CZECH TECHNICAL UNIVERSITY IN PRAGUE
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Review work

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Schrödinger operators with moving point perturbation

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1. Introduction

First, let me start with an example. Let us consider a two-dimensional electric particle of charge e (i.e. electron) and mass m in the presence of a uniform magnetic field perpendicular to the system plane. At this point solving the eigenvalue/eigenfunction problem is well analysed in many quantum mechanics books. But provided that such particle interacts with a point perturbation with varying support, we come to remarkable spectral problem.

The Hamiltonian of the system mentioned above can be chosen in the form

$$H = -[(\partial_x - iBy)^2 + \partial_y^2] + \beta\delta(y) \ , \tag{1}$$

where B is the strength of the magnetic field, $\delta(y)$ is the Dirac delta function and β is the strength of the perturbation potential. By virtue of the Fourier transform we can rewrite (1) in the momentum representation as follows

$$H = -\partial_y^2 + B^2(y+q)^2 + \beta\delta(y) \ , \tag{2}$$

where $q = \frac{p_x}{B}$ is the location of the perturbation on the y axis. The Hamiltonian of the unperturbed system

$$H_0 = -\partial_y^2 + V(y) \ , V(y) = B^2(y+q)^2 \tag{3}$$

has a discrete spectrum. The principal aim of this work is to analyse the q-dependence of the eigenvalues of the operator H.

There are more examples of systems for which the investigation of eigenvalues reduces to spectral properties of operators having the form (2). The operator (2) can be chosen to describe a short-range impurity in a potential well, where $V(y)$ would be the confining potential of the well and β would characterize the impurity potential ($\beta < 0$ for attractive impurities and $\beta > 0$ for repulsive ones). Besides, taking $\beta > 0$ we obtain Hamiltonian of a charged particle tunneling through moving barrier.

2. Spectral analysis of perturbed Hamiltonian H

First, let us summarize basic results of the spectral properties of the operator H_0 in $L^2(\mathcal{R})$ denoted by the differential equation

$$H = -\partial_y^2 + V(y) \ , \tag{4}$$

where the potential V is bounded from below and $V \in L^2_{loc}(\mathcal{R})$. Such operator belongs to class of so-called Schrödinger operators.In this case the expression (4) defines an essentially self-adjoint operator on $C_0^\infty(\mathcal{R})$. It's obvious that in our case $lim_{y\to\pm\infty} V(y) = +\infty$

The spectrum of H_0 consists of so-called Landau levels, that is $\sigma(H_0) = \{\lambda_n = B(2n+1) \,|\, n \in \mathcal{N}_0\}$. We see that $\sigma(H_0)$ is discrete, bounded from below and $\lambda_j < \lambda_k$ for all $j < k$, $j, k \in \mathcal{N}_0$. The multiplicity of each eigenvalue is 1. The eigenfunctions can be written in the form

$$\Phi_n = e^{-\frac{\xi^2}{2}} H_n(\xi) \ ,$$

where $H_n = (-1)^n e^{\xi^2} (\frac{d}{d\xi})^n e^{-\xi^2}$ are the Hermitean polynomials and $\xi = \sqrt{B}(y + q)$.

These solutions can be written [1] in terms of confluent hypergeometric functions

$$\psi_\pm = \sqrt{\pi} e^{-\frac{B(y+q)^2}{2}} \Big[\frac{M(\frac{B-\lambda}{4B}, \frac{1}{2}; B(y+q)^2)}{\Gamma(\frac{3B-\lambda}{4B})} \mp$$

$$\mp 2\sqrt{B}(y+q) \frac{M(\frac{3B-\lambda}{4B}, \frac{3}{2}; B(y+q)^2)}{\Gamma(\frac{B-\lambda}{4B})} \Big] \ , \tag{5}$$

where

$$M(a, b; z) = \sum_{j=0}^{+\infty} \frac{(a)_j}{(b)_j} \frac{z^j}{j!} \ ,$$

$$(a)_n = a(a+1)(a+2)...(a+n-1) \ , a_0 = 1$$

and $\psi_\pm$ is $L^2(\mathcal{R})$ in $\pm\infty$.

At this point we have an orthonormal system Φ_n of eigenfunctions of the operator H_0 corresponding to the eigenvalue λ_n. The resolvent $R_0(\zeta)$ of the operator H_0: $R_0(\zeta) = (H_0 - \zeta)^{-1}$ has an integral kernel $G_0(y, z; \zeta)$ which we call the Green function of H_0. We can write [4]

$$G_0(y, z; \zeta) = \sum_{n=0}^{\infty} \sum_{k=1}^{\kappa(n)} (\lambda_n - \zeta)^{-1} \Phi_{n,k}(y) \overline{\Phi_{n,k}(z)} \ , \tag{6}$$

where $\kappa(n)$ is the multiplicity of eigenvalue λ_n. In our case $\kappa(n) = 1$ for all $n \in \mathcal{N}$. By virtue of the Mercer theorem the series (6) converges locally uniformly on $\mathcal{R}^2 \times \rho(H_0)$.

Now we shall consider the self-adjoint operator H in the space $L^2(\mathcal{R})$ defined by

$$H = H_0 + \beta\delta(y - q) \ , \tag{7}$$

which we obtain by argument shift in (2). According to the *Krein resolvent formula* (see *appendix*) H is defined by its resolvent $R(\zeta)$ with the integral kernel

$$G(y, z; \zeta) = G_0(y, z; \zeta) - [Q(\zeta) + \beta^{-1}]^{-1} G_0(y, q; \zeta) G_0(q, z; \zeta) \ , \tag{8}$$

where $Q(\zeta)$ is the so-called Krein function. From the condition

$$\int_{\mathcal{R}} |G_0(y, z; \zeta)|^2 dy < \infty \ , \forall y \in \mathcal{R} \ and \, \zeta \in \rho(H_0)$$

we get $Q(\zeta) \equiv Q(\zeta, q) = G_0(q, q; \zeta)$. Hence

$$Q(\zeta) = \sum_{n=0}^{\infty} \sum_{k=1}^{\kappa(n)} (\lambda_n - \zeta)^{-1} |\Phi_{n,k}(q)|^2. \tag{9}$$

The following lemmas will be useful for finding $\sigma(H)$.

Lemma 1. *For every $y_0 \in \mathcal{R}$ the relation $\Phi_{n,k}(y_0) \neq 0$ is valid for infinitely many values of the index n.*

Lemma 2. *For a fixed $q \in \mathcal{R}$ the following statements are true:*

(i) *$\zeta \mapsto Q(\zeta, q))$ is a meromorphic function of $\zeta \in \mathcal{C}$ with infinitely many simple poles. The poles of this function are exactly those points λ_n for which there exists $k \in \{1, ..., \kappa(n)\}$ such that $\Phi_{n,k}(q) \neq 0$.*

(ii) *$\frac{\partial Q(\zeta,q)}{\partial \zeta} > 0$ if $\zeta \in \rho(H_0) \bigcap \mathcal{R}$.*

(iii) *For real E the function $E \mapsto Q(E, q)$ increases from $-\infty$ to $+\infty$ as E varies between any two neighbouring poles.*

(iv) *For every $\zeta \in H_0$ the function $y \mapsto G_0(y, q; \zeta)$ does not vanish identically in $\mathcal{R}$.*

(v) *$Q(\zeta, q) \to 0$ as $\mathcal{R}\zeta \to -\infty$ locally uniformly with respect to $q \in \mathcal{R}$.*

Proofs of this lemmas can be found in [4].

Proceeding further, let us fix $q \in \mathcal{R}$. Considering the function $\zeta \mapsto Q(\zeta, q)$ which has the form (9) we have the set of all its poles. Let us arrange these poles in an increasing sequence $\lambda_{n_0} < \lambda_{n_1} < ... < \lambda_{n_k} < ...$. Further we shall assume that if a point $\zeta = \lambda_n$ does not belong to the sequence, then the function $Q(\zeta, q)$ is equal to its continuous extension to the point. The resolvent $R(\zeta) = (H - \zeta)^{-1}$ is a bounded operator for all $\zeta \in \rho(H) = \mathcal{C} \setminus \sigma(H)$ and has the integral kernel $G(y, z; \zeta)$. According to *Lemma* 1 and *Lemma* 2 for any $\zeta \neq \lambda_{n_k}$ the function $G_0(., q; \zeta)$ does not vanish identically on $\mathcal{R}$. From here we get that $\sigma(H)$ is determined by the pole of the resolvent, that is

$$Q(\zeta, q) + \beta^{-1} = 0. \tag{10}$$

The equation (10) is known as *dispersion equation*. Said in other words, every ζ satisfying the dispersion equation belongs to the spectrum $\sigma(H)$. H being the self-adjoint operator results in $\sigma(H) \subset \mathcal{R}$. Thus equation (10) has only real solutions.

Lemma 2 says that for real E the function $E \mapsto Q(E, q)$ increases from $-\infty$ to $+\infty$ as E varies between any two neighbouring poles, which means that in each interval $(\lambda_{n_{j-1}}, \lambda_{n_j})$, $j \in \mathcal{N}$, the equation $Q(E, q) = -\beta^{-1}$ has a solution. These solutions can be arranged in an increasing sequence $\varepsilon_1(q) < \varepsilon_2(q) < ...$. If $\beta > 0$, equation (10) has no other solutions; otherwise it has an additional solution $\varepsilon_0(q)$ lying in the interval $(-\infty, \lambda_{n_0})$ [4].

In the case of Schrödinger operator we will use the following representation of the Green function G_0. Let $\zeta_0 \in \rho(H_0)$ and $\Psi_\pm(y; \zeta_0)$ are functions (5). Then

$$G_0(y, z; \zeta_0) = \frac{\Psi_+(max(x, y); \zeta_0)\Psi_-(min(y, z); \zeta_0)}{\omega(\zeta_0)}, \tag{11}$$

where

$$\omega(\zeta_0) = W(\Psi_+(y; \zeta_0), \Psi_-(y; \zeta_0))$$

is the Wronskian of $\Psi_+(y; \zeta_0)$ and $\Psi_-(y; \zeta_0)$. In a neighbourhood of the point ζ_0 the functions $\Psi_\pm(y; \zeta)$ can be chosen to be analytical ones of ζ. It follows from the elementary properties of the resolvent of H_0 which has the integral kernel, the Green function G_0, that the function $\omega(\zeta_0)$ has only simple zeros which coincide with the eigenvalues λ_n of the operator H_0. From (9) and (11) we obtain

$$Q(\zeta, q) = \frac{\Psi_+(q; \zeta)\Psi_-(q; \zeta)}{\omega(\zeta)}. \tag{12}$$

Because for $\zeta \in \mathcal{R}$ and $\zeta < \lambda_0$ the functions $\Psi_\pm(y;\zeta)$ can be chosen to be strictly positive for all $y \in \mathcal{R}$, $\frac{\partial Q}{\partial \zeta} > 0$ and $lim_{\zeta \to -\infty} Q(\zeta, q) = 0$ for all $q \in \mathcal{R}$, we have

$$\omega(\zeta) > 0. \tag{13}$$

In our case the Wronskian has the form

$$\omega(\zeta_0) = \frac{4\pi\sqrt{B}}{\Gamma(\frac{3B-\zeta_0}{4B})\Gamma(\frac{B-\zeta_0}{4B})}.$$

Because $\Gamma(z)\Gamma(z+\frac{1}{n})...\Gamma(z+\frac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-nz}\Gamma(nz)$ (see [1]), taking $z = \frac{B-\zeta_0}{2B}$ and $n = 2$, we can rewrite the Wronskian as follows

$$\omega(\zeta_0) = \frac{\sqrt{\pi B} 2^{\frac{3B-\zeta_0}{2B}}}{\Gamma(\frac{B-\zeta_0}{2B})}.$$

The spectrum of H consists of all $\varepsilon_n = \varepsilon_n(q)$ that satisfy the dispersion equation of the form

$$\sqrt{\frac{\pi}{B}} 2^{\frac{\varepsilon_n(q)-3B}{2B}} \Gamma\Big(\frac{B-\varepsilon_n(q)}{2B}\Big) \times$$

$$\times\, e^{-Bq^2}\Big[\frac{M(\frac{B-\varepsilon_n(q)}{4B}, \frac{1}{2}; Bq^2)}{\Gamma(\frac{3B-\varepsilon_n(q)}{4B})} - 4Bq^2 \frac{M(\frac{3B-\varepsilon_n(q)}{4B}, \frac{3}{2}; Bq^2)}{\Gamma(\frac{B-\varepsilon_n(q)}{4B})}\Big] + \beta^{-1} = 0. \tag{14}$$

Theorem 1. *The spectrum of the operator H is a discrete one and consists of four non-intersecting parts σ_k $(k = 1, 2, 3, 4)$ which are described as follows.*

(i) *The part σ_1 consists of all the solutions $\varepsilon_j(q)$ of equation* (14) *which are different from the eigenvalues λ_n of the operator H_0. Each solution $\varepsilon_j(q)$ is a simple eigenvalue of the operator H, the corresponding normalized eigenfunction has the form*

$$\widehat{\Phi}_j(y) = G_0(y, q; \varepsilon_j(q))\Big[\frac{\partial Q}{\partial E}(\varepsilon_j(q), q))\Big]^{-\frac{1}{2}}.$$

(ii) *The part σ_2 consists of all the eigenvalues λ_n of the operator H_0 satisfying equation* (14) *and such that $\Phi_{m,k}(q) = 0$ for all $k = 1, ..., \kappa(m)$. Each point of σ_2 is an eigenvalue of the operator H of multiplicity $\kappa(m)+1$; an orthonormal bases of the corresponding eigensubspace is spanned by the functions $\Phi_{m,k}$ and the function*

$$\Psi_m(y) = G_0(y, q; \lambda_m)\Big[\frac{\partial Q}{\partial E}(\lambda_m, q)\Big]^{-\frac{1}{2}}.$$

(iii) *The part* σ_3 *consists of all the eigenvalues* λ_m *of the operator* H_0 *which do not satisfy equation* (14) *and such that* $\Phi_{m,k}(q) = 0$ *for all* $k = 1, ..., \kappa(m)$*. Each point of* σ_3 *is an eigenvalue of the operator* H *of multiplicity* $\kappa(m)$*; an orthonormal bases of the corresponding eigensubspace is spannes by the functions* $\Phi_{m,k}$*.*

(iv) *The part* σ_4 *consists of all the eigenvalues* λ_m *of the operator* H_0 *for which* $\kappa(m) \geq 2$ *and which are poles of the function* $\zeta \mapsto Q(\zeta, q)$*. (i.e., for which* $\Phi_{m,k}(q) \neq 0$ *at least for one* k*). Each point of* σ_4 *is an eigenvalue of the operator* H *of multiplicity* $\kappa(m)-1$*; the corresponding eigensubspace is the orthogonal complement of the function*

$$\widehat{F}_m(y) = \overline{\Phi_{m,1}(q)}\Phi_{m,1}(y) + ... + \overline{\Phi_{m,\kappa(m)}(q)}\Phi_{m,\kappa(m)}(y)$$

in the eigensubspace of the operator H_0 *associated with* λ_m*.*

Proof of *Theorem* 1 can be found in [4].

Lemma 3. *Let the potential* $V(y)$ *be an even function. If* $\Phi_m(q) = 0$*, then* $Q(\lambda_m, q) = 0$*.*

Proof of *Lemma* 3 can be found in [4].

In our case the eigenfunctions $\Phi_{m,k}$ of the Schrödinger operator H_0 satisfy $\Phi_{m,k} = \Phi_{m,1}$. That is $k \in \{1, ..., \kappa(m)\} = \{1\}$ (each eigenvalue λ_m, $m \in \mathcal{N}$ has multiplicity $\kappa(m) = 1$) . Therefore Lemma 3 and Theorem 1 both imply that the set σ_4 from Theorem 1 is empty.

Lemma 3 and Theorem 1 also imply:

Corollary. *Let the potential* $V(y)$ *be an even function. Then:*

(a) *For* $\beta \neq +\infty$ *all the eigenvalues of the operator* H *are simple and belong either to the set* σ_1 *or to the set* σ_3 *from Theorem* 1*.*

(a) *For* $\beta = +\infty$ *the set* σ_3 *is empty, and all the eigenvalues* λ_m *from the spectrum of* H *are doubly degenerate.*

(b) λ_0 *does not belong to the spectrum of all the operators* H*, while every eigenvalue of the form* λ_{2m+1} *belongs to the spectrum of any operator* $H = H(q = 0, \beta)$*.*

Our next aim is to analyse the q-dependence of the eigenvalues $\varepsilon_j(q)$. Let us introduce the notations first:

$$\lambda_{-1} = -\infty$$

$$X_{-1} = \mathcal{R}$$

$$X_n = \{q \in \mathcal{R} : \Phi_{n,j}(q) \neq 0 \text{ for all } j = 1, ..., \kappa(n)\} \quad (n \geq 0).$$

The following theorem will be useful for us:

Implicit function theorem. *If $f_1, ..., f_n$ are continuously differentiable functions on a neighbourhood of the point $(x_0, y_0) = (x_1^0, ..., x_n^0, y_1^0, ..., y_m^0) \in \mathcal{R}^n \times \mathcal{R}^m$, if $f_1(x_0, y_0) = f_2(x_0, y_0) = ... = f_n(x_0, y_0) = 0$, and if the $n \times n$ matrix $(\mathcal{J})_{i,j} = \frac{\partial f_i}{\partial x_j}$ is nonsingular at (x_0, y_0), then there is a neighbourhood U of the point $y_0 = (y_1^0, ..., y_m^0)$ in $\mathcal{R}^m$, there is a neighbourhood V of the point $x_0 = (x_1^0, ..., x_n^0)$ in $\mathcal{R}^n$, and there is a unique mapping $\varphi : U \mapsto V$ such that $\varphi(y_0) = x_0$ and $f_1(\varphi(y), y) = ... = f_n(\varphi(y), y) = 0$ for all $y \in U$. Furthermore, φ is continuously differentiable.*

In our case $m, n = 1$, $f(x_0, y_0) = Q(x_0, y_0) + \beta^{-1} = 0$, $(E_k(q), q) = (x_0, y_0) \in (\lambda_{k-1}, \lambda_k) \times (X_{k-1} \bigcap X_k)$. The function $E \mapsto Q(E, q)$ is continuously differentiable on each set $(\lambda_{k-1}, \lambda_k) \times (X_{k-1} \bigcap X_k)$. For any fixed $q \in X_{k-1} \bigcap X_k$, the function $E \mapsto Q(E, Q)$ is real-analytic on the interval $(\lambda_{k-1}, \lambda_k)$ as a function of E. Further we have

$$\frac{\partial f}{\partial x} \equiv \frac{\partial}{\partial E}\Big[Q(E(q), q) + \beta^{-1}\Big] = \frac{\partial Q}{\partial E}(E(q), q).$$

It follows from Lemma 2 that $\frac{\partial Q}{\partial E} \neq 0$ on the set $(\lambda_{k-1}, \lambda_k) \times (X_{k-1} \bigcap X_k)$, which implies that $det|\mathcal{J}| \neq 0$ therefore the matrix $\mathcal{J}$ is non-singular. Thus the premises of the Implicit function theorem are satisfied. Therefore the equation (14) for any $q \in X_{k-1} \bigcap X_k$ has a unique solution $E_k(q) \in (\lambda_{k-1}, \lambda_k)$. Let us recall that $k \in \mathcal{N}$ for $\beta > 0$ or $k \in \mathcal{N}_0 = \mathcal{N} \bigcup \{0\}$ if $\beta < 0$.

Proposition 1. *Each function $E_k(q)$ has a continuous extension on the whole real line $\mathcal{R}$.*

The result of the Implicit function theorem and Proposition 1 is the following theorem:

Theorem 2. *For each fixed* $0 \neq \beta \in (-\infty, +\infty]$ *there is a sequence* $(E_n(q))$ *of continuous functions of* $q \in \mathcal{R}$ *such that the following properties are satisfied:*

(i) $\lambda_{n-1} \leq E_n(q) \leq \lambda_n$ *for all* n.

(ii) $E_n(q) \leq E_{n+1}(q)$ *for all admissible values of* n.

(iii) *For each* q *the set consisting of all* $E_n(q)$ *and all the numbers* λ_m *with* $\kappa(m) \geq 2$ *is the complete collection of all the eigenvalues of the operator* $H = H(q, \beta)$.

(iv) *If* $\lambda_{n-1} < E_n(q) < \lambda_n$, *then* $E_n(q)$ *is the unique solution of the dispersion equation* (14).

(v) *If* λ_n *is a pole of the function* $\zeta \mapsto Q(\zeta, q)$ *(i.e. if* $\Phi_{n,k}(q) \neq 0$ *at least for one* k*), then* $\lambda_{n-1} < E_n(q) < \lambda_n$.

(vi) *Provided that* λ_n *is not a pole of the function* $\zeta \mapsto Q(\zeta, q)$ *(i.e.* $\Phi_{n,k}(q) = 0$ *for all* k*), we have the following:*

(a) *If* $Q(\lambda_n, q) + \beta^{-1} < 0$, *then* $E_n(q) = \lambda_n < E_{n+1}(q)$.

(b) *If* $Q(\lambda_n, q) + \beta^{-1} > 0$, *then* $E_n(q) < \lambda_n = E_{n+1}(q)$.

(c) *If* $Q(\lambda_n, q) + \beta^{-1} = 0$, *then* $E_n(q) = \lambda_n = E_{n+1}(q)$.

Definition. The points for which condition (iv) of Theorem 2 is satisfied we shall call the *non-singular points* of the function $E_n(q)$. If condition (vi)/(a), (vi)/(b) or (vi)/(c) is satisfied, then q will be called *singular point of kind* (a), (b) or (c), respectively.

In the next step we will evaluate the energy of bound state of the perturbed Hamiltonian H. It can be proved that for $\beta \neq +\infty$ we have $Dom(H_0) = Dom(H) \subset C(\mathcal{R})$. Further if $\psi \in Dom(H)$, then $q(\psi) = q_0(\psi) + \beta|\psi(q)|^2$ where q and q_0 are the quadratic forms associated with the operators H and H_0, respectively; $|\psi(q)|^2 \ll h_0(\psi)$ for any $\psi \in Dom(H_0)$. Thus for all $q \in \mathcal{R}$ we obtain [4]

$$-\frac{\beta^2}{4} \leq E_0(q)\,. \tag{15}$$

Let us denote the asymptotics of the so-called Krein function (as a function of $q \in \mathcal{R}$). The potential $V(y)$ satisfies $\lim_{y \to \pm\infty} V(y) = +\infty$. The set of all discontinuity points of the function $V(y)$ is empty. It's obvious that $V(y)$ decreases on the half-line $y \leq 0$ and increases on the half-line

$y \geq 0$. $V(y)$ is differentiable and there exists a point $y_0 > 0$ such that the derivative $V'(y)$ is a locally absolutely continuous function for $|y| > y_0$, and $\int_{|y|>y_0} |\frac{5V'^2(y)}{4V^3(y)} - \frac{V''(y)}{V^2(y)}| \sqrt{V(y)} dx < \infty$. Then for any $E \in \mathcal{R} \setminus \sigma(H_0)$ $Q(E,q) \sim V(q)^{-\frac{1}{2}}$ for $q \to \pm\infty$ which implies that $lim_{q \to \pm\infty} Q(q,E) = 0$.

Moreover, provided that the set of all discontinuity points of the function $V(y)$ is finite (or empty), then if $\lambda_n \neq E \in \mathcal{R}$ for every n, $\Psi_+(s;E) \neq 0$ and $Q'(E,s) = 0$ ($s \in \mathcal{R}$) we have $Q''(E, s \pm 0) < 0$ [4].

The previous hypothesis and Theorem 2 lead to a final result of this work: for the Schrödinger operator H_0 the functions $E_k(q)$ have the following properties:

(i) Let $\beta < 0$ and $k \geq 1$. Then the function $E_k(q)$ attains the maximum λ_k at those points $q \in \mathcal{R}$ for which $\Phi_k(q) = 0$ and has no other points of local maximum. Besides, $E_k(q) > \lambda_{k-1}$ for all $q \in \mathcal{R}$.

(ii) Let $0 < \beta < \infty$ and $k \geq 1$. Then the function $E_k(q)$ attains its minimum λ_{k-1} at those points $q \in \mathcal{R}$ for which $\Phi_{k-1}(q) = 0$ and has no other points of local minimum. Besides, $E_k(q) < \lambda_k$ for all $q \in \mathcal{R}$.

(iii) Let $\beta = \infty$ and $k \geq 1$. Then:

(a) $\lambda_0 < E_1(q)$ for all $q \in \mathcal{R}$.

(b) The function $E_1(q)$ attains its maximum which is equal to λ_1 at that point q for which $\Phi_1(q) = 0$ and has no other points of local maximum or local minimum.

(c) For $k \geq 2$ the function $E_k(q)$ attains its minimum λ_{k-1} at those points $q \in \mathcal{R}$ for which $\Phi_{k-1}(q) = 0$, and attains its maximum λ_k at those points $q \in \mathcal{R}$ for which $\Phi_k(q) = 0$; this function has no other points of local maximum or local minimum.

(iv) Let the potential $V(y)$ fulfills the properties as discussed above. Then:

(a) If $0 < \beta \leq \infty$, then $lim_{q \to \pm\infty} E_k(q) = \lambda_{k-1}$ for $k \geq 1$.

(b) If $\beta < 0$, then $lim_{q \to \pm\infty} E_k(q) = \lambda_k$ for $k \geq 0$. For the same values of β, there exists a unique point q_0 such that $E_0'(q_0) = 0$, moreover, $E_0'(q) > 0$ if $q > q_0$, and $E_0'(q) < 0$ if $q < q_0$.

3. Conclusion

In this work we have showed that the spectrum of perturbed Hamiltonian $H = H(q, \beta)$ apart from so-called Landau levels consists of absolutely continuous bands varying between two adjacent Landau levels without crossing them. For $q \to \pm\infty$ these bands adhere to corresponding Landau levels, which depends on the parameter β. Moreover, there exists unique point at which the bound state of H attains its maximum/minimum according to the value of β.

4. Appendix

Self-adjoint extensions of symmetric operators

Assume $\dot{A}$ to be a densely defined, closed, symmetric operator in some Hilbert space $\mathcal{H}$ with deficiency indices (1,1). If

$$\dot{A}^*\phi(z) = z\phi(z), \quad \phi(z) \in \mathcal{D}(\dot{A}^*), \;\; z \in \mathcal{C} \setminus \mathcal{R}, \tag{16}$$

we have

Theorem 4.1. *All self-adjoint extensions A_θ of $\dot{A}$ may be parametrized by a real parameter $\theta \in [0, 2\pi)$ where*

$$\mathcal{D}(A_\theta) = \{g + c\phi_+ + ce^{i\theta}\phi_- \,|\, g \in \mathcal{D}(\dot{A}), c \in \mathcal{C}\},$$

$$A_\theta(g + c\phi_+ + ce^{i\theta}\phi_-) = \dot{A}g + ic\phi_+ - ice^{i\theta}\phi_-, \quad 0 \leq \theta < 2\pi, \tag{17}$$

and

$$\phi_\pm = \phi(\pm i), \quad \|\phi_+\| = \|\phi_-\|. \tag{18}$$

Concerning resolvents of self-adjoint extensions of $\dot{A}$ we state

Theorem 4.2 (Krein's formula) *Let B and C denote any self-adjoint extensions of $\dot{A}$. Then we have that*

$$(B - z)^{-1} - (C - z)^{-1} = \lambda(z)(\phi(\overline{z}), .)\phi(z), \quad z \in \rho(B) \cap \rho(C), \tag{19}$$

where $\lambda(z) \neq 0$ for $z \in \rho(B) \cap \rho(C)$ and λ and ϕ may be chosen to be analytic in $z \in \rho(B) \cap \rho(A)$. In fact, $\phi(z)$ may be defined as

$$\phi(z) = \phi(z_0) + (z - z_0)(C - z)^{-1}\phi(z_0), \quad z \in \rho(C), \tag{20}$$

where $\phi(z_0)$, $y_0 \in \mathcal{C} \setminus \mathcal{R}$, is a solution of (26) *for $z = z_0$ and $\lambda(z)$ satisfies*

$$\lambda(z)^{-1} = \lambda(z')^{-1} - (z - z')(\phi(\overline{z})\phi(y')) \quad z, z' \in \rho(B) \cap \rho(C)\,, \tag{21}$$

if $\phi(z)$ is chosen according to (20)*.*

Next we turn to the general case and assume that $\dot{A}$ is densely defined, closed symmetric operator in $\mathcal{H}$ with deficiency indices (N, N), $N \in \mathcal{N}$. Let B and C be two self-adjoint extensions of $\dot{A}$ and denote by $\mathring{A}$ the maximal common part of B and C (i.e., $\mathring{A}$ obeys $\mathring{A} \subseteq B$, $\mathring{A} \subseteq C$ and $\mathring{A}$ extends any operator A' that fulfills $A' \subseteq B$, $A' \subseteq C$). Let M, $0 < M \leq N$, be the deficiency indices of $\mathring{A}$ and let $\{\phi_1(z), ..., \phi_M(z)\}$ span the corresponding deficiency subspace of $\mathring{A}$, i.e.,

$$\mathring{A}^* \phi_m(z) = z\phi_m(z), \quad \phi_m(z) \in \mathcal{D}(\mathring{A}^*), \ \ m = 1, ..., M, \ \ z \in \mathcal{C} \setminus \mathcal{R}, \tag{22}$$

and $\{\phi_1(z), ..., \phi_M(z)\}$ are linearly independent. Then the analog of theorem 4.2 reads

Theorem 4.3 (Krein's formula for deficiency indices $N > 1$) *Let B, C, $\mathring{A}$, and $\dot{A}$ be as above. Then*

$$(B-z)^{-1} - (C-z)^{-1} = \sum_{m,n=1}^{M} \lambda_{mn}(z)(\phi_n(\overline{z}), .)\phi_m(z), \quad z \in \rho(B) \cap \rho(C), \tag{23}$$

where the matrix $\lambda(z)$ is nonsingular for $z \in \rho(B) \cap \rho(C)$ and $\lambda_{mn}(z)$ and $\phi_m(z)$, $m, n = 1, ..., M$, may be chosen to be analytic in $z \in \rho(B) \cap \rho(C)$. In fact, $\phi_m(z)$ may be defined as

$$\phi_m(z) = \phi_m(z_0) + (z - z_0)(C - z)^{-1}\phi_m(z_0), \quad m = 1, ..., M, \ \ z \in \rho(C), \tag{24}$$

where $\phi_m(z_0), m = 1, ..., M$, $z_0 \in \mathcal{C} \setminus \mathcal{R}$, are linearly independent solutions of (22) *for $z = z_0$ and the matrix $\lambda(z)$ satisfies*

$$[\lambda(z)]^{-1}_{mn} = [\lambda(z')]^{-1}_{mn} - (z - z')(\phi_n(\overline{z}), \phi_m(z')), \quad m, n = 1, ..., M,$$
$$z, z' \in \rho(B) \cap \rho(C), \tag{25}$$

if the $\phi_m(z), m = 1, ..., M$, are defined according to (24).

In general, we have

$$(B-z)^{-1} - (C-z)^{-1} = \sum_{m,n=1}^{N} \widetilde{\lambda}_{mn}(z)(\widetilde{\phi}_n(\overline{z}), .)\widetilde{\phi}_m(z), \ \ z \in \rho(B) \cap \rho(C), \tag{26}$$

where now $\widetilde{\phi}_m(z)$, $m = 1, ..., N$, are linearly independent solutions of

$$\mathring{A}^* \widetilde{\phi}_m(z) = z\widetilde{\phi}_m(z), \quad \widetilde{\phi}_m(z) \in \mathcal{D}(\mathring{A}^*), \ \ m = 1, ..., N, \ z \in \mathcal{C} \setminus \mathcal{R}, \tag{27}$$

and, in general, $det\, \widetilde{\lambda}(z) \equiv 0$.

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