^* = jj^* = -(1) \,, \quad (ij)(ij)^* = (1) \,.$$

III. Octonions

Octonions were discovered by Graves and Cayley in 1844-1845. For the construction of the octonion algebra with the general element

$$O = a_0 e_0 + a_n e_n \,, \quad n = 1, 2, \ldots, 7$$

where e_0 is the unit element and a_0, a_n are real numbers, the multiplication law of its eight basis units e_0, e_n usually is given. For the case of ordinary octonions

$$e_0^2 = e_0 \,, \quad e_n^2 = -e_0 \,, \quad e_0^* = e_0 \,, \quad e_n^* = -e_n \,,$$

and norms are positively defined

$$N = OO^* = O^*O = a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \,.$$

The multiplication rules obey the form

$$(e_n e_m) = -\delta_{nm} e_0 + \epsilon_{nmk} \epsilon_k \,,$$

where δ_{nm} is the Kronecker symbol and ϵ_{nmk} is the fully anti-symmetric tensor with the value $\epsilon_{nmk} = +1$ for the following values of indices

$$nmk = 123, 145, 176, 246, 257, 347, 365 \,.$$

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